Graduate Cosmology Spring 2025
Homework 8

due by 11:59pm on Friday 5/2, 2025.

Problem 1: Chaotic inflation: background [16 points]

Adapted from a problem designed by Matteo Braglia

Due to its simplicity, one of the most well-known models of inflation is **chaotic inflation**, with potential

$$V(\phi) = M_{\rm pl}^4 \frac{\lambda_n}{(2n)!} (\phi/M_{\rm pl})^{2n}, \tag{1}$$

where $n = 1, 2, , 3, ...$ is an integer and the coupling λ_n as defined here is dimensionless. While this potential is ruled out by current observations (as you will see in Problem 2), it remains instructive to study it, as it provides a good opportunity to practice calculations in the inflationary context.

1) [1 point] Compute the potential slow-roll (SR) parameters ϵ_V and η_V, as a function of n and ϕ.

2) [3 points] Using the slow-roll approximation, approximate the background Klein-Gordon equation for ϕ by a first-order ordinary differential equation (ODE), and rewrite it as and ODE in terms of the number of e-folds, $N \equiv \ln(a/a_i)$. Combining with the slow-roll approximation of Friedmann's equation, derive the following explicit solution for $\phi(N)$:

$$\phi(N) = \sqrt{\phi_i^2 - 4nNM_{\rm pl}^2}.$$

3) [1 point] Compute the value of the field at the end of inflation (defined when $\epsilon_V = 1$). Determine the initial field value required to generate 60 e-folds of inflation.

4) This question applies to any potential V (not just the one given above), so keep your dependence on V general.

a) [1 point] Rewrite the *exact* Friedmann's equation as an equation for H^2 in terms of $V(\phi)$ and $\phi_{,N} \equiv \frac{d\phi}{dN}$.

b) [3 points] Show that the *exact* background Klein-Gordon equation can be rewritten as the following second-order (and non-linear) differential equation for $\phi(N)$:

$$\phi_{,NN} + \left(3 - \frac{\phi_{,N}^2}{2\,M_{\rm pl}^2}\right)\phi_{,N} + \frac{V_{,\phi}}{V}\left(3M_{\rm pl}^2 - \frac{\phi_{,N}^2}{2}\right) = 0,$$

where $\phi_{,N} = d\phi/dN$ and $\phi_{,NN} = d^2\phi/dN^2$.

c) [1 point] Write the *exact* slow-roll parameter $\epsilon \equiv -\dot{H}/H^2$ in terms of $\phi_{,N}$.

5) [4 points] Write a Python (or Mathematica) code to solve the *exact* background Klein–Gordon equation for the scalar field $\phi(N)$, for the chaotic inflation potential with arbitrary n. Use the initial conditions obtained from the SR approximation in the earlier equations (with 60 e-folds of inflation). For $n = 1$, plot the evolution of $\phi(N)/M_{\rm pl}$ from the initial time to the end of inflation, and overlay it with the analytic expressions for $\phi(N)$ in the slow-roll approximation. On a separate plot, show the evolution of the slow-roll parameter ϵ and of the potential slow-roll parameter ϵ_V.

6) [2 points] Now change the initial conditions for the derivative $\phi_{,N}(N_i = 0)$ to be a factor 0, 2, 5, or 10 times the slow-roll value. For each different initial condition, plot the phase space evolution $\phi_{,N}$ vs. ϕ. What do you observe?

Problem 2: Chaotic inflation: perturbations [4 points]

The end result of the calculation presented in class is that single-field slow-roll inflation predicts a near scale-invariant primordial curvature perturbation,

$$\Delta^2_{\mathcal{R}}(k) = A_s (k/k_*)^{n_s - 1},$$

with amplitude and tilt given by

$$A_s = \frac{1}{8\pi^2 \epsilon_{V,*}} \frac{H_*^2}{M_{\rm pl}^2}, \qquad n_s - 1 = -2\epsilon_* - \kappa_* \approx (2\eta_V - 6\epsilon_V)_*,$$

where the stars mean "to be evaluated at horizon exit" of the pivot scale k_*, i.e. when $k_* = (aH)_*$. In addition, primordial tensor perturbations are generated, with near scale-invariant power spectrum

$$\Delta^2_h(k) = A_t (k/k_*)^{n_t},$$

where the tensor-to-scalar ratio $r \equiv A_t/A_s \approx 16\ \epsilon_{V,*}$ and the tensor spectral index is $n_t \approx -2\epsilon_{V,*}$.

1) [3 points] Assume that the scale k_* exits the horizon at $N_* = N_{\rm end} - 50$, i.e. 50 e-folds before the end of inflation. Compute A_s, n_s, r and n_t, as a function of n, for the chaotic inflation potential given in Problem 1, assuming the slow-roll regime holds. *Hint:* start by rewriting $\phi(N)$ in terms of $N_{\rm end} - N$ instead of just N.

2) [1 point] The latest measurement of the scalar spectral index from Planck is $n_s = 0.9652 \pm 0.0042$. BICEP-Keck set a 95% confidence limit on $r < 0.035$. Show that this rules of chaotic inflation.

Problem 3: Tensor modes [10 points]

In this problem we consider pure tensor modes, i.e. a perturbed FLRW metric of the form

$$ds^2 = a^2(\eta)[-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j],$$

where the *gravitational-wave* metric perturbation h_{ij} is symmetric, trace-free ($\delta^{ij}h_{ij} = 0$) and transverse ($\partial_i h_{ij} = 0$).

1) [1 point] What components of the Einstein tensor do you expect to be non-zero?

2) [1 point] Which fluid variables do you expect to source tensor modes (and why?), out of δ, V^i, Π_{ij}?

3) [2 points] In vacuum, tensor perturbations satisfy the following differential equation in Fourier space:

$$h''_{ij} + 2\mathcal{H}h'_{ij} + k^2 h_{ij} = 0.$$

Solve analytically for the evolution of h_{ij} in radiation and matter domination separately (use Mathematica if needed). Explain how modes with wavenumber $k \ll k_{\rm eq}$ and $k \gg k_{\rm req}$, respectively, evolve, as quantitatively as possible.

4) [3 points] Compute the Christoffel symbols of this metric. Write the null geodesic equation and show that the variable $p \equiv aE_{\rm obs}$, where $E_{\rm obs}$ is the energy measured by comoving observers (compute their 4-velocity!), evolves according to

$$\frac{dp}{d\eta} = -\frac{p}{2} h'_{ij}\hat{p}^i\hat{p}^j,$$

where $\hat{p}$ is the photon's direction of propagation ($\hat{p}^i = P^i/\sqrt{\delta_{jk}P^jP^k}$).

5) [3 points] Assuming instantaneous decoupling at η_*, and neglecting anisotropic stress, show (in detail!) that the line-of-sight solution for the photon temperature anisotropy is

$$\Theta(\eta_0, \vec{x} = 0, \hat{p}) = -\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\int_{\eta_*}^{\eta_0} d\eta\ h'_{ij}(\eta, \vec{k})\hat{p}^i\hat{p}^j e^{-ik\chi\hat{p}\cdot\hat{k}}, \qquad \chi \equiv \eta_0 - \eta_*.$$

The main point of this exercise was to show how **tensor modes also generate temperature anisotropies**. If you want to delve deeper in this question, I encourage you to do Baumann's problem 7.6. This should give you a qualitative understanding of Fig. 7.24 in Baumann showing the C_ℓ's for scalar and tensor modes.