Extragalactic Astrophysics HW5

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Structure Formation, Analytic Exercise #5.

> Consider a spherical region with mean overdensity $\bar{\delta} > 0$, within an expanding universe with no cosmological constant. As long as there is no *shell crossing* — that is, material at one radius does not catch up to material at another radius — the equations governing the radius of this sphere over time are
>
> $$\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -\frac{GM(< R)}{R^2} = -\frac{4\pi G}{3}\bar{\rho}\left(1 + \bar{\delta}\right) R \tag{1}$$

part (a)

> In terms of Ω_m at the present time, what is the condition that the spherical region will collapse on itself?

Let's begin with the first equality in Equation 1. We can massage it slightly to obtain

$$\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} + \frac{GM}{R^2} = 0. \tag{2}$$

Note that we drop the $(< R)$ for the enclosed mass because the mass in this spherical region is conserved (due to the assumption of no shell crossing), and so M is a constant.

We integrate this equation over R. The first term integrates as

$$\int \frac{\mathrm{d}^2 R}{\mathrm{d}t^2}\,\mathrm{d}R = \int \frac{\mathrm{d}^2 R}{\mathrm{d}t^2}\frac{\mathrm{d}R}{\mathrm{d}t}\,\mathrm{d}t = \frac{1}{2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2. \tag{3}$$

The second term gives

$$\int \frac{GM}{R^2}\,\mathrm{d}R = -\frac{GM}{R}, \tag{4}$$

and the integral of 0 is a constant, which we'll call E. Thus,

$$\frac{1}{2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - \frac{GM}{R} = E. \tag{5}$$

Mukhanov §1.2.3 points out that this is the same form as the calculation of the escape velocity for an object to escape the gravitational field of some body. In particular, R can be seen as describing the height of the object. For that scenario, $E > 0$ means that the object is able to escape, where $E < 0$ describes the object gaining some height before turning around and falling back down. In our case, R increasing to some maximum before turning around and crashing to zero is exactly the behavior of collapse, so our condition is that $E < 0$.

For $E < 0$ to be true, we must have $\dot{R}^2 < 2GM/R$. We can use the fact that we know the mass M in terms of the mean density of the universe $\bar{\rho}$, the mean overdensity of the sphere $\bar{\delta}$ and the radius of the sphere R to re-arrange this into

$$\begin{aligned} \dot{R}^2 &< \frac{2G}{R}\frac{4\pi}{3}\bar{\rho}\left(1+\bar{\delta}\right)R^3 \\ &< \frac{8\pi G}{3}\bar{\rho}\left(1+\bar{\delta}\right)R^2. \end{aligned} \tag{6}$$

We also know that the matter density parameter today is given by

$$\Omega_m = \frac{8\pi G}{3H^2}\bar{\rho}, \tag{7}$$

which we can use to replace $\bar{\rho}$, yielding

$$\dot{R}^2 < H^2\Omega_m\left(1+\bar{\delta}\right)R^2. \tag{8}$$

Rearranging gives

$$\frac{(\dot{R}/R)^2}{H^2} < \Omega_m\left(1+\bar{\delta}\right). \tag{9}$$

Initially, the spherical overdensity will be expanding due to the expansion of the universe, so $\dot{R}/R = H$ at $t = 0$. Since $\bar{\delta}$ describes the initial overdensity, the entire inequality is technically being considered at the initial time, so we can use this! Cancelling these terms, we finally obtain

$$\Omega_m\left(1+\bar{\delta}\right) > 1, \tag{10}$$

as was seen in the notes.

part (b)

> Demonstrate that the solutions to the above equation can be expressed as
>
> $$\begin{aligned} R/R_m &= (1-\cos\eta)/2, \\ t/t_m &= (\eta-\sin\eta)/\pi, \end{aligned} \tag{11}$$
>
> where at time t_m the sphere reaches its maximum radius R_m, before collapsing.

First, let's analyze the ansatz. t/t_m is a monotonically increasing function of η, which on average increases by 1 for an increase in η of π. As such, we can think of η as a parameterization of time. Then, R/R_m is a simple cosine varying from 0 to 1 with a period of 2π. It starts at zero, increases to 1 at $\eta = \pi$ ($t = t_m$), then decreases back to zero at $\eta = 2\pi$ ($t = 2t_m$). I don't think it really makes sense to continue the solution beyond $\eta = 2\pi$, so we'll limit our analysis to this first period only.

OK, now let's try out the ansatz. Given the behavior we described above, we can compute E. Because we know that R turns around when $R = R_m$, $\mathrm{d}R/\mathrm{d}\eta = 0$ at that point.[1] Since E is conserved, we can just compute it at this moment, and we know that it will hold for all η. We find

$$E = -\frac{GM}{R_m}, \tag{12}$$

which is certainly negative.

Next, we can evaluate the expression which we defined to be equal to E. To start, let's evaluate $\mathrm{d}R/\mathrm{d}t$.

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}t}\frac{\mathrm{d}R}{\mathrm{d}\eta} = \frac{R'}{t'}, \tag{13}$$

where we are now considering R and t to be functions of η alone, and the prime denotes a derivative (w.r.t. η). Taking these derivatives of the ansatz, we find

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{(R_m/2)\sin\eta}{(t_m/\pi)(1-\cos\eta)} = \frac{\pi R_m}{2t_m}\frac{\sin\eta}{1-\cos\eta}. \tag{14}$$

Now we can evaluate

$$\begin{aligned} E &= \frac{1}{2}\left(\frac{\pi R_m}{2t_m}\frac{\sin\eta}{1-\cos\eta}\right)^2 - \frac{2GM}{R_m}\frac{1}{1-\cos\eta} \\ &= \frac{1}{(1-\cos\eta)^2}\left[\frac{\pi^2 R_m^2}{8t_m^2}\sin^2\eta - \frac{2GM}{R_m}(1-\cos\eta)\right]. \end{aligned} \tag{15}$$

We require this to reduce to $E = -GM/R_m$. Let's keep manipulating this expression until we get something obvious.

$$\begin{aligned} E &= \frac{1}{(1-\cos\eta)^2}\left[\frac{\pi^2 R_m^2}{8t_m^2}\sin^2\eta - \frac{2GM}{R_m}(1-\cos\eta)\right] \\ &= \frac{GM}{R_m}\frac{1}{(1-\cos\eta)^2}\left[\frac{\pi^2 R_m^3}{8GMt_m^2}(1-\cos^2\eta) - 2(1-\cos\eta)\right] \end{aligned} \tag{16}$$

Now, if $(\pi^2 R_m^3)/(8GMt_m^2)$ were equal to 1, we would find

$$\begin{aligned} E &= \frac{GM}{R_m}\frac{1}{(1-\cos\eta)^2}\left[(1-\cos^2\eta) - 2(1-\cos\eta)\right] \\ &= -\frac{GM}{R_m}\frac{1}{(1-\cos\eta)^2}\left[1 - 2\cos\eta + \cos^2\eta\right] \\ &= -\frac{GM}{R_m}. \end{aligned} \tag{17}$$

[1] Actually, regardless of the ansatz we choose to take, this must hold for any solution which is initially expanding, then collapsing, and the maximum radius is labeled R_m.

So, the given ansatz describes a one-parameter family of collapsing solutions with

$$E = -GM/R_m, \quad t_m^2 = \frac{\pi^2 R_m^3}{8GM}. \tag{18}$$

I'm not sure how to (or if one can) show this is the only family of collapsing solutions, but we've certainly verified that is one such family.

part (c)

> Show that at time t_m, the density of the sphere relative to the mean density of the universe will be $\rho_m/\bar{\rho}(t_m) = 9\pi^2/16$.

Taking R_m as given, we can compute the (mean) density of the sphere at its maximum radius by just dividing the constant mass M by the volume:

$$\rho_m = \frac{M}{V_m} = \frac{M}{(4\pi/3)R_m^3} = \frac{3M}{4\pi}\frac{\pi^2}{8GMt_m^2} = \frac{3\pi}{32Gt_m^2}. \tag{19}$$

Next we need to handle the evolution of the mean density of the universe. During matter domination, which is when I assume this problem is taking place, we have $\bar{\rho} \propto a^{-3}$, and $a \propto t^{2/3}$. Further, we know

$$H^2 = \frac{8\pi G}{3}\bar{\rho} \tag{20}$$

under the assumption of a flat universe ($\bar{\rho} = \rho_c$), which is (I think) the same as assuming matter domination. As usual, $H = \dot{a}/a$, and we can take our proportionality relation for a above to find $\dot{a} = 2a/3t$. Plugging this in, we find

$$\frac{4}{9t^2} = \frac{8\pi G}{3}\bar{\rho} \tag{21}$$

which reduces to

$$\bar{\rho} = \frac{1}{6\pi G t^2}. \tag{22}$$

Putting these together gives

$$\frac{\rho_m}{\bar{\rho}(t_m)} = \frac{3\pi}{32Gt_m^2}\frac{6\pi G t_m^2}{1} = \frac{9\pi^2}{16} \tag{23}$$

as desired.

part (d)

> The collapse of the sphere will proceed in reverse, and will therefore take t_m to do so. However, upon full collapse shell-crossing will occur, because the collisionless dark matter will pass through the origin and oscillate around it. This process can be modeled to derive the detailed structure

of the resulting halo mass profile, but the virial theorem ($U = -2K$) can tell us about its overall size. Show that the final characteristic radius of the resulting *virialized* halo is $R_{\rm vir} = R_m/2$.

The total energy E is conserved. It's easy to compute at t_m when $\mathrm{d}R/\mathrm{d}t = 0$ and so the kinetic energy is zero. The potential energy is proportional to $-GM/R_m$; for our purposes we can just drop this constant of proportionality. Thus the total energy is $E = -GM/R_m$.

After virialization, we'll have $E = K + U$ and $U = -2K$, but $E = -GM/R_m$ still. Thus, $K = GM/R_m$ and $U = -2GM/R_m$. We also know that the gravitational potential energy of this final configuration will be $U = -GM/R_{\rm vir}$, where $R_{\rm vir}$ is the final radius of the virialized halo. Comparing these two values of U, we see

$$-2GM/R_m = -GM/R_{\rm vir} \implies R_{\rm vir} = R_m/2. \tag{24}$$

part (e)

Show that the mean overdensity within the resulting halo is $\delta_{\rm vir} = 18\pi^2$.

We can directly compute the mean density of the resulting halo as

$$\rho = \frac{M}{V} = \frac{3M}{4\pi R_{\rm vir}^3} = \frac{6M}{\pi R_m^3} = \frac{3\pi}{4Gt_m^2}. \tag{25}$$

We can use equation 22 again to find the mean density of the universe after virialization:

$$\bar{\rho} = \bar{\rho}(2t_m) = \frac{1}{24\pi G t_m^2}. \tag{26}$$

The ratio of these two gives

$$1 + \delta_{\rm vir} = \frac{\rho}{\bar{\rho}} = \frac{3\pi}{4Gt_m^2}(24\pi G t_m^2) = 18\pi^2, \tag{27}$$

which isn't exactly what we were asked to show but looks really close!

part (f)

By linearizing the equations 11, show that the linearly extrapolated overdensity at the time of collapse is $\delta_{\rm lin}(2t_m) \approx 1.686$.

We begin by series expanding equations 11 to obtain

$$R/R_m = \frac{1}{2}(1 - \cos\eta) = \frac{\eta^2}{4} - \frac{\eta^4}{48} + \cdots, \tag{28}$$

$$t/t_m = \frac{1}{\pi}(\eta - \sin\eta) = \frac{1}{\pi}\left(\frac{\eta^3}{6} - \frac{\eta^5}{120} + \cdots\right). \tag{29}$$

I plugged the second series into Mathematica's `InvertSeries` function to obtain

$$\eta^2 = (6\pi t/t_m)^{2/3}\left[1 + \frac{1}{30}(6\pi t/t_m)^{2/3} + \cdots\right]. \tag{30}$$

Plugging this into the R/R_m series gives

$$\begin{aligned} R/R_m &= \frac{1}{4}(6\pi t/t_m)^{2/3}\left[1 + \frac{1}{30}(6\pi t/t_m)^{2/3}\right] - \frac{1}{48}(6\pi t/t_m)^{4/3} + \cdots \\ &= \frac{1}{4}(6\pi t/t_m)^{2/3}\left[1 - \frac{1}{20}(6\pi t/t_m)^{2/3}\right] + \cdots . \end{aligned} \tag{31}$$

Now we want to repeat part (e) using the linearized R/R_m. The mean density is

$$1 + \delta_{\rm lin}(t) = \frac{\rho}{\bar{\rho}(t)} = \frac{3M}{4\pi R(t)^3}(6\pi G t^2) = \frac{9GMt^2}{2R(t)^3}. \tag{32}$$

Using the series expansion, we estimate

$$\begin{aligned} R^{-3} &= R_m^{-3}\frac{64}{(6\pi t/t_m)^2}\left[1 - \frac{1}{20}(6\pi t/t_m)^{2/3}\right]^{-3} \\ &= \frac{2}{9GMt^2}\left[1 + \frac{3}{20}(6\pi t/t_m)^{2/3}\right. \end{aligned} \tag{33}$$

which can be inserted readily into $1 + \delta_{\rm lin}$ to yield

$$\delta_{\rm lin}(t) = \frac{3}{20}(6\pi t/t_m)^{2/3}. \tag{34}$$

Evaluating this at $t = 2t_m$, we find

$$\delta_{\rm lin}(2t_m) = \frac{3}{20}(12\pi)^{2/3} \approx 1.686, \tag{35}$$

as desired.