Probability and Statistics Notes series - Note Set 0B

Basic Combinatorics

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Abstract

This is a companion note to accompany my *Probability and Statistics* notes series. We review some basic combinatorics in this note, including the methods of counting and recurrence relations.

Contents

1 Counting

Counting problems arise everywhere. Therefore we would need a more efficient and systematic counting method. The goal of this first section would be on developing algorithms for counting.

1.1 Basic counting principles

The most important yet fundamental theorem is as follows:

Theorem 1.1 (Fundamental theorem of counting). If a job contains k separate tasks, where the i^{th} task can be done in n_i ways, then the whole job can be done in $n_1 n_2 \cdots n_k$ ways.

Proof. By induction, it is suffice to prove for the case $k = 2$. We enumerate all possible ways to complete the tasks as follows:

$$\begin{matrix} (1,1), & (1,2), & \cdots, & (1,n_2), \\ (2,1), & (2,2), & \cdots, & (2,n_2), \\ \vdots & \vdots & \ddots & \vdots \\ (n_1,1), & (n_1,2), & \cdots, & (n_1,n_2) \end{matrix}$$

where we say the whole job can be done by the $(i,j)^{\text{th}}$ way if task 1 can be done by its i^{th} way and task 2 can be done by its j^{th} way. Hence the set of possible ways consists of n_1 rows, each containing n_2 elements. Therefore there are $n_1 n_2$ ways to do the whole job. ■

Example 1.1. A standard license plate contains 6 places, where the first 2 places are to be occupied by letters other than '`I`','`O`' or '`Q`', and the last 4 places are to be occupied by numbers.

(a) What is the number of possible plates?

(b) If repetition among letters or numbers is not allowed, what is the number of possible plates?

Solution.

(a) $23 \times 23 \times 10 \times 10 \times 10 \times 10 = 5,290,000$.

(b) $23 \times 22 \times 10 \times 9 \times 8 \times 7 = 2,550,240$. ■

Example 1.2.

(a) How many functions are there from a set with m elements to a set with n elements?

(b) How many *injective* functions are there from a set with m elements to a set with n elements?

It is much harder to count surjective functions, so let's study it later.

Solution.

(a) A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain, So there are $\underbrace{n \times n \times \cdots \times n}_{m\ n\text{'s}} = n^m$ functions.

(b) When $m > n$, no injective function exists. For $m \leq n$, if we label to elements in the domain as $a_1, a_2, \cdots, a_m$, then there are n ways to assign a function value to a_1, $n-1$ ways to assign a function value to a_2, ..., and finally $n - m + 1$ ways to assign a function value to a_m. In total there are $n(n-1)\cdots(n-m+1)$ such functions. ■

1.1.1 Permutation

An ordered arrangement without repetition is also called **permutation**. Example 1.1(b) already provided us with the first example of permutation. In general, when we have n objects, by fundamental theorem of counting, there are $n(n-1)\cdots 1$ different permutations. It would be convenient to define the following function:

Definition 1.2. For $n \in \mathbb{N}$, we define the **factorial** of n by

$$n! = n(n-1)\cdots 1.$$

We also define $0! = 1$.

It makes sense to define $0! = 1$ because we have exactly one way to permute 0 objects - which is to do nothing.

Example 1.3. Cookies is a girl group of 9 girls including Stephy, Theresa, and Kary.

(a) (i) In how many ways can the girls stand in a row?
  (ii) In how many ways if Stephy and Kary must stand together?
  (iii) In how many ways if Stephy and Theresa must not stand together?

(b) (i) In how many ways can the girls stand in *two* rows, with 4 in the first row and 5 in the second row?
  (ii) In how many ways if Stephy and Kary must stand together?
  (iii) In how many ways if Stephy and Theresa must not stand together?

Solution.

(a) (i) $9! = 362,880$.
  (ii) We first consider Stephy and Kary as a bundle and permute them with the 7 other teammates, this accounts for the $8!$ factor. Then in the S-K bundle, there are $2!$ ways to permute them. Therefore the number of ways is $8! \times 2! = 80640$.
  (iii) We just deduct the case that they stand together from the unrestricted case. Therefore the number of ways is $9! - 8! \times 2! = 282,240$.

(b) (i) The answer is still $9! = 362,880$. This is the same as arranging them in a row.
  (ii) We first consider arranging them in a row, which accounts for the $8! \times 2!$ term. Then we subtract the number of ways that they stand in two rows, which is $7! \times 2!$. Therefore the number of ways is $8! \times 2! - 7! \times 2! = 70560$.
  (iii) We just deduct the case that they stand together from the unrestricted cases. Therefore the number of ways is $9! - (8! \times 2! - 7! \times 2!) = 292,320$. ∎

Sometimes a proportion of our objects are indistinguishable, for example, we cannot distinguish between the `E`'s in the word `MELEE`.

Example 1.4. How many different letter arrangements can be formed from the letters

(a) `MELEE`? (b) `NIGERIAN`? (c) `MISSISSIPPI`?

Solution.

(a) We label the `E`'s in `MELEE` as $\mathtt{ME_1LE_2E_3}$ first, as if the `E`'s are distinguishable. This accounts for the $5!$ factor. But actually the `E`'s are indistinguishable, so

$$\begin{array}{ccc} \mathtt{ME_1LE_2E_3} & \mathtt{ME_2LE_1E_3} & \mathtt{ME_3LE_1E_2} \\ \mathtt{ME_1LE_3E_2} & \mathtt{ME_2LE_3E_1} & \mathtt{ME_3LE_2E_1} \end{array}$$

are the same - they are just `MELEE`. So each possible letter arrangement is counted for $3!$ times. Therefore the total number of arrangement is $\dfrac{5!}{3!} = 20$.

(b) $\dfrac{8!}{2!2!} = 10080$.

(c) $\dfrac{11!}{4!4!2!} = 34650$. ∎

1.1.2 Combination

We want to determine the number of different subsets of r objects that could be formed from a set of n objects. There are $n(n-1)\cdots(n-r+1) = \dfrac{n!}{(n-r)!}$ ways to select r objects out one by one. But since the order of selection is irrelevant, noting that each subset of r objects chosen are counted $r!$ times, it follows that the number of different subsets of r objects that could be formed from a set of n objects is given by

$$\frac{n!}{(n-r)!r!}.$$

Definition 1.3. Define the **binomial coefficient** of n choose r by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

whenever $0 \leq r \leq n$. We also define $\dbinom{n}{r} = 0$ if $r > n$.

Note that $\dbinom{n}{0} = \dfrac{n!}{n!0!} = 1$ and $\dbinom{n}{n} = \dfrac{n!}{0!n!} = 1$. What are their meanings?

Lemma 1.4. For $0 \leq r \leq n$, we have

$$\binom{n}{r} = \binom{n}{n-r}.$$

Proof.

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)![n-(n-r)]!} = \binom{n}{n-r}.$$

■

Example 1.5.

(a) A committee of 7 is to be formed from a class of 35 students. How many different committees are possible?

(b) A president, a treasurer, a secretary and 4 admins, all different, are to be chosen from a class of 35 students. How many different committees are possible?

Solution.

(a) $\dbinom{35}{7} = 6,724,520.$

(b) $35 \times 34 \times 33 \times \dbinom{32}{4} = 1,412,149,200.$

Alternatively: $\dbinom{35}{7} \times 7 \times 6 \times 5 = 1,412,149,200.$ ■

Example 1.6.

(a) From a group of 6 men and 9 women, how many different committees consisting of 3 men and 6 women can be formed?

(b) What if two of the women will either serve together or not at all?

(c) What if two of the women refuse to serve together?

(d) What if one of the women refuses to serve together with one of the men?

Solution.

(a) $\dbinom{6}{3}\dbinom{9}{6} = 1680$

(b) The number of ways if they serve together is $\dbinom{6}{3}\dbinom{7}{4} = 700$. The number of ways if none of them serve is $\dbinom{6}{3}\dbinom{7}{6} = 140$. So the total number is $700 + 140 = 840$.

(c) $1680 - 700 = 980$.

(d) $1680 - \dbinom{5}{2}\dbinom{8}{5} = 1120.$ ■

Example 1.7. 4 girls and 7 boys are standing in a row. How many possible ways are there if no two girls are standing together?

Solution. We first arrange the 7 boys, which accounts for a 7! factor. Now, if no two girls are standing together, then the space between two boys must contain at most 1 girl. We choose 4 of the 8 spaces to place a girl there. This accounts for a $\binom{8}{4} \times 4!$ factor. So the total number is $7! \times \binom{8}{4} \times 4! = 8,467,200$. ■

$$\wedge \text{ B } \wedge \text{ B } \wedge \text{ B } \wedge \text{ B } \wedge \text{ B } \wedge \text{ B } \wedge \text{ B } \wedge$$
$$\text{B} = \text{boy}$$
$$\wedge = \text{place for at most one girl}$$

1.1.3 Properties of binomial coefficients and combinatorial proofs

In this subsubsection we present some combinatorial identities including the famous binomial formula. For each identity, we only present a combinatorial proof. It is noteworthy that every identity here has an algebraic proof, and readers are strongly recommended to formulate their own algebraic proofs as an exercise.

The philosophy of combinatorial proof is simple - we count the way of choosing object in two different ways, and they have to be equal.

Example 1.8. Reprove lemma 1.4 by a combinatorial proof: for $0 \le r \le n$, we have

$$\binom{n}{r} = \binom{n}{n-r}.$$

Proof. We have $\binom{n}{r}$ ways to choose r objects from n to form a subset. Alternatively, we can choose r objects from n by not choosing $n-r$ objects. We have $\binom{n}{n-r}$ ways not choosing $n-r$ objects, which establishes the identity. ■

Lemma 1.5 (Pascal rule)**.** If $0 < r \le n$, then

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}.$$

Proof. Consider a set of $n+1$ objects, where we fix attention on some particular one of these objects, say we call it object X. There are $\binom{n+1}{r}$ ways to choose r objects from this set of objects. On the other hand, there are $\binom{n}{r-1}$ subsets of size r that contains object X (since each subset is formed by selecting $r-1$ from the remaining n objects), and $\binom{n}{r}$ subsets of size r that does not contain object X. Thus the identity holds. ■

Theorem 1.6 (Binomial formula)**.** Let x, y be objects s.t. $xy = yx$. Then for any $n \in \mathbb{N}$, it holds

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

Proof. Upon expansion, the terms are of the form $x^r y^{n-r}$. To determine the coefficient of $x^r y^{n-r}$, note that to obtain such a term amounts to choose r x's from the n $(x+y)$ terms. Thus the coefficient is $\binom{n}{r}$ as desired. ■

Corollary 1.7. A set of n elements has 2^n subsets.

Proof. Since there are $\binom{n}{r}$ subsets containing r elements, the desired number is

$$\sum_{r=0}^{n} \binom{n}{r} = (1+1)^n = 2^n.$$

■

At this stage, readers are expected to be highly familiar with binomial formula. We reprove the formula in order to illustrate the power of combinatorial proofs. We present two more examples of combinatorial proof:

Example 1.9. Let $r \leq \max\{m, n\}$. Prove **Vandermonde's identity**

$$\binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$$

algebraically to persuade yourself this is true first. Then present a combinatorial proof.
Hints: For an algebraic proof, consider the binomial expansion of $(1+x)^{m+n}$. While for a combinatorial proof, consider forming a group of r people from n boys and m girls.

Proof.

(a) **Algebraic proof**:
Note that $(1+x)^{m+n} = (1+x)^m(1+x)^n$. Looking at the LHS, the coefficient of x^r is $\binom{n+m}{r}$. While looking at the RHS, the coefficient of x^r is $\sum_{i+j=r} \binom{n}{i}\binom{m}{j} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$ as desired.

(b) **Combinatorial proof**:
Consider forming a group of r people from n boys and m girls. Clearly we have $\binom{n+m}{r}$ ways to do so. On the other hand, we can also form $\binom{n}{k}\binom{m}{r-k}$ groups of exactly k boys, so summing through $0, 1, 2 \cdots, r$ gives the number of possible groups of r people. In other words, it holds $\binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$. ■

Example 1.10. Let $n \geq r$. Prove the **hockey stick identity**

$$\binom{n+1}{r+1} = \sum_{k=r}^{n} \binom{k}{r}$$

algebraically to persuade yourself this is true first. Then present a combinatorial proof.

Proof.

(a) **Algebraic proof**:
We perform induction on n:

(i) When $n = r$, we have

$$\binom{r+1}{r+1} = 1 = \binom{r}{r}.$$

(ii) Suppose $\binom{n+1}{r+1} = \sum_{k=r}^{n} \binom{k}{r}$. Then

$$\begin{aligned} \sum_{k=r}^{n+1} \binom{k}{r} &= \sum_{k=r}^{n} \binom{k}{r} + \binom{n+1}{r} \\ &= \binom{n+1}{r+1} + \binom{n+1}{r} \\ &= \binom{n+2}{r+1} && \text{by Pascal rule} \end{aligned}$$

By induction, the proof is completed.

(b) **Combinatorial proof**:
Consider the set $\{1, 2, \cdots, n, n+1\}$. We can form $\binom{n+1}{r+1}$ subsets of size $r+1$. On the other hand, if the greatest integer in the subset is $k+1$, then we have $\binom{k}{r}$ ways to form such a subset. Summing through $r, r+1, \cdots, n$ gives the total number of subsets of size $r+1$ (no need to sum through $1, 2, \cdots, r-1$ because our set has $r+1$ elements). ■

1.1.4 Multinomial coefficients

The problem of interest in this subsubsection is:

- Given a set of n objects, how many ways can we partition it into r subsets of sizes $n_1, n_2, \cdots, n_r$ respectively, where $\sum_{i=1}^{r} n_i = n$?

This is not hard to answer:

(i) We have $\binom{n}{n_1}$ ways to form the first subset of n_1 objects.

(ii) We have $\binom{n-n_1}{n_2}$ ways to form the second subset of n_2 objects from the $n-n_1$ remained objects.

(iii) We have $\binom{n-n_1-n_2}{n_3}$ ways to form the third subset of n_3 objects from the $n-n_1-n_2$ remained objects.

We can repeat in this fashion until we have formed the $(r-1)^{\text{th}}$ subset. The r^{th} subset is automatically formed at this step. By fundamental theorem of counting, there are

$$\begin{aligned}
&\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{r-2}}{n_{r-1}} \\
&= \frac{n!}{(n-n_1)!n_1!}\cdot\frac{(n-n_1)!}{(n-n_1-n_2)!n_2!}\cdot\frac{(n-n_1-n_2)!}{(n-n_1-n_2-n_3)!n_3!}\cdots\frac{(n-n_1-n_2-\cdots-n_{r-2})!}{(n-n_1-n_2-\cdots-n_{r-2}-n_{r-1})!n_{r-1}!} \\
&= \frac{n!}{n_1!n_2!n_3!\cdots n_{r-1}!(n-n_1-n_2-\cdots-n_{r-2}-n_{r-1})!} \\
&= \frac{n!}{n_1!n_2!n_3!\cdots n_{r-1}!n_r!}
\end{aligned}$$

possible partitions.

It makes sense to define the following notation:

Definition 1.8. If $\sum_{i=1}^{r} n_i = n$, we define the **multinomial coefficient** $\binom{n}{n_1, n_2, \cdots, n_r}$ by

$$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1!n_2!n_3!\cdots n_{r-1}!n_r!}.$$

Lemma 1.9. For $0 \le r \le n$, we have

$$\binom{n}{r, n-r} = \binom{n}{r},$$

where on left hand side we have a multinomial coefficient, and on right hand side we have a binomial coefficient.

Proof. Clear. ■

Example 1.11. A police department in a small city consists of 10 officers. If the department policy is to have 5 officers patrolling the streets, 2 officers working full time at the station, and 3 officers on reserve, how many different divisions of the 10 officers into the 3 groups are possible?

Solution. $\binom{10}{5, 2, 3} = 2520$ ■

We can also redo the word permutation problems using multinomial coefficients:

Example 1.12. How many different letter arrangements can be formed from the letters `MISSISSIPPI`?

Solution. This amounts to partition 11 places to the 1 `M`, 4 `I`'s, 4 `S`'s and 2 `P`'s. Hence there are $\binom{11}{1, 4, 4, 2} = 34650$ possible ways. ■

1.1.5 Number of integral solutions

In this subsubsection we will study the number of nonnegative integral solutions of equations of the form

$$x_1 + x_2 + \cdots + x_r = n$$

where $r, n \in \mathbb{N}$. This equation is closely related to a problem called **combination with repetition**.

It would be easier to consider positive solutions first:

Lemma 1.10. If $r, n \in \mathbb{N}$, then the equation

$$x_1 + x_2 + \cdots + x_r = n$$

has $\binom{n-1}{r-1}$ positive integral solutions.

Proof. Suppose we are given n objects. To count the number of positive integral solutions of the equation, we have to count the ways to divide the objects into r nonempty subsets. To do so, we can select $r-1$ of the $n-1$ spaces between adjacent objects as our partition points, so there are $\binom{n-1}{r-1}$ such solutions. ■

$$1 \ \wedge \ 1 \ \wedge \ 1 \ \wedge \ \cdots \ \wedge \ 1 \ \wedge \ 1 \ \wedge \ 1$$
$1 =$ object
$\wedge =$ place for partition points

Theorem 1.11. If $r, n \in \mathbb{N}$, then the equation

$$x_1 + x_2 + \cdots + x_r = n$$

has $\binom{n+r-1}{r-1}$ nonnegative integral solutions.

Proof. The number of nonnegative integral solutions of $x_1 + x_2 + \cdots + x_r = n$ is the same as the number of positive integral solutions of $y_1 + y_2 + \cdots + y_r = n + r$ (which can be seen by taking $y_i = x_i + 1$). By the lemma, the latter equation has $\binom{n+r-1}{r-1}$ positive integral solutions, so the former equation also has $\binom{n+r-1}{r-1}$ nonnegative integral solutions. ■

We now illustrate the use of the above theorem to solve real-life problems related to combinations with repetition.

Example 1.13. An investor has 20 thousand dollars to invest among 4 possible investments. Each investment must be in units of a thousand dollars.

(a) If the total of 20 thousand is to be invested, how many different investment strategies are possible?

(b) What if not all the money needs to be invested?

Proof.

(a) Let x_i be the number of thousands invested in investment i. Then x_i's satisfy the equation

$$x_1 + x_2 + x_3 + x_4 = 20, \quad x_i \geq 0.$$

Thus there are $\binom{20+4-1}{4-1} = 1771$ different investment strategies.

(b) We further let x_5 be the number of thousands not invested. Then x_i's satisfy the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20, \quad x_i \geq 0.$$

Thus there are $\binom{20+5-1}{5-1} = 10626$ different investment strategies. ■

Example 1.14. From an unlimited selection of five types of soda, one of which is Dr. Pepper, you are putting 25 cans on a table. Determine the number of ways you can select 25 cans of soda if

(a) there are no restrictions.

(b) you must include at least seven Dr. Peppers.

(c) it turns out there are only three Dr. Peppers available.

Solution. Suppose we have selected x_i cans of soda of type i, where x_5 counts the number of cans of Dr. Pepper selected.

(a) Note that the x_i's satisfy the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 25, \quad x_i \geq 0.$$

Thus there are $\binom{25+5-1}{5-1} = 23751$ different ways.

(b) Note that the x_i's satisfy the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 25, \quad x_i \geq 0 \text{ and } x_5 \geq 7.$$

We transform the equation by setting $y_5 = x_5 - 7$, so

$$x_1 + x_2 + x_3 + x_4 + y_5 = 18, \quad x_i, y_5 \geq 0.$$

Thus there are $\binom{18+5-1}{5-1} = 7315$ different ways.

(c) Note that the x_i's satisfy the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 25, \quad x_i \geq 0 \text{ and } 0 \leq x_5 \leq 3.$$

The desired number can be obtained by subtracting the number of solutions of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 25, \quad x_i \geq 0 \text{ and } x_5 \geq 4$$

from the total number of unrestricted ways. We transform the equation by setting $y_5 = x_5 - 4$, so

$$x_1 + x_2 + x_3 + x_4 + y_5 = 21, \quad x_i, y_5 \geq 0.$$

Thus there are $23751 - \binom{21+5-1}{5-1} = 11101$ different ways. ■

The following final computational example would be an important concept sharpener:

Example 1.15. Suppose you have 100 each of the following six types of tea bags: Black, Chamomile, Earl Grey, Green, Jasmine and Rose. Determine the number of ways to perform the following tasks:

(a) You are making a cup of tea for the Provost, a math professor and a student.

(b) You are making a cup of tea for the Provost, a math professor and a student. Each person will have a different flavour.

(c) You are making a pot of tea with four tea bags.

(d) You are making a pot of tea with four tea bags, each a different flavour.

(e) There are 10 people at a party and each person wants a different flavour of tea.

Solution.

(a) $6 \times 6 \times 6 = 216$

(b) $6 \times 5 \times 4 = 120$

(c) Let x_i be the number of tea bags of type i, where we assume x_6 is the number of Rose tea bags. Then the x_i's satisfy the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 30, \quad x_i \geq 0.$$

Thus there are $\binom{4+6-1}{6-1} = 126$ different ways.

(d) $\binom{6}{4} = 15$

(e) 0. We got only 6 types of tea. ■

1.2 Pigeonhole principle

In this subsection, we present the famous pigeonhole principle. The pigeonhole principle states something like water is wet, yet it is very useful.

Theorem 1.12 (Pigeonhole principle). If n items are put into m containers, with $n > m$, then at least one container contains more than one item.

Proof. We prove the contrapositive. If every container holds no more than one item, then the maximum number of items cannot exceed the number of containers, i.e. $n \le m$. ■

Corollary 1.13. If $m > n$, then there are no injective functions from a set with m elements to a set with n elements.

Proof. In this case the m items are each x in the domain, and the n containers are each y in the codomain. Since we have more items than containers, by pigeonhole principle, at least one container contains more than one item. In other words, there exists a element y in the codomain corresponds to at least two elements in the domain, say x_1, x_2. Then $f(x_1) = f(x_2) = y$, so f is not injective. ■

Example 1.16. Some simple applications of pigeonhole principle:

- In any group of 367 people, there must be at least two with the same birthday.

  Remark. However, in a class of 23 students, the probability that at least two students share the same birthday is surprisingly high, exceeding 50%. This is the famous **birthday paradox**.

- An average human head has $150,000$ hairs, so it makes sense to assume that no one has more than $1,000,000$ hairs. The population of Hong Kong is more than $7,500,000$ as of 2025, so there are at least 2 Hong Kong citizens having the same number of hairs.
- If a Martian has an infinite number of red, blue, yellow, and black socks in a drawer, s/he needs to draw at least 5 socks to guarantee a pair.

We can even have more than a qualitative statement:

Theorem 1.14 (Generalized pigeonhole principle). If N items are put into k containers, then there is at least one container containing at least $\left\lceil \frac{N}{k} \right\rceil$ items.

Here $\lceil \cdot \rceil$ is the **ceiling function** where $\lceil x \rceil$ is the smallest integer greater than or equal to x. For example, $\lceil 7 \rceil = 7$, $\lceil 6.9 \rceil = 7$ and $\lceil -6.9 \rceil = -6$. Obviously

$$x \le \lceil x \rceil < x + 1.$$

Proof. We prove it by contradiction. Suppose on the contrary that no container contains more than $\left\lceil \frac{N}{k} \right\rceil - 1$ objects. Then there are a total of

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N$$

items, which contradicts our assumption that we have N items. ■

Example 1.17. In a class of 100 students, there are at least $\left\lceil \frac{100}{12} \right\rceil = 9$ students who were born in the same month.

The philosophy of pigeonhole principle is about thinking of the worst case, which we illustrate as in the following example:

Example 1.18.

(a) How many cards must be drawn from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

(b) How many must be drawn to guarantee that at least three hearts ♥ are selected?

Solution.

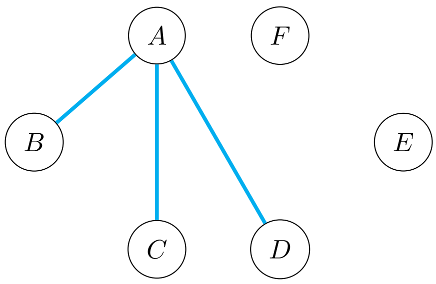
(a) In the worst case, when we have drawn 8 cards, we have 2 cards of each suit. Even so, we cannot avoid having a third card of the same suit when we draw the 9^{th} card, so 9 cards suffice.

(b) In the worst case, when when we have drawn 39 cards, we have all the diamonds ♦, clubs ♣, and spades ♠, before drawing a single heart ♥. The next three cards must be all hearts ♥, so 42 cards suffice. ■

In my opinion, the most interesting example of the pigeonhole principle is as follows:

Theorem 1.15 (Party theorem)**.** Assume that in a party of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.

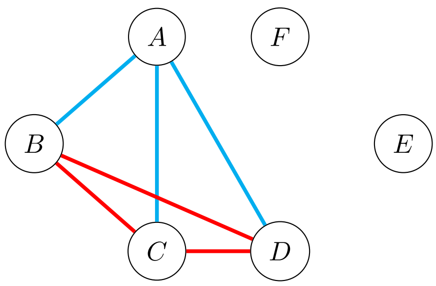
Proof. Let's focus on an individual A. By generalized pigeonhole principle, there are either three or more who are friends of A, or three or more who are enemies of A. Without loss of generality we suppose B, C, D are friends of A. Let's visualize it by a **graph**:



A blue line represents a friendship, and a red line represents an enemyship. So three mutual friends are represented by a blue triangle, while three mutual enemies are represented by a red triangle. Suppose on the contrary that there are no three mutual friends or three mutual enemies in the group. Then

- B and C are enemies, otherwise A, B and C are mutual friends, contradicting our assumption.
- B and D are enemies, otherwise A, B and D are mutual friends, contradicting our assumption.
- C and D are enemies, otherwise A, C and D are mutual friends, contradicting our assumption.

We update these information into our graph:



Thus a red triangle ΔBCD is formed, meaning that B, C and D are mutual enemies. Contradiction arises. ■

1.3 Principle of inclusion-exclusion

In this subsection, we will introduce the tasty pie in combinatorics - the princple of inclusion-exclusion. The baby case is as follows:

Proposition 1.16. $|A \cup B| = |A| + |B| - |A \cap B|$ for any finite sets A, B.

Proof. $|A| + |B|$ counts elements in $A \setminus B$ and $B \setminus A$ exactly once, and elements in $A \cap B$ twice, so $|A| + |B| - |A \cap B|$ would count every element in $A \cup B$ exactly twice. ∎

We can use the idea of this proof to generalize the pie to finitely many sets:

Theorem 1.17 (Principle of inclusion-exclusion)**.** For any finite sets $A_1, A_2, \cdots, A_n$, it holds

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{n+1} \left|\bigcap_{i=1}^{n} A_i\right|.$$

We can even write it more compactly as

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \ne S \subset \{1,2,\cdots,n\}} (-1)^{|S|+1} \left|\bigcap_{i \in S} A_i\right|,$$

but this would be too compact for our usage. In our discussion, we will most frequently use the formula for $n = 2, 3, 4$.

- When $n = 3$, the formula reads

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

- When $n = 4$, the formula reads

$$\begin{aligned} |A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| \\ & - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & - |A \cap B \cap C \cap D|. \end{aligned}$$

Proof of principle of inclusion-exclusion. We will show that every element of $\bigcup_{i=1}^{n} A_i$ is counted exactly once by the RHS. Suppose a is a member of exactly r of the sets $A_1, A_2, \cdots, A_n$. Note that a is counted $\binom{r}{k}$ times in the summmation involving intersections of k of the sets. Thus a is counted

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{r}{k} = -\sum_{k=0}^{n} (-1)^k \binom{r}{k} + 1 = -(1-1)^n + 1 = 1$$

time by the RHS. ∎

Sometimes we may also use the following form:

Corollary 1.18 (Principle of inclusion-exclusion)**.** For any finite set S with $A_1, A_2, \cdots, A_n \subset S$, it holds

$$\left|\bigcap_{i=1}^{n} A_i^c\right| = |S| - \sum_{i=1}^{n} |A_i| + \sum_{1 \le i < j \le n} |A_i \cap A_j| - \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n \left|\bigcap_{i=1}^{n} A_i\right|$$

where $A^c := S \setminus A \ \forall A \subset S$.

Proof. Simply note that $\bigcap_{i=1}^{n} A_i^c = \left(\bigcup_{i=1}^{n} A_i\right)^c$ by de Morgan's law, so by the baby case,

$$|S| = \left|\bigcap_{i=1}^{n} A_i^c\right| + \left|\bigcup_{i=1}^{n} A_i\right|.$$

Now plug in the original form of the principle, done. ∎

Example 1.19. In a high school, there are 1000 students taking at least one of the physics, chemistry, and biology courses. It is given that:

- The total number of students who take physics is 310.
- The total number of students who take chemistry is 650.
- The total number of students who take biology is 440.
- The total number of students who take physics and chemistry is 170.
- The total number of students who take physics and biology is 150.
- The total number of students who take chemistry and biology is 180.

How many students takes all three courses?

Solution. Let P, K, B denote the set of all physics, chemistry, and biology students respectively. Then by principle of inclusion-exclusion,

$$|P \cup K \cup B| = |P| + |K| + |B| - |P \cap K| - |K \cap B| - |B \cap P| + |P \cap K \cap B|.$$

So $1000 = 310 + 650 + 440 - 170 - 150 - 180 + |P \cap K \cap B|$, i.e. $|P \cap K \cap B| = 100$. So 100 students take all three courses. ■

1.3.1 Number of surjective functions.

The next application is to count the number of surjective functions $\{1, 2, \cdots, m\} \to \{1, 2, \cdots, n\}$. It is clear that $m \geq n$.

First recall there are n^m functions $\{1, 2, \cdots, m\} \to \{1, 2, \cdots, n\}$. This is obviously overshooting, so we shall discount the $\binom{n}{1}(n-1)^m$ functions that misses at least one value in $\{1, 2, \cdots, n\}$. But this is then undershooting because functions that miss at least two values are counted twice. Thus, we need to add back these $\binom{n}{2}(n-2)^m$ functions. This again overshoots; we need to subtract $\binom{n}{3}(n-3)^m$ functions that miss at least three values. Repeat in this fashion, we can deduce that:

- For $m \geq n$, there are $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^m$ surjective functions $\{1, 2, \cdots, m\} \to \{1, 2, \cdots, n\}$.

By a simple change of variables, we have

Theorem 1.19. For $m \geq n$, there are $\displaystyle\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^m$ surjective functions $\{1, 2, \cdots, m\} \to \{1, 2, \cdots, n\}$.

Example 1.20. How many ways are there to assign 5 different tasks to 4 different trainees if each trainee is assigned at least 1 task?

Solution. We view the assignment of tasks as a function from the set of 5 different tasks to the set of 4 different trainees. An assignment that each trainee is assigned at least 1 task is the same as a surjective function from the set of 5 different tasks to the set of 4 different trainees. So by the above formula, the desired number is

$$\sum_{k=0}^{4}(-1)^{n-k}\binom{4}{k}k^5 = 240.$$

■

1.3.2 Derangements

A **derangement** is a permutation in which no objects are left in their original positions. In other words, if we derange $1, 2, \cdots, n$ to $i_1, i_2, \cdots, i_n$, then $i_k \neq k$. Our goal is to count the number D_n of derangements of $\{1, 2, \cdots, n\}$.

First recall there are $n!$ permutations of $\{1, 2, \cdots, n\}$. This is obviously overshooting, so we shall discount the $\binom{n}{1}(n-1)!$ permutations that fix at least one value in $\{1, 2, \cdots, n\}$. But this is then undershooting because permutations that fix at least two values are discounted twice. Thus, we need to add back these $\binom{n}{2}(n-2)!$ permutations. This again overshoots; we need to subtract $\binom{n}{3}(n-3)!$ permutations that fix at least three values. Repeat in this fashion, we can deduce the following formula:

$$\begin{aligned} D_n &= \sum_{k=0}^{n}(-1)^k \binom{n}{k}(n-k)! \\ &= \sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!} \cdot (n-k)! \\ &= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \end{aligned}$$

Theorem 1.20. There are $n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ derangements of $\{1, 2, \cdots, n\}$.

The formula of derangement is not easy to compute. But we still have ways to facilitate their estimations/ computations.

The first interesting observation is that

$$\lim_{n\to\infty} \frac{D_n}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1}.$$

Recall an alternating series $L = \sum_{n=0}^{\infty}(-1)^n a_n$ with partial sums $S_n = \sum_{k=0}^{n}(-1)^k a_k$ satisfy the following estimation formula:

$$|S_n - L| \leq |S_n - S_{n+1}| = a_{n+1}$$

In the case of the derangement, we have

$$\left|\frac{D_n}{n!} - e^{-1}\right| \leq \frac{1}{(n+1)!}.$$

When $n \geq 7$, the error bound $\frac{1}{(n+1)!} \leq \frac{1}{8!} = \frac{1}{40320}$ is very small, and $\frac{D_n}{n!}$ and e^{-1} agree to at least 3 decimal places. Hence the following estimation makes sense:

Corollary 1.21. $D_n \approx n!e^{-1}$ with error bound $\left|D_n - n!e^{-1}\right| \leq \frac{1}{n+1}$.

n	D_n	$n!e^{-1}$ (5 d.p.)
1	0	0.36788
2	1	0.73576
3	2	2.20728
4	9	8.82911
5	44	44.14553
6	265	264.87320
7	1854	1854.11238
8	14833	14832.89907
9	133496	133496.09161
10	1334961	1334960.91612

Two more formulas for derangement are displayed as follows:

Proposition 1.22. $D_n = (n-1)(D_{n-1} + D_{n-2})\ \forall n \geq 3$.

Proof. Consider the D_n derangments of $\{1, 2, \cdots, n\}$. We can naturally partition them into $n-1$ classes of equal sizes based on the first position. Thus $D_n = (n-1)d_n$ where d_n counts the derangements in which 2 is in the first position. These derangements are of the form

$$2, i_2, i_3, \cdots, i_n, \quad i_k \neq k.$$

Note that we can further partition the derangements of such form into two classes: $i_2 = 1$ and $i_2 \neq 1$. Let d_n' be the number of derangements of the form

$$2, 1, i_3, \cdots, i_n, \quad i_k \neq k,$$

and d_n'' be the number of derangements of the form

$$2, i_2, i_3, \cdots, i_n, \quad i_2 \neq 1 \text{ and } i_k \neq k.$$

Thus $d_n = d_n' + d_n''$ and hence $D_n = (n-1)(d_n' + d_n'')$. It remains to compute the numbers d_n' and d_n''.

- Note that the number of derangements of the form

  $$2, 1, i_3, \cdots, i_n, \quad i_k \neq k$$

  naturally induces an derangement of $\{3, 4, \cdots, n\}$, so there are D_{n-2} derangement of such form, i.e. $d_n' = D_{n-2}$.
- Note that the number of derangements of the form

  $$2, i_2, i_3, \cdots, i_n, \quad i_2 \neq 1 \text{ and } i_k \neq k$$

  naturally induces a permutation $i_2, i_3, \cdots, i_n$ of $\{1, 3, 4, \cdots, n\}$ in which $1 \neq i_2, 3 \neq i_3, \cdots, i_n \neq n$. By renaming $1 \mapsto 2$, we can see there are D_{n-1} such permutations, so $d_n'' = D_{n-1}$. ■

Proposition 1.23. $D_n = nD_{n-1} + (-1)^n\ \forall n \geq 2$

Proof. Rewrite $D_n - nD_{n-1} = (-1)(D_{n-1} - (n-1)D_{n-2})$ for $n > 2$. Then we iterate the formula as follows:

$$\begin{aligned}
D_n - nD_{n-1} &= (-1)(D_{n-1} - (n-1)D_{n-2}) \\
&= (-1)^2(D_{n-2} - (n-2)D_{n-3}) \\
&= (-1)^3(D_{n-3} - (n-3)D_{n-4}) \\
&= \cdots \\
&= (-1)^{n-2}(D_2 - 2D_1)
\end{aligned}$$

Put $D_2 = 1$ and $D_1 = 0$, we have $D_n - nD_{n-1} = (-1)^{n-2} = (-1)^n$ for $n > 2$ as desired. Finally, it is easy to check that for $n = 2$, we have $D_1 + (-1)^2 = 0 + 1 = 1 = D_2$. ■

These are two basic examples of **recurrence relation**. In the next section, we will study the theory of recurrence relations in depth.

Example 1.21. At a party there are n men and n women. In how many ways can the n women choose male partners for the first dance? How many ways are there for the second dance if everyone has to change partners?

Proof. For the first dance there are $n!$ ways. For the second dance each woman has to choose as a partner a man other than the one with whom she first danced. So there are D_n ways. ■

2 Recurrence relations and generating functions

2.1 Recurrence relations

Consider a sequence $\{a_n\}_{n\in\omega}$. We can describe a_n in terms of lower terms $a_{n-1}, a_{n-2}, \cdots, a_0$. For example, we can write an **arithmetic sequence** with the **common difference** d in the form of

$$a_n = a_{n-1} + d,$$

and a **geometric sequence** with the **common ratio** r in the form of

$$a_n = ra_{n-1}.$$

These kind of expressions are called **recurrence relations**.

On the other hand, let's suppose the only thing we know about a certain sequence is a recurrence relation. Sometimes we may be able to look for a general term of the sequence. For example, suppose the only thing we know about an arithmetic sequence is the above recurrence relation. Then we can use the **method of iteration** to find that

$$\begin{aligned} a_n &= a_{n-1} + d \\ &= (a_{n-2} + d) + d = a_{n-2} + 2d \\ &= (a_{n-3} + d) + 2d = a_{n-3} + 3d \\ &= \cdots \\ &= a_1 + (n-1)d \\ &= (a_0 + d) + (n-1)d = a_0 + nd \end{aligned}$$

Similarly, for a geometric sequence:

$$\begin{aligned} a_n &= ra_{n-1} \\ &= r(ra_{n-2}) = r^2 a_{n-2} \\ &= r^2(ra_{n-3}) = r^3 a_{n-3} \\ &= \cdots \\ &= r^{n-1} a_1 \\ &= r^{n-1}(ra_0) = r^n a_0 \end{aligned}$$

Example 2.1. An **arithmetico-geometric sequence** is a sequence $\{a_n\}_{n\geq 0}$ satisfying the recurrence relation

$$a_n = ra_{n-1} + d$$

where r, d are constants called the **common ratio** and **common difference** respectively. What is its general term?

Solution. If $r = 1$ then $a_n = a_0 + nd\ \forall n$ by the general term of arithmetic sequence. Hence suppose $r \neq 1$. Let k be such that $a_n - k = r(a_{n-1} - k)$, so $\{a_n - k\}$ is a geometric sequence with common ratio r. Then by the general term of geometric sequence, we know

$$a_n - k = r^n(a_0 - k) \implies a_n = r^n(a_0 - k) + k.$$

It remains to determine the value of k. Note that $(1-r)k = d$, so $k = \dfrac{d}{1-r}$. Therefore the general term is

$$a_n = r^n\left(a_0 - \frac{d}{1-r}\right) + \frac{d}{1-r}.$$

■

Remark. The label of arithmetico-geometric sequence may also be given to different objects combining characteristics of both arithmetic and geometric sequences. The one we gave above is the **French notion**. Another commonly used notion is the product of an arithmetic sequence and a geometric sequence, namely if $\{a_n\}$ is an arithmetic sequence and $\{b_n\}$ is a geometric sequence, then $\{a_nb_n\}$ is called a arithmetico-geometric sequence.

2.1.1 Modeling with recurrence relations

Recurrence relations are useful discrete models, as we would illustrate in the following examples.

Example 2.2 (Compound interest)**.** Suppose that a person deposits $\$P$ in a savings account at a bank at an interest rate of $r\%$ per annum compounded annually. How much will be in the account after t years?

Solution. Let P_t denote the amount in the account after t years. Because the amount in the account after t years equals the amount in the account after $t-1$ years plus interest for the t^{th} year, we see that the sequence $\{P_t\}$ satisfies the recurrence relation

$$P_t = P_{t-1} + P_{t-1} \times r\% = (1 + r\%)P_{t-1} \ \forall t > 0,$$

with $P_0 = P$. By the general term of geometric sequence, we have $P_t = (1 + r\%)^t P_0 = P(1 + r\%)^t$. ■

Example 2.3 (Fibonnacci sequence)**.** A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair (with one of each sex) every month. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.

Solution. Let f_n be the number of pairs of rabbits at the end of the n^{th} month.

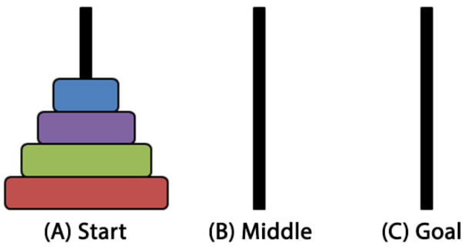
- At the end of the first month, there is 1 pair of rabbits, so $f_1 = 1$.
- At the end of the second month, there is still only 1 pair of rabbits, because they don't breed in the second month. Thus $f_2 = 1$.
- At the end of the n^{th} month, we have f_{n-1} pairs of rabbits from the previous month, as well as the f_{n-2} pairs of newborn rabbits breed by the f_{n-2} pairs of rabbits at least 2 months old. Thus $f_n = f_{n-1} + f_{n-2}$.

Therefore the desired recurrence relation is

$$\begin{cases} f_n = f_{n-1} + f_{n-2} \ \forall n \geq 3 \\ f_1 = f_2 = 1. \end{cases}$$

■

Example 2.4 (Tower of Hanoi)**.** A popular puzzle, called the Tower of Hanoi, consists of three pegs mounted on a board together with disks of different sizes. Initially, these disks are placed on peg A in order of size, with the largest on the bottom (as shown in figure). The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The goal of the puzzle is to have all the disks on peg C in order of size, with the largest on the bottom. Let H_n denote the minimal number of moves needed to solve the Tower of Hanoi problem with n disks. Find the general term of the sequence $\{H_n\}$.



Solution.

- To solve a Hanoi tower with only 1 disk, we need exactly 1 move, so $H_1 = 1$.
- To solve a Hanoi tower with n disks ($n > 1$), we follow this scheme:
  (i) Transfer the top $n-1$ disks from peg A to peg B. We need H_{n-1} moves.
  (ii) Transfer the bottom disk from peg A to peg C. We need 1 move.
  (iii) Transfer the top $n-1$ disks from peg B to peg C. We need H_{n-1} moves.
  So $H_n = 2H_{n-1} + 1$.

To apply the general term of arithmetico-geometric sequence, we may define $H_0 = 0$ (by extending the recurrence relation to $H_1 = 2H_0 + 1$). Thus we have $H_n = 2^n - 1$. ■

2.2 Second order linear constant coefficient recurrence relations

2.2.1 Basic theory

To understand what we need to do in this subsection, let's classify recurrence relations first:

- A **linear** recurrence relation of order k is a recurrence relation of the form

$$a_n = \alpha_1(n)a_{n-1} + \alpha_2(n)a_{n-2} + \cdots + \alpha_k(n)a_{n-k} + \beta(n),$$

where $\alpha_1(n), \alpha_2(n), \cdots, \alpha_k(n)$ are functions of n and $\alpha_k(n) \neq 0$. Furthermore,

- if $\beta(n) = 0\ \forall n$, we say the linear recurrence relation is **homogeneous**.
- if $\alpha_i(n)$ are all constants, we say the linear recurrence relation have **constant coefficients**.

Note that if the recurrence relation is non-linear, we cannot further classify if it is homogeneous or have constant coefficients.

Example 2.5.

Recurrence relation	Linear?	Homogeneous?	Constant coefficient?	Order?
$P_t = (1 + r\%)P_{t-1}$	✓	✓	✓	1
$f_n = f_{n-1} + f_{n-2}$	✓	✓	✓	2
$H_n = 2H_{n-1} + 1$	✓	✗	✓	1
$D_n = (n-1)(D_{n-1} + D_{n-2})$	✓	✓	✗	2
$D_n = nD_{n-1} + (-1)^n$	✓	✗	✗	1

So now we know our main goal is to solve recurrence relations of the form

$$a_n = Aa_{n-1} + Ba_{n-2} + \beta(n),$$

where A, B are constants with $B \neq 0$, and $\beta(n)$ is a function. Here we develop a minimal amount of basic theory under an extra homogeneous assumption that $\beta \equiv 0$.

Theorem 2.1 (Principle of superposition)**.** Consider the second order linear homogenous recurrence relation

$$a_n = Aa_{n-1} + Ba_{n-2}.$$

If $\{a_n\}$ and $\{b_n\}$ are solutions of the recurrence relations, then $\{C_1a_n + C_2b_n\}$ is also a solution, where C_1, C_2 are constants.

Proof. Since $\{a_n\}$ and $\{b_n\}$ are solutions, we have

$$a_n = Aa_{n-1} + Ba_{n-2} \quad \text{and} \quad b_n = Ab_{n-1} + Bb_{n-2}.$$

Hence

$$\begin{aligned} &A(C_1a_{n-1} + C_2b_{n-1}) + B(C_1a_{n-2} + C_2b_{n-2}) \\ &= C_1Aa_{n-1} + C_2Ab_{n-1} + C_1Ba_{n-2} + C_2Bb_{n-2} \\ &= C_1(Aa_{n-1} + Ba_{n-2}) + C_2(Ab_{n-1} + Bb_{n-2}) \\ &= C_1a_n + C_2b_n \end{aligned}$$

So $\{C_1a_n + C_2b_n\}$ is also a solution. ■

Now suppose we already yield two solutions $\{a_n^{(1)}\}, \{a_n^{(2)}\}$. We need to determine the solution $a_n = C_1a_n^{(1)} + C_2a_n^{(2)}$ that satisfy the given values of a_0, a_1. In other words,

$$\begin{cases} C_1a_0^{(1)} + C_2a_0^{(2)} = a_0 \\ C_1a_1^{(1)} + C_2a_1^{(2)} = a_1 \end{cases}$$

where C_1, C_2 are constants to be determined. Now we can write the equation in matrix form

$$\begin{bmatrix} a_0^{(1)} & a_0^{(2)} \\ a_1^{(1)} & a_1^{(2)} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}.$$

This motivates us to define the **Wronskian** of $\{a_n^{(1)}\}, \{a_n^{(2)}\}$:

$$W_n := \begin{vmatrix} a_n^{(1)} & a_n^{(2)} \\ a_{n+1}^{(1)} & a_{n+1}^{(2)} \end{vmatrix} = a_n^{(1)}a_{n+1}^{(2)} - a_{n+1}^{(1)}a_n^{(2)}$$

When $W_0 \neq 0$, we know the above linear system has a unique solution.

In the language of linear algebra, when $W \neq 0$, the solutions $\{a_n^{(1)}\}, \{a_n^{(2)}\}$ are linearly independent, and span the solution space of the second order linear constant coefficient homogeneous recurrence relation. In other words, The solution space of the recurrence relation satisfies the conditions of a vector space and the two solutions $\{a_n^{(1)}\}, \{a_n^{(2)}\}$ are the basis of the solution space. The dimension of the solution space is 2 which corresponds to the order of the recurrence relation.

Example 2.6. Show that $a_n^{(1)} = r_1^n$ and $a_n^{(2)} = r_2^n$ have a nonzero Wronskian when $r_1 \neq r_2$ and r_1, r_2 are nonzero.

Proof.

$$W_n = \begin{vmatrix} r_1^n & r_2^n \\ r_1^{n+1} & r_2^{n+1} \end{vmatrix} = r_1^n r_2^{n+1} - r_1^{n+1} r_2^n = r_1^n r_2^n (r_2 - r_1) \neq 0$$

■

2.2.2 Homogeneous case

In the following we solve the recurrence relations

$$a_n = Aa_{n-1} + Ba_{n-2}, \tag{*}$$

where A, B are constants with $B \neq 0$. Recall the solution of the recurrence relations $a_n = ra_{n-1}$ is of the form $a_n = a_0 r^n$, so an educated guess of a solution of (*) would be

$$a_n = r^n$$

where r is a constant to be determined. We can put it into (*) to get

$$r^n = Ar^{n-1} + Br^{n-2}$$

Under the assumption that $r \neq 0$, we can obtain

$$r^2 = Ar + B \quad \Longrightarrow \quad r^2 - Ar - B = 0.$$

This is called the **characteristic equation** of (*). By quadratic formula we know the roots of the characteristic equation are

$$r_\pm = \frac{A \pm \sqrt{A^2 + 4B}}{2}.$$

As usual, the solutions would depend on the discriminant $D = A^2 + 4B$. Furthermore, note that $B \neq 0$, so $r_\pm \neq 0$.

Case I: $D \neq 0$.

In this case $r_\pm$ are distinct. By principle of superposition, the general solution of (*) is

$$a_n = C_1 r_+^n + C_2 r_-^n,$$

where C_1, C_2 are constants to be determined.

Example 2.7. Solve the recurrence relation $\begin{cases} f_n = f_{n-1} + f_{n-2} \ \forall n \geq 2 \\ f_0 = 0, f_1 = 1 \end{cases}$.

Solution. The characteristic equation is $r^2 = r + 1$. Solving, the roots are

$$\varphi = \frac{1 + \sqrt{5}}{2}, \quad \psi = \frac{1 - \sqrt{5}}{2}$$

So $f_n = C_1 \varphi^n + C_2 \psi^n$ for some constants C_1, C_2. By using the initial conditions $f_0 = 0, f_1 = 1$, we have

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 \varphi + C_2 \psi = 1 \end{cases}.$$

Solving, we have $C_1 = -C_2 = \dfrac{1}{\sqrt{5}}$. Therefore

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

■

Remark. The Fibonacci numbers form an integer sequence, but it looks like a sequence of irrational numbers from its general term above.

Example 2.8. Solve the recurrence relation $\begin{cases} a_n = 2a_{n-1} - 5a_{n-2} \ \forall n \geq 2 \\ a_0 = 1, a_1 = 5 \end{cases}$.

Solution. The characteristic equation is $r^2 = 2r - 5$. Solving, the roots are

$$r_1 = 1 + 2i, \quad r_2 = 1 - 2i.$$

So $a_n = C_1 r_1^n + C_2 r_2^n$ for some constants C_1, C_2. By using the initial conditions $a_0 = 1, a_1 = 5$, we have

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 r_1 + C_2 r_2 = 5 \end{cases}.$$

Solving, we have $C_1 = \overline{C_2} = \dfrac{1-2i}{2}$. Therefore

$$\begin{aligned} a_n &= \frac{1-2i}{2}(1+2i)^n + \frac{1+2i}{2}(1-2i)^n \\ &= \frac{5}{2}(1+2i)^{n-1} + \frac{5}{2}(1-2i)^{n-1} \end{aligned}$$

■

Remark. The sequence is clearly real, but its general term involves complex numbers.

Case II: $D = 0$.

In this case $r_- = r_+$, let's just call them r. We know $a_n = r^n$ would be a solution, but then how to find another solution? Let's try $a_n = nr^n$:

$$\begin{aligned} Aa_{n-1} + Ba_{n-2} &= A(n-1)r^{n-1} + B(n-2)r^{n-2} \\ &= Anr^{n-1} + Bnr^{n-2} - Ar^{n-1} - 2Br^{n-2} \\ &= nr^{n-2}(Ar + B) - r^{n-2}(Ar + 2B) \end{aligned}$$

By the condition $D = A^2 + 4B = 0$, we know

$$r = \frac{A \pm \sqrt{D}}{2} = \frac{A}{2} \quad \text{and} \quad B = -\frac{A^2}{4}.$$

Put it back will imply

$$\begin{aligned} Aa_{n-1} + Ba_{n-2} &= nr^{n-2}(Ar + B) - r^{n-2}(Ar + 2B) \\ &= nr^{n-2}\left(\frac{A^2}{2} - \frac{A^2}{4}\right) - r^{n-2}\left(\frac{A^2}{2} - \frac{A^2}{2}\right) \\ &= nr^{n-2}\left(\frac{A^2}{4}\right) \\ &= nr^n = a_n. \end{aligned}$$

Now let's check r^n and nr^n are independent:

$$\begin{vmatrix} r^n & nr^n \\ r^{n+1} & (n+1)r^{n+1} \end{vmatrix} = (n+1)r^{2n+1} - nr^{2n+1} = r^{2n+1} \neq 0$$

because $r \neq 0$. Therefore they are independent solutions, and the general solution of (*) is

$$a_n = C_1 r^n + C_2 n r^n,$$

where C_1, C_2 are constants to be determined.

Example 2.9. Solve the recurrence relation $\begin{cases} a_n = 6a_{n-1} - 9a_{n-2} \ \forall n \geq 2 \\ a_0 = 2, a_1 = 3 \end{cases}$.

Solution. The characteristic equation is $x^2 = 6x - 9$. Solving, the only root is 3 with multiplicity 2. So $a_n = C_1 3^n + C_2 n 3^n$ for some constants C_1, C_2. By using the initial conditions $a_0 = 2, a_1 = 3$, we have

$$\begin{cases} C_1 = 2 \\ 3C_1 + 3C_2 = 3 \end{cases}$$

Solving, we have $C_1 = 2, C_2 = -1$. Therefore

$$a_n = 2 \times 3^n - n3^n = (2-n)3^n.$$

■

2.2.3 Inhomogeneous case

Now we move on to inhomogeneous case and study recurrence relations of the form

$$a_n = Aa_{n-1} + Ba_{n-2} + \beta(n)$$

We have a three-step solution method when the inhomogeneous term $\beta(n) \neq 0$:

1. Find the general solution of the homogeneous part $a_n = Aa_{n-1} + Ba_{n-2}$. Let's denote the homogeneous solution $a_n^{(h)}$ by:
$$a_n^{(h)} = C_1 a_n^{(1)} + C_2 a_n^{(2)}$$
2. Find a particular solution $a_n^{(p)}$ of the original inhomogeneous equation. We will guess a solution of a similar form with $\beta(n)$. Some easier guesses are as follows:
   (i) If $f(t) = kr^n$, we would guess a particular solution of the form $Cn^m r^n$ where C is a constant to be determined, and m is the multiplicity of r as a root of the characteristic equation $r^2 = Ar + B$.
   (ii) If $\beta(n)$ is a polynomial of degree k, we would guess a particular solution of the same form, a polynomial of degree k.
3. The general solution of the inhomogeneous equation is $a_n = a_n^{(h)} + a_n^{(p)}$. We can now determine the constants C_1, C_2.

Step 3 requires a bit justification: we need to check that $a_n = a_n^{(h)} + a_n^{(p)}$ is a solution. Indeed, since $a_n^{(h)}$ is the general solution of the homogeneous part, we have

$$a_n^{(h)} = Aa_{n-1}^{(h)} + Ba_{n-2}^{(h)},$$

and similarly since $a_n^{(p)}$ is a solution of the original inhomogeneous equation, we have

$$a_n^{(p)} = Aa_{n-1}^{(p)} + Ba_{n-2}^{(p)} + \beta(n).$$

Now we check that

$$\begin{aligned} Aa_{n-1} + Ba_{n-2} + \beta(n) &= A\left(a_{n-1}^{(h)} + a_{n-1}^{(p)}\right) + B\left(a_{n-2}^{(h)} + a_{n-2}^{(p)}\right) + \beta(n) \\ &= \left(Aa_{n-1}^{(h)} + Ba_{n-2}^{(h)}\right) + \left(Aa_{n-1}^{(p)} + Ba_{n-2}^{(p)} + \beta(n)\right) \\ &= a_n^{(h)} + a_n^{(p)} \\ &= a_n \end{aligned}$$

So this means $a_n = a_n^{(h)} + a_n^{(p)}$ is indeed a solution.

Example 2.10. Solve the recurrence relation $\begin{cases} a_n = 3a_{n-1} + 10a_{n-2} + 7 \times 5^n \ \forall n \geq 2 \\ a_0 = 4, a_1 = 3 \end{cases}$.

Solution.

1. We first figure out the general solution of the homogeneous part. The characteristic equation is
$$r^2 - 3r - 10 = 0.$$
Solving, we have $r = 5$ or $r = -2$. So the general solution of the homogeneous part is of the form
$$a_n^{(h)} = C_1 5^n + C_2(-2)^n.$$
2. Since 5 is a root of the characteristic equation of multiplicity 1, we guess the solution $a_n^{(p)} = Cn5^n$. Put it back into the original recurrence relation, we have
$$Cn5^n = 3C(n-1)5^{n-1} + 10C(n-2)5^{n-2} + 7 \times 5^n.$$
Cancelling 5^{n-2} on both sides and rearranging the equation, we have
$$175 - 35C = 0 \implies C = 5$$
So $a_n^{(p)} = n5^{n+1}$.
3. The general solution of the recurrence relation is
$$a_n = C_1 5^n + C_2(-2)^n + n5^{n+1}.$$
The initial conditions $a_0 = 4, a_1 = 3$ implies $C_1 = -2$ and $C_2 = 6$. So the solution is
$$a_n = -2 \times 5^n + 6(-2)^n + n5^{n+1}.$$

■

Example 2.11. Find a particular solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2} + 8n^2 - 24n$$

Solution. We guess the solution $a_n^{(p)} = An^2 + Bn + C$. Put it into the recurrence relation, we have

$$An^2 + Bn + C = 6(A(n-1)^2 + B(n-1) + C) - 9(A(n-2)^2 + B(n-2) + C) + 8n^2 - 24n.$$

Expanding and rearranging gives

$$4An^2 + (-24A + 4B)n + (30A - 12B + 4C) = 8n^2 - 24n.$$

Solving, we have $A = 2$, $B = 6$ and $C = 3$. So a particular solution is

$$a_n^{(p)} = 2n^2 + 6n + 3$$

■

2.3 Recursive functions

A function is said to be **defined recursively** or to be a **recursive function** if its rule of definition refers to itself. Unlike recurrence relations, which only refer to previous terms, recursive functions may refer to later terms, so the well-definedness is not immediate from induction. Therefore, it is often hard to tell if a recursive function is truly well-defined. For example, it is still unknown whether the following 'function' `Col` $: \mathbb{N} \to \mathbb{N}$ is well-defined:

$$\texttt{Col}(n) = \begin{cases} 1 & \text{if } n = 1 \\ \texttt{Col}\left(\dfrac{n}{2}\right) & \text{if } n \text{ is even} \\ \texttt{Col}(3n+1) & \text{if } n > 1 \text{ is odd} \end{cases}$$

This is the infamous, notorious **Collatz conjecture**, and if this `Col` is well-defined, then $\texttt{Col}(n) = 1\ \forall n \in \mathbb{N}$ What makes it infamous is its unnaturalness: Imagine if you go to another galaxy and meet another intelligent civilization, and they might not even think of this particular problem. They might know about primes, the twin prime conjecture, or even a proof for it, but it is doubtful for them even to formulate such a conjecture. In fact, Shizuo Kakutani, a Japanese and American mathematician, even joked that 'this problem was part of a conspiracy to slow down mathematical research in the U.S.'

Example 2.12. **McCarthy's 91 function** $M : \mathbb{N} \to \mathbb{Z}$ is defined by

$$M(n) = \begin{cases} n - 10 & \text{if } n > 100 \\ M(M(n+11)) & \text{if } n \leq 100 \end{cases}.$$

Compute $M(99)$.

Solution.

$$\begin{aligned} M(99) &= M(M(110)) \\ &= M(100) \\ &= M(M(111)) \\ &= M(101) \\ &= 91 \end{aligned}$$

■

Example 2.13. Consider the following attempt to define a recursive function $g : \mathbb{N} \to \mathbb{Z}$

$$g(n) = \begin{cases} 1 & \text{if } n = 1 \\ 1 + g\left(\dfrac{n}{2}\right) & \text{if } n \text{ is even} \\ g(3n-1) & \text{if } n > 1 \text{ is odd} \end{cases}$$

Check if g is well-defined.

Solution. We first check some small numbers:

(i) $g(2) = 1 + g(1) = 1 + 1 = 2$

(ii) $g(3) = g(8) = 1 + g(4) = 2 + g(2) = 4$

(iii) $g(4) = 1 + g(2) = 3$

(iv) $g(5) = g(14) = 1 + g(7) = 1 + g(20) = 2 + g(10) = 3 + g(5)$ which implies $0 = 3$

Therefore g is not well-defined. ■

2.4 Generating functions

In this subsection, we will discuss another important device in combinatorics and many other fields called the **generating functions**.

2.4.1 Ordinary generating functions

Definition 2.2. Given a sequence $\{a_n\}$, its **ordinary generating function** (abbrev. ogf) is the formal power series

$$f = \sum_{n=0}^{\infty} a_n x^n.$$

We also write $f \overset{ogf}{\longleftrightarrow} \{a_n\}$ to indicate that f is the ogf of $\{a_n\}$.

Here ogf are **formal power series**. The word formal here means that we do not have to worry about the convergence issues, because we will never assign a value to x except putting $x = 0$, where in this case

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + x \sum_{n=1}^{\infty} a_n x^{n-1}$$

and so $f(0) = a_0$. Here we only care about the sequence of coefficients $\{a_n\}$.

Recall the following formal power series operations:

- Equality: $\displaystyle\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \iff \{a_n\} = \{b_n\}$.
- Addition: $\displaystyle\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n$
- Cauchy product: $\displaystyle\left\{\sum_{n=0}^{\infty} a_n x^n\right\}\left\{\sum_{n=0}^{\infty} b_n x^n\right\} = \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} a_k b_{n-k}\right\} x^n$

Example 2.14. The ogf of the sequence $1, 1, \cdots$ is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Example 2.15. The ogf of the sequence $\binom{n}{0}, \binom{n}{1}, \cdots, \binom{n}{n}, 0, 0, \cdots$ is

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$

Example 2.16. Determine the ogf of the sequence $\{n^2\}_{n\geq 0}$.

Solution. Recall $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Differentiating both sides gives

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}.$$

Multiplying both sides by x gives

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n.$$

Differentiating again and multiplying both sides by x again gives

$$\frac{x(1+x)}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^n = \sum_{n=0}^{\infty} n^2 x^n.$$

Therefore $\displaystyle\frac{x(1+x)}{(1-x)^3} \overset{ogf}{\longleftrightarrow} \{n^2\}_{n\geq 0}$. ■

To use generating functions to solve many important counting problems, we will need to apply the binomial theorem for exponents that are not positive integers.Before we state an extended version of the binomial theorem, we need to define **extended binomial coefficients**.

Definition 2.3. Let $\alpha \in \mathbb{R}$ and $r \in \omega$. The **generalise binomial coefficient** of α over r is defined by

$$\binom{\alpha}{r} = \begin{cases} \dfrac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!} = \dfrac{1}{r!}\displaystyle\prod_{k=0}^{r-1}(\alpha-k) & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases}.$$

Note that when $\alpha \in \omega$, $\binom{\alpha}{r}$ reduces to usual binomial coefficients, in particular if $\alpha > r$ then $\binom{\alpha}{r} = 0$.

Example 2.17.

- $\binom{-2}{3} = \frac{1}{3!}(-2)(-3)(-4) = -4$
- $\binom{\frac{1}{2}}{3} = \frac{1}{3!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) = \frac{1}{16}$

Lemma 2.4. For $n \in \mathbb{N}$ and $r \in \omega$, then $\binom{-n}{r} = (-1)^r\binom{n+r-1}{r}$.

Proof. If $r = 0$ then clearly both sides reduce to 1. Otherwise,

$$\begin{aligned} \binom{-n}{r} &= \frac{1}{r!}\prod_{k=0}^{r-1}(-n-k) \\ &= (-1)^r\frac{1}{r!}\prod_{k=0}^{r-1}(n+k) \\ &= (-1)^r\frac{1}{r!}\prod_{k=0}^{r-1}(n+r-1-k) \\ &= (-1)^r\binom{n+r-1}{r}. \end{aligned}$$

■

Example 2.18. $\binom{-2}{3} = (-1)^3\binom{2+3-1}{3} = -4$ as computed above.

We can now extend the binomial theorem:

Theorem 2.5 (Newton's binomial theorem)**.** If $\alpha \in \mathbb{R}$ and $|x| < 1$, then

$$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k.$$

Therefore for $\alpha \in \mathbb{R}$, the ogf of the sequence $\binom{\alpha}{0}, \binom{\alpha}{1}, \cdots$ is

$$\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n = (1+x)^\alpha.$$

by Newton's binomial theorem.

2.4.2 Application to recurrence relations

An important usage of ogf is to solve recurrence relations, in particular linear nonhomogeneous recurrence relations with constant coefficients. The philosophy is to compress the ogf into a closed-form expression, and then re-expand the ogf to give a formal power series.

Example 2.19. Solve the recurrence relation $\begin{cases} a_n = 3a_{n-1} + 3^n \ \forall n \geq 1 \\ a_0 = 2 \end{cases}$.

Solution. The ogf of $\{a_n\}$ is

$$\begin{aligned} G(x) = \sum_{n=0}^{\infty} a_n x^n &= 2 + \sum_{n=1}^{\infty} (3a_{n-1} + 3^n) x^n \\ &= 2 + \sum_{v=0}^{\infty} (3a_n + 3^{n+1}) x^{n+1} \\ &= 2 + 3 \left\{ \sum_{n=0}^{\infty} a_n x^{n+1} \right\} + \left\{ \sum_{n=0}^{\infty} 3^{n+1} x^{n+1} \right\} \\ &= 2 + 3x \left\{ \sum_{n=0}^{\infty} a_n x^n \right\} + 3x \left\{ \sum_{n=0}^{\infty} 3^n x^n \right\} \\ &= 2 + 3xG(x) + \frac{3x}{1-3x} \end{aligned}$$

Thus $(1-3x)G(x) = 2 + \dfrac{3x}{1-3x} = 1 + \dfrac{1}{1-3x} \implies G(x) = \dfrac{1}{1-3x} + \dfrac{1}{(1-3x)^2}$. To determine the formal power series expansion of the ogf $\dfrac{1}{(1-3x)^2}$, we do the following:

- Recall $\dfrac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} 3^n x^n$. Differentiate both sides gives

$$\frac{3}{(1-3x)^2} = \sum_{n=1}^{\infty} n 3^n x^{n-1} = \sum_{n=0}^{\infty} (n+1) 3^{n+1} x^n$$

  and therefore $\dfrac{1}{(1-3x)^2} = \sum_{n=0}^{\infty} (n+1) 3^n x^n$.

Now we expand

$$G(x) = \frac{1}{1-3x} + \frac{1}{(1-3x)^2} = \sum_{n=0}^{\infty} 3^n x^n + \sum_{n=0}^{\infty} (n+1) 3^n x^n = \sum_{n=0}^{\infty} (n+2) 3^n x^n.$$

Therefore $a_n = (n+2)3^n \ \forall n \in \omega$. Note that this coincides with our previous solution as desired. ■

Example 2.20. Solve the recurrence relation $\begin{cases} f_n = f_{n-1} + f_{n-2} \ \forall n \geq 2 \\ f_0 = 0, f_1 = 1 \end{cases}$.

Solution. The ogf of $\{f_n\}$ is

$$\begin{aligned} G(x) = \sum_{n=0}^{\infty} f_n x^n &= 0 + 1 + \sum_{n=2}^{\infty} (f_{n-1} + f_{n-2}) x^n \\ &= 1 + \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n \\ &= 1 + \sum_{n=1}^{\infty} f_n x^{n+1} + \sum_{n=0}^{\infty} f_n x^{n+2} \\ &= 1 + xG(x) + x^2 G(x) \end{aligned}$$

Thus $(1 - x - x^2)G(x) = 1 \Longrightarrow G(x) = \dfrac{1}{1 - x - x^2}$. We break it down into partial fractions:

$$G(x) = \frac{1}{1 - x - x^2} = \frac{1}{\sqrt{5}} \frac{1}{1 - \varphi x} - \frac{1}{\sqrt{5}} \frac{1}{1 - \psi x}$$

where $\varphi = \dfrac{1 - \sqrt{5}}{2}$ and $\psi = \dfrac{1 - \sqrt{5}}{2}$ are the roots of the polynomial $1 - x - x^2$. Thus

$$\begin{aligned} G(x) &= \frac{1}{\sqrt{5}} \frac{1}{1 - \varphi x} - \frac{1}{\sqrt{5}} \frac{1}{1 - \psi x} \\ &= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} (\varphi x)^n - \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} (\psi x)^n \\ &= \sum_{n=0}^{\infty} \left\{ \frac{1}{\sqrt{5}} \varphi^n - \frac{1}{\sqrt{5}} \psi^n \right\} x^n \end{aligned}$$

Therefore $f_n = \dfrac{1}{\sqrt{5}} \varphi^n - \dfrac{1}{\sqrt{5}} \psi^n \ \forall n \in \omega$, again this coincides with our previous solution as desired. ■

2.4.3 Application to counting

Let $r \in \mathbb{N}$ be fixed. Suppose the sequence $\{h_n\}$ is defined by letting h_n to be the number of nonnegative integral solutions of the equation

$$x_1 + x_2 + \cdots + x_r = n.$$

In an earlier subsection, we have already shown that

$$h_n = \binom{n + r - 1}{r - 1} = \binom{n + r - 1}{n}$$

So the ogf of this sequence is $h(x) = \displaystyle\sum_{n=0}^{\infty} \binom{n + r - 1}{n} x^n$. But then what is its closed form? We recall from lemma 2.4 that

$$\binom{n + r - 1}{n} = (-1)^n \binom{-r}{n}$$

(yes this is inverted!) and so by Newton's binomial theorem, we have

$$h(x) = \sum_{n=0}^{\infty} (-1)^n \binom{-r}{n} x^n = \sum_{n=0}^{\infty} \binom{-r}{n} (-x)^n = (1 - x)^{-r} = \frac{1}{(1 - x)^r}.$$

It would be instructive to go another way round:

$$h(x) = \frac{1}{(1 + x)^r} = \underbrace{\frac{1}{1 + x} \times \frac{1}{1 + x} \times \cdots \times \frac{1}{1 + x}}_{r \text{ factors}} = \left\{ \sum_{e_1 = 0}^{\infty} x^{e_1} \right\} \left\{ \sum_{e_2 = 0}^{\infty} x^{e_2} \right\} \cdots \left\{ \sum_{e_r = 0}^{\infty} x^{e_r} \right\} \qquad (*)$$

where x^{e_k} is a typical term of the k^{th} factor. Note that $x^{e_1} x^{e_2} \cdots x^{e_r} = x^n$ provided that $e_1 + e_2 + \cdots + e_r = n$, thus the coefficient of x^n in the expansion (*) counts the number of nonnegative integral solutions of the equation $x_1 + x_2 + \cdots + x_r = n$ as we expected.

The above discussion illustrated the idea of using generating functions to count.

Example 2.21. What is the sequence that has the ogf

$$(1 + x + x^2)(1 + x + \cdots + x^4)(1 + x + \cdots + x^5)?$$

Solution. Let x^{e_1} $(0 \le e_1 \le 2)$, x^{e_2} $(0 \le e_2 \le 4)$ and x^{e_3} $(0 \le e_3 \le 5)$ denote a typical term in the first, second and third factor respectively. Multiplying them we obtain $x^{e_1}x^{e_2}x^{e_3} = x^n$ provide that

$$e_1 + e_2 + e_3 = n.$$

Thus the coefficient of x^n counts the number h_n of integral solutions of $e_1 + e_2 + e_3 = n$ in which $0 \le e_1 \le 2$, $0 \le e_2 \le 4$ and $0 \le e_3 \le 5$. Observe $h_n = 0$ whenever $n > 2 + 4 + 5 = 11$. ■

Example 2.22. In Hong Kong, there are coins of face values \$1, \$2, \$5 and \$10. Find the ogf for the number h_n of ways of making \$$n$ using these coins.

Solution. h_n counts the number of nonnegative integral solutions of the equation

$$e_1 + 2e_2 + 5e_5 + 10e_{10} = n$$

where e_k counts the number of \$$k$ coins. The ogf of $\{h_n\}$ is

$$\begin{aligned} g(x) &= (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^5 + x^{10} + \cdots)(1 + x^{10} + x^{20} + \cdots) \\ &= \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^5}\frac{1}{1-x^{10}} \end{aligned}$$

■

Example 2.23. Find the number h_n of bags of n fruits that can be made out of apples, bananas, oranges and pears, where in each bag,

- the number of apples is even,
- the number of bananas is a multiple of 5,
- the number of oranges is at most 4, and
- the number of pears is either 0 or 1.

Solution. This amounts to counting the number h_n of nonnegative integral solutions of the equation

$$e_1 + e_2 + e_3 + e_4 = n$$

where $e_1 \ge 0$ is even which counts the number of apples, $e_2 \ge 0$ is a multiple of 5 which counts the number of bananas, $0 \le e_3 \le 4$ which counts the number of oranges and $0 \le e_4 \le 1$ which counts the number of pears. We create a factor of the generating function for each fruit, where the exponents are the allowable numbers for that type of fruit:

$$g(x) = (1 + x^2 + x^4 + \cdots)(1 + x^5 + x^{10} + \cdots)(1 + x + \cdots + x^4)(1 + x)$$

where the first factor is the ‘apple factor’, the second factor is the ‘banana factor’ etc. Now we note that

$$\begin{aligned} 1 + x^2 + x^4 + \cdots &= \frac{1}{1-x^2} \\ 1 + x^5 + x^{10} + \cdots &= \frac{1}{1-x^5} \\ 1 + x + \cdots + x^4 &= \frac{1-x^5}{1-x} \end{aligned}$$

and thus $g(x) = \dfrac{1}{1-x^2}\dfrac{1}{1-x^5}\dfrac{1-x^5}{1-x}(1+x) = \dfrac{1}{(1-x)^2} = \displaystyle\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n$, so $h_n = n + 1$. ■

2.4.4 Exponential generating functions

Sometimes it would be more convenient to solve problems using **exponential generating functions**:

Definition 2.6. Given a sequence $\{a_n\}$, its **exponential generating function** (abbrev. egf) is the formal series

$$f = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

We also write $f \overset{egf}{\longleftrightarrow} \{a_n\}$ to indicate that f is the egf of $\{a_n\}$.

Example 2.24. The egf of $\{n!\}$ is

$$\sum_{n=0}^{\infty} n! \frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Example 2.25. Let $r \in \mathbb{R}$. The egf of $\{r^n\}$ is

$$\sum_{n=0}^{\infty} r^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(rx)^n}{n!} = e^{rx}.$$

The following example captures the power of egf when handling problems about permutations:

Example 2.26. Define

$$P(n,r) = \frac{n!}{(n-r)!} = r!\binom{n}{r}.$$

For fixed $n \in \mathbb{N}$, the egf of the sequence $P(n,0), P(n,1), \cdots, P(n,n), 0, 0, \cdots$ is

$$\sum_{k=0}^{n} P(n,k)\frac{x^k}{k!} = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

Of course egf behaves like ogf. They are just formal power series!

Example 2.27. Let B_n be the sequence defined by

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \ \forall n \in \omega$$

and $B_0 := 1$. What is the egf of $\{B_n\}$?

Solution. Let $\beta(x) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$ be the egf of $\{B_n\}$. Then

$$\begin{aligned}
\beta'(x) = \sum_{n=1}^{\infty} B_n \frac{x^{n-1}}{(n-1)!} &= \sum_{n=0}^{\infty} B_{n+1} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{n}{k} B_k \right\} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \frac{B_k}{k!} \frac{1}{(n-k)!} \right\} x^n \\
&= \left\{ \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \right\} \left\{ \sum_{n=0}^{\infty} \frac{x^n}{n!} \right\} \\
&= \beta(x) e^x
\end{aligned}$$

Therefore $e^x = \frac{\beta'(x)}{\beta(x)} = (\log \beta(x))'$. So $e^x + C = \log \beta(x)$ and hence $\beta(x) = e^{e^x + C}$. Recall $\beta(0) = B_0 = 1$, so $C = -1$. Therefore $\beta(x) = e^{e^x - 1}$. ∎

Similar to ogf, egf can also be used to solve recurrence relations.

Example 2.28. Solve the recurrence relation $\begin{cases} a_n = na_{n-1} + 2 \ \forall n \geq 1 \\ a_0 = 1 \end{cases}$.

Solution. The egf of $\{a_n\}$ is

$$\begin{aligned} G(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} &= 1 + \sum_{n=1}^{\infty} (na_{n-1} + 2) \frac{x^n}{n!} \\ &= 1 + \sum_{n=0}^{\infty} ((n+1)a_n + 2) \frac{x^{n+1}}{(n+1)!} \\ &= 1 + x \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} + 2 \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \\ &= 1 + xG(x) + 2(e^x - 1) \\ &= xG(x) + 2e^x - 1 \end{aligned}$$

Thus $(1-x)G(x) = 2e^x - 1 \Longrightarrow G(x) = \dfrac{2e^x - 1}{1-x}$ and thus

$$\begin{aligned} G(x) &= \frac{2}{1-x} e^x - \frac{1}{1-x} \\ &= 2 \left\{ \sum_{n=0}^{\infty} x^n \right\} \left\{ \sum_{n=0}^{\infty} \frac{x^n}{n!} \right\} - \sum_{n=0}^{\infty} x^n \\ &= 2 \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \frac{1}{k!} \right\} x^n - \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} n! \left\{ \sum_{k=0}^{n} \frac{2}{k!} - 1 \right\} \frac{x^n}{n!} \end{aligned}$$

and therefore $a_n = n! \left\{ \sum_{k=0}^{n} \frac{2}{k!} - 1 \right\}$. ■