Extragalactic Astrophysics HW7

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Dynamics, Analytic Exercise #5.

Using the spherical symmetric first-order Jeans Equation,

$$\partial_r(n\sigma_{rr}^2) + \frac{n}{r}\left[2\sigma_{rr}^2 - (\sigma_{\theta\theta}^2 + \sigma_{\phi\phi}^2)\right] = -n\partial_r\Phi, \tag{1}$$

show that

$$M(<r) = -\frac{r\sigma_{rr}^2}{G}\left[\frac{\mathrm{d}\ln n}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln\sigma_{rr}^2}{\mathrm{d}\ln r} + 2\beta\right]. \tag{2}$$

Note that, due to spherical symmetry, n and σ_{rr}^2 and Φ are functions only of r, and thus we can switch between partial and total derivatives w.r.t. r without an issue.

Let's begin from the Jeans Equation. Under spherical symmetry, we have $\sigma_{\theta\theta}^2 = \sigma_{\phi\phi}^2$, so

$$\partial_r(n\sigma_{rr}^2) + \frac{2n}{r}\left(\sigma_{rr}^2 - \sigma_{\theta\theta}^2\right) = -n\partial_r\Phi. \tag{3}$$

We can also introduce the anisotropy parameter $\beta \equiv 1 - \sigma_{\theta\theta}^2/\sigma_{rr}^2$. Plugging this in simplifies the Jeans Equation to

$$\partial_r(n\sigma_{rr}^2) + \frac{2n}{r}\sigma_{rr}^2\beta = -n\partial_r\Phi. \tag{4}$$

Now we want to derive an expression for the total enclosed mass, which requires us consider the relation between the gravitational potential Φ and the mass density ρ. This relation is

$$\nabla^2\Phi = 4\pi G\rho. \tag{5}$$

The enclosed mass can then be written in terms of Φ as

$$\begin{aligned}
M(<r) &= \int_0^r \mathrm{d}r'\ 4\pi r'^2\rho &(6)\\
&= \int_0^r \mathrm{d}r'\ \frac{r'^2}{G}\nabla'^2\Phi &(7)\\
&= \frac{1}{G}\int_0^r \mathrm{d}r'\ \partial_{r'} r'^2 \partial_{r'}\Phi &(8)\\
&= \frac{r^2}{G}\partial_r\Phi. &(9)
\end{aligned}$$

Now $\partial_r \Phi$ is a term in the Jeans Equation, which we can solve for and plug in. This gives

$$M(<r) = -\frac{1}{G}\left[2r\sigma_{rr}^2\beta + \frac{r^2}{n}\partial_r(n\sigma_{rr}^2)\right] \tag{10}$$

$$= -\frac{r\sigma_{rr}^2}{G}\left[2\beta + \frac{r}{n\sigma_{rr}^2}\partial_r(n\sigma_{rr}^2)\right]. \tag{11}$$

Finally, we can identify the logarithmic derivative:

$$\frac{\mathrm{d}\ln f}{\mathrm{d}\ln x} = \frac{\mathrm{d}x}{\mathrm{d}\ln x}\frac{\mathrm{d}\ln f}{\mathrm{d}x} = \frac{x}{f}\frac{\mathrm{d}f}{\mathrm{d}x}. \tag{12}$$

Making this replacement yields

$$M(<r) = -\frac{r\sigma_{rr}^2}{G}\left[2\beta + \frac{\mathrm{d}\ln(n\sigma_{rr}^2)}{\mathrm{d}\ln r}\right] \tag{13}$$

or, expanding using log rules,

$$M(<r) = -\frac{r\sigma_{rr}^2}{G}\left[2\beta + \frac{\mathrm{d}\ln n}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln \sigma_{rr}^2}{\mathrm{d}\ln r}\right], \tag{14}$$

as desired.