Cosmology HW7 Connor Hainje

Problem 1:

part 1. $0 = P^\alpha \nabla_\alpha P^\mu = P^\alpha \left(\partial_\alpha P^\mu + \Gamma^\mu_{\alpha\beta} P^\beta \right) \rightarrow P^\alpha \partial_\alpha P^\mu = - \Gamma^\mu_{\alpha\beta} P^\alpha P^\beta$

note that $\frac{d}{d\lambda} P^\mu = \frac{dx^\alpha}{d\lambda} \frac{\partial P^\mu}{\partial x^\alpha} = P^\alpha \partial_\alpha P^\mu$ so $\frac{d}{d\lambda} P^\mu = - \Gamma^\mu_{\alpha\beta} P^\alpha P^\beta$.

λ is a function of η alone so $\frac{d}{d\lambda} P^\mu = \frac{d\eta}{d\lambda} \frac{dP^\mu}{d\eta} = P^0 \frac{dP^\mu}{d\eta}$.

then $P^0 \frac{dP^\mu}{d\eta} = - \Gamma^\mu_{\alpha\beta} P^\alpha P^\beta$. $\longrightarrow$ $\mu = 0$: $P^0 \frac{dP^0}{d\eta} = - \Gamma^0_{\alpha\beta} P^\alpha P^\beta$

by $-m^2 = P^\mu P_\mu$, we have

$$0 = g_{\mu\nu} P^\mu P^\nu = g_{00} (P^0)^2 + 2 g_{0i} P^i P^0 + g_{ij} P^i P^j = - a^2 (1 + 2\psi) (P^0)^2 + a^2 (1 - 2\phi) \delta_{ij} P^i P^j$$

so $\delta_{ij} P^i P^j = \frac{1 + 2\psi}{1 - 2\phi} (P^0)^2 \approx [1 + 2(\phi + \psi)] (P^0)^2$

and $\sqrt{\delta_{ij} P^i P^j} = P^0 \left[1 + 2(\phi + \psi) \right]^{1/2} \approx P^0 (1 + \phi + \psi) \rightarrow P^0 \approx \sqrt{\delta_{ij} P^i P^j} \, (1 - \phi - \psi)$

$$\begin{aligned}
\Gamma^0_{\alpha\beta} P^\alpha P^\beta &= \Gamma^0_{\alpha 0} P^\alpha P^0 + \Gamma^0_{\alpha j} P^\alpha P^j \\
&= \Gamma^0_{00} (P^0)^2 + \Gamma^0_{i0} P^i P^0 + \Gamma^0_{0j} P^0 P^j + \Gamma^0_{ij} P^i P^j \\
&= \Gamma^0_{00} (P^0)^2 + 2 \Gamma^0_{i0} P^i P^0 + \Gamma^0_{ij} P^i P^j \\
&= (\mathcal{H} + \psi') (P^0)^2 + 2 \partial_i \psi \, P^i P^0 + \left(\mathcal{H} - \phi' - 2\mathcal{H}(\phi + \psi) \right) \delta_{ij} P^i P^j \\
&= \quad \text{''} \qquad\qquad \text{''} \qquad + \left(\mathcal{H} - \phi' - 2\mathcal{H}(\phi + \psi) \right) \left(1 + 2(\phi + \psi) \right) (P^0)^2 \\
&= \quad \text{''} \qquad\qquad \text{''} \qquad + (\mathcal{H} - \phi') (P^0)^2 \\
&= (2\mathcal{H} + \psi' - \phi') (P^0)^2 + 2 \partial_i \psi \, P^i P^0
\end{aligned}$$

plugging in... $\frac{dP^0}{d\eta} = - \frac{1}{P^0} \Gamma^0_{\alpha\beta} P^\alpha P^\beta = - \left[(2\mathcal{H} + \psi' - \phi') P_0 + 2 \partial_i \psi \, P^i \right]$

$$= \left(-2\mathcal{H} - \psi' + \phi' - 2 \partial_i \psi \, P^i \frac{1}{P^0} \right) P^0$$

$\frac{P^i}{P^0} = \frac{P^i}{\sqrt{\delta_{ij} P^i P^j}} (1 + \phi + \psi) = \hat{p}^i (1 + \phi + \psi) \rightarrow 2 \partial_i \psi \frac{P^i}{P^0} \approx 2 \partial_i \psi \, \hat{p}^i (1 + \phi + \psi) \approx 2 \partial_i \psi \, \hat{p}^i$

$\rightarrow \frac{dP^0}{d\eta} = \left(-2\mathcal{H} + \phi' - \psi' - 2 \partial_i \psi \, \hat{p}^i \right) P^0$

part 2. compute the components of the 4velocity of comoving observers:

$$U^\mu_{com} = \left(U^0_{com} ,\ \vec{0} \right) \quad \text{since } dx^i = 0 \text{ for } \underline{\text{comoving}}$$

$$-1 = U^\mu U_\mu = g_{\mu\nu} U^\mu U^\nu = g_{00} (U^0)^2 = -a^2 (1+2\Psi) (U^0)^2$$

$$\Rightarrow (U^0)^2 = a^{-2} (1-2\Psi) \quad \Rightarrow U^0 = \frac{1}{a} (1-\Psi)$$

$$\longrightarrow \quad U^\mu_{com} = \left(\frac{1-\Psi}{a} ,\ \vec{0} \right).$$

photon energy measured by comoving observers is

$$E_{obs} = -g_{\mu\nu} U^\mu_{com} P^\nu = -g_{00} U^0_{com} P^0 = a^2 (1+2\Psi) \frac{1}{a} (1-\Psi) P^0 = a (1+\Psi) P^0$$

so $p = a E_{obs} = a^2 (1+\Psi) P^0$.

$$\frac{dp}{d\eta} = 2a \frac{da}{d\eta} (1+\Psi) P^0 + a^2 \frac{d\Psi}{d\eta} P^0 + a^2 (1+\Psi) \frac{dP^0}{d\eta}$$

$$= a^2 \Big(2\mathcal{H} (1+\Psi) + (\partial_\eta + \partial_\eta x^i \partial_i) \Psi + (1+\Psi)(-2\mathcal{H} + \Phi' - \Psi' - 2\hat{p}^i \partial_i \Psi) \Big) P^0$$

$$= a^2 \Big(\Psi' + \hat{p}^i \partial_i \Psi + (1+\Psi)(\Phi' - \Psi' - 2\hat{p}^i \partial_i \Psi) \Big) P^0$$

$$= a^2 \left(\Phi' - \hat{p}^i \partial_i \Psi \right) P^0$$

$$= \left(\Phi' - \hat{p}^i \partial_i \Psi \right) (1-\Psi)\, p$$

$$\approx \left(\Phi' - \hat{p}^i \partial_i \Psi \right) p$$

Problem 2: code on GitHub

part 1. can use last week's code to compute $F_0(k) = \frac{1}{4}\delta_\gamma(k) + \phi(k)$, $V_b(k)$ at η_*

want Sachs-Wolfe and Doppler contributions to $D_\ell = T_0^2 \frac{\ell^2}{2\pi} C_\ell$

SW $\frac{D_\ell^{SW}}{T_0^2} \simeq \left| T_{F_0}(\eta, k = \ell/\chi_*) \right|^2 \Delta_R^2 (k = \ell/\chi_*)$

Doppler $\frac{D_\ell^D}{T_0^2} \simeq \frac{1}{3} \left| T_{V_b}(\eta, k = \ell/\chi_*) \right|^2 \Delta_R^2 (k = \ell/\chi_*)$

$F_0(\eta, \vec{k}) = T_{F_0}(\eta, k)\, R_i(\vec{k}) \qquad V_b(\eta, \vec{k}) = T_{V_b}(\eta, k)\, R_i(\vec{k})$

in code, we picked $\phi_i = 1 \ \forall \vec{k}$, so $R_i = \frac{3}{2}$

$\rightarrow T_{F_0}(\eta, k) = \frac{2}{3} F_0(k)$, sim for V_b.

$\Delta_R^2 = A_s \left(\frac{k}{k_p}\right)^{n_s - 1} \qquad A_s = 2\times10^{-9}, \ n_s = 0.96, \ k_p = 0.05 \text{ Mpc}^{-1}$

part 2.

$\langle \Theta^2 \rangle = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell \approx \int_{\ell_{min}}^{\infty} d\log\ell \ \frac{\ell^2}{2\pi} C_\ell$

so I numerically integrated D_ℓ / T_0^2 over $\log \ell$

$\rightarrow \langle \Theta^2 \rangle^{1/2} \simeq 4.9 \times 10^{-5}$ which is roughly the "1 part in 25,000" anisotropy stated on Wikipedia

Problem 3:

$$\frac{d}{d\eta}\hat{p}^i \approx 2\left(\delta^{ij} - \hat{p}^i\hat{p}^j\right)\partial_j\phi$$

$$\Delta\hat{p}^i(\eta_0) = 2\left(\delta^{ij} - \hat{p}^i\hat{p}^j\right)\int_{\eta_*}^{\eta_0} d\eta \;\; \partial_j\phi\Big|_{\eta,\vec{x}=-\chi\hat{p}} \qquad \chi = \eta_0 - \eta$$

part 1. $\left\langle |\Delta\hat{p}|^2\right\rangle = \left\langle \Delta\hat{p}^i\,\Delta\hat{p}_i\right\rangle = 4\left\langle (\delta^{ij}-\hat{p}^i\hat{p}^j)(\delta_{ik}-\hat{p}_i\hat{p}_k)\int_{\eta_*}^{\eta_0} d\eta \int_{\eta_*}^{\eta_0} d\eta'\; \partial_j\phi(\eta)\,\partial^k\phi(\eta')\right\rangle$

$$(\delta^{ij}-\hat{p}^i\hat{p}^j)(\delta_{ik}-\hat{p}_i\hat{p}_k) = \delta^{ij}\delta_{ik} - \delta^{ij}\hat{p}_i\hat{p}_k - \delta_{ik}\hat{p}^i\hat{p}^j + \hat{p}^i\hat{p}^j\hat{p}_i\hat{p}_k = \delta^j{}_k - \hat{p}^j\hat{p}_k$$

$$\rightarrow \left\langle |\Delta\hat{p}|^2\right\rangle = 4\int_{\eta_*}^{\eta_0} d\eta\int_{\eta_*}^{\eta_0} d\eta'\left\langle (\delta^j{}_k - \hat{p}^j\hat{p}_k)\,\partial_j\phi\big|_\eta\,\partial^k\phi\big|_{\eta'}\right\rangle$$

$$= 4\int_{\eta_*}^{\eta_0} d\eta\int_{\eta_*}^{\eta_0} d\eta'\left\langle (\delta^{jk} - \hat{p}^j\hat{p}^k)\,\partial_j\phi\big|_\eta\,\partial_k\phi\big|_{\eta'}\right\rangle$$

swapping ${}_k{}^k \to {}^k{}_k$

$$= 4\int_{\eta_*}^{\eta_0} d\eta\int_{\eta_*}^{\eta_0} d\eta'\left\langle \partial_j\phi\big|_\eta\,\partial^j\phi\big|^{\eta'} - \hat{p}^j\partial_j\phi\big|_\eta\,\hat{p}^k\partial_k\phi\big|_{\eta'}\right\rangle$$

$$\phi(\vec{k},\eta) = T_\phi(k,\eta)\,R_i(\vec{k})$$

$$\Rightarrow \left\langle \phi(\vec{k},\eta)\,\phi^*(\vec{k}',\eta')\right\rangle = T_\phi(k,\eta)\,T_\phi^*(k',\eta')\,(2\pi)^3\delta(\vec{k}-\vec{k}')\,P_R(k)$$

$$\left\langle \phi(\eta,\vec{x})\,\phi(\eta',\vec{x}')\right\rangle = \int d^3k\,d^3k'\,e^{+i\vec{k}\cdot\vec{x}}e^{+i\vec{k}'\cdot\vec{x}'}\left\langle \phi(\eta,\vec{k})\,\phi(\eta',\vec{k}')\right\rangle$$

$$= \int d^3k\,d^3k'\,e^{+i\vec{k}\cdot\vec{x}}e^{+i\vec{k}'\cdot\vec{x}'}(2\pi)^3\delta(\vec{k}+\vec{k}')\,P_R(k)\,T_\phi(k,\eta)\,T_\phi^*(k',\eta')$$

$$= \int d^3k\;e^{+i\vec{k}\cdot(\vec{x}-\vec{x}')}\,P_R(k)\,T_\phi(k,\eta)\,T_\phi^*(k,\eta')$$

$$(\partial_j\phi)(\eta,\vec{k}) = \mathcal{F}[\partial_j\phi] = \int d^3x\,e^{-i\vec{k}\cdot\vec{x}}\,\partial_j\phi = -\int d^3x\,\partial_j e^{-i\vec{k}\cdot\vec{x}}\,\phi = i\vec{k}\int d^3x\,e^{-i\vec{k}\cdot\vec{x}}\phi = i\vec{k}\,\phi(\vec{k})$$

$$\left\langle \partial_j\phi(\eta,\vec{x})\,\partial^j\phi(\eta',\vec{x}')\right\rangle = \int d^3k\,d^3k'\,e^{+i\vec{k}\cdot\vec{x}}e^{+i\vec{k}'\cdot\vec{x}'}\left\langle ik_j\,\phi(\eta,\vec{k})\;ik'^j\,\phi(\eta',\vec{k}')\right\rangle$$

$$= \int d^3k\,d^3k'\,e^{+i\vec{k}\cdot\vec{x}}e^{+i\vec{k}'\cdot\vec{x}'}(-k_jk'^j)\left\langle \phi(\eta,\vec{k})\,\phi(\eta',\vec{k}')\right\rangle$$

$$= \int d^3k\;k^2\,e^{+i\vec{k}\cdot(\vec{x}-\vec{x}')}\,P_R(k)\,T_\phi(k,\eta)\,T_\phi^*(k,\eta')$$

$$\left\langle \partial_i\phi(\eta,\vec{x})\,\partial_j\phi(\eta',\vec{x}')\right\rangle = \int d^3k\;k_ik_j\,e^{+i\vec{k}\cdot(\vec{x}-\vec{x}')}\,P_R(k)\,T_\phi(k,\eta)\,T_\phi^*(k,\eta')$$

so $\left\langle |\Delta\hat{p}|^2\right\rangle = 4\int_{\eta_*}^{\eta_0} d\eta\int_{\eta_*}^{\eta_0} d\eta'\left[\left\langle \partial_j\phi\big|_\eta\,\partial^j\phi\big|^{\eta'}\right\rangle - \hat{p}^j\hat{p}^k\left\langle\partial_j\phi\big|_\eta\,\partial_k\phi\big|_{\eta'}\right\rangle\right] \qquad \vec{x} = \chi\hat{p}$

$$= 4\int_{\eta_*}^{\eta_0} d\eta\int_{\eta_*}^{\eta_0} d\eta'\int d^3k\left[k^2 - (\hat{p}\cdot\vec{k})^2\right]e^{+i\vec{k}\cdot\hat{p}(\chi-\chi')}\,P_R(k)\,T_\phi(k,\eta)\,T_\phi^*(k,\eta')$$

$$= 4\int d^3k\left[k^2 - (\hat{p}\cdot\vec{k})^2\right]P_R(k)\int_{\eta_*}^{\eta_0} d\eta\int_{\eta_*}^{\eta_0} d\eta'\;e^{+i\vec{k}\cdot\hat{p}(\chi-\chi')}\,T_\phi(k,\eta)\,T_\phi^*(k,\eta')$$

assume between η_* and η_0 the Universe is matter dominated.

then $\phi \sim$ const, so $T_\phi(k,\eta) \approx T_\phi(k,\eta_*)$.

so $\left\langle |\Delta\hat{p}|^2\right\rangle = 4\int d^3k\left[k^2 - (\hat{p}\cdot\vec{k})^2\right]P_R(k)\,|T_\phi(k,\eta_*)|^2\int_{\eta_*}^{\eta_0} d\eta\int_{\eta_*}^{\eta_0} d\eta'\;e^{+i\vec{k}\cdot\hat{p}(\chi-\chi')}$

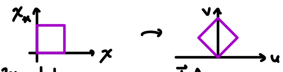
$$= 4\int d^3k\left[k^2 - (\hat{p}\cdot\vec{k})^2\right]P_R(k)\,|T_\phi(k,\eta_*)|^2\;I(\vec{k}\cdot\hat{p})$$

introduce $\Delta_R^2(k) = \frac{k^3}{2\pi^2}P_R(k)$. $\rightarrow \left\langle |\Delta\hat{p}|^2\right\rangle = 8\pi^2\int d^3k\left[k^2 - (\hat{p}\cdot\vec{k})^2\right]\frac{\Delta_R^2(k)}{k^3}\,|T_\phi(k,\eta_*)|^2\;I(\vec{k}\cdot\hat{p})$

part 2. $I(\vec{k}\cdot\hat{p}) = \int_0^{\chi_*} d\chi \int_0^{\chi_*} d\chi' \; e^{i\vec{k}\cdot\hat{p}(\chi'-\chi)}$

let $u = \chi' - \chi$, $v = \chi' + \chi$. then $\left|\frac{\partial(u,v)}{\partial(\chi,\chi')}\right| = \left|\begin{matrix} \partial u/\partial\chi & \partial u/\partial\chi' \\ \partial v/\partial\chi & \partial v/\partial\chi' \end{matrix}\right| = \left|\begin{matrix} -1 & +1 \\ +1 & +1 \end{matrix}\right| = |-1-1| = 2.$

so $d\chi\, d\chi' = \left|\frac{\partial(u,v)}{\partial(\chi,\chi')}\right|^{-1} du\, dv = \frac{1}{2} du\, dv.$

limits of integration:  : $u \in [-\chi_*, +\chi_*]$, $v \in [|u|, 2\chi_* - |u|]$.

so $I(\vec{k}\cdot\hat{p}) = \int_{-\chi_*}^{+\chi_*} du \int_{|u|}^{2\chi_*-|u|} dv \; \frac{1}{2}\, e^{i\vec{k}\cdot\hat{p}\,u} = \int_{-\chi_*}^{+\chi_*} du\, (\chi_* - |u|)\, e^{i\vec{k}\cdot\hat{p}\,u}$

if $k \gg \eta_*^{-1}$, then $I(\vec{k}\cdot\hat{p}) \approx \int_{-\infty}^{+\infty} du\, (\chi_* - |u|)\, e^{i\vec{k}\cdot\hat{p}\,u}$

$$= \mathcal{F}[\chi_* - |u|](\vec{k}\cdot\hat{p})$$

$$= 2\pi\, \chi_*\, \delta(\vec{k}\cdot\hat{p}) + \frac{2}{(\vec{k}\cdot\hat{p})^2}$$

$$\approx 2\pi\, \eta_0\, \delta(\vec{k}\cdot\hat{p})$$ using $\chi_* \approx \eta_0$, large k.

can't figure out how to get rid of this.

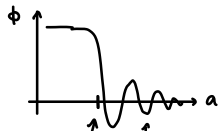
part 3. $\langle |\Delta p|^2 \rangle = 8\pi^2 \int d^3k \left[k^2 - (\hat{p}\cdot\vec{k})^2\right] \frac{\Delta_R^2(k)}{k^3} \left|T_\phi(k,\eta_*)\right|^2 \pi\eta_0\, \delta(\vec{k}\cdot\hat{p})$

$$\approx \eta_0 \int dk_x\, dk_y\, \frac{\Delta_R^2(k)}{k} \left|T_\phi(k,\eta_*)\right|^2$$ picking coords st $\hat{p} = \hat{z}$, $\delta(\vec{k}\cdot\hat{p})$ enforces $\vec{k} \perp \hat{z}$

$$= 2\pi\, \eta_0 \int dk\, \Delta_R^2(k) \left|T_\phi(k,\eta_*)\right|^2$$

$$= 2\pi \int d\ln k\; \eta_0 k\, \Delta_R^2(k) \left|T_\phi(k,\eta_*)\right|^2$$

part 4. for $k \gtrsim k_{eq}$, mode enters horizon during rad. dom. and decays as $\sim \frac{1}{\eta} \sim \frac{1}{a}$.



$k \sim \mathcal{H} \sim a^{-1}$ $\quad a_{eq}$ somewhere over here

so, by η_*, ϕ is highly suppressed

$\rightarrow T_\phi(k \gtrsim k_{eq}, \eta_*) \sim \frac{1}{\eta_* - 1/k} \sim \frac{1}{\eta_*}$

for $k < k_{eq}$, mode does not evolve during rad. dom. and ϕ remains $= \phi_i$

$\phi_i = \frac{2}{3} R_i$ so $T_\phi \simeq \frac{2}{3}$

then $\langle |\Delta p|^2 \rangle \simeq \frac{8\pi}{9} \eta_0 \int_0^{k_{eq}} d\ln k\; k\, \Delta_R^2(k)$

$$= \frac{8\pi}{9} \eta_0 \int_0^{k_{eq}} dk\; A_s \left(\frac{k}{k_p}\right)^{n_s - 1}$$

$$= \frac{8\pi}{9} \eta_0 A_s \frac{k_p}{n_s} \left(\frac{k_{eq}}{k_p}\right)^{n_s}$$

Baumann eq 4.6 $\eta_0 \approx \frac{2}{H_0\sqrt{\Omega_m}}$

$A_s = 2 \times 10^{-9}$, $n_s = 0.96$, $k_p = 0.05\ \mathrm{Mpc}^{-1}$, $\eta_0 \approx 16\ \mathrm{Gpc}$, $k_{eq} \approx 0.01\ \mathrm{Mpc}^{-1}$ $\leftarrow k_{eq} \sim \mathcal{H}_{eq} = \frac{H(z=3400)}{3400}$

$\rightarrow \langle |\Delta p|^2 \rangle \approx 9.9 \times 10^{-7}$ rad $\approx$ 0.003 arcmin not order 1 !

Problem 4:   define $\bar{d}^3k = \frac{d^3k}{(2\pi)^3}$.

part 1.

$$R(\vec{x}) = G(\vec{x}) + f\left(G(\vec{x})^2 - \langle G^2\rangle\right) \qquad \langle G^2\rangle = \int \bar{d}^3k \; P_G(\vec{k})$$

$$\begin{aligned} R(\vec{k}) &= \mathcal{F}[R(\vec{x})] \\ &= \mathcal{F}[G(\vec{x})] + f\left(\mathcal{F}[G(\vec{x})^2] - \mathcal{F}[1]\langle G^2\rangle\right) \\ &= \boxed{G(\vec{k}) + f\left(\int \bar{d}^3k' \, G(\vec{k}')\, G(\vec{k}-\vec{k}') - (2\pi)^3 \delta(\vec{k}) \langle G^2\rangle\right)} \end{aligned}$$

↑ by convolution theorem.

part 2.

$$\begin{aligned} \langle R(\vec{k}_1) R(\vec{k}_2) R(\vec{k}_3)\rangle &= \left\langle \left(G(\vec{k}_1) + f\, G_{NL}(\vec{k}_1)\right)\left(G(\vec{k}_2) + f\, G_{NL}(\vec{k}_2)\right)\left(G(\vec{k}_3) + f\, G_{NL}(\vec{k}_3)\right)\right\rangle \\ &= \cancel{\langle G_1 G_2 G_3\rangle}^{\,0} + f\langle G_{NL1} G_2 G_3\rangle + f\langle G_1 G_{NL2} G_3\rangle + f\langle G_1 G_2 G_{NL3}\rangle \\ &\quad + f^2\langle G_{NL1} G_{NL2} G_3\rangle + f^2 \langle G_{NL1} G_2 G_{NL3}\rangle + f^2\langle G_1 G_{NL2} G_{NL3}\rangle \\ &\quad + f^3 \cancel{\langle G_{NL1} G_{NL2} G_{NL3}\rangle}^{\,0 \text{ by Isserlis' Thm}} \end{aligned}$$

$$\begin{aligned} \langle G_{NL1} G_2 G_3\rangle &= \int \bar{d}^3k' \left\langle G(\vec{k}') G(\vec{k}_1-\vec{k}') G(\vec{k}_2) G(\vec{k}_3)\right\rangle - (2\pi)^3 \delta(\vec{k}_1)\langle G^2\rangle \left\langle G(\vec{k}_2) G(\vec{k}_3)\right\rangle \\ &= \int \bar{d}^3k' \left[\langle G(\vec{k}') G(\vec{k}_1-\vec{k}')\rangle \langle G_2 G_3\rangle + \langle G(\vec{k}') G_2\rangle\langle G(\vec{k}_1-\vec{k}') G_3\rangle + \langle G(\vec{k}') G_3\rangle \langle G(\vec{k}_1-\vec{k}') G_2\rangle\right] \text{ by Isserlis' Thm} \\ &\quad - (2\pi)^3 \delta(\vec{k}_1) \langle G^2\rangle \langle G(\vec{k}_2) G(\vec{k}_3)\rangle \end{aligned}$$

$$\langle G(\vec{k})\, G(\vec{p})\rangle = \langle G(\vec{k})\, G^*(-\vec{p})\rangle = (2\pi)^3 \delta(\vec{k}+\vec{p})\, P(\vec{k})$$

$$\begin{aligned} &= \int \bar{d}^3k' (2\pi)^6 \left[\delta(\vec{k}_1)\delta(\vec{k}_2+\vec{k}_3) P(k') P(k_2) + \delta(\vec{k}'+\vec{k}_2)\delta(\vec{k}_1+\vec{k}_3-\vec{k}') P(k') P(k_3) \right. \\ &\qquad \left. + \delta(\vec{k}'+\vec{k}_3)\delta(\vec{k}_1+\vec{k}_2-\vec{k}') P(k') P(k_2)\right] - (2\pi)^3 \delta(\vec{k}_1) \langle G^2\rangle (2\pi)^3 \delta(\vec{k}_2+\vec{k}_3) P(k_2) \\ &= 2(2\pi)^3 \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3) P(k_2) P(k_3) \end{aligned}$$

$$\begin{aligned} \langle G_{NL1} G_{NL2} G_3\rangle &= \int \bar{d}^3k_1' \, \bar{d}^3k_2' \cancel{\left\langle G(\vec{k}_1') G(\vec{k}_1-\vec{k}_1') G(\vec{k}_2') G(\vec{k}_2-\vec{k}_2') G(\vec{k}_3)\right\rangle}^{\,0} \\ &\quad - \int \bar{d}^3k_1' (2\pi)^3 \delta(\vec{k}) \langle G^2\rangle \cancel{\left\langle G(\vec{k}_1') G(\vec{k}_1-\vec{k}_1') G(\vec{k}_3)\right\rangle}^{\,0} \\ &\quad - \int \bar{d}^3k_2' (2\pi)^3 \delta(\vec{k}_1) \langle G^2\rangle \cancel{\left\langle G(\vec{k}_2') G(\vec{k}_2-\vec{k}_2') G(\vec{k}_3)\right\rangle}^{\,0} \\ &\quad + (2\pi)^6 \delta(\vec{k}_1)\delta(\vec{k}_2) \langle G^2\rangle^2 \cancel{\langle G(\vec{k}_3)\rangle}^{\,0} \\ &= 0. \end{aligned}$$

all $\langle \cdot \rangle \to 0$ since they all have an odd # factors of G

So,

$$\begin{aligned} \langle R(\vec{k}_1) R(\vec{k}_2) R(\vec{k}_3)\rangle &= 2(2\pi)^3 f\, \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)\left[P(k_1)P(k_2) + (k_1,k_3) + (k_2,k_3)\right] \\ &= (2\pi)^3 \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)\; B(k_1,k_2,k_3) \end{aligned}$$

for $\boxed{B = 2f\left[P(k_1)P(k_2) + (k_1,k_3) + (k_2,k_3)\right] = f\left[P(k_1)P(k_2) + \text{all perms}_{k_1,k_2,k_3}\right]}$

part 3. $R(\vec{k}) = G(\vec{k}) + g\,\mathcal{F}[G^3]$

$$\mathcal{F}[G^3] = (2\pi)^{-6}\,\mathcal{F}[G] * \mathcal{F}[G] * \mathcal{F}[G] \quad \text{by conv thm, associativity of } *$$

$$= (2\pi)^{-3}\left[\int d^3k_1\; G(\vec{k}_1)\, G(\vec{k}-\vec{k}_1)\right] * \mathcal{F}[G]$$

$$= \int d^3k_1\, d^3k_2\;\; G(\vec{k}_1)\, G(\vec{k}_2-\vec{k}_1)\, G(\vec{k}-\vec{k}_2)$$

$$\rightarrow\; \boxed{R(\vec{k}) = G(\vec{k}) + g\,\underbrace{\int d^3k_1\, d^3k_2\;\; G(\vec{k}_1)\, G(\vec{k}_2-\vec{k}_1)\, G(\vec{k}-\vec{k}_2)}_{\equiv\, G_{NL}}\;.}$$

part 4.

$$\langle R_1R_2R_3R_4\rangle = \left\langle (G_1+gG_1^{NL})(G_2+gG_2^{NL})(G_3+gG_3^{NL})(G_4+gG_4^{NL})\right\rangle$$

$$= \langle G_1G_2G_3G_4\rangle + g\langle G_1^{NL}G_2G_3G_4\rangle + g\langle G_1G_2^{NL}G_3G_4\rangle + g\langle G_1G_2G_3^{NL}G_4\rangle + g\langle G_1G_2G_3G_4^{NL}\rangle + \mathcal{O}(g^2)$$

$$\langle R_1R_2\rangle\langle R_3R_4\rangle = \left\langle (G_1+gG_1^{NL})(G_2+gG_2^{NL})\right\rangle\left\langle (G_3+gG_3^{NL})(G_4+gG_4^{NL})\right\rangle$$

$$= \left(\langle G_1G_2\rangle + g\langle G_1^{NL}G_2\rangle + g\langle G_1G_2^{NL}\rangle\right)\left(\langle G_3G_4\rangle + g\langle G_3^{NL}G_4\rangle + g\langle G_3G_4^{NL}\rangle\right) + \mathcal{O}(g^2)$$

$$= \langle G_1G_2\rangle\langle G_3G_4\rangle + g\langle G_1^{NL}G_2\rangle\langle G_3G_4\rangle + g\langle G_1G_2^{NL}\rangle\langle G_3G_4\rangle + g\langle G_1G_2\rangle\langle G_3^{NL}G_4\rangle + g\langle G_1G_2\rangle\langle G_3G_4^{NL}\rangle + \mathcal{O}(g^2)$$

$$\langle G_1^{NL}G_2\rangle = \int d^3p\, d^3q\; \langle G(\vec{p})\, G(\vec{q}-\vec{p})\;\; G(\vec{k}_1-\vec{q})\, G(\vec{k}_2)\rangle$$

$$= \int d^3p\, d^3q\Big[\langle G(\vec{p})\, G(\vec{q}-\vec{p})\rangle\langle G(\vec{k}_1-\vec{q})\, G(\vec{k}_2)\rangle + \langle G(\vec{p})\, G(\vec{k}_1-\vec{q})\rangle\langle G(\vec{q}-\vec{p})\, G(\vec{k}_2)\rangle$$
$$+ \langle G(\vec{p})\, G(\vec{k}_2)\rangle\langle G(\vec{k}_1-\vec{q})\, G(\vec{q}-\vec{p})\rangle\Big]$$

$$= (2\pi)^6\int d^3p\, d^3q\,\Big[\delta(q)\,\delta(\vec{k}_1+\vec{k}_2-\vec{q})\, P(p)\, P(k_2) + \delta(\vec{k}_1+\vec{p}-\vec{q})\,\delta(\vec{k}_2+\vec{q}-\vec{p})\, P(p)\, P(k_2) + \delta(\vec{k}_2+\vec{p})\,\delta(\vec{k}_1-\vec{p})\, P(p)\, P(\vec{q}-\vec{p})\Big]$$

$$= 3\,(2\pi)^6\,\delta(\vec{k}_1+\vec{k}_2)\, P(k_2)\,\langle G^2\rangle$$

$$\langle G_1^{NL}G_2G_3G_4\rangle = \int d^3p\; d^3q\;\; \langle G(\vec{p})\, G(\vec{q}-\vec{p})\, G(\vec{k}_1-\vec{q})\, G(\vec{k}_2)\, G(\vec{k}_3)\, G(\vec{k}_4)\rangle$$

$$= \int d^3p\; d^3q\;\; \langle p, q-p\rangle\langle k_1-q, k_2\rangle\langle k_3,k_4\rangle + \langle p, k_1-q\rangle\langle q-p, k_2\rangle\langle k_3,k_4\rangle + \langle p,k_2\rangle\langle k_1-q, q-p\rangle\langle k_3,k_4\rangle$$
$$+ \langle p,k_2\rangle\langle k_1-q, k_3\rangle\langle q-p, k_4\rangle + \langle p,k_2\rangle\langle k_1-q, k_4\rangle\langle q-p, k_3\rangle + \underset{2,3,4}{\text{cyclic}}$$

$$= \left(\langle G_1^{NL}G_2\rangle\langle G_3G_4\rangle + \underset{2,3,4}{\text{cyclic}}\right) + \left(\int d^3p\;\; d^3q\;\; \langle p,k_2\rangle\langle k_1-q, k_3\rangle\langle q-p, k_4\rangle + \underset{2,3,4}{\text{perms}}\right)$$

$$= \left(\langle G_1^{NL}G_2\rangle\langle G_3G_4\rangle + \underset{2,3,4}{\text{cyclic}}\right) + 6\,(2\pi)^3\,\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\;\; P(k_2)\, P(k_3)\;\; P(k_4)$$

$$\frac{1}{g}\langle R_1R_2R_3R_4\rangle_c = \left[\langle G_1^{NL}G_2G_3G_4\rangle - \left(\langle G_1^{NL}G_2\rangle\langle G_3G_4\rangle + \underset{2,3,4}{\text{cyclic}}\right) + \underset{1,2,3,4}{\text{cyclic}}\right]$$

$$= 6\,(2\pi)^3\,\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\;\; P(k_2)\, P(k_3)\;\; P(k_4) \;+\; \underset{1,2,3,4}{\text{cyclic}}$$

$$\rightarrow\; \langle R_1R_2R_3R_4\rangle_c = (2\pi)^3\,\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\;\; T(k_1,k_2,k_3,k_4)$$

$$\text{where}\;\; \boxed{T = 6g\left(P(k_1)\,P(k_2)\,P(k_3) + \underset{1,2,3,4}{\text{cyclic}}\right) = g\left(P(k_1)\,P(k_2)\,P(k_3) + \underset{1,2,3,4}{\text{all perms}}\right)}$$