Cosmology Homework 4

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using $\hbar = 1$. code at github.com/cmhainje/cosmo-hw

Problem 1:

equilibrium number density: $$n_s^0 = \begin{cases} g_s \left(\frac{m_s T}{2\pi}\right)^{3/2} e^{-m_s/T} & (T \ll m_s) \\ g_s \frac{T^3}{\pi^2} & (T \gg m_s) \end{cases}$$

$$f_{He} = n_{He}/n_H. \qquad n_{He} \equiv n_{He^0} + n_{He^+} + n_{He^{++}}, \qquad n_H = n_{H^0} + n_p = n_p \qquad x_e \equiv \frac{n_e}{n_H}.$$

$He^0 + \gamma \rightarrow He^+ + e^-$:

$$\tilde{m}_e = \frac{m_{He^+}}{m_{He^0}} m_e$$

Saha equation: $$\frac{n_{He^+} n_e}{n_{He^0}} = \frac{n^0_{He^+} n^0_e}{n^0_{He^0}} = \frac{g_{He^+} g_e}{g_{He^0}} \left(\frac{\tilde{m}_e T}{2\pi}\right)^{3/2} e^{-(m_{He^+} + m_e - m_{He^0})/T}$$
$$= 4 \left(\frac{\tilde{m}_e T}{2\pi}\right)^{3/2} e^{-E_1/T} \qquad m_{He^+} + m_e - m_{He^0} = E_1$$

assume $n_{He^{++}} = 0 \rightarrow n_{He} = n_{He^0} + n_{He^+}$.

enforce charge neutrality: $n_e = n_p + n_{He^+} = n_H + n_{He^+}$

$$\rightarrow n_{He^+} = n_e - n_H = x_e n_H - n_p = n_H(x_e - 1)$$

$$f_{He} = \frac{n_{He^0} + n_{He^+}}{n_H} \rightarrow n_{He^0} = n_H f_{He} - n_{He^+} = n_H(f_{He} + 1 - x_e)$$

$$\frac{n_{He^+} n_e}{n_{He^0}} = \frac{n_H(x_e - 1)\, n_H x_e}{n_H(f_{He} + 1 - x_e)} = \frac{x_e(x_e - 1)}{f_{He} + 1 - x_e} n_H.$$

so $$\frac{(x_e - 1)x_e}{1 + f_{He} - x_e} = \frac{4}{n_H}\left(\frac{\tilde{m}_e T}{2\pi}\right)^{3/2} e^{-E_1/T}$$

$He^+ + \gamma \rightarrow He^{++} + e^-$

Saha equation $$\frac{n_{He^{++}} n_e}{n_{He^+}} = \left(\frac{\tilde{\tilde{m}}_e T}{2\pi}\right)^{3/2} e^{-E_2/T} \qquad \tilde{\tilde{m}}_e = \frac{m_{He^{++}}}{m_{He^+}} m_e.$$

assume $n_{He^0} = 0 \rightarrow n_{He} = n_{He^+} + n_{He^{++}}$.

enforce charge neutrality $\rightarrow n_e = n_p + n_{He^+} + 2n_{He^{++}}$

$$f_{He} = \frac{n_{He^+} + n_{He^{++}}}{n_H} \Rightarrow n_{He^+} + n_{He^{++}} = f_{He} n_H.$$

$$\rightarrow n_e = n_H + f_{He} n_H + n_{He^{++}} \rightarrow n_{He^{++}} = n_H(x_e - 1 - f_{He})$$

$$n_{He^+} = f_{He} n_H - n_{He^{++}} = n_H(1 + 2f_{He} - x_e)$$

$$\frac{n_{He^{++}} n_e}{n_{He^+}} = \frac{n_H(x_e - 1 - f_{He})\, n_H x_e}{n_H(1 + 2f_{He} - x_e)} = \frac{(x_e - 1 - f_{He})x_e}{1 + 2f_{He} - x_e} n_H.$$

so $$\frac{(x_e - 1 - f_{He})x_e}{1 + 2f_{He} - x_e} = \frac{1}{n_H}\left(\frac{\tilde{\tilde{m}}_e T}{2\pi}\right)^{3/2} e^{-E_2/T}$$

part 2: see notebook

Problem 2:

part 1. $\vec{F} = m_e \vec{a} = \frac{e^2}{r^2}\hat{r} \longrightarrow \vec{a} = \frac{e^2}{m_e r^2}\hat{r}$

$\vec{D} = e\vec{r} \Rightarrow \ddot{\vec{D}} = e\vec{a} = \frac{e^3}{m_e r^2}\hat{r}$

$\frac{dE_{rad}}{dt} = \frac{2}{3}\ddot{\vec{D}}^2 = \frac{2}{3}\frac{e^6}{m_e^2 r^4}$

part 2. (a) assuming straight line trajectory : $r^2 = x^2 + b^2$. $v = \dot{x}$

$$E_{rad} = \int_{-\infty}^{+\infty} dt\, \frac{2}{3}\frac{e^6}{m_e^2 r^4} = \frac{2}{3}\frac{e^6}{m_e^2}\int_{-\infty}^{+\infty} dt\, r^{-4}$$

$$\approx \frac{2}{3}\frac{e^6}{m_e^2}\, 2\int_0^{\infty}\frac{dx}{v(x^2+b^2)^2}$$

v approximately constant

$$\approx \frac{4e^6}{3m_e^2 v}\int_0^{\infty}\frac{dx}{(x^2+b^2)^2}$$

let $u = x/b$. $dx = b\,du$. $(x^2+b^2)^2 = (u^2+1)^2 b^4$.

$$= \frac{4e^6}{3m_e^2 v b^3}\int_0^{\infty}\frac{du}{(1+u^2)^2}$$

$$= \frac{\pi}{3}\frac{e^6}{m_e^2 v b^3}$$

(b) radiative capture occurs if $E_{rad} = \frac{\pi}{3}\frac{e^6}{m_e^2 v b^3} > KE_{init} = \frac{1}{2}m_e v^2$.

critical b : $\frac{\pi}{3}\frac{e^6}{m_e^2 v b^3} = \frac{1}{2}m_e v^2 \rightarrow b^3 = \frac{2\pi}{3}\frac{e^6}{m_e^3 v^3} \longrightarrow b_{max,cl}(v) = \left(\frac{2\pi}{3}\right)^{1/3}\frac{e^2}{m_e v}$.

for $b < b_{max,cl}$, $KE_{init} = \frac{1}{2}m_e v^2$, $PE_{closest} = \frac{e^2}{b} > \left(\frac{3}{2\pi}\right)^{1/3} m_e v$

because the electron is nonrelativistic, $v^2 < v$. also, $\frac{1}{2} < \left(\frac{3}{2\pi}\right)^{1/3} \approx 0.78$

thus $KE_{init} < PE_{closest}$, which definitely violates the straight-line assumption

part 3. (a) $r^2\dot{\theta} = bv$ $\qquad r(\theta) = \frac{2r_0}{1+\cos(\theta-\pi)} = \frac{2r_0}{1-\cos\theta}$

$$E_{rad} = \frac{2}{3}\frac{e^6}{m_e^2}\int_{-\infty}^{+\infty}\frac{dt}{r^4} = \frac{2}{3}\frac{e^6}{m_e^2}\int_0^{2\pi}\frac{d\theta}{r^4\dot{\theta}} = \frac{2}{3}\frac{e^6}{m_e^2 bv}\int_0^{2\pi}\frac{d\theta}{r^2}$$

$$= \frac{1}{6}\frac{e^6}{m_e^2 bv r_0^2}\int_0^{2\pi} d\theta\,[1-\cos\theta]^2$$

$$= \frac{1}{6}\frac{e^6}{m_e^2 bv r_0^2}\int_0^{2\pi} d\theta\,[1-\cancelto{0}{2\cos\theta}+\cos^2\theta]$$

$$= \frac{1}{6}\frac{e^6}{m_e^2 bv r_0^2}[2\pi+\pi]$$

$$= \frac{\pi}{2}\frac{e^6}{m_e^2 bv r_0^2}$$

now $\frac{1}{2}m_e v_0^2 = \frac{e^2}{r_0}$ and $r_0 v_0 = bv$

$$v_0^2 = \frac{2e^2}{m_e r_0} \rightarrow r_0^2\frac{2e^2}{m_e r_0} = (bv)^2 \longrightarrow r_0 = \frac{m_e(bv)^2}{2e^2}$$

then $E_{rad} = \frac{\pi}{2}\frac{e^6}{m_e^2(bv)}\frac{4e^4}{m_e^2(bv)^4} = 2\pi\frac{e^{10}}{m_e^4(bv)^5}$ ✓

(b) $E_{rad} = 2\pi \dfrac{e^{10}}{m_e^4 (bv)^5} = KE_{init} = \dfrac{1}{2} m_e v^2$

$$4\pi \frac{e^{10}}{m_e^5 v^7} = b^5 \quad \rightarrow \quad b_{max,cl}(v) = (4\pi)^{1/5} \frac{e^2}{m v^{7/5}} .$$

for $b < b_{max,cl}$, $KE_{init} = \frac{1}{2} m_e v^2$, $PE \simeq \frac{e^2}{b} > (4\pi)^{-1/5} m_e v^{7/5}$.

$$\frac{1}{2} < (4\pi)^{-1/5} \approx 0.60 \quad \text{and} \quad v < 1 \Rightarrow v^2 < v^{7/5}$$

hence $\frac{KE}{PE} \lesssim v^{3/5} \ll 1$ for $v \ll 1$ so the parabolic assumption holds.

part 4. (a) $\dfrac{d\vec{L}_{rad}}{dt} = \dfrac{2}{3} \dot{\vec{D}} \times \ddot{\vec{D}}$

$$\ddot{\vec{D}} = \frac{e^3}{m_e r^2} \hat{r} = \frac{e^3}{m_e r^3} \vec{r}, \quad \dot{\vec{D}} = e \dot{\vec{r}}$$

so $\dot{\vec{D}} \times \ddot{\vec{D}} = \dfrac{e^4}{m_e r^3} \dot{\vec{r}} \times \vec{r} = \dfrac{e^4}{m_e^2 r^3} \vec{L}$ (negative?)

hence $\dfrac{d\vec{L}_{rad}}{dt} = \dfrac{2}{3} \dfrac{e^4}{m_e^2 r^3} \vec{L}$

(b) $L_{rad} = \displaystyle\int_{-\infty}^{+\infty} dt \, \frac{2}{3} \frac{e^4}{m_e^2 r^3} \vec{L} = \frac{2}{3} \frac{e^4 bv}{m_e} \int_0^{2\pi} \frac{d\theta}{\dot{\theta} r^3} = \frac{2}{3} \frac{e^4}{m_e} \int_0^{2\pi} \frac{d\theta}{r}$

$$= \frac{1}{3} \frac{e^4}{m_e r_0} \int_0^{2\pi} d\theta \, (1 - \cancel{\cos}\,\theta) = \frac{2\pi}{3} \frac{e^4}{m_e r_0} = \frac{4\pi}{3} \frac{e^6}{m_e^2 (bv)^2}$$

(c) require $L_{rad} > \hbar$. critical point: $\dfrac{4\pi}{3} \dfrac{e^6}{m_e^2 (bv)^2} = \hbar = 1$

$$\rightarrow b^2 = \frac{4\pi}{3} \frac{e^6}{m_e^2 v^2} \quad \rightarrow \quad b_{max} = \left(\frac{4\pi}{3}\right)^{1/2} \frac{e^3}{m_e v} \approx (4.9 \times 10^{-16} \, m) \, v^{-1} .$$

compare to classical $b_{max,cl}(v) = (4\pi)^{1/5} \dfrac{e^2}{m v^{7/5}} \approx (4.7 \times 10^{-15} \, m) \, v^{-7/5}$.

hence $\dfrac{b_{max}}{b_{max,cl}} \approx 0.1 \, v^{2/5} \ll 1$ for $v \ll 1$.

(d) $\sigma(v) = \pi \, b_{max}^2(v)$

$$\mathcal{A}_{cl}(T) = \langle \sigma v \rangle = \int d^3v \, f_{MB}(v) \, \sigma(v) \, v$$

$$= \int_0^\infty dv \, 4\pi v^2 \left(\frac{m_e}{2\pi T}\right)^{3/2} \exp\left(-\frac{m_e v^2}{2T}\right) \pi \, \frac{4\pi}{3} \frac{e^6}{m_e^2 v^2} \, v$$

$$= 4\pi \left(\frac{m_e}{2\pi T}\right)^{3/2} \pi \, \frac{4\pi}{3} \frac{e^6}{m_e^2} \int_0^\infty dv \, \exp\left(-\frac{m_e v^2}{2T}\right) v$$

$$= \frac{2(2\pi)^{3/2}}{3} \frac{e^6}{m_e^{1/2} T^{3/2}} \frac{T}{m_e}$$

$$= \frac{2(2\pi)^{3/2}}{3} \frac{e^6}{m_e^{3/2} T^{1/2}}$$

redimensionalizing, $= \dfrac{2(2\pi)^{3/2}}{3} \dfrac{e^6}{(m_e c^2)^{3/2} T^{1/2} \hbar} = (1.4 \times 10^{-13} \, cm^3/s) \left(\dfrac{T}{10^4 \, K}\right)^{-1/2}$.

plot in notebook

Problem 3: in notebook