Graduate Cosmology Spring 2025
Homework 1 solutions

How to self-grade: for each question, full credit for correct answer with correct reasoning; half-credit for correct reasoning but incorrect answer due to algebra error; no credit for incorrect reasoning (even if final answer is correct out of luck).

Problem 1

1) The line element $ds^2 = a^2(\eta)[-\eta^2 + \gamma_{ij}(\vec{x})dx^i dx^j]$ corresponds to metric components $g_{00} = -a^2(\eta)$, $g_{0i} = 0$, $g_{ij} = a^2(\eta)\gamma_{ij}(\vec{x})$, hence inverse-metric components $g^{00} = -1/a^2(\eta)$, $g^{0i} = 0$, $g^{ij} = \gamma^{ij}(\vec{x})/a^2(\eta)$. Let's plug this into the formula for the Christoffel symbols,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right).$$

First, with $\lambda = 0$, we find

$$\Gamma^0_{\mu\nu} = -\frac{1}{2}\frac{1}{a^2} \left(\partial_\mu g_{0\nu} + \partial_\nu g_{0\mu} - \partial_0 g_{\mu\nu}\right).$$

This implies

$$\begin{aligned}
\Gamma^0_{00} &= -\frac{1}{2}\frac{1}{a^2}\partial_\eta g_{00} = \frac{a'(\eta)}{a(\eta)},\\
\Gamma^0_{0i} &= 0\\
\Gamma^0_{ij} &= \frac{1}{2}\frac{1}{a^2}\partial_\eta g_{ij} = \frac{a'(\eta)}{a(\eta)}\gamma_{ij}.
\end{aligned}$$

Next, with $\lambda = i$, we get

$$\Gamma^i_{\mu\nu} = \frac{1}{2}\frac{1}{a^2}\gamma^{ik} \left(\partial_\mu g_{k\nu} + \partial_\nu g_{\mu k} - \partial_k g_{\mu\nu}\right).$$

This implies

$$\begin{aligned}
\Gamma^i_{00} &= 0,\\
\Gamma^i_{0j} &= \frac{1}{2}\frac{1}{a^2}\gamma^{ik} 2a'a\gamma_{kj} = \frac{a'}{a}\delta^i_k,\\
\Gamma^i_{jl} &= \frac{1}{2}\gamma^{ik}\left(\partial_j\gamma_{kl} + \partial_l\gamma_{jk} - \partial_k\gamma_{jl}\right).
\end{aligned}$$

We see that the $\Gamma^i_{\mu\nu}$ coefficients take the same form as in the $(t, \vec{x})$ coordinates, up to the replacement $\dot{a}/a \to a'/a$. In contrast, the $\Gamma^0_{\mu\nu}$ coefficients are different, in particular because $\Gamma^0_{00} \neq 0$ in the conformal coordinates.

2) The geodesic equation is

$$0 = P^\mu \nabla_\mu P^\nu = P^\mu \left(\partial_\mu P^\nu + \Gamma^\nu_{\mu\lambda} P^\lambda\right).$$

Consider the $\nu = 0$ component first. The geodesic equation gives

$$0 = P^\mu \partial_\mu P^0 + \Gamma^0_{\mu\lambda} P^\mu P^\lambda = P^\mu \partial_\mu P^0 + \left(\frac{a'}{a} P^0 P^0 + \gamma_{ij} P^i P^j\right).$$

Dividing everything by P^0 and recalling that $P^\mu/P^0 = dx^\mu/dx^0 = dx^\mu/d\eta$, we arrive at

$$\frac{dP^0}{d\eta} + \frac{a'}{a}\left(P^0 + \frac{1}{P^0}\gamma_{ij} P^i P^j\right) = 0.$$

Now, for a massless particle, $a^2(\eta)\left[-(P^0)^2 + \gamma_{ij}P^iP^j\right] = 0$, implying $\gamma_{ij}P^iP^j = (P^0)^2$. Hence the geodesic equation becomes

$$\frac{dP^0}{d\eta} + 2\frac{a'}{a}P^0 = 0.$$

This has a simple solution $P^0 \propto 1/a^2(\eta)$. **Beware that in these coordinates P^0 is not the energy measured by comoving observers! It does not have immediate physical meaning.**

Now consider comoving observers. By definition, this means they have constant (spatial) comoving coordinates x^i. Hence their 4-velocity takes the form $U^\mu_{\rm obs} = dx^\mu/d\tau = (U^0_{\rm obs}, 0, 0, 0)$. To find the value of $U^0_{\rm obs}$, we impose the normalization condition $-1 = g_{\mu\nu}U^\mu_{\rm obs}U^\nu_{\rm obs} = g_{00}\left(U^0_{\rm obs}\right)^2 = -a^2(\eta)\left(U^0_{\rm obs}\right)^2$. So we conclude that $U^0_{\rm obs} = 1/a(\eta)$ in conformal coordinates.

The energy measured by comoing observers is $E_{\rm obs} = -g_{\mu\nu}P^\mu U^\nu_{\rm obs} = -g_{00}P^0U^0_{\rm obs} = a^2 \times 1/a \times P^0 = a(\eta)P^0$. Hence we conclude again that $E_{\rm obs} \propto 1/a(\eta)$, as we already found in $(t, \vec{x})$ coordinates.

3) Let us start by writing the Christoffel symbols explicitly in the case of flat spatial geometry, in cartesian coordinates:

$$\Gamma^0_{00} = \frac{a'}{a}, \quad \Gamma^0_{0i} = 0, \quad \Gamma^0_{ij} = \frac{a'}{a}\delta_{ij}$$
$$\Gamma^i_{00} = 0, \quad \Gamma^i_{0j} = \frac{a'}{a}\delta^i_j, \quad \Gamma^i_{jl} = 0.$$

Let us apply the equation $\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda}A^\lambda$ to $A^\nu = \nabla^\nu\phi$, and contract the indices:

$$0 = \nabla_\mu\nabla^\mu\phi = \partial_\mu\nabla^\mu\phi + \Gamma^\mu_{\mu\lambda}\nabla^\lambda\phi = \partial_\eta\nabla^0\phi + \partial_i\nabla^i\phi + \Gamma^\mu_{\mu 0}\nabla^0\phi + \Gamma^\mu_{\mu i}\nabla^i\phi.$$

Next, recall that $\nabla^\mu\phi = g^{\mu\nu}\nabla_\nu\phi = g^{\mu\nu}\partial_\nu\phi$, so that $\nabla^0\phi = g^{00}\partial_\eta\phi = -\frac{1}{a^2}\partial_\eta\phi$ and $\nabla^i\phi = \frac{1}{a^2}\gamma^{ij}\partial_j\phi = \frac{1}{a^2}\partial_i\phi$. We also need the sum of Christoffel symbols,

$$\Gamma^\mu_{\mu 0} = \Gamma^0_{00} + \Gamma^i_{0i} = \frac{a'}{a} + \frac{a'}{a}\delta^i_i = 4\frac{a'}{a},$$
$$\Gamma^\mu_{\mu i} = \Gamma^0_{0i} + \Gamma^j_{ji} = 0 + 0 = 0.$$

Plugging in the equation of motion, we obtain

$$0 = -\partial_\eta\left(\frac{1}{a^2}\partial_\eta\phi\right) + \frac{1}{a^2}\partial_i(\partial_i\phi) - 4\frac{a'}{a}\frac{1}{a^2}\partial_\eta\phi.$$

Simplifying and multiplying by a^2, we arrive at

$$\phi'' + 2\frac{a'}{a}\phi' - \partial_i\partial_i\phi = 0.$$

4) As we saw earlier, comoving observers have 4-velocity $U^\mu_{\rm obs} = (1/a(\eta), 0, 0, 0)$ in conformal coordinates. Hence they observe $\rho_{\rm obs} = U^\mu_{\rm obs}U^\nu_{\rm obs}T_{\mu\nu} = \frac{1}{a^2}T_{00}$. Plugging in the equation given to us:

$$T_{00} = (\nabla_0\phi)^2 - \frac{1}{2}g_{00}(\nabla_\alpha\phi\nabla^\alpha\phi) = (\phi')^2 + \frac{1}{2}a^2\left(-\frac{1}{a^2}(\phi')^2 + \frac{1}{a^2}(\vec{\nabla}\phi)^2\right) = \frac{1}{2}(\phi')^2 + \frac{1}{2}(\vec{\nabla}\phi)^2,$$

hence

$$\rho_{\rm obs} = \frac{1}{2a^2}\left((\phi')^2 + (\vec{\nabla}\phi)^2\right).$$

On to the pressure. Let us first compute the trace of the stress-energy tensor:

$$T^\mu_{\ \mu} = g^{\mu\nu}T_{\mu\nu} = g^{00}T_{00} + g^{ij}T_{ij} = -\frac{1}{a^2}T_{00} + \frac{1}{a^2}\delta^{ij}T_{ij}.$$

We see that we need to compute

$$T_{ij} = \nabla_i\phi\nabla_j\phi - \frac{1}{2}g_{ij}(\nabla_\alpha\phi\nabla^\alpha\phi) = \partial_i\phi\partial_j\phi - \frac{1}{2}a^2\delta_{ij}\left(-\frac{1}{a^2}(\phi')^2 + \frac{1}{a^2}(\vec{\nabla}\phi)^2\right) = \frac{1}{2}\delta_{ij}(\phi')^2 + \partial_i\phi\partial_j\phi - \frac{1}{2}\delta_{ij}(\vec{\nabla}\phi)^2.$$

We thus get $\delta^{ij}T_{ij} = \frac{3}{2}(\phi')^2 - \frac{1}{2}(\vec{\nabla}\phi)^2$, implying $T^\mu_{\ \mu} = \frac{1}{a^2}\left((\phi')^2 - (\vec{\nabla}\phi)^2\right)$, hence

$$P_{\rm obs} = \frac{1}{2a^2}\left((\phi')^2 - \frac{1}{3}(\vec{\nabla}\phi)^2\right).$$

For a homogeneous scalar field, $\vec{\nabla}\phi = 0$, hence the equation of motion is $\phi'' + 2(a'/a)\phi' = 0$, implying $\phi'(\eta) \propto 1/a^2(\eta)$. A homogeneous scalar field thus has

$$P_{\rm obs} = \rho_{\rm obs} = (\phi')^2/2a^2 \propto 1/a^6.$$

5) The conservation of the stress-energy tensor reads (using metric-compatibility of covariant derivatives)

$$0 = \nabla_\mu T^{\mu\nu} = \nabla_\mu\left(\nabla^\mu\phi\nabla^\nu\phi\right) - \frac{1}{2}g^{\mu\nu}\nabla_\mu(\nabla^\lambda\phi\nabla_\lambda\phi) = (\nabla_\mu\nabla^\mu\phi)\nabla^\nu\phi + \nabla^\mu\phi\nabla_\mu\nabla^\nu\phi - \nabla^\lambda\phi\nabla^\mu\nabla_\lambda\phi.$$

Using the commutation of covariant derivatives of scalar fields, we see that the second and last terms cancel out, hence

$$0 = \nabla_\mu T^{\mu\nu} = (\nabla_\mu\nabla^\mu\phi)\nabla^\nu\phi.$$

Hence unless $\nabla^\nu\phi = 0$ everywhere, which implies a boring constant scalar field, we conclude that $\nabla_\mu\nabla^\mu\phi = 0$.

Problem 2

This is just a matter of plugging in and using the Christoffel symbols we gave in class for FLRW in (t, x^i) coordinates. First, let us compute the $\nu = 0$ component of $\nabla_\mu G^\mu_{\ \nu}$:

$$\begin{aligned}
\nabla_\mu G^\mu_{\ 0} &= \partial_\mu G^\mu_{\ 0} + \Gamma^\mu_{\mu\sigma}G^\sigma_{\ 0} - \Gamma^\sigma_{\mu 0}G^\mu_{\ \sigma} \\
&= \partial_0 G^0_{\ 0} + \partial_i \cancel{G^i_{\ 0}} + \Gamma^\mu_{\mu 0}G^0_{\ 0} + \Gamma^\mu_{\mu i}\cancel{G^i_{\ 0}} - \cancel{\Gamma^0_{00}}G^0_{\ 0} - \Gamma^i_{00}\cancel{G^0_{\ i}} - \Gamma^0_{0i}\cancel{G^i_{\ 0}} - \Gamma^i_{j0}G^j_{\ i} \\
&= \partial_t G^0_{\ 0} + 3\frac{\dot{a}}{a}G^0_{\ 0} - \frac{\dot{a}}{a}G^i_{\ i} = a^{-3}\partial_t(a^3 G^0_{\ 0}) - \frac{\dot{a}}{a}G^i_{\ i},
\end{aligned}$$

where we have kept only the nonvanishing terms and replaced the Christoffel symbols by the values we gave in class. Now substituting with the given expressions, we get

$$\nabla_\mu G^\mu_{\ 0} = -\frac{3}{a^3}\left[2\dot{a}\ddot{a}a + \dot{a}^3 + k\dot{a}/R_0^2\right] + 3\frac{\dot{a}}{a}\left[2\ddot{a}/a + \dot{a}^2/a^2 + k/(a^2R_0^2)\right] = 0.$$

Next, let us consider the $\nu = i$ component:

$$\begin{aligned}
\nabla_\mu G^\mu_{\ i} &= \partial_\mu G^\mu_{\ i} + \Gamma^\mu_{\mu\sigma}G^\sigma_{\ i} - \Gamma^\sigma_{\mu i}G^\mu_{\ \sigma} \\
&= \partial_t\cancel{G^0_{\ i}} + \cancel{\partial_j G^j_{\ i}} + \Gamma^\mu_{\mu 0}\cancel{G^0_{\ i}} + \Gamma^\mu_{\mu j}G^j_{\ i} - \cancel{\Gamma^0_{0i}}G^0_{\ 0} - \Gamma^0_{ji}\cancel{G^j_{\ 0}} - \Gamma^j_{0i}\cancel{G^0_{\ j}} - \Gamma^j_{ki}G^k_{\ j} = \Gamma^k_{kj}G^j_{\ i} - \Gamma^j_{ki}G^k_{\ j}.
\end{aligned}$$

Now, with $G^k_{\ j} = \frac{1}{3}(G^l_{\ l})\delta^k_j$, we get

$$\nabla_\mu G^\mu_{\ i} = \frac{1}{3}G^l_{\ l}\ \left[\Gamma^k_{ki} - \Gamma^k_{ki}\right] = 0.$$

Problem 3

1. For a perfect fluid with positive density and pressure, Raychaudhuri's Equation gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) < 0.$$

Hence there can be no static solution, with constant a.

2. We now consider a Universe with non-relativistic matter with density ρ_m and negligible pressure, and a cosmological constant Λ, which can be equivalently cast as an ideal fluid with $\rho_\Lambda = \Lambda/(8\pi G)$ and $P_\Lambda = -\rho_\Lambda$. Raychaudhuri's equation is then

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_m + \rho_\Lambda + 3P_\Lambda\right) = -\frac{4\pi G}{3}\left(\rho_m - 2\rho_\Lambda\right).$$

The right-hand-side vanishes if $\rho_\Lambda = \frac{1}{2}\rho_m$, which translates to $\Lambda = 4\pi G\rho_m$.

But to get a static solution we also need $\dot{a} = 0$. So, from Friedmann's equation, this implies

$$0 = (\dot{a}/a)^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda) - \frac{k}{a^2 R_0^2} = 4\pi G\rho_m - \frac{k}{a^2 R_0^2}.$$

Thus this requires $k = +1$ (spherical 3-D slices) with a radius of curvature $R_0 = \frac{1}{\sqrt{4\pi G\rho_m}}$, where we have set $a = 1$ since this solution is static, hence has $a(t) = \text{constant}$.

3. Let us now consider small perturbations: $\rho_m(t) = \rho_{m,0}(1+\delta(t))$ and $a(t) = 1+\epsilon(t)$, with $\rho_{m,0} = \Lambda/(4\pi G)$. Since matter density scales as $\rho_m \propto a^{-3}$, we conclude that, at linear order, $\delta(t) = -3\epsilon(t)$. Let us now plug into Raychaudhuri's equation:

$$\ddot{\epsilon} = -\frac{4\pi G}{3}\rho_{m,0}\delta(t) = 4\pi G\rho_{m,0}\;\epsilon(t).$$

We see that ϵ grows exponentially on a timescale $\tau = 1/\sqrt{4\pi G\rho_{m,0}}$.

Problem 4

We saw that $\rho \propto a^{-3(1+w)}$. So if $w < -1$, the energy density of this strange fluid actually increases with scale factor. Friedmann's equation then reads

$$(\dot{a}/a)^2 = \frac{8\pi G}{3}\rho_0 a^{-3(1+w)} = H_0^2 a^{-3(1+w)},$$

implying (for $da/dt > 0$)

$$\frac{da}{dt} = H_0 a^{1-3(1+w)/2}.$$

This can be solved as

$$dt = H_0^{-1}a^{3(1+w)/2-1}da \quad\Rightarrow\quad t - t_0 = \frac{2H_0^{-1}}{3(1+w)}\left(a^{3(1+w)/2}-1\right) = \frac{2H_0^{-1}}{3|1+w|}\left(1 - a^{-3|1+w|/2}\right),$$

where we have written $(1+w) = -|1+w|$ since $w < -1$. We see that $a \to +\infty$ at finite time

$$t_{\rm rip} - t_0 = \frac{2H_0^{-1}}{3|1+w|}.$$

We may then rewrite the equation above as

$$t_{\rm rip} - t = (t_{\rm rip} - t_0)a^{-3|1+w|/2} \qquad\Rightarrow\qquad a(t) = \left(\frac{t_{\rm rip}-t_0}{t_{\rm rip}-t}\right)^{\frac{2}{3|1+w|}}.$$

If you want to read more about this strange idea, see the famous short paper by Caldwell, Kamionkowski and Nevin (`https://arxiv.org/abs/astro-ph/0302506`).