Cosmology HW2

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code for this HW at github.com/cmhainje/cosmo-hw/blob/main/hw2/homework2.ipynb

PROBLEM 1:

part 1: $\left(\frac{\dot a}{a}\right)^2 = H_0^2 \sum_n \Omega_n a^n \rightarrow \dot a^2 = H_0^2 \sum_n \Omega_n a^{n+2}$

$$t(a) = \int \frac{dt}{da}\, da = \int \frac{1}{\dot a}\, da = H_0^{-1} \int \frac{da}{(\sum_n \Omega_n a^{n+2})^{1/2}}$$

part 2:

(a) neglecting curvature $(\Omega_k = 0)$ and taking $a \ll a_{m\Lambda}$,

$$t(a) = H_0^{-1} \int \frac{da}{\sqrt{\Omega_m a^{-1} + \Omega_r a^{-2}}}$$

$a_{eq} = \frac{\Omega_r}{\Omega_m}$ so $\Omega_m a^{-1} + \Omega_r a^{-2} = a_{eq}^{-2}\left(\Omega_m a_{eq}(a_{eq}/a) + \Omega_r (a_{eq}/a)^2\right)$

$$= a_{eq}^{-2}\left[\Omega_r (a/a_{eq})^{-1} + \Omega_r (a/a_{eq})^{-2}\right]$$

then $t(a) = \frac{a_{eq}}{H_0\sqrt{\Omega_r}} \int \frac{da}{\sqrt{(a/a_{eq})^{-1} + (a/a_{eq})^{-2}}}$

let $x = a/a_{eq} \rightarrow da = a_{eq}\, dx$ so $t(a) = \frac{a_{eq}^2}{H_0\sqrt{\Omega_r}} \int \frac{dx}{\sqrt{x^{-1} + x^{-2}}}$

$$= \frac{a_{eq}^2}{H_0\sqrt{\Omega_r}} \int \frac{x\, dx}{\sqrt{1+x}}$$

$$= \frac{2a_{eq}^2}{3H_0\sqrt{\Omega_r}} (x-2)\sqrt{1+x} + \text{const}$$

let $t(0) = 0 \rightarrow 0 = \frac{-4a_{eq}^2}{3H_0\sqrt{\Omega_r}} + C$ so $t(a) = \frac{2a_{eq}^2}{3H_0\sqrt{\Omega_r}}\left[2 + (x-2)\sqrt{1+x}\right]$

(b) $t(a) = \frac{2a_{eq}^2}{3H_0\sqrt{\Omega_r}} (x-2)\sqrt{1+x}$

$a \ll a_{eq} \longrightarrow x \ll 1$: $(x-2)\sqrt{1+x} = (-2+x)(1 + x/2 - \frac{1}{8}x^2 + O(x^3))$

$$= -2 + x - x + \tfrac{1}{2}x^2 + \tfrac{1}{4}x^2 + O(x^3)$$

$$= -2 + \tfrac{3}{4}x^2 + O(x^3)$$

$a \gg a_{eq} \rightarrow x \gg 1$: let $y = \frac{1}{x}$. limit is then $y \ll 1$.

$$(x-2)\sqrt{1+x} = \left(\tfrac{1}{y} - 2\right)\sqrt{1 + 1/y} = y^{-3/2}(1-2y)\sqrt{1+y}$$

$$= y^{-3/2}(1-2y)(1 + y/2 - \tfrac{1}{8}y^2 + O(y^3))$$

$$= y^{-3/2}(1 - 2y + y/2 - y^2 - \tfrac{1}{8}y^2 + O(y^3))$$

$$= y^{-3/2} - \tfrac{3}{2}y^{-1/2} + O(y^0)$$

$$= x^{3/2} - \tfrac{3}{2}x^{1/2} + \Omega(x^0)$$

I guess obviously it's just $x^{3/2}$ to leading order hence the limits are

$$t(a) \approx \begin{cases} \dfrac{a_{eq}^2}{2H_0\sqrt{\Omega_r}} \left(a/a_{eq}\right)^2 & \text{if } a \ll a_{eq} \\ \dfrac{2a_{eq}^2}{3H_0\sqrt{\Omega_r}} \left(a/a_{eq}\right)^{3/2} & a \gg a_{eq} \end{cases}$$

(c) $t_{eq} = t(a_{eq}) = \dfrac{2a_{eq}^2}{3H_0\sqrt{\Omega_r}}\left[2 + (1-2)\sqrt{1+1}\right] = \dfrac{2(2-\sqrt{2})}{3}\dfrac{a_{eq}^2}{H_0\sqrt{\Omega_r}}$

$H_0 = 67.66$ km/s/Mpc from Planck 18 $\quad \Omega_r = 9.0 \times 10^{-5}$ from lecture

$\Omega_m = 0.30966$ from Planck 18 $\longrightarrow a_{eq} = 2.9 \times 10^{-4}$

then $t_{eq} = 5.1 \times 10^4$ year $\quad$ agrees with naive estimate from lecture!

part 3.

(a) neglecting curvature and taking $a \gg a_{eq}$, $\quad t(a) = H_0^{-1}\displaystyle\int \frac{da}{\sqrt{\Omega_m a^{-1} + \Omega_\Lambda a^{+2}}}$

defining $a_{m\Lambda} = \left(\dfrac{\Omega_m}{\Omega_\Lambda}\right)^{1/3}$ and $x = a/a_{m\Lambda}$, $\quad dx\, a_{m\Lambda} = da$

$$\Omega_m a^{-1} + \Omega_\Lambda a^{+2} = \Omega_\Lambda a_{m\Lambda}^3 a^{-1} + \Omega_\Lambda a^{+2} = \Omega_\Lambda a_{m\Lambda}^2 \left(x^{-1} + x^{+2}\right)$$

then $t(a) = \dfrac{1}{H_0\sqrt{\Omega_\Lambda}\, a_{m\Lambda}} \displaystyle\int \frac{da}{\sqrt{x^{-1}+x^2}} = \frac{1}{H_0\sqrt{\Omega_\Lambda}} \int \frac{dx}{\sqrt{x^{-1}+x^2}} = \frac{1}{H_0\sqrt{\Omega_\Lambda}} \int \sqrt{\frac{x}{1+x^3}}\, dx = \frac{2}{3H_0\sqrt{\Omega_\Lambda}} \operatorname{arcsinh}(x^{3/2})$

this time, setting $t(0) = 0$ sets the constant to zero.

Baumann 2.177 gives $a(t) = \left(\dfrac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left(\dfrac{3}{2} H_0\sqrt{\Omega_\Lambda}\, t\right)$

inverting... $\quad x^{3/2} = \sinh\left(\dfrac{3}{2} H_0\sqrt{\Omega_\Lambda}\, t\right)$

$$\operatorname{arcsinh}(x^{3/2}) = \frac{3}{2} H_0\sqrt{\Omega_\Lambda}\, t$$

$$\frac{2}{3H_0\sqrt{\Omega_\Lambda}} \operatorname{arcsinh}(x^{3/2}) = t \quad ✓$$

(b) $\Omega_m = 0.30966$, $\Omega_\Lambda = 0.68885 \quad a_{m\Lambda} = 0.766$. $\quad t_0$ is when $a=1$.

$t_0 = t(1) = t\left(x = \dfrac{1}{0.766}\right) = \dfrac{2}{3H_0\sqrt{\Omega_\Lambda}} \operatorname{arcsinh}\left(\left(\dfrac{1}{0.766}\right)^{3/2}\right) = 1.38 \times 10^{10}$ years $\quad$ (13.8 Gyr)

part 4.

$dt^2 = a^2 d\eta^2 \Rightarrow \dfrac{d\eta}{dt} = a^{-1} \Rightarrow \eta(t) - \cancel{\eta(0)}^{\,0} = \displaystyle\int_0^t \frac{dt'}{a(t')}$ $\quad$ so $\eta_0 = \displaystyle\int_0^{t_0} \frac{dt}{a(t)}$.

using the $a(t)$ from above, $\eta_0 = \displaystyle\int_0^{13.8\text{ Gyr}} \Big[\underbrace{a_{m\Lambda}}_{0.766} \sinh^{2/3}\Big(\frac{3}{2} \underbrace{H_0\sqrt{\Omega_\Lambda}}_{5.743\times10^{-2}\text{ Gyr}}\, t\Big)\Big]^{-1} dt = 47.2$ Gyr $= 14500$ Mpc

PROBLEM 2:

part 1. $\chi(z) = \eta_0 - \eta(z)$

$$\eta(t) = \int_0^t \frac{dt}{a(t)} \longrightarrow \eta(a) = \int_0^a \frac{1}{a}\frac{dt}{da}\,da = \int_0^a \frac{da}{a\,\dot{a}(a)}$$

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\left[\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda\right] \longrightarrow \dot a = H_0\left(\Omega_r a^{-2} + \Omega_m a^{-1} + \Omega_k a^0 + \Omega_\Lambda a^2\right)^{1/2}$$

$$\rightarrow \eta(a) = \int_0^a \frac{da}{H_0\, a\,(\Omega_r a^{-2} + \Omega_m a^{-1} + \Omega_k a^0 + \Omega_\Lambda a^2)^{1/2}}$$

$a = \frac{1}{1+z}$ so

$$\chi(z) = \eta_0 - H_0^{-1}\int_0^{\frac{1}{1+z}} \frac{da}{a\,(\Omega_r a^{-2} + \Omega_m a^{-1} + \Omega_k a^0 + \Omega_\Lambda a^2)^{1/2}}$$

$$= H_0^{-1}\int_{1/1+z}^{1} \frac{da}{a\,(\Omega_r a^{-2} + \Omega_m a^{-1} + \Omega_k a^0 + \Omega_\Lambda a^2)^{1/2}}$$

part 2. see notebook.

part 3. $\Omega_k = \frac{-k}{(R_0 H_0)^2} \Rightarrow R_0 = \frac{1}{H_0\sqrt{|\Omega_k|}}$ for $k \neq 0$, undefined if $\Omega_k = k = 0$.

$$S_k = R_0\begin{cases}\sin(\chi/R_0) & \text{if } k=1\\ \chi/R_0 & k=0\\ \sinh(\chi/R_0) & k=-1\end{cases} = \begin{cases}\frac{1}{H_0\sqrt{|\Omega_k|}}\sin\left(H_0\sqrt{|\Omega_k|}\,\chi\right) & k=1\\ \chi & k=0\\ \frac{1}{H_0\sqrt{|\Omega_k|}}\sinh\left(H_0\sqrt{|\Omega_k|}\,\chi\right) & k=-1\end{cases}$$

$d_L = (1+z)\,S_k(\chi(z))$, $\quad d_A = (1+z)^{-1}\,S_k(\chi(z))$

see notebook.

part 4. taking $\Omega_k = 0$, $z \ll z_{eq}$ $(a \gg a_{eq})$

in this case, $\chi(z) = H_0^{-1}\int_{\frac{1}{1+z}}^{1}\frac{da}{a\,(\Omega_m a^{-1} + \Omega_\Lambda a^2)^{1/2}}$

then $d_L(z) = (1+z)\,S_0(\chi(z)) = (1+z)\,\chi(z) = H_0^{-1}(1+z)\int_{\frac{1}{1+z}}^{1}\frac{da}{a\,(\Omega_m a^{-1} + \Omega_\Lambda a^2)^{1/2}}$

using our usual tricks, $a_{m\Lambda} = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3}$ and $x = a/a_{m\Lambda}$, we can rewrite the integrand as

$$\frac{da}{a\,(\Omega_m a^{-1} + \Omega_\Lambda a^2)^{1/2}} = \frac{da}{a\sqrt{\Omega_\Lambda}\,a_{m\Lambda}\sqrt{x^{-1} + x^2}} = \frac{1}{\sqrt{\Omega_\Lambda}\,a_{m\Lambda}}\frac{dx}{\sqrt{x + x^4}}$$

then
$$d_L(z) = H_0^{-1}\frac{1+z}{a_{m\Lambda}\sqrt{\Omega_\Lambda}}\int_{1/a_{m\Lambda}(1+z)}^{1/a_{m\Lambda}}\frac{dx}{\sqrt{x+x^4}}$$

$$= \underbrace{H_0^{-1}\frac{1+z}{\sqrt{\Omega_\Lambda}}\left[(1+z)\,{}_2F_1\left(\tfrac{1}{3},\tfrac{1}{2},\tfrac{4}{3},-a_{m\Lambda}^3(1+z)^3\right) - {}_2F_1\left(\tfrac{1}{3},\tfrac{1}{2},\tfrac{4}{3},-a_{m\Lambda}^3\right)\right]}_{\equiv F(z;\,\Omega_\Lambda)}$$

where ${}_2F_1$ is the hypergeometric function

in the $\Omega_r = \Omega_k = 0$ limit, we can take $\Omega_m = 1 - \Omega_\Lambda$ and $a_{m\Lambda}^3 = \frac{1}{\Omega_\Lambda} - 1$

see notebook.

part 5.

(a) $$\chi^2(\Omega_\Lambda, H_0, M) = \sum_i \sigma_i^{-2}\left[5\log_{10} F(z_i;\Omega_\Lambda) - \mu_i + K(H_0, M)\right]^2$$

we seek $\tilde{K} = \underset{K}{\text{argmin}}\ \chi^2(\Omega_\Lambda, K)\Big|_{\Omega_\Lambda}$.

$$\frac{\partial}{\partial K}\chi^2 = \sum_i \sigma_i^{-2}\, 2\left[5\log F_i - \mu_i + \tilde{K}\right] = 0 \qquad \text{letting } F_i \equiv F(z_i;\Omega_\Lambda).$$

$$\rightarrow \sum_i \sigma_i^{-2}(5\log F_i - \mu_i) + \tilde{K}\sum_i \sigma_i^{-2} = 0$$

$$\rightarrow \tilde{K} = \frac{-1}{\sum_i \sigma_i^{-2}}\sum_i \sigma_i^{-2}(5\log F_i - \mu_i)$$

plugging this in,

$$\tilde{\chi}^2(\Omega_\Lambda) = \sum_i \sigma_i^{-2}\left[5\log F(z_i;\Omega_\Lambda) - \mu_i + \tilde{K}\right]^2$$

$$= \sum_i \sigma_i^{-2}(5\log F(z_i;\Omega_\Lambda) - \mu_i)^2 + \tilde{K}^2\sum_i \sigma_i^{-2} + 2\tilde{K}\sum_i \sigma_i^{-2}(5\log F(z_i;\Omega_\Lambda) - \mu_i)$$

$$= \sum_i \sigma_i^{-2}(5\log F_i - \mu_i)^2 + \frac{1}{\sum_i \sigma_i^{-2}}\left[\sum_i \sigma_i^{-2}(5\log F_i - \mu_i)\right]^2$$

$$- \frac{2}{\sum_i \sigma_i^{-2}}\sum_i \sigma_i^{-2}(5\log F_i - \mu_i)\sum_i \sigma_i^{-2}(5\log F_i - \mu_i)$$

$$= \sum_i \sigma_i^{-2}(5\log F_i - \mu_i)^2 - \frac{1}{\sum_i \sigma_i^{-2}}\left[\sum_i \sigma_i^{-2}(5\log F_i - \mu_i)\right]^2$$

(b) condensing notation, $\tilde{\chi}^2 = \sum_i \sigma_i^{-2}\Delta_i^2 - \frac{1}{\sum_i \sigma_i^{-2}}\left(\sum_i \sigma_i^{-2}\Delta_i\right)^2$