Graduate Cosmology Spring 2025
Homework 6

due by 11:59pm on Monday 4/14, 2025.

Problem 1: Baryon temperature and pressure [14 points]

So far we have assumed that the baryon temperature follows exactly the photon temperature, while in reality they are only coupled to one another while Thomson scattering is effective. In this problem you will derive the relevant equation. The first 5 questions only require considering *background* quantities.

1) [2 points] Using the first law of thermodynamics $dU = -PdV + \delta Q$, applied to baryons, show that the background baryon temperature T_b evolves according to $\dot{T}_b + 2HT_b = \frac{2}{3n_b}\dot{q}$, where n_b is the (background) total number density of all "baryon" particles, and $\dot{q}$ is the net rate of heating (minus cooling) of baryons per unit proper volume per unit time. To simplify, assume the ionization state of baryons is fixed, so that $a^3 n_b$ is constant.

2) [1 point] Consider Thomson scattering in the initial electron's rest frame. Assuming the magnitude of the photon's final and initial momenta are approximately equal, compute the electron's recoil velocity (by conservation of momentum), and show that the electron gains energy $\Delta E_e \approx \frac{E_\gamma^2}{m_e}(1 - \cos\theta)$, where θ is the angle between initial and final photon momenta.

3) [3 points] The differential rate of scattering per unit volume per photon energy interval per $\cos\theta$ interval is

$$\frac{dN_{\rm scat}}{dtdV dE_\gamma d\cos\theta} = n_e \frac{d\sigma_{\rm T}}{d\cos\theta}\frac{dn_\gamma}{dE_\gamma}(1 + f_\gamma(E_\gamma)),$$

where dn_γ/dE_γ is the number density of photons per energy interval, and the last term accounts for stimulated scatterings. Neglecting the thermal motions of electrons (i.e. assuming the iniial electron's rest-frame is the cosmic rest-frame), show that the rate at which photons heat electrons per unit volume is

$$\dot{q}(T_b = 0) = \frac{4 n_e \sigma_{\rm T} \rho_\gamma}{m_e} T_\gamma.$$

In reality baryons have a finite temperature, and the *net* heating minus cooling rate must vanish when $T_b = T_\gamma$. At lowest order in T_b/m_e, the cooling rate has to be linear in T_b. Therefore, the net heating minus cooling rate is obtained from the above expression with the last factor T_γ replaced by $(T_\gamma - T_b)$.

4) [1 point] Show that the total baryon particle number density, including electrons, neutral and ionized hydrogen and helium, is $n_b = n_{\rm H}(1 + x_e + f_{\rm He})$, where $f_{\rm He}$ is the ratio of helium-to-hydrogen by number. Do not make any assumption about Helium being neutral or ionized.

5) [3 points] Putting it all together, we have arrived at the following equation for the baryon temperature:

$$\dot{T}_b + 2HT_b = \frac{8\sigma_{\rm T}\rho_\gamma x_e}{3m_e(1 + x_e + f_{\rm He})}(T_\gamma - T_b).$$

Using your recombination code (extend it all the way to $z = 50$), plot the ratio of the rate multiplying $(T_\gamma - T_b)$ over the Hubble rate, as a function of redshift, and find the approximate redshift z_T at which it falls below unity. Explain qualitatively how T_b evolves with scale factor before and after z_T.

6) [4 points] For this last question, we consider perturbations. Using the continuity equation to relate density perturbations to the velocity field (neglecting ϕ'), and neglecting baryon temperature fluctuations to simplify, estimate the ratio of the baryon pressure term $\nabla P_b/\overline{\rho}_b$ to the Hubble friction term $\mathcal{H}V_b$ in the baryon momentum equation, as a function of wavenumber k and scale factor (2 points). Considering the epoch after recombination (during which baryon perturbations start growing on a Hubble timescale), estimate the wavenumber beyond which the pressure term dominates, as a function of redshift $z \gg 1$. Express your result as $k_P \sim \#(z/z_T)^\alpha$ Mpc^{-1}, and give two values of α, one for $z \gtrsim z_T$, and one for $z \lesssim z_T$ (2 points). *Hint:* make a simple approximation to relate n_b to ρ_b.

Problem 2: Silk damping reloaded [4 points]

In this problem you will derive the exponential damping to the finite photon mean-free path more rigorously, starting from the Boltzmann hierarchy. We combine photon and baryon momentum equations to eliminate the Thomson drag force, and to simplify, neglect the baryon inertia $R \ll 1$. Putting back the anisotropic stress (but neglecting the next photon moment), the equations satisfied by the photon-baryon fluid are then

$$\begin{aligned}
&\delta_\gamma' - \frac{4}{3} k V_\gamma = 4\phi' \\
&V_\gamma' = -\frac{1}{4} k\delta_\gamma + k\Pi_\gamma - k\psi, \\
&\Pi_\gamma' = -\frac{4}{15} kV_\gamma - \frac{9}{10} a n_e \sigma_\mathrm{T} \Pi_\gamma.
\end{aligned}$$

1) [1 point] Considering the limit $a n_e \sigma_\mathrm{T} \gg \mathcal{H}, k$, find a quasi-stationary approximation for Π_γ from the last equation, substitute back in the 2nd one, and combine with the first one to arrive at the following equation for δ_γ:

$$\delta_\gamma'' + \frac{8}{27} \frac{k^2}{a n_e \sigma_\mathrm{T}} \delta_\gamma' + \frac{k^2}{3} \delta_\gamma = \mathrm{rhs}(\phi, \psi).$$

2) [3 points] Seek a WKB homogeneous solution of the form $\delta_\gamma = A(\eta) e^{\pm ik\eta/\sqrt{3}}$ with $A'/A \ll k$, and show that the leading-order solution is

$$A(\eta) \propto \exp\left(-\alpha\, k^2 \int^\eta \frac{d\eta'}{a n_e \sigma_\mathrm{T}(\eta')}\right),$$

where α is a numerical constant that you should give explicitly. Note that to get an even more accurate prefactor α, one would have to account for non-vanishing baryon inertia, compute the photon-baryon slip $V_b - V_\gamma$ at the same order as Π_γ, and moreover account for photon polarization. But the general procedure would remain identical.

Problem 3: Simplified cosmological perturbation code [12 points]

Write a numerical code to solve the relativistic scalar cosmological perturbation equations with adiabatic initial conditions, for photons, baryons, neutrinos, and dark matter, with the following approximations:

- Assume neutrinos are relativistic at all times, and neglect their anisotropic stress, hence treating them as an ideal fluid with sound speed $c_s = 1/\sqrt{3}$.

- Treat photons and baryons as a tightly-coupled ideal fluid, accounting for the non-zero baryon inertia R, until redshift $z_d = 10^3$.

- After z_d, assume baryons and photons are fully decoupled, and follow them separately (in addition to DM and neutrinos). Neglect baryon pressure, and neglect photon anisotropic stress, i.e. treat them as an ideal fluid with sound speed $c_s = 1/\sqrt{3}$. Make sure to match all overdensities and velocities at z_d.

Your code should thus evolve 9 different variables: $\Phi = \Psi$, and δ_s and V_s for 4 species s = photons, baryons, neutrinos, CDM. For each wavenumber k, make sure to initialize your code deep in the super-horizon era.

Plot, as a function of scale factor, the evolution of the overdensities of photons, baryons and CDM (divided by some initial conditions) from the initial scale factor to the present time, for wavenumbers $k = 0.001, 0.01$, and 0.1 Mpc^{-1} – make a separate plot for each k, and split the plot in before and after z_d and/or use a logarithmic scale if necessary.

Lastly, plot a snapshot of $\Theta_0 + \Phi$ at z_d, as a function of wavenumber.

Cosmology HW6 Connor Hainje

Problem 1:

part 1. $dU = -P\,dV + \delta Q$, $dU = \rho\, dV + V\, d\rho \rightarrow \delta Q = V\,d\rho + (\rho + P)\,dV$

$\frac{\delta Q}{dt} = V\dot{\rho} + (\rho+P)\dot{V}$. $V \sim a^3$ so $\frac{dV}{V} = 3\frac{da}{a} \Rightarrow \dot{V} = 3HV$

$\Rightarrow \frac{\delta Q}{dt} = [\dot{\rho} + 3H(\rho+P)]\,V \Rightarrow \dot{q} = \frac{\partial}{\partial V}\frac{\delta Q}{dt} = \dot{\rho} + 3H(\rho+P)$.

$\rho = mn + \frac{3}{2}nT$, $P = nT$. $\dot{\rho} = \left(m + \frac{3}{2}T\right)\dot{n} + \frac{3}{2}n\dot{T}$. assuming $a^3 n$ is constant, $\dot{n} = -3Hn$.

thus $\dot{\rho} = -3H\rho + \frac{3}{2}n\dot{T}$. $\Rightarrow \dot{q} = \frac{3}{2}n\dot{T} + 3HnT$

rearranging, $\dot{T} + 2HT = \frac{2}{3}\frac{\dot{q}}{n}$ $\square$

part 2. in e^- rest frame: initially, e^- has $\vec{v}_i = 0$, γ has $\vec{p}_i$

finally, e^- has $\vec{v}_f$, γ has $\vec{p}_f$

conservation of momentum: $\vec{p}_f + m_e\vec{v}_f = \vec{p}_i \Rightarrow \vec{v}_f = \frac{\vec{p}_i - \vec{p}_f}{m_e}$

assuming $|\vec{p}_i| = |\vec{p}_f|$, let $\vec{p}_i = p\,\hat{p}_i$, $\vec{p}_f = p\,\hat{p}_f$. then $\vec{v}_f = \frac{p}{m_e}(\hat{p}_i - \hat{p}_f)$

final electron energy is $E_f = \frac{1}{2}m_e\vec{v}_f^{\,2} = \frac{1}{2}m_e\frac{p^2}{m_e^2}(\hat{p}_i - \hat{p}_f)^2 = \frac{p^2}{m_e}(1-\cos\theta)$

since initial e^- energy is 0 and photon energy $= p$ (up to c^2), then $\Delta E_e = \frac{E_\gamma^2}{m_e}(1-\cos\theta)$ $\square$

part 3. a scatter with photon of energy E_γ at angle θ imparts ΔE_e on the electron

the scattering rate is $dN_s/dt\,dV\,dE_\gamma\,d\cos\theta$. the heating rate is $\dot{q} = \frac{dQ}{dV\,dt} = \frac{dN_s\,\Delta E_e}{dt\,dV}$

$$\longrightarrow \dot{q} = \int dE_\gamma\, d\cos\theta\, d\varphi\, \frac{dN_s}{dt\,dV\,dE_\gamma\,d\cos\theta}\,\Delta E$$

$$= \int_0^\infty dE_\gamma \int_{-1}^{+1} d\cos\theta \int_0^{2\pi} d\varphi\; n_e \frac{d\sigma_T}{d\cos\theta}\frac{dn_\gamma}{dE_\gamma}\left(1 + f_\gamma(E_\gamma)\right)\frac{E_\gamma^2}{m_e}(1-\cos\theta)$$

components: $d\sigma_T/d\cos\theta = \frac{3}{16\pi}\sigma_T\,(1+\cos^2\theta)$, $f_\gamma(E_\gamma) = \left[\exp E_\gamma/T_\gamma - 1\right]^{-1}$

$$n_\gamma = \frac{2}{h^3}\int d^3p\, f_\gamma(E(p)) = \frac{2}{h^3}\int dp\, 4\pi p^2 f_\gamma(E(p)) = \frac{8\pi}{h^3}\int dE\, E^2 f(E) \Rightarrow \frac{dn}{dE} = \frac{8\pi}{h^3}E^2 f(E)$$

so
$$\dot{q} = \int_0^\infty dE_\gamma \int_{-1}^{+1} dc \int_0^{2\pi} d\varphi\; n_e \frac{3\sigma_T}{16\pi}(1+c^2)\frac{8\pi}{h^3}E_\gamma^2 f_\gamma (1+f_\gamma)\frac{E_\gamma^2}{m_e}(1-c)$$

$$= \frac{n_e}{m_e}\,2\pi \int_{-1}^{+1} dc\, \frac{3\sigma_T}{16\pi}(1+c^2)(1-c) \int_0^\infty dE\, \frac{8\pi}{h^3}E^4 f(1+f)$$

$$= \frac{n_e\sigma_T}{m_e}\frac{8\pi}{h^3}T^5 \underbrace{\int_0^\infty du\, u^4 e^u (e^u - 1)^{-2}}_{4\pi^4/15}$$

$$= \frac{n_e\sigma_T}{m_e}\frac{32\pi^5}{15h^3}T_\gamma^5$$

$$= \frac{4n_e\sigma_T}{m_e}\rho_\gamma T_\gamma \quad \text{since } \rho_\gamma = \frac{8\pi^5}{15h^3}T_\gamma^4$$ $\square$

part 4. $n_b = n_e + \underbrace{n_{H^0} + n_{H^+}}_{\equiv n_H} + \underbrace{n_{He^0} + n_{He^+} + n_{He^{++}}}_{\equiv n_{He}}$. recall $x_e \equiv \frac{n_e}{n_H}$ and $f_{He} \equiv \frac{n_{He}}{n_H}$.

thus $n_b = x_e n_H + n_H + f_{He} n_H = n_H (1 + x_e + f_{He})$ $\square$

part 5. see code on GitHub. I found $z_T \approx 105$.

this means for $z > z_T$, Thomson scattering remains effective enough to keep the baryon temperature equal to the photon temperature so $T_b \sim T_\gamma \sim a^{-1}$.

afterward, $\dot{T}_b \approx -2HT_b$ so the baryon temperature evolves independent of the photon temperature with $T_b \sim a^{-2}$.

part 6. continuity equations: $\delta_\gamma' = -\frac{4}{3}\nabla\cdot\vec{V}_\gamma$, $\quad \delta_b' = -\nabla\cdot\vec{V}_b$

momentum eqns:

$$\vec{V}_\gamma' = -\frac{1}{4}\nabla\delta_\gamma + a n_e \sigma_T (\vec{V}_b - \vec{V}_\gamma)$$

$$\vec{V}_b' = -\mathcal{H}\vec{V}_b - \frac{\nabla P_b}{\bar{\rho}_b} - \frac{4}{3}\frac{\bar{\rho}_\gamma}{\bar{\rho}_b} a n_e \sigma_T (\vec{V}_b - \vec{V}_\gamma)$$

in Fourier space:

$$\delta_\gamma' = -\frac{4}{3}(i\vec{k})\cdot(i\hat{k}V_\gamma) = +\frac{4}{3}kV_\gamma$$

$$\delta_b' = +kV_b$$

$$V_\gamma' = -\frac{1}{4}k\delta_\gamma + a n_e\sigma_T (V_b - V_\gamma)$$

$$V_b' = -\mathcal{H}V_b - \frac{kP_b}{\bar{\rho}_b} - \frac{4}{3}\frac{\bar{\rho}_\gamma}{\bar{\rho}_b} a n_e \sigma_T (V_b - V_\gamma)$$

since $\Theta_0 \sim \delta$ and $\Theta_1 \sim V$, "neglecting baryon temperature fluctuations" means $\delta_b, V_b \to 0$? using this to solve the photon equations, then

then: $\delta_\gamma' = \frac{4}{3}kV_\gamma$, $\quad V_\gamma' = -\frac{1}{4}k\delta_\gamma - a n_e \sigma_T V_\gamma$

taking $a n_e \sigma_T \propto a^{-2}$, $\frac{d}{d\eta}(a n_e\sigma_T) = -2\mathcal{H}\, a n_e \sigma_T$ so

$$V_\gamma'' + a n_e\sigma_T V_\gamma' + \left(\frac{k^2}{3} - 2\mathcal{H}\, a n_e\sigma_T\right)V_\gamma = 0.$$

if $\mathcal{H}$, $a n_e \sigma_T$ are slow compared to V_γ, then

$$V_\gamma = \left(\exp{-\frac{a n_e\sigma_T}{2}\eta}\right)\left[c_1 \exp\sqrt{(a n_e\sigma_T)^2 + 8\mathcal{H}\, a n_e\sigma_T - \frac{4}{3}k^2}\,\frac{\eta}{2} + c_2 \exp -\sqrt{(a n_e\sigma_T)^2 + 8\mathcal{H}\, a n_e\sigma_T - \frac{4}{3}k^2}\,\frac{\eta}{2}\right]$$

$$\sim \exp\left(2\mathcal{H} - \frac{k^2/3}{a n_e\sigma_T}\right)\eta$$

then $0 = -\mathcal{H}V_b - \frac{\nabla P_b}{\bar{\rho}_b} + \frac{4}{3}\frac{\bar{\rho}_\gamma}{\bar{\rho}_b} a n_e\sigma_T V_\gamma$

$$\rightarrow \frac{\nabla P_b/\bar{\rho}_b}{\mathcal{H}V_b} \approx \frac{-\mathcal{H}V_b + \frac{4}{3}\bar{\rho}_\gamma/\bar{\rho}_b\, a n_e\sigma_T V_\gamma}{\mathcal{H}V_b}$$

$$= \frac{4}{3}\frac{\bar{\rho}_\gamma}{\bar{\rho}_b} a n_e\sigma_T - 1$$

Problem 2:

(1) $\delta_\gamma' - \frac{4}{3} k V_\gamma = 4\phi'$

(2) $V_\gamma' = -\frac{1}{4} k \delta_\gamma + k \Pi_\gamma - k\Psi$

(3) $\Pi_\gamma' = -\frac{4}{15} k V_\gamma - \frac{9}{10} a n_e \sigma_T \Pi_\gamma$

part 1. taking the quasistationary approximation for Π_γ $(\Pi_\gamma' \to 0)$

we get $\Pi_\gamma \approx \left(-\frac{4}{15} k V_\gamma\right)\left(\frac{10}{9} \frac{1}{a n_e \sigma_T}\right) = -\frac{8}{27} \frac{k}{a n_e \sigma_T} V_\gamma$

plug into eq 2: $V_\gamma' = -\frac{1}{4} k \delta_\gamma + k \Pi_\gamma - k\Psi$

$= -\frac{1}{4} k \delta_\gamma - \frac{8}{27} \frac{k^2}{a n_e \sigma_T} V_\gamma - k\Psi$

eq 1: $\frac{4}{3} k V_\gamma = \delta_\gamma' - 4\phi' \Rightarrow k V_\gamma = \frac{3}{4}\delta_\gamma' - 3\phi'$

$\Rightarrow V_\gamma' = -\frac{1}{4} k \delta_\gamma - \frac{8}{27} \frac{k}{a n_e \sigma_T}\left(\frac{3}{4}\delta_\gamma' - 3\phi'\right) - k\Psi$

$\frac{d}{d\eta}(\text{eq } 1)$: $\delta_\gamma'' - \frac{4}{3} k V_\gamma' = 4\phi''$

$\Rightarrow \delta_\gamma'' + \frac{4}{3} k \left[\frac{1}{4} k \delta_\gamma + \frac{8}{27} \frac{k}{a n_e \sigma_T}\left(\frac{3}{4}\delta_\gamma' - 3\phi'\right) + k\Psi\right] = 4\phi''$

$$\delta_\gamma'' + \frac{8}{27} \frac{k^2}{a n_e \sigma_T} \delta_\gamma' + \frac{1}{3} k^2 \delta_\gamma = 4\phi'' + \frac{32}{27} \frac{k^2}{a n_e \sigma_T} \phi' - \frac{4}{3} k^2 \Psi$$

part 2. seek WKB soln $\delta_\gamma = A(\eta)\, e^{\pm i k\eta/\sqrt{3}} \equiv A e^{\pm i\varphi}$, assuming $\frac{A'}{A} \ll k$

$\delta = A e^{\pm i\varphi}$ $\quad\quad\quad \varphi' = k/\sqrt{3}$, $\varphi'' = 0$.

$\delta' = \pm i\varphi' A e^{\pm i\varphi} + A' e^{\pm i\varphi}$

$\delta'' = \pm i \cancelto{0}{\varphi''} A e^{\pm i\varphi} \pm 2i\varphi' A' e^{\pm i\varphi} + (\pm i\varphi')^2 A e^{\pm i\varphi} + A'' e^{\pm i\varphi}$

plugging in: $0 = A'' \pm 2i\varphi' A' - \varphi'^2 A$

$+ \frac{8}{27} \frac{k^2}{a n_e \sigma_T} A' \pm \frac{8}{27} \frac{k^2}{a n_e \sigma_T} i\varphi' A + \frac{1}{3} k^2 A$

$= A'' \pm 2i\varphi' A' + \frac{8}{27} \frac{k^2}{a n_e \sigma_T}(A' \pm i\varphi' A) + \cancelto{0}{\left(\frac{k^2}{3} - \varphi'^2\right)} A$

$= A'' \pm \frac{2ik}{\sqrt{3}} A' + \frac{8}{27} \frac{k^2}{a n_e \sigma_T}\left(A' \pm \frac{ik}{\sqrt{3}} A\right)$

$\sim \varkappa^2 A \quad \varkappa k A \quad \frac{k^2}{a n_e \sigma_T}(\varkappa + k) A$

to leading order, $0 = 2A' + \frac{8}{27} \frac{k^2}{a n_e \sigma_T} A$

$\to \frac{A'}{A} = -\frac{4}{27} \frac{k^2}{a n_e \sigma_T}$

$\log A = -\frac{4k^2}{27} \int \frac{d\eta}{a n_e \sigma_T}$

$$\to A(\eta) \propto \exp\left[-\frac{4}{27} k^2 \int^\eta \frac{d\eta'}{a n_e \sigma_T(\eta')}\right]$$ ✓

Problem 3:

here I'll collect all the equations:

neutrinos decoupled at all (relevant) times,

relativistic at all times (so ~massless $\rightarrow$ $P = \rho/3$)

and no anisotropic stress ($\rightarrow \Pi = 0,\ c_s^2 = 1/3$)

then:
$$\delta_\nu' = \frac{4}{3} k V_\nu + 4\phi'$$
$$V_\nu' = -\frac{1}{4} k \delta_\nu - k\phi$$

cdm pressureless perfect fluid, decoupled at all times
$$\delta_c' = k V_c + 3\phi'$$
$$V_c' = -\mathcal{H} V_c - k\phi$$

photons & baryons for $z > z_d = 10^3$, tight coupling
$$\delta_\gamma' = \frac{4}{3} k V_\gamma + 4\phi'$$
$$V_\gamma' = -\frac{k}{4(1+R)}\delta_\gamma - k\phi - \frac{R}{1+R}\mathcal{H} V_\gamma \qquad R \equiv \frac{3}{4}\frac{\bar\rho_b}{\bar\rho_\gamma}.$$
$$\delta_b = \frac{3}{4}\delta_\gamma, \quad V_b = V_\gamma.$$

for $z < z_d$, total decoupling

photons assume $\Pi_\gamma = 0,\ c_s^2 = 1/3$
$$\delta_\gamma' = \frac{4}{3} k V_\gamma + 4\phi'$$
$$V_\gamma' = -\frac{1}{4} k \delta_\gamma - k\phi$$

baryons assume $P_b = 0$. note $\Pi_b = 0$ always.
$$\delta_b' = k V_b + 3\phi'$$
$$V_b' = -\mathcal{H} V_b - k\phi$$

potential evolves under $\phi'' + 3(1+w)\mathcal{H}\phi' + wk^2\phi = 4\pi G a^2(\delta P - w\,\delta\rho) \qquad w \equiv \bar P/\bar\rho$.

want to split into two first-order ODEs for dimensionless variables.

let $\varphi_1 = \phi,\ \varphi_2 = d\phi/da$. then
$$\phi' = \frac{d\phi}{d\eta} = \frac{da}{d\eta}\frac{d\phi}{da} = \mathcal{H} a \varphi_2.$$
$$\phi'' = \frac{d}{d\eta}(\mathcal{H} a \varphi_2) = \mathcal{H}' a \varphi_2 + \mathcal{H}^2 a \varphi_2 + \mathcal{H} a \varphi_2' = \frac{1-3w}{2}\mathcal{H}^2 a \varphi_2 + \mathcal{H} a \varphi_2'$$

then our system of first-order ODEs is

$$\varphi_1' = \mathcal{H} a \varphi_2$$

$$\varphi_2' = -\frac{7+3w}{2} \mathcal{H} \varphi_2 - w \frac{k^2}{\mathcal{H} a} \varphi_1 + 4\pi G \frac{a}{\mathcal{H}} (\delta P - w \delta\rho)$$

it's more convenient to consider log a the integration variable

note $\displaystyle \frac{d}{d\eta} = \frac{da}{d\eta} \frac{d}{da}$

$$= \frac{da}{d\eta} \frac{d\log a}{da} \frac{d}{d\log a}$$

$$= \mathcal{H} a \frac{1}{a} \frac{d}{d\log a}$$

$$= \mathcal{H} \frac{d}{d\log a}$$