Graduate Cosmology Spring 2025
Homework 5

due by 11:59pm on Thursday 3/20, 2025.

Problem 1: Thomson visibility function; reionization [6 points]

1) [2 points] Using your recombination code, compute the visibility function $g_z(z)$ (which integrates to unity over redshift), and plot it for the Planck 2018 best-fit cosmological parameters.

Long after recombination, the first stars and galaxies formed, produced UV photons which eventually fully reionized the Universe. This is called **reionization**[1], and happened at $z_{\rm reio} \lesssim 10$.

2) [1 point] Assuming that the source terms $(\Theta_0, ...)$ in the line-of-sight integral for temperature anisotropies are negligible at $z \lesssim z_{\rm reio}$ [see next problem for why that is], show that the main effect of reionization is to decrease the observed CMB anisotropies by a factor $e^{-\tau_{\rm reio}}$, whre $\tau_{\rm reio} \equiv \int_{t_{\rm reio}}^{t_0} dt \; n_e \sigma_{\rm T}$ is the **reionization optical depth**.

The previous result implies that it is very hard to constrain reionization from CMB temperature anisotropies alone, as its effect is degenerate with (i.e. equivalent to) merely changing the amplitude of initial perturbations. However, it turns out that large-angular-scale CMB *polarization* anisotropies get *enhanced* by reionization. This has allowed Planck (and WMAP before it) to measure $\tau_{\rm reio} = 0.054 \pm 0.007$.

3) [3 points] Assuming reionization happens instantaneously at $z_{\rm reio}$ and that after that hydrogen and helium are fully ionized, write the correspondence between $\tau_{\rm reio}$ and $z_{\rm reio}$, and plot $\tau_{\rm reio}(z_{\rm reio})$ for $z_{\rm reio} \in (0, 15)$. Solve for $z_{\rm reio}$ (and give an error bar!) given Planck's measurement of $\tau_{\rm reio}$.

Problem 2: free streaming and projection [8 points]

The goal of this problem is to gain some intuition into the effect of free streaming of photons after last scattering, and how fluctuations at the surface of last scattering translate to observed anisotropies today.

To simplify, we assume that photons are tightly coupled to electrons until η_*, and that they then free-stream – in other words, we are assuming that recombination happened intantaneously and that the visibility function is a Dirac delta function. In that approximation, the photon anisotropy at some later time $\eta > \eta_*$ (and at the origin of spatial coordinates, to simplify) is given by

$$\Theta(\eta, \vec{x} = \vec{0}, \hat{p}) \approx \Theta(\eta_*, \vec{x} = -(\eta - \eta_*)\hat{p}, \hat{p}) = \Theta_0(\eta_*, \vec{x} = -(\eta - \eta_*)\hat{p}) + \hat{p} \cdot \vec{V}_e(\eta_*, \vec{x} = -(\eta - \eta_*)\hat{p}).$$

Consider a single Fourier mode with comoving wavenumber $\vec{k}$, so that $\Theta_0(\eta_*, \vec{x}) = \Theta_* \; e^{i\vec{k}\cdot\vec{x}}$, and, assuming purely scalar modes, $\vec{V}_e = i\hat{k} \; V_* \; e^{i\vec{k}\cdot\vec{x}}$.

1) [3 points] Calculate the photon monopole Θ_0 and dipole $\vec{V}_\gamma = i\hat{k}V_\gamma$ at the origin of coordinates at $\eta > \eta_*$. Show that they are suppressed for $k(\eta - \eta_*) \gg 1$.

2) [2 points] Compute the *comoving* wavenumber k_* corresponding to perturbations with *physical* wavelength equal to the Hubble length $1/H(\eta_*)$ at last-scattering. Give your numerical answer in inverse Mpc. Assuming perturbations at last scattering have typical wavenumbers $k \gtrsim k_*$, compute the suppression of the monopole and dipole at $z \lesssim 10$ relative to their amplitudes at recombination.

3) [3 points] A plane wave can be expanded in Legendre polynomials as follows:

$$e^{-i\vec{k}\cdot\vec{r}} = \sum_{\ell=0}^{\infty} (2\ell + 1)(-i)^\ell j_\ell(kr) P_\ell(\hat{k} \cdot \hat{r}),$$

[1] Please do not mix reionization with recombination!

where j_ℓ is the ℓ-th order **Spherical Bessel function**. For a given $\vec{k}$, the temperature anisotropy can also be expanded in terms of Legendre polynomials:

$$\Theta(\eta, \vec{x} = \vec{0}, \hat{p}) = \sum_\ell (2\ell + 1)\Theta_\ell(\eta)\ P_\ell(\hat{p} \cdot \hat{k}).$$

Write down the Legendre coefficients $\Theta_\ell(\eta)$ in terms of Θ_*, V_*, and in terms of spherical Bessel functions and their derivatives. Confirm that Θ_0 defined in question 1) and in this question are identical, and find the mapping between V_γ and Θ_1. *Hint:* $i(\hat{k} \cdot \hat{r})e^{-i\lambda\hat{k}\cdot\hat{r}} = -\frac{d}{d\lambda}e^{-i\lambda\hat{k}\cdot\hat{r}}$.

Problem 3: Gauge transformation of the stress-energy tensor [5 points]

Like any tensor field, the stess-energy tensor transforms as follows under a coordinate transformation $\tilde{x}^\lambda = x^\lambda + \xi^\lambda$:

$$\tilde{T}^\mu_{\ \nu} = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} T^\alpha_{\ \beta},$$

where both sides are to be evaluated at the *same event*, described either with coordinates x^λ or coordinates $\tilde{x}^\lambda = x^\lambda + \xi^\lambda$. Note that we work with the "one index up, one index down" version of the stress-energy tensor as these components take a particularly simple form.

1) [1 point] By writing $T^\mu_{\ \nu}(x) = \overline{T}^\mu_{\ \nu}(x) + \delta T^\mu_{\ \nu}$ and likewise $\tilde{T}^\mu_{\ \nu}(\tilde{x}) = \overline{T}^\mu_{\ \nu}(\tilde{x}) + \delta\tilde{T}^\mu_{\ \nu}$, where the background tensors are identical in both cases, and are evaluated at the same *coordinate values* as the left-hand sides, derive the relation between $\delta\tilde{T}^\mu_{\ \nu}$ and $\delta T^\mu_{\ \nu}$ at linear order in gauge transformations.

2) [3 points] Using $T^0_{\ 0} = -(\overline{\rho}+\delta\rho)$, $\quad T^i_{\ 0} = -(\overline{\rho}+\overline{P})V^i$, $\quad T^i_{\ j} = (\overline{P}+\delta P)\delta^i_j + \Pi^i_{\ j}$, derive the gauge transformations of $\delta\rho, \delta P, V^i$, and $\Pi^i_{\ j}$ (write your results in terms of ξ^μ, not using Baumann's notation). In particular, you should have obtained that the anisotropic stress tensor $\Pi^i_{\ j}$ is gauge invariant.

3) [1 point] Prove that the so-called curvature perturbation

$$\mathcal{R} = -C - \frac{1}{3}k^2 E + \mathcal{H}\frac{\delta\rho}{\overline{\rho}'},$$

where C and E are the scalar parts of the metric perturbation h_{ij} defined in the notes, is gauge invariant.

Problem 4: Vector modes [9 points]

In this problem we study pure (transverse) vector modes. Since gauge freedom allows us to set one of the two vector components of the metric to zero, we shall only keep the vector part of h_{0i}. The perturbed metric we study is then

$$ds^2 = a^2(\eta)[-d\eta^2 + 2h_{0i}\ d\eta dx^i + d\vec{x}^2], \qquad \partial_i h_{0i} = 0.$$

1) [1 point] Show that, at linear order in perturbations, the inverse metric is $g^{\mu\nu} = \frac{1}{a^2}[\eta^{\mu\nu} - h^{\mu\nu}]$, where $h^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$ (note that this applies not just to the specific metric above).

For the remainder of this problem, you may either write your own Mathematica code, or download Baumann's notebook at this link. Note that you will have to understand and adapt his notebook, and re-express the output in a clearer way.

2) [4 points] Compute the pertubed Einstein tensor components G_{00}, G_{0i}, G_{ij} of this metric at linear order in h_{0i}. Make sure to fully simplify the result using the fact that h_{0i} is transverse (divergence-free).

3) [1 point] From the $0i$ Einstein field equation, derive a differential equation satisfied by h_{0i} and (the transverse part of) V^i, defined through $T^i_{\ 0} = -(\overline{\rho} + \overline{P})V^i$, where $T^{\mu\nu}$ is the *total* stress-energy tensor.

4) [3 points] Assume that the total anisotropic stress vanishes (it always does for some time in the early Universe). From the conservation of the total stress-energy tensor (specifically from $\nabla_\mu T^{\mu i} = 0$), derive a simple equation for the evolution of V_i. Show that this implies that h_{0i} decays. Note: you may need to use Baumann's code for the Christoffel symbols.

Cosmology Homework 5 $\quad a = \frac{1}{1+z} \quad da = -(1+z)^{-2}\, dz$ $\quad$ Connor Hainje

code on GitHub at github.com/cmhainje/cosmo-hw

Problem 1:

part 1. see notebook

part 2. $\Theta(\eta_0, \vec{x}=0, \hat{p}) = \int_0^{\eta_0} d\eta\; g(\eta)\; S(\eta, \vec{x}=(\eta-\eta_0)\hat{p}, \hat{p})$

where $g(\eta) \equiv a\, n_e\, \sigma_T \exp\left[-\int_\eta^{\eta_0} d\eta'\, a' n_e' \sigma_T\right]$ $\qquad S \equiv \Theta_0 + \vec{V}_e \cdot \hat{p} + \frac{3}{20} Q_{ij}\, \hat{p}_i \hat{p}_j$

for $z \lesssim z_{reio}$, we assume S to be negligible

$$\Rightarrow \Theta(\eta_0, \vec{x}=0, \hat{p}) \simeq \int_0^{\eta_{reio}} d\eta\; g(\eta)\; S(\eta, \vec{x}=(\eta-\eta_0)\hat{p}, \hat{p})$$

define $\tau_{reio} = \int_{t_{reio}}^{t_0} dt\; n_e \sigma_T = \int_{\eta_{reio}}^{\eta_0} d\eta\; a\, n_e\, \sigma_T$

$$g(\eta_{reio}) = a\, n_e\, \sigma_T\, e^{-\tau_{reio}}$$

$$g(\eta < \eta_{reio}) = a\, n_e\, \sigma_T\, e^{-\tau_{reio}} \exp\left[-\int_\eta^{\eta_{reio}} d\eta'\, a' n_e' \sigma_T'\right]$$

$$\equiv e^{-\tau_{reio}}\, \tilde{g}(\eta),$$ defining $\tilde{g}$ to be the value of g ignoring reio

$$\Rightarrow \Theta(\eta_0, \vec{x}=0, \hat{p}) = e^{-\tau_{reio}} \int_0^{\eta_{reio}} d\eta\; \tilde{g}(\eta)\; S(\eta, \vec{x}=(\eta-\eta_0)\hat{p}, \hat{p})$$

part 3. given Y_{He} and assuming H, He are fully ionized, $n_p = \frac{1-Y_{He}}{m_p} \rho_b$ and $n_{He^{++}} = \frac{Y_{He}}{m_{He^{++}}} \rho_b$.

assume charge neutrality $\Rightarrow n_e = n_H + 2 n_{He} = \left(\frac{1-Y_{He}}{m_p} + \frac{2Y_{He}}{m_{He^{++}}}\right) \rho_b$

$$\rho_b = \rho_{b,0}\, a^{-3}, \quad \rho_{b,0} = \rho_{crit,0}\, \Omega_{b,0} = \rho_{crit,0}\, \omega_b h^2 = \frac{3\,(100\ \mathrm{km/s/Mpc})^2}{8\pi G}\, \omega_b$$

then $\tau_{reio} = \int_{t_{reio}}^{t_0} dt\; n_e \sigma_T$

$$= \left(\frac{1-Y_{He}}{m_p} + \frac{2Y_{He}}{m_{He^{++}}}\right) \rho_{b,0}\, \sigma_T \int_{t_{reio}}^{t_0} dt\; a^{-3}$$

$$= \left(\frac{1-Y_{He}}{m_p} + \frac{2Y_{He}}{m_{He^{++}}}\right) \rho_{b,0}\, \sigma_T \int_{a_{reio}}^{1} \frac{da}{\dot{a}\, a^3}$$

assuming $\omega_k = 0$ and $\omega_r \ll \omega_m, \omega_\Lambda$ then $\dot{a}\, a^3 = (100\ \frac{\mathrm{km/s}}{\mathrm{Mpc}}) \sqrt{\omega_\Lambda}\, a_{m\Lambda}^4\, x^2 \sqrt{x + x^4}$ $\quad (x \equiv a/a_{m\Lambda})$

$$\approx \left(\frac{1-Y_{He}}{m_p} + \frac{2Y_{He}}{m_{He^{++}}}\right) \frac{\rho_{b,0}\, \sigma_T}{(100\ \mathrm{km/s/Mpc}) \sqrt{\omega_\Lambda}\, a_{m\Lambda}^3} \int_{a_{reio}/a_{m\Lambda}}^{1/a_{m\Lambda}} \frac{dx}{x^2 \sqrt{x + x^4}}$$

$$= \left(\frac{1-Y_{He}}{m_p} + \frac{2Y_{He}}{m_{He^{++}}}\right) \frac{\rho_{b,0}\, \sigma_T}{(100\ \mathrm{km/s/Mpc}) \sqrt{\omega_\Lambda}\, a_{m\Lambda}^3}\, \frac{2}{3} \left[\sqrt{1 + \left(\frac{a_{m\Lambda}}{a_{reio}}\right)^3} - \sqrt{1 + \left(\frac{a_{m\Lambda}}{1}\right)^3}\right]$$

$$= \left(\frac{1-Y_{He}}{m_p} + \frac{2Y_{He}}{m_{He^{++}}}\right) \frac{\rho_{b,0}\, \sigma_T}{(100\ \mathrm{km/s/Mpc}) \sqrt{\omega_m}}\, \frac{2}{3} \left[\sqrt{a_{m\Lambda}^{-3} + a_{reio}^{-3}} - \sqrt{a_{m\Lambda}^{-3} + 1}\right]$$

$$= \alpha \left[\sqrt{a_{m\Lambda}^{-3} + (1+z_{reio})^3} - \sqrt{a_{m\Lambda}^{-3} + 1}\right]$$ where $\alpha = 2.415 \times 10^{-3}$ for Planck 18 parameters

inverting: $z_{reio} = \left[\left(\frac{\tau_{reio}}{\alpha}\right)^2 + 2\frac{\tau_{reio}}{\alpha} \sqrt{a_{m\Lambda}^{-3} + 1} + 1\right]^{1/3} - 1 = 11.88$ $\quad$ (using $\tau_{reio} = 0.054 \pm 0.007$)

error prop: $\sigma_z = \frac{\partial z}{\partial \tau} \sigma_\tau = \frac{1}{3} \left[\left(\frac{\tau_{reio}}{\alpha}\right)^2 + 2\frac{\tau_{reio}}{\alpha} \sqrt{a_{m\Lambda}^{-3} + 1} + 1\right]^{-2/3} \left[\frac{2\tau_{reio}}{\alpha^2} + \frac{2}{\alpha}\sqrt{a_{m\Lambda}^{-3} + 1}\right] \sigma_\tau = 0.43$

Problem 2:

part 1. $\Theta(\eta, \vec{x}=0, \hat{p}) = \Theta_0(\eta_*, -(\eta-\eta_*)\hat{p}) + \hat{p}\cdot\vec{V}_e(\eta_*, -(\eta-\eta_*)\hat{p})$

Fourier: $\Theta_0(\eta_*, \vec{x}) = \Theta_* e^{i\vec{k}\cdot\vec{x}}$, scalar V_e: $\vec{V}_e = i\hat{k} V_* e^{i\vec{k}\cdot\vec{x}}$

then $\Theta(\eta, \vec{0}, \hat{p}) = \left(\Theta_* + i\hat{k}\cdot\hat{p}\, V_*\right) \exp\left[-ik(\eta-\eta_*)(\hat{k}\cdot\hat{p})\right]$

so $\Theta_0 = \Theta_* \exp\left[-ik(\eta-\eta_*)(\hat{k}\cdot\hat{p})\right]$ and $V_\gamma = V_* \exp\left[-ik(\eta-\eta_*)(\hat{k}\cdot\hat{p})\right]$?

can be written $\Theta_0 = \Theta_*\left(e^{-i\hat{k}\cdot\hat{p}}\right)^{k(\eta-\eta_*)}$, $V_\gamma = V_*\left(e^{-i\hat{k}\cdot\hat{p}}\right)^{k(\eta-\eta_*)}$

$e^{-i\hat{k}\cdot\hat{p}} < 1$ except when $\hat{k}\perp\hat{p}$, so $k(\eta-\eta_*) \gg 1$ suppresses

$(\mathcal{H} = aH)$

part 2. $\frac{1}{H(\eta_*)}$ is a physical length, $\frac{1}{\mathcal{H}(\eta_*)}$ is the corresponding comoving length

taking $z_* = 1100$, comoving wavenumber $k_* = \frac{1}{c\,\text{length}} = aH(\eta_*) = 0.0048\ \text{Mpc}^{-1} = (208\ \text{Mpc})^{-1}$

part 3. $\Theta = \sum_\ell (2\ell+1)\,\Theta_\ell(\eta)\, P_\ell(\hat{p}\cdot\hat{k})$

but also $\Theta = \left(\Theta_* + i\hat{k}\cdot\hat{p}\, V_*\right) \exp\left[ik\chi_*(\hat{k}\cdot\hat{p})\right]$ defining $\chi_* = -(\eta-\eta_*)$

$$= \left(\Theta_* + V_* \frac{d}{d(k\chi_*)}\right) \sum_\ell (2\ell+1)\, i^\ell j_\ell(k\chi_*)\, P_\ell(\hat{k}\cdot\hat{p})$$

$$= \sum_\ell (2\ell+1)\, i^\ell \left[\Theta_*\, j_\ell(k\chi_*) + V_*\, j_\ell'(k\chi_*)\right] P_\ell(\hat{k}\cdot\hat{p})$$

then $\Theta_\ell = i^\ell\left[\Theta_*\, j_\ell(k\chi_*) + V_*\, j_\ell'(k\chi_*)\right]$

Problem 3:

part 1.

$$\tilde{T}^\mu{}_\nu = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} T^\alpha{}_\beta(x^\lambda)$$

$$= (\delta^\mu{}_\alpha + \partial_\alpha \xi^\mu)(\delta^\beta{}_\nu - \partial_\nu \xi^\beta)\ T^\alpha{}_\beta(\tilde{x}^\lambda - \xi^\lambda)$$

$$= (\delta^\mu{}_\alpha \delta^\beta{}_\nu - \delta^\mu{}_\alpha \partial_\nu \xi^\beta + \partial_\alpha \xi^\mu \delta^\beta{}_\nu)\left[T^\alpha{}_\beta(\tilde{x}^\lambda) - \xi^\lambda \partial_\lambda T^\alpha{}_\beta\right]$$

$$= T^\mu{}_\nu(\tilde{x}^\lambda) - \left(\partial_\nu \xi^\beta T^\mu{}_\beta - \partial_\alpha \xi^\mu T^\alpha{}_\nu + \xi^\lambda \partial_\lambda T^\mu{}_\nu\right)$$

$$\tilde{T}^\mu{}_\nu = \bar{T}^\mu{}_\nu(\tilde{x}^\lambda) + \delta\tilde{T}^\mu{}_\nu(\tilde{x}^\lambda) \quad \text{and} \quad T^\mu{}_\nu(\tilde{x}^\lambda) = \bar{T}^\mu{}_\nu(\tilde{x}^\lambda) + \delta T^\mu{}_\nu(\tilde{x}^\lambda)$$

so $$\delta\tilde{T}^\mu{}_\nu - \delta T^\mu{}_\nu\Big|_{\tilde{x}^\lambda} = -\left(\partial_\nu \xi^\beta \bar{T}^\mu{}_\beta - \partial_\alpha \xi^\mu \bar{T}^\alpha{}_\nu + \xi^\lambda \partial_\lambda \bar{T}^\mu{}_\nu\right)$$

part 2.

$$T^0{}_0 = -(\bar{\rho} + \delta\rho) \qquad T^i{}_0 = -(\bar{\rho} + \bar{P})v^i \qquad T^i{}_j = (\bar{P} + \delta P)\delta^i{}_j + \Pi^i{}_j$$

$$\Rightarrow \delta T^0{}_0 = -\delta\rho \qquad \delta T^i{}_0 = -(\bar{\rho} + \bar{P})v^i \qquad \delta T^i{}_j = \delta P\, \delta^i{}_j + \Pi^i{}_j$$

thus

$^0{}_0$:
$$(-\widetilde{\delta\rho}) - (-\delta\rho) = -\left(\partial_0 \xi^\beta \bar{T}^0{}_\beta - \partial_\alpha \xi^0 \bar{T}^\alpha{}_0 + \xi^\lambda \partial_\lambda \bar{T}^0{}_0\right)$$
$$\delta\rho - \widetilde{\delta\rho} = -\left(-\partial_0 \xi^0 \bar{\rho} + \partial_0 \xi^0 \bar{\rho} - \xi^\lambda \partial_\lambda \bar{\rho}\right)$$
$$\widetilde{\delta\rho} = \delta\rho - \xi^0 \partial_0 \bar{\rho}$$

$^i{}_0$:
$$\delta\tilde{T}^i{}_0 - \delta T^i{}_0 = -\left(\partial_0 \xi^\beta \bar{T}^i{}_\beta - \partial_\alpha \xi^i \bar{T}^\alpha{}_0 + \xi^\lambda \partial_\lambda \bar{T}^i{}_0\right)$$
$$-(\bar{\rho} + \bar{P})(\tilde{v}^i - v^i) = -\partial_0 \xi^i (\bar{\rho} + \bar{P})$$
$$\tilde{v}^i = v^i + \partial_0 \xi^i$$

$^i{}_j$:
$$\delta\tilde{T}^i{}_j - \delta T^i{}_j = -\left(\partial_j \xi^\beta \bar{T}^i{}_\beta - \partial_\alpha \xi^i \bar{T}^\alpha{}_j + \xi^\lambda \partial_\lambda \bar{T}^i{}_j\right)$$
$$\widetilde{\delta P}\, \delta^i{}_j + \tilde{\Pi}^i{}_j - \delta P\, \delta^i{}_j - \Pi^i{}_j = -\xi^\lambda \partial_\lambda \bar{P} \delta^i{}_j$$

grouping terms proportional to $\delta^i{}_j$ and $\perp$ to it gives:

$$\widetilde{\delta P} = \delta P - \xi^0 \partial_0 \bar{P} \quad \text{and} \quad \tilde{\Pi}^i{}_j = \Pi^i{}_j .$$

part 3.

$$\mathcal{R} = -C - \frac{1}{3}k^2 E + \mathcal{H}\frac{\delta\rho}{\bar{\rho}'}$$

$$\tilde{\mathcal{R}} = -\tilde{C} - \frac{1}{3}k^2 \tilde{E} + \mathcal{H}\frac{\widetilde{\delta\rho}}{\bar{\rho}'}$$

$$= -\left[C - \mathcal{H}\xi^0 + \frac{1}{3}k^2 L\right] - \frac{1}{3}k^2\left[E - L\right] + \mathcal{H}\frac{1}{\bar{\rho}'}\left[\delta\rho - \xi^0 \partial_0 \bar{\rho}\right]$$

$$= -C + \cancel{\mathcal{H}\xi^0} - \cancel{\frac{1}{3}k^2 L} - \frac{1}{3}k^2 E + \cancel{\frac{1}{3}k^2 L} + \mathcal{H}\frac{\delta\rho}{\bar{\rho}'} - \cancel{\mathcal{H}\xi^0}$$

$$= \mathcal{R}$$

□

Problem 4:

part 1. $$g_{\mu\nu} = a^2\left[\eta_{\mu\nu} + h_{\mu\nu}\right]$$

consider $A^{\mu\nu} = a^{-2}\left[\eta^{\mu\nu} - h^{\mu\nu}\right] = a^{-2}\left[\eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}\right]$

if $A^{\mu\nu}g_{\nu\rho} = \delta^{\mu}{}_{\rho}$, then $A^{\mu\nu}$ is the inverse metric i.e. $g^{\mu\nu} = A^{\mu\nu}$.

let's evaluate it :

$$\begin{aligned} A^{\mu\nu}g_{\nu\rho} &= a^{-2}\left[\eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}\right]a^2\left[\eta_{\nu\rho} + h_{\nu\rho}\right] \\ &= \left[\eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}\right]\left[\eta_{\nu\rho} + h_{\nu\rho}\right] \\ &= \delta^{\mu}{}_{\rho} + \eta^{\mu\nu}h_{\nu\rho} - \eta^{\mu\alpha}\delta^{\beta}{}_{\rho}h_{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}h_{\nu\rho} \\ &= \delta^{\mu}{}_{\rho} + \cancel{\eta^{\mu\nu}h_{\nu\rho}} - \cancel{\eta^{\mu\alpha}h_{\alpha\rho}} + O(h^2) \\ &= \delta^{\mu}{}_{\rho} + O(h^2) \end{aligned}$$

So indeed $g^{\mu\nu} = a^{-2}\left[\eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}\right]$ to linear order in perturbation h. □

part 2. note that $h^{\mu\nu} = -h_{\mu\nu}$ for our metric

$$G_{00} = 3\left(\frac{a'}{a}\right)^2 - 2\frac{a'}{a}\partial_i h_{0i} = 3\left(\frac{a'}{a}\right)^2$$
$$= 3\mathcal{H}^2$$

$$G_{0i} = \left(\frac{a'}{a}\right)^2 h_{0i} - 2\frac{a''}{a}h_{0i} + \partial_j\partial_{(i}h_{j)0} = \left(\frac{a'}{a}\right)^2 h_{0i} - 2\frac{a''}{a}h_{0i} - \frac{1}{2}\partial_j\partial_j h_{0i}$$
$$= -\left(2\mathcal{H}' + \mathcal{H}^2 + \frac{1}{2}\vec{\nabla}^2\right)h_{0i}$$

$$\begin{aligned} G_{ij} = \delta_{ij}&\left[\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a} + 2\frac{a'}{a}\partial_k h_{0k} + \partial_0\partial_k h_{0k}\right] \\ &- \frac{a'}{a}\partial_j h_{0i} - \frac{a'}{a}\partial_i h_{0j} - \frac{1}{2}\partial_0\partial_j h_{0i} - \frac{1}{2}\partial_0\partial_i h_{0j} \end{aligned}$$
$$= \delta_{ij}\left[-2\mathcal{H}' - \mathcal{H}^2\right] - \mathcal{H}\partial_j h_{0i} - \mathcal{H}\partial_i h_{0j} - \frac{1}{2}\partial_0\partial_j h_{0i} - \frac{1}{2}\partial_0\partial_i h_{0j}$$

part 3. EFEs : $G_{\mu\nu} = 8\pi G\, T_{\mu\nu} = 8\pi G\, T_{\nu\mu} = 8\pi G\, g_{\nu\alpha}T^{\alpha}{}_{\mu}$

EFE_{0i} : $$G_{0i} = 8\pi G\, g_{i\alpha}T^{\alpha}{}_{0} = 8\pi G\left[g_{i0}T^{0}{}_{0} + g_{ij}T^{j}{}_{0}\right]$$
$$= -8\pi G\left[h_{0i}(\bar{\rho} + \delta\rho) + (\bar{\rho} + \bar{P})V_i\right]$$

$$-\left(2\mathcal{H}' + \mathcal{H}^2 + \frac{1}{2}\vec{\nabla}^2\right)h_{0i} = -8\pi G\left(\bar{\rho}h_{0i} + (\bar{\rho} + \bar{P})V_i\right)$$

re-arranging: $$\left(\mathcal{H}^2 + 2\mathcal{H}' + \frac{1}{2}\nabla^2 - 8\pi G\bar{\rho}\right)h_{0i} = 8\pi G\,(\bar{\rho} + \bar{P})V_i$$

part 4.

$$0 = \nabla_\mu T^\mu{}_i = \nabla_0 T^0{}_i + \nabla_j T^j{}_i$$

$$= -\nabla_0\left((\bar\rho+\bar P)V_i\right) + \nabla_j\left((\bar P+\delta P)\delta^j{}_i\right)$$

$$= -\nabla_0\left((\bar\rho+\bar P)V_i\right) + \nabla_i\,\delta P$$

$$\nabla_0\left[\left(\mathcal{H}^2+2\mathcal{H}'+\tfrac{1}{2}\vec\nabla^2-8\pi G\bar\rho\right)h_{0i}\right] = 8\pi G\,\nabla_i\,\delta P \qquad ?$$