Graduate Cosmology Spring 2025
Homework 4

due by 11:59pm on Thursday 3/6, 2025.

Problem 1: Helium recombinations [8 pts]

The first and second ionization energies of Helium, required to unbind Helium's first electron ($\mathrm{He}^0 + \gamma \to \mathrm{He}^+ + e^-$) and second electron ($\mathrm{He}^+ + \gamma \to \mathrm{He}^{++} + e^-$), are $E_1 = 24.59$ eV and $E_2 = 54.42$ eV, respectively.

1) [4 pts] Write down the Saha ionization equilibrium equations for Helium, being very explicit about your notation. Enforcing charge neutrality, and denoting by $f_{\mathrm{He}} = n_{\mathrm{He}}/n_{\mathrm{H}}$ the ratio of the number densities of Helium (in all ionization stages) to Hydrogen (in all ionization stages), translate these equations as equations involving $x_e \equiv n_e/n_{\mathrm{H}}$ alone, each valid during the relevant phase of helium recombination. To do so, assume that Hydrogen is fully ionized at all relevant times, and that the two recombination stages of helium are well separated, so that Helium is **either** present in neutral and singly ionized forms **or** in singly and doubly-ionized forms (but never in all 3 ionization forms simultaneously). [2 points for each Saha equation]

2) [4 pts] Write a code that solve for $x_e(z)$ during Helium recombinations, assumed to proceed in equilibrium, given ω_b and Y_{He} (the helium-to-baryon mass fraction) as inputs – you'll have to convert Y_{He} into f_{He}. Remember that $x_e = n_e/n_{\mathrm{H}}$ is the ratio of free-electron abundance to total (neutral and ionized) Hydrogen abundance, so it can go above unity: it should go from $1 + 2f_{\mathrm{He}}$ to $1 + f_{\mathrm{He}}$ during the first stage of helium recombination, then from $1 + f_{\mathrm{He}}$ to 1 during the second stage of helium recombination. Your code should switch from one Saha equation to the other when x_e is sufficiently close to $1 + f_{\mathrm{He}}$ (say, within 10^{-4}). Assuming the Planck 2018 best-fit value for ω_b and $Y_{\mathrm{He}} = 0.24$, plot the ionization "fraction" $x_e(z)$ as a function of redshift $z \in (1500, 8000)$.

Problem 2: Semi-classical estimate of the recombination coefficient [10 pts]

In this problem we will make a semi-classical estimate of the radiative recombination coefficient for the process $e^- + p^+ \to H^0 + \gamma$, using classical EM and a touch of quantum mechanics. Consider an electron approaching a proton (taken to be very heavy and at rest at the origin of coordinates) with velocity at infinity v and impact parameter b.

1) [1 pt] The power radiated by a time-varying electric dipole moment $\vec{D}(t)$ is (in Gaussian units, and with $c = 1$) $dE_{\mathrm{rad}}/dt = \frac{2}{3}(\ddot{\vec{D}})^2$, where $\ddot{\vec{D}} = d^2\vec{D}/dt^2$ is the second derivative of the electric dipole moment. By using Newton's law of acceleration, show that the power radiated by the electron along its trajectory is

$$\frac{dE_{\mathrm{rad}}}{dt} = \frac{2}{3}\frac{e^6}{m_e^2}\frac{1}{r^4},$$

where e is the elementary charge and r is the electron-proton separation. *Hint: what is $\vec{D}$ in terms of $\vec{r}$?*

2) As a first pass let us approximate the electron's trajectory as a straight line with constant velocity v and impact parameter b. This approximation is only accurate if $\frac{1}{2}m_e v^2 \gg e^2/b$, i.e. if the initial kinetic energy is much greater than the electrostatic potential energy at closest approach, so that the trajectory is not deflected much by the electrostatic field from the proton.

(a) [1 pt] Show that in this limit, the total energy radiated over the entire trajectory is

$$E_{\mathrm{rad}} = \frac{\pi}{3}\frac{e^6}{m_e^2 b^3 v}.$$

(b) [1 pt] Classically, radiative capture occurs if the total energy radiated exceeds the initial kinetic energy of the electron. For a given velocity v, calculate the maximum impact parameter $b_{\mathrm{max,cl}}(v)$ for which radiative capture occurs classically. Show that $b < b_{\mathrm{max,cl}}(v)$ is *inconsistent* with the straight-line trajectory assumption. *Hint: remember that we are using units where $c = 1$.*

3) Let us now consider the opposite regime, $\frac{1}{2}m_e v^2 \gg e^2/b$. This is the regime in which trajectories are most deflected, and are quasi-parabolic (both the straight line and the parabola being limiting cases of hyperbolic trajectories). Let us denote the separation and velocity at closest approach by r_0 and v_0, respectively. The quasi-parabolic limit holds when $v_0^2 \gg v^2$ and $r_0 \ll b$, implying that $\frac{1}{2}m_e v_0^2 \approx e^2/r_0$. Defining θ as the polar angle of the trajectory, ranging from 0 at $t \to -\infty$ to 2π at $t \to +\infty$, the radial separation for a parabolic orbit takes the form

$$r(\theta) = \frac{2r_0}{1 + \cos(\theta - \pi)}, \qquad \theta \in (0, 2\pi).$$

(a) [2 pt] Using conservation of angular momentum $r^2\dot{\theta} = bv$, write the total energy radiated $E_{\rm rad}$ as an integral over θ, calculate it explicitly, and express $E_{\rm rad}$ in terms of r_0, b, v. Next, combining $\frac{1}{2}m_e v_0^2 \approx e^2/r_0$ with conservation of angular momentum at closest approach $r_0 v_0 = bv$, solve for r_0 as a function of b and v, and show that

$$E_{\rm rad} = 2\pi \frac{e^{10}}{m_e^4 (bv)^5}.$$

(b) [1 pt] Compute the maximum impact parameter $b_{\rm max,cl}(v)$ for classical radiative capture. Check that $b < b_{\rm max,cl}(v)$ is consistent with the assumption of quasi-parabolic trajectory.

From here, we could go on and estimate the classical recombination coefficient $\mathcal{A}_{\rm cl}(T) = \langle \sigma(v) v\rangle = \langle \pi b_{\rm max}^2(v) \times v\rangle$, where the average is over the Maxwell-Boltzmann distribution of the electron velocity with temperature T. What we would find, however, is that this *significantly overestimates* the quantum-mechanical result. Indeed, so far our calculation is neglecting the important fact that photons are quantized, and carry angular momentum $\hbar$, hence the electron cannot be captured unless it radiates sufficient energy *and* sufficient angular momentum $L_{\rm rad} \geq \hbar$ to emit a photon. Let's add this ingredient to our calculation, by accounting for the the radiation of angular momentum by a classical time-varying electric dipole moment $\vec{D}(t)$, at rate $d\vec{L}_{\rm rad}/dt = \frac{2}{3}\dot{\vec{D}} \times \ddot{\vec{D}}$.

4) (a) [1 pt] Show that we may rewrite the rate of angular momentum radiation as

$$\frac{d\vec{L}_{\rm rad}}{dt} = \frac{2}{3}\frac{e^4}{m_e^2}\frac{\vec{L}}{r^3},$$

where $\vec{L}$ is the electron's orbital angular momentum with magnitude $L = m_e bv$.

(b) [1 pt] Assuming again a nearly parabolic orbit, show that the total angular momentum radiated along the electron's trajectory (which is assumed unperturbed) is

$$L_{\rm rad} = \frac{4\pi}{3}\frac{e^6}{m_e^2}\frac{1}{(bv)^2}.$$

(c) [1 pt] Compute the maximum impact parameter $b_{\rm max}(v)$ below which $L_{\rm rad} > \hbar$. Show that this is significantly smaller than the maximum impact parameter $b_{\rm max,cl}(v)$ we found from the purely classical energy loss condition.

(d) [2 pts] From this calculation we can obtain a semi-classical estimate of the cross-section for radiative capture $\sigma(v) = \pi b_{\rm max}^2(v)$. From this, compute the semi-classical recombination coefficient $\mathcal{A}_{\rm cl}(T) = \langle \sigma(v) \times v\rangle$, where the average is over the Maxwell-Boltzmann distribution of the electron velocity with temperature T. Write your result, first, as an analytic function of e, m_e, T. Re-instate all the c's explicitly, and then put $\mathcal{A}_{\rm cl}$ in the form $\mathcal{A}_{\rm cl}(T) = A(T/10^4{\rm K})^p$, giving the power law index p, and the coefficient A in ${\rm cm^3 s^{-1}}$. You should find $A \sim 10^{-13}$ ${\rm cm^{-3}}$ ${\rm s^{-1}}$.

The case-B recombination coefficient was computed quantum-mechanically and tabulated as a function of electron temperature (assuming zero photon temperature) in Pequignot, Petitjean & Boisson, A&A 251, 680 (1991). They fit their results with the following analytic function:

$$\mathcal{A}_B(T) \approx 4.309\frac{T_4^{-0.6166}}{1 + 0.6703\ T_4^{0.53}} \times 10^{-13}\ {\rm cm^3 s^{-1}}, \qquad T_4 \equiv T/10^4{\rm K}.$$

Plot the ratio of this coefficient to your estimate $\mathcal{A}_{\rm cl}$ for $T \in (10^3, 10^4)$ K. You should find that the semi-classical result under-estimates the quantum result by a factor 2 to 4. Given the sweeping approximations we made, this is not too bad!

Problem 3: Hydrogen recombination code [12 pts]

Write your own Hydrogen recombination code in Python, which takes as inputs the parameters T_0 (the CMB temperature today), $\omega_b, \omega_m, \omega_\Lambda, N_{\rm eff}$, and $Y_{\rm He}$, and outputs the free-electron fraction $x_e(z)$. Do *not* assume a spatially flat Universe; compute the curvature consistently given the inputs.

You have all the ingredients you need in the lecture notes, and the case-B recombination coefficient is given in Problem 2 above (use the exact, quantum result, not your semi-classical estimate). Initialize your code at $z = 1500$ with $x_e = x_e^{\rm Saha}$ (the *Hydrogen* Saha-equilibrium value, which you should also implement in a function).

Download Hyrec2 (https://github.com/nanoomlee/HYREC-2), run it and your code for the same parameters (be careful that the inputs may not be in the same format, e.g. Ω_b vs ω_b), and plot the resulting $x_e(z)$ for $z \in (200, 1500)$ on the same figure, along the Saha equilibrium result, for Planck's best-fit cosmology. Note that, in order to go beyond $z = 200$, we would have to account for the fact that baryons and photons no longer share the same temperature.