Graduate Cosmology Spring 2025
Homework 3

due by 11:59pm on Monday 2/24, 2025.

Problem 1: H-theorem and equilibrium [12 points]

In this problem we will explore how the Bose-Einstein and Fermi-Dirac distributions arise in a simplified case. We neglect cosmological expansion (or equivalently work in a locally inertial frame), and assume that there are no gravitational / electromagnetic fields, i.e. that particles move along straight lines between collisions. We moreover consider a homogeneous Universe. We consider two species of particles, A and B, which can annihilate into one another in reactions of the form $A + A \leftrightarrow B + B$. We assume that A is a fermion and B is a boson.

1. [2 points] Write the general form of the collision operators $C[\mathcal{N}_A](\vec{p})$ and $C[\mathcal{N}_B](\vec{q})$ for the phase-space densities of A and of B as integrals involving their respective occupation numbers $f_A(\vec{p}), f_B(\vec{q})$. Make sure to use the same notation for things which are identical between the two collision operators. Show that the total number density $n_A + n_B$ of particles of type A and B is conserved.

2. [2 points] Show that the total energy density $\rho_A + \rho_B$ is conserved. *Hint*: $\int d^3p d^3p' S(\vec{p}, \vec{p}~') E(\vec{p}) = \frac{1}{2}\int d^3p d^3p' S(\vec{p}, \vec{p}~')[E(\vec{p}) + E(\vec{p}~')]$ if S is symmetric in its two arguments.

Consider the so-called H-functional $H = H_A + H_B$ (which has nothing to do with Hubble in this context!), where

$$H_A \equiv \frac{g_A}{h^3}\int d^3p \ \left[f_A(\vec{p})\ln(f_A(\vec{p})) + (1 - f_A(\vec{p}))\ln(1 - f_A(\vec{p}))\right]$$
$$H_B \equiv \frac{g_B}{h^3}\int d^3q \ \left[f_B(\vec{q})\ln(f_B(\vec{q})) - (1 + f_B(\vec{q}))\ln(1 + f_B(\vec{q}))\right],$$

where g_A and g_B are the degeneracy factors of each species.

3. [2 points] Compute the time derivative of H_A (make sure to convert $d\mathcal{N}_A/dt$ to df_A/dt first). Your result should be an integral over 4 momenta. By using the symmetry of part of the integrand under exchange of the two A particles, rewrite your result in a form fully symmetric under exchange of the two A particles (see hint for question 2 above).

4. [2 points] Compute the time drivative of H_B, and combine your results to write the total time derivative of H in the form

$$\frac{dH}{dt} = \int_{\text{momenta}} \Gamma \times \ (X - Y)(\ln Y - \ln X),$$

specifying X and Y in terms of f_A and f_B.

5. [1 point] Show that $(X - Y)(\ln Y - \ln X) \leq 0$ for any two strictly positive numbers X, Y. This is a simple functional analysis exercise, do it rigorously, don't just plot the function in Mathematica! This implies that $dH/dt \leq 0$.

Equilibrium is defined as a stationary (time-independent) and stable solution of the Boltzmann equations for A and B. Hence, at equilibrium, $dH/dt = 0$. Moreover, to enforce stability, the equilibrium phase-space distributions must *minimize* H (otherwise a small departure from equilibrium would drive H lower since $dH/dt < 0$).

6. [3 points] Find the occupation numbers f_A and f_B that minimize H while preserving the total number density $n_A + n_B$ as well as the total energy density $\rho_A + \rho_B$ (since collisions conserve total energy). This a problem of *constrained minimization*. To solve it, you want to minimize $H + \alpha(n_A + n_B) + \beta(\rho_A + \rho_B)$, with respect to all possible variations of f_A and f_B, where α and β are constant Lagrange multipliers. *Hint*: you will need to take functional derivatives of integrals of f_A and f_B. You should have done so when doing Lagrangian mechanics. Confirm that you obtain the Bose-Einstein and Fermi-Dirac solutions for B and A, respectively, and relate the temperature and chemical potentials to the Lagrange multipliers.

Problem 2: Massive neutrinos [12 points]

In class we showed that, after they decouple around $T \approx 1$ MeV, the neutrino occupation number takes the form $f_\nu(p,a) = (\exp(p/T_\nu(a)) + 1)^{-1}$, where p is the momentum measured by comoving observers, and $T_\nu(a) \propto 1/a$. Today the neutrino temperature is $T_{\nu,0} \approx 1.95$ K. There are 3 neutrino mass eigenstates (masses m_1, m_2, m_3), and each has 2 internal degrees of freedom.

1. [1 point] Compute the total number density of neutrinos today, per cubic centimeter.

2. [1 point] Assuming a neutrino has mass m_ν, calculate approximately at which redshift it becomes non-relativistic, in the sense that T_ν falls below m_ν. You should write your result in the form $1 + z_{\rm nr} = \#(m_\nu/0.1\text{eV})^\#$. Describe the behavior of the energy density of that type of neutrino as a function of redshift.

3. [2 points] What minimum mass must neutrinos have to be non-relativistic today? Suppose that is the case, since they do not interact they effectively behave like dark matter. Compute the total energy density in all neutrinos today, and translate it to a dimensionless density parameter $\omega_\nu \equiv \Omega_\nu h^2 = \#(\sum m_\nu/\text{eV})^\#$, where $\sum m_\nu = m_1 + m_2 + m_3$ is the sum of neutrino masses. Derive a limit on the sum of neutrino masses such that their energy density today does not exceed the dark matter energy density as measured by Planck.

4. [1 point] CMB anisotropies are sensitive to the redshift of matter-radiation equality. What is the maximum mass that individual neutrinos can have so they are still relativisitic around that epoch (hence do not change the redshift of matter-radiation equality, which assumes relativistic neutrinos)?

5. a. [4 points] Assuming a spatically-flat FLRW spacetime, calculate the radial comoving distance χ from which a neutrino reaching us today with observed momentum p_0 and mass m started traveling at decoupling (assuming it did not interact since). Your result should be an integral over scale factor. *Hint: there is no need to use the geodesic equation. Relate p and E (the momentum and energy observed by comoving observers) to $P^i = m dx^i/d\tau$ and $P^0 = m dt/d\tau$, so you can find an equation for dx^i/dt.*

b. [2 points] Now assume that p_0 and m are such that the neutrino becomes non-relativistic well within the matter era (i.e. well after matter-radiation equality, and well before today). Approximate the integral you found above by two pieces (before and after the non-relativistic transition at $a_{\rm nr}$). Argue that each piece is dominated by $a \sim a_{\rm nr}$ contributions, use this result to approximate the integrands and compute the integrals explicitly. You should find that both pieces (before and after $a_{\rm nr}$) end up contributing equally to χ.

c. [1 point] For a typical neutrino (explain what that means), in what range should the mass be so that the transition falls well within the matter era? Give a numerical value of your result above for such a typical neutrino in the form $\chi(m) = \#(m/0.1\text{eV})^\#$ Gpc.

Problem 3: Baryon abundance [4 points]

1. [2 points] Assuming protons have strictly vanishing chemical potential, and that they remained in thermal equilibrium with photons until today, show that on average, today there would not even be a single proton in the entire Hubble volume $V = 4\pi H_0^{-3}/3$.

2. [2 points] Given the measured baryon density $\omega_b \approx 0.022$, and assuming that Helium makes up 24% of baryons by mass (and protons the other 76%), calculate the *actual* number density of protons today.