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Cosmology Homework 3 — Connor Hainje

Problem 1: 12/12

part 1. 2/2

$$C[\mathcal{N}_A](\vec{p}) = \int d^3\vec{p}'\, d^3\vec{q}\, d^3\vec{q}'\ \delta^4(p^\mu + p'^\mu - q^\mu - q'^\mu)\ \Gamma(p^\mu, p'^\mu, q^\mu, q'^\mu)$$
$$\left[- f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right) + f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right)\right] ✓$$

$$C[\mathcal{N}_B](\vec{q}) = \int d^3\vec{q}'\, d^3\vec{p}\, d^3\vec{p}'\ \delta^4(p^\mu + p'^\mu - q^\mu - q'^\mu)\ \Gamma(p^\mu, p'^\mu, q^\mu, q'^\mu)$$
$$\left[- f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right) + f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right)\right] ✓$$

$d_t(a^3 n) = a^3\, \dot{n}|_{coll}$ in a static universe $\Rightarrow$ $d_t n = \dot{n}|_{coll} = \int d^3p\ C[\mathcal{N}]$

$$d_t(n_A + n_B) = \int d^3p\ C[\mathcal{N}_A](\vec{p}) + \int d^3q\ C[\mathcal{N}_B](\vec{q})$$
$$= \int d^3p\, d^3p'\, d^3q\, d^3q'\ \delta^4(\Sigma p_i^\mu)\ \Gamma(p_i^\mu)$$
$$\left[\cancel{- f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right)} + \cancel{f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right)} \right.$$
$$\left. \cancel{- f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right)} + \cancel{f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right)}\right]$$
$$= 0 \quad ✓ ✓$$

part 2. 2/2

$\dot{a} = 0 \Rightarrow d_t\rho = \dot{\rho}|_{coll} = \int d^3p\ C[\mathcal{N}](\vec{p})\ E(\vec{p})$

$$d_t(\rho_A + \rho_B) = \int d^3p\ C[\mathcal{N}_A](\vec{p})\ E(\vec{p}) + \int d^3q\ C[\mathcal{N}_B](\vec{q})\ E(\vec{q})$$
$$= \int d^3p\, d^3p'\, d^3q\, d^3q'\ \delta^4(\Sigma p_i^\mu)\ \Gamma(p_i^\mu) \longleftarrow \text{for brevity, we'll write this } \int_{p_i}\Gamma$$
$$\left\{ \left[- f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right) + f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right)\right] E(\vec{p}) \right.$$
$$\left. + \left[- f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right) + f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right)\right] E(\vec{q}) \right\}$$

$\int d^3p\, d^3p'\ S(\vec{p},\vec{p}')\ E(\vec{p}) = \int d^3p\, d^3p'\ S(\vec{p},\vec{p}')\ \frac{E(\vec{p}) + E(\vec{p}')}{2}$ if $S(p,p') = S(p',p)$ so

$$d_t(\rho_A+\rho_B) = \int_{p_i}\Gamma \left\{ \left[- f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right) + f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right)\right] \frac{E(\vec{p}) + E(\vec{p}')}{2} \right.$$
$$\left. + \left[- f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right) + f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right)\right] \frac{E(\vec{q}) + E(\vec{q}')}{2} \right\}$$

by conservation of energy (e.g. the 0-component of the δ^4), $E(\vec{p}) + E(\vec{p}') = E(\vec{q}) + E(\vec{q}')$. hence,

$$d_t(\rho_A+\rho_B) = \int_{p_i}\Gamma\ \frac{E(\vec{p}) + E(\vec{p}')}{2} \left[\cancel{- f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right)} + \cancel{f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right)} \right.$$
$$\left. \cancel{- f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right)} + \cancel{f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right)}\right]$$
$$= 0. ✓$$

part 3. 2/2

$$C[\mathcal{N}_A] = \frac{d}{dt}\mathcal{N}_A = \frac{d}{dt}\left(\frac{g_A}{h^3} f_A\right) = \frac{g_A}{h^3}\frac{d}{dt} f_A$$

$$\frac{d}{dt}H_A = \frac{g_A}{h^3}\int d^3p \;\frac{d}{dt}\left[f_A(\vec{p}) \ln f_A(\vec{p}) + (1-f_A(\vec{p}))\ln(1-f_A(\vec{p}))\right]$$

$$\begin{aligned}\frac{d}{dt}\left[f\ln f + (1-f)\ln(1-f)\right] &= \dot{f}\ln f + f\frac{1}{f}\dot{f} + (-\dot{f})\ln(1-f) + (1-f)\frac{1}{1-f}(-\dot{f}) \\ &= \dot{f}(1+\ln f) - \dot{f}\left[1+\ln(1-f)\right] \\ &= \dot{f}\left[1+\ln f - 1 - \ln(1-f)\right] \\ &= \dot{f}\left[\ln f - \ln(1-f)\right]\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}H_A &= \frac{g_A}{h^3}\int d^3p \;\dot{f}_A(\vec{p})\left[\ln f_A(\vec{p}) - \ln(1-f_A(\vec{p}))\right] \\ &= \int d^3p \; C[\mathcal{N}_A](\vec{p})\left[\ln f_A(\vec{p}) - \ln(1-f_A(\vec{p}))\right] \\ &= \int_{p_i}\Gamma\left[\ln f_A(\vec{p}) - \ln(1-f_A(\vec{p}))\right]\left[-f_A(\vec{p})f_A(\vec{p}')(1+f_B(\vec{q}))(1+f_B(\vec{q}')) + f_B(\vec{q})f_B(\vec{q}')(1-f_A(\vec{p}))(1-f_A(\vec{p}'))\right] \\ &= \int_{p_i}\Gamma\,\frac{\ln f_A(\vec{p}) - \ln(1-f_A(\vec{p})) + \ln f_A(\vec{p}') - \ln(1-f_A(\vec{p}'))}{2}\left[\cdots\right]\end{aligned}$$ ✓

part 4. 2/2

$$\frac{d}{dt}H_B = \frac{g_B}{h^3}\int d^3q \;\frac{d}{dt}\left[f_B(\vec{q}) \ln f_B(\vec{q}) - (1+f_B(\vec{q}))\ln(1+f_B(\vec{q}))\right]$$

$$\frac{d}{dt}\left[f\ln f - (1+f)\ln(1+f)\right] = \dot{f}(1+\ln f) - \dot{f}\left[1+\ln(1+f)\right] = \dot{f}\left[\ln f - \ln(1+f)\right]$$

$$\begin{aligned}\frac{d}{dt}H_B &= \frac{g_B}{h^3}\int d^3q \;\dot{f}_B(\vec{q})\left[\ln f_B(\vec{q}) - \ln(1+f_B(\vec{q}))\right] \\ &= \int d^3q \; C[\mathcal{N}_B](\vec{q})\left[\ln f_B(\vec{q}) - \ln(1+f_B(\vec{q}))\right] \\ &= \int_{p_i}\Gamma\left[\ln f_B(\vec{q}) - \ln(1+f_B(\vec{q}))\right] \\ &\quad\times\left[-f_B(\vec{q})f_B(\vec{q}')(1-f_A(\vec{p}))(1-f_A(\vec{p}')) + f_A(\vec{p})f_A(\vec{p}')(1+f_B(\vec{q}))(1+f_B(\vec{q}'))\right] \\ &= \int_{p_i}\Gamma\,\frac{\ln f_B(\vec{q}) - \ln(1+f_B(\vec{q})) + \ln f_B(\vec{q}') - \ln(1+f_B(\vec{q}'))}{2} \\ &\quad\times\left[-f_B(\vec{q})f_B(\vec{q}')(1-f_A(\vec{p}))(1-f_A(\vec{p}')) + f_A(\vec{p})f_A(\vec{p}')(1+f_B(\vec{q}))(1+f_B(\vec{q}'))\right]\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}H &= \frac{d}{dt}H_A + \frac{d}{dt}H_B \\ &= \int_{p_i}\Gamma\left[-f_A(\vec{p})f_A(\vec{p}')(1+f_B(\vec{q}))(1+f_B(\vec{q}')) + f_B(\vec{q})f_B(\vec{q}')(1-f_A(\vec{p}))(1-f_A(\vec{p}'))\right] \\ &\quad \frac{1}{2}\Big[\underbrace{\ln f_A(\vec{p}) + \ln f_A(\vec{p}') - \ln(1-f_A(\vec{p})) - \ln(1-f_A(\vec{p}')) - \ln f_B(\vec{q}) - \ln f_B(\vec{q}') + \ln(1+f_B(\vec{q})) + \ln(1+f_B(\vec{q}'))}_{\ln f_A(\vec{p}) f_A(\vec{p}')(1+f_B(\vec{q}))(1+f_B(\vec{q}')) \;-\; \ln f_B(\vec{q}) f_B(\vec{q}')(1-f_A(\vec{p}))(1-f_A(\vec{p}'))}\Big]\end{aligned}$$

$$= \int_{p_i}\frac{1}{2}\Gamma\,(X-Y)(\ln Y - \ln X) \quad \text{for} \quad \begin{aligned} X &= f_B(\vec{q})f_B(\vec{q}')(1-f_A(\vec{p}))(1-f_A(\vec{p}')) \\ Y &= f_A(\vec{p})f_A(\vec{p}')(1+f_B(\vec{q}))(1+f_B(\vec{q}'))\end{aligned}$$ ✓

part 5. 1/1

show $(X-Y)(\ln Y - \ln X) \le 0$ for $X, Y > 0$.

if $X=Y$, it's trivially zero. further, it's symmetric wrt $X \leftrightarrow Y$ so WLOG assume $X > Y$.

$\ln$ is monotonically increasing wrt its argument, so $X > Y \Rightarrow \ln X > \ln Y$.

hence $X > Y \Rightarrow (X-Y) > 0, \; (\ln Y - \ln X) < 0$

$\Rightarrow (X-Y)(\ln Y - \ln X) < 0$

so generally $(X-Y)(\ln Y - \ln X) \le 0$ with equality only if $X=Y$. ✓

part 6. 3/3

obj function: $H + \alpha(n_A + n_B) + \beta(\rho_A + \rho_B)$.

$$n = \int d^3p \, \mathcal{N} = \frac{g}{h^3}\int d^3p \, f, \quad \rho = \int d^3p \, \mathcal{N} E(p) = \frac{g}{h^3}\int d^3p \, f \, E(p)$$

$$H_A + H_B = \frac{g_A}{h^3}\int d^3p \left[f_A(\vec{p}) \ln f_A(\vec{p}) + (1-f_A(\vec{p}))\ln(1-f_A(\vec{p}))\right] + \frac{g_B}{h^3}\int d^3q \left[f_B(\vec{q}) \ln f_B(\vec{q}) - (1+f_B(\vec{q}))\ln(1+f_B(\vec{q}))\right]$$

$$n_A + n_B = \frac{g_A}{h^3}\int d^3p \, f_A(\vec{p}) + \frac{g_B}{h^3}\int d^3q \, f_B(\vec{q}) \qquad \rho_A + \rho_B = \frac{g_A}{h^3}\int d^3p \, f_A(\vec{p}) E(\vec{p}) + \frac{g_B}{h^3}\int d^3q \, f_B(\vec{q}) E(\vec{q})$$

$$\text{obj} = \frac{g_A}{h^3}\int d^3p \left[f_A(\vec{p})\left(\ln f_A(\vec{p}) + \alpha + \beta E(\vec{p})\right) + (1-f_A(\vec{p}))\ln(1-f_A(\vec{p}))\right]$$
$$+ \frac{g_B}{h^3}\int d^3q \left[f_B(\vec{q})\left(\ln f_B(\vec{q}) + \alpha + \beta E(\vec{q})\right) - (1+f_B(\vec{q}))\ln(1+f_B(\vec{q}))\right]$$

so, we need to solve $\frac{\delta}{\delta f_A}\text{obj} = 0 \qquad \frac{\delta}{\delta f_B}\text{obj} = 0$

$$\delta \text{obj}_A = \frac{g_A}{h^3}\int d^3p \left[\log f_A(\vec{p}) - \log(1-f_A(\vec{p})) + \alpha + \beta E(\vec{p})\right] \delta f_A .$$

setting to zero $\Rightarrow \log f_A(\vec{p}) - \log(1-f_A(\vec{p})) + \alpha + \beta E(\vec{p}) = 0.$ ✓

$$\log \frac{f}{1-f} = -\alpha - \beta E \longrightarrow f_A(\vec{p}) = \left(e^{\alpha + \beta E(\vec{p})} + 1\right)^{-1}.$$ ✓

$$\delta \text{obj}_B = \frac{g_B}{h^3}\int d^3q \left[\log f_B(\vec{q}) - \log(1+f_B(\vec{q})) + \alpha + \beta E(\vec{q})\right] \delta f_B$$

setting to zero $\Rightarrow \log f_B(\vec{q}) - \log(1+f_B(\vec{q})) + \alpha + \beta E(\vec{q}) = 0.$ ✓

$$\log \frac{f}{1+f} = -\alpha - \beta E \longrightarrow f_B(\vec{q}) = \left(e^{\alpha + \beta E(\vec{q})} - 1\right)^{-1}.$$ ✓

these are the Fermi-Dirac (A) and Bose-Einstein (B) distributions

if we identify $\alpha = -\mu/T$ and $\beta = 1/T$. ✓

Problem 2: 8/12

using $\hbar = 1$, $h = 2\pi$.

part 1. before decoupling, ν have $n(T) = \frac{9}{h^3} 4\pi \frac{3}{2} \zeta(3) T^3 = \frac{12\pi}{h^3} \zeta(3) T^3 = \frac{3\zeta(3)}{2\pi^2} T^3$

1/1

after, $n = \frac{3\zeta(3)}{2\pi^2} T_\nu^3$ so today, $n = \frac{3\zeta(3)}{2\pi^2} T_{\nu 0}^3 = 113$ neutrinos/cm^3 ✓

this applies for each species individually, making a total of 338 neutrinos/cm^3

part 2. suppose neutrino mass is m_ν. want to find $z(T_\nu = m_\nu)$.

0.5/1

$T_\nu \propto a^{-1}$ so $a T_\nu(a) = (1) T_\nu(1)$. hence $a_{nr} = \frac{T_{\nu 0}}{T_{nr}} = \frac{T_{\nu 0}}{m_\nu}$

$1 + z_{nr} = \frac{1}{a_{nr}} = \frac{m_\nu}{T_{\nu 0}} = \frac{m_\nu}{1.68 \times 10^{-4}\,eV} = (595)\left(\frac{m_\nu}{0.1\ eV}\right)$ ✓

missing $\rho_\nu(z)$ -0.5

part 3. minimum mass to be nonrelativistic today: $m_\nu \sim T_{\nu 0} = 0.168$ meV.

2/2

or, to be safe, $m_\nu \sim T_{\gamma,0} = 0.235$ meV.

assume $m_\nu > T_{\nu 0}$ st ν are nonrelativistic

$$\omega_\nu = \Omega_\nu h^2 = \frac{h^2}{\rho_{crit}} \rho_{\nu,0} = \frac{h^2}{\rho_{crit}} \Sigma_i m_i n_i$$

$$= \frac{h^2}{\rho_{crit}} \left(\frac{3\zeta(3)}{2\pi^2} T_{\nu 0}^3\right) \Sigma_i m_i = \frac{\Sigma_i m_i}{93.4\ eV} = 0.0107 \frac{\Sigma_i m_i}{1\ eV}$$ ✓

Planck $\Omega_{DM,0} = 0.26069 \Rightarrow \omega_{DM} = 0.1193$.

$\omega_\nu < \omega_{DM} \Rightarrow \Sigma_i m_i < (0.1193)(93.4\ eV) = 11.1$ eV. ✓

part 4. in order for ν to be relativistic at z_{eq}, $m_\nu < T_{eq}$

1/1

from Baumann Table 3.1, $z_{eq} = 3400$, $T_{eq} = 0.80$ eV

so $m_\nu < 0.80$ eV ✓

part 5. (a) spatially flat: $ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + \chi^2 d\Omega^2\right)$

3/4

ν traveling on a straight line $\Rightarrow d\Omega = 0$.

detected with physical momentum p_0, energy $E = \sqrt{p_0^2 + m^2}$.

saw in lecture that $p^2(t) = a^2(t) P_i P^i$, $E = P^0 = \sqrt{p^2(t) + m^2}$

$P^1 = m \frac{d\chi}{d\tau} = \frac{p(t)}{a(t)}$. $P^0 = m \frac{dt}{d\tau} = E = \sqrt{p^2(t) + m^2}$

hence $\frac{d\chi}{dt} = \frac{1}{a(t)} \frac{p(t)}{E(t)}$ ✓. we know $p \propto a^{-1}$ so $a(t)\,p(t) = 1\, p_0 \Rightarrow p(t) = \frac{p_0}{a}$.

then $\frac{d\chi}{dt} = \frac{1}{a} \frac{(p_0/a)}{\sqrt{(p_0/a)^2 + m^2}}$ ✓. thus $\chi_{dec} = \int_{t_{dec}}^{t_0} \frac{1}{a} \frac{(p_0/a)}{\sqrt{(p_0/a)^2 + m^2}}\, dt$ ✓ $\quad dt = \frac{dt}{da} da = \frac{da}{\dot a} = \frac{da}{aH}$

$$= \int_{a_{dec}}^{1} \frac{1}{a} \frac{(p_0/a)}{\sqrt{(p_0/a)^2 + m^2}} \frac{da}{H_0\sqrt{(\Omega_r a^{-2} + \Omega_m a^{-1} + \Omega_\Lambda a^2)}}$$

dropped this :(-1

(b) a_{nr} is such that $p(a) = \frac{p_0}{a_{nr}} \sim m$ and Ω_m dominates

0/2

letting $x = \frac{a}{p_0/m}$

taking matter domination, the integrand becomes $\frac{1}{H_0 \sqrt{\Omega_m}} \frac{p_0/a}{\sqrt{(p_0/a)^2 + m^2}\ \sqrt{a^{-1}}} = \frac{\sqrt{p_0/m}}{H_0 \sqrt{\Omega_m}} \frac{1}{\sqrt{x^{-1} + x^{+1}}}$

this function has a peak at $x = 1$ i.e. $a \sim a_{nr}$ and rapidly falls off on either side

we can write the integral as $\chi_{dec} = \frac{\sqrt{p_0/m}}{H_0 \sqrt{\Omega_m}} \left[\int_{\frac{a_{dec}}{p_0/m}}^{1} \frac{dx}{\sqrt{x^{-1} + x^{+1}}} + \int_{1}^{m/p_0} \frac{dx}{\sqrt{x^{-1} + x^{+1}}} \right]$

I don't know how to do this integral expansion

(c) a "typical" neutrino has $p_{dec} \sim T_{\nu,dec}$ ✓ +0.5

0.5/1

this would give $p_0 \sim a_{dec} T_{\nu,dec} = \frac{1}{6 \times 10^9} (1\ \text{MeV}) = 0.167$ meV

and hence $\chi(m) = \frac{\sqrt{p_0/m}}{H_0 \sqrt{\Omega_m}} \int_{\frac{a_{dec}}{p_0/m}}^{m/p_0} \frac{dx}{\sqrt{x^{-1} + x^{+1}}}$

$$= (7.96\ \text{Gpc}) \sqrt{\frac{0.167\ \text{meV}}{m}} \int_{\frac{1}{6\times10^9}\frac{m}{0.167\ \text{meV}}}^{m/0.167\ \text{meV}} \frac{dx}{\sqrt{x^{-1} + x^{+1}}}$$

$$\approx (7.96\ \text{Gpc}) \left(\frac{m}{0.251\ \text{meV}} \right)$$

wrong integral ...

$$= (31.7\ \text{Gpc}) \left(\frac{m}{1\ \text{meV}} \right)$$

$$= 3170\ \text{Gpc}\ \frac{m}{0.1\ \text{eV}} \quad ?$$

I ran out of time...

Problem 3: 4/4

part 1. 2/2

if $\mu_b = 0$ and protons remained in equilibrium with photons, protons would have an FD distribution and the number density would go like

$$n_p = \frac{8\pi}{h^3} T^3 I_+(m_p/T) \approx \frac{2}{h^3}(2\pi m_p T)^{3/2} e^{-m_p/T}$$

today $T = 2.73\,K$, so $n_p = (1.71 \times 10^{27}\ m^{-3}) \exp(-3.99 \times 10^{12})$

$$V = \frac{4\pi}{3} H_0^{-3} = 1.07 \times 10^{79}\ m^3$$

so $\#_p = n_p V = (1.83 \times 10^{106}) \exp(-3.99 \times 10^{12})$

$= \exp(244 - 3.99 \times 10^{12})$

$= \exp(-3.99 \times 10^{12})$

$\simeq 0.$ and certainly $\ll 1.$

part 2. 2/2

$$\omega_b = 0.022$$

$$\Omega_b = (0.022)(0.6766)^{-2} = 0.0481$$

$$\rho_{b,0} = (0.0481)\,\rho_{crit} = 4.14 \times 10^{-31}\ g/cm^3$$

$$\rho_p = 0.76\,\rho_{b,0} = 3.14 \times 10^{-31}\ g/cm^3$$

$$n_p = \rho_p/m_p = 0.188\ m^{-3}$$

if we choose to include the protons in He nuclei,

$$\rho_{He} = 0.24\,\rho_{b,0} = 9.93 \times 10^{-32}\ g/cm^3$$

$$n_{He} = \rho_{He}/m_{He} \leftarrow 6.646 \times 10^{-24}\,g = 0.0149\ m^{-3}$$

two protons per He: $n_p + 2n_{He} = 0.218\ m^{-3}$