Graduate Cosmology Spring 2025
Homework 2 solutions

How to self-grade: full points for correct answer with correct reasoning; half-points for correct reasoning but incorrect answer due to algebra error; zero point for incorrect reasoning (even if final answer is correct out of luck).

Problem 1

1. We rewrite $dt/da = 1/\dot{a} = \frac{1}{aH}$. Using Friedmann's equation, and the boundary condition $t = 0$ at $a = 0$, we get

$$t(a) = H_0^{-1} \int_0^a \frac{db}{b(\Omega_r b^{-4} + \Omega_m b^{-3} + \Omega_k b^{-2} + \Omega_\Lambda)^{1/2}}.$$

2. (a) For $a \ll a_{m\Lambda}$, since $b < a$ in the integral, we may neglect Ω_Λ (as well as Ω_k) in the integrand, so we have

$$t \approx H_0^{-1} \int_0^a \frac{db}{b\sqrt{\Omega_r b^{-4} + \Omega_m b^{-3}}} = H_0^{-1} \int_0^a \frac{b db}{\sqrt{\Omega_r + \Omega_m b}} = \frac{a_{\rm eq}^{3/2}}{H_0\sqrt{\Omega_m}} \int_0^{a/a_{\rm eq}} \frac{x dx}{\sqrt{1+x}}.$$

Using the hint, we get

$$t \approx \frac{a_{\rm eq}^{3/2}}{H_0\sqrt{\Omega_m}} \frac{2}{3}\left(2 + (a/a_{\rm eq} - 2)\sqrt{1 + a/a_{\rm eq}}\right),$$

where we set the constant adequately so that $t(0) = 0$.

(b) Let us now compute the result in the limit $a \ll a_{\rm eq}$. We first Taylor-expand $\sqrt{1 + a/a_{\rm eq}} \approx 1 + \frac{1}{2}(a/a_{\rm eq}) - \frac{1}{8}(a/a_{\rm eq})^2$. We see that the constant term and the term linear in $a/a_{\rm eq}$ cancel out, so the leading order is quadratic in $a/a_{\rm eq}$):

$$t(a \ll a_{\rm eq}) \approx \frac{a_{\rm eq}^{3/2}}{2H_0\sqrt{\Omega_m}}(a/a_{\rm eq})^2.$$

The $a \gg a_{\rm eq}$ expansion is easier:

$$t(a_{\rm eq} \ll a \ll a_{m\Lambda}) \approx \frac{2}{3}\frac{a^{3/2}}{2H_0\sqrt{\Omega_m}}.$$

Both of these solutions match the $a(t)$ results we found in class, using $\Omega_r = a_{\rm eq}\Omega_m$.

(c) We plug in $a = a_{\rm eq}$ and find

$$t_{\rm eq} = \frac{2}{3}\frac{a_{\rm eq}^{3/2}}{H_0\sqrt{\Omega_m}}(2 - \sqrt{2})$$

With $H_0^{-1} \approx 14$ Gyr, we obtain $t_{\rm eq} \approx 50$ kyr.

3. (a) For $a \gg a_{\rm eq}$, the integral is dominated by $b \approx a$, i.e. we may neglect matter and radiation:

$$t(a \gg a_{\rm eq}) \approx H_0^{-1} \int_0^a \frac{db}{b\sqrt{\Omega_m b^{-3} + \Omega_\Lambda}} = \frac{1}{H_0\sqrt{\Omega_\Lambda}} \int_0^{a/a_{m\Lambda}} \frac{dx}{x\sqrt{1 + 1/x^3}} = \frac{1}{H_0\sqrt{\Omega_\Lambda}} \int_0^{a/a_{m\Lambda}} \frac{x^{1/2}dx}{\sqrt{1+x^3}}$$
$$= \frac{2}{3}\frac{1}{H_0\sqrt{\Omega_\Lambda}} \text{Arcsinh}((a/a_{m\Lambda})^{3/2}),$$

where we recall that $a_{m\Lambda} = (\Omega_m/\Omega_\Lambda)^{1/3}$. It is straightforward to see that this precisely the inverse of (2.177) in Baumann.

(b) We now plug in $a = 1$ in this result to find the age of the Universe:

$$t_0 = \frac{2}{3}\frac{1}{H_0\sqrt{\Omega_\Lambda}}\text{Arcsinh}(\sqrt{\Omega_\Lambda/\Omega_m}) \approx 0.95\ H_0^{-1} \approx 13.7\text{ Gyr}$$

where we used $H_0 \approx 68$ km/s/Mpc.

4) The conformal time η satisfies the ODE $d\eta = dt/a$. So it can be expressed as an integral in a similar to $t(a)$, but with an additional factor of $1/a$. In the matter-Λ era, this is

$$\eta(a) \approx \frac{1}{H_0}\int_0^a \frac{db}{b^2\sqrt{\Omega_m b^{-3} + \Omega_\Lambda}}.$$

This integral is not analytic, so we just do it numerically, for $a = 1$, and find $\eta_0 \approx 3.24 H_0^{-1} \approx 47$ Gyr ≈ 14 Gpc.

Problem 2:

1) By definition $d\eta = dt/a(t) = dt/da\ da/a = da/a\dot{a} = da/(a^2H(a))$. Since we are asked to express things in terms of redshift $z = 1/a - 1$, we get $dz = -da/a^2$, hence $d\eta = -dz/H(z)$. So, putting it together, we get

$$\chi(z) = \int_{\eta(z)}^{\eta_0} d\eta = \int_0^z \frac{dz'}{H(z')} = \frac{1}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_r(1+z')^4 + \Omega_m(1+z')^3 + \Omega_k(1+z')^2 + \Omega_\lambda}}$$
$$= \frac{1}{100\text{km/s/Mpc}}\int_0^z \frac{dz'}{\sqrt{\omega_r(1+z')^4 + \omega_m(1+z')^3 + \omega_k(1+z')^2 + \omega_\lambda}}.$$

2. and 3.: see attached Python notebook.

4. For a spatially flat Universe, $S_k(\chi) = \chi$, and the luminosity distance is therefore just $d_L(z) = (1+z)\chi(z)$. Neglecting radiation, and since the Universe is spatially flat, $\Omega_m = 1 - \Omega_\Lambda$, so that

$$d_L(z) = H_0^{-1}(1+z)\int_0^z \frac{dz'}{\sqrt{\Omega_\Lambda + (1-\Omega_\Lambda)(1+z')^3}}.$$

This can be expressed in terms of hypergeometric functions but that is not particularly useful, so we compute it numerically.

5. (a) We first find the parameter K that minimizes the χ^2 by taking the derivative of χ^2 with respect to K and setting it to zero. This gives

$$K_{\min}(\Omega_\Lambda) = \left(\sum_i \frac{\mu_i - 5\log_{10}(F(z_i,\Omega_\Lambda))}{\sigma_i^2}\right)/(\sum_i 1/\sigma_i^2).$$

We then plug this back into the original χ^2 to find the desired expression.

(b) See attached notebook. The best-fit is $\Omega_\Lambda = 0.715$.