Graduate Cosmology Spring 2025
Homework 6 solutions

How to self-grade: full points for correct answer with correct reasoning; half-points for correct reasoning but incorrect answer due to algebra error; zero point for incorrect reasoning (even if final answer is correct out of luck).

Problem 1

1) Consider a volume $V = a^3 V_c$, where V_c is a fixed comoving volume. The total energy density of baryons in that volume is $U = (\rho_b + \frac{3}{2} n_b T_b)V$, which includes the rest-mass and thermal energy density. SInce $\rho_b a^3$ and $n_b a^3$ are constant, we get $dU = \frac{3}{2} n_b V dT_b$. The pressure is $P_b = n_b T_b$. So the first law of thermodynamics becomes

$$\frac{3}{2} n_b V dT_b + n_b T_b dV = \delta Q$$

Dividing by dt, using $dV/dt = 3HV$, and substituting $\delta Q = \dot{q} dt V$, we get, after diving by V,

$$\frac{3}{2} n_b \dot{T}_b + 3H n_b T_b = \dot{q} \quad \Rightarrow \dot{T}_b + 2HT_b = \frac{2}{3}\frac{\dot{q}}{n_b}$$

2) Conservation of momentum in the electron's initial rest-frame implies $m_e \vec{v}_{e,f} = \vec{p}_i - \vec{p}_f \approx E_\gamma(\hat{p}_i - \hat{p}_f)$. The electron therefore gains energy $\Delta E_e = \frac{1}{2} m_e v_{e,f}^2 = E_\gamma^2/m_e(1 - \hat{p}_i \cdot \hat{p}_f) = E_\gamma^2/m_e(1 - \cos\theta)$.

3) To get the heating rate per unit volume, we multiply ΔE_e by $dN_{\rm scat}/dt dE_\gamma dV d\cos\theta$ and integrate:

$$\dot{q} = n_e \int dE_\gamma d\cos\theta \frac{dn_\gamma}{dE_\gamma} \frac{d\sigma_{\rm T}}{d\cos\theta}(1 + f_\gamma(E_\gamma)) \times \Delta E_e(E_\gamma, \cos\theta).$$

First, since $d\sigma_{\rm T}/d\cos\theta \propto (1 + \cos^2\theta)$, the $\cos\theta$ term in ΔE_γ integrates to zero, and we are simply left with

$$\dot{q} = \frac{n_e \sigma_{\rm T}}{m_e} \int dE_\gamma \frac{dn_\gamma}{dE_\gamma}(1 + f_\gamma(E_\gamma))E_\gamma^2.$$

To compute the integral, we substitute

$$dE_\gamma \frac{dn_\gamma}{dE_\gamma} = d^3p \frac{dn_\gamma}{d^3p} = d^3p \mathcal{N}_\gamma(p) = d^3p \frac{2}{h^3} f_\gamma(p),$$

with $f_\gamma(p) = (e^{p/T_\gamma} - 1)^{-1}$. Hence we find

$$\dot{q} = \frac{n_e \sigma_{\rm T}}{m_e} \frac{2}{h^3} \int dp 4\pi p^4 \frac{e^{p/T_\gamma}}{(e^{p/T_\gamma} - 1)^2}.$$

We recognize that $e^{p/T_\gamma}/(e^{p/T_\gamma} - 1)^2 = -T_\gamma \frac{d}{dp}(e^{p/T_\gamma} - 1)^{-1}$. Integrating by parts, we thus get

$$\dot{q} = \frac{n_e \sigma_{\rm T}}{m_e} 4T_\gamma \times \frac{2}{h^3} \int dp 4\pi p^3 \frac{1}{e^{p/T_\gamma} - 1}.$$

We recognize the last integral as ρ_γ, hence obtain the desired result.

4) The total number density of particles is

$$n_b = n_e + n_p + n_{\rm H^0} + n_{\rm He^{++}} + n_{\rm He^+} + n_{\rm He^0}.$$

Now, by definition, $n_p + n_{\rm H^0} = n_{\rm H}$, and $n_{\rm He^{++}} + n_{\rm He^+} + n_{\rm He^0} = n_{\rm He,tot} = f_{\rm He} n_{\rm H}$. With $n_e = n_{\rm H} x_e$, we arrive at $n_b = n_{\rm H}(1 + x_e + f_{\rm He})$.

5) See jupyter notebook on github for the plot. The qualitative evolution of T_b is as follows. While $\sigma_{\rm T}\rho_\gamma x_e/m_e \gg H$, at $z \gtrsim z_T$, Thomson scattering forces $T_b \approx T_\gamma \propto 1/a$. Once Thomson scattering becomes negligible, at $z \lesssim z_T$, $\dot{T}_b \approx -2HT_b$, implying that $T_b \propto 1/a^2$. This is the pure adiabatic decay of baryon temperature, which can be understood from the fact that $T_b \sim \langle V_b^2 \rangle \propto 1/a^2$ when no interactions are present.

6) Neglecting the potential term, the continuity equation for baryons, in Fourier space, is $\delta_b' - kV_b = 0$, using $\vec{V}_b = i\hat{k}V_b$ for scalar modes. After baryons decouple from photons, $\delta_b' \sim \mathcal{H}\delta_b$ hence $V_b \sim (\mathcal{H}/k)\delta_b$. The baryon pressure is $P_b = n_b T_b \sim \frac{\bar{\rho}_b}{m_p}(1+\delta_b)T_b$, where we neglected temperature perturbations, and approximated n_b as if all baryons were protons. Therefore $|\vec{\nabla}P_b|/\rho_b \sim T_b/m_p k\delta_b$. Hence, we have

$$\frac{|\vec{\nabla}P_b/\rho_b|}{|\mathcal{H}\vec{V}_b|} \sim \frac{T_b}{m_p}\frac{k\delta_b}{\mathcal{H}^2\delta_b/k} \sim \frac{T_b}{m_p}\frac{k^2}{\mathcal{H}^2}.$$

Now, after recombination, we may approximate the Universe as matter-dominated, so $\mathcal{H} \approx H_0\sqrt{\Omega_m}\, z^{1/2}$. The baryon temperature is approximately

$$T_b \approx \begin{cases} 0.25 \text{ eV}(z/10^3) & \text{for} \quad z \gtrsim z_T, \\ 0.25 \text{ eV}(z_T/10^3)(z/z_T)^2 & \text{for} \quad z \lesssim z_T. \end{cases}$$

Using $m_p \approx 10^9$ eV, the scale at which pressure dominates is therefore

$$\begin{aligned} k_P \sim \mathcal{H}\sqrt{\frac{m_p}{T_b}} &\sim 6\times 10^4 \times H_0\sqrt{\Omega_m}\, z^{1/2} \times \begin{cases} \sqrt{10^3/z}, & z \gtrsim z_T \\ \sqrt{10^3 z_T}/z, & z \lesssim z_T \end{cases} \\ &\sim 3\times 10^2 \text{ Mpc}^{-1} \times \begin{cases} 1, & z \gtrsim z_T, \\ \sqrt{z_T/z}, & z \lesssim z_T. \end{cases} \end{aligned}$$

This is a *much* smaller scale than the scales we've been discussing so far (typically $k \lesssim 1$ Mpc^{-1}). Hence baryon pressure is negligible on all but the very smallest scales.

Problem 2

1) From the last equation, we have, when the Thomson scattering rate is large,

$$\Pi_\gamma \approx -\frac{8}{27}\frac{k}{a n_e \sigma_{\rm T}} V_\gamma.$$

Plugging back into the momentum equation, it is then just algebra to obtain the desired equation.

2) We compute the derivatives of the ansatz $\delta_\gamma = A(\eta)e^{\pm ik\eta/\sqrt{3}}$:

$$\begin{aligned} \delta_\gamma' &= (A' \pm i(k/\sqrt{3})A)e^{\pm ik\eta/\sqrt{3}}, \\ \delta_\gamma'' &= \left(A'' \pm 2i(k/\sqrt{3})A' - (k^2/3)A\right) e^{\pm ik\eta/\sqrt{3}}. \end{aligned}$$

Plugging back into the homogeneous equation, we get the following equation for A:

$$A'' \pm 2i(k/\sqrt{3})A' + \frac{8}{27}\frac{k^2}{a n_e \sigma_{\rm T}}(A' \pm i(k/\sqrt{3})A) = 0.$$

Keeping the leading-order terms, we obtain

$$A' + \frac{4}{27}\frac{k^2}{a n_e \sigma_{\rm T}} A = 0,$$

which has the solution

$$A(\eta) \propto \exp\left(-\frac{4}{27}k^2 \int^\eta \frac{d\eta'}{a n_e \sigma_{\rm T}(\eta')}\right).$$

Problem 3

See Jupyter notebook on github, where I gathered all the equations. It is important to understand that the system is *very* unphysical for neutrinos after horizon entry and for photons after decoupling. Indeed, both species should free-stream in these regimes rather than be treated as an ideal fluid.