Graduate Cosmology Spring 2025
Homework 4 solutions

How to self-grade: full points for correct answer with correct reasoning; half-points for correct reasoning but incorrect answer due to algebra error; zero point for incorrect reasoning (even if final answer is correct out of luck).

Problem 1

The Saha equilibrium equation for the process $\mathrm{He}^{++} + e^- \leftrightarrow \mathrm{He}^+ + \gamma$ is a simple generalization of the one we wrote for hydorgen, *before* making any simplification for charge neutrality:

$$\frac{n_e n_{\mathrm{He}^{++}}}{n_{\mathrm{He}^+}} = \frac{(2\pi m_e T)^{3/2}}{h^3} e^{-E_2/T}.$$

Likewise, the Saha equilibrium equation for the process $\mathrm{He}^+ + e^- \leftrightarrow \mathrm{He}^0 + \gamma$ is

$$\frac{n_e n_{\mathrm{He}^+}}{n_{\mathrm{He}^0}} = (2\pi m_e T)^{3/2} e^{-E_1/T}.$$

We now enforce charge neutrality by writing that the total number density of positive charges equals the number density of free electrons: $n_e = n_p + n_{\mathrm{He}^+} + 2n_{\mathrm{He}^{++}}$. Since hydrogen is fully ionized at the relevant epochs, the number density of protons is $n_p = n_{\mathrm{H}}$.

• During the first[1] helium recombination $\mathrm{He}^{++} + e^- \leftrightarrow \mathrm{He}^+ + \gamma$, we assume that the abundance of neutral helium is negligible, so that $n_{\mathrm{He}^{++}} + n_{\mathrm{He}^+} = n_{\mathrm{He}} = f_{\mathrm{He}} n_{\mathrm{H}}$. Charge neutrality then gives $n_{\mathrm{He}^{++}} = n_e - (n_p + n_{\mathrm{He}^+} + n_{\mathrm{He}^{++}}) = n_e - n_{\mathrm{H}}(1 + f_{\mathrm{He}})$. We moreover write $n_{\mathrm{He}^+} = n_{\mathrm{He}} - n_{\mathrm{He}^{++}} = f_{\mathrm{He}} n_{\mathrm{H}} - n_e + n_{\mathrm{H}}(1 + f_{\mathrm{He}}) = n_{\mathrm{H}}(1 + 2f_{\mathrm{He}}) - n_e$. Dividing everything by n_{H}, the Saha equation becomes

$$\frac{x_e(x_e - (1 + f_{\mathrm{He}}))}{1 + 2f_{\mathrm{He}} - x_e} = \frac{(2\pi m_e T)^{3/2}}{h^3 n_{\mathrm{H}}} e^{-E_2/T}$$

• During the second helium recombination $\mathrm{He}^+ + e^- \leftrightarrow \mathrm{He}^0 + \gamma$, we assume that the abundance of doubly-ionized helium is negligible, so that $n_{\mathrm{He}^+} + n_{\mathrm{He}^0} = n_{\mathrm{He}} = f_{\mathrm{He}} n_{\mathrm{H}}$. Charge neutrality then gives $n_{\mathrm{He}^+} = n_e - n_p = n_e - n_{\mathrm{H}}$, and the abundance of neutral helium is then $n_{\mathrm{He}^0} = f_{\mathrm{He}} n_{\mathrm{H}} - n_{\mathrm{He}^+} = (1 + f_{\mathrm{He}}) n_{\mathrm{H}} - n_e$. Again, dividing everything by n_{H}, the Saha equation becomes

$$\frac{x_e(x_e - 1)}{1 + f_{\mathrm{He}} - x_e} = \frac{(2\pi m_e T)^{3/2}}{h^3 n_{\mathrm{H}}} e^{-E_1/T}$$

The last thing we need to figure out is the conversion from Y_{He} (the fraction of Helium to baryons by mass) to f_{He} (the ratio of Helium to Hydrogen number densities). This is given by

$$f_{\mathrm{He}} = \frac{n_{\mathrm{He}}}{n_{\mathrm{H}}} = \frac{m_{\mathrm{H}}}{m_{\mathrm{He}}} \frac{\rho_{\mathrm{He}}}{\rho_{\mathrm{H}}} = \frac{m_{\mathrm{H}}}{m_{\mathrm{He}}} \frac{Y_{\mathrm{He}}}{1 - Y_{\mathrm{He}}} \approx 0.08 \text{ for } Y_{\mathrm{He}} = 0.24.$$

See github repository for the implementation of theses eqautions and plots: `https://github.com/alihaimoud/Cosmology/blob/main/HW4.ipynb`

Problem 2

1) Newton's law tells us that $m_e \ddot{\vec{r}} = -(e^2/r^2)\hat{r}$, hence $\ddot{\vec{D}} = -e\ddot{\vec{r}} = \frac{e^3}{m_e r^2}\hat{r}$. The power radiated is then $P = \frac{2}{3}(\ddot{\vec{D}})^2 = \frac{2}{3} e^6/(m_e^2 r^4)$, as advertized.

[1] Note that it is the *second* ionization energy that is relevant to the "first" (i.e. earliest) helium recombination, and the *first* ionization energy that is relevant to the "second" (i.e. latest) helium recombination...

2) a) The total energy radiated is

$$E_{\rm rad} = \int_{-\infty}^{\infty} dt \frac{dE_{\rm rad}}{dt} = \frac{2}{3}\frac{e^6}{m_e^2}\int_{-\infty}^{\infty}\frac{dt}{r^4(t)} = \frac{2}{3}\frac{e^6}{m_e^2}\int_{-\infty}^{\infty}\frac{dt}{(b^2+(vt)^2)^2} = \frac{2}{3}\frac{e^6}{m_e^2 b^4}\frac{b}{v}\int_{-\infty}^{\infty}\frac{du}{(1+u^2)^2}.$$

where we used $r^2 = b^2 = (vt)^2$ along a straight-line trajectory with contant velocity v and impact parameter b. The last integral is $\pi/2$, so we find, as advertized,

$$E_{\rm rad} = \frac{\pi}{3}\frac{e^6}{m_e^2 b^3 v}.$$

(b) The condition $E_{\rm rad} > \frac{1}{2}m_e v^2$ translates to

$$b < b_{\max}(v) = (2\pi/3)^{1/3}\frac{e^2}{m_e v}.$$

If this condition is satisfied, then

$$\frac{\frac{1}{2}m_e v^2}{e^2/b} = \frac{1}{2}m_e v^2 b/e^2 < \frac{1}{2}(2\pi/3)^{1/3} v = \frac{1}{2}(2\pi/3)^{1/3}(v/c),$$

where I explicitly put back the c dependence in the last equation. Since electrons are non-relativistic, $v/c \ll 1$, hence the condition for straight-line trajectories is not satisfied for impact parameters below $b_{\max}(v)$.

3) (a) Again, the total energy radiated is the integral of P over time, but now we will substitute $dt = d\theta/\dot\theta = \frac{r^2}{bv}d\theta$, so we get

$$E_{\rm rad} = \frac{2}{3}\frac{e^6}{m_e^2 bv}\int_0^{2\pi}\frac{d\theta}{r^2(\theta)} = \frac{1}{6}\frac{e^6}{m_e^2 bv}\frac{1}{r_0^2}\int_0^{2\pi}(1+\cos(\theta-\pi))^2 = \frac{\pi}{2}\frac{e^6}{m_e^2 bv r_0^2},$$

since the last integral is 3π. Next, substituting $v_0 = bv/r_0$ into $\frac{1}{2}m_e v_0^2 = e^2/r_0$, we obtain $\frac{1}{2}m_e(bv/r_0)^2 = e^2/r_0$, implying $r_0 = \frac{1}{2}m_e(bv)^2/e^2$, hence the advertized result

$$E_{\rm rad} = 2\pi\frac{e^{10}}{m_e^4(bv)^5}.$$

(b) Requiring $E_{\rm rad} > \frac{1}{2}m_e v^2$ implies

$$b < b_{\max}(v) = (4\pi)^{1/5}\frac{e^2}{m_e v^{7/5}}.$$

Let us now check consistency with the parabolic trajectory assumption if $b < b_{\max}$:

$$\frac{\frac{1}{2}m_e v^2}{e^2/b} = \frac{1}{2}\frac{m_e v^2 b}{e^2} < \frac{1}{2}(4\pi)^{1/5}v^{3/5} = \frac{1}{2}(4\pi)^{1/5}(v/c)^{3/5} \ll 1.$$

So, this time, the calculation is indeed self-consistent.

4) (a) We have again $\ddot{\vec{D}} = -e\ddot{\vec{r}} = e^3/(m_e r^2)\hat{r}$, and $\dot{\vec{D}} = -e\vec{v}$, so

$$\frac{d\vec{L}_{\rm rad}}{dt} = \frac{2}{3}\frac{e^4}{m_e r^3}\vec{r}\times\vec{v} = \frac{e^4}{m_e^2 r^3}\vec{L}.$$

(b) We proceed similarly as for the energy calculation (and use the fact that $\vec{L}_{\rm rad}$ has constant direction along $\vec{L}$)

$$L_{\rm rad} = \int dt\frac{dL_{\rm rad}}{dt} = \frac{2}{3}\frac{e^4}{m_e^2}L\int\frac{dt}{r^3} = \frac{2}{3}\frac{e^4}{m_e^2}L\int\frac{d\theta}{\dot\theta r^3} = \frac{2}{3}\frac{e^4}{m_e^2}\frac{m_e bv}{bv}\int\frac{d\theta}{r} = \frac{2}{3}\frac{e^4}{m_e}\frac{2\pi}{2r_0} = \frac{4\pi}{3}\frac{e^6}{m_e^2}\frac{1}{(bv)^2}.$$

The condition $L_{\rm rad} > \hbar$ translates to

$$b < b_{\max}(v) = (4\pi/3)^{1/2}\frac{e^3}{\sqrt{\hbar}\, m_e v}$$

The ratio of this maximum impact parameter to the previously found $b_{\rm max,cl}(v)$ is of order

$$\frac{b_{\rm max}}{b_{\rm max,cl}} \sim \frac{e^3/(\sqrt{\hbar} m_e v)}{e^2/(m_e v^{7/5})} = \frac{e}{\sqrt{\hbar}} v^{2/5}$$

Now, the ratio $e^2/\hbar$ is the famous fine-structure constant $\alpha \approx 1/137$. Hence, we see that the ratio of maximum impact parameters if of order $\sqrt{\alpha} v^{2/5} \ll 1$, since both α and v (in units of the speed of light) are small compared to unity.

(c) The Maxwell-Boltzmann distribution of electron velocities at temperature T is

$$P(\vec{v}) = \frac{1}{(2\pi T/m_e)^{3/2}} \exp\left(-\frac{m_e v^2}{2T}\right),$$

which integrates to unity over $d^3 v$. Hence the semi-classical recombination coefficient is

$$\begin{aligned}\mathcal{A}_{\rm cl}(T) &= \int d^3 v\ \pi b_{\rm max}^2(v) v P(\vec{v}) = \frac{4\pi^2}{3} \frac{e^6}{m_e^2 \hbar} \int d^3 v\ \frac{1}{v} P(\vec{v}) \\ &= \frac{4\pi^2}{3} \frac{e^6}{m_e^2 \hbar} \frac{1}{(2\pi T/m_e)^{3/2}} \int 4\pi v dv \exp\left(-\frac{m_e v^2}{2T}\right) = \frac{2}{3}(2\pi)^{3/2} \frac{e^6}{m_e^2 \hbar} (m_e/T)^{1/2} = \frac{2}{3}(2\pi)^{3/2} \alpha \frac{e^4}{m_e^2} (T/m_e)^{-1/2}.\end{aligned}$$

We now put the coefficient in the desired form

$$\mathcal{A}_{\rm cl}(T) = A T_4^{-1/2}, \qquad A \equiv \frac{2}{3}(2\pi)^{3/2} \alpha \frac{e^4}{(m_e c^2)} c (10^4 {\rm K}/m_e)^{-1/2} \approx 1.4 \times 10^{-13} {\rm cm^3 s^{-1}}.$$

where I re-instated all the c's in the final expression.

Problem 3

See code in github repository `https://github.com/alihaimoud/Cosmology/blob/main/HW4.ipynb`
Note that in order to get the curvature, you actually need h (or to input the big Ω's rather than the small ω's.