Cosmology HW6 Connor Hainje

Problem 1:

part 1. $dU = -P\,dV + \delta Q$, $dU = \rho\,dV + V\,d\rho \rightarrow \delta Q = V\,d\rho + (\rho+P)\,dV$

$\frac{\delta Q}{dt} = V\dot{\rho} + (\rho+P)\dot{V}$. $V \sim a^3$ so $\frac{dV}{V} = 3\frac{da}{a} \Rightarrow \dot{V} = 3HV$

$\Rightarrow \frac{\delta Q}{dt} = [\dot{\rho} + 3H(\rho+P)]\,V \Rightarrow \dot{q} = \frac{\partial}{\partial V}\frac{\delta Q}{dt} = \dot{\rho} + 3H(\rho+P)$.

$\rho = mn + \frac{3}{2}nT$, $P = nT$. $\dot{\rho} = \left(m + \frac{3}{2}T\right)\dot{n} + \frac{3}{2}n\dot{T}$. assuming $a^3 n$ is constant, $\dot{n} = -3Hn$.

thus $\dot{\rho} = -3H\rho + \frac{3}{2}n\dot{T}$. $\Rightarrow \dot{q} = \frac{3}{2}n\dot{T} + 3HnT$

rearranging, $\dot{T} + 2HT = \frac{2}{3}\frac{\dot{q}}{n}$ $\square$

part 2. in e^- rest frame: initially, e^- has $\vec{v}_i = 0$, γ has $\vec{p}_i$

finally, e^- has $\vec{v}_f$, γ has $\vec{p}_f$

conservation of momentum: $\vec{p}_f + m_e\vec{v}_f = \vec{p}_i \Rightarrow \vec{v}_f = \frac{\vec{p}_i - \vec{p}_f}{m_e}$

assuming $|\vec{p}_i| = |\vec{p}_f|$, let $\vec{p}_i = p\,\hat{p}_i$, $\vec{p}_f = p\,\hat{p}_f$. then $\vec{v}_f = \frac{p}{m_e}(\hat{p}_i - \hat{p}_f)$

final electron energy is $E_f = \frac{1}{2}m_e\vec{v}_f^{\,2} = \frac{1}{2}m_e\frac{p^2}{m_e^2}(\hat{p}_i - \hat{p}_f)^2 = \frac{p^2}{m_e}(1-\cos\theta)$

since initial e^- energy is 0 and photon energy $= p$ (up to c^2), then $\Delta E_e = \frac{E_\gamma^2}{m_e}(1-\cos\theta)$ $\square$

part 3. a scatter with photon of energy E_γ at angle θ imparts ΔE_e on the electron

the scattering rate is $dN_s / dt\,dV\,dE_\gamma\,d\cos\theta$. the heating rate is $\dot{q} = \frac{dQ}{dV\,dt} = \frac{dN_s\,\Delta E_e}{dt\,dV}$

$$\rightarrow \dot{q} = \int dE_\gamma\,d\cos\theta\,d\varphi\,\frac{dN_s}{dt\,dV\,dE_\gamma\,d\cos\theta}\,\Delta E$$

$$= \int_0^\infty dE_\gamma \int_{-1}^{+1} d\cos\theta \int_0^{2\pi} d\varphi\; n_e \frac{d\sigma_T}{d\cos\theta}\frac{dn_\gamma}{dE_\gamma}\left(1 + f_\gamma(E_\gamma)\right)\frac{E_\gamma^2}{m_e}(1-\cos\theta)$$

components: $d\sigma_T / d\cos\theta = \frac{3}{16\pi}\sigma_T\,(1+\cos^2\theta)$, $f_\gamma(E_\gamma) = \left[\exp E_\gamma/T_\gamma - 1\right]^{-1}$

$$n_\gamma = \frac{2}{h^3}\int d^3p\,f_\gamma(E(p)) = \frac{2}{h^3}\int dp\,4\pi p^2 f_\gamma(E(p)) = \frac{8\pi}{h^3}\int dE\,E^2 f(E) \Rightarrow \frac{dn}{dE} = \frac{8\pi}{h^3}E^2 f(E)$$

so
$$\dot{q} = \int_0^\infty dE_\gamma \int_{-1}^{+1} dc \int_0^{2\pi} d\varphi\; n_e \frac{3\sigma_T}{16\pi}(1+c^2)\frac{8\pi}{h^3}E_\gamma^2 f_\gamma(1+f_\gamma)\frac{E_\gamma^2}{m_e}(1-c)$$

$$= \frac{n_e}{m_e}2\pi \int_{-1}^{+1} dc\,\frac{3\sigma_T}{16\pi}(1+c^2)(1-c)\int_0^\infty dE\,\frac{8\pi}{h^3}E^4 f(1+f)$$

$$= \frac{n_e\sigma_T}{m_e}\frac{8\pi}{h^3}T^5 \underbrace{\int_0^\infty du\,u^4 e^u (e^u-1)^{-2}}_{4\pi^4/15}$$

$$= \frac{n_e\sigma_T}{m_e}\frac{32\pi^5}{15h^3}T_\gamma^5$$

$$= \frac{4n_e\sigma_T}{m_e}\rho_\gamma T_\gamma \quad \text{since } \rho_\gamma = \frac{8\pi^5}{15h^3}T_\gamma^4$$ $\square$

part 4. $n_b = n_e + \underbrace{n_{H^0} + n_{H^+}}_{\equiv n_H} + \underbrace{n_{He^0} + n_{He^+} + n_{He^{++}}}_{\equiv n_{He}}$. recall $x_e \equiv \frac{n_e}{n_H}$ and $f_{He} \equiv \frac{n_{He}}{n_H}$.

thus $n_b = x_e n_H + n_H + f_{He} n_H = n_H (1 + x_e + f_{He})$ $\square$

part 5. see code on GitHub. I found $z_T \approx 105$.

this means for $z > z_T$, Thomson scattering remains effective enough to keep the baryon temperature equal to the photon temperature so $T_b \sim T_\gamma \sim a^{-1}$.

afterward, $\dot{T}_b \approx -2HT_b$ so the baryon temperature evolves independent of the photon temperature with $T_b \sim a^{-2}$.

part 6. continuity equations: $\delta_\gamma' = -\frac{4}{3}\nabla\cdot\vec{V}_\gamma$, $\delta_b' = -\nabla\cdot\vec{V}_b$

momentum eqns:

$$\vec{V}_\gamma' = -\frac{1}{4}\nabla\delta_\gamma + a n_e \sigma_T (\vec{V}_b - \vec{V}_\gamma)$$

$$\vec{V}_b' = -\mathcal{H}\vec{V}_b - \frac{\nabla P_b}{\bar{\rho}_b} - \frac{4}{3}\frac{\bar{\rho}_\gamma}{\bar{\rho}_b} a n_e \sigma_T (\vec{V}_b - \vec{V}_\gamma)$$

in Fourier space:

$$\delta_\gamma' = -\frac{4}{3}(i\vec{k})\cdot(i\hat{k}V_\gamma) = +\frac{4}{3}kV_\gamma$$

$$\delta_b' = +kV_b$$

$$V_\gamma' = -\frac{1}{4}k\delta_\gamma + a n_e \sigma_T (V_b - V_\gamma)$$

$$V_b' = -\mathcal{H}V_b - \frac{kP_b}{\bar{\rho}_b} - \frac{4}{3}\frac{\bar{\rho}_\gamma}{\bar{\rho}_b} a n_e \sigma_T (V_b - V_\gamma)$$

since $\Theta_0 \sim \delta$ and $\Theta_1 \sim V$, "neglecting baryon temperature fluctuations" means $\delta_b, V_b \to 0$? using this to solve the photon equations, then

then: $\delta_\gamma' = \frac{4}{3}kV_\gamma$, $V_\gamma' = -\frac{1}{4}k\delta_\gamma - a n_e \sigma_T V_\gamma$

taking $a n_e \sigma_T \propto a^{-2}$, $\frac{d}{d\eta}(a n_e \sigma_T) = -2\mathcal{H}\, a n_e \sigma_T$ so

$$V_\gamma'' + a n_e \sigma_T V_\gamma' + \left(\frac{k^2}{3} - 2\mathcal{H}\, a n_e \sigma_T\right)V_\gamma = 0.$$

if $\mathcal{H}$, $a n_e \sigma_T$ are slow compared to V_γ, then

$$V_\gamma = \left(\exp \frac{-a n_e \sigma_T}{2}\eta\right)\left[c_1 \exp \sqrt{(a n_e \sigma_T)^2 + 8\mathcal{H}\, a n_e \sigma_T - \frac{4}{3}k^2}\,\frac{\eta}{2} + c_2 \exp -\sqrt{(a n_e \sigma_T)^2 + 8\mathcal{H}\, a n_e \sigma_T - \frac{4}{3}k^2}\,\frac{\eta}{2}\right]$$

$$\sim \exp\left(2\mathcal{H} - \frac{k^2/3}{a n_e \sigma_T}\right)\eta$$

then $0 = -\mathcal{H}V_b - \frac{\nabla P_b}{\bar{\rho}_b} + \frac{4}{3}\frac{\bar{\rho}_\gamma}{\bar{\rho}_b} a n_e \sigma_T V_\gamma$

$$\rightarrow \frac{\nabla P_b/\bar{\rho}_b}{\mathcal{H}V_b} \approx \frac{-\mathcal{H}V_b + \frac{4}{3}\bar{\rho}_\gamma/\bar{\rho}_b\, a n_e \sigma_T V_\gamma}{\mathcal{H}V_b}$$

$$= \frac{4}{3}\frac{\bar{\rho}_\gamma}{\bar{\rho}_b} a n_e \sigma_T - 1$$

Problem 2:

(1) $\delta_\gamma' - \frac{4}{3} k V_\gamma = 4\phi'$

(2) $V_\gamma' = -\frac{1}{4} k \delta_\gamma + k \Pi_\gamma - k\Psi$

(3) $\Pi_\gamma' = -\frac{4}{15} k V_\gamma - \frac{9}{10} a n_e \sigma_T \Pi_\gamma$

part 1. taking the quasistationary approximation for Π_γ $(\Pi_\gamma' \to 0)$

we get $\Pi_\gamma \approx \left(-\frac{4}{15} k V_\gamma\right)\left(\frac{10}{9} \frac{1}{a n_e \sigma_T}\right) = -\frac{8}{27} \frac{k}{a n_e \sigma_T} V_\gamma$

plug into eq 2: $V_\gamma' = -\frac{1}{4} k \delta_\gamma + k \Pi_\gamma - k\Psi$

$$= -\frac{1}{4} k \delta_\gamma - \frac{8}{27} \frac{k^2}{a n_e \sigma_T} V_\gamma - k\Psi$$

eq 1: $\frac{4}{3} k V_\gamma = \delta_\gamma' - 4\phi' \Rightarrow k V_\gamma = \frac{3}{4} \delta_\gamma' - 3\phi'$

$$\Rightarrow V_\gamma' = -\frac{1}{4} k \delta_\gamma - \frac{8}{27} \frac{k}{a n_e \sigma_T} \left(\frac{3}{4} \delta_\gamma' - 3\phi'\right) - k\Psi$$

$\frac{d}{d\eta}(\text{eq } 1)$: $\delta_\gamma'' - \frac{4}{3} k V_\gamma' = 4\phi''$

$$\Rightarrow \delta_\gamma'' + \frac{4}{3} k \left[\frac{1}{4} k \delta_\gamma + \frac{8}{27} \frac{k}{a n_e \sigma_T} \left(\frac{3}{4} \delta_\gamma' - 3\phi'\right) + k\Psi\right] = 4\phi''$$

$$\delta_\gamma'' + \frac{8}{27} \frac{k^2}{a n_e \sigma_T} \delta_\gamma' + \frac{1}{3} k^2 \delta_\gamma = 4\phi'' + \frac{32}{27} \frac{k^2}{a n_e \sigma_T} \phi' - \frac{4}{3} k^2 \Psi$$

part 2. seek WKB soln $\delta_\gamma = A(\eta)\, e^{\pm i k\eta/\sqrt{3}} \equiv A e^{\pm i\varphi}$, assuming $\frac{A'}{A} \ll k$

$\delta = A e^{\pm i\varphi}$ $\qquad \varphi' = k/\sqrt{3}, \quad \varphi'' = 0.$

$\delta' = \pm i\varphi' A e^{\pm i\varphi} + A' e^{\pm i\varphi}$

$\delta'' = \pm i \cancelto{0}{\varphi''} A e^{\pm i\varphi} \pm 2i\varphi' A' e^{\pm i\varphi} + (\pm i\varphi')^2 A e^{\pm i\varphi} + A'' e^{\pm i\varphi}$

plugging in: $0 = A'' \pm 2i\varphi' A' - \varphi'^2 A$

$$+ \frac{8}{27} \frac{k^2}{a n_e \sigma_T} A' \pm \frac{8}{27} \frac{k^2}{a n_e \sigma_T} i\varphi' A + \frac{1}{3} k^2 A$$

$$= A'' \pm 2i\varphi' A' + \frac{8}{27} \frac{k^2}{a n_e \sigma_T} \left(A' \pm i\varphi' A\right) + \cancelto{0}{\left(\frac{k^2}{3} - \varphi'^2\right)} A$$

$$= A'' \pm \frac{2ik}{\sqrt{3}} A' + \frac{8}{27} \frac{k^2}{a n_e \sigma_T} \left(A' \pm \frac{ik}{\sqrt{3}} A\right)$$

$\sim \quad \varkappa^2 A \qquad \varkappa k A \qquad \frac{k^2}{a n_e \sigma_T} (\varkappa + k) A$

to leading order, $0 = 2A' + \frac{8}{27} \frac{k^2}{a n_e \sigma_T} A$

$$\to \frac{A'}{A} = -\frac{4}{27} \frac{k^2}{a n_e \sigma_T}$$

$$\log A = -\frac{4k^2}{27} \int \frac{d\eta}{a n_e \sigma_T}$$

$$\to A(\eta) \propto \exp\left[-\frac{4}{27} k^2 \int^\eta \frac{d\eta'}{a n_e \sigma_T(\eta')}\right] \checkmark$$

Problem 3:

here I'll collect all the equations:

<u>neutrinos</u> decoupled at all (relevant) times,

relativistic at all times (so ~massless $\rightarrow P = \rho/3$)

and no anisotropic stress ($\rightarrow \Pi = 0,\ c_s^2 = 1/3$)

then: $$\delta_\nu' = \frac{4}{3} k V_\nu + 4\phi'$$

$$V_\nu' = -\frac{1}{4} k \delta_\nu - k\phi$$

<u>cdm</u> pressureless perfect fluid, decoupled at all times

$$\delta_c' = kV_c + 3\phi'$$

$$V_c' = -\mathcal{H} V_c - k\phi$$

<u>photons & baryons</u> for $z > z_d = 10^3$, <u>tight coupling</u>

$$\delta_\gamma' = \frac{4}{3} k V_\gamma + 4\phi'$$

$$V_\gamma' = -\frac{k}{4(1+R)}\delta_\gamma - k\phi - \frac{R}{1+R}\mathcal{H} V_\gamma \qquad R \equiv \frac{3}{4}\frac{\bar\rho_b}{\bar\rho_\gamma}.$$

$$\delta_b = \frac{3}{4}\delta_\gamma, \quad V_b = V_\gamma.$$

for $z < z_d$, <u>total decoupling</u>

<u>photons</u> assume $\Pi_\gamma = 0,\ c_s^2 = 1/3$

$$\delta_\gamma' = \frac{4}{3} k V_\gamma + 4\phi'$$

$$V_\gamma' = -\frac{1}{4} k\delta_\gamma - k\phi$$

<u>baryons</u> assume $P_b = 0$. note $\Pi_b = 0$ always.

$$\delta_b' = kV_b + 3\phi'$$

$$V_b' = -\mathcal{H} V_b - k\phi$$

<u>potential</u> evolves under $\phi'' + 3(1+w)\mathcal{H}\phi' + wk^2\phi = 4\pi G a^2(\delta P - w\,\delta\rho)$ $\quad w \equiv \bar P/\bar\rho$.

want to split into two first-order ODEs for dimensionless variables.

let $\varphi_1 = \phi$, $\varphi_2 = d\phi/da$. then

$$\phi' = \frac{d\phi}{d\eta} = \frac{da}{d\eta}\frac{d\phi}{da} = \mathcal{H} a \varphi_2.$$

$$\phi'' = \frac{d}{d\eta}(\mathcal{H} a \varphi_2) = \mathcal{H}' a \varphi_2 + \mathcal{H}^2 a \varphi_2 + \mathcal{H} a \varphi_2' = \frac{1-3w}{2}\mathcal{H}^2 a \varphi_2 + \mathcal{H} a \varphi_2'$$

then our system of first-order ODEs is

$$\varphi_1' = \mathcal{H} a \varphi_2$$

$$\varphi_2' = -\frac{7+3w}{2}\mathcal{H}\varphi_2 - w\frac{k^2}{\mathcal{H}a}\varphi_1 + 4\pi G\frac{a}{\mathcal{H}}(\delta P - w\delta\rho)$$

it's more convenient to consider log a the integration variable

note
$$\frac{d}{d\eta} = \frac{da}{d\eta}\frac{d}{da}$$
$$= \frac{da}{d\eta}\frac{d\log a}{da}\frac{d}{d\log a}$$
$$= \mathcal{H}a\,\frac{1}{a}\frac{d}{d\log a}$$
$$= \mathcal{H}\frac{d}{d\log a}$$