Graduate Cosmology Spring 2025
Homework 2

due by 11:59pm on Wednesday 2/12, 2025.

Problem 1: age of the Universe [13 points]

1. [1 point] For a general Universe with arbitrary Ω_i, express the function $t(a)$ as an integral.

2. (a) [2 points] Neglecting curvature, and focusing on $a \ll a_{m\Lambda}$ (matter-radiation era), simplify the integral, and express $t(a)$ explicitly as a function of $a/a_{\rm eq}$. Do *not* use Mathematica: instead, first simplify the integrand so it becomes independent of cosmological parameters, and then use the result $\int dx x/\sqrt{1+x} = (2/3)(x-2)\sqrt{1+x}+\text{const}$.

(b) [4 points] Find the leading-order asymptotic solutions at $a \ll a_{\rm eq}$ and $a \gg a_{\rm eq}$, by doing a Taylor expansion if needed (again, do it by hand, not with Mathematica). Make sure that your results are consistent with the expressions we derived in class for $a(t)$ in both matter and radiation domination. You may need the Taylor expansion $\sqrt{1+x} = 1 + x/2 - x^2/8 + \mathcal{O}(x^3)$. [2 points for each limit].

(c) [1 point] Compute $t_{\rm eq} = t(a_{\rm eq})$ in years, showing your reasoning.

3. (a) [2 points] Now focus on $a \gg a_{\rm eq}$ (matter-Λ era), and find an explicit equation for $t(a)$. Again, do not use Mathematica: first write the integral as a function of $a/a_{m\Lambda}$, with an integrand that is indepenent of cosmological parameters, then use $\int\ x^{1/2}dx/\sqrt{1+x^3} = \frac{2}{3}\text{Arcsinh}(x^{3/2}) + \text{const}$. Check that your result is consistent with Eq. (2.177) of Baumann.

(b) [1 point] Compute the age of the Universe t_0 in years, showing your reasoning.

4. [2 points] What is the conformal time today, η_0, expressed both in years and in Mpc? Give a detailed reasoning. This requires performing a numerical integral, feel free to use the software of your choice.

Problem 2: cosmological distances [15 points]

1. [1 point] Express the comoving radial distance $\chi(z) = \eta_0 - \eta(z)$ as a function of redshift in the form of an integral, as a function of H_0 and of the dimensionless density parameters $\Omega_r, \Omega_m, \Omega_\Lambda, \Omega_k$.

2. [3 points] Write a function in Python (or programming language of your choice) for $\chi(z; H_0, \Omega_i)$. Assuming Planck 2018 best-fit values of the cosmological parameters (look them up!), and $\Omega_k = 0$, plot $\chi(z)$ in units of Gpc, as a function of $(1+z)$ for $z \in (0, 10^4)$. Use a log-scale for $1+z$ and a linear scale for $\chi(z)$.

3. [4 points] Write functions in Python (or programming language of your choice) for the angular-diameter distance $d_A(z; H_0, \Omega_i)$ and the luminosity distance $d_L(z; H_0, \Omega_i)$. Note that you will first need to express the radius of curvature R_0 (for non-flat space cases) as a function of H_0 and Ω_k. On two separate figures (on for d_A, one for d_L), plot each distance in Gpc as a function of redsfhit for $z \in (0, 10)$, for the following cosmological parameters:
- $\Omega_k = 0$ and Planck 2018 best-fit cosmology
- $\Omega_k = +0.5$, $H_0 = H_0^{\rm Planck}$, and other $\Omega_i = 0.5\ \Omega_i^{\rm Planck}$
- $\Omega_k = -0.5$, $H_0 = H_0^{\rm Planck}$, and other $\Omega_i = 1.5\ \Omega_i^{\rm Planck}$
- $\Omega_k = \Omega_\Lambda = \Omega_r = 0$, $\Omega_m = 1$, $H_0 = H_0^{\rm Planck}$.

4. [1 point] In the remainder of this problem, we assume a spatially flat Universe, and focus on low redshifts $z \ll z_{\rm eq}$ so that we may safely neglect the contribution of radiation to the Hubble rate. Show that in that case the luminosity distance takes the form $d_L(z) = H_0^{-1} F(z; \Omega_\Lambda)$, and write F explicitly. Write a separate Python function for $F(z; \Omega_\Lambda)$.

5. The Pantheon dataset consists of a large number of supernovae, with known redshifts z_i, and an estimate of the (log) of their luminosity distance [called distance modulus] μ_i, up to an overall additive constant corresponding to a

global multiplicative offset in supernovae luminosities; explicitly, the distance modulus for each supernova i is

$$\mu_i = 5\log_{10}(d_{L,i}/\mathrm{Mpc}) + M,$$

where M is the global additive constant. Each measurement of μ_i is assorted with an error bar σ_i. Our goal is to find the cosmological parameters (and global constant M) that minimize the χ^2:

$$\chi^2(\Omega_\Lambda, H_0, M) = \sum_i \frac{\left(5\log_{10}\left[d_L(z_i, H_0, \Omega_\Lambda)/\mathrm{Mpc}\right] + M - \mu_i\right)^2}{\sigma_i^2} = \sum_i \frac{\left(5\log_{10}[F(z_i;\Omega_\Lambda)] - \mu_i + K\right)^2}{\sigma_i^2},$$

where $K \equiv M + 5\log_{10}(H_0^{-1}/\mathrm{Mpc})$. We see that M and H_0 are fully degenerate, in that they always appear through the single combination K.

(a) [2 point] For a given value of Ω_Λ, calculate the parameter $K(\Omega_\Lambda)$ that minimizes the χ^2. Plug back in the expression for χ^2 and show that the partially minimized χ^2 is now

$$\tilde{\chi}^2(\Omega_\Lambda) = \sum_i \frac{\left(5\log_{10}[F(z_i;\Omega_\Lambda)] - \mu_i\right)^2}{\sigma_i^2} - \frac{1}{\sum_i 1/\sigma_i^2}\left(\sum_i \frac{5\log_{10}[F(z_i;\Omega_\Lambda)] - \mu_i}{\sigma_i^2}\right)^2$$

(b) [4 points] Download the luminosity-distance data from the Pantheon sample (Scolnic et al. 2018) given in the "Pantheon SN Parameters (.txt)" file at this link: `https://archive.stsci.edu/prepds/ps1cosmo/scolnic_datatable.html`. The columns are (1) i-th supernova name, (2) z_i, (3) z_i again (4) Δz (neglect these last two columns), (5) distance modulus μ_i, and (6) error σ_i on the distance modulus. Plot $\tilde{\chi}^2(\Omega_\Lambda)$ as a function of $\Omega_\Lambda \in (0, 1)$. Use your favorite minimization routine to find the parameter Ω_Λ minimizing $\tilde{\chi}^2$.