Cosmology HW8 Connor Hainje

Problem 1:

$$V(\phi) = M_{Pl}^4 \frac{\lambda_n}{(2n)!}\left(\frac{\phi}{M_{Pl}}\right)^{2n} \qquad M_{Pl}^2 = \frac{1}{8\pi G}$$

part 1. the potential slow-roll parameters are

$$\varepsilon_V = \frac{1}{2} M_{Pl}^2 \left(\frac{V_{,\phi}}{V}\right)^2 \qquad \eta_V = M_{Pl}^2 \frac{V_{,\phi\phi}}{V}$$

for our potential,

$$V = M_{Pl}^4 \frac{\lambda_n}{(2n)!}\left(\frac{\phi}{M_{Pl}}\right)^{2n}$$
$$V_{,\phi} = M_{Pl}^3 \frac{\lambda_n}{(2n-1)!}\left(\frac{\phi}{M_{Pl}}\right)^{2n-1}$$
$$V_{,\phi\phi} = M_{Pl}^2 \frac{\lambda_n}{(2n-2)!}\left(\frac{\phi}{M_{Pl}}\right)^{2n-2}$$

so

$$\varepsilon_V = \frac{1}{2} M_{Pl}^2 \left[\frac{M_{Pl}^{4-2n} \frac{\lambda_n}{(2n-1)!} \phi^{2n-1}}{M_{Pl}^{4-2n} \frac{\lambda_n}{(2n)!} \phi^{2n}}\right]^2 = \frac{1}{2} M_{Pl}^2 \left(\frac{(2n)!}{(2n-1)!}\phi^{-1}\right)^2 = 2n^2 M_{Pl}^2 \phi^{-2}.$$

$$\eta_V = M_{Pl}^2 \frac{M_{Pl}^{4-2n} \frac{\lambda_n}{(2n-2)!} \phi^{2n-2}}{M_{Pl}^{4-2n} \frac{\lambda_n}{(2n)!} \phi^{2n}} = M_{Pl}^2 (2n)(2n-1)\phi^{-2}.$$

part 2. KG eqn: $\ddot\phi + 3H\dot\phi + V_{,\phi} = 0.$

in the slow roll approximation, we take $3H\dot\phi + V_{,\phi} \simeq 0.$

let $N = \ln a/a_i$. $\frac{d}{dt} = \frac{da}{dt}\frac{dN}{da}\frac{d}{dN} = (aH)\left(\frac{1}{a}\right)\frac{d}{dN} = H\frac{d}{dN}.$

$$0 = V_{,\phi} + 3H\dot\phi = V_{,\phi} + 3H^2 \phi_{,N}.$$

$$H^2 = \frac{1}{3M_{Pl}^2}\left[\frac{1}{2}\dot\phi^2 + V\right] \approx \frac{1}{3M_{Pl}^2} V.$$

then
$$0 = M_{Pl}^{4-2n}\frac{\lambda_n}{(2n-1)!}\phi^{2n-1} + \frac{1}{M_{Pl}^2} M_{Pl}^{4-2n}\frac{\lambda_n}{(2n)!}\phi^{2n}\phi_{,N}$$
$$= \phi_{,N}\phi + 2n M_{Pl}^2$$

$$-2n M_{Pl}^2 = \phi\frac{d\phi}{dN} = \frac{d}{dN}\left(\frac{1}{2}\phi^2\right)$$

$$\rightarrow -2n M_{Pl}^2 (N-0) = \frac{1}{2}\left(\phi^2 - \phi_i^2\right) \rightarrow \phi = \sqrt{\phi_i^2 - 4n M_{Pl}^2 N}$$

part 3. $\varepsilon_V = 2n^2 M_{Pl}^2 \phi^{-2} = 1 \implies \phi = \sqrt{2}\, n M_{Pl}$ at end of inflation

so $2n^2 M_{Pl}^2 = \phi_i^2 - 4n M_{Pl}^2 N \rightarrow \phi_i = \sqrt{2M_{Pl}^2\, n(n+2N)}$

for $N = 60$: $\phi_i = \sqrt{2M_{Pl}^2\, n(n+120)}$

part 4. (a) $H^2 = \frac{1}{3M_{Pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V\right]$

$$\dot{\phi} = H\phi_{,N} \rightarrow 3M_{Pl}^2 H^2 = \frac{1}{2}H^2\phi_{,N}^2 + V$$

$$\rightarrow H^2\left(3M_{Pl}^2 - \frac{1}{2}\phi_{,N}^2\right) = V$$

$$\rightarrow \boxed{H^2 = \frac{V}{3M_{Pl}^2 - \phi_{,N}^2/2}}$$

part 4. (b) KG : $\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0.$

$$\frac{d}{dt} = H\frac{d}{dN}$$

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(H\frac{d}{dN}\right) = \dot{H}\frac{d}{dN} + H\frac{d}{dt}\frac{d}{dN} = \dot{H}\frac{d}{dN} + H^2\frac{d^2}{dN^2}$$

$$\rightarrow \ddot{\phi} = \dot{H}\phi_{,N} + H^2\phi_{,NN}$$

$$\rightarrow 0 = H^2\phi_{,NN} + \dot{H}\phi_{,N} + 3H^2\phi_{,N} + V_{,\phi}$$

$$0 = \phi_{,NN} + \left(\frac{\dot{H}}{H^2} + 3\right)\phi_{,N} + \frac{1}{H^2}V_{,\phi}$$

$$H^2 = \frac{V}{3M_{Pl}^2 - \phi_{,N}^2/2} \qquad \dot{H} = -\frac{1}{2M_{Pl}^2}\dot{\phi}^2 = -\frac{1}{2M_{Pl}^2}H^2\phi_{,N}^2$$

$$\rightarrow \boxed{0 = \phi_{,NN} + \left(3 - \frac{1}{2M_{Pl}^2}\phi_{,N}^2\right)\phi_{,N} + \frac{V_{,\phi}}{V}\left(3M_{Pl}^2 - \frac{1}{2}\phi_{,N}^2\right)}$$

part 4. (c) $\boxed{\varepsilon = -\frac{\dot{H}}{H^2} = \frac{1}{2M_{Pl}^2}\phi_{,N}^2}$

part 5. code

$$\phi_{,N} = \frac{d}{dN}\left(\phi_i^2 - 4n M_{Pl}^2 N\right)^{1/2}$$

$$= \frac{1}{2}\left(\phi_i^2 - 4nM_{Pl}^2 N\right)^{-1/2}\left(-4nM_{Pl}^2\right)$$

$$= -2n M_{Pl}^2/\phi(N)$$

part 6. code

Problem 2:

part 1. $\phi(N) = \sqrt{\phi_i^2 - 4M_{Pl}^2\, n\, N}$ $\qquad$ N: # folds elapsed

$N = N_{end} - N_{before}$

$\rightarrow \phi(N_{before}) = \sqrt{\phi_i^2 - 4M_{Pl}^2\, n\,(N_{end} - N_{before})}$

$N_* = N_{end} - 50$ i.e. $N_{before} = 50$

$\rightarrow \phi_* = \sqrt{\phi_i^2 - 4M_{Pl}^2\, n\,(N_{end} - 50)}$ $\qquad$ $k_* = aH\big|_{N=N_{end}-50}$

$$\epsilon_{V*} = 2n^2\left(\phi_*/M_{Pl}\right)^{-2} \qquad \eta_{V*} = 2n(2n-1)\left(\phi_*/M_{Pl}\right)^{-2}$$

$$n_s - 1 = 2\eta_{V*} - 6\epsilon_{V*}$$
$$= \left(-20n^2 + 12n\right)\left(\phi_*/M_{Pl}\right)^{-2}$$

$$H_*^2 = \frac{1}{3M_{Pl}^2}\left[\frac{1}{2}\dot\phi^2 + V\right]_* \approx \frac{1}{3M_{Pl}^2}V_* = \frac{1}{3}M_{Pl}^{2-2n}\frac{\lambda_n}{(2n)!}\phi_*^{2n}$$

$$A_s = \frac{1}{8\pi^2\epsilon_{V*}}\frac{H_*^2}{M_{Pl}^2} = \frac{1}{8\pi^2 M_{Pl}^4\, 2n^2\phi_*^{-2}}\,\frac{1}{3}M_{Pl}^{2-2n}\frac{\lambda_n}{(2n)!}\phi_*^{2n}$$
$$= \frac{\lambda_n}{48\pi^2 n^2 (2n)!}\left(\frac{\phi_*}{M_{Pl}}\right)^{2n+2}$$

$$r = A_t / A_s \approx 16\,\epsilon_{V*}$$
$$= 32n^2\left(\phi_*/M_{Pl}\right)^{-2}$$

$$n_t \approx -2\,\epsilon_{V*}$$
$$= -4n^2\left(\phi_*/M_{Pl}\right)^{2}$$

part 2. $n_s = 0.9652 \pm 0.0042 \quad \Rightarrow \quad \left(\phi_*/M_{Pl}\right)^2 = \dfrac{12n - 20n^2}{1 - 0.9652}$

$r < 0.035 \quad \Rightarrow \quad \left(\phi_*/M_{Pl}\right)^2 > \dfrac{32n^2}{0.035}$

$$-20n^2 + 12n > \frac{1-0.9652}{0.035}\,32n^2 = 31.82\,n^2$$

$$n > 4.32\,n^2$$

$$\Rightarrow 0 < n < 0.232$$

there is no integer n which satisfies this inequality $\rightarrow$ ruled out!

Problem 3:

part 1. only the diagonal components are nonzero when $h_{ij} = 0$

with nonzero h_{ij}, I expect perturbations to G_{ij}, particularly when $i \neq j$.

part 2. I expect Π_{ij} to source tensor modes since it's the only tensor

also since h^{TT}_{ij} and Π_{ij} are both gauge invariant

part 3. $h''_{ij} + 2\mathcal{H} h'_{ij} + k^2 h_{ij} = 0.$

$$\frac{d}{d\eta} = \frac{da}{d\eta}\frac{d}{da} = a\mathcal{H}\frac{d}{da}$$

$$\frac{d^2}{d\eta^2} = \frac{d}{d\eta}\left(a\mathcal{H}\frac{d}{da}\right) = a'\mathcal{H}\frac{d}{da} + a\mathcal{H}'\frac{d}{da} + a\mathcal{H}\frac{d}{d\eta}\frac{d}{da}$$

$$= a\mathcal{H}^2\frac{d}{da} + a\mathcal{H}'\frac{d}{da} + (a\mathcal{H})^2\frac{d^2}{da^2}$$

<u>rad dom</u> $\mathcal{H} = aH = a H_0\sqrt{\Omega_r a^{-4}} = \dfrac{H_0\sqrt{\Omega_r}}{a}$

$$\mathcal{H}' = \frac{d\mathcal{H}}{d\eta} = H_0\sqrt{\Omega_r}\frac{d}{d\eta}\frac{1}{a} = -H_0\sqrt{\Omega_r}\, a^{-2}\frac{da}{d\eta} = -H_0\sqrt{\Omega_r}\,\frac{1}{a}\mathcal{H} = -\mathcal{H}^2$$

$$\Rightarrow \frac{d^2}{d\eta^2} = (a\mathcal{H})^2\frac{d^2}{da^2}$$

$$\longrightarrow a^2\mathcal{H}^2 h_{ij,aa} + 2a\mathcal{H}^2 h_{ij,a} + k^2 h_{ij} = 0.$$

$$H_0^2\Omega_r h_{ij,aa} + 2H_0^2\Omega_r\frac{h_{ij,a}}{a} + k^2 h_{ij} = 0.$$

$$\rightarrow h_{ij,aa} + 2\frac{1}{a}h_{ij,a} + \frac{k^2}{H_0^2\Omega_r}h_{ij} = 0.$$

$$\rightarrow h_{ij} = C_1\frac{1}{a}e^{-ia\,k/H_0\sqrt{\Omega_r}} - i\,C_2\frac{1}{2a\,k/H_0\sqrt{\Omega_r}}e^{+ia\,k/H_0\sqrt{\Omega_r}}$$

a_{eq}: $\quad \Omega_m a_{eq}^{-3} = \Omega_r a_{eq}^{-4} \rightarrow a_{eq} = \Omega_r/\Omega_m$

H_{eq}: $\quad H_{eq} = H_0\sqrt{\Omega_m a_{eq}^{-3} + \Omega_r a_{eq}^{-4}}$

$$= H_0\sqrt{\Omega_m^4\Omega_r^{-3} + \Omega_m^4\Omega_r^{-3}}$$

$$= H_0\,\Omega_m^2\,\Omega_r^{-3/2}\sqrt{2}$$

$$k_{eq} = H_0\,\Omega_m^2\,\Omega_r^{-3/2}\sqrt{2}\;\Omega_r\,\Omega_m^{-1}$$

$$= H_0\,\Omega_m\,\Omega_r^{-1/2}\sqrt{2}$$

$$H_0\sqrt{\Omega_r} = \frac{k_{eq}\Omega_r}{\sqrt{2}\,\Omega_m} = k_{eq}a_{eq}/\sqrt{2}$$

$$H_0\sqrt{\Omega_m} = k_{eq}\sqrt{a_{eq}/2}$$

$$h_{ij} = C_1\frac{a_{eq}}{a}\cos\left(\sqrt{2}\frac{a\,k}{a_{eq}k_{eq}}\right) + C_2\frac{a_{eq}k_{eq}}{a\,k}\sin\left(\sqrt{2}\frac{a\,k}{a_{eq}k_{eq}}\right)$$

limits:

$k \ll k_{eq}$: $h_{ij} \to C_1 \frac{a_{eq}}{a} 1 + C_2 \frac{a_{eq} k_{eq}}{a k} \sqrt{2} \frac{a k}{a_{eq} k_{eq}} + O(k^2)$

$= C_1 \frac{a_{eq}}{a} + C_2 \sqrt{2}$

as a function of a, $\boxed{h_{ij} \sim \text{constant.}}$

$k \gg k_{eq}$: $h_{ij} \to C_1 \frac{a_{eq}}{a} \cos\left(\sqrt{2} \frac{a k}{a_{eq} k_{eq}}\right) + C_2 \cancel{\frac{a_{eq} k_{eq}}{a k}}^{\text{small}} \sin\left(\sqrt{2} \frac{a k}{a_{eq} k_{eq}}\right)$

$= C_1 \frac{a_{eq}}{a} \cos\left(\sqrt{2} \frac{ak}{a_{eq} k_{eq}}\right)$

as a function of a, $\boxed{h_{ij} \sim \frac{1}{a} \cos \omega a}$ for large ω

matter dom. $\mathcal{H} = aH = a H_0 \sqrt{\Omega_m a^{-3}} = H_0 \sqrt{\Omega_m}\, a^{-1/2}$.

$\mathcal{H}' = \frac{d}{d\eta}\left(H_0 \sqrt{\Omega_m}\, a^{-1/2}\right) = -H_0 \sqrt{\Omega_m} \frac{1}{2} a^{-3/2} a' = -\frac{1}{2} \mathcal{H}^2$

$\to \frac{d^2}{d\eta^2} = a \mathcal{H}^2 \frac{d}{da} + a \mathcal{H}' \frac{d}{da} + (a\mathcal{H})^2 \frac{d^2}{da^2}$

$= \frac{1}{2} a \mathcal{H}^2 \frac{d}{da} + (a\mathcal{H})^2 \frac{d^2}{da^2}$

$a \mathcal{H}^2 = H_0^2 \Omega_m$

$h_{ij}'' + 2\mathcal{H} h_{ij}' + k^2 h_{ij} = 0.$

$0 = (a\mathcal{H})^2 h_{ij,aa} + \frac{1}{2} a \mathcal{H}^2 h_{ij,a} + 2 a \mathcal{H}^2 h_{ij,a} + k^2 h_{ij}$

$= (a\mathcal{H})^2 h_{ij,aa} + \frac{5}{2} a \mathcal{H}^2 h_{ij,a} + k^2 h_{ij}$

$= H_0^2 \Omega_m a\, h_{ij,aa} + \frac{5}{2} H_0^2 \Omega_m h_{ij,a} + k^2 h_{ij}$

$= h_{ij,aa} + \frac{5}{2} \frac{1}{a} h_{ij,a} + \frac{k^2}{H_0^2 \Omega_m a} h_{ij}$

$$\boxed{h_{ij}(a) = \left[C_1 \frac{2c}{a} + C_2 \frac{1}{8(ac^2)^{3/2}} \right] \cos\left(2\sqrt{ac^2}\right) + \left[-C_1 \frac{1}{a^{3/2}} + C_2 \frac{1}{4ac^2} \right] \sin\left(2\sqrt{ac^2}\right)}$$

$c \equiv \frac{k}{H_0 \sqrt{\Omega_m}} = \frac{k}{k_{eq}} \sqrt{\frac{2}{a_{eq}}}$

$k \ll k_{eq}$ $(c \ll 1)$: $h_{ij} \to \left[C_1 \cancel{\frac{2c}{a}} + C_2 \frac{1}{8(ac^2)^{3/2}} \right] 1 + \left[-C_1 \cancel{\frac{1}{a^{3/2}}} + C_2 \frac{1}{4ac^2} \right] 2\sqrt{ac^2}$

$\sim C_2 \frac{1}{8 a^{3/2} c^3} + C_2 \frac{1}{2\sqrt{a}\, c}$

as a function of a, $\boxed{h_{ij} \sim 1/\sqrt{a}}$ not constant?

$$k \gg k_{eq}\ (c \gg 1): \quad h_{ij} \to \left[C_1 \frac{2c}{a} + C_2 \cancel{\frac{1}{8(ac^2)^{3/2}}} \right] \cos\left(2\sqrt{ac^2}\right)$$

$$+ \left[-C_1 \frac{1}{a^{3/2}} + C_2 \cancel{\frac{1}{4ac^2}} \right] \sin\left(2\sqrt{ac^2}\right)$$

$$\sim C_1 \frac{2c}{a} \cos\left(2\sqrt{ac^2}\right) - C_1 \frac{1}{a^{3/2}} \sin\left(2\sqrt{ac^2}\right)$$

as a function of a, $h_{ij} \sim \frac{1}{a} \cos(\omega \sqrt{a})$ for large ω

part 4. Christoffels:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left[\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right]$$

$$g_{\mu\nu} = a^2 \left[\eta_{\mu\nu} + h_{\mu\nu} \right], \qquad g^{\mu\nu} = a^{-2} \left[\eta^{\mu\nu} - h^{\mu\nu} \right]$$ (proved on a past homework)

$$h_{\mu\nu} = h_{ij} : \quad h_{ij} = h_{ji}, \ \delta^{ij} h_{ij} = 0, \ \partial_i h_{ij} = 0. \qquad h^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta} = h_{\mu\nu}$$

$$\Gamma^0_{00} = \frac{1}{2} g^{0\rho} \left[\partial_0 g_{0\rho} + \partial_0 g_{\rho 0} - \partial_\rho g_{00} \right]$$

$$= \frac{1}{2} g^{00} \partial_0 g_{00} = \frac{1}{2} a^{-2} \partial_\eta a^2$$

$$= \frac{1}{2} a^{-2}\, 2 a a' = \frac{a'}{a} = \mathcal{H}$$

$$\Gamma^0_{0i} = \frac{1}{2} g^{0\rho} \left[\partial_0 g_{i\rho} + \partial_i g_{\rho 0} - \partial_\rho g_{0i} \right]$$

$$= \frac{1}{2} g^{00} \partial_i g_{00} = 0.$$

$$\Gamma^0_{ij} = \frac{1}{2} g^{0\rho} \left[\partial_i g_{j\rho} + \partial_j g_{\rho i} - \partial_\rho g_{ij} \right]$$

$$= -\frac{1}{2} g^{00} \partial_0 g_{ij} = \frac{1}{2} a^{-2} \partial_\eta a^2 (\delta_{ij} + h_{ij})$$

$$= \mathcal{H} \delta_{ij} + \mathcal{H} h_{ij} + \frac{1}{2} h'_{ij}$$

$$\Gamma^i_{00} = \frac{1}{2} g^{i\rho} \left[\partial_0 g_{0\rho} + \partial_0 g_{\rho 0} - \partial_\rho g_{00} \right]$$

$$= -\frac{1}{2} g^{ij} \partial_j g_{00} = 0.$$

$$\Gamma^i_{0j} = \frac{1}{2} g^{i\rho} \left[\partial_0 g_{j\rho} + \partial_j g_{\rho 0} - \partial_\rho g_{0j} \right]$$

$$= \frac{1}{2} g^{ik} \partial_0 g_{jk} = \frac{1}{2} a^{-2} (\delta_{ik} - h_{ik}) \partial_0 \left[a^2 (\delta_{jk} + h_{jk}) \right]$$

$$= \frac{1}{2} a^{-2} (\delta_{ik} - h_{ik}) \partial_0 \left[a^2 (\delta_{jk} + h_{jk}) \right] = \mathcal{H} \delta_{ij} + \frac{1}{2} h'_{ij}$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell} \left[\partial_j g_{k\ell} + \partial_k g_{\ell j} - \partial_\ell g_{jk} \right]$$

$$= \frac{1}{2} (\delta_{i\ell} - h_{i\ell}) \left[\partial_j h_{k\ell} + \partial_k h_{\ell j} - \partial_\ell h_{jk} \right]$$

$$= \frac{1}{2} \left[\partial_j h_{ik} + \partial_k h_{ij} - \partial_i h_{jk} \right]$$

comoving observers have $U^\mu_{obs} = (U^0_{obs}, \vec{0})$ since $d\vec{x} = 0$ when comoving

$$-1 = U^\mu_{obs} U_{obs\,\mu} = g_{\mu\nu} U^\mu_{obs} U^\nu_{obs} = g_{00} (U^0_{obs})^2 = -a^2 {U^0_{obs}}^2 \rightarrow U^0_{obs} = \frac{1}{a}.$$

let $p = a E_{obs}$, $\quad E_{obs} = -U^\mu_{obs} P_\mu = -g_{\mu\nu} U^\mu_{obs} P^\nu = -g_{0\nu} U^0_{obs} P^\nu$

$$= -g_{00} U^0_{obs} P^0 = a^2 \frac{1}{a} P^0 = a P^0 \rightarrow p = a^2 P^0.$$

null geodesic equation: $\quad P^0 \frac{d}{d\eta} P^\mu = -\Gamma^\mu_{\alpha\beta} P^\alpha P^\beta$

$$\begin{aligned}
\rightarrow P^0 \frac{d}{d\eta} P^0 &= -\Gamma^0_{\alpha\beta} P^\alpha P^\beta \\
&= -\left[\Gamma^0_{00} P^0 P^0 + \cancel{2\Gamma^0_{0i} P^0 P^i} + \Gamma^0_{ij} P^i P^j \right] \\
&= -\left[\mathcal{H} (P^0)^2 + \left(\mathcal{H} \delta_{ij} + \mathcal{H} h_{ij} + \tfrac{1}{2} h'_{ij} \right) P^i P^j \right]
\end{aligned}$$

by $-m^2 = P^\mu P_\mu$, we have

$$0 = g_{\mu\nu} P^\mu P^\nu = g_{00} (P^0)^2 + \cancel{2 g_{0i} P^0 P^i} + g_{ij} P^i P^j = -a^2 (P^0)^2 + a^2 (\delta_{ij} + h_{ij}) P^i P^j$$

$$\rightarrow (P^0)^2 = \delta_{ij} P^i P^j + h_{ij} P^i P^j = |\vec{p}|^2 + h_{ij} P^i P^j$$

$$\begin{aligned}
\frac{dP^0}{d\eta} &= -\frac{1}{P^0} \left[\mathcal{H} (P^0)^2 + \mathcal{H} (\delta_{ij} + h_{ij}) P^i P^j + \tfrac{1}{2} h'_{ij} P^i P^j \right] \\
&= -2\mathcal{H} P^0 - \tfrac{1}{2} h'_{ij} P^i P^j \frac{1}{P^0}
\end{aligned}$$

$$\rightarrow \frac{dP^0}{d\eta} + 2\mathcal{H} P^0 = -\frac{1}{2} h'_{ij} \frac{P^i P^j}{P^0}.$$

$$\frac{dp}{d\eta} = \frac{d}{d\eta}(a^2 P^0) = 2 a a' P^0 + a^2 \frac{d}{d\eta} P^0 = a^2 \left[2\mathcal{H} P^0 + \frac{d}{d\eta} P^0 \right]$$

$$\begin{aligned}
\rightarrow \frac{dp}{d\eta} &= -\frac{a^2}{2} h'_{ij} \frac{P^i P^j}{P^0} = -\frac{a^2}{2} \frac{1}{P^0} h'_{ij} \hat{p}^i \hat{p}^j \delta_{kl} P^k P^l \\
&= -\frac{a^2}{2} \frac{1}{P^0} h'_{ij} \hat{p}^i \hat{p}^j \left((P^0)^2 + \overbrace{h_{kl} P^k P^l}^{\text{neglect}} \right) \\
&= -\frac{p}{2} h'_{ij} \hat{p}^i \hat{p}^j .
\end{aligned}$$

$\square$

part 5.

$$f_0(p) = \left[e^{p/\bar{T}} - 1 \right]^{-1}$$

$$f(\vec{p}) = f_0\left(\frac{p}{1 + \Theta(\eta, \vec{x}, \hat{p})} \right)$$

$$= f_0(p) - p \frac{\partial f_0}{\partial p} \Theta(\eta, \vec{x}, \hat{p})$$

Boltzmann equation for f:

$$\left.\frac{df}{dt}\right|_{traj} = 0 \quad \leftarrow \text{ neglecting all sources of } \Theta \text{ except } h_{ij}.$$

$$= \frac{\partial f}{\partial \eta} + \left.\frac{d\vec{x}}{d\eta}\right|_{traj} \cdot \frac{\partial f}{\partial \vec{x}} + \left.\frac{dp}{d\eta}\right|_{traj} \frac{\partial f}{\partial p} + \left.\frac{d\hat{p}}{d\eta}\right|_{traj} \cdot \frac{\partial f}{\partial \hat{p}}$$

$$= (-)p \frac{\partial f_0}{\partial p} \frac{\partial \Theta}{\partial \eta} + \left.\frac{d\vec{x}}{d\eta}\right|_{traj} \cdot (-)p \frac{\partial f_0}{\partial p} \frac{\partial \Theta}{\partial \vec{x}} + \left.\frac{dp}{d\eta}\right|_{traj} \left[\frac{\partial f_0}{\partial p} + O(\Theta) \right] + \left.\frac{d\hat{p}}{d\eta}\right|_{traj} \cdot (-)p \frac{\partial f_0}{\partial p} \frac{\partial \Theta}{\partial \hat{p}}.$$

$$\rightarrow \quad 0 = \frac{\partial \Theta}{\partial \eta} + \left.\frac{d\vec{x}}{d\eta}\right|_{traj} \cdot \frac{\partial \Theta}{\partial \vec{x}} - \left.\frac{dp}{d\eta}\right|_{traj} \frac{1}{p} + \left.\frac{d\hat{p}}{d\eta}\right|_{traj} \cdot \frac{\partial \Theta}{\partial \hat{p}}$$

$$\left.\frac{d\vec{x}}{d\eta}\right|_{traj} = \hat{p} \quad \text{for photons}$$

$$\left.\frac{d\hat{p}}{d\eta}\right|_{traj} = 0 \quad \text{to zeroth order in perturbations (in bkg universe)}$$

$$\left.\frac{dp}{d\eta}\right|_{traj} = -\frac{p}{2} h'_{ij} \hat{p}^i \hat{p}^j \quad \text{by last problem}$$

$$\rightarrow \quad 0 = \frac{\partial \Theta}{\partial \eta} + \hat{p} \cdot \frac{\partial \Theta}{\partial \vec{x}} + \frac{1}{2} h'_{ij} \hat{p}^i \hat{p}^j$$

note: solutions with different values of $\hat{p}$ are independent (no $\partial\Theta/\partial\hat{p}$)

let $\Theta_{\hat{p}}(\eta) = \Theta(\eta, \vec{x} = (\eta_0 - \eta)\hat{p}, \hat{p})$

$$\rightarrow \quad 0 = \frac{d}{d\eta} \Theta_{\hat{p}} + \frac{1}{2} h'_{ij} \hat{p}^i \hat{p}^j$$

$$\rightarrow \quad \Theta_{\hat{p}}(\eta) - \Theta_{\hat{p}}(\eta_*) = \int_{\eta_*}^{\eta} d\eta' \left[\frac{-1}{2} h'_{ij}(\eta', \vec{x} = (\eta_0 - \eta')\hat{p}) \hat{p}^i \hat{p}^j \right]$$

↳ take to zero under assumption of instant decoupling?

$$h'_{ij}(\eta, \vec{x}) = \int d^3k \, e^{-i\vec{x}\cdot\vec{k}} h'_{ij}(\eta, \vec{k})$$

so
$$\Theta(\eta_0, \vec{x} = 0, \hat{p}) = \int_{\eta_*}^{\eta_0} d\eta \left[\frac{-1}{2} \int d^3k \, e^{-i(\eta_0 - \eta')\hat{p}\cdot\vec{k}} h'_{ij}(\eta, \vec{k}) \hat{p}^i \hat{p}^j \right]$$

$$= -\frac{1}{2} \int d^3k \int_{\eta_*}^{\eta_0} d\eta \; h'_{ij}(\eta, \vec{k}) \hat{p}^i \hat{p}^j e^{-i\chi\hat{p}\cdot\vec{k}}$$