Graduate Cosmology Spring 2025
Homework 7 solutions

How to self-grade: full points for correct answer with correct reasoning; half-points for correct reasoning but incorrect answer due to algebra error; zero point for incorrect reasoning (even if final answer is correct out of luck).

Problem 1

1) The geodesic equation is

$$0 = P^\mu \nabla_\mu P^\nu = P^\mu \partial_\mu P^\nu + \Gamma^\nu_{\mu\lambda} P^\mu P^\lambda.$$

We recall that $P^i/P^0 = dx^i/d\eta$, hence

$$P^\mu \partial_\mu P^\nu = P^0 \left(\partial_\eta + \frac{dx^i}{d\eta}\partial_i\right) P^\nu = P^0 \frac{dP^\nu}{d\eta}\Big|_{\rm traj}.$$

Taking the $\nu = 0$ component of the geodesic equation and dividing by P^0, we thus get

$$\frac{dP^0}{d\eta}\Big|_{\rm traj} = -\frac{1}{P^0}\Gamma^0_{\mu\lambda}P^\mu P^\lambda = -\Gamma^0_{00}P^0 - 2\Gamma^0_{0i}P^i - \frac{1}{P^0}\Gamma^0_{ij}P^iP^j.$$

We now look up the relevant Christoffel symbols in the textbook:

$$\Gamma^0_{00} = \mathcal{H} + \Psi', \qquad \Gamma^0_{0i} = \partial_i \Psi, \qquad \Gamma^0_{ij} = \left[\mathcal{H} - \Phi' - 2\mathcal{H}(\Phi + \Psi)\right]\delta_{ij}.$$

Substituting in the geodesic equation, we get

$$\frac{dP^0}{d\eta}\Big|_{\rm traj} = -(\mathcal{H} + \Psi')P^0 - 2P^i\partial_i\Psi + \frac{1}{P^0}\left[-\mathcal{H} + \Phi' + 2\mathcal{H}(\Phi + \Psi)\right]\delta_{ij}P^iP^j.$$

So far this holds for general geodesics. Let us now specialize to *null* geodesics, for which $g_{\mu\nu}P^\mu P^\nu = 0$, which implies

$$(1 - 2\Phi)\delta_{ij}P^iP^j = (1 + 2\Psi)(P^0)^2 \quad \Rightarrow \quad \sqrt{\delta_{ij}P^iP^j} = (1 + \Psi + \Phi)P^0,$$

at linear order in perturbations, and assuming $P^0 > 0$. Substituting $P^i = \sqrt{\delta_{jk}P^jP^k}\,\hat{p}^i$ in the geodesic equation, and working at linear order in perturbations, we obtain the desired equation,

$$\frac{dP^0}{d\eta}\Big|_{\rm traj} = \left[-2\mathcal{H} + \Phi' - \Psi' - 2\hat{p}^i\partial_i\Psi\right]P^0.$$

2) The first order of business is to compute the comoving observers' 4-velocity U^μ. By definition, comoving observers have fixed comoving spatial coordinates, so $U^i = 0$. Normalizing their 4-velocity, $-1 = g_{\mu\nu}U^\mu U^\nu = -a^2(1+2\Psi)(U^0)^2$ implies $U^0 = (1-\Psi)/a$ at linear order in perturbations. Therefore, the observed energy is $E_{\rm obs} = -g_{\mu\nu}U^\mu P^\nu = -g_{00}U^0P^0 = a^2(1+2\Psi)(1-\Psi)/aP^0 = a(1+\Psi)P^0$. Hence, $p \equiv aE_{\rm obs} = a^2(1+\Psi)P^0$ at linear order in perturbations. So

$$\frac{dp}{d\eta}\Big|_{\rm traj} = a^2(1+\Psi)\frac{dP^0}{d\eta}\Big|_{\rm traj} + P^0\frac{d}{d\eta}\Big|_{\rm traj}(a^2(1+\Psi)).$$

Recalling the meaning of $d/d\eta|_{\rm traj} \equiv \partial_\eta + \hat{p}\cdot\vec{\nabla}$, we see that the term $-2\mathcal{H}$ is canceled by $d(a^2)/d\eta$ and $d\Psi/d\eta = \Psi' + \hat{p}^i\partial_i\Psi$ partially cancels some of the other terms, so we are left with

$$\frac{dp}{d\eta}\Big|_{\rm traj} = (\Phi' - \hat{p}^i\partial_i\Psi)p.$$

Problem 2: See updated jupyter notebook on github.

Problem 3

1) Let us start by substituting Φ with its Fourier expansion,

$$\Phi(\eta,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \mathcal{T}_\Phi(\eta,k)\mathcal{R}(\vec{k}),$$

so that

$$\Delta\hat{p}^i = 2(\delta^{ij} - \hat{p}^i\hat{p}^j)\int_{\eta_*}^{\eta_0} d\eta \int \frac{d^3k}{(2\pi)^3} ik_j \; e^{-i\chi\vec{k}\cdot\hat{p}} \mathcal{T}_\Phi(\eta,k)\mathcal{R}(\vec{k}).$$

Therefore, we get, using the fact that the deflection angle is real (so we can replace it with its complex conjugate),

$$\langle|\Delta\hat{p}|^2\rangle = \langle\Delta\hat{p}^i\Delta\hat{p}^i\rangle = 4(\delta^{ij}-\hat{p}^i\hat{p}^j)(\delta^{il}-\hat{p}^i\hat{p}^l)\iint_{\eta_*}^{\eta_0} d\eta d\eta' \iint \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3} k_j k'_l \; e^{-i\chi\vec{k}\cdot\hat{p}} e^{i\chi'\vec{k}'\cdot\hat{p}} \mathcal{T}_\Phi(\eta,k)\mathcal{T}^*_\Phi(\eta',k')\langle\mathcal{R}(\vec{k})\mathcal{R}^*(\vec{k})\rangle.$$

We now insert

$$\langle\mathcal{R}(\vec{k})\mathcal{R}^*(\vec{k})\rangle = P_\mathcal{R}(k)(2\pi)^3\delta_\mathrm{D}(\vec{k}'-\vec{k}) = \frac{2\pi^2}{k^3}\Delta^2_\mathcal{R}(k)(2\pi)^3\delta_\mathrm{D}(\vec{k}'-\vec{k}),$$

so one of the k integrals collapses and we obtain

$$\langle|\Delta\hat{p}|^2\rangle = 4(\delta^{jl} - \hat{p}^j\hat{p}^l)\iint_{\eta_*}^{\eta_0} d\eta \; d\eta' \int \frac{d^3k}{(2\pi)^3} \; k_j k_l \; \frac{2\pi^2}{k^3}\Delta^2_\mathcal{R}(k)\mathcal{T}_\Phi(\eta,k)\mathcal{T}^*_\Phi(\eta',k)e^{i(\chi'-\chi)\vec{k}\cdot\hat{p}}.$$

Assuming matter domination, the gravitational potential remains approximately constant in time, so $\mathcal{T}_\Phi(\eta,k) \approx \mathcal{T}_\Phi(\eta_*,k)$. Hence, we may approximate the variance of the deflection angle by

$$\begin{aligned}\langle|\Delta\hat{p}|^2\rangle &\approx 4(\delta^{jl} - \hat{p}^j\hat{p}^l)\int \frac{d^3k}{(2\pi)^3} \; k_j k_l \; \frac{2\pi^2}{k^3}\Delta^2_\mathcal{R}(k)|\mathcal{T}_\Phi(\eta_*,k)|^2\mathcal{I}(\vec{k}\cdot\hat{p}),\\ \mathcal{I}(\vec{k}\cdot\hat{p}) &\equiv \iint_{\eta_*}^{\eta_0} d\eta \; d\eta' \; e^{i(\chi'-\chi)\vec{k}\cdot\hat{p}}.\end{aligned}$$

Since $\chi = \eta_0 - \eta$, we can rewrite the integrals over η, η' as integrals over $\chi, \chi' \in (0, \chi_* \equiv \eta_0 - \eta_*)$.

2) Changing variables to $u \equiv \chi' - \chi$ and $v \equiv \chi' + \chi$, and computing the Jacobian of the transformation, one finds $dudv = 2d\chi d\chi'$. In addition, $v + u = 2\chi' \in (0, \chi_*)$ and $v - u = 2\chi \in (0, \chi_*)$, so that, for a given $u \in (-\chi_*, \chi_*)$, the variable v is constained to $|u| \leq v \leq 2\chi_* - |u|$. Hence we may rewrite $\mathcal{I}$ as a function of $\mu \equiv \hat{k}\cdot\hat{p}$ as follows

$$\mathcal{I}(\mu \equiv \hat{k}\cdot\hat{p}) = \frac{1}{2}\int_{-\chi_*}^{\chi_*} du \int_{|u|}^{2\chi_*-|u|} dv \; e^{-iuk\mu} = \int_{-\chi_*}^{\chi_*} du \; (\chi_* - |u|)e^{-iuk\mu}.$$

The integral can be computed analytically:

$$\mathcal{I}(\mu) = 2\frac{1-\cos(\chi_* k\mu)}{(k\mu)^2}.$$

For $k\chi_* \gg 1$, this function is sharply peaked at $\mu = 0$, reaching χ_*^2 at $\mu \ll 1/(k\chi_*)$ and decaying as $\sim 1/\mu^2$ for $\mu \gtrsim 1/(k\chi_*)$. We may therefore approximate it as a Dirac delta of μ, with a proportionality constant to be determined by the integral of the function over μ:

$$\mathcal{I}(\mu) \approx C\delta_\mathrm{D}(\mu), \qquad C = \int d\mu \; \mathcal{I}(\mu) = 2\pi\chi_*/k.$$

Since $\delta_\mathrm{D}(\hat{k}\cdot\hat{p}) = k \; \delta_\mathrm{D}(\vec{k}\cdot\hat{p})$ and $\chi_* \approx \eta_0$ we conclude that (note the missing factor 2 in the original problem)

$$\mathcal{I}(\vec{k}\cdot\hat{p}) \approx 2\pi\eta_0\delta_\mathrm{D}(\vec{k}\cdot\hat{p}).$$

3) We first rewrite $(\delta_{jl} - \hat{p}^j\hat{p}^l)k_j k_l = k^2 - (\vec{k}\cdot\hat{p})^2$. Since the approximate Dirac delta picks $\vec{k}\cdot\hat{p} = 0$, we are left with

$$\langle|\Delta\hat{p}|^2\rangle \approx 4\eta_0 \int \frac{dk_x dk_z}{(2\pi)^2} k^2 \frac{2\pi^2}{k^3}\Delta^2_{\mathcal{R}}(k)|\mathcal{T}_\Phi(\eta_*, k)|^2, \qquad k \equiv \sqrt{k_x^2 + k_y^2}.$$

Using cylindrical coordinates, and the fact that the integrand only depends on k but not on the polar angle, we replace $dk_x dk_y = 2\pi k dk$, and arrive at

$$\langle|\Delta\hat{p}|^2\rangle \approx 4\pi \int d\ln k \; k\eta_0 \Delta^2_{\mathcal{R}}(k)|\mathcal{T}_\Phi(\eta_*, k)|^2,$$

again a factor 2 larger than what was originally written in the problem.

4) We saw in Lecture 18 that during radiation domination, the potential decays as $\Phi(k,\eta) \propto 1/(k\eta)^2$ after horizon entry. Therefore, at $\eta_{\rm eq} \approx k_{\rm eq}^{-1}$, the potential has decayed relative to initial conditions by a factor of order $(k_{\rm eq}/k)^2$. As the potential remains roughly constant after matter-radiation equality, this means that, for $k \gtrsim k_{\rm eq}$, corresponding to modes that have entered the horizon in the radiation era, $|\mathcal{T}_\Phi(\eta_*, k)|^2 \propto (k_{\rm eq}/k)^4$. So we will neglect the contribution of $k \gtrsim k_{\rm eq}$ to the integral.

Modes with $k \lesssim k_{\rm eq}$ enter the horizon during matter domination, and Φ remains approximately constant and equal to its initial value, i.e. $\Phi(\eta, k) \approx \frac{2}{3}\mathcal{R}_i$ (lecture 21), the primordial curvature perturbation. Hence $\mathcal{T}_\Phi(\eta_*, k \lesssim k_{\rm eq}) \approx 2/3$. So, we obtain

$$\langle|\Delta\hat{p}|^2\rangle \approx 4\pi(2/3)^2 \int_0^{k_{\rm eq}} \frac{dk}{k}\; k\eta_0 \Delta^2_{\mathcal{R}}(k) = 4\pi(2/3)^2 A_s \int_0^{k_{\rm eq}} \frac{dk}{k}\; k\eta_0 (k/k_P)^{n_s - 1} = \frac{16\pi}{9} A_s \frac{1}{n_s}(k_P\eta_0)(k_{\rm eq}/k_P)^{n_s}.$$

Plugging $A_s = 2\times 10^{-9}$, $n_s = 0.96$, $\eta_0 \approx 14$ Gpc, $k_{\rm eq} \approx 0.01$ Mpc^{-1} and $k_P = 0.05$ Mpc^{-1}, we arrive at

$$\langle|\Delta\hat{p}|^2\rangle \approx 1.7\times 10^{-6}.$$

This is in radians squared. Converting to arcminutes (1 arcmin = (1/60) degree = (1/60) $\times\pi/180$ = $\pi/60/180$ radians), we find

$$\langle|\Delta\hat{p}|^2\rangle^{1/2} \approx 4.5 \text{ arcmin}.$$

A more accurate calculation gives a rms deflection angle of 2.7 arcmin. If you are interested in learning more about lensing of the CMB, see this review article by Lewis and Challinor: `https://arxiv.org/abs/astro-ph/0601594`.

Problem 4

1) In Fourier space, products become convolutions, and constants transform to $(2\pi)^3\delta_{\rm D}$:

$$\mathcal{R}(\vec{k}) = G(\vec{k}) + f\left[\int \frac{d^3k'}{(2\pi)^3} G(\vec{k}')G(\vec{k}-\vec{k}') - \langle G^2\rangle(2\pi)^3\delta_{\rm D}(\vec{k})\right]$$

2) We rewrite the equation above as $\mathcal{R}(\vec{k}) = G(\vec{k}) + f\ H(\vec{k})$, where H is quadratic in G. Since the three point function of a Gaussian random field vanishes, we conclude that, at lowest order in f,

$$\langle\mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_2)\mathcal{R}(\vec{k}_3)\rangle = f\left[\langle H(\vec{k}_1)G(\vec{k}_2)G(\vec{k}_3)\rangle + (\vec{k}_1 \leftrightarrow \vec{k}_2) + (\vec{k}_1 \leftrightarrow \vec{k}_3)\right].$$

Let us compute the first term:

$$\langle H(\vec{k}_1)G(\vec{k}_2)G(\vec{k}_3)\rangle = \int \frac{d^3k'}{(2\pi)^3}\langle G(\vec{k}')G(\vec{k}_1-\vec{k}')G(\vec{k}_2)G(\vec{k}_3)\rangle - \langle G^2\rangle(2\pi)^3\delta_{\rm D}(\vec{k}_1)\langle G(\vec{k}_2)G(\vec{k}_3)\rangle. \tag{1}$$

Since G is Gaussian, we may use Wick's theorem:

$$\begin{aligned}\big\langle G(\vec{k}')G(\vec{k}_1-\vec{k}')G(\vec{k}_2)G(\vec{k}_3)\big\rangle = &\big\langle G(\vec{k}')G(\vec{k}_1-\vec{k}')\big\rangle\big\langle G(\vec{k}_2)G(\vec{k}_3)\big\rangle\\ &+\big\langle G(\vec{k}')G(\vec{k}_2)\big\rangle\big\langle G(\vec{k}_1-\vec{k}')G(\vec{k}_3)\big\rangle\\ &+\big\langle G(\vec{k}')G(\vec{k}_3)\big\rangle\big\langle G(\vec{k}_1-\vec{k}')G(\vec{k}_2)\big\rangle.\end{aligned}$$

We now u the reality of $G(\vec{x})$, implying

$$\langle G(\vec{k})G(\vec{k}')\rangle = \langle G(\vec{k})G^*(-\vec{k})\rangle = (2\pi)^3\delta_{\rm D}(\vec{k}+\vec{k}')P_G(k).$$

Hence, we obtain

$$\begin{aligned}\langle G(\vec{k}')G(\vec{k}_1-\vec{k}')G(\vec{k}_2)G(\vec{k}_3)\rangle &= P_G(k')(2\pi)^3\delta_{\rm D}(\vec{k}_1)\langle G(\vec{k}_2)G(\vec{k}_3)\rangle \\ &+(2\pi)^6 P_G(k_2)P_G(k_3)\left[\delta_{\rm D}(\vec{k}'+\vec{k}_2)\delta_{\rm D}(\vec{k}_1-\vec{k}'+\vec{k}_3)+\delta_{\rm D}(\vec{k}'+\vec{k}_3)\delta_{\rm D}(\vec{k}_1-\vec{k}'+\vec{k}_2)\right] \quad (2)\end{aligned}$$

Upon integrating over d^3k', the first term in Eq. (2) precisely cancels the last term in Eq. (1). The integral of the second line of (2) over d^3k' is straightforward due to the Dirac delta function, and both terms give the same result, evenutally arriving at

$$\langle H(\vec{k}_1)G(\vec{k}_2)G(\vec{k}_3)\rangle = 2\times(2\pi)^3 P_G(k_2)P_G(k_3)\delta_{\rm D}(\vec{k}_1+\vec{k}_2+\vec{k}_3).$$

Putting everything together, we arrive at the desired form,

$$\langle \mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_2)\mathcal{R}(\vec{k}_3)\rangle = (2\pi)^2\delta_{\rm D}(\vec{k}_1+\vec{k}_2+\vec{k}_3)B(k_1,k_2,k_3),$$

with

$$B(k_1,k_2,k_3) = 2f\left[P_G(k_2)P_G(k_3)+P_G(k_1)P_G(k_3)+P_G(k_1)P_G(k_2)\right].$$

3) In Fourier space, we get

$$\mathcal{R}(\vec{k}) = G(\vec{k}) + g\int\frac{d^3k'}{(2\pi)^3}\frac{d^3k''}{(2\pi)^3}G(\vec{k}')G(\vec{k}'')G(\vec{k}-\vec{k}'-\vec{k}'').$$

4) Again, we rewrite the equation above as $\mathcal{R}(\vec{k}) = G(\vec{k}) + g\ H(\vec{k})$, where H is cubic in G. By Wick's theorem, the connected 4-point function of G vanishes, hence, to linear order in g, the trispectrum is

$$\langle \mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_2)\mathcal{R}(\vec{k}_3)\mathcal{R}(\vec{k}_4)\rangle_c = g\langle H(\vec{k}_1)G(\vec{k}_2)G(\vec{k}_3)G(\vec{k}_4)\rangle_c + 3\text{ permutations},$$

where the permutations exchange to which wavenumber H is evaluated. We now plug in the convolution for H, and we will put square brackets around the products of 3 G's inside to indicate that they are to be taken as a block when removing the disconnected part of the 4-point function:

$$\langle H(\vec{k}_1)G(\vec{k}_2)G(\vec{k}_3)G(\vec{k}_4)\rangle_c = \int\frac{d^3k'}{(2\pi)^3}\frac{d^3k''}{(2\pi)^3}\langle[G(\vec{k}')G(\vec{k}'')G(\vec{k}_1-\vec{k}'-\vec{k}'')]G(\vec{k}_2)G(\vec{k}_3)G(\vec{k}_4)\rangle_c,$$

where

$$\begin{aligned}\langle[G(\vec{k}')G(\vec{k}'')G(\vec{k}_1-\vec{k}'-\vec{k}'')]G(\vec{k}_2)G(\vec{k}_3)G(\vec{k}_4)\rangle_c \equiv &\ \langle G(\vec{k}')G(\vec{k}'')G(\vec{k}_1-\vec{k}'-\vec{k}'')G(\vec{k}_2)G(\vec{k}_3)G(\vec{k}_4)\rangle \\ &-\langle[G(\vec{k}')G(\vec{k}'')G(\vec{k}_1-\vec{k}'-\vec{k}'')]G(\vec{k}_2)\rangle\ \langle G(\vec{k}_3)G(\vec{k}_4)\rangle \\ &-\langle[G(\vec{k}')G(\vec{k}'')G(\vec{k}_1-\vec{k}'-\vec{k}'')]G(\vec{k}_3)\rangle\ \langle G(\vec{k}_2)G(\vec{k}_4)\rangle \\ &-\langle[G(\vec{k}')G(\vec{k}'')G(\vec{k}_1-\vec{k}'-\vec{k}'')]G(\vec{k}_4)\rangle\ \langle G(\vec{k}_2)G(\vec{k}_3)\rangle.\end{aligned}$$

The first term is a 6-point function (!!), which, according to Wick's theorem, contains $5\times3=15$ distinct terms. Each of the 4-point functions contain 3 different terms. I let you convince yourselves that the 9 terms contributed by lines 2-4 of the expression above are part of the 15 terms contributed by the 6-point function in the first line. So we are left with 6 distinct terms contributed by the 6-point function, which are obtained when each one of the 3 G's inside the square brackets is paired with a G that is oustide the square brackets (i.e. $G(\vec{k}_2), G(\vec{k}_3)$ or $G(\vec{k}_4)$). There are precisely 6 ways to do this pairing, so we have the correct total number of terms. So, we get

$$\begin{aligned}\langle[G(\vec{k}')G(\vec{k}'')G(\vec{k}_1-\vec{k}'-\vec{k}'')]G(\vec{k}_2)G(\vec{k}_3)G(\vec{k}_4)\rangle_c &= \langle G(\vec{k}')G(\vec{k}_2)\rangle\ \langle G(\vec{k}'')G(\vec{k}_3)\rangle\ \langle G(\vec{k}_1-\vec{k}'-\vec{k}'')G(\vec{k}_4)\rangle + 5\text{ permutations} \\ &= (2\pi)^9\ P_G(k_2)P_G(k_3)P_G(k_4)\left[\delta_{\rm D}(\vec{k}'+\vec{k}_2)\delta_{\rm D}(\vec{k}''+\vec{k}_3)\delta_{\rm D}(\vec{k}_1-\vec{k}'-\vec{k}''+\vec{k}_4)+5\text{ permutations}\right] \\ &= (2\pi)^9\ P_G(k_2)P_G(k_3)P_G(k_4)\left[\delta_{\rm D}(\vec{k}'+\vec{k}_2)\delta_{\rm D}(\vec{k}''+\vec{k}_3)\delta_{\rm D}(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)+5\text{ permutations}\right],\end{aligned}$$

where we substituted $\vec{k}' = -\vec{k}_1$ and $\vec{k}'' = -\vec{k}_3$ in the last Dirac delta function. We see that all 6 permutations of (k_2, k_3, k_3) actually give the same result. Upon integrating over d^3k' and d^3k'', we thus get

$$\langle H(\vec{k}_1)G(\vec{k}_2)G(\vec{k}_3)G(\vec{k}_4)\rangle_c = 6 \times (2\pi)^3 P_G(k_2)P_G(k_3)P_G(k_4)\delta_{\rm D}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4).$$

So, our end result is

$$\langle \mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_2)\mathcal{R}(\vec{k}_3)\mathcal{R}(\vec{k}_4)\rangle_c = (2\pi)^3\delta_{\rm D}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)T(k_1, k_2, k_3, k_4),$$

with

$$\begin{aligned} T(k_1, k_2, k_3, k_4) &= 6g\left[P_G(k_2)P_G(k_3)P_G(k_4) + 3 \text{ permutations}\right] \\ &= 6gP_G(k_1)P_G(k_2)P_G(k_3)P_G(k_4)\left[\frac{1}{P_G(k_1)} + \frac{1}{P_G(k_2)} + \frac{1}{P_G(k_3)} + \frac{1}{P_G(k_4)}\right]. \end{aligned}$$