1 Introduction to combinatorics

Reference.

1. Ross, *A First Course in Probability*, Ch. 1
2. Rosen, *Discrete Mathematics and its Applications*, Ch. 6,8
3. Wade, *An Introduction to Analysis*, Ch.5.4, 7.4, 12.6

Everyone can count, but can you really count? Let's do a sanity check:

- Suppose Michael slept at 3 am and woke up at 10 am. How long did he sleep?
- Michael had a vacation from May 9 to May 31 (inclusive) of the same year. How long is his vacation?

If your answers are 7 hours and 23 days, congratulations! You are well equipped to study this note! In this note, we will study basic notions and applications of combinatorics, the mathematics of counting. Unless otherwise stated, every number is assumed to be a nonnegative integer.

1.1 Basics of counting

The most important yet fundamental theorem is as follows:

Theorem 1.1 (Fundamental theorem of counting). If a job contains k separate tasks, where the i^{th} task can be done in n_i ways, then the whole job can be done in $n_1 n_2 \cdots n_k$ ways.

Proof. By induction, it is suffice to prove for the case $k = 2$. We enumerate all possible ways to complete the tasks as follows:

$$\begin{array}{cccc} (1,1), & (1,2), & \cdots, & (1,n_2), \\ (2,1), & (2,2), & \cdots, & (2,n_2), \\ \vdots & \vdots & \ddots & \vdots \\ (n_1,1), & (n_1,2), & \cdots, & (n_1,n_2) \end{array}$$

where we say the whole job can be done by the $(i,j)^{\text{th}}$ way if task 1 can be done by its i^{th} way and task 2 can be done by its j^{th} way. Hence the set of possible ways consists of n_1 rows, each containing n_2 elements. Therefore there are $n_1 n_2$ ways to do the whole job. ■

Example 1. A standard license plate contains 6 places, where the first 2 places are to be occupied by letters other than '`I`','`O`' or '`Q`', and the last 4 places are to be occupied by numbers.

(a) What is the number of possible plates?

(b) If repetition among letters or numbers is not allowed, what is the number of possible plates?

Solution.

(a) $23 \times 23 \times 10 \times 10 \times 10 \times 10 = 5,290,000$.

(b) $23 \times 22 \times 10 \times 9 \times 8 \times 7 = 2,550,240$. ■

Example 2.

(a) How many functions are there from a set with m elements to a set with n elements?

(b) How many *injective* functions are there from a set with m elements to a set with n elements?

Solution.

(a) A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain, So there are $\underbrace{n \times n \times \cdots \times n}_{m\ n\text{'s}} = n^m$ functions.

(b) When $m > n$, no injective function exists[1]. For $m \leq n$, if we label to elements in the domain as $a_1, a_2, \cdots, a_m$, then there are n ways to assign a function value to a_1, $n-1$ ways to assign a function value to a_2, ..., and finally $n-m+1$ ways to assign a function value to a_m. In total there are $n(n-1)\cdots(n-m+1)$ such functions. ■

[1]This is intuitively true, but a rigorous justification requires the use of pigeonhole principle, which will be introduced in the next subsection. See corollary 1.14.

1.1.1 Permutation

An ordered arrangement without repetition is also called **permutation**. Example 1(b) already provided us with the first example of permutation. In general, when we have n objects, by fundamental theorem of counting, there are $n(n-1)\cdots 1$ different permutations. It would be convenient to define the following function:

Definition 1.2. For $n \in \mathbb{N}$, we define the **factorial** of n by

$$n! = n(n-1)\cdots 1.$$

We also define $0! = 1$.

It makes sense to define $0! = 1$ because we have exactly one way to permute 0 objects - which is to do nothing.

Example 3. Cookies is a girl group of 9 girls including Stephy, Theresa, and Kary.

(a) (i) In how many ways can the girls stand in a row?
(ii) In how many ways if Stephy and Kary must stand together?
(iii) In how many ways if Stephy and Theresa must not stand together?

(b) (i) In how many ways can the girls stand in *two* rows, with 4 in the first row and 5 in the second row?
(ii) In how many ways if Stephy and Kary must stand together?
(iii) In how many ways if Stephy and Theresa must not stand together?

Solution.

(a) (i) $9! = 362,880$.
(ii) We first consider Stephy and Kary as a bundle and permute them with the 7 other teammates, this accounts for the $8!$ factor. Then in the S-K bundle, there are $2!$ ways to permute them. Therefore the number of ways is $8! \times 2! = 80640$.
(iii) We just deduct the case that they stand together from the unrestricted case. Therefore the number of ways is $9! - 8! \times 2! = 282,240$.

(b) (i) The answer is still $9! = 362,880$. This is the same as arranging them in a row.
(ii) We first consider arranging them in a row, which accounts for the $8! \times 2!$ term. Then we subtract the number of ways that they stand in two rows, which is $7! \times 2!$. Therefore the number of ways is $8! \times 2! - 7! \times 2! = 70560$.
(iii) We just deduct the case that they stand together from the unrestricted cases. Therefore the number of ways is $9! - (8! \times 2! - 7! \times 2!) = 292,320$. ∎

Sometimes a proportion of our objects are indistinguishable, for example, we cannot distinguish between the `E`'s in the word `MELEE`.

Example 4. How many different letter arrangements can be formed from the letters
(a) `MELEE`? (b) `NIGERIAN`? (c) `MISSISSIPPI`?

Solution.

(a) We label the `E`'s in `MELEE` as $\mathtt{ME_1LE_2E_3}$ first, as if the `E`'s are distinguishable. This accounts for the $5!$ factor. But actually the `E`'s are indistinguishable, so

$$\begin{array}{ccc} \mathtt{ME_1LE_2E_3} & \mathtt{ME_2LE_1E_3} & \mathtt{ME_3LE_1E_2} \\ \mathtt{ME_1LE_3E_2} & \mathtt{ME_2LE_3E_1} & \mathtt{ME_3LE_2E_1} \end{array}$$

are the same - they are just `MELEE`. So each possible letter arrangement is counted for $3!$ times. Therefore the total number of arrangement is $\dfrac{5!}{3!} = 20$.

(b) $\dfrac{8!}{2!2!} = 10080$.

(c) $\dfrac{11!}{4!4!2!} = 34650$. ∎

1.1.2 Combination

We want to determine the number of different subsets of r objects that could be formed from a set of n objects. There are $n(n-1)\cdots(n-r+1) = \dfrac{n!}{(n-r)!}$ ways to select r objects out one by one. But since the order of selection is irrelevant, noting that each subset of r objects chosen are counted $r!$ times, it follows that the number of different subsets of r objects that could be formed from a set of n objects is given by

$$\frac{n!}{(n-r)!r!}.$$

Definition 1.3. Define the **binomial coefficient** of n choose r by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

whenever $0 \leq r \leq n$. We also define $\binom{n}{r} = 0$ if $r > n$.

Note that $\binom{n}{0} = \dfrac{n!}{n!0!} = 1$ and $\binom{n}{n} = \dfrac{n!}{0!n!} = 1$. What are their meanings?

Lemma 1.4. For $0 \leq r \leq n$, we have

$$\binom{n}{r} = \binom{n}{n-r}.$$

Proof.

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)![n-(n-r)]!} = \binom{n}{n-r}.$$

■

Example 5.

(a) A committee of 7 is to be formed from a class of 35 students. How many different committees are possible?

(b) A president, a treasurer, a secretary and 4 admins, all different, are to be chosen from a class of 35 students. How many different committees are possible?

Solution.

(a) $\binom{35}{7} = 6,724,520$.

(b) $35 \times 34 \times 33 \times \binom{32}{4} = 1,412,149,200$.

Alternatively: $\binom{35}{7} \times 7 \times 6 \times 5 = 1,412,149,200$. ■

Example 6.

(a) From a group of 6 men and 9 women, how many different committees consisting of 3 men and 6 women can be formed?

(b) What if two of the women will either serve together or not at all?

(c) What if two of the women refuse to serve together?

(d) What if one of the women refuses to serve together with one of the men?

Solution.

(a) $\binom{6}{3}\binom{9}{6} = 1680$

(b) The number of ways if they serve together is $\binom{6}{3}\binom{7}{4} = 700$. The number of ways if none of them serve is $\binom{6}{3}\binom{7}{6} = 140$. So the total number is $700 + 140 = 840$.

(c) $1680 - 700 = 980$.

(d) $1680 - \binom{5}{2}\binom{8}{5} = 1120$. ■

Example 7. 4 girls and 7 boys are standing in a row. How many possible ways are there if no two girls are standing together?

Solution. We first arrange the 7 boys, which accounts for a 7! factor. Now, if no two girls are standing together, then the space between two boys must contain at most 1 girl. We choose 4 of the 8 spaces to place a girl there. This accounts for a $\binom{8}{4} \times 4!$ factor. So the total number is $7! \times \binom{8}{4} \times 4! = 8,467,200$. ∎

$$\wedge \text{ B } \wedge \text{ B } \wedge \text{ B } \wedge \text{ B } \wedge \text{ B } \wedge \text{ B } \wedge \text{ B } \wedge$$
$$\text{B} = \text{boy}$$
$$\wedge = \text{place for at most one girl}$$

1.1.3 Properties of binomial coefficients and combinatorial proofs

In this subsubsection we present some combinatorial identities including the famous binomial formula. For each identity, we only present a combinatorial proof. It is noteworthy that every identity here has an algebraic proof, and readers are strongly recommended to formulate their own algebraic proofs as an exercise.

The philosophy of combinatorial proof is simple - we count the way of choosing object in two different ways, and they have to be equal.

Example 8. Reprove lemma 1.4 by a combinatorial proof: for $0 \leq r \leq n$, we have

$$\binom{n}{r} = \binom{n}{n-r}.$$

Proof. We have $\binom{n}{r}$ ways to choose r objects from n to form a subset. Alternatively, we can choose r objects from n by not choosing $n-r$ objects. We have $\binom{n}{n-r}$ ways not choosing $n-r$ objects, which establishes the identity. ∎

Lemma 1.5 (Pascal rule)**.** If $0 < r \leq n$, then

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}.$$

Proof. Consider a set of $n+1$ objects, where we fix attention on some particular one of these objects, say we call it object X. There are $\binom{n+1}{r}$ ways to choose r objects from this set of objects. On the other hand, there are $\binom{n}{r-1}$ subsets of size r that contains object X (since each subset is formed by selecting $r-1$ from the remaining n objects), and $\binom{n}{r}$ subsets of size r that does not contain object X. Thus the identity holds. ∎

Theorem 1.6 (Binomial formula)**.** Let R be a commutative ring and $x, y \in R$. Then for any $n \in \mathbb{N}$, it holds

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

Proof. Upon expansion, the terms are of the form $x^r y^{n-r}$. To determine the coefficient of $x^r y^{n-r}$, note that to obtain such a term amounts to choose r x's from the n $(x+y)$ terms. Thus the coefficient is $\binom{n}{r}$ as desired. ∎

Corollary 1.7. A set of n elements has 2^n subsets.

Proof. Since there are $\binom{n}{r}$ subsets containing r elements, the desired number is

$$\sum_{r=0}^{n} \binom{n}{r} = (1+1)^n = 2^n.$$

∎

At this stage, readers are expected to be highly familiar with binomial formula. We reprove the formula in order to illustrate the power of combinatorial proofs only. We present two more examples of combinatorial proof:

Example 9. Let $r \leq \max\{m, n\}$. Prove **Vandermonde's identity**

$$\binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$$

algebraically to persuade yourself this is true first. Then present a combinatorial proof.

Hints: For an algebraic proof, consider the binomial expansion of $(1+x)^{m+n}$. While for a combinatorial proof, consider forming a group of r people from n boys and m girls.

Proof.

(a) **Algebraic proof**:

Note that $(1+x)^{m+n} = (1+x)^m(1+x)^n$. Looking at the LHS, the coefficient of x^r is $\binom{n+m}{r}$. While looking at the RHS, the coefficient of x^r is $\displaystyle\sum_{i+j=r} \binom{n}{i}\binom{m}{j} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$ as desired.

(b) **Combinatorial proof**:

Consider forming a group of r people from n boys and m girls. Clearly we have $\binom{n+m}{r}$ ways to do so. On the other hand, we can also form $\binom{n}{k}\binom{m}{r-k}$ groups of exactly k boys, so summing through $0, 1, 2 \cdots, r$ gives the number of possible groups of r people. In other words, it holds $\displaystyle\binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$. ■

Example 10. Let $n \geq r$. Prove the **hockey stick identity**

$$\binom{n+1}{r+1} = \sum_{k=r}^{n} \binom{k}{r}$$

algebraically to persuade yourself this is true first. Then present a combinatorial proof.

Proof.

(a) **Algebraic proof**:

We perform induction on n:

(i) When $n = r$, we have

$$\binom{r+1}{r+1} = 1 = \binom{r}{r}.$$

(ii) Suppose $\displaystyle\binom{n+1}{r+1} = \sum_{k=r}^{n} \binom{k}{r}$. Then

$$\begin{aligned}
\sum_{k=r}^{n+1} \binom{k}{r} &= \sum_{k=r}^{n} \binom{k}{r} + \binom{n+1}{r} \\
&= \binom{n+1}{r+1} + \binom{n+1}{r} \\
&= \binom{n+2}{r+1} && \text{by Pascal rule}
\end{aligned}$$

By induction, the proof is completed.

(b) **Combinatorial proof**:

Consider the set $\{1, 2, \cdots, n, n+1\}$. We can form $\binom{n+1}{r+1}$ subsets of size $r+1$. On the other hand, if the greatest integer in the subset is $k+1$, then we have $\binom{k}{r}$ ways to form such a subset. Summing through $r, r+1, \cdots, n$ gives the total number of subsets of size $r+1$ (no need to sum through $1, 2, \cdots, r-1$ because our set has $r+1$ elements). ■

1.1.4 Multinomial coefficients

The problem of interest in this subsubsection is:

- Given a set of n objects, how many ways can we partition it into r subsets of sizes $n_1, n_2, \cdots, n_r$ respectively, where $\sum_{i=1}^{r} n_i = n$?

This is not hard to answer:

(i) We have $\binom{n}{n_1}$ ways to form the first subset of n_1 objects.

(ii) We have $\binom{n-n_1}{n_2}$ ways to form the second subset of n_2 objects from the $n-n_1$ remained objects.

(iii) We have $\binom{n-n_1-n_2}{n_3}$ ways to form the third subset of n_3 objects from the $n-n_1-n_2$ remained objects.

We can repeat in this fashion until we have formed the $(r-1)^{\text{th}}$ subset. The r^{th} subset is automatically formed at this step. By fundamental theorem of counting, there are

$$\begin{aligned}
&\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{r-2}}{n_{r-1}} \\
&= \frac{n!}{(n-n_1)!n_1!}\cdot\frac{(n-n_1)!}{(n-n_1-n_2)!n_2!}\cdot\frac{(n-n_1-n_2)!}{(n-n_1-n_2-n_3)!n_3!}\cdots\frac{(n-n_1-n_2-\cdots-n_{r-2})!}{(n-n_1-n_2-\cdots-n_{r-2}-n_{r-1})!n_{r-1}!} \\
&= \frac{n!}{n_1!n_2!n_3!\cdots n_{r-1}!(n-n_1-n_2-\cdots-n_{r-2}-n_{r-1})!} \\
&= \frac{n!}{n_1!n_2!n_3!\cdots n_{r-1}!n_r!}
\end{aligned}$$

possible partitions.

It makes sense to define the following notation:

Definition 1.8. If $\sum_{i=1}^{r} n_i = n$, we define the **multinomial coefficient** $\binom{n}{n_1, n_2, \cdots, n_r}$ by

$$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1!n_2!n_3!\cdots n_{r-1}!n_r!}.$$

Lemma 1.9. For $0 \le r \le n$, we have

$$\binom{n}{r, n-r} = \binom{n}{r},$$

where on left hand side we have a multinomial coefficient, and on right hand side we have a binomial coefficient.

Proof. Clear. ■

Example 11. A police department in a small city consists of 10 officers. If the department policy is to have 5 officers patrolling the streets, 2 officers working full time at the station, and 3 officers on reserve, how many different divisions of the 10 officers into the 3 groups are possible?

Solution. $\binom{10}{5, 2, 3} = 2520$ ■

We can also redo the word permutation problems using multinomial coefficients:

Example 12. How many different letter arrangements can be formed from the letters `MISSISSIPPI`?

Solution. This amounts to partition 11 places to the 1 `M`, 4 `I`'s, 4 `S`'s and 2 `P`'s. Hence there are $\binom{11}{1, 4, 4, 2} = 34650$ possible ways. ■

Similar to binomials, we can also formulate **multinomial formula**:

Theorem 1.10 (Multinomial formula)**.** Let R be a commutative ring and $x_1, x_2, \cdots, x_m \in R$. Then for any $n \in \mathbb{N}$, it holds

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \cdots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}.$$

Proof. We prove it by induction on m.

(a) When $m = 2$, this is just our binomial formula.

(b) Suppose it holds $(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \cdots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$. Then

$$\begin{aligned}
&(x_1 + x_2 + \cdots + x_m + x_{m+1})^n \\
&= (x_1 + x_2 + \cdots + (x_m + x_{m+1}))^n \\
&= \sum_{k_1+k_2+\cdots+k_{m-1}+k=n} \binom{n}{k_1, k_2, \cdots, k_{m-1}, k} x_1^{k_1} x_2^{k_2} \cdots x_{m-1}^{k_{m-1}} (x_m + x_{m+1})^k \\
&= \sum_{k_1+k_2+\cdots+k_{m-1}+k=n} \binom{n}{k_1, k_2, \cdots, k_{m-1}, k} x_1^{k_1} x_2^{k_2} \cdots x_{m-1}^{k_{m-1}} \left\{ \sum_{j=0}^{k} \binom{k}{j} x_m^j x_{m+1}^{k-j} \right\} \\
&= \sum_{k_1+k_2+\cdots+k_{m-1}+k=n} \binom{n}{k_1, k_2, \cdots, k_{m-1}, k} x_1^{k_1} x_2^{k_2} \cdots x_{m-1}^{k_{m-1}} \left\{ \sum_{k_m+k_{m+1}=k} \binom{k}{k_m, k_{m+1}} x_m^{k_m} x_{m+1}^{k_{m+1}} \right\} \\
&= \sum_{k_1+k_2+\cdots+k_{m-1}+k_m+k_{m+1}=n} \binom{n}{k_1, k_2, \cdots, k_{m-1}, k_m + k_{m+1}} \binom{k_m + k_{m+1}}{k_m, k_{m+1}} x_1^{k_1} x_2^{k_2} \cdots x_{m-1}^{k_{m-1}} x_m^{k_m} x_{m+1}^{k_{m+1}} \\
&= \sum_{k_1+k_2+\cdots+k_m+k_{m+1}=n} \binom{n}{k_1, k_2, \cdots, k_{m-1}, k_m, k_{m+1}} x_1^{k_1} x_2^{k_2} \cdots x_{m-1}^{k_{m-1}} x_m^{k_m} x_{m+1}^{k_{m+1}}
\end{aligned}$$

By induction, the proof is completed. ∎

Of course, one can prove this formula by a combinatorial argument, but an inductive argument would be cleaner.

Example 13.

$$\begin{aligned}
&(x + y + z)^2 \\
&= \sum_{a+b+c=2} \binom{2}{a, b, c} x^a y^b z^c \\
&= \binom{2}{2,0,0} x^2 y^0 z^0 + \binom{2}{0,2,0} x^0 y^2 z^0 + \binom{2}{0,0,2} x^0 y^0 z^2 + \binom{2}{1,1,0} x^1 y^1 z^0 + \binom{2}{1,0,1} x^1 y^0 z^1 + \binom{2}{0,1,1} x^0 y^1 z^1 \\
&= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz \\
&= x^2 + y^2 + z^2 + 2(xy + yz + zx)
\end{aligned}$$

Example 14. By putting $x_1 = x_2 = \cdots = x_m = 1$ into the multinomial formula, it holds

$$\sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \cdots, k_m} = m^n.$$

1.1.5 Number of integral solutions

In this subsubsection we will study the number of nonnegative integral solutions of equations of the form

$$x_1 + x_2 + \cdots + x_r = n$$

where $r, n \in \mathbb{N}$. This equation is closely related to a problem called **combination with repetition**.

It would be easier to consider positive solutions first:

Lemma 1.11. If $r, n \in \mathbb{N}$, then the equation

$$x_1 + x_2 + \cdots + x_r = n$$

has $\binom{n-1}{r-1}$ positive integral solutions.

Proof. Suppose we are given n objects. To count the number of positive integral solutions of the equation, we have to count the ways to divide the objects into r nonempty subsets. To do so, we can select $r-1$ of the $n-1$ spaces between adjacent objects as our partition points, so there are $\binom{n-1}{r-1}$ such solutions. ■

$$1 \ \wedge \ 1 \ \wedge \ 1 \ \wedge \ \cdots \ \wedge \ 1 \ \wedge \ 1 \ \wedge \ 1$$
$$1 = \text{object}$$
$$\wedge = \text{place for partition points}$$

Theorem 1.12. If $r, n \in \mathbb{N}$, then the equation

$$x_1 + x_2 + \cdots + x_r = n$$

has $\binom{n+r-1}{r-1}$ nonnegative integral solutions.

Proof. The number of nonnegative integral solutions of $x_1 + x_2 + \cdots + x_r = n$ is the same as the number of positive integral solutions of $y_1 + y_2 + \cdots + y_r = n + r$ (which can be seen by taking $y_i = x_i + 1$). By the lemma, the latter equation has $\binom{n+r-1}{r-1}$ positive integral solutions, so the former equation also has $\binom{n+r-1}{r-1}$ nonnegative integral solutions. ■

We now illustrate the use of the above theorem to solve real-life problems related to combinations with repetition.

Example 15. An investor has 20 thousand dollars to invest among 4 possible investments. Each investment must be in units of a thousand dollars.

(a) If the total of 20 thousand is to be invested, how many different investment strategies are possible?

(b) What if not all the money needs to be invested?

Proof.

(a) Let x_i be the number of thousands invested in investment i. Then x_i's satisfy the equation

$$x_1 + x_2 + x_3 + x_4 = 20, \quad x_i \geq 0.$$

Thus there are $\binom{20+4-1}{4-1} = 1771$ different investment strategies.

(b) We further let x_5 be the number of thousands not invested. Then x_i's satisfy the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20, \quad x_i \geq 0.$$

Thus there are $\binom{20+5-1}{5-1} = 10626$ different investment strategies. ■

Example 16. From an unlimited selection of five types of soda, one of which is Dr. Pepper, you are putting 25 cans on a table. Determine the number of ways you can select 25 cans of soda if

(a) there are no restrictions.

(b) you must include at least seven Dr. Peppers.

(c) it turns out there are only three Dr. Peppers available.

Solution. Suppose we have selected x_i cans of soda of type i, where x_5 counts the number of cans of Dr. Pepper selected.

(a) Note that the x_i's satisfy the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 25, \quad x_i \geq 0.$$

Thus there are $\binom{25+5-1}{5-1} = 23751$ different ways.

(b) Note that the x_i's satisfy the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 25, \quad x_i \geq 0 \text{ and } x_5 \geq 7.$$

We transform the equation by setting $y_5 = x_5 - 7$, so

$$x_1 + x_2 + x_3 + x_4 + y_5 = 18, \quad x_i, y_5 \geq 0.$$

Thus there are $\binom{18+5-1}{5-1} = 7315$ different ways.

(c) Note that the x_i's satisfy the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 25, \quad x_i \geq 0 \text{ and } 0 \leq x_5 \leq 3.$$

The desired number can be obtained by subtracting the number of solutions of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 25, \quad x_i \geq 0 \text{ and } x_5 \geq 4$$

from the total number of unrestricted ways. We transform the equation by setting $y_5 = x_5 - 4$, so

$$x_1 + x_2 + x_3 + x_4 + y_5 = 21, \quad x_i, y_5 \geq 0.$$

Thus there are $23751 - \binom{21+5-1}{5-1} = 11101$ different ways. ■

The following final computational example would be an important concept sharpener:

Example 17. Suppose you have 100 each of the following six types of tea bags: Black, Chamomile, Earl Grey, Green, Jasmine and Rose. Determine the number of ways to perform the following tasks:

(a) You are making a cup of tea for the Provost, a math professor and a student.

(b) You are making a cup of tea for the Provost, a math professor and a student. Each person will have a different flavour.

(c) You are making a pot of tea with four tea bags.

(d) You are making a pot of tea with four tea bags, each a different flavour.

(e) There are 10 people at a party and each person wants a different flavour of tea.

Solution.

(a) $6 \times 6 \times 6 = 216$

(b) $6 \times 5 \times 4 = 120$

(c) Let x_i be the number of tea bags of type i, where we assume x_6 is the number of Rose tea bags. Then the x_i's satisfy the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 30, \quad x_i \geq 0.$$

Thus there are $\binom{4+6-1}{6-1} = 126$ different ways.

(d) $\binom{6}{4} = 15$

(e) 0. We got only 6 types of tea. ■

1.2 Pigeonhole principle

In this subsection, we present the famous pigeonhole principle. The pigeonhole principle states something like water is wet, yet it is very useful.

Theorem 1.13 (Pigeonhole principle)**.** If n items are put into m containers, with $n > m$, then at least one container contains more than one item.

Proof. We prove the contrapositive. If every container holds no more than one item, then the maximum number of items cannot exceed the number of containers, i.e. $n \leq m$. ■

Corollary 1.14. If $m > n$, then there are no injective functions from a set with m elements to a set with n elements.

Proof. In this case the m items are each x in the domain, and the n containers are the preimages sets $f^{-1}(\{y\})$ for each y in the codomain. Since we have more items than containers, by pigeonhole principle, at least one container contains more than one item. In other words, there exists a preimage $f^{-1}(\{y\})$ containing at least two elements, say x_1, x_2. Then $f(x_1) = f(x_2) = y$, so f is not injective. ■

Corollary 1.15. If $m < n$, then there are no surjective functions from a set with m elements to a set with n elements.

Proof. Let S, T be sets such that $|S| = m$ and $|T| = n$. Suppose $f : S \to T$ is a surjective function. Then $f(S) = T$. Since $|S| = m$, it holds $|f(S)| \leq m$. Thus

$$n = |T| = |f(S)| \leq m,$$

which is nonsense. ■

Example 18. Some simple applications of pigeonhole principle:

- In any group of 367 people, there must be at least two with the same birthday. [2]
- An average human head has $150,000$ hairs, so it makes sense to assume that no one has more than $1,000,000$ hairs. The population of Hong Kong is more than $7,500,000$ as of 2025, so there are at least 2 Hong Kong citizens having the same number of hairs.
- If a Martian has an infinite number of red, blue, yellow, and black socks in a drawer, s/he needs to draw at least 5 socks to guarantee a pair.

1.2.1 Some sophisticated applications of pigeonhole principle

We will first present several applications to statements about divisibility.

Example 19. Let S be a set of $n + 1$ integers. Prove that there exists distinct $a, b \in S$ s.t. $a - b$ is a multiple of n.

Proof. Reduce every element in S mod n. Since there exists only n congruence classes mod n, by pigeonhole principle, at least one of the congruence classes contains at least 2 members of S. So there exists $a, b \in S$ s.t. $a \equiv b \pmod{n}$, i.e. $n \mid (a - b)$. ■

Example 20. Let $S = \{1, 2, \cdots, 2n\}$. Show that if we choose $n + 1$ numbers from S, then there exist two numbers such that one is a multiple of the other.

Proof. For every element in S, we can write it as $2^a b$ where $a \geq 0$ and b is an odd number. Clearly we have at most n possible values of b, which are all odd numbers in S. But we have $n + 1$ chosen numbers, so pigeonhole principle guarantees that there are at least two chosen numbers that share the same b, call them $2^i b$ and $2^j b$ respectively. Obviously they are multiples, because if we assume without loss of generality that $i > j$, then we can see $2^i b = 2^{i-j}(2^j b)$ is a multiple of $2^j b$. ■

[2] However, in a class of 23 students, the probability that at least two students share the same birthday is surprisingly high, exceeding 50%. This is the famous **birthday paradox**.

In the next theorem, we will study strictly monotonic subsequences of finite sequence. Their definitions are parallel to infinite sequences.

Theorem 1.16 (Erdös-Szekeres theorem)**.** Every finite sequence of $n^2 + 1$ distinct real numbers has a monotonic subsequence of length $n + 1$.

Example 21. The finite sequence $3, 66, 12, 65, 2, -2, 13, 96, 69, 0$ of $10 = 3^2 + 1$ numbers has an increasing subsequence $3, 12, 65, 96$ of length $3 + 1 = 4$.

Proof. Let the sequence be $\{a_\nu\}_{\nu=1}^{n^2+1}$. For each a_ν, we assign an ordered pair (i_ν, d_ν) to it, where i_ν is the length of the longest increasing subsequence beginning with a_ν, and d_ν is the length of the longest decreasing subsequence beginning with a_ν.
Now suppose on the contrary that no monotonic subsequence of length $n+1$ exists. Then $i_\nu, d_\nu \le n \ \forall \nu$. In particular, there are n^2 possible ordered pairs (i_ν, d_ν). But we have $n^2 + 1$ ordered pairs, so by pigeonhole principle, there exist terms a_s, a_t s.t. $(i_s, d_s) = (i_t, d_t)$. Without loss of generality, we suppose $s < t$. Since the terms are distinct, we have $a_s < a_t$ or $a_s > a_t$.

(i) Suppose $a_s < a_t$. Then by definition of i_t, we can find an increasing subsequence S of length i_t starting at a_t. Then we can build an increasing subsequence of length $i_t + 1$ starting at a_s by taking a_s followed by S. This contradicts the fact that $i_s = i_t$.

(ii) Suppose $a_s > a_t$. Then by definition of d_t, we can find a decreasing subsequence S of length d_t starting at a_t. Then we can build a decreasing subsequence of length $d_t + 1$ starting at a_s by taking a_s followed by S. This contradicts the fact that $d_s = d_t$. ■

1.2.2 Dirichlet approximation theorem

Theorem 1.17 (Dirichlet approximation theorem)**.** Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$. Then there exist $a, b \in \mathbb{Z}$ with $1 \le a \le n$ s.t.

$$|a\alpha - b| < \frac{1}{n}.$$

Clearly $|a\alpha - b|$ would be something way smaller than 1. If we write $a\alpha = i + d$ where $i \in \mathbb{Z}$ and $d \in [0, 1)$, then it would make sense to take $b = i$ and hence $a\alpha - b = d \in [0, 1)$. We formalized these notations:

Definition 1.18. Let $x \in \mathbb{R}$. We define its **integral part** $\lfloor x \rfloor$ to be the greatest integer less than or equal to x. We also define the **decimal part** $\{x\}$ of x by $x - \lfloor x \rfloor$.

Note that $x - 1 < \lfloor x \rfloor \le x$ and $0 \le \{x\} < 1$.

Example 22.

(i) $\lfloor 7 \rfloor = 7$, $\{7\} = 0$

(ii) $\lfloor 6.9 \rfloor = 6$, $\{6.9\} = 0.9$

(iii) $\lfloor -6.9 \rfloor = -7$, $\{-6.9\} = 0.1$

Proof of Dirichlet approximation theorem. Consider the $n + 1$ numbers $0, \{\alpha\}, \{2\alpha\}, ..., \{n\alpha\}$ and the n disjoint intervals $\left[0, \frac{1}{n}\right), \left[\frac{1}{n}, \frac{2}{n}\right), ..., \left[\frac{n-1}{n}, 1\right)$. Since the intervals are disjoint, each number can lie in exactly one interval only. By pigeonhole principle, at least two of the numbers lie in the same interval, so there exists $0 \le j < k \le n$ s.t.

$$|\{j\alpha\} - \{k\alpha\}| < \frac{1}{n}.$$

Let $a := k - j$ where $1 \le a \le n$. Then $a\alpha = k\alpha - j\alpha = \lfloor k\alpha \rfloor - \lfloor j\alpha \rfloor + \{k\alpha\} - \{j\alpha\}$. Taking $b := \lfloor k\alpha \rfloor - \lfloor j\alpha \rfloor$ completes the proof. ■

1.2.3 Generalized pigeonhole principle

We can even have more than a qualitative statement:

Theorem 1.19 (Generalized pigeonhole principle)**.** If N items are put into k containers, then there is at least one container containing at least $\left\lceil \frac{N}{k} \right\rceil$ items.

Here $\lceil \cdot \rceil$ is the **ceiling function** where $\lceil x \rceil$ is the smallest integer greater than or equal to x. For example, $\lceil 7 \rceil = 7$, $\lceil 6.9 \rceil = 7$ and $\lceil -6.9 \rceil = -6$. Obviously

$$x \leq \lceil x \rceil < x + 1.$$

Proof. We prove it by contradiction. Suppose on the contrary that no container contains more than $\left\lceil \frac{N}{k} \right\rceil - 1$ objects. Then there are a total of

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N$$

items, which contradicts our assumption that we have N items. ■

Example 23. In a class of 100 students, there are at least $\left\lceil \frac{100}{12} \right\rceil = 9$ students who were born in the same month.

Example 24.

(a) How many cards must be drawn from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

(b) How many must be drawn to guarantee that at least three hearts ♥ are selected?

Solution.

(a) If N cards are drawn, by generalized pigeonhole principle, there are at least $\left\lceil \frac{N}{4} \right\rceil$ cards of the same suit. Consequently, if $\left\lceil \frac{N}{4} \right\rceil \geq 3$, then we are guaranteed that at least three cards of the same suit are drawn. The smallest integer satisfying $\left\lceil \frac{N}{4} \right\rceil \geq 3$ is 9, so 9 cards suffice.
Alternatively, we can solve it without directly applying generalized pigeonhole principle (but we are still using the philosophy of pigeonhole principle). In the worst case, when we have drawn 8 cards, we have 2 cards of each suit. Even so, we cannot avoid having a third card of the same suit when we draw the 9^{th} card, so 9 cards suffice.

(b) We will use the philosophy in the alternative solution to (a). In the worst case, when when we have drawn 39 cards, we have all the diamonds ♦, clubs ♣, and spades ♠, before drawing a single heart ♥. The next three cards must be all hearts ♥, so 42 cards suffice. ■

1.3 Principle of inclusion-exclusion

In this subsection, we will introduce the tasty pie in combinatorics - the princple of inclusion-exclusion. The baby case is as follows:

Proposition 1.20. $|A \cup B| = |A| + |B| - |A \cap B|$ for any finite sets A, B.

Proof. $|A| + |B|$ counts elements in $A \setminus B$ and $B \setminus A$ exactly once, and elements in $A \cap B$ twice, so $|A| + |B| - |A \cap B|$ would count every element in $A \cup B$ exactly twice. ■

We can use the idea of this proof to generalize the pie to finitely many sets:

Theorem 1.21 (Principle of inclusion-exclusion)**.** For any finite sets $A_1, A_2, \cdots, A_n$, it holds

$$\left|\bigcup_{i=1}^n A_i\right| = \sum_{i=1}^n |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{n+1} \left|\bigcap_{i=1}^n A_i\right|.$$

We can even write it more compactly as

$$\left|\bigcup_{i=1}^n A_i\right| = \sum_{\emptyset \ne S \subset \{1,2,\cdots,n\}} (-1)^{|S|+1} \left|\bigcap_{i \in S} A_i\right|,$$

but this would be too compact for our usage. In our discussion, we will most frequently use the formula for $n = 2, 3, 4$.

- When $n = 3$, the formula reads
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$
- When $n = 4$, the formula reads
$$\begin{aligned} |A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| \\ & - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & - |A \cap B \cap C \cap D|. \end{aligned}$$

Proof of principle of inclusion-exclusion. We will show that every element of $\bigcup_{i=1}^n A_i$ is counted exactly once by the RHS. Suppose a is a member of exactly r of the sets $A_1, A_2, \cdots, A_n$. Note that a is counted $\binom{r}{k}$ times in the summmation involving intersections of k of the sets. Thus a is counted

$$\sum_{k=1}^n (-1)^{k+1} \binom{r}{k} = -\sum_{k=0}^n (-1)^k \binom{r}{k} + 1 = -(1-1)^n + 1 = 1$$

time by the RHS. ■

Sometimes we may also use the following form:

Corollary 1.22 (Principle of inclusion-exclusion)**.** For any finite set S with $A_1, A_2, \cdots, A_n \subset S$, it holds

$$\left|\bigcap_{i=1}^n A_i^c\right| = |S| - \sum_{i=1}^n |A_i| + \sum_{1 \le i < j \le n} |A_i \cap A_j| - \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n \left|\bigcap_{i=1}^n A_i\right|$$

where $A^c := S \setminus A \ \forall A \subset S$.

Proof. Simply note that $\bigcap_{i=1}^n A_i^c = \left(\bigcup_{i=1}^n A_i\right)^c$ by de Morgan's law, so by the baby case,

$$|S| = \left|\bigcap_{i=1}^n A_i^c\right| + \left|\bigcup_{i=1}^n A_i\right|.$$

Now plug in the original form of the principle, done. ■

Example 25. In a high school, there are 1000 students taking at least one of the physics, chemistry, and biology courses. It is given that:

- The total number of students who take physics is 310.
- The total number of students who take chemistry is 650.
- The total number of students who take biology is 440.
- The total number of students who take physics and chemistry is 170.
- The total number of students who take physics and biology is 150.
- The total number of students who take chemistry and biology is 180.

How many students takes all three courses?

Solution. Let P, K, B denote the set of all physics, chemistry, and biology students respectively. Then by principle of inclusion-exclusion,

$$|P \cup K \cup B| = |P| + |K| + |B| - |P \cap K| - |K \cap B| - |B \cap P| + |P \cap K \cap B|.$$

So $1000 = 310 + 650 + 440 - 170 - 150 - 180 + |P \cap K \cap B|$, i.e. $|P \cap K \cap B| = 100$. So 100 students take all three courses. ∎

1.3.1 Sieving primes

We will present a method to count all primes not exceeding n. The following lemma is useful in solving this kind of problem:

Lemma 1.23 (Sieve of Eratosthenes)**.** If n is a composite number, then n has a prime divisor not exceeding $\sqrt{n}$.

Proof. Write $n = ab$ where $1 < a \leq b < n$. Suppose all prime divisors of n exceed $\sqrt{n}$. Then $b \geq a > \sqrt{n}$. Hence $n = ab > (\sqrt{n})^2 = n$, contradiction arises. ∎

Example 26. What is the number of primes not exceeding 100?

Solution.

- By the above lemma, a composite integer not exceeding 100 has a prime divisor not exceeding $\sqrt{100} = 10$.
- Note that the primes below not exceeding 10 are $2, 3, 5, 7$. Thus the primes not exceeding 100 are $2, 3, 5, 7$, and those positive integers greater than 1 and not exceeding 100 that is divisible by none of $2, 3, 5, 7$.
- Denote by P_k the set of integers not exceeding 100 that is divisible by k. By the second bullet point, the total number of primes not exceeding 100 is given by $|P_2^c \cap P_3^c \cap P_5^c \cap P_7^c| + 4$.
- Note that $|P_k| = \left\lfloor \dfrac{100}{k} \right\rfloor$. By principle of inclusion-exclusion,

$$\begin{aligned}
|P_2^c \cap P_3^c \cap P_5^c \cap P_7^c| &= 99 - |P_2| - |P_3| - |P_5| - |P_7| \\
&\quad + |P_2 \cap P_3| + |P_2 \cap P_5| + |P_2 \cap P_7| + |P_3 \cap P_5| + |P_3 \cap P_7| + |P_5 \cap P_7| \\
&\quad - |P_2 \cap P_3 \cap P_5| - |P_2 \cap P_3 \cap P_7| - |P_2 \cap P_5 \cap P_7| - |P_3 \cap P_5 \cap P_7| \\
&\quad + |P_2 \cap P_3 \cap P_5 \cap P_7| \\
&= 99 - |P_2| - |P_3| - |P_5| - |P_7| \\
&\quad + |P_6| + |P_{10}| + |P_{14}| + |P_{15}| + |P_{21}| + |P_{35}| \\
&\quad - |P_{30}| - |P_{42}| - |P_{70}| - |P_{105}| + |P_{210}| \\
&= 99 - 50 - 33 - 20 - 14 + 16 + 10 + 7 + 6 + 4 + 2 - 3 - 2 - 1 - 0 + 0 \\
&= 21
\end{aligned}$$

- So there are 25 primes not exceeding 100. ∎

1.3.2 Number of surjective functions.

The next application is to count the number of surjective functions $\{1, 2, \cdots, m\} \to \{1, 2, \cdots, n\}$. By corollary 1.15, we know $m \geq n$.

First recall there are n^m functions $\{1, 2, \cdots, m\} \to \{1, 2, \cdots, n\}$. This is obviously overshooting, so we shall discount the $\binom{n}{1}(n-1)^m$ functions that misses at least one value in $\{1, 2, \cdots, n\}$. But this is then undershooting because functions that miss at least two values are counted twice. Thus, we need to add back these $\binom{n}{2}(n-2)^m$ functions. This again overshoots; we need to subtract $\binom{n}{3}(n-3)^m$ functions that miss at least three values. Repeat in this fashion, we can deduce that:

- For $m \geq n$, there are $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^m$ surjective functions $\{1, 2, \cdots, m\} \to \{1, 2, \cdots, n\}$.

By a simple change of variables, we have

Theorem 1.24. For $m \geq n$, there are $\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^m$ surjective functions $\{1, 2, \cdots, m\} \to \{1, 2, \cdots, n\}$.

Example 27. How many ways are there to assign 5 different tasks to 4 different trainees if each trainee is assigned at least 1 task?

Solution. We view the assignment of tasks as a function from the set of 5 different tasks to the set of 4 different trainees. An assignment that each trainee is assigned at least 1 task is the same as a surjective function from the set of 5 different tasks to the set of 4 different trainees. So by the above formula, the desired number is

$$\sum_{k=0}^{4}(-1)^{n-k}\binom{4}{k}k^5 = 240.$$

■

1.3.3 Derangements

A **derangement** is a permutation in which no objects are left in their original positions. In other words, if we derange $1, 2, \cdots, n$ to $i_1, i_2, \cdots, i_n$, then $i_k \neq k$. Our goal is to count the number D_n of derangements of $\{1, 2, \cdots, n\}$.

First recall there are $n!$ permutations of $\{1, 2, \cdots, n\}$. This is obviously overshooting, so we shall discount the $\binom{n}{1}(n-1)!$ permutations that fix at least one value in $\{1, 2, \cdots, n\}$. But this is then undershooting because permutations that fix at least two values are discounted twice. Thus, we need to add back these $\binom{n}{2}(n-2)!$ permutations. This again overshoots; we need to subtract $\binom{n}{3}(n-3)!$ permutations that fix at least three values. Repeat in this fashion, we can deduce the following formula:

$$\begin{aligned} D_n &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)! \\ &= \sum_{k=0}^{n}(-1)^k\frac{n!}{k!(n-k)!}\cdot(n-k)! \\ &= n!\sum_{k=0}^{n}\frac{(-1)^k}{k!} \end{aligned}$$

Theorem 1.25. There are $n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$ derangements of $\{1, 2, \cdots, n\}$.

The formula of derangement is not easy to compute. But we still have ways to facilitate their estimations/ computations.

The first interesting observation is that

$$\lim_{n\to\infty}\frac{D_n}{n!}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}=e^{-1}.$$

Recall an alternating series $L=\sum_{n=0}^{\infty}(-1)^n a_n$ with partial sums $S_n=\sum_{k=0}^{n}(-1)^k a_k$ satisfy the following estimation formula:

$$|S_n-L|\le|S_n-S_{n+1}|=a_{n+1}$$

In the case of the derangement, we have

$$\left|\frac{D_n}{n!}-e^{-1}\right|\le\frac{1}{(n+1)!}.$$

When $n\ge 7$, the error bound $\frac{1}{(n+1)!}\le\frac{1}{8!}=\frac{1}{40320}$ is very small, and $\frac{D_n}{n!}$ and e^{-1} agree to at least 3 decimal places. Hence the following estimation makes sense:

Corollary 1.26. $D_n\approx n!e^{-1}$ with error bound $\left|D_n-n!e^{-1}\right|\le\frac{1}{n+1}$.

n	D_n	$n!e^{-1}$ (5 d.p.)
1	0	0.36788
2	1	0.73576
3	2	2.20728
4	9	8.82911
5	44	44.14553
6	265	264.87320
7	1854	1854.11238
8	14833	14832.89907
9	133496	133496.09161
10	1334961	1334960.91612

Two more formulas for derangement are displayed as follows:

Proposition 1.27. $D_n=(n-1)(D_{n-1}+D_{n-2})\ \forall n\ge 3.$

Proof. Consider the D_n derangments of $\{1,2,\cdots,n\}$. We can naturally partition them into $n-1$ classes of equal sizes based on the first position. Thus $D_n=(n-1)d_n$ where d_n counts the derangements in which 2 is in the first position. These derangements are of the form

$$2,i_2,i_3,\cdots,i_n,\quad i_k\ne k.$$

Note that we can further partition the derangements of such form into two classes: $i_2=1$ and $i_2\ne 1$. Let d_n' be the number of derangements of the form

$$2,1,i_3,\cdots,i_n,\quad i_k\ne k,$$

and d_n'' be the number of derangements of the form

$$2,i_2,i_3,\cdots,i_n,\quad i_2\ne 1 \text{ and } i_k\ne k.$$

Thus $d_n=d_n'+d_n''$ and hence $D_n=(n-1)(d_n'+d_n'')$. It remains to compute the numbers d_n' and d_n''.

- Note that the number of derangements of the form

  $$2,1,i_3,\cdots,i_n,\quad i_k\ne k$$

  naturally induces an derangement of $\{3,4,\cdots,n\}$, so there are D_{n-2} derangement of such form, i.e. $d_n'=D_{n-2}$.
- Note that the number of derangements of the form

  $$2,i_2,i_3,\cdots,i_n,\quad i_2\ne 1 \text{ and } i_k\ne k$$

  naturally induces a permutation $i_2,i_3,\cdots,i_n$ of $\{1,3,4,\cdots,n\}$ in which $1\ne i_2,3\ne i_3,\cdots,i_n\ne n$. By renaming $1\mapsto 2$, we can see there are D_{n-1} such permutations, so $d_n''=D_{n-1}$. ■

Proposition 1.28. $D_n = nD_{n-1} + (-1)^n \ \forall n \geq 2$

Proof. Rewrite $D_n - nD_{n-1} = (-1)(D_{n-1} - (n-1)D_{n-2})$ for $n > 2$. Then we iterate the formula as follows:

$$\begin{aligned} D_n - nD_{n-1} &= (-1)(D_{n-1} - (n-1)D_{n-2}) \\ &= (-1)^2(D_{n-2} - (n-2)D_{n-3}) \\ &= (-1)^3(D_{n-3} - (n-3)D_{n-4}) \\ &= \cdots \\ &= (-1)^{n-2}(D_2 - 2D_1) \end{aligned}$$

Put $D_2 = 1$ and $D_1 = 0$, we have $D_n - nD_{n-1} = (-1)^{n-2} = (-1)^n$ for $n > 2$ as desired. Finally, it is easy to check that for $n = 2$, we have $D_1 + (-1)^2 = 0 + 1 = 1 = D_2$. ■

These are two basic examples of **recurrence relation**. In the next note, we will study the theory of recurrence relations in depth.

Example 28. At a party there are n men and n women. In how many ways can the n women choose male partners for the first dance? How many ways are there for the second dance if everyone has to change partners?

Proof. For the first dance there are $n!$ ways. For the second dance each woman has to choose as a partner a man other than the one with whom she first danced. So there are D_n ways. ■

1.4 Analytic generalities

In this subsection, we will generalize the binomial formula and factorial to real values. They are useful in our future discussions.

1.4.1 Newton's binomial theorem

The first task is to extend the binomial formula to real exponents.

Definition 1.29. Let $\alpha \in \mathbb{R}$ and $r \in \omega$. The **generalise binomial coefficient** of α over r is defined by

$$\binom{\alpha}{r} = \begin{cases} \dfrac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!} = \dfrac{1}{r!}\displaystyle\prod_{k=0}^{r-1}(\alpha-k) & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases}.$$

Note that when $\alpha \in \omega$, $\binom{\alpha}{r}$ reduces to usual binomial coefficients, in particular if $\alpha > r$ then $\binom{\alpha}{r} = 0$.

Example 29.

- $\binom{-2}{3} = \dfrac{1}{3!}(-2)(-3)(-4) = -4$
- $\binom{\frac{1}{2}}{3} = \dfrac{1}{3!}\left(\dfrac{1}{2}\right)\left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right) = \dfrac{1}{16}$

Lemma 1.30. For $n \in \mathbb{N}$ and $r \in \omega$, then $\binom{-n}{r} = (-1)^r\binom{n+r-1}{r}$.

Proof. If $r = 0$ then clearly both sides reduce to 1. Otherwise,

$$\binom{-n}{r} = \frac{1}{r!}\prod_{k=0}^{r-1}(-n-k) = (-1)^r\frac{1}{r!}\prod_{k=0}^{r-1}(n+k) = (-1)^r\frac{1}{r!}\prod_{k=0}^{r-1}(n+r-1-k) = (-1)^r\binom{n+r-1}{r}.$$

■

Example 30. $\binom{-2}{3} = (-1)^3\binom{2+3-1}{3} = -4$ as computed above.

Now we can start extending binomial's formula. Note that Vandermonde's identity still holds for generalized binomial coefficients, and it turns out to be an important ingredient in our proof of Newton's binomial theorem.

Lemma 1.31 (Vandermonde's identity)**.** For $\alpha, \beta \in \mathbb{R}$ and $r \in \omega$, it holds

$$\sum_{k=0}^{r} \binom{\alpha}{r-k}\binom{\beta}{k} = \binom{\alpha+\beta}{r}.$$

Proof. We prove it by induction.

(i) When $r = 0$ it is trivial. When $r = 1$, then $\binom{\alpha}{1}\binom{\beta}{0} + \binom{\alpha}{0}\binom{\beta}{1} = \alpha + \beta = \binom{\alpha+\beta}{1}$ as desired.

(ii) Suppose it holds $\sum_{k=0}^{r} \binom{\alpha}{r-k}\binom{\beta}{k} = \binom{\alpha+\beta}{r}$. Then

$$\begin{aligned}
\binom{\alpha+\beta}{r+1} &= \binom{\alpha+\beta}{r} \times \frac{\alpha+\beta-r}{r+1} \\
&= \sum_{k=0}^{r} \binom{\alpha}{r-k}\binom{\beta}{k} \frac{\alpha+\beta-r}{r+1} \\
&= \sum_{k=0}^{r} \binom{\alpha}{r-k}\binom{\beta}{k} \left\{ \frac{\alpha-r+k}{r+1} + \frac{\beta-k}{r+1} \right\} \\
&= \left\{ \sum_{k=0}^{r} \binom{\alpha}{r-k}\binom{\beta}{k} \frac{\alpha-r+k}{r+1} \right\} + \left\{ \sum_{k=0}^{r} \binom{\alpha}{r-k}\binom{\beta}{k} \frac{\beta-k}{r+1} \right\} \\
&= \left\{ \sum_{k=0}^{r} \binom{\alpha}{r+1-k}\binom{\beta}{k} \frac{r-k+1}{r+1} \right\} + \left\{ \sum_{k=0}^{r} \binom{\alpha}{r-k}\binom{\beta}{k+1} \frac{k+1}{r+1} \right\} \\
&= \left\{ \sum_{k=0}^{r} \binom{\alpha}{r+1-k}\binom{\beta}{k} \frac{r-k+1}{r+1} \right\} + \left\{ \sum_{k=1}^{r+1} \binom{\alpha}{r-k+1}\binom{\beta}{k} \frac{k}{r+1} \right\} \\
&= \binom{\alpha}{r+1} + \sum_{k=0}^{r} \binom{\alpha}{r+1-k}\binom{\beta}{k} \left\{ \frac{r-k+1}{r+1} + \frac{k}{r+1} \right\} + \binom{\beta}{r+1} \\
&= \sum_{k=0}^{r+1} \binom{\alpha}{r+1-k}\binom{\beta}{k}
\end{aligned}$$

By induction, the proof is completed. ∎

The following exercise in analysis would also be used in our proof:

Exercise (Wade, An Introduction to Analysis, Ex 3.3.9).
Suppose $f : \mathbb{R} \to (0, \infty)$ satisfies $f(x+y) = f(x)f(y)$.

(a) Prove that $f(nx) = [f(x)]^n \ \forall x \in \mathbb{R}, n \in \mathbb{Z}$.

(b) Prove that $f(rx) = [f(x)]^r \ \forall x \in \mathbb{R}, r \in \mathbb{Q}$.

(c) Prove that f is continuous at 0 **iff** f is continuous on $\mathbb{R}$

(d) Prove that if f is continuous at 0, then there exists $a \in (0, +\infty)$ s.t. $f(x) = a^x \ \forall x \in \mathbb{R}$.

Proof.

(a) Note that by induction, we can easily prove that $f(nx) = [f(x)]^n \ \forall x \in \mathbb{R}, n \in \mathbb{N}$. Also,

$$f(0) = f(2 \times 0) = [f(0)]^2$$

which implies $f(0) = 0$ or $f(0) = 1$. By looking at the range of f, we know $f(0) = 1$, and hence $f(0x) = [f(x)]^0 \ \forall x \in \mathbb{R}$. Next note that for any $x \in \mathbb{R}$,

$$f(x)f(-x) = f(x-x) = f(0) = 1$$

so $f(-x) = [f(x)]^{-1}$. Thus for any $n \in \mathbb{N}$, it holds

$$f(-nx) = [f(-x)]^n = [f(x)]^{-n}.$$

(b) Note that for any $n \in \mathbb{N}$, $x \in \mathbb{R}$, we have

$$f(x) = f\left(\frac{nx}{n}\right) = \left[f\left(\frac{x}{n}\right)\right]^n \implies f\left(\frac{x}{n}\right) = [f(x)]^{1/n}.$$

Therefore for any $r = \dfrac{m}{n}$ where $m \in \mathbb{Z}, n \in \mathbb{N}$, we have

$$f(rx) = f\left(\frac{mx}{n}\right) = \left[f\left(\frac{x}{n}\right)\right]^m = [f(x)]^{m/n} = [f(x)]^r.$$

(c) ($\Longleftarrow$) is trivial. For ($\Longrightarrow$), suppose f is continuous at 0. For any $x \in \mathbb{R}$, pick any $\{x_n\} \subset \mathbb{R}$ s.t. $\lim\limits_{n\to\infty} x_n = x$. Then

$$\lim_{n\to\infty} (x_n - x) = 0.$$

By sequential criterion,

$$\lim_{n\to\infty} f(x_n - x) = f(0) = 1.$$

Thus for any $n \in \mathbb{N}$,

$$\begin{aligned} |f(x_n) - f(x)| &= |f(x)| \left| \frac{f(x_n)}{f(x)} - 1 \right| \\ &= |f(x)||f(x_n)f(-x) - 1| \\ &= |f(x)||f(x_n - x) - 1| \to 0 \quad \text{as } n \to \infty \end{aligned}$$

Therefore $\lim\limits_{n\to\infty} f(x_n) = f(x)$. By sequential criterion, f is continuous at $x \in \mathbb{R}$.

(d) For $x \in \mathbb{R}$, let $\{q_n\} \subset \mathbb{Q}$ be s.t. $x = \lim\limits_{n\to\infty} q_n$. Then

$$f(x) = f\left(\lim_{n\to\infty} q_n\right) = \lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} [f(1)]^{q_n} = [f(1)]^x.$$

■

Theorem 1.32 (Newton's binomial theorem). If $\alpha \in \mathbb{R}$ and $|x| < 1$, then

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

Proof. We first verify the convergence of the series on $|x| < 1$. Fix $|x| < 1$ and define $f(\alpha) = \sum\limits_{k=0}^{\infty} \dbinom{\alpha}{k} x^k$. Then

$$\lim_{k\to\infty} \left| \frac{\dbinom{\alpha}{k+1} x^{k+1}}{\dbinom{\alpha}{k} x^k} \right| = \lim_{k\to\infty} \left| \frac{\alpha - k}{k+1} \right| |x| = |x| < 1$$

which is independent of α. So by ratio test, f converges absolutely and absolutely on $\mathbb{R}$. Hence f is continuous. Now note that for $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} f(\alpha)f(\beta) &= \left\{ \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \right\} \left\{ \sum_{k=0}^{\infty} \binom{\beta}{k} x^k \right\} \\ &= \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k} \binom{\alpha}{k-j} \binom{\beta}{k} \right\} x^k \\ &= \sum_{k=0}^{\infty} \binom{\alpha+\beta}{k} x^k = f(\alpha+\beta) && \text{Vandermonde's identity} \end{aligned}$$

Thus it follows from the above exercise that $f(\alpha) = f(1)^\alpha$. But then

$$f(1) = \sum_{k=0}^{\infty} \binom{1}{k} x^k = 1 + x.$$

Therefore $f(\alpha) = (1+x)^\alpha$. ■

1.4.2 Gamma function

We can extend factorial continuously to a function on $(0, \infty)$ as follows:

Definition 1.33. Let the **gamma function** $\Gamma : (0, \infty) \to \mathbb{R}$ be defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, \mathrm{d}t$$

Example 31.

$$\Gamma(1) = \lim_{R\to\infty} \int_0^R e^{-t} \, \mathrm{d}t = \lim_{R\to\infty} (1 - e^{-R}) = 1$$

Since the gamma function is defined by an improper integral, we are obligated to verify its well-definedness.

Theorem 1.34. For any $x \in (0, \infty)$, $\Gamma(x)$ exists and is finite.

Proof. Split the integral into two:

$$\Gamma(x) = \int_0^1 t^{x-1}e^{-t} \, \mathrm{d}t + \int_1^\infty t^{x-1}e^{-t} \, \mathrm{d}t =: I + J.$$

Note that for fixed $x \in \mathbb{R}$, it holds

$$\lim_{t\to\infty} t^{x-1}e^{-t/2} = 0,$$

so $0 \leq t^{x-1}e^{-t} \leq e^{-t/2}$ for sufficiently large t. Note that $e^{-t/2}$ is improperly integrable on $[1, \infty)$ (this is a part of $\Gamma(1)$). so by comparison theorem, J exists and is finite.
To show that I is finite for $x > 0$, we split it into two cases:
Case I: Suuppose $x \geq 1$. Then $t^{x-1} \leq 1 \; \forall t \in [0, 1]$. Hence

$$I = \int_0^1 t^{x-1}e^{-t} \, \mathrm{d}t \leq \int_0^1 e^{-t} \, \mathrm{d}t = 1 - \frac{1}{e}.$$

Thus $\Gamma(x)$ is finite for $x \geq 1$.
Case II: Suppose $x \in (0, 1)$. Then $x + 1 \geq 1$, so $\Gamma(x + 1)$ is finite by Case I. Now using integration by parts, we have

$$\begin{aligned}\Gamma(x) &= \int_0^\infty t^{x-1}e^{-t} \, \mathrm{d}t \\ &= \left[\frac{t^x e^{-t}}{x}\right]_0^\infty + \int_0^\infty \frac{t^x e^{-t}}{x} \, \mathrm{d}t \\ &= \frac{1}{x}\Gamma(x + 1)\end{aligned}$$

So $\Gamma(x)$ is also finite for $x \in (0, 1)$. ■

More properties of the gamma function is recorded as follows:

Corollary 1.35.

(a) $\Gamma(x + 1) = x\Gamma(x) \; \forall x > 0$,

(b) $\Gamma(n) = (n - 1)! \; \forall n \in \mathbb{N}$,

Proof.

(a) This is already proved in the above theorem - note that the proof there does not depend on the assumption that $x \in (0, 1)$. The part where we used this assumption is the finiteness of $\Gamma(x + 1)$ only.

(b) Since $\Gamma(1) = 1 = 0!$, by induction we are done. ■

As an exercise, let's compute $\Gamma\left(\frac{1}{2}\right)$. To do so, an integral formula is needed:

Proposition 1.36 (Gaussian integral). $\displaystyle\int_0^\infty e^{-x^2}\ \mathrm{d}x = \frac{\sqrt{\pi}}{2}$.

Proof. Let $I = \displaystyle\int_0^\infty e^{-x^2}\ \mathrm{d}x$. Note that $0 \le e^{-x^2} \le e^{-x}\ \forall x \in [1, +\infty)$, so by comparison theorem, e^{-x^2} is improperly integrable on $[1, +\infty)$, hence on $[0, +\infty)$. Having the convergence, we can compute the integral by

$$\begin{aligned}
I^2 &= \left\{\int_0^\infty e^{-x^2}\ \mathrm{d}x\right\}\left\{\int_0^\infty e^{-y^2}\ \mathrm{d}y\right\}\\
&= \int_0^\infty \int_0^\infty e^{-x^2}e^{-y^2}\ \mathrm{d}x\ \mathrm{d}y\\
&= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\ \mathrm{d}x\ \mathrm{d}y\\
&= \int_0^{\pi/2} \int_0^\infty e^{-r^2}r\ \mathrm{d}r\ \mathrm{d}\theta\\
&= \int_0^{\pi/2} \left[-\frac{1}{2}e^{-r^2}\right]_0^{+\infty}\ \mathrm{d}\theta\\
&= \frac{\pi}{4}
\end{aligned}$$

■

Example 32. By the substitution $t = u^2$, we have

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-1/2}e^{-t}\ \mathrm{d}t = \int_0^{+\infty} \frac{e^{-u^2}}{u}(2u\ \mathrm{d}u) = 2\int_0^{+\infty} e^{-u^2}\ \mathrm{d}u = \sqrt{\pi}.$$

1.4.3 Stirling's formula

In this subsubsection we shall establish the following approximation of the factorial:

Theorem 1.37 (Stirling's formula). $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1$.

Since the proof is rather lengthy, we shall break it down into some lemmas and propositions.

Lemma 1.38. If $n \in \mathbb{N}$, then

$$\int_0^n \log(x)\ \mathrm{d}x < \log(n!) < \int_1^{n+1} \log(x)\ \mathrm{d}x.$$

Proof. By maximum-minimum inequality of integral, since log is increasing, for $k \in \mathbb{N}$, we have

$$\int_{k-1}^k \log(x)\ \mathrm{d}x < \log(k)\{k-(k-1)\} = \log(k),$$
$$\int_k^{k+1} \log(x)\ \mathrm{d}x > \log(k)\{(k+1)-k)\} = \log(k),$$

where the inequality is strict because log is strictly increasing. By combining the inequalities, we have

$$\int_{k-1}^k \log(x)\ \mathrm{d}x < \log(k) < \int_k^{k+1} \log(x)\ \mathrm{d}x.$$

Summing over $k = 1, 2, \cdots, n$ gives

$$\int_0^n \log(x)\ \mathrm{d}x < \sum_{k=1}^n \log(k) = \log(n!) < \int_1^{n+1} \log(x)\ \mathrm{d}x.$$

A subtlety is that we need to show the convergence of the improper integral $\int_0^n \log(x)\ \mathrm{d}x$. Indeed,

$$\int_0^1 \log(x)\ \mathrm{d}x = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \log(x)\ \mathrm{d}x = \lim_{\varepsilon\to 0^+}\Big[x\log(x)-x\Big]_\varepsilon^1 = -1-\lim_{\varepsilon\to 0^+}(\varepsilon\log(\varepsilon)-\varepsilon),$$

and by L'Hôpital's rule, we have

$$\lim_{\varepsilon\to 0^+}\varepsilon\log(\varepsilon) = \lim_{\varepsilon\to 0^+}\frac{\log(\varepsilon)}{1/\varepsilon} = \lim_{\varepsilon\to 0^+}\frac{1/\varepsilon}{-1/\varepsilon^2} = -\lim_{\varepsilon\to 0^+}\varepsilon = 0.$$

Therefore we have $\int_0^1 \log(x)\ \mathrm{d}x = -1$, concluding the convergence. ■

Evaluate the integrals in lemma 1.38 gives

$$n\log(n)-n<\log(n!)<(n+1)\log(n+1)-n$$

Now observe the average of the upper and lower bounds is

$$\frac{\{(n+1)\log(n+1)-n\}+\{n\log(n)-n\}}{2} = \frac{(n+1)\log(n+1)+n\log(n)}{2}-n.$$

Note that when n is sufficient large, $\log(n+1)\approx\log(n)$. This approximation gives

$$\frac{(n+1)\log(n+1)+n\log(n)}{2}-n \approx \frac{(2n+1)\log(n)}{2}-n = \left(n+\frac{1}{2}\right)\log(n)-n.$$

Hence it makes sense to define the sequence

$$d_n = \log(n!)-\left\{\left(n+\frac{1}{2}\right)\log(n)-n\right\} = n+\log(n!)-\left(n+\frac{1}{2}\right)\log(n).$$

Then we can see that

$$\begin{aligned}
d_{n+1}-d_n &= (n+1)+\log((n+1)!)-\left(n+1+\frac{1}{2}\right)\log(n+1)-n-\log(n!)+\left(n+\frac{1}{2}\right)\log(n)\\
&= 1+\log(n+1)+\log(n!)-\left(n+1+\frac{1}{2}\right)\log(n+1)-\log(n!)+\left(n+\frac{1}{2}\right)\log(n)\\
&= 1-\left(n+\frac{1}{2}\right)\log(n+1)+\left(n+\frac{1}{2}\right)\log(n)\\
&= 1-\left(n+\frac{1}{2}\right)\log\left(\frac{n}{n+1}\right)
\end{aligned}$$

or equivalently $d_n-d_{n+1} = \left(n+\frac{1}{2}\right)\log\left(\frac{n+1}{n}\right)-1$.

Lemma 1.39. The sequences $\{d_n\}_{n\in\mathbb{N}}$ and $\left\{d_n-\frac{1}{12n}\right\}_{n\in\mathbb{N}}$ converge to the same limit, which we shall call it C.

Proof. Recall the Taylor series expansion of $\log(1-x)$ is

$$\log(1-x) = -\sum_{k=1}^{\infty}\frac{x^k}{k}\ \forall|x|<1.$$

By a change of variable, we also have

$$\log(1+x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}\ \forall|x|<1.$$

Then we have

$$\log(1+x)-\log(1-x) = 2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}\ \forall|x|<1,$$

in other words, we have $\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}\ \forall|x|<1$

Now we note that

$$\frac{n+1}{n} = \frac{1+\dfrac{1}{2n+1}}{1-\dfrac{1}{2n+1}}$$

where $-1 < \dfrac{1}{2n+1} < 1$. Thus by the above Taylor series expansion, we have

$$\begin{aligned}
d_n - d_{n+1} &= \left(n+\frac{1}{2}\right)\log\left(\frac{n+1}{n}\right) - 1 \\
&= (2n+1)\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{1}{2n+1}\right)^{2k+1} - 1 \\
&= \sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{1}{2n+1}\right)^{2k} - 1 \\
&= \sum_{k={\color{red}1}}^{\infty}\frac{1}{2k+1}\left(\frac{1}{2n+1}\right)^{2k} \\
&< \frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{1}{2n+1}\right)^{2k} \\
&= \frac{1}{3}\times\frac{(2n+1)^{-2}}{1-(2n+1)^{-2}} \\
&= \frac{1}{3}\times\frac{1}{(2n+1)^2-1} \\
&= \frac{1}{12}\left(\frac{1}{n}-\frac{1}{n+1}\right)
\end{aligned}$$

Rearranging gives $d_{n+1} - \dfrac{1}{12(n+1)} > d_n - \dfrac{1}{12n}$. This implies $\left\{d_n - \dfrac{1}{12n}\right\}$ is an increasing sequence. But what about $\{d_n\}$ itself? Note that in the above manipulation, we have

$$d_n - d_{n+1} = \sum_{k=1}^{\infty}\frac{1}{2k+1}\left(\frac{1}{2n+1}\right)^{2k} > 0,$$

so $\{d_n\}$ is a decreasing sequence. Finally, observe

$$d_n > d_n - \frac{1}{12n} \ \forall n \in \mathbb{N}.$$

Together with the monotonicities of the two sequences, we have

$$d_1 > d_2 > \cdots > d_n > d_n - \frac{1}{12n} > \cdots > d_2 - \frac{1}{24} > d_1 - \frac{1}{12} \ \forall n \in \mathbb{N}.$$

So we can see

- $\{d_n\}$ is a decreasing sequence that is bounded below by $d_1 - \dfrac{1}{12}$, and
- $\left\{d_n - \dfrac{1}{12n}\right\}$ is an increasing sequence that is bounded above by d_1.

By monotone convergence theorem, $\{d_n\}$ and $\left\{d_n - \dfrac{1}{12n}\right\}$ are convergent. Now if $\lim_{n\to\infty} d_n = C$, then we have

$$\lim_{n\to\infty}\left(d_n - \frac{1}{12n}\right) = C.$$

■

Knowing the convergence of $\{d_n\}$ is already enough. The role of $\left\{d_n - \frac{1}{12n}\right\}$ here is a tool for us to establish the convergence of $\{d_n\}$. Anyway, knowing the convergence of $\{d_n\}$, we can prove the existence of the desired limit, marking the halfway milestone of our lengthy proof.

Proposition 1.40. $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{n}} = e^C$.

Proof. We know $\displaystyle\lim_{n\to\infty}\left\{n + \log(n!) - \left(n+\frac{1}{2}\right)\log(n)\right\} = \lim_{n\to\infty} d_n = C$. This implies

$$\begin{aligned}
e^C &= \exp\left\{\lim_{n\to\infty}\left\{n + \log(n!) - \left(n+\frac{1}{2}\right)\log(n)\right\}\right\}\\
&= \lim_{n\to\infty}\exp\left\{n + \log(n!) - \left(n+\frac{1}{2}\right)\log(n)\right\}\\
&= \lim_{n\to\infty}\frac{n!}{n^n e^{-n}\sqrt{n}}
\end{aligned}$$

■

It remains to find out that $e^C = \sqrt{2\pi}$. An easy way would be to use the **Wallis formula**:

Theorem 1.41 (Wallis formula). $\displaystyle\frac{\pi}{2} = \prod_{k=1}^{\infty}\left\{\frac{2k}{2k-1}\cdot\frac{2k}{2k+1}\right\}$

Proof. Define the integral

$$I_n = \int_0^\pi \sin^n(x)\ \mathrm{d}x.$$

Our experience from calculus tells us that we need to prove a reduction formula. Well that's fucking true mate.

$$\begin{aligned}
I_n = \int_0^\pi \sin^n(x)\ \mathrm{d}x &= -\int_0^\pi \sin^{n-1}(x)\ \mathrm{d}\cos(x)\\
&= -\Big[\sin^{n-1}(x)\cos(x)\Big]_0^\pi + \int_0^\pi \cos(x)\ \mathrm{d}\sin^{n-1}(x)\\
&= (n-1)\int_0^\pi \sin^{n-2}(x)\cos^2(x)\ \mathrm{d}x\\
&= (n-1)\int_0^\pi \sin^{n-2}(x)(1-\sin^2(x))\ \mathrm{d}x\\
&= (n-1)\int_0^\pi \sin^{n-2}(x) - \sin^n(x)\ \mathrm{d}x\\
&= (n-1)\int_0^\pi \sin^{n-2}(x)\ \mathrm{d}x - (n-1)\int_0^\pi \sin^n(x)\ \mathrm{d}x\\
&= (n-1)I_{n-2} - (n-1)I_n
\end{aligned}$$

Rearranging, we have $I_n = \frac{n-1}{n} I_{n-2}\ \forall n \geq 2$. It would be convenient to split into two cases, based on the parity of the indices.

- **Case I**: If $n = 2m+1$, then we have
$$I_{2m+1} = \frac{2m}{2m+1} I_{2m-1} = \frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1} I_{2m-3} = \cdots = \frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdots\frac{2}{3} I_1.$$
Now compute $I_1 = \int_0^\pi \sin(x)\ \mathrm{d}x = 2$. Thus
$$I_{2m+1} = \frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdots\frac{2}{3}\cdot 2 = 2\prod_{k=1}^{m}\frac{2k}{2k+1}.$$

- **Case II**: If $n = 2m$, then we have
$$I_{2m} = \frac{2m-1}{2m} I_{2m-2} = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} I_{2m-4} = \cdots = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdots \frac{1}{2} I_0.$$
Now compute $I_0 = \int_0^\pi \, \mathrm{d}x = \pi$. Thus
$$I_{2m} = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdots \frac{1}{2} \cdot \pi = \pi \prod_{k=1}^{m} \frac{2k-1}{2k}.$$

Now we can see
$$\frac{I_{2m}}{I_{2m+1}} = \frac{\pi}{2} \prod_{k=1}^{m} \left\{ \frac{2k-1}{2k} \cdot \frac{2k+1}{2k} \right\}$$
It remains to compute $\lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}}$. To do so, note that $0 \le \sin(x) \le 1 \ \forall x \in [0, \pi]$, so
$$\sin^{2m+1}(x) \le \sin^{2m}(x) \le \sin^{2m-1}(x) \ \forall x \in [0, \pi].$$
By integrating over $[0, \pi]$, we have
$$I_{2m+1} \le I_{2m} \le I_{2m-1} \quad \implies \quad 1 \le \frac{I_{2m}}{I_{2m+1}} \le \frac{I_{2m-1}}{I_{2m+1}} = \frac{2m+1}{2m}.$$
By sandwich theorem, we have
$$\lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}} = 1.$$
Therefore we have
$$1 = \frac{\pi}{2} \prod_{k=1}^{\infty} \left\{ \frac{2k-1}{2k} \cdot \frac{2k+1}{2k} \right\} \quad \implies \quad \frac{\pi}{2} = \prod_{k=1}^{\infty} \left\{ \frac{2k}{2k-1} \cdot \frac{2k}{2k+1} \right\}$$
as desired. ■

We can rewrite the Wallis formula as
$$\begin{aligned}
\frac{\pi}{2} &= \prod_{k=1}^{\infty} \left\{ \frac{2k}{2k-1} \cdot \frac{2k}{2k+1} \right\} \\
&= \lim_{n\to\infty} \left\{ \frac{2}{1} \cdot \frac{2}{3} \right\} \left\{ \frac{4}{3} \cdot \frac{4}{5} \right\} \cdots \left\{ \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right\} \\
&= \lim_{n\to\infty} \frac{2^2 \cdot 4^2 \cdots (2n)^2}{1 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2 \cdot (2n+1)}
\end{aligned}$$
Now we can finally conclude our proof by deducing $e^C = \sqrt{2\pi}$:

Proof of Stirling's formula. Let $u_n = \frac{n!}{n^n e^{-n} \sqrt{n}}$, so $\lim_{n\to\infty} u_n = e^c$. Then
$$\begin{aligned}
e^c = \frac{(e^c)^2}{e^c} = \frac{\left(\lim_{n\to\infty} u_n\right)^2}{\lim_{n\to\infty} u_{2n}} &= \lim_{n\to\infty} \frac{u_n^2}{u_{2n}} \\
&= \lim_{n\to\infty} \frac{(n!)^2}{n^{2n+1} e^{-2n}} \cdot \frac{(2n)^{2n} e^{-2n} \sqrt{2n}}{(2n)!} \\
&= \lim_{n\to\infty} \frac{2^{2n} (n!)^2}{(2n)!} \cdot \sqrt{\frac{2}{n}} \\
&= \lim_{n\to\infty} \frac{2^{2n} (1^2 \cdot 2^2 \cdots n^2)}{1 \cdot 2 \cdot 3 \cdots (2n-1) \cdot (2n)} \cdot \sqrt{\frac{2}{n}} \\
&= \lim_{n\to\infty} \frac{2^2 \cdot 4^2 \cdots (2n)^2}{1 \cdot 2 \cdot 3 \cdots (2n-1) \cdot (2n)} \cdot \sqrt{\frac{2}{n}} \\
&= \lim_{n\to\infty} \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)} \cdot \sqrt{\frac{2}{n}} \\
&= \lim_{n\to\infty} \sqrt{\frac{2^2 \cdot 4^2 \cdots (2n)^2}{1 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2 \cdot (2n+1)}} \cdot \sqrt{2n+1} \cdot \sqrt{\frac{2}{n}} \\
&= \sqrt{\frac{\pi}{2}} \cdot \sqrt{2} \cdot \sqrt{2} = \sqrt{2\pi}
\end{aligned}$$
as desired. ■