# Graduate Cosmology Spring 2025
# Homework 7

due by 11:59pm on Thursday 4/24, 2025.

**Problem 1: Null geodesics in conformal Newtonian gauge [4 points]**

For this problem we consider pure scalar modes, in the conformal Newtonian gauge:

$$ds^2 = a^2(\eta)[-(1+2\Psi)d\eta^2 + (1-2\Phi)d\vec{x}^2].$$

For all calculations below, you can use the Christoffel symbols for this metric given in Eqs. (6.64)-(6.69) of Baumann.

1) [2 points] Show that, along null geodesics, the zero-th component of a photon's 4-momentum satisfies the following equation, at linear order in perturbations

$$\frac{dP^0}{d\eta}\Big|_{\text{traj}} = \left(-2\mathcal{H} + \Phi' + \Psi' - 2\hat{p}^i\partial_i\Psi\right)P^0, \qquad \hat{p}^i \equiv \frac{P^i}{\sqrt{\delta_{jk}P^jP^k}}.$$

2) [2 points] Compute the components of the 4-velocity of comoving observers at linear order in perturbations, and show that the quantity $p \equiv aE_{\text{obs}}$, where $E_{\text{obs}}$ is the photon energy measured by comoving observers, evolves as follows along geodesics:

$$\frac{dp}{d\eta}\Big|_{\text{traj}} = (\Phi' - \hat{p}^i\partial_i\Psi)p.$$

This was the last ingredient needed for the photon Boltzmann equation, which we had not derived in class.

**Problem 2: Approximate $C_\ell$'s from your code [5 points]**

For this problem you may use either your own cosmological perturbation code, or the one I posted on github – so, not having written or finished your own code before should not prevent you from doing this problem!

1) [3 points] When including baryon inertia and polarization, the Silk damping scale at decoupling is $k_D \approx 7.2$ Mpc. Multiply the output of your perturbation code at $\eta_*$ (for $F_0 \equiv \Theta_0 + \Psi$ and $V_b$) by the diffusion damping factor $e^{-k^2/k_D^2}$, and compute the Sachs Wolfe and Doppler contributions to the CMB temperature angular power spectrum, as approximated in the lecture notes. Your plots should show $D_\ell \equiv T_0^2\ell^2 C_\ell/(2\pi)$ in units of $\mu\text{K}^2$, where $T_0 \approx 2.73$ K is the CMB temperature today, as a function of $\ell$, for $20 \le \ell \le 2000$, and should look like the left figure of Fig. 7.7. of Baumann, i.e. show the SW and Doppler contributions individually, and summed.

2) [2 points] Assuming a nearly scale invariant primordial power spectrum with amplitude $A_s = 2 \times 10^{-9}$ and scalar spectral index $n_s = 0.96$ at pivot scale $k_P = 0.05$ Mpc$^{-1}$, estimate the rms temperature anisotropy $\langle\Theta^2\rangle^{1/2}$ from the above result (this should be a dimensionless number, which is the *fractional* temperature fluctuation).

**Problem 3: Gravitational lensing of the CMB [7 points]**

I will spare you the derivation, but the spatial part of the null geodesic equation in the conformal Newtonian gauge implies the following evolution of photon propagation direction along geodesics:

$$\frac{d\hat{p}^i}{d\eta}\Big|_{\text{traj}} = (\delta^{ij} - \hat{p}^i\hat{p}^j)\partial_j(\Phi + \Psi) \approx 2(\delta^{ij} - \hat{p}^i\hat{p}^j)\partial_j\Phi,$$

where the second equality holds when anisotropic stresses are negligible. This implies that, for an unperturbed direction of propagation $\hat{p}$, the total deflection angle of a photon between last scattering and today is obtained by the following integral along the line of sight:

$$\Delta\hat{p}^i(\eta_0) = 2(\delta^{ij} - \hat{p}^i\hat{p}^j)\int_{\eta_*}^{\eta_0} d\eta\ (\partial_j\Phi)(\eta, \vec{x} = -\chi\hat{p}), \qquad \chi \equiv \eta_0 - \eta.$$

1) [3 points] For a fixed $\hat{p}$, express the variance of the deflection angle $\langle|\Delta\hat{p}|^2\rangle$, as an integral involving the primordial curvature power spectrum $P_{\mathcal{R}}(k)$ and the transfer function of the gravitational potential $\mathcal{T}_\Phi(\eta, k)$. Approximating the Universe as matter dominated between $\eta_*$ and $\eta_0$ (*hint*: how does $\Phi$ evolve as a function of $\eta$ then?), re-express your integral in terms of $\Delta^2_{\mathcal{R}}(k), \mathcal{T}_\Phi(\eta_*, k)$ and the quantity

$$\mathcal{I}(\vec{k}\cdot\hat{p}) \equiv \int_0^{\chi_*} d\chi \int_0^{\chi_*} d\chi' \; e^{i\vec{k}\cdot\hat{p}(\chi'-\chi)}, \qquad \chi_* \equiv \eta_0 - \eta_* \approx \eta_0.$$

2) [1 point] Change variables to $u \equiv \chi' - \chi$ and $v \equiv \chi' + \chi$ in the integal above, and show that, in the limit of modes well inside the horizon today ($k \gg 1/\eta_0$), we may approximate

$$\mathcal{I}(\vec{k}\cdot\hat{p}) \approx \pi\eta_0\delta_{\rm D}(\vec{k}\cdot\hat{p}).$$

If we align the coordinate system such that the $z$ axis is along $\hat{p}$, this means that we may substitute

$$\int d^3k \; \mathcal{I}(\vec{k}\cdot\hat{p}) f(k_x, k_y, k_z) \approx \pi\eta_0 \int dk_y dk_z \; f(k_x, k_y, k_z = 0).$$

3) [1 point] Combining the two previous questions show that the variance of the deflection angle is

$$\langle|\Delta\hat{p}|^2\rangle \approx 2\pi \int d\ln k \; \eta_0 k \; \Delta^2_{\mathcal{R}}(k) \; |\mathcal{T}_\Phi(\eta_*, k)|^2.$$

4) [2 points] Argue that the integrand decays quickly (say how quickly) for $k \gtrsim k_{\rm eq}$, and approximate $\mathcal{T}_\Phi(\eta_*, k)$ for $k \lesssim k_{\rm eq}$ by a simple number, making your reasoning very clear (*hint*: what is the relation between $\Phi_i$ and $\mathcal{R}_i$?). Assuming a nearly scale invariant primordial power spectrum with amplitude $A_s = 2 \times 10^{-9}$ and scalar spectral index $n_s = 0.96$ at pivot scale $k_P = 0.05$ Mpc$^{-1}$, estimate the rms deflection angle due to lensing in arcmin. *Hint*: your answer should be of order unity in these units.

**Problem 4: local-type primordial non-Gaussianity [9 points]**

The goal of this exercise is to have you practice Isserlis' (or Wick's) theorem for Gaussian random fields.

The standard initial conditions are assumed to be a perfect random Gaussian field. However, non-standard inflationay physics could imply a small amount of primordial non-Gaussianity. Here we consider two examples of a "local-type" non-Gaussianity, for which the primordial curvature perturbation $\mathcal{R}$ can be expressed as a non-linear function of a statistically isotropic and homogeneous Gaussian random field $G$ (with zero mean) at the same position.

- First, we consider quadratic local-type non-Gaussianity:

$$\mathcal{R}(\vec{x}) = G(\vec{x}) + f\left(G^2(\vec{x}) - \langle G^2\rangle\right), \qquad \text{where } f \text{ is a constant.}$$

1) [1 point] Rewrite the equation above in Fourier space. Don't forget Dirac delta functions where relevant.

2) [3 points] Compute the *bispectrum* or *3-point function* of $\mathcal{R}$ at linear order in $f$; show that it takes the form

$$\langle\mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_2)\mathcal{R}(\vec{k}_3)\rangle = (2\pi)^2\delta_{\rm D}(\vec{k}_1+\vec{k}_2+\vec{k}_3)B(k_1, k_2, k_3),$$

and express $B$ explicitly in terms of the power spectrum $P_G(k)$ of the Gaussian random field $G$. *Hint*: for a real field, $G^*(\vec{k}) = G(-\vec{k})$; express your result as $B(k_1, k_2, k_3) = ... + \text{permutations}(k's)$

- Next, consider cubic local-type non-Gaussianity:

$$\mathcal{R}(\vec{x}) = G(\vec{x}) + g\; G^3(\vec{x}), \qquad \text{where } g \text{ is a constant.}$$

3) [1 point] Rewrite the equation above in Fourier space.

4) [4 points] The *trispectrum* or *connected 4-point function* of $\mathcal{R}$ is defined as

$$\begin{aligned}\langle\mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_2)\mathcal{R}(\vec{k}_3)\mathcal{R}(\vec{k}_4)\rangle_c \equiv\; &\langle\mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_2)\mathcal{R}(\vec{k}_3)\mathcal{R}(\vec{k}_4)\rangle \\ &- \langle\mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_2)\rangle\;\langle\mathcal{R}(\vec{k}_3)\mathcal{R}(\vec{k}_4)\rangle \\ &- \langle\mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_3)\rangle\;\langle\mathcal{R}(\vec{k}_2)\mathcal{R}(\vec{k}_4)\rangle \\ &- \langle\mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_4)\rangle\;\langle\mathcal{R}(\vec{k}_2)\mathcal{R}(\vec{k}_3)\rangle.\end{aligned}$$

In other words, it is the 4-point function from which we have subtracted the value it would have if $\mathcal{R}$ were Gaussian. Compute the trispectrum of $\mathcal{R}$ at linear order in $g$; show that it takes the form

$$\langle \mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_2)\mathcal{R}(\vec{k}_3)\mathcal{R}(\vec{k}_4)\rangle_c = (2\pi)^3\delta_{\rm D}(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)T(k_1,k_2,k_3,k_4),$$

and express $T$ in terms of $P_G(k)$. Again, express your result in the form $T(k_1,k_2,k_3,k_4) = ... +$ permutations$(k's)$.