Graduate Cosmology Spring 2025
Homework 3 solutions

How to self-grade: full points for correct answer with correct reasoning; half-points for correct reasoning but incorrect answer due to algebra error; zero point for incorrect reasoning (even if final answer is correct out of luck).

Problem 1

1. This is a special case of what we wrote in class. The collision operator for A takes the form

$$C[\mathcal{N}_A](\vec{p}) = \int d^3p'\ d^3q\ d^3q' \delta^4(p_\mu + p'_\mu - q_\mu - q'_\mu)\ \Gamma(p's) \\ \times \left[f_B(\vec{q})f_B(\vec{q}\,')(1 - f_A(\vec{p}))(1 - f_A(\vec{p}\,')) - f_A(\vec{p})f_A(\vec{p}\,')(1 + f_B(\vec{q}))(1 + f_B(\vec{q}\,'))\right],$$

where Γ depends on (invariant combinations) of all four momenta. Similarly, we have

$$C[\mathcal{N}_B](\vec{q}) = \int d^3p\ d^3p'\ d^3q' \delta^4(p_\mu + p'_\mu - q_\mu - q'_\mu)\ \Gamma(p's) \\ \times \left[f_A(\vec{p})f_A(\vec{p}\,')(1 + f_B(\vec{q}))(1 + f_B(\vec{q}\,')) - f_B(\vec{q})f_B(\vec{q}\,')(1 - f_A(\vec{p}))(1 - f_A(\vec{p}\,'))\right],$$

with the same Γ. The Boltzmann equation in the non-expanding, homogeneous Universe, is

$$\frac{d\mathcal{N}_A}{dt}\Big|_{\text{free traj}} = \frac{\partial \mathcal{N}_A}{\partial t} = C[\mathcal{N}_A],$$

since $\mathcal{N}_A$ is independent of $\vec{x}$, and $d\vec{p}/dt|_{\text{free traj}} = \vec{0}$ in a non-expanding Universe with no forces. The number density of particles A then changes according to

$$\frac{dn_A}{dt} = \frac{d}{dt}\int d^3p\ \mathcal{N}_A(t, \vec{p}) = \int d^3p\ \partial_t \mathcal{N}_A = \int d^3p\ C[\mathcal{N}_A](\vec{p}) \\ = \int_{\text{all p's}} \Gamma(p's) \times \left[f_B(\vec{q})f_B(\vec{q}\,')(1 - f_A(\vec{p}))(1 - f_A(\vec{p}\,')) - f_A(\vec{p})f_A(\vec{p}\,')(1 + f_B(\vec{q}))(1 + f_B(\vec{q}\,'))\right],$$

where from here on we denote

$$\int_{\text{all p's}} \cdots \equiv \int d^3p\ d^3p'\ d^3q\ d^3q' \delta^4(p_\mu + p'_\mu - q_\mu - q'_\mu) \cdots$$

It is easy to see that the number density of B satisfies $dn_B/dt = -dn_A/dt$.

2. The energy density of particles A evolves according to

$$\frac{d\rho_A}{dt} = \frac{d}{dt}\int d^3p\ E(p; m_A)\ \mathcal{N}_A(t, \vec{p}) = \int d^3p\ E(p; m_A)\ \partial_t \mathcal{N}_A = \int d^3p\ E(p; m_A)\ C[\mathcal{N}_A](\vec{p}) \\ = \int_{\text{all p's}} \Gamma(p's)\ E(p; m_A)\ \left[f_B(\vec{q})f_B(\vec{q}\,')(1 - f_A(\vec{p}))(1 - f_A(\vec{p}\,')) - f_A(\vec{p})f_A(\vec{p}\,')(1 + f_B(\vec{q}))(1 + f_B(\vec{q}\,'))\right].$$

Using the hint, we symmetrize the integrand in p, p':

$$\frac{d\rho_A}{dt} = \frac{1}{2}\int_{\text{all p's}} \Gamma\left[E(p; m_A) + E(p'; m_A)\right] \left[f_B(\vec{q})f_B(\vec{q}\,')(1 - f_A(\vec{p}))(1 - f_A(\vec{p}\,')) - f_A(\vec{p})f_A(\vec{p}\,')(1 + f_B(\vec{q}))(1 + f_B(\vec{q}\,'))\right].$$

We get an identical expression for $d\rho_B/dt$ with a minus sign, and $E(p, m_A) \to E(q; m_B)$. So $d\rho_A/dt + d\rho_B/dt$ in an integral whose integrand is proportional to $E(p, m_A) + E(p', m_A) - E(q, m_B) - E(q', m_B)$, but the Dirac delta enforces this factor to vanish (energy conservation for each collision). Hence $d\rho_A/dt + d\rho_B/dt = 0$.

3. We note that $f_A = (h^3/g_A)\mathcal{N}_A$, so the Boltzmann equation for f_A becomes $\partial_t f_A = (h^3/g_A)C[\mathcal{N}_A]$.

We then have, using the chain rule, and $d/dx[x \ln x + (1-x)\ln(1-x)] = \ln(x) - \ln(1-x)$

$$\begin{aligned}\frac{dH_A}{dt} &= \frac{g_A}{h^3}\int d^3p\,\frac{\partial f_A}{\partial t}\left[\ln f_A(\vec{p}) - \ln(1-f_A(\vec{p}))\right]\\ &= \int_{\text{all p's}} \Gamma\ \left[f_B(\vec{q})f_B(\vec{q}\,')(1-f_A(\vec{p}))(1-f_A(\vec{p}\,')) - f_A(\vec{p})f_A(\vec{p}\,')(1+f_B(\vec{q}))(1+f_B(\vec{q}\,'))\right]\left[\ln f_A(\vec{p}) - \ln(1-f_A(\vec{p}))\right].\end{aligned}$$

Since the first term in brackets is fully symmetric in exchange of $\vec{p}$ and $\vec{p}\,'$, using the hint, we arrive at

$$\begin{aligned}\frac{dH_A}{dt} &= \frac{1}{2}\int_{\text{all p's}} \Gamma\ \left[f_B(\vec{q})f_B(\vec{q}\,')(1-f_A(\vec{p}))(1-f_A(\vec{p}\,')) - f_A(\vec{p})f_A(\vec{p}\,')(1+f_B(\vec{q}))(1+f_B(\vec{q}\,'))\right]\\ &\quad\times\left[\ln f_A(\vec{p}) - \ln(1-f_A(\vec{p})) + \ln f_A(\vec{p}\,') - \ln(1-f_A(\vec{p}\,'))\right]\\ &= \frac{1}{2}\int_{\text{all p's}} \Gamma\ \left[f_B(\vec{q})f_B(\vec{q}\,')(1-f_A(\vec{p}))(1-f_A(\vec{p}\,')) - f_A(\vec{p})f_A(\vec{p}\,')(1+f_B(\vec{q}))(1+f_B(\vec{q}\,'))\right]\\ &\quad\times\left[\ln[f_A(\vec{p})f_A(\vec{p}\,')] - \ln[(1-f_A(\vec{p}))(1-f_A(\vec{p}\,'))]\right]\end{aligned}$$

4. To compute the derivative of H_B we follow the same steps, and use $d/dx[x\ln x - (1+x)\ln(1+x)] = \ln(x) - \ln(1+x)$, so that

$$\begin{aligned}\frac{dH_B}{dt} &= -\frac{1}{2}\int_{\text{all p's}} \Gamma\ \left[f_B(\vec{q})f_B(\vec{q}\,')(1-f_A(\vec{p}))(1-f_A(\vec{p}\,')) - f_A(\vec{p})f_A(\vec{p}\,')(1+f_B(\vec{q}))(1+f_B(\vec{q}\,'))\right]\\ &\quad\times\left[\ln[f_B(\vec{q})f_B(\vec{q}\,')] - \ln[(1+f_B(\vec{q}))(1+f_B(\vec{q}\,'))]\right].\end{aligned}$$

Summing the two, we arrive at

$$\frac{dH}{dt} = \frac{1}{2}\int_{\text{all p's}} \Gamma \times (X - Y)(\ln Y - \ln X),$$

where $X \equiv f_B(\vec{q})f_B(\vec{q}\,')(1-f_A(\vec{p}))(1-f_A(\vec{p}\,'))$ and $Y \equiv f_A(\vec{p})f_A(\vec{p}\,')(1+f_B(\vec{q}))(1+f_B(\vec{q}\,'))$.

5. We rewrite $(X-Y)(\ln Y - \ln X) = X(1-r)\ln r$, where $r \equiv Y/X$. Let us consider the function $F(r) \equiv (1-r)\ln r$. It has derivative $F'(r) = (1-r)/r - \ln r = 1/r - \ln r + 1$, and second derivative $F''(r) = -1/r^2 - 1/r$. Since $F'(r) < 0$ we conclude that $F'(r)$ is strictly decreasing. Given that $F'(1) = 0$, we conclude that $F' > 0$ for $r < 1$ and $F' < 0$ for $r > 1$. Therefore F increases for $r < 1$, reaches a maximum $F(1) = 0$, and decreases for $r > 1$. So we conclude that $F(r) \le F(1) = 0$ for all r. Hence $(X-Y)(\ln Y - \ln X) \le 0$.

6. We want to miminize the following functional of f_A and f_B:

$$\begin{aligned}\mathcal{F}[f_A, f_B] \equiv\ & H_A[f_A] + H_B[f_B]\\ &+\alpha\left[\frac{g_A}{h^3}\int d^3p\ f_A(\vec{p}) + \frac{g_B}{h^3}\int d^3q\ f_B(\vec{q})\right]\\ &+\beta\left[\frac{g_A}{h^3}\int d^3p\ E(p;m_A)\ f_A(\vec{p}) + \frac{g_B}{h^3}\int d^3q\ E(q;m_B)\ f_B(\vec{q})\right].\end{aligned}$$

Taking functional derivatives:

$$\begin{aligned}\frac{\delta\mathcal{F}}{\delta f_A(\vec{p})} &= \frac{g_A}{h^3}\left[\ln\left(\frac{f_A(\vec{p})}{1-f_A(\vec{p})}\right) + \alpha + \beta E(p;m_A)\right]\\ \frac{\delta\mathcal{F}}{\delta f_B(\vec{q})} &= \frac{g_B}{h^3}\left[\ln\left(\frac{f_B(\vec{p})}{1+f_B(\vec{q})}\right) + \alpha + \beta E(q;m_B)\right].\end{aligned}$$

Both must vanish for the equilibrium solution. This implies

$$\begin{aligned}\frac{f_A(\vec{p})}{1-f_A(\vec{p})} &= \exp(-\alpha - \beta E(p;m_A)),\\ \frac{f_B(\vec{q})}{1+f_B(\vec{q})} &= \exp(-\alpha - \beta E(q;m_A)),\end{aligned}$$

which, upon solving, gives us

$$f_A(p) = \frac{1}{\exp(\beta E(p;m_A)+\alpha)+1}, \qquad f_B(p) = \frac{1}{\exp(\beta E(p;m_B)+\alpha)-1}$$

This is precisely the Fermi-Dirac and Bose-Einstein distributions, with a common temperature $T = \beta^{-1}$ and equal chemical potentials $\mu_A = \mu_B = -T\alpha$.

Problem 2

1. The number density of neutrinos is

$$n_\nu = \frac{6}{h^3}\int d^3p\left(e^{p/T_\nu}+1\right)^{-1} = \frac{24\pi}{h^3}T_\nu^3\underbrace{\int dx x^2(e^x+1)^{-1}}_{3\zeta(3)/2} = 36\pi\zeta(3)(T_\nu/h)^3.$$

Remember that we have set $c = 1$. When we put it back in, we need to replace $T_\nu/h \to T_\nu/(hc)$, which has dimensions of inverse length. With $T_{\nu,0} = 1.95$ K $\approx 1.7\times10^{-4}$ eV and $hc \approx 1.24\times10^{-4}$ eV cm, we obtain

$$n_{\nu,0} \approx 338 \text{ cm}^{-3}.$$

2. We find the non-relativistic transition by solving $m_\nu = T_\nu(z_{\rm nr}) \approx 1.7\times10^{-4}(1+z_{\rm nr})$ eV, implying

$$1+z_{\rm nr} \approx \frac{m_\nu}{1.7\times10^{-4}\text{eV}} \approx 590\frac{m_\nu}{0.1\text{ eV}}.$$

The energy density of a single sepcies of neutrinos (and antineutrinos) of mass m_ν is given by

$$\rho_\nu = \frac{2}{h^3}\int d^3p\ \sqrt{p^2+m_\nu^2}f_\nu(p) = \frac{2}{h^3}\int d^3p\ \frac{\sqrt{p^2+m_\nu^2}}{e^{p/T_\nu}+1}.$$

Let us emphasize that the neutrino distriution is the **relativistic Fermi-Dirac distribution even after they are no longer relativistic**. That is because this is a "frozen" distribution, from back when neutrinos were actually in thermal equilibrium, which was when they were relativistic. We then obtain

$$\rho_\nu = \frac{2}{h^3}T_\nu^4\int 4\pi x^2 dx\ \frac{\sqrt{x^2+(m_\nu/T_\nu)^2}}{e^x+1}.$$

We see that when $m_\nu \ll T_\nu$, which corresponds to $z \gg z_{\rm nr}$, the integral is a constant, and $\rho_\nu \propto T_\nu^4 \propto a^{-4}$. When $m_\nu \gg T_\nu$, which corresponds to $z \ll z_{\rm nr}$, we may neglect the x^2 term in the square root, and find $\rho_\nu = \frac{1}{3}m_\nu n_\nu$ (recalling that n_ν is the number density of all species of neutrinos, but ρ_ν is just for one species, since different "species" may have different masses). So in this limit, $\rho_\nu \propto T_\nu^3 \propto a^{-3}$.

3. We want $z_{\rm nr} \geq 0$, implying $m_\nu \gtrsim T_{\nu,0} \approx 1.7\times10^{-4}$ eV.
The rest of the question is just an exercise in unit conversion. If neutrinos are non-relativistic today, their energy density is $\rho_\nu = \sum_i m_i n_{\nu,i} = \sum_i m_i \frac{1}{3}n_{\nu,\rm tot} = \frac{1}{3}(\sum m_\nu)n_\nu$, since they all have the same number density. The density parameter ω_ν is defined as

$$\omega_\nu = \frac{8\pi G\rho_\nu}{3(100\text{km/s/Mpc})^2} \approx 0.011\frac{\sum m_\nu}{1\text{ eV}}$$

Planck 2018 dark matter density is $\omega_c = 0.12\pm0.001$, hence, setting $\omega_\nu < 0.12$ implies $\sum m_\nu \lesssim 11$ eV.

4. We want $z_{\rm nr} \approx 590\frac{m_\nu}{0.1\text{ eV}} \leq 3500$, this implies $m_\nu \leq 0.6$ eV.

5. a. The first order of business is to use the hint and figure out dx^i/dt. In the $(t,\vec{x})$ FLRW coordinates, comoving observers have 4-velocity $U^\mu = (1,0,0,0)$, hence measure $E = -U^\mu P_\mu = -P_0 = P^0$. Moreover, the 3-momentum they measure can be defined as $p^2 = E^2 - m^2$. But we also have $m^2 = -g_{\mu\nu}P^\mu P^\nu = (P^0)^2 - a^2(t)P^iP^i$, where we assumed flat spacetime. Therefore, $p^2 = a^2(t)P^iP^i$, so $P^i = p^i/a(t)$. Hence the coordinate velocity is

$$\frac{dx^i}{dt} = \frac{P^i}{P^0} = \frac{p^i}{a\sqrt{p^2+m^2}}.$$

For a radial neutrino, we thus have

$$\frac{d\chi}{dt} = -\frac{p}{a\sqrt{p^2+m^2}}.$$

Note that this implies $d\chi/d\eta = -p/\sqrt{p^2+m^2}$, which reduces to $d\chi/d\eta = -1$ for relativistic neutrinos.

Let us denote by p_0 the momentum of the neutrino *today* (not to be mixed up with the zero-th lower component of p_μ). We then have $p(t) = p_0/a(t)$. Hence, the radial comoving coordinate is

$$\chi(p_0, m) = \int_{t_{\rm dec}}^{t_0} dt \frac{p_0}{a^2(t)\sqrt{p_0^2/a(t)^2 + m^2}} = \int \frac{dt}{a(t)\sqrt{1 + a^2(t)m^2/p_0^2}} = \int_{a_{\rm dec}}^{1} \frac{da}{a^2 H(a)\sqrt{1 + a^2 m^2/p_0^2}}.$$

Let us note right away that since $a_{\rm dec} \ll p_0/m$ and $a_{\rm dec} \ll a_{\rm eq}$, we may take $a_{\rm dec} = 0$ in the integral with no loss of accuracy. We see that this integral reduces to the standard radial comoving distance if $p_0 \gg m$, i.e. in the ultra-relativistic limit, but is modified for $p_0/m \ll 1$ as is the case here.

b. We split the integral as follows. For $a < a_{\rm nr} = p_0/m$, we may approximate the square root term in the denominator by 1 (the neutrino is relativistic then). For $a > a_{\rm nr}$, we approximate the square root term by $am/p_0 = a/a_{\rm nr}$. So we have

$$\chi(p_0, m) \approx \int_0^{a_{\rm nr}} \frac{da}{a^2 H(a)} + a_{\rm nr} \int_{a_{\rm nr}}^1 \frac{da}{a^3 H(a)}.$$

Now, since $a_{\rm eq} \ll a_{\rm nr} \ll a_{m\Lambda}$ by assumption, in the first integral we may neglect dark energy, and in the second we may neglect radiation, so that

$$\chi(p_0, m) \approx \frac{1}{H_0} \int_0^{a_{\rm nr}} d\ln a \frac{a}{\sqrt{\Omega_m a + \Omega_r}} + \frac{1}{H_0} a_{\rm nr} \int_{a_{\rm nr}}^1 \frac{d\ln a}{a^{1/2}\sqrt{\Omega_\Lambda a^3 + \Omega_m}},$$

where we have written the integrals in terms of $d\ln a$ to highlight the regions that contribute most. We see that the first integral is dominated by the largest $a \sim a_{\rm nr}$, hence in the matter era, and the second integral is dominated by the smallest $a \sim a_{\rm nr}$, also in the matter era. So we may neglect the contributions of radiation and dark energy in both integrals:

$$\chi(p_0, m) \approx \frac{1}{H_0\sqrt{\Omega_m}} \left[\int_0^{a_{\rm nr}} \frac{da}{\sqrt{a}} + a_{\rm nr} \int_{a_{\rm nr}}^1 \frac{da}{a^{3/2}}\right] \approx \frac{4\sqrt{a_{\rm nr}}}{H_0\sqrt{\Omega_m}} = \sqrt{p_0/m}\frac{4}{H_0\sqrt{\Omega_m}},$$

where we neglected corrections of order $\sqrt{a_{\rm nr}} \ll 1$. As advertized, the two pieces contribute equally.

c. A typical neutrino has momentum today $p_0 \sim T_{\nu,0}$. Requiring $1 \ll z_{\rm nr} \ll 3500$ implies $2 \times 10^{-4} \ll m/\text{eV} \ll 0.6$. Our result above translates to $\chi \approx 1.3\ \sqrt{0.1\ \text{eV}/m}$ Gpc.

Problem 3

1. Protons have a mass $m_p \approx 938$ MeV, much greater than the CMB temperature today $T_0 \approx 2.5 \times 10^{-4}$ eV, hence they are very non-relativistic today. With vanishing chemical potential, their equilibrium abundance today would be

$$n_p = \frac{2}{h^3}(2\pi m_p T_0)^{3/2} e^{-m_p/T_0}.$$

Hence the total number of protons within a Hubble volume would be

$$N \sim \frac{8\pi}{3}(2\pi m_p T_0/H_0^2/h^2)^{3/2} e^{-m_p/T_0}.$$

The exponent is of order $m_p/T_0 \sim 10^9/(2.5 \times 10^{-4}) \sim 4 \times 10^{12}$, so no code will ever be able to compute this number... Instead, let us compute the $\log_{10}$:

$$\log_{10}(e^{-m_p/T_0}) = -m_p/T_0/\ln(10) \approx -1.6 \times 10^{12}.$$

So, the exponential term is of order $10^{-1.6\times10^{12}}$, i.e. 0.000... ($\sim 10^{12}$ zeros)...1. No matter what the prefactor is, this will dwarf it, and the result is that there would be much much much less than one proton per Hubble volume.

2. We know that $\omega_b = \frac{8\pi G\rho_b}{3(100\mathrm{km/s/Mpc})^2}$, so

$$\rho_b = \frac{3\omega_b}{8\pi G}(100\mathrm{km/s/Mpc})^2 \approx 4.2\times10^{-31}\mathrm{g.cm}^{-3}$$

The mass density of protons is 76% of that. Each proton weighs $m_p \approx 1.7\times10^{-24}$ g, hence we get

$$n_p \approx 0.76\times\frac{4.2\times10^{-31}}{1.7\times10^{-24}}\mathrm{cm}^{-3} \approx 1.9\times10^{-7}\mathrm{cm}^{-3}.$$