Graduate Cosmology Spring 2025
Homework 5 solutions

How to self-grade: full points for correct answer with correct reasoning; half-points for correct reasoning but incorrect answer due to algebra error; zero point for incorrect reasoning (even if final answer is correct out of luck).

Problem 1

1) See jupyter notebook on github. Please note that I had an error in my recombination code (missing factor 4 in the denominator of the Peebles C factor), which is now corrected. The visibility function peaks at $z \approx 1080$.

2) Recall that the temperature anisotroopy observed today takes the form

$$\Theta_{\rm obs}(\hat{p}) = \int d\eta \ g(\eta) \ S(\eta, -(\eta_0 - \eta)\hat{p}, \hat{p}), \qquad g(\eta) \equiv a n_e \sigma_{\rm T} \exp\left(-\int_\eta^{\eta_0} d\eta' a' n_e' \sigma_{\rm T}\right), \qquad n_e = x_e n_{\rm H}.$$

If the Universe gets re-ionized at $z = z_{\rm reio} \sim 10$, the free-electron fraction at $z \leq z_{\rm reio}$ changes from $x_e \sim 10^{-4} \ll 1$ to $x_e = 1 + 2f_{\rm He} \approx 1.16$. Although the ionization history is only changed at $z \leq z_{\rm reio}$, the visibility function is actually affected at all redshifts due to the exponential term, which includes an integral all the way to the present time. In particular, at $z \geq z_{\rm reio}$, the visibility function is multiplied by a factor $e^{-\tau_{\rm reio}}$ (neglecting the very small contribution from $x_e \sim 10^{-4}$ that would be there without reionization). Since we are told to neglect the source term contributions from low redshift, we conclude that the main effect of reionization is to damp anisotropies by this factor.

3) We write explicitly, using $x_e = 1 + 2f_{\rm He}$,

$$\tau_{\rm reio} = \int_{t_{\rm reio}}^{t_0} (1 + 2f_{\rm He}) n_{\rm H} \sigma_{\rm T} dt = (1 + 2f_{\rm He})\sigma_{\rm T} \int_{a_{\rm reio}}^1 da \frac{n_{\rm H}(a)}{aH(a)} = (1 + 2f_{\rm He})\sigma_{\rm T} \frac{n_{\rm H,0}}{H_0} \int_{a_{\rm reio}}^1 da \frac{a^{-3}}{a\sqrt{\Omega_m a^{-3} + \Omega_\Lambda}},$$

where we neglected the contribution of radiation at these low redshifts, as well as that of curvature, and where $n_{\rm H,0}$ is the hydrogen number density today. The integral is actually analytic, and we get

$$\tau_{\rm reio} = (1 + 2f_{\rm He})\sigma_{\rm T} \frac{n_{\rm H,0}}{H_0} \frac{2}{3\Omega_m} \left(\sqrt{\Omega_m (1 + z_{\rm reio})^3 + \Omega_\Lambda} - \sqrt{\Omega_m + \Omega_\Lambda}\right).$$

With $z_{\rm reio} \sim 10$, we see that the first term largely dominates (in other words, we could have assumed pure matter domination), hence we have the very simple approximate analytic result

$$\tau_{\rm reio} \approx \frac{2}{3}(1 + 2f_{\rm He})\sigma_{\rm T} \frac{n_{\rm H,0}}{H_0\sqrt{\Omega_m}} (1 + z_{\rm reio})^{3/2} \approx 0.00236 \ (1 + z_{\rm reio})^{3/2}.$$

where we used the Planck best-fit cosmological parameters. We can easily invert this relation to

$$z_{\rm reio} \approx (\tau_{\rm reio}/0.00236)^{2/3} - 1,$$

which we can furthermore differentiate to obtain the error in $z_{\rm reio}$ given the error in $\tau_{\rm reio}$:

$$\Delta z_{\rm reio} \approx \frac{2}{3} (\tau_{\rm reio}/0.00236)^{-1/3} \Delta\tau_{\rm reio}/0.00236.$$

With the given Planck measurement of $\tau_{\rm reio} = 0.054 \pm 0.007$, we conclude that reionization happened at

$$z_{\rm reio} = 7.1 \pm 0.70.$$

Problem 2

1) Let us plug in the single Fourier mode:

$$\Theta(\eta, \vec{x} = 0, \hat{p}) = \left(\Theta_* + iV_*(\hat{k} \cdot \hat{p})\right) e^{-ik(\eta - \eta_*)\hat{k}\cdot\hat{p}}.$$

To find the monopole, we must average over the direction $\hat{p}$:

$$\Theta_0(\eta, \vec{x}=0) = \int \frac{d^2\hat{p}}{4\pi}\Theta(\eta, \vec{x}=0, \hat{p}) = \Theta_* \int \frac{d^2\hat{p}}{4\pi} e^{-ik(\eta-\eta_*)\hat{k}\cdot\hat{p}} + iV_* \int \frac{d^2\hat{p}}{4\pi}(\hat{k}\cdot\hat{p}) e^{-ik(\eta-\eta_*)\hat{k}\cdot\hat{p}}$$
$$= \Theta_* \frac{1}{2}\int_{-1}^{1} d\mu\; e^{-ik(\eta-\eta_*)\mu} + iV_* \frac{1}{2}\int_{-1}^{1} d\mu\; \mu\; e^{-ik(\eta-\eta_*)\mu},$$

where in the second line we used spherical polar coordinates with $\hat{k}$ along the z-axis, and performed the φ integral (since nothing depends on that angle), so the only integral left is over $\mu \equiv \hat{k}\cdot\hat{p}$. The first integral can easily be done, and the second with slightly more work (integrating by parts), so we obtain, in the end,

$$\Theta_0 = \Theta_* \frac{\sin X}{X} + V_* \frac{\sin X - X\cos X}{X^2}, \qquad X \equiv k(\eta - \eta_*).$$

We see that this converges to Θ_* at $\eta = \eta_*$, and is then an oscillating function with magnitude decreasing as $1/k(\eta-\eta_*)$ when this product gets large.

To find the dipole, defined as $\Theta(\hat{p}) = \Theta_0 + \vec{V}_\gamma \cdot \hat{p}$, we must average $3\hat{p}\Theta(\hat{p})$ over angles:

$$\vec{V}_\gamma = 3\int \frac{d^2\hat{p}}{4\pi}\hat{p}\;\Theta(\hat{p}) = 3\Theta_* \int \frac{d^2\hat{p}}{4\pi}\;\hat{p}\; e^{-iX\hat{k}\cdot\hat{p}} + 3iV_* \int \frac{d^2\hat{p}}{4\pi}\;\hat{p}\;(\hat{k}\cdot\hat{p}) e^{-iX\hat{k}\cdot\hat{p}}.$$

There is only one preferred direction, $\hat{k}$, so it must be that $\vec{V}_\gamma \propto \hat{k}$. We write $\vec{V}_\gamma = i\hat{k}V_\gamma$, where $V_\gamma = -i\hat{k}\cdot\vec{V}_\gamma$ is thus given by

$$V_\gamma = -3i\Theta_* \frac{1}{2}\int_{-1}^{1} d\mu\; \mu e^{-iX\mu} + 3V_* \frac{1}{2}\int_{-1}^{1} d\mu\; \mu^2 e^{-iX\mu}.$$

Upon calculating the integrals, we get

$$V_\gamma = 3\Theta_* \frac{X\cos X - \sin X}{X^2} + 3V_* \frac{\left(X^2-2\right)\sin X + 2X\cos X}{X^3}.$$

Again, this converges to V_* at $\eta \to \eta_*$, and decays as $1/k(\eta-\eta_*)$ at large values of this product.

2) The desired wavenumbers is just $k_* = \mathcal{H}(a_*) = a_* H(a_*)$. In the matter-radiation era, we have

$$k_* = H_0\sqrt{\Omega_m}\sqrt{a_*^{-1}(1 + a_{\rm eq}/a_*)}.$$

Plugging $a_* \approx 10^{-3}$ and $a_{\rm eq}/a_* \approx 0.3$, we get $k_* \approx 0.005$ Mpc^{-1}.

To compute the desired suppression, we first need to calculate the conformal time η corresponding to $z = 10$. To do so, use

$$\frac{d\eta}{dz} = -a^2\frac{d\eta}{da} = -a\frac{dt}{da} = -\frac{1}{H(z)} = -\frac{1}{H_0\sqrt{\Omega_m(1+z)^3 + \Omega_r(1+z)^4}},$$

implying

$$\eta(z) = \int_z^\infty \frac{dz'}{H_0\sqrt{\Omega_m(1+z')^3 + \Omega_r(1+z')^4}}.$$

The integral is dominated by the smallest z', so we may compute it neglecting Ω_r alltogether:

$$\eta(z) \approx \frac{1}{H_0\sqrt{\Omega_m}}\int_z^\infty \frac{dz'}{(1+z')^{3/2}} = \frac{2}{H_0\sqrt{\Omega_m}}\frac{1}{\sqrt{1+z}}.$$

Therefore, for $z \ll z_*$, $\eta \gg \eta_*$, and we get $k(\eta-\eta_*) \approx k\eta$. Hence, for $k \gtrsim k_*$, we find

$$k(\eta(z) - \eta_*) \approx k\eta(z) \gtrsim k_*\eta(z) \approx 2\sqrt{(1+z_*)/(1+z)} \approx 2\sqrt{10^3/10} \approx 20.$$

where I neglected the corrections of order $a_{\rm eq}/a_* \approx 0.3$. So we conclude that the photon monopole and dipole are suppressed by at least a factor ~ 20 at reionization relative to their values at recombination/ last scattering.

3) We rewrite the temperature anisotropy as

$$\Theta(\eta, \vec{x}=0, \hat{p}) = \left(\Theta_* - V_* \frac{d}{dX}\right) e^{-iX\hat{k}\cdot\hat{p}}, \qquad X \equiv k(\eta - \eta_*).$$

Then, we plug in the plane-wave expansion, so we obtain

$$\begin{aligned}\Theta(\eta, \vec{x}=0, \hat{p}) &= \left(\Theta_* - V_* \frac{d}{dX}\right) \sum_\ell (2\ell+1)(-i)^\ell j_\ell(X) P_\ell(\hat{k}\cdot\hat{p}) \\ &= \sum_\ell (2\ell+1)(-i)^\ell \Theta_\ell P_\ell(\hat{k}\cdot\hat{p}).\end{aligned}$$

Note that there was a missing $(-i)^\ell$ in the homework, so your result may differ by this factor because of that. We can then read off the coefficients:

$$\Theta_\ell = \left(\Theta_* - V_* \frac{d}{dX}\right) j_\ell(X).$$

Problem 3

1) We use the same trick as in the lecture notes, and write $x^\lambda \approx \tilde{x}^\lambda - \xi(\tilde{x}^\lambda)$. We then have, to first order in ξ,

$$\tilde{T}^\mu_{\ \nu}(\tilde{x}^\lambda) = \left(\delta^\mu_{\ \alpha} + \partial_\alpha \xi^\mu\right)\left(\delta^\beta_{\ \nu} - \partial_\nu \xi^\beta\right) T^\alpha_{\ \beta}(x^\lambda).$$

Expanding the last term as $T^\alpha_{\ \beta}(x^\lambda) = T^\alpha_{\ \beta}(\tilde{x}^\lambda) - \xi^\lambda \partial_\lambda T^\alpha_{\ \beta}$, so we can express everything at the same values of the coordinates, and then substituting $T = \overline{T} + \delta T$, we obtain

$$\delta\tilde{T}^\mu_{\ \nu} = \delta T^\mu_{\ \nu} + \partial_\alpha \xi^\mu \ \overline{T}^\alpha_{\ \ \nu} - \partial_\nu \xi^\beta \overline{T}^\mu_{\ \ \beta} - \xi^\lambda \partial_\lambda \overline{T}^\mu_{\ \ \nu}.$$

where all quantities are expressed at the same values of the coordinates. Since the background stress energy tensor only depends on $\eta = x^0$, the last term simplifes to $\xi^0 \partial_0 \overline{T}^\mu_{\ \ \nu}$.

2) We start by applying the above to $\mu = \nu = 0$:

$$-\delta\tilde{\rho} = -\delta\rho + \partial_\alpha \xi^0 \ \overline{T}^\alpha_{\ \ 0} - \partial_0 \xi^\beta \overline{T}^0_{\ \ \beta} + \xi^0 \partial_0 \overline{\rho}.$$

Since the background stress-energy tensor is diagonal, the 2nd and last term are identical up to a minus sign, and thus cancel. So we are left with

$$\delta\tilde{\rho} = \delta\rho - \xi^0 \overline{\rho}'.$$

Next, applying to $\mu = i, \nu = 0$, we get

$$-(\overline{\rho} + \overline{P})\tilde{V}^i = -(\overline{\rho} + \overline{P})V^i + \partial_\alpha \xi^i \ \overline{T}^\alpha_{\ \ 0} - \partial_0 \xi^\beta \ \overline{T}^i_{\ \ \beta},$$

where we used $\overline{T}^i_{\ \ 0} = 0$. Again, using the fact that the background stress-energy tensor is diagonal, we get

$$-(\overline{\rho} + \overline{P})\tilde{V}^i = -(\overline{\rho} + \overline{P})V^i + \partial_0 \xi^i (-\overline{\rho}) - \partial_0 \xi^i \ \overline{P} \quad \Rightarrow \tilde{V}^i = V^i + \partial_0 \xi^i.$$

Lastly, we apply this to $\mu = i, \nu = j$, and obtain

$$\delta\tilde{T}^i_{\ j} = \delta T^i_{\ j} + \partial_\alpha \xi^i \overline{T}^\alpha_{\ \ j} - \partial_j \xi^\beta \overline{T}^i_{\ \ \beta} - \xi^0 \partial_0 \overline{T}^i_{\ \ j}.$$

Again, using the diagonality of $\overline{T}^\mu_{\ \ \nu}$, we see that the 2nd and third terms are identical up to a sign, and cancel out, so we are left with

$$\delta\tilde{T}^i_{\ j} = \delta T^i_{\ j} - \delta^i_{\ j} \xi^0 \ \overline{P}'.$$

In other words, a gauge transformation only adds a component proportional to the identity matrix to $T^i{}_j$, hence does not change the anisotropic stress. We conclude that $\Pi^i{}_j$, **with this specific, one-up, one-down arrangement of indices**, is gauge invariant – it would not be the case for Π_{ij} for instance.

3) In the notes (Lecture 15), we already saw how C and E transform:

$$\tilde{C} = C + \frac{1}{3}k^2 L - \mathcal{H}\xi^0, \qquad \tilde{E} = E - L,$$

where L is the longitudianl part of ξ^i. Therefore, we find

$$\tilde{\mathcal{R}} = \mathcal{R} + \xi^0 \left(\mathcal{H} - \mathcal{H}\frac{\overline{\rho}'}{\overline{\rho}'} \right) = \mathcal{R}.$$

In other words, this combination of metric and matter perturbations is indeed gauge-invariant.

Problem 4

1) Consider a perturbed metric $g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu})$. We look for its inverse in the form $g^{\mu\nu} = \frac{1}{a^2}(\eta^{\mu\nu} + \epsilon^{\mu\nu})$. Requiring that these two are indeed inverse of one another, we get

$$\delta^\alpha_\beta = g^{\alpha\mu} g_{\mu\beta} = (\eta^{\alpha\mu} + \epsilon^{\alpha\mu})(\eta_{\mu\beta} + h_{\mu\beta}) = \delta^\alpha_\beta + \epsilon^{\alpha\mu}\eta_{\mu\beta} + \eta^{\alpha\mu} h_{\mu\beta} + \mathcal{O}(h\epsilon) \qquad \Rightarrow \epsilon^{\alpha\mu}\eta_{\mu\beta} = -\eta^{\alpha\mu} h_{\mu\beta}.$$

Multiplying by the inverse-Minkowski metric, and using its symmetry, we get the desired result.

2) See the attached notebook. The only modification is to the metric. Then, the main difficulty is rewriting the output in a general form, accounting for the divergence-less-ness of h_{0i} (which at least I didn't figure out how to easily enforce in the code). This is what the code's output is for a few specific coefficients:

$$\begin{aligned}
G_{00} &= 3\mathcal{H}^2 - 2\mathcal{H}(\partial_x h_{01} + \partial_y h_{02} + \partial_z h_{03}) \\
G_{01} &= -(\mathcal{H}^2 + 2\mathcal{H}')h_{01} - \frac{1}{2}(\partial_y^2 h_{01} + \partial_z^2 h_{01}) + \frac{1}{2}(\partial_{xy} h_{02} + \partial_{xz} h_{03}) \\
G_{11} &= 2\mathcal{H}\left[\partial_y h_{02} + \partial_z h_{03}\right] + \partial_y \partial_\eta h_{02} + \partial_z \partial_\eta h_{03} - 2\mathcal{H}' - \mathcal{H}^2, \\
G_{12} &= -\frac{1}{2}\partial_y \partial_\eta h_{01} - \frac{1}{2}\partial_x \partial_\eta h_{02} - \mathcal{H}(\partial_y h_{01} + \partial_x h_{02}).
\end{aligned}$$

The first step is to rewrite these in a more general way (and cross-check that it matches the other coefficients):

$$\begin{aligned}
G_{00} &= 3\mathcal{H}^2 - 2\mathcal{H}\partial_i h_{0i} \\
G_{0i} &= -(\mathcal{H}^2 + 2\mathcal{H}')h_{0i} - \frac{1}{2}\nabla^2 h_{0i} + \frac{1}{2}\partial_i \partial_j h_{0j} \\
G_{ij} &= -(2\mathcal{H}' + \mathcal{H}^2)\delta_{ij} - \mathcal{H}(\partial_i h_{0j} + \partial_j h_{0i}) + 2\mathcal{H}\delta_{ij}\partial_k h_{0k} - \frac{1}{2}\left(\partial_\eta \partial_i h_{0j} + \partial_\eta \partial_j h_{0i}\right) + \delta_{ij}\partial_\eta \partial_k h_{0k}.
\end{aligned}$$

You should double check that these expressions (especially for G_{ij}) indeed match the output of the code when $i = j$ and $i \neq j$. Next, we simplify these expressions using $\partial_i h_{0i} = 0$, and arrive at

$$\begin{aligned}
G_{00} &= 3\mathcal{H}^2, \\
G_{0i} &= -(\mathcal{H}^2 + 2\mathcal{H}')h_{0i} - \frac{1}{2}\nabla^2 h_{0i} \\
G_{ij} &= -(2\mathcal{H}' + \mathcal{H}^2)\delta_{ij} - \mathcal{H}(\partial_i h_{0j} + \partial_j h_{0i}) - \frac{1}{2}\left(\partial_\eta \partial_i h_{0j} + \partial_\eta \partial_j h_{0i}\right).
\end{aligned}$$

Always check that your expressions are dimensionally correct, and have the right symmetries.

3) First, since we computed $G_{\mu\nu}$, with both components down, we need $T_{0i} = T_{i0}$, with both components down:

$$T_{i0} = g_{ij} T^j{}_0 = a^2 T^i{}_0 = -a^2(\overline{\rho} + \overline{P})V^i.$$

So, the $0i$ EFE tells us

$$\boxed{(\mathcal{H}^2 + 2\mathcal{H}')h_{0i} + \frac{1}{2}\nabla^2 h_{0i} = 8\pi G a^2(\overline{\rho} + \overline{P})V^i}.$$

4) I didn't actually use the notebook for this problem. In the absence of anisotropic stress, $T^{\mu\nu} = (\rho + P)U^\mu U^\nu + P g^{\mu\nu}$. We then have, using metric compatibility $\nabla_\mu g^{\mu\nu} = 0$,

$$\nabla_\mu T^{\mu\nu} = U^\mu U^\nu \nabla_\mu(\rho + P) + (\rho + P)\left[U^\mu \nabla_\mu U^\nu + U^\nu \nabla_\mu U^\mu\right] + g^{\mu\nu}\nabla_\mu P = 0.$$

We start by projecting this equation along U_ν and obtain, with $U_\nu U^\nu = -1$, that

$$U^\mu \nabla_\mu \rho + (\rho + P)\nabla_\mu U^\mu = 0.$$

Subracting out this term (time U^ν) from the original equation, we are left with the term perpendicular to U^ν:

$$(\rho + P)U^\mu \nabla_\mu U^\nu + (g^{\mu\nu} + U^\mu U^\nu)\partial_\mu P = 0,$$

where we used $\nabla_\mu P = \partial_\mu P$ since P is a scalar field. This is simply the relativistic equivalent of $\rho d\vec{V}/dt = -\vec{\nabla} P$.

So far this was very general, up to neglecting the anisotropic stress. Let us now specialize to pure vector perturbations. Since ρ and P are scalars, they are unperturbed, so $\rho = \overline{\rho}$ and $P = \overline{P}$. Since $\overline{P}$ only depends on $\eta = x^0$, we thus obtain

$$(\overline{\rho} + \overline{P})\left[U^\mu \partial_\mu U^\nu + \Gamma^\nu_{\mu\lambda} U^\mu U^\lambda\right] + (g^{0\nu} + U^0 U^\nu)\partial_0 \overline{P} = 0,$$

where we wrote explicitly the covariant derivative of U^ν. Let us now expand this to linear order in perturbations. Recall that $\overline{U}^0 = 1/a$ and $\overline{U}^i = 0$. Since we are considering pure vector modes, $\delta U^0 = 0$. Moreover, $\overline{\Gamma}^i_{00} = 0$ since the background is isotropic. Let us set $\nu = i$; we obtain, at linear order in perturbations,

$$(\overline{\rho} + \overline{P})\left[\frac{1}{a}\partial_0 \delta U^i + 2\frac{1}{a}\overline{\Gamma}^i_{0j}\delta U^j\right] + \left(\delta g^{0i} + \frac{1}{a}\delta U^i\right)\partial_0 \overline{P} = 0.$$

We are almost there. The only Christoffel we need is the background $\overline{\Gamma}^i_{0j} = \mathcal{H}\delta^i_j$. Moreover, using question 1),

$$\delta g^{0i} = -\frac{1}{a^2}\eta^{0\mu}\eta^{i\nu}h_{\mu\nu} = +\frac{1}{a^2}h_{0i}.$$

Lastly, We need to relate δU^i to V^i, which is defined through $T^i_{\ 0} = -(\overline{\rho} + \overline{P})V^i$. From our definition, $T^i_{\ 0} = -a^2 T^{i0} = -a^2\left[\frac{1}{a}(\overline{\rho} + \overline{P})\delta U^i\right] = -a(\overline{\rho} + \overline{P})\delta U^i$, hence $\delta U^i = V^i/a$. Putting it all together, we arrive at

$$\boxed{\partial_\eta V^i + \left(\mathcal{H} + \frac{\overline{P}'}{\overline{\rho} + \overline{P}}\right)V^i = -\frac{\overline{P}'}{\overline{\rho} + \overline{P}}h_{0i}}.$$

We can now in principle combine the 2 boxed equations and solve them (probably numerically). To simplify our lives, we will focus on radiation domination, during which $\overline{P} = \frac{1}{3}\overline{\rho}$ and $\overline{\rho}' = -4\mathcal{H}\overline{\rho}$. In this case the term multiplying V^i cancels out, and we are left with

$$h_{0i} = \frac{1}{\mathcal{H}}\partial_\eta V^i.$$

Substituting this into the $0i$ EFE, and working in Fourier space, we obtain the following equation for V^i:

$$(\mathcal{H} + 2\mathcal{H}'/\mathcal{H} - k^2/2\mathcal{H})\partial_\eta V^i = 4\mathcal{H}^2 V^i,$$

where we substituted $8\pi G a^2(\overline{\rho} + \overline{P}) = 4\mathcal{H}^2$ during radiation domination. Lastly, substitute $\mathcal{H} = 1/\eta$, so we arrive at

$$\left(1 + \frac{1}{2}k^2\eta^2\right)\partial_\eta V^i = -\frac{4}{\eta}V^i.$$

The solution is

$$V^i(\eta, \vec{k}) = C\frac{(2 + k^2\eta^2)^2}{\eta^4}, \quad h_{0i} = -8C\frac{2 + k^2\eta^2}{\eta^4}.$$

We see that both V^i and h_{0i} decay rapidly as $1/\eta^4$ until horizon entry; after that, V^i asymptotes to a constant and h_{0i} keeps on decaying as $1/\eta^2$.