Cosmology Homework 3 Connor Hainje

Problem 1:

part 1.

$$C[\mathcal{N}_A](\vec{p}) = \int d^3\vec{p}'\, d^3\vec{q}\, d^3\vec{q}'\ \delta^4(p^\mu + p'^\mu - q^\mu - q'^\mu)\ \Gamma(p^\mu, p'^\mu, q^\mu, q'^\mu)$$
$$\left[ - f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right) + f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right)\right]$$

$$C[\mathcal{N}_B](\vec{q}) = \int d^3\vec{q}'\, d^3\vec{p}\, d^3\vec{p}'\ \delta^4(p^\mu + p'^\mu - q^\mu - q'^\mu)\ \Gamma(p^\mu, p'^\mu, q^\mu, q'^\mu)$$
$$\left[ - f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right) + f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right)\right]$$

$d_t(a^3 n) = a^3 \dot{n}\big|_{coll}$ in a static universe $\Rightarrow$ $d_t n = \dot{n}\big|_{coll} = \int d^3p\ C[\mathcal{N}]$

$$d_t(n_A + n_B) = \int d^3p\ C[\mathcal{N}_A](\vec{p}) + \int d^3q\ C[\mathcal{N}_B](\vec{q})$$
$$= \int d^3p\, d^3p'\, d^3q\, d^3q'\ \delta^4(\Sigma p_i^\mu)\ \Gamma(p_i^\mu)$$
$$\Big[ \cancel{- f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right)} + \cancel{f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right)}$$
$$\cancel{- f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right)} + \cancel{f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right)}\Big]$$
$$= 0 \quad \checkmark$$

part 2.

$\dot{a} = 0 \quad \Rightarrow \quad d_t\rho = \dot{\rho}\big|_{coll} = \int d^3p\ C[\mathcal{N}](\vec{p})\ E(\vec{p})$

$$d_t(\rho_A + \rho_B) = \int d^3p\ C[\mathcal{N}_A](\vec{p})\ E(\vec{p}) + \int d^3q\ C[\mathcal{N}_B](\vec{q})\ E(\vec{q})$$
$$= \int d^3p\, d^3p'\, d^3q\, d^3q'\ \delta^4(\Sigma p_i^\mu)\ \Gamma(p_i^\mu) \longleftarrow \text{for brevity, we'll write this } \int_{p_i}\Gamma$$
$$\Big\{ \left[ - f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right) + f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right)\right] E(\vec{p})$$
$$+ \left[ - f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right) + f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right)\right] E(\vec{q}) \Big\}$$

$$\int d^3p\, d^3p'\ S(\vec{p},\vec{p}')\ E(\vec{p}) = \int d^3p\, d^3p'\ S(\vec{p},\vec{p}')\ \frac{E(\vec{p}) + E(\vec{p}')}{2} \quad \text{if } S(p,p') = S(p',p) \quad \text{so}$$

$$d_t(\rho_A + \rho_B) = \int_{p_i}\Gamma \Big\{ \left[ - f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right) + f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right)\right] \frac{E(\vec{p}) + E(\vec{p}')}{2}$$
$$+ \left[ - f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right) + f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right)\right] \frac{E(\vec{q}) + E(\vec{q}')}{2} \Big\}$$

by conservation of energy (e.g. the 0-component of the $\delta^4$), $E(\vec{p}) + E(\vec{p}') = E(\vec{q}) + E(\vec{q}')$. hence,

$$d_t(\rho_A + \rho_B) = \int_{p_i}\Gamma\ \frac{E(\vec{p}) + E(\vec{p}')}{2} \Big[ \cancel{- f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right)} + \cancel{f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right)}$$
$$\cancel{- f_B(\vec{q}) f_B(\vec{q}')\left(1-f_A(\vec{p})\right)\left(1-f_A(\vec{p}')\right)} + \cancel{f_A(\vec{p}) f_A(\vec{p}')\left(1+f_B(\vec{q})\right)\left(1+f_B(\vec{q}')\right)}\Big]$$
$$= 0.$$

part 3.

$$C[\mathcal{N}_A] = \frac{d}{dt}\mathcal{N}_A = \frac{d}{dt}\left(\frac{g_A}{h^3} f_A\right) = \frac{g_A}{h^3}\frac{d}{dt} f_A$$

$$\frac{d}{dt}H_A = \frac{g_A}{h^3}\int d^3p \;\frac{d}{dt}\Big[ f_A(\vec{p}) \ln f_A(\vec{p}) + \big(1-f_A(\vec{p})\big)\ln\big(1-f_A(\vec{p})\big)\Big]$$

$$\begin{aligned}\frac{d}{dt}\big[ f\ln f + (1-f)\ln(1-f)\big] &= \dot{f}\ln f + f\,\frac{1}{f}\,\dot{f} + (-\dot{f})\ln(1-f) + (1-f)\,\frac{1}{1-f}\,(-\dot{f})\\ &= \dot{f}\big(1+\ln f\big) - \dot{f}\big[1+\ln(1-f)\big]\\ &= \dot{f}\big[1+\ln f - 1 - \ln(1-f)\big]\\ &= \dot{f}\big[\ln f - \ln(1-f)\big]\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}H_A &= \frac{g_A}{h^3}\int d^3p \;\dot{f}_A(\vec{p})\,\big[\ln f_A(\vec{p}) - \ln\big(1-f_A(\vec{p})\big)\big]\\ &= \int d^3p\; C[\mathcal{N}_A](\vec{p})\,\big[\ln f_A(\vec{p}) - \ln\big(1-f_A(\vec{p})\big)\big]\\ &= \int_{p_i}\Gamma\,\big[\ln f_A(\vec{p}) - \ln\big(1-f_A(\vec{p})\big)\big]\Big[- f_A(\vec{p})\, f_A(\vec{p}\,')\big(1+f_B(\vec{q})\big)\big(1+f_B(\vec{q}\,')\big) + f_B(\vec{q})\, f_B(\vec{q}\,')\big(1-f_A(\vec{p})\big)\big(1-f_A(\vec{p}\,')\big)\Big]\\ &= \int_{p_i}\Gamma\;\frac{\ln f_A(\vec{p}) - \ln(1-f_A(\vec{p})) + \ln f_A(\vec{p}\,') - \ln(1-f_A(\vec{p}\,'))}{2}\,\big[\cdots\big]\end{aligned}$$

part 4.

$$\frac{d}{dt}H_B = \frac{g_B}{h^3}\int d^3q \;\frac{d}{dt}\Big[ f_B(\vec{q})\ln f_B(\vec{q}) - \big(1+f_B(\vec{q})\big)\ln\big(1+f_B(\vec{q})\big)\Big]$$

$$\frac{d}{dt}\big[ f\ln f - (1+f)\ln(1+f)\big] = \dot{f}\big(1+\ln f\big) - \dot{f}\big[1+\ln(1+f)\big] = \dot{f}\big[\ln f - \ln(1+f)\big]$$

$$\begin{aligned}\frac{d}{dt}H_B &= \frac{g_B}{h^3}\int d^3q\;\dot{f}_B(\vec{q})\,\big[\ln f_B(\vec{q}) - \ln\big(1+f_B(\vec{q})\big)\big]\\ &= \int d^3q\; C[\mathcal{N}_B](\vec{q})\,\big[\ln f_B(\vec{q}) - \ln\big(1+f_B(\vec{q})\big)\big]\\ &= \int_{p_i}\Gamma\,\big[\ln f_B(\vec{q}) - \ln\big(1+f_B(\vec{q})\big)\big]\\ &\qquad\times\Big[- f_B(\vec{q})\, f_B(\vec{q}\,')\big(1-f_A(\vec{p})\big)\big(1-f_A(\vec{p}\,')\big) + f_A(\vec{p})\, f_A(\vec{p}\,')\big(1+f_B(\vec{q})\big)\big(1+f_B(\vec{q}\,')\big)\Big]\\ &= \int_{p_i}\Gamma\;\frac{\ln f_B(\vec{q}) - \ln(1+f_B(\vec{q})) + \ln f_B(\vec{q}\,') - \ln(1+f_B(\vec{q}\,'))}{2}\\ &\qquad\times\Big[- f_B(\vec{q})\, f_B(\vec{q}\,')\big(1-f_A(\vec{p})\big)\big(1-f_A(\vec{p}\,')\big) + f_A(\vec{p})\, f_A(\vec{p}\,')\big(1+f_B(\vec{q})\big)\big(1+f_B(\vec{q}\,')\big)\Big]\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}H &= \frac{d}{dt}H_A + \frac{d}{dt}H_B\\ &= \int_{p_i}\Gamma\,\Big[- f_A(\vec{p})\, f_A(\vec{p}\,')\big(1+f_B(\vec{q})\big)\big(1+f_B(\vec{q}\,')\big) + f_B(\vec{q})\, f_B(\vec{q}\,')\big(1-f_A(\vec{p})\big)\big(1-f_A(\vec{p}\,')\big)\Big]\\ &\qquad \frac{1}{2}\Big[\ln f_A(\vec{p}) + \ln f_A(\vec{p}\,') - \ln\big(1-f_A(\vec{p})\big) - \ln\big(1-f_A(\vec{p}\,')\big) - \ln f_B(\vec{q}) - \ln f_B(\vec{q}\,') + \ln\big(1+f_B(\vec{q})\big) + \ln\big(1+f_B(\vec{q}\,')\big)\Big]\end{aligned}$$

$$\ln f_A(\vec{p})\, f_A(\vec{p}\,')\big(1+f_B(\vec{q})\big)\big(1+f_B(\vec{q}\,')\big) - \ln f_B(\vec{q})\, f_B(\vec{q}\,')\big(1-f_A(\vec{p})\big)\big(1-f_A(\vec{p}\,')\big)$$

$$= \int_{p_i}\frac{1}{2}\Gamma\,\big(X-Y\big)\big(\ln Y - \ln X\big) \quad \text{for}\quad \begin{aligned} X &= f_B(\vec{q})\, f_B(\vec{q}\,')\big(1-f_A(\vec{p})\big)\big(1-f_A(\vec{p}\,')\big)\\ Y &= f_A(\vec{p})\, f_A(\vec{p}\,')\big(1+f_B(\vec{q})\big)\big(1+f_B(\vec{q}\,')\big)\end{aligned}$$

part 5.

show $(X-Y)(\ln Y - \ln X) \leq 0$ for $X, Y > 0$.

if $X=Y$, it's trivially zero. further, it's symmetric wrt $X \leftrightarrow Y$ so WLOG assume $X > Y$.

ln is monotonically increasing wrt its argument, so $X > Y \Rightarrow \ln X > \ln Y$.

hence $X > Y \Rightarrow (X-Y) > 0, \; (\ln Y - \ln X) < 0$

$\Rightarrow (X-Y)(\ln Y - \ln X) < 0$

so generally $(X-Y)(\ln Y - \ln X) \leq 0$ with equality only if $X=Y$.

part 6.

obj function: $H + \alpha(n_A + n_B) + \beta(\rho_A + \rho_B)$.

$$n = \int d^3p \, \mathcal{N} = \frac{g}{h^3}\int d^3p \, f, \quad \rho = \int d^3p \, \mathcal{N} \, E(p) = \frac{g}{h^3}\int d^3p \, f \, E(p)$$

$$H_A + H_B = \frac{g_A}{h^3}\int d^3p\left[f_A(\vec{p})\ln f_A(\vec{p}) + (1-f_A(\vec{p}))\ln(1-f_A(\vec{p}))\right] + \frac{g_B}{h^3}\int d^3q\left[f_B(\vec{q})\ln f_B(\vec{q}) - (1+f_B(\vec{q}))\ln(1+f_B(\vec{q}))\right]$$

$$n_A + n_B = \frac{g_A}{h^3}\int d^3p \, f_A(\vec{p}) + \frac{g_B}{h^3}\int d^3q \, f_B(\vec{q}) \qquad \rho_A + \rho_B = \frac{g_A}{h^3}\int d^3p \, f_A(\vec{p}) \, E(\vec{p}) + \frac{g_B}{h^3}\int d^3q \, f_B(\vec{q}) \, E(\vec{q})$$

$$\text{obj} = \frac{g_A}{h^3}\int d^3p\left[f_A(\vec{p})\left(\ln f_A(\vec{p}) + \alpha + \beta E(\vec{p})\right) + (1-f_A(\vec{p}))\ln(1-f_A(\vec{p}))\right]$$
$$+ \frac{g_B}{h^3}\int d^3q\left[f_B(\vec{q})\left(\ln f_B(\vec{q}) + \alpha + \beta E(\vec{q})\right) - (1+f_B(\vec{q}))\ln(1+f_B(\vec{q}))\right]$$

so, we need to solve $\frac{\delta}{\delta f_A}\text{obj} = 0 \qquad \frac{\delta}{\delta f_B}\text{obj} = 0$

$$\delta\text{obj}_A = \frac{g_A}{h^3}\int d^3p\left[\log f_A(\vec{p}) - \log(1-f_A(\vec{p})) + \alpha + \beta E(\vec{p})\right]\delta f_A.$$

setting to zero $\Rightarrow \log f_A(\vec{p}) - \log(1-f_A(\vec{p})) + \alpha + \beta E(\vec{p}) = 0.$

$$\log\frac{f}{1-f} = -\alpha - \beta E \longrightarrow f_A(\vec{p}) = \left(e^{\alpha+\beta E(\vec{p})} + 1\right)^{-1}.$$

$$\delta\text{obj}_B = \frac{g_B}{h^3}\int d^3q\left[\log f_B(\vec{q}) - \log(1+f_B(\vec{q})) + \alpha + \beta E(\vec{q})\right]\delta f_B$$

setting to zero $\Rightarrow \log f_B(\vec{q}) - \log(1+f_B(\vec{q})) + \alpha + \beta E(\vec{q}) = 0.$

$$\log\frac{f}{1+f} = -\alpha - \beta E \longrightarrow f_B(\vec{q}) = \left(e^{\alpha+\beta E(\vec{q})} - 1\right)^{-1}.$$

these are the Fermi-Dirac (A) and Bose-Einstein (B) distributions

if we identify $\alpha = -\mu/T$ and $\beta = 1/T$.

Problem 2:

part 1. before decoupling, $\nu$ have $n(T) = \frac{g}{h^3} 4\pi \frac{3}{2} \zeta(3) T^3 = \frac{12\pi}{h^3}\zeta(3) T^3 \overset{\text{using } \hbar=1,\ h=2\pi.}{=} \frac{3\zeta(3)}{2\pi^2} T^3$

after, $n = \frac{3\zeta(3)}{2\pi^2} T_\nu^3$ so today, $n = \frac{3\zeta(3)}{2\pi^2} T_{\nu 0}^3 = 113$ neutrinos/cm$^3$

this applies for each species individually, making a total of 338 neutrinos/cm$^3$

part 2. suppose neutrino mass is $m_\nu$. want to find $z(T_\nu = m_\nu)$.

$T_\nu \propto a^{-1}$ so $a\,T_\nu(a) = (1)\,T_\nu(1)$. hence $a_{nr} = \frac{T_{\nu 0}}{T_{nr}} = \frac{T_{\nu 0}}{m_\nu}$

$$1 + z_{nr} = \frac{1}{a_{nr}} = \frac{m_\nu}{T_{\nu 0}} = \frac{m_\nu}{1.68\times 10^{-4}\,\text{eV}} = (595)\left(\frac{m_\nu}{0.1\ \text{eV}}\right)$$

part 3. minimum mass to be nonrelativistic today: $m_\nu \sim T_{\nu 0} = 0.168$ meV.

or, to be safe, $m_\nu \sim T_{\gamma,0} = 0.235$ meV.

assume $m_\nu > T_{\nu 0}$ st $\nu$ are nonrelativistic

$$\omega_\nu = \Omega_\nu h^2 = \frac{h^2}{\rho_{crit}} \rho_{\nu,0} = \frac{h^2}{\rho_{crit}} \Sigma_i m_i n_i$$

$$= \frac{h^2}{\rho_{crit}} \left(\frac{3\zeta(3)}{2\pi^2} T_{\nu 0}^3\right) \Sigma_i m_i = \frac{\Sigma_i m_i}{93.4\ \text{eV}} = 0.0107 \frac{\Sigma_i m_i}{1\ \text{eV}}$$

Planck $\Omega_{DM,0} = 0.26069 \Rightarrow \omega_{DM} = 0.1193$.

$\omega_\nu < \omega_{DM} \Rightarrow \Sigma_i m_i < (0.1193)(93.4\ \text{eV}) = 11.1$ eV.

part 4. in order for $\nu$ to be relativistic at $z_{eq}$, $m_\nu < T_{eq}$

from Baumann Table 3.1, $z_{eq} = 3400$, $T_{eq} = 0.80$ eV

so $m_\nu < 0.80$ eV

part 5. (a) spatially flat: $ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + \chi^2 d\Omega^2\right)$

$\nu$ traveling on a straight line $\Rightarrow d\Omega = 0$.

detected with physical momentum $p_0$, energy $E = \sqrt{p_0^2 + m^2}$.

saw in lecture that $p^2(t) = a^2(t) P_i P^i$, $E = P^0 = \sqrt{p^2(t) + m^2}$

$P^1 = m \frac{d\chi}{d\tau} = \frac{p(t)}{a(t)}$. $P^0 = m\frac{dt}{d\tau} = E = \sqrt{p^2(t) + m^2}$

hence $\frac{d\chi}{dt} = \frac{1}{a(t)} \frac{p(t)}{E(t)}$. we know $p \propto a^{-1}$ so $a(t)\,p(t) = 1\,p_0 \Rightarrow p(t) = \frac{p_0}{a}$.

then $\frac{d\chi}{dt} = \frac{1}{a}\frac{(p_0/a)}{\sqrt{(p_0/a)^2 + m^2}}$. thus $\chi_{dec} = \int_{t_{dec}}^{t_0} \frac{1}{a}\frac{(p_0/a)}{\sqrt{(p_0/a)^2 + m^2}}\,dt$ $\qquad dt = \frac{dt}{da}da = \frac{da}{\dot a}$

$$= \int_{a_{dec}}^{1} \frac{(p_0/a)}{\sqrt{(p_0/a)^2 + m^2}} \frac{da}{H_0\sqrt{(\Omega_r a^{-2} + \Omega_m a^{-1} + \Omega_\Lambda a^2)}}$$

(b) $a_{nr}$ is such that $p(a) = \frac{p_0}{a_{nr}} \sim m$ and $\Omega_m$ dominates

letting $x = \frac{a}{p_0/m}$

taking matter domination, the integrand becomes $\frac{1}{H_0\sqrt{\Omega_m}} \frac{p_0/a}{\sqrt{(p_0/a)^2 + m^2}\ \sqrt{a^{-1}}} = \frac{\sqrt{p_0/m}}{H_0\sqrt{\Omega_m}} \frac{1}{\sqrt{x^{-1} + x^{+1}}}$

this function has a peak at $x=1$ i.e. $a \sim a_{nr}$ and rapidly falls off on either side

we can write the integral as $\chi_{dec} = \frac{\sqrt{p_0/m}}{H_0\sqrt{\Omega_m}} \left[ \int_{\frac{a_{dec}}{p_0/m}}^{1} \frac{dx}{\sqrt{x^{-1} + x^{+1}}} + \int_{1}^{m/p_0} \frac{dx}{\sqrt{x^{-1} + x^{+1}}} \right]$

I don't know how to do this integral expansion

(c) a "typical" neutrino has $p_{dec} \sim T_{\nu,dec}$

this would give $p_0 \sim a_{dec} T_{\nu,dec} = \frac{1}{6 \times 10^9} (1\ \text{MeV}) = 0.167$ meV

and hence $\chi(m) = \frac{\sqrt{p_0/m}}{H_0\sqrt{\Omega_m}} \int_{\frac{a_{dec}}{p_0/m}}^{m/p_0} \frac{dx}{\sqrt{x^{-1} + x^{+1}}}$

$$= (7.96\ \text{Gpc}) \sqrt{\frac{0.167\ \text{meV}}{m}} \int_{\frac{1}{6\times10^9}\frac{m}{0.167\,\text{meV}}}^{m/0.167\,\text{meV}} \frac{dx}{\sqrt{x^{-1} + x^{+1}}}$$

$$\approx (7.96\ \text{Gpc}) \left(\frac{m}{0.251\ \text{meV}}\right)$$

$$= (31.7\ \text{Gpc}) \left(\frac{m}{1\ \text{meV}}\right)$$

$$= 3170\ \text{Gpc}\ \frac{m}{0.1\ \text{eV}} \quad ?$$

I ran out of time...

Problem 3:

part 1. if $\mu_b = 0$ and protons remained in equilibrium with photons, protons would have an FD distribution and the number density would go like

$$n_p = \frac{8\pi}{h^3} T^3 I_+(m_p/T) \approx \frac{2}{h^3} (2\pi m_p T)^{3/2} e^{-m_p/T}$$

today $T = 2.73\,K$, so $n_p = (1.71 \times 10^{27}\ m^{-3}) \exp(-3.99 \times 10^{12})$

$V = \frac{4\pi}{3} H_o^{-3} = 1.07 \times 10^{79}\ m^3$

so $\#_p = n_p V = (1.83 \times 10^{106}) \exp(-3.99 \times 10^{12})$

$= \exp(244 - 3.99 \times 10^{12})$

$= \exp(-3.99 \times 10^{12})$

$\simeq 0.$ and certainly $\ll 1.$

part 2. $\omega_b = 0.022$

$\Omega_b = (0.022)(0.6766)^{-2} = 0.0481$

$\rho_{b,0} = (0.0481)\, \rho_{crit} = 4.14 \times 10^{-31}\ g/cm^3$

$\rho_p = 0.76\, \rho_{b,0} = 3.14 \times 10^{-31}\ g/cm^3$

$n_p = \rho_p / m_p = 0.188\ m^{-3}$

if we choose to include the protons in He nuclei,

$\rho_{He} = 0.24\, \rho_{b,0} = 9.93 \times 10^{-32}\ g/cm^3$

$n_{He} = \rho_{He} / m_{He} \leftarrow 6.646 \times 10^{-24}\ g \quad = 0.0149\ m^{-3}$

two protons per He: $n_p + 2 n_{He} = 0.218\ m^{-3}$