Cosmology HW1 (30/30) Connor Hainje

PROBLEM 1: 17/17

part 1. 3/3

$$ds^2 = a(\eta)^2 \left[-d\eta^2 + \gamma_{ij}(\vec{x})\, dx^i dx^j \right]$$

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left[\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right]$$

$$g_{00} = -a^2. \quad g_{0i} = g_{i0} = 0. \quad g_{ij} = a^2 \gamma_{ij}$$

$$g^{00} = -a^{-2}. \quad g^{0i} = g^{i0} = 0. \quad g^{ij} = a^{-2} \gamma^{ij}$$

$$\Gamma^0_{\mu\nu} = \frac{1}{2} g^{0\rho} \left[\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right]$$

$$= \frac{1}{2}(-a^{-2}) \left[\partial_\mu g_{\nu 0} + \partial_\nu g_{\mu 0} - \partial_0 g_{\mu\nu} \right]$$

$$\partial_\alpha g_{\beta 0} = \partial_\alpha (-a^2)\, \mathbb{1}(\beta = 0) = \partial_\eta (-a^2)\, \mathbb{1}(\alpha=\beta=0) = -2a'a\, \mathbb{1}(\alpha=\beta=0)$$

$$\partial_0 g_{\mu\nu} = 2 \frac{a'}{a} g_{\mu\nu}$$

$$= \frac{-1}{2a^2} \left[-4a'a\, \mathbb{1}(\mu=\nu=0) - 2\frac{a'}{a} g_{\mu\nu} \right]$$

* $\hookrightarrow \Gamma^0_{00} = \frac{-1}{2a^2} \left[-4a'a + 2a'a \right] = \frac{a'}{a}$ ✓ +1/2

* $\hookrightarrow \Gamma^0_{i0} = \frac{a'}{a^3} g_{i0} = 0.$ ✓ +1/2

$\hookrightarrow \Gamma^0_{ij} = \frac{a'}{a^3} g_{ij} = \frac{a'}{a} \gamma_{ij}$ ✓ +1/2

$$\Gamma^i_{\mu\nu} = \frac{1}{2} g^{i\rho} \left[\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right]$$

$$= \frac{1}{2a^2} \gamma^{ij} \left[\partial_\mu g_{\nu j} + \partial_\nu g_{\mu j} - \partial_j g_{\mu\nu} \right]$$

$$\partial_\alpha g_{\beta j} = \partial_\alpha a^2 \gamma_{ij}\, \mathbb{1}(\beta = i) = 2a'a\, \gamma_{ij}\, \mathbb{1}(\alpha=0, \beta=i) + a^2 \partial_k \gamma_{ij}\, \mathbb{1}(\alpha=k, \beta=i)$$

$$\partial_j g_{\mu\nu} = a^2 \partial_j \gamma_{ik}\, \mathbb{1}(\mu=i, \nu=k)$$

* $\hookrightarrow \Gamma^i_{00} = 0.$ ✓ +1/2

* $\hookrightarrow \Gamma^i_{j0} = \Gamma^i_{0j} = \frac{1}{2a^2} \gamma^{ik} \left[2a'a\, \gamma_{jk} + 0 - 0 \right] = \frac{a'}{a} \gamma^{ik} \gamma_{kj} = \frac{a'}{a} \delta^i{}_j.$ ✓ +1/2

$\hookrightarrow \Gamma^i_{jk} = \frac{1}{2a^2} \gamma^{il} \left[\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk} \right]$

$$= \frac{1}{2} \gamma^{il} \left[\partial_j \gamma_{kl} + \partial_k \gamma_{jl} - \partial_l \gamma_{jk} \right]$$ ✓ +1/2

many of these can be unified into $\Gamma^\mu_{\nu 0} = \Gamma^\mu_{0\nu} = \frac{a'}{a} \delta^\mu{}_\nu$.

part 2. 4/4

the geodesic equation is $P^\mu \nabla_\mu P^\nu = 0$. consider the $\nu = 0$ component:

$$0 = P^\mu \nabla_\mu P^0$$
$$= P^\mu \partial_\mu P^0 + \Gamma^0_{\mu\rho} P^\mu P^\rho$$
$$= P^\mu \partial_\mu P^0 + \frac{a'}{a} P^0 P^0 + \frac{a'}{a} \gamma_{ij} P^i P^j$$

define $a^2 \gamma_{ij} P^i P^j = p^2$.

$$P_\mu P^\mu = g_{\mu\nu} P^\mu P^\nu = -a^2 P^0 P^0 + a^2 \gamma_{ij} P^i P^j = 0 \quad \text{since massless}$$
$$\text{so} \quad P^0 P^0 = \gamma_{ij} P^i P^j \quad \text{and} \quad a^2 (P^0)^2 = p^2.$$

thus $0 = P^\mu \partial_\mu P^0 + 2 \frac{a'}{a^3} p^2$

$$(P^0)^2 = a^{-2} p^2$$

∂_μ : $\quad P^0 \partial_\mu P^0 = a^{-2} p \, \partial_\mu p + - a' a^{-3} p^2 \, \mathbb{1}(\mu = 0)$

P^μ / P^0 : $\quad P^\mu \partial_\mu P^0 = a^{-2} \frac{P^\mu}{P^0} p \, \partial_\mu p - a' a^{-3} p^2$

$$= a^{-2} p \frac{dx^\mu}{d\eta} \frac{\partial}{\partial x^\mu} p - a' a^{-3} p^2$$
$$= a^{-2} p p' - a' a^{-3} p^2$$

thus $0 = p' + \frac{a'}{a} p \rightarrow \frac{p'}{p} = -\frac{a'}{a} \rightarrow p(\eta) \propto a(\eta)^{-1}$. ✓ +1

what is U^μ_{obs} in conformal coordinates? soln: $P^0 \propto a^{-2}$. above: $P^0 = a^{-1} p \propto a^{-2}$

comoving observer still has $dx^i = 0 \rightarrow U^\mu_{obs} = (U^0_{obs}, 0, 0, 0)$

still have $U^\mu_{obs} U^{obs}_\mu = -1$.

in conformal coordinates, $U^\mu_{obs} U^{obs}_\mu = g_{\mu\nu} U^\mu_{obs} U^\nu_{obs}$

$$= g_{00} (U^0_{obs})^2$$
$$= -a^2 (U^0_{obs})^2 = -1.$$

hence $U^0_{obs} = a^{-1}$ and $U^\mu_{obs} = (a^{-1}, 0, 0, 0)$ ✓ +2

energy measured by a comoving observer is then

$$E_{obs} = -U^\mu_{obs} P_\mu = -g_{\mu\nu} U^\mu_{obs} P^\nu = -g_{00} U^0_{obs} P^0 = a^2 a^{-1} \frac{p}{a} = p$$

recall that $p \propto a^{-1}$, so $E_{obs} = p \propto a^{-1}$ ✓ +1

part 3. 4/4

$$\nabla_\mu \nabla^\mu \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = 0.$$

$$\begin{aligned}
0 &= g^{\mu\nu} \nabla_\mu \nabla_\nu \phi \\
&= g^{\mu\nu} \nabla_\mu \partial_\nu \phi \\
&= g^{\mu\nu} \left(\partial_\mu \partial_\nu \phi - \Gamma^\lambda_{\mu\nu} \partial_\lambda \phi \right) \\
&= g^{00} \left(\partial_0 \partial_0 \phi - \Gamma^\lambda_{00} \partial_\lambda \phi \right) + g^{ij} \left(\partial_i \partial_j \phi - \Gamma^\lambda_{ij} \partial_\lambda \phi \right) \\
&= -a^{-2} \left(\phi'' - \frac{a'}{a} \phi' \right) + a^{-2} \gamma^{ij} \left[\partial_i \partial_j \phi - \frac{a'}{a} \gamma_{ij} \phi' \right] \\
&\quad - \frac{1}{2} a^{-2} \gamma^{ij} \gamma^{kl} \left(\partial_i \gamma_{jk} + \partial_j \gamma_{ki} - \partial_k \gamma_{ij} \right) \partial_l \phi \\
&= -a^{-2} \phi'' + a^{-3} a' \phi' + a^{-2} \gamma^{ij} \partial_i \partial_j \phi - a^{-3} a' \gamma^{ij} \gamma_{ij} \phi' \\
&\quad - \frac{1}{2} a^{-2} \gamma^{ij} \gamma^{kl} \left(\partial_i \gamma_{jk} + \partial_j \gamma_{ki} - \partial_k \gamma_{ij} \right) \partial_l \phi
\end{aligned}$$

Carroll eq 3.16

$$\nabla_\alpha A_\beta = \partial_\alpha A_\beta - \Gamma^\lambda_{\alpha\beta} A_\lambda .$$

re-arranging,

$$\phi'' + 2 \frac{a'}{a} \phi' = \gamma^{ij} \partial_i \partial_j \phi - \frac{1}{2} \gamma^{ij} \gamma^{kl} \left(\partial_i \gamma_{jk} + \partial_j \gamma_{ki} - \partial_k \gamma_{ij} \right) \partial_l \phi$$

I didn't specialize to spatially flat... $\gamma_{ij} = \delta_{ij}$.

$\phi'' + 2\frac{a'}{a}\phi' = \partial_i \partial_i \phi - 0$ matches solution ✓ +4

suppose ϕ was homogeneous: $\phi = \phi(\eta)$. then, $\phi'' + 2\frac{a'}{a}\phi' = 0$

looks very much like $\ddot{\phi} + 3H\dot{\phi} = 0$!

$$\phi' = \frac{d}{d\eta}\phi = \frac{dt}{d\eta}\frac{d\phi}{dt} = a\dot{\phi}, \qquad a' = a\dot{a}$$

$$\phi'' = \frac{d}{d\eta}\frac{d}{d\eta}\phi = \frac{dt}{d\eta}\frac{d}{dt}\frac{dt}{d\eta}\frac{d}{dt}\phi = a\frac{d}{dt}\left(a\dot{\phi}\right) = a^2\ddot{\phi} + a\dot{a}\dot{\phi}$$

thus $\phi'' + 2\frac{a'}{a}\phi' = a^2\ddot{\phi} + a\dot{a}\dot{\phi} + 2\dot{a}a\dot{\phi} = a^2\left(\ddot{\phi} + 3H\dot{\phi}\right)$ they are in fact equivalent!

part 4. 4/4

energy density: $\rho_{obs} = U^{\mu}_{obs} U^{\nu}_{obs} T_{\mu\nu}$.

$$U^{\mu}_{obs} = (a^{-1}, 0, 0, 0). \qquad T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla_\lambda \phi \nabla^\lambda \phi)$$

$$\nabla_\lambda \phi \nabla^\lambda \phi = g^{\alpha\beta}(\partial_\alpha \phi)(\partial_\beta \phi) = -a^{-2}(\partial_0 \phi)^2 + a^{-2} \gamma^{ij} \partial_i \phi \partial_j \phi$$

$$\rightarrow \rho_{obs} = a^{-2} T_{00} = a^{-2}\left[(\phi')^2 + \frac{1}{2} a^2 a^{-2} \left(\gamma^{ij} \partial_i \phi \partial_j \phi - (\phi')^2 \right) \right]$$

$$= \frac{1}{2} a^{-2} \left[(\phi')^2 + \gamma^{ij} \partial_i \phi \partial_j \phi \right]$$ ✓ +1

pressure: $P_{obs} = \frac{1}{3} (T^{\mu}{}_{\mu} + \rho_{obs})$

$$T^{\mu}{}_{\mu} = g^{\mu\nu} T_{\mu\nu}$$

$$= g^{00} T_{00} + g^{ij} T_{ij}$$

$$= -a^{-2} \frac{1}{2} \left[(\phi')^2 + \gamma^{ij} \partial_i \phi \partial_j \phi \right]$$

$$+ a^{-2} \gamma^{ij} \left[\partial_i \phi \partial_j \phi - \frac{1}{2} g_{ij} a^{-2} \left(\gamma^{kl} \partial_k \phi \partial_l \phi - (\phi')^2 \right) \right]$$

$$= -\frac{1}{2} a^{-2} \left((\phi')^2 + \gamma^{ij} \partial_i \phi \partial_j \phi \right)$$

$$+ a^{-2} \gamma^{ij} \partial_i \phi \partial_j \phi - \frac{3}{2} a^{-2} \left(\gamma^{kl} \partial_k \phi \partial_l \phi - (\phi')^2 \right)$$

$$= a^{-2} \phi'^2 - a^{-2} \gamma^{ij} \partial_i \phi \partial_j \phi$$

$$\rightarrow P_{obs} = \frac{1}{3} \left[a^{-2} \left(\phi'^2 - \gamma^{ij} \partial_i \phi \partial_j \phi \right) + \frac{1}{2} a^{-2} \left(\phi'^2 + \gamma^{ij} \partial_i \phi \partial_j \phi \right) \right]$$

$$= \frac{1}{2} a^{-2} \phi'^2 - \frac{1}{6} a^{-2} \gamma^{ij} \partial_i \phi \partial_j \phi$$ ✓ +2

if the field is homogeneous, $\partial_i \phi = 0$. then

$$\rho_{obs} = \frac{1}{2} a^{-2} \phi'^2, \qquad P_{obs} = \rho_{obs} = \frac{1}{2} a^{-2} \phi'^2$$

showed in the last part that the EOM for homogeneous ϕ is $\phi'' + 2\frac{a'}{a}\phi' = 0$

$$\phi''/\phi' = -2\, a'/a \quad \Rightarrow \quad \phi' \propto a^{-2}.$$ ✓ +1

$$\rightarrow \rho = P = \frac{1}{2} a^{-2} \phi'^2 \propto a^{-2} (a^{-2})^2 = a^{-6} \ !$$

part 5. 2/2

$$
\begin{aligned}
0 &= g^{\mu\alpha} \nabla_\mu T_{\alpha\beta} \\
&= g^{\mu\alpha} \nabla_\mu \left[\nabla_\alpha \phi \, \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} \left(\nabla_\lambda \phi \, \nabla^\lambda \phi \right) \right] \\
&= g^{\mu\alpha} \nabla_\mu \left(\nabla_\alpha \phi \, \nabla_\beta \phi \right) - \frac{1}{2} g^{\mu\alpha} g_{\alpha\beta} \nabla_\mu \left(\nabla_\lambda \phi \, \nabla^\lambda \phi \right) \\
&= g^{\mu\alpha} \left(\nabla_\mu \nabla_\alpha \phi \, \nabla_\beta \phi + \nabla_\alpha \phi \, \nabla_\mu \nabla_\beta \phi \right) - \frac{1}{2} \nabla_\beta \left(\nabla_\lambda \phi \, \nabla^\lambda \phi \right) \\
&= \nabla_\mu \nabla^\mu \phi \, \nabla_\beta \phi + \nabla^\mu \phi \, \nabla_\mu \nabla_\beta \phi - \nabla_\beta \nabla_\lambda \phi \, \nabla^\lambda \phi \\
&= \left(\nabla_\mu \nabla^\mu \phi \right) \nabla_\beta \phi
\end{aligned}
$$

cancel by commutation of ∇ on ϕ

suppose $\nabla_\beta \phi = 0$ everywhere. then $\nabla^\mu \nabla_\mu \phi = 0$ trivially.

otherwise, $0 = \left(\nabla_\mu \nabla^\mu \phi \right) \nabla_\beta \phi \Rightarrow \nabla_\mu \nabla^\mu \phi = 0.$

hence $\nabla^\alpha T_{\alpha\beta} = 0 \Rightarrow \nabla_\mu \nabla^\mu \phi = 0.$

PROBLEM 2: 4/4

$$\Gamma^0_{ij} = a\dot{a}\,\gamma_{ij} \qquad \Gamma^i_{0j} = \frac{\dot{a}}{a}\,\delta^i{}_j \qquad \Gamma^k_{ij} = \frac{1}{2}\gamma^{k\ell}\left[\partial_i \gamma_{j\ell} + \partial_j \gamma_{\ell i} - \partial_\ell \gamma_{ij}\right]$$

$$G^0{}_0 = -3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right] \qquad G^i{}_j = -\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right]\delta^i{}_j\,.$$

$$\nabla_\mu G^\mu{}_\nu = \partial_\mu G^\mu{}_\nu + \Gamma^\mu_{\mu\lambda}\,G^\lambda{}_\nu - \Gamma^\lambda_{\mu\nu}\,G^\mu{}_\lambda$$

$$\Gamma^\mu_{\mu\lambda} = \Gamma^0_{0\lambda} + \Gamma^i_{i\lambda} = \Gamma^i_{i\lambda}$$

$\nu = 0:$

$$\partial_\mu G^\mu{}_0 = \partial_0 G^0{}_0 = -3\,\partial_t\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right]$$

$$= -6\left(\frac{\dot{a}}{a}\right)\left[\left(\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right) - \frac{k}{a^2R_0^2}\right] = -6\,\frac{\dot{a}}{a}\frac{\ddot{a}}{a} + 6\left(\frac{\dot{a}}{a}\right)\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right]$$

$$\Gamma^i_{i\lambda}G^\lambda{}_0 = \Gamma^i_{i0}G^0{}_0 = \left(\frac{\dot{a}}{a}\delta^i{}_i\right)\left(-3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right]\right)$$

$$= -9\left(\frac{\dot{a}}{a}\right)\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right]$$

$$\Gamma^\lambda_{\mu 0}G^\mu{}_\lambda = \cancel{\Gamma^\lambda_{00}}\,G^0{}_\lambda + \Gamma^\lambda_{i0}\,G^i{}_\lambda = \Gamma^0_{i0}\,\cancel{G^i{}_0} + \Gamma^j_{i0}\,G^i{}_j$$

$$= -\left(\frac{\dot{a}}{a}\delta^j{}_i\right)\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right]\delta^i{}_j\,.$$

$$= -3\left(\frac{\dot{a}}{a}\right)\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right]$$

$$= -6\left(\frac{\dot{a}}{a}\right)\left(\frac{\ddot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right]$$

so
$$\nabla_\mu G^\mu{}_0 = \left\{-6\left(\frac{\dot{a}}{a}\right)\left(\frac{\ddot{a}}{a}\right) + 6\left(\frac{\dot{a}}{a}\right)\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right]\right\} + \left\{-9\left(\frac{\dot{a}}{a}\right)\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right]\right\}$$

$$- \left\{-6\left(\frac{\dot{a}}{a}\right)\left(\frac{\ddot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right]\right\}$$

$$= (-6+6)\left(\frac{\dot{a}}{a}\right)\left(\frac{\ddot{a}}{a}\right) + (6-9+3)\left(\frac{\dot{a}}{a}\right)\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right] = 0.$$ ✓

$\nu = j \neq 0:$

$$\partial_\mu G^\mu{}_j = \partial_i G^i{}_j = -\partial_j\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right] = 0$$ since $a = a(t)$, k & R_0 constants

$$\Gamma^i_{i\lambda}G^\lambda{}_j = \Gamma^i_{ik}G^k{}_j = -\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right]\Gamma^i_{ij}$$

$$\Gamma^\lambda_{\mu j}\,G^\mu{}_\lambda = \Gamma^\lambda_{0j}\,G^0{}_\lambda + \Gamma^\lambda_{ij}\,G^i{}_\lambda$$

$$= \Gamma^k_{0j}\,\cancel{G^0{}_k} + \cancel{\Gamma^0_{0j}}\,G^0{}_0 + \Gamma^k_{ij}\,G^i{}_k + \Gamma^0_{ij}\,\cancel{G^i{}_0}$$

$$= -\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right]\Gamma^i_{ij}$$

so
$$\nabla_\mu G^\mu{}_j = \{0\} + \left\{-\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right]\Gamma^i_{ij}\right\} - \left\{-\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right]\Gamma^i_{ij}\right\}$$

$$= 0.$$ ✓ □

PROBLEM 3: [Baumann 2.8] 5/5

part 1. 1/1

for a perfect fluid with $\rho, P > 0$, the EFEs reduce to the Friedmann equations:

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2 R_0^2} \qquad \frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

if $\rho, P > 0$, then $\ddot a / a < 0$ and, in particular, $\ddot a \neq 0$ so $\dot a \neq 0$.

no solution with $\dot a = 0$ $\Rightarrow$ no static solution! ✓

part 2. 2/2

pressureless matter $(\rho > 0,\ P = 0)$ and cosmological constant Λ

Friedmann equations with Λ can be written

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2 R_0^2} + \frac{\Lambda}{3} \qquad \frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}$$

we seek a static solutions with $\dot a = 0 \Rightarrow \ddot a = 0$.

eq 2 then implies $\quad 0 = -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3} \quad \Rightarrow \quad \Lambda = 4\pi G \rho$. ✓

plugging this in to eq 1 gives

$$0 = \frac{8\pi G}{3}\rho - \frac{k}{a^2 R_0^2} + \frac{\Lambda}{3} = 4\pi G \rho - \frac{k}{a^2 R_0^2}$$

$$\Rightarrow k = 4\pi G \rho\, a^2 R_0^2 = \Lambda\, a^2 R_0^2 .$$

ρ is positive, as is a^2 and R_0^2, so $k > 0$. thus the spatial curvature is <u>spherical</u>. ✓

part 3. 2/2

perturbation: $\quad \rho_m = \rho_0[1 + \delta(t)], \quad a(t) = 1 + \varepsilon(t)$.

$$\frac{\ddot a}{a} = \ddot\varepsilon(1-\varepsilon) \approx \ddot\varepsilon = -\frac{4\pi G}{3}\rho_0(1+\delta) + \underbrace{\frac{\Lambda}{3}}_{= 4\pi G \rho_0} = -\frac{4\pi G}{3}\rho_0 \delta . \checkmark$$

$$\left(\frac{\dot a}{a}\right)^2 \approx 0 = \frac{8\pi G}{3}\rho_0(1+\delta) - \frac{k}{R_0^2 a^2} + \frac{\Lambda}{3}$$

$$= \frac{8\pi G}{3}\rho_0(1+\delta) - 4\pi G \rho_0 (1 - 2\varepsilon) + \frac{4\pi G}{3}\rho_0$$

$$= 4\pi G \rho_0 + \frac{8\pi G}{3}\rho_0 \delta - 4\pi G \rho_0 + 8\pi G \rho_0 \varepsilon$$

$$= \frac{8\pi G}{3}\rho_0[\delta + 3\varepsilon] \quad \Rightarrow \quad \delta = -3\varepsilon . \checkmark$$

$$0 = \ddot\varepsilon + \frac{4\pi G}{3}\rho_0 \delta = \ddot\varepsilon - 4\pi G \rho_0 \varepsilon \checkmark \quad \text{with solution} \quad \varepsilon(t) = \underbrace{A e^{+\sqrt{4\pi G \rho_0}\, t}}_{\text{unstable!}} \checkmark + B e^{-\sqrt{4\pi G \rho_0}\, t}$$

PROBLEM 4: 4/4

$k=0$, $w<-1$. for constant w, $\rho \propto a^{-3(1+w)}$.

let $w = -1 - \gamma/3$ for $\gamma > 0$ since $w<-1$. then $\rho \propto a^{+\gamma}$.

hence $\left(\frac{\dot a}{a}\right)^2 = H_0^2\, \Omega\, a^{+\gamma} \rightarrow \dot a^2 = H_0^2\, \Omega\, a^{\gamma+2}$. (defining $\Omega = \frac{\rho_0}{\rho_{crit,0}} = \frac{8\pi G}{3H_0^2}\rho_0$)

$$\dot a = H_0\sqrt{\Omega}\, a^{1+\gamma/2} \longrightarrow \int_1^a a^{-1-\gamma/2}\, da = \int_{t_0}^t H_0\sqrt{\Omega}\, dt$$

$$-\frac{2}{\gamma}\left(a^{-\gamma/2} - 1\right) = H_0\sqrt{\Omega}\,(t-t_0)$$

$$a(t) = \left(1 - \frac{\gamma}{2} H_0\sqrt{\Omega}\,(t-t_0)\right)^{-2/\gamma}$$

a diverges when the argument to $(\ldots)^{-2/\gamma}$ is equal to zero since $\gamma>0$ and thus $-2/\gamma < 0$.

this happens at t_{rip}, $\frac{\gamma}{2} H_0\sqrt{\Omega}\,(t_{rip}-t_0) = 1. \rightarrow t_{rip} - t_0 = \frac{2/\gamma}{H_0\sqrt{\Omega}}$

we can put this back in terms of the original components $\gamma = -3(1+w)$.

$$t_{rip} - t_0 = \frac{-1}{\sqrt{6\pi G\rho_0}\,(1+w)} = \frac{1}{\sqrt{6\pi G\rho_0}\,(|w|-1)} \quad (>0)$$

note $\Omega = 1$ since this Universe has just the one component.

thus $\rho_0 = \frac{3H_0^2}{8\pi G}$ and so $t_{rip} - t_0 = \frac{2}{3H_0(|w|-1)}$ as in the solution

they agree! ✓