Extragalactic Astrophysics HW3

Connor Hainje

For this homework, I've decided to do **Detectors, Analytic Exercise #2**. Here is the problem statement.

> It is a commonly used practice to combine different observations of a quantity (say flux) by using a weighted mean, using weights equal to the inverse variance of each observation. The advantage of doing so is that if the inverse variance is known, this weighted mean is the minimum variance estimator of the quantity itself. For imaging conducted with photon counting detectors, when the background is minimal, the noise in a data set is the Poisson noise in the expectation value of the signal. However, observationally we often only have access to one realization of the signal, and often the quoted errors are based on the Poisson noise estimated from the signal itself. If we take multiple observations and combine them together weighting by the inverse variance estimated in this way, it leads to a bias. Ignoring any background contribution to the noise, estimate this bias as a function of the true expected number of photons $\bar{n}$.

Suppose we have observations n_i which are sampled from a Poisson distribution of true mean $\bar{n}$. What is the signal we estimate from these observations if we take the noise estimate for each observation to be $\sigma_i^2 = n_i$?

As in the problem statement, we combine the observations by a weighted mean, with observations weighted by the inverse variance. The combined signal estimate is then given by

$$\begin{aligned} \text{signal} &= \frac{\sum_i n_i \sigma_i^{-2}}{\sum_i \sigma_i^{-2}} && (1) \\ &= \frac{\sum_i n_i n_i^{-1}}{\sum_i n_i^{-1}} && (2) \\ &= \frac{N}{\sum_i n_i^{-1}} && (3) \\ &= \left(\frac{1}{N} \sum_i n_i^{-1} \right)^{-1} && (4) \end{aligned}$$

The bias is given by signal $- \bar{n}$. We can find the expected value of the bias by computing the expected value of the signal, given that n_i are observations of random variable $n \sim \text{Pois}(\bar{n})$. Note then that the signal above is an estimate of

$$\left(\mathbb{E}[n^{-1}] \right)^{-1} \tag{5}$$

which we can compute. This expectation value is given by

$$\mathbb{E}[n^{-1}] = \sum_{k=0}^{\infty} \left(\frac{1}{k}\right) \frac{\bar{n}^k e^{-\bar{n}}}{k!} \tag{6}$$

$$= e^{-\bar{n}} \sum_{k=0}^{\infty} \frac{\bar{n}^k}{k\ k!}. \tag{7}$$

As stated, this is ill posed, as the summand contains $1/0$ for $k = 0$. We'll try to patch this up by assuming $\sigma_i^2 = 1$ if $n_i = 0$, but this will probably cause us to incur an error. Making this substitution, we have

$$\mathbb{E}[n^{-1}] = e^{-\bar{n}} \sum_{k=0}^{\infty} \frac{\bar{n}^k}{k\ k!} \tag{8}$$

$$= e^{-\bar{n}} \left[1 + \sum_{k=1}^{\infty} \frac{\bar{n}^k}{k\ k!}\right]. \tag{9}$$

Mathematica allows us to evaluate the series, obtaining

$$\sum_{k=1}^{\infty} \frac{\bar{n}^k}{k\ k!} = -\gamma + \mathrm{Ei}(\bar{n}) - \log(\bar{n}), \tag{10}$$

where γ is the Euler Gamma constant ($\gamma \approx 0.577$) and Ei is the exponential integral function. Hence, the expected bias can be written

$$\mathrm{bias} = \left(\mathbb{E}[n^{-1}]\right)^{-1} - \bar{n} \tag{11}$$

$$= \left(e^{-\bar{n}}\left[1 - \gamma + \mathrm{Ei}(\bar{n}) - \log(\bar{n})\right]\right)^{-1} - \bar{n} \tag{12}$$

$$= \frac{e^{\bar{n}}}{1 - \gamma + \mathrm{Ei}(\bar{n}) - \log(\bar{n})} - \bar{n}. \tag{13}$$

We can plot this for a range of $\bar{n}$ values. The result is shown in Figure 1. The figure shows that the bias starts at 1 and decreases sharply with $\bar{n}$ until around $\bar{n} \sim 6$, after which is plateaus around -1.

We can also estimate the bias numerically. We'll do this as follows.

- Given a value of $\bar{n}$, draw 1 000 samples from $\mathrm{Pois}(\bar{n})$.
- Assign each sample a variance estimate $\sigma_i^2 = \max(n_i, 1)$ and compute the bias of the combined signal estimate bias $= \sum_i n_i \sigma_i^{-2} / \sum_i \sigma_i^{-2} - \bar{n}$.
- Repeat this procedure 1 000 times to get 1 000 estimates of the bias, and take the median of the resulting distribution.
- Repeat for additional values of $\bar{n}$.

Carrying out this procedure for 20 values of $\bar{n}$ between 10^{-4} and 20 (using 10^{-4} to avoid divisions by zero) gives a plot of median numerical bias versus $\bar{n}$,

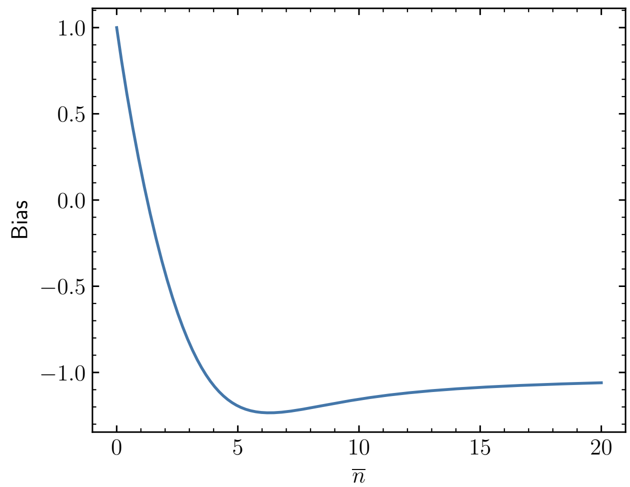

Figure 1: Analytic estimate of the expected bias for a range of the true mean $\bar{n}$.

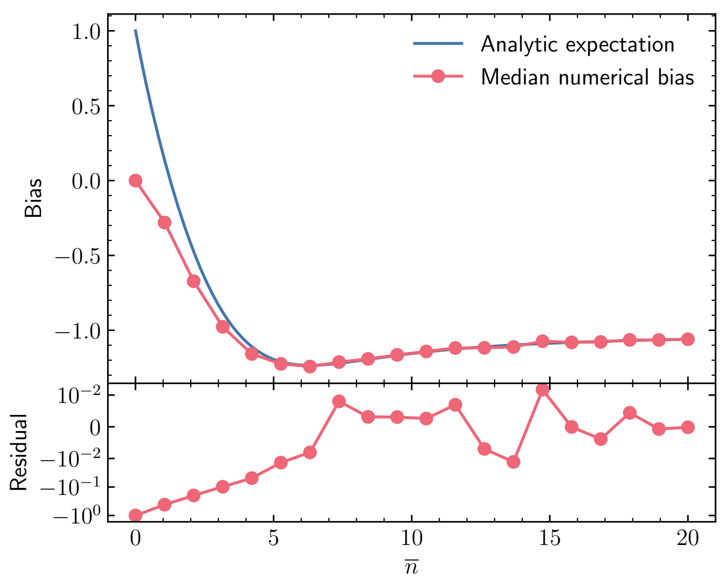

Figure 2: Numerical bias estimate computed by drawing samples from Poisson distributions many times and computing the actual average bias from combining the signals in the given way. Also plotted is the same curve as Figure 1 for comparison. The two methods agree well for $\bar{n} \gtrsim 6$, but disagree strongly at small $\bar{n}$.

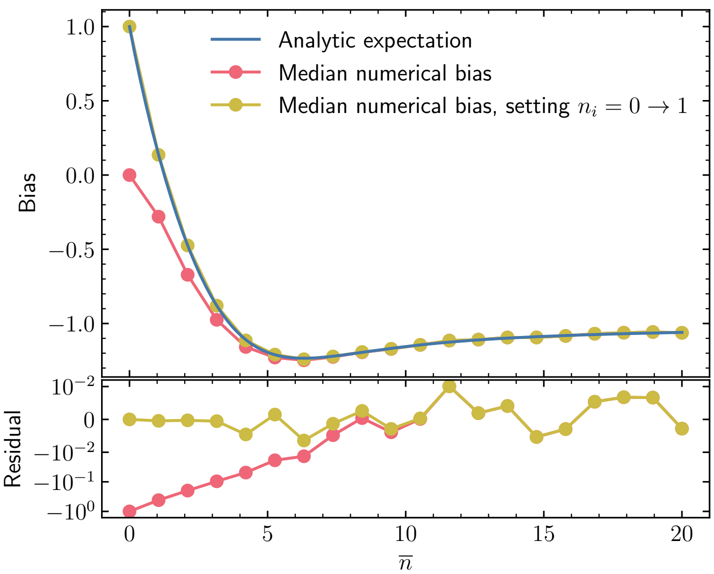

Figure 3: Figure 2 reproduced with an additional curve showing the result of setting all samples $n_i = 0$ to 1 (not just their associated variance estimates). This verifies that the difference between our analytic and numerical estimates is due to this, and as such the numerical estimate is "more" correct, probably.

which is shown in Figure 2. Plotted alongside these results are the numerical estimates.

We see that for $\bar{n} \gtrsim 6$, the two methods agree to better than a percent, supporting our calculation. For small $\bar{n}$, though, we become sensitive to the $n_i = 0$ issue, and the two methods diverge.

This is because using a value of $\sigma_i^2 = 1$ when $n_i = 0$ (as is done in the numerical estimate) is somewhat different from computing

$$\mathbb{E}[1/n] \approx \sum_{k=0}^{\infty} P(k|\overline{n}) \max(k,1)^{-1}, \tag{14}$$

as this second one is actually changing the $n_i = 0 \to 1$ as well. We can verify this by re-running the numerical estimate, and setting both n_i and σ_i^2 to 1 whenever our sample of $n_i = 0$. The result is given in Figure 3 and verifies the claim. In this way, the numerical estimate is probably "more" correct, as it makes a weaker assumption about what to do when $n_i = 0$.

The code is available online at the following links: Mathematica notebook and Jupyter notebook.