# Graduate Cosmology Spring 2025
# Homework 1

due by 11:59pm on Wednesday, February 5th 2025.

**Problem 1: conformal coordinates and scalar fields [17 points]**

1) In class we computed the Christoffel symbols of the FLRW metric in $(t, x^i)$ coordinates, i.e. with $x^0 = t$ such that the metric is $ds^2 = -dt^2 + a^2(t)\gamma_{ij}(\vec{x})dx^i dx^j$. Compute the Christoffel symbols of the FLRW metric in conformal coordinates $(\eta, x^i)$, i.e. with $x^0 = \eta$, such that $ds^2 = a^2(\eta)[-d\eta^2 + \gamma_{ij}(\vec{x})dx^i dx^j]$. To keep your notation compact, use a prime to denote partial derivatives with respect to $\eta$, i.e. $f' \equiv \frac{\partial f}{\partial \eta}$. *[0.5 point per coefficient; 3 points total]*

2) Explicitly write the geodesic equation $P^\mu \nabla_\mu P^\nu = 0$ for a massless particle and show that it implies that the energy observed by comoving observers $E_{\rm obs} = -U^\mu_{\rm obs} P^\mu$ scales as $E_{\rm obs} \propto 1/a(\eta)$. Make sure to explicitly write the 4-velocity $U^\mu_{\rm obs}$ of comoving observers in conformal coordinates. *[2 points for comoving observer's 4-velocity, 2 points for proof that $E_{\rm obs} \propto 1/a$; 4 points total]*

For the remainder of this Problem we specialize to **spatially flat** FLRW spacetime.

3) A free, massless scalar field $\phi$ satisfies the equation of motion $\nabla_\mu \nabla^\mu \phi \equiv g^{\mu\nu}\nabla_\mu \nabla_\nu \phi = 0$. Write the equation of motion of such a scalar field in FLRW as a partial differential equation for $\phi(\eta, \vec{x})$ in terms of conformal time and cartesian comoving coordinates. Hint: the covariant derivative of a scalar field is equal to its partial derivative, but the covariant derivative of a vector field (such as $\nabla^\mu \phi \equiv g^{\mu\nu}\nabla_\nu \phi$) is not just its partial derivative... *[4 points]*

4) The stress-energy tensor of a free massless scalar field is $T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu}(\nabla_\lambda \phi \nabla^\lambda \phi)$. Recall that the energy density as measured by observers with 4-velocity $U^\mu_{\rm obs}$ is $\rho_{\rm obs} = U^\mu_{\rm obs} U^\nu_{\rm obs} T_{\mu\nu}$. In addition, the observed pressure is $P_{\rm obs} = \frac{1}{3}(T^\mu_{\ \mu} + \rho_{\rm obs})$, where $T^\mu_{\ \mu} \equiv g^{\mu\nu}T_{\mu\nu}$. Compute the energy density and pressure of a free massless scalar field as measured by comoving observers. Then show that, if the field is homogeneous (i.e. independent of spatial coordinates), $P_{\rm obs} = \rho_{\rm obs} \propto 1/a^6$. *[1 point for $\rho_{\rm obs}$, 2 points for $P_{\rm obs}$, 1 point for homogeneous limit; 4 points total]*

5) Show that the conservation of the stress-energy tensor of a free massless scalar field implies the equation of motion given in part 3). Hint: for scalar fields, covariant derivatives commute, $\nabla_\mu \nabla_\nu \phi = \nabla_\nu \nabla_\mu \phi$. *[2 points]*.

**Problem 2: Einstein tensor in FLRW [4 points]**

The nonvanishing components of the Einstein tensor of the homogeneous and isotropic FLRW spacetime with $(t, x^i)$ coordinates (and metric $ds^2 = -dt^2 + a^2(t)\gamma_{ij}(\vec{x})dx^i dx^j$) are

$$G^0_{\ 0} = -3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2 R_0^2}\right],$$

$$G^i_{\ j} = -\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2 R_0^2}\right]\delta^i_j,$$

where $k = 0, \pm 1$ depending on whether 3D spatial slices are flat, spherical or hyperbolic, and $R_0$ is the constant radius of curvature in the latter two cases. Show that the Einstein tensor satisfies the contracted Bianchi identity, $\nabla_\mu G^\mu_{\ \nu} = 0$. Recall that for a mixed-index tensor, $\nabla_\lambda T^\alpha_{\ \beta} = \partial_\lambda T^\alpha_{\ \beta} + \Gamma^\alpha_{\lambda\sigma} T^\sigma_{\ \beta} - \Gamma^\sigma_{\lambda\beta} T^\alpha_{\ \sigma}$.

**Problem 3: Einstein's biggest blunder [5 points]**

Problem 2.8 of Baumann. *1 point for question 1, 2 points for question 2, 2 points for question 3.*

**Problem 4: The Big Rip [4 points]**

Suppose the Universe is spatially flat and made entirely of an ideal fluid with equation of state $P = w\rho$, with constant $w < -1$. Show that the scale factor diverges at a finite time $t_{\rm rip}$. Express $t_{\rm rip} - t_0$ in terms of $H_0$ and $w$.