19/28

Cosmology Homework 5 $\quad a = \frac{1}{1+z} \quad da = -(1+z)^{-2}\,dz$ Connor Hainje

code on GitHub at github.com/cmhainje/cosmo-hw

Problem 1: 5/6

part 1. see notebook 2/2

part 2. 1/1

$$\Theta(\eta_0, \vec{x}=0, \hat{p}) = \int_0^{\eta_0} d\eta \; g(\eta) \; S(\eta, \vec{x}=(\eta-\eta_0)\hat{p}, \hat{p})$$

where $g(\eta) \equiv a\, n_e\, \sigma_T \exp\left[-\int_\eta^{\eta_0} d\eta'\, a' n_e' \sigma_T\right]$ $\qquad S \equiv \Theta_0 + \vec{V}_e \cdot \hat{p} + \frac{3}{20} Q_{ij}\, \hat{p}_i \hat{p}_j$

for $z \lesssim z_{reio}$, we assume S to be negligible

$$\Rightarrow \Theta(\eta_0, \vec{x}=0, \hat{p}) \simeq \int_0^{\eta_{reio}} d\eta \; g(\eta) \; S(\eta, \vec{x}=(\eta-\eta_0)\hat{p}, \hat{p})$$

define $\tau_{reio} = \int_{t_{reio}}^{t_0} dt\, n_e \sigma_T = \int_{\eta_{reio}}^{\eta_0} d\eta\, a\, n_e\, \sigma_T$

$$g(\eta_{reio}) = a\, n_e\, \sigma_T\, e^{-\tau_{reio}}$$

$$g(\eta < \eta_{reio}) = a\, n_e\, \sigma_T\, e^{-\tau_{reio}} \exp\left[-\int_\eta^{\eta_{reio}} d\eta'\, a' n_e' \sigma_T'\right]$$

$\equiv e^{-\tau_{reio}}\, \tilde{g}(\eta)$, defining $\tilde{g}$ to be the value of g ignoring reio

$$\Rightarrow \Theta(\eta_0, \vec{x}=0, \hat{p}) = e^{-\tau_{reio}} \int_0^{\eta_{reio}} d\eta \; \tilde{g}(\eta) \; S(\eta, \vec{x}=(\eta-\eta_0)\hat{p}, \hat{p})$$ ✓

part 3. 2/3 given Y_{He} and assuming H, He are fully ionized, $n_p = \frac{1-Y_{He}}{m_p}\rho_b$ and $n_{He^{++}} = \frac{Y_{He}}{m_{He^{++}}}\rho_b$.

assume charge neutrality $\Rightarrow n_e = n_H + 2n_{He} = \left(\frac{1-Y_{He}}{m_p} + \frac{2Y_{He}}{m_{He^{++}}}\right)\rho_b$

$$\rho_b = \rho_{b,0}\, a^{-3}, \quad \rho_{b,0} = \rho_{crit,0}\, \Omega_{b,0} = \rho_{crit,0}\, \omega_b h^2 = \frac{3\,(100\text{ km/s/Mpc})^2}{8\pi G}\,\omega_b$$

then
$$\tau_{reio} = \int_{t_{reio}}^{t_0} dt\, n_e\, \sigma_T$$
$$= \left(\frac{1-Y_{He}}{m_p} + \frac{2Y_{He}}{m_{He^{++}}}\right)\rho_{b,0}\,\sigma_T \int_{t_{reio}}^{t_0} dt\, a^{-3}$$
$$= \left(\frac{1-Y_{He}}{m_p} + \frac{2Y_{He}}{m_{He^{++}}}\right)\rho_{b,0}\,\sigma_T \int_{a_{reio}}^{1} \frac{da}{\dot{a}\, a^3}$$

assuming $\omega_k = 0$ and $\omega_r \ll \omega_m, \omega_\Lambda$ then $\dot{a}\, a^3 = \left(100\,\frac{\text{km/s}}{\text{Mpc}}\right)\sqrt{\omega_\Lambda}\, a_{m\Lambda}^4\, x^2 \sqrt{x + x^4}$ $\quad (x \equiv a/a_{m\Lambda})$

$$\approx \left(\frac{1-Y_{He}}{m_p} + \frac{2Y_{He}}{m_{He^{++}}}\right)\frac{\rho_{b,0}\,\sigma_T}{(100\text{ km/s/Mpc})\sqrt{\omega_\Lambda}\, a_{m\Lambda}^3} \int_{a_{reio}/a_{m\Lambda}}^{1/a_{m\Lambda}} \frac{dx}{x^2\sqrt{x+x^4}}$$ ✓ $\quad a_{m\Lambda} = \left(\omega_m/\omega_\Lambda\right)^{1/3}$

$$= \left(\frac{1-Y_{He}}{m_p} + \frac{2Y_{He}}{m_{He^{++}}}\right)\frac{\rho_{b,0}\,\sigma_T}{(100\text{ km/s/Mpc})\sqrt{\omega_\Lambda}\, a_{m\Lambda}^3}\, \frac{2}{3}\left[\sqrt{1 + \left(\frac{a_{m\Lambda}}{a_{reio}}\right)^3} - \sqrt{1 + \left(\frac{a_{m\Lambda}}{1}\right)^3}\right]$$ ✓

$$= \left(\frac{1-Y_{He}}{m_p} + \frac{2Y_{He}}{m_{He^{++}}}\right)\frac{\rho_{b,0}\,\sigma_T}{(100\text{ km/s/Mpc})\sqrt{\omega_m}}\, \frac{2}{3}\left[\sqrt{a_{m\Lambda}^{-3} + a_{reio}^{-3}} - \sqrt{a_{m\Lambda}^{-3} + 1}\right]$$ ✓

$$= \alpha\left[\sqrt{a_{m\Lambda}^{-3} + (1+z_{reio})^3} - \sqrt{a_{m\Lambda}^{-3} + 1}\right]$$ where $\alpha = 2.415 \times 10^{-3}$ for Planck 18 parameters

inverting: $z_{reio} = \left[\left(\frac{\tau_{reio}}{\alpha}\right)^2 + 2\,\frac{\tau_{reio}}{\alpha}\sqrt{a_{m\Lambda}^{-3} + 1} + 1\right]^{1/3} - 1 = 11.88$ $\quad$ (using $\tau_{reio} = 0.054 \pm 0.007$)

error prop: $\sigma_z = \frac{\partial z}{\partial \tau}\sigma_\tau = \frac{1}{3}\left[\left(\frac{\tau_{reio}}{\alpha}\right)^2 + 2\,\frac{\tau_{reio}}{\alpha}\sqrt{a_{m\Lambda}^{-3}+1} + 1\right]^{-2/3}\left[\frac{2\tau_{reio}}{\alpha^2} + \frac{2}{\alpha}\sqrt{a_{m\Lambda}^{-3}+1}\right]\sigma_\tau = 0.43$

I had a small bug in my code to compute z_{reio}: accidentally wrote $a_{m\Lambda} = \left(\Omega_m/\Omega_\Lambda\right)^3$. $\boxed{-1}$

after fix, I got $z_{reio} = 9.8 \pm 0.29$. I am dubious about the assumption of matter domination, though.

Problem 2: 4/8

part 1. $\Theta(\eta, \vec{x}=0, \hat{p}) = \Theta_0(\eta_*, -(\eta-\eta_*)\hat{p}) + \hat{p}\cdot\vec{V}_e(\eta_*, -(\eta-\eta_*)\hat{p})$

0/3

Fourier: $\Theta_0(\eta_*, \vec{x}) = \Theta_* e^{i\vec{k}\cdot\vec{x}}$, scalar V_e: $\vec{V}_e = i\hat{k}\, V_* e^{i\vec{k}\cdot\vec{x}}$

then $\Theta(\eta, \vec{0}, \hat{p}) = (\Theta_* + i\,\hat{k}\cdot\hat{p}\, V_*)\exp[-ik(\eta-\eta_*)(\hat{k}\cdot\hat{p})]$

I was confused and didn't see that I needed to integrate over $\hat{p}$.

so $\Theta_0 = \Theta_* \exp[-ik(\eta-\eta_*)(\hat{k}\cdot\hat{p})]$ and $V_\gamma = V_* \exp[-ik(\eta-\eta_*)(\hat{k}\cdot\hat{p})]$?

can be written $\Theta_0 = \Theta_*\left(e^{-i\hat{k}\cdot\hat{p}}\right)^{k(\eta-\eta_*)}$, $V_\gamma = V_*\left(e^{-i\hat{k}\cdot\hat{p}}\right)^{k(\eta-\eta_*)}$

$e^{-i\hat{k}\cdot\hat{p}} < 1$ except when $\hat{k}\perp\hat{p}$, so $k(\eta-\eta_*) \gg 1$ suppresses

$(\mathcal{H} = aH)$

part 2. $\frac{1}{H(\eta_*)}$ is a physical length, $\frac{1}{\mathcal{H}(\eta_*)}$ is the corresponding comoving length

1/2

taking $z_* = 1100$, comoving wavenumber $k_* = \frac{1}{c\text{ length}} = aH(\eta_*) = \boxed{0.0048\ \text{Mpc}^{-1}} = (208\ \text{Mpc})^{-1}$ ✓

didn't calculate suppression

part 3. $\Theta = \sum_\ell (2\ell+1)\,\Theta_\ell(\eta)\, P_\ell(\hat{p}\cdot\hat{k})$

but also $\Theta = (\Theta_* + i\,\hat{k}\cdot\hat{p}\, V_*)\exp[ik\chi_*(\hat{k}\cdot\hat{p})]$ defining $\chi_* = -(\eta-\eta_*)$

3/3

$= \left(\Theta_* + V_* \frac{d}{d(k\chi_*)}\right)\sum_\ell (2\ell+1)\, i^\ell j_\ell(k\chi_*)\, P_\ell(\hat{k}\cdot\hat{p})$

$= \sum_\ell (2\ell+1)\, i^\ell \left[\Theta_* j_\ell(k\chi_*) + V_* j_\ell'(k\chi_*)\right] P_\ell(\hat{k}\cdot\hat{p})$

then $\boxed{\Theta_\ell = i^\ell\left[\Theta_* j_\ell(k\chi_*) + V_* j_\ell'(k\chi_*)\right]}$ ✓

sign wrong due to $(-i)^\ell$, I'm counting it

Problem 3: 5/5

part 1. 1/1

$$\tilde{T}^\mu{}_\nu = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} T^\alpha{}_\beta(x^\lambda)$$

$$= (\delta^\mu{}_\alpha + \partial_\alpha \xi^\mu)(\delta^\beta{}_\nu - \partial_\nu \xi^\beta)\; T^\alpha{}_\beta(\tilde{x}^\lambda - \xi^\lambda)$$

$$= (\delta^\mu{}_\alpha \delta^\beta{}_\nu - \delta^\mu{}_\alpha \partial_\nu \xi^\beta + \partial_\alpha \xi^\mu \delta^\beta{}_\nu)\left[T^\alpha{}_\beta(\tilde{x}^\lambda) - \xi^\lambda \partial_\lambda T^\alpha{}_\beta\right]$$

$$= T^\mu{}_\nu(\tilde{x}^\lambda) - \left(\partial_\nu \xi^\beta T^\mu{}_\beta - \partial_\alpha \xi^\mu T^\alpha{}_\nu + \xi^\lambda \partial_\lambda T^\mu{}_\nu\right)$$

$\tilde{T}^\mu{}_\nu = \bar{T}^\mu{}_\nu(\tilde{x}^\lambda) + \delta\tilde{T}^\mu{}_\nu(\tilde{x}^\lambda)$ and $T^\mu{}_\nu(\tilde{x}^\lambda) = \bar{T}^\mu{}_\nu(\tilde{x}^\lambda) + \delta T^\mu{}_\nu(\tilde{x}^\lambda)$

so $$\delta\tilde{T}^\mu{}_\nu - \delta T^\mu{}_\nu\Big|_{\tilde{x}^\lambda} = -\left(\partial_\nu \xi^\beta \bar{T}^\mu{}_\beta - \partial_\alpha \xi^\mu \bar{T}^\alpha{}_\nu + \xi^\lambda \partial_\lambda \bar{T}^\mu{}_\nu\right)$$ ✓

part 2. 3/3

$T^0{}_0 = -(\bar{\rho} + \delta\rho)$ $\quad T^i{}_0 = -(\bar{\rho} + \bar{P})V^i$ $\quad T^i{}_j = (\bar{P} + \delta P)\delta^i{}_j + \Pi^i{}_j$

$\Rightarrow \delta T^0{}_0 = -\delta\rho$ $\quad \delta T^i{}_0 = -(\bar{\rho} + \bar{P})V^i$ $\quad \delta T^i{}_j = \delta P\,\delta^i_j + \Pi^i{}_j$

thus

${}^0{}_0$: $$(-\tilde{\delta\rho}) - (-\delta\rho) = -\left(\partial_0 \xi^\beta \bar{T}^0{}_\beta - \partial_\alpha \xi^0 \bar{T}^\alpha{}_0 + \xi^\lambda \partial_\lambda \bar{T}^0{}_0\right)$$

$$\delta\rho - \tilde{\delta\rho} = -\left(-\partial_0 \xi^0 \bar{\rho} + \partial_0 \xi^0 \bar{\rho} - \xi^\lambda \partial_\lambda \bar{\rho}\right)$$

$$\tilde{\delta\rho} = \delta\rho - \xi^0 \partial_0 \bar{\rho}$$ ✓

${}^i{}_0$: $$\delta\tilde{T}^i{}_0 - \delta T^i{}_0 = -\left(\partial_0 \xi^\beta \bar{T}^i{}_\beta - \partial_\alpha \xi^i \bar{T}^\alpha{}_0 + \xi^\lambda \partial_\lambda \bar{T}^i{}_0\right)$$

$$-(\bar{\rho} + \bar{P})(\tilde{V}^i - V^i) = -\partial_0 \xi^i (\bar{\rho} + \bar{P})$$

$$\tilde{V}^i = V^i + \partial_0 \xi^i$$ ✓

${}^i{}_j$: $$\delta\tilde{T}^i{}_j - \delta T^i{}_j = -\left(\partial_j \xi^\beta \bar{T}^i{}_\beta - \partial_\alpha \xi^i \bar{T}^\alpha{}_j + \xi^\lambda \partial_\lambda \bar{T}^i{}_j\right)$$

$$\tilde{\delta P}\,\delta^i_j + \tilde{\Pi}^i{}_j - \delta P\,\delta^i_j - \Pi^i{}_j = -\xi^\lambda \partial_\lambda \bar{P}\,\delta^i{}_j$$

grouping terms proportional to $\delta^i{}_j$ and $\perp$ to it gives:

$$\tilde{\delta P} = \delta P - \xi^0 \partial_0 \bar{P}$$ ✓ and $$\tilde{\Pi}^i{}_j = \Pi^i{}_j$$ ✓.

part 3. 1/1

$$\mathcal{R} = -C - \tfrac{1}{3}k^2 E + \mathcal{H}\frac{\delta\rho}{\bar{\rho}'}$$

$$\tilde{\mathcal{R}} = -\tilde{C} - \tfrac{1}{3}k^2 \tilde{E} + \mathcal{H}\frac{\tilde{\delta\rho}}{\bar{\rho}'}$$

$$= -\left[C - \mathcal{H}\xi^0 + \tfrac{1}{3}k^2 L\right] - \tfrac{1}{3}k^2[E - L] + \mathcal{H}\frac{1}{\bar{\rho}'}\left[\delta\rho - \xi^0\partial_0\bar{\rho}\right]$$

$$= -C + \cancel{\mathcal{H}\xi^0} - \cancel{\tfrac{1}{3}k^2 L} - \tfrac{1}{3}k^2 E + \cancel{\tfrac{1}{3}k^2 L} + \mathcal{H}\frac{\delta\rho}{\bar{\rho}'} - \cancel{\mathcal{H}\xi^0}$$

$$= \mathcal{R}$$

$\square$

Problem 4: 4/9

part 1. $g_{\mu\nu} = a^2\left[\eta_{\mu\nu} + h_{\mu\nu}\right]$

1/1

consider $A^{\mu\nu} = a^{-2}\left[\eta^{\mu\nu} - h^{\mu\nu}\right] = a^{-2}\left[\eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}\right]$

if $A^{\mu\nu} g_{\nu\rho} = \delta^{\mu}{}_{\rho}$, then $A^{\mu\nu}$ is the inverse metric i.e. $g^{\mu\nu} = A^{\mu\nu}$.

let's evaluate it:
$$\begin{aligned}
A^{\mu\nu} g_{\nu\rho} &= a^{-2}\left[\eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}\right] a^2\left[\eta_{\nu\rho} + h_{\nu\rho}\right] \\
&= \left[\eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}\right]\left[\eta_{\nu\rho} + h_{\nu\rho}\right] \\
&= \delta^{\mu}{}_{\rho} + \eta^{\mu\nu}h_{\nu\rho} - \eta^{\mu\alpha}\delta^{\beta}{}_{\rho}h_{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}h_{\nu\rho} \\
&= \delta^{\mu}{}_{\rho} + \cancel{\eta^{\mu\nu}h_{\nu\rho}} - \cancel{\eta^{\mu\alpha}h_{\alpha\rho}} + O(h^2) \\
&= \delta^{\mu}{}_{\rho} + O(h^2)
\end{aligned}$$

So indeed $g^{\mu\nu} = a^{-2}\left[\eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}\right]$ to linear order in perturbation h. $\square$

part 2. note that $h^{\mu\nu} = -h_{\mu\nu}$ for our metric

2/2

$$G_{00} = 3\left(\frac{a'}{a}\right)^2 - 2\frac{a'}{a}\partial_i h_{0i} = 3\left(\frac{a'}{a}\right)^2$$
$$= \boxed{3\mathcal{H}^2} \checkmark$$

$$G_{0i} = \left(\frac{a'}{a}\right)^2 h_{0i} - 2\frac{a''}{a}h_{0i} + \partial_j\partial_{(i}h_{j)0} = \left(\frac{a'}{a}\right)^2 h_{0i} - 2\frac{a''}{a}h_{0i} - \frac{1}{2}\partial_j\partial_j h_{0i}$$
$$= \boxed{-\left(2\mathcal{H}' + \mathcal{H}^2 + \frac{1}{2}\vec{\nabla}^2\right)h_{0i}} \checkmark$$

$$G_{ij} = \delta_{ij}\left[\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a} + 2\frac{a'}{a}\partial_k h_{0k} + \partial_0\partial_k h_{0k}\right]$$
$$- \frac{a'}{a}\partial_j h_{0i} - \frac{a'}{a}\partial_i h_{0j} - \frac{1}{2}\partial_0\partial_j h_{0i} - \frac{1}{2}\partial_0\partial_i h_{0j}$$
$$= \boxed{\delta_{ij}\left[-2\mathcal{H}' - \mathcal{H}^2\right] - \mathcal{H}\partial_j h_{0i} - \mathcal{H}\partial_i h_{0j} - \frac{1}{2}\partial_0\partial_j h_{0i} - \frac{1}{2}\partial_0\partial_i h_{0j}} \checkmark$$

part 3. EFEs: $G_{\mu\nu} = 8\pi G\, T_{\mu\nu} = 8\pi G\, T_{\nu\mu} = 8\pi G\, g_{\nu\alpha}T^{\alpha}{}_{\mu}$

2/3

EFE$_{0i}$:
$$G_{0i} = 8\pi G\, g_{i\alpha}T^{\alpha}{}_{0} = 8\pi G\left[\underline{g_{i0}T^{0}{}_{0}} + g_{ij}T^{j}{}_{0}\right]$$
$$= -8\pi G\left[\underline{h_{0i}(\bar{\rho} + \delta\rho)} + (\bar{\rho} + \bar{P})V_i\right]a^2 \quad -1$$
$$-\left(2\mathcal{H}' + \mathcal{H}^2 + \frac{1}{2}\vec{\nabla}^2\right)h_{0i} = -8\pi G\left(\bar{\rho}h_{0i} + (\bar{\rho} + \bar{P})V_i\right)$$

re-arranging: $$\boxed{\left(\mathcal{H}^2 + 2\mathcal{H}' + \frac{1}{2}\nabla^2 - \underline{8\pi G\bar{\rho}}\right)h_{0i} = 8\pi G\,(\bar{\rho} + \bar{P})V_i}$$

this $\bar{\rho}$ term is not in the solution: why?

isn't $T_{i0} = g_{i\mu}T^{\mu}{}_{0}$ not there
$$= g_{i0}T^{0}{}_{0} + g_{ij}T^{j}{}_{0}$$
$$= (a^2 h_{0i})(-(\bar{\rho} + \delta\rho)) + (a^2\delta_{ij})(-(\bar{\rho} + \bar{P})V^j)$$
$$= -a^2\left(h_{0i}\bar{\rho} + (\bar{\rho} + \bar{P})V^i\right) \; ?$$

part 4.

0/3

$$0 = \nabla_\mu T^\mu{}_i = \nabla_0 T^0{}_i + \nabla_j T^j{}_i$$

$$= -\nabla_0\left((\bar\rho+\bar P)V_i\right) + \nabla_j\left((\bar P+\delta P)\delta^j{}_i\right)$$

$$= -\nabla_0\left((\bar\rho+\bar P)V_i\right) + \nabla_i \delta P$$

$$\nabla_0\left[\left(\mathcal{H}^2 + 2\mathcal{H}' + \tfrac{1}{2}\vec\nabla^2 - 8\pi G\bar\rho\right)h_{0i}\right] = 8\pi G\, \nabla_i \delta P \qquad ?$$

I couldn't figure this out