

Mini-course on Sparse estimation off-the-grid **Sparsistency**

Q: Given $y = \Phi\mu_{a,x} + w$, does the solution to $P_\lambda(y)$ consist of precisely s spikes?

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Support stability

Recall: if $p_\lambda = \operatorname{argmax} D_\lambda(y)$ and $\eta_\lambda = \Phi^* p_\lambda$, then $\operatorname{Supp}(\mu_\lambda) \subset \{x : |\eta_\lambda(x)| = 1\}$

What is the behaviour of η_λ when λ and $\|w\|$ are small?

Limit of η_λ : Suppose $y = \Phi\mu_{a,x} + w$.

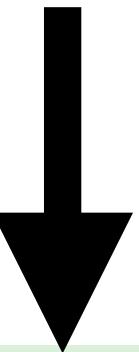
If $D_0(y)$ has a solution, then as $\lambda \rightarrow 0$, $\|w\| \rightarrow 0$,

$$\|p_\lambda - p_0\| \rightarrow 0, \quad p_0 = \operatorname{argmin} \left\{ \|p\| : p \in \operatorname{argmax} D_0(\Phi\mu_{a,x}) \right\}$$

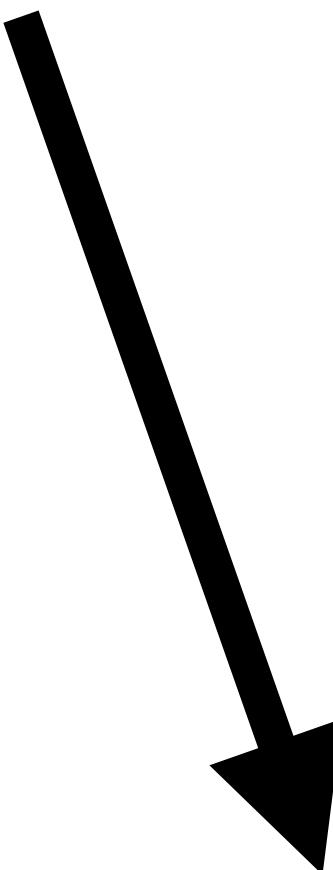
The limit dual problem

- Recall $p_\lambda = \operatorname{argmax}_{\|\Phi^* p\|_\infty \leq 1} \langle p, y \rangle - \lambda \|p\|^2/2$
- Let p_0 be of minimal norm such that $p_0 \in \operatorname{argmax}_{\|\Phi^* p\|_\infty \leq 1} \langle p, y \rangle$

$$\langle p_\lambda, y \rangle - \lambda \|p_\lambda\|^2/2 \geq \langle p_0, y \rangle - \lambda \|p_0\|^2/2 \geq \langle p_\lambda, y \rangle - \lambda \|p_0\|^2/2$$



- $\|p_\lambda\| \leq \|p_0\|$ for all λ .
- $(p_\lambda)_\lambda$ converges (up to subseq) to \bar{p} with $\|p_0\| \geq \|\bar{p}\|$ and $\|\Phi^* \bar{p}\|_\infty \leq 1$



Take limit $\lambda \rightarrow 0$

$\langle \bar{p}, y \rangle \geq \langle p_0, y \rangle$, so $\bar{p} = p_0$

Minimal norm certificate

We say that η is non degenerate if:

- $\eta''(x_i) \neq 0$
- $\eta(x_i) = \text{sign}(a_i)$
- $\forall x \notin \{x_i\}, |\eta(x)| < 1$

Minimal norm certificate

$$\eta_\lambda \xrightarrow{L^\infty} \eta_0 = \Phi^* p_0$$
$$\eta_0 = \underset{\eta=\Phi^* p}{\operatorname{argmin}} \|p\| \quad \text{s.t.} \quad \begin{cases} \forall i, \eta(x_i) = \text{sign}(a_i) \\ \|\eta\|_\infty \leq 1 \end{cases}$$

If η_0 is non-degenerate, then η_λ is also non degenerate when λ is sufficiently small.

Theorem (Duval and Peyre, 2015):

If η_0 is non-degenerate, then for $\|w\|/\lambda = \mathcal{O}(1)$ and $\lambda = \mathcal{O}(1)$, the solution to

$P_\lambda(y)$ is unique, $\mu_\lambda = \sum_{i=1}^s a_{\lambda,i} \delta_{x_{\lambda,i}}$ and $\|(x_\lambda, a_\lambda) - (x_0, a_0)\| = \mathcal{O}(\|w\|)$

Computing the minimal norm certificate

Minimal norm certificate

$$\eta_0 = \Phi^* p_0 = \operatorname{argmin}_{\eta=\Phi^* p} \|p\| \quad \text{s.t.} \quad \begin{cases} \forall i, \eta(x_i) = \operatorname{sign}(a_i) \\ \|\eta\|_\infty \leq 1 \end{cases}$$

Necessary:

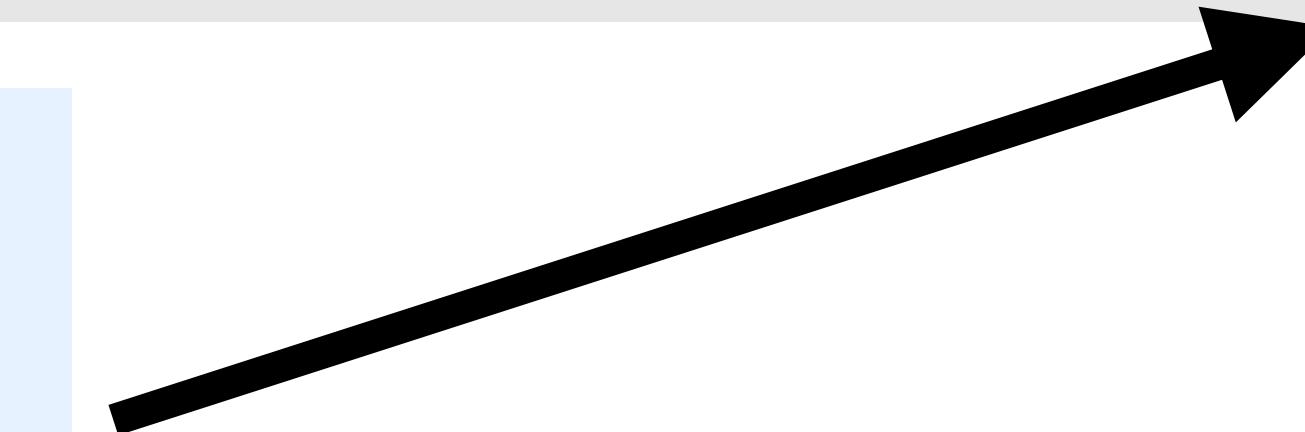
$$\begin{cases} \operatorname{sign}(a_i) = \langle p, \phi(x_i) \rangle \\ 0 = \langle p, \phi'(x_i) \rangle \end{cases}$$

Pre-certificate

$$\eta_V = \Phi^* p_V = \operatorname{argmin}_{\eta=\Phi^* p} \|p\| \quad \text{s.t.} \quad \begin{cases} \forall i, \eta(x_i) = \operatorname{sign}(a_i) \\ \forall i, \eta'(x_i) = 0 \end{cases}$$

$$\Gamma = [(\phi(x_i))_i, (\nabla \phi(x_i))_i]$$

$$\Gamma^* p = \begin{pmatrix} \operatorname{sign}(a) \\ 0_{sd} \end{pmatrix}$$



Linear system of $ds + s$ equations

Computing the minimal norm certificate

η_V can be computed by solving a linear system

$$\begin{pmatrix} [K(x_i, x_j)]_{i,j} & [K^{(1,0)}(x_i, x_j)]_{i,j} \\ [K^{(0,1)}(x_i, x_j)]_{i,j} & [K^{(1,1)}(x_i, x_j)]_{i,j} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \text{sign}(a) \\ 0_n \end{pmatrix}$$

$$\eta_V(x) = \sum_{i=1}^n u_i K(x_i, x) + \sum_{i=1}^n v_i K^{(10)}(x_i, x) \quad K(x, x') = \langle \phi(x), \phi(x') \rangle$$

Useful checks for analysing support stability:

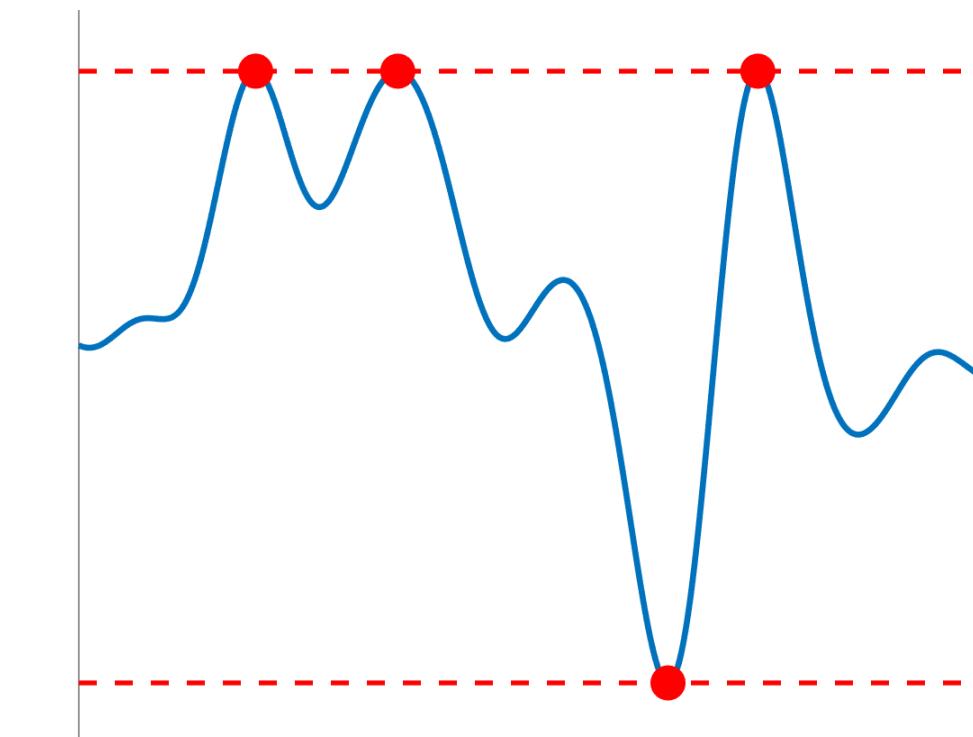
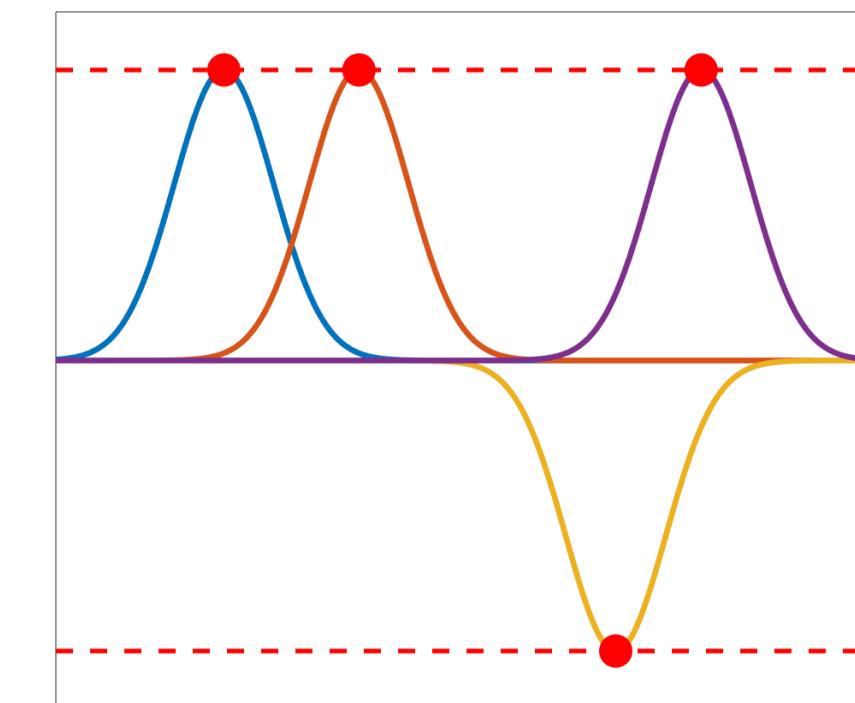
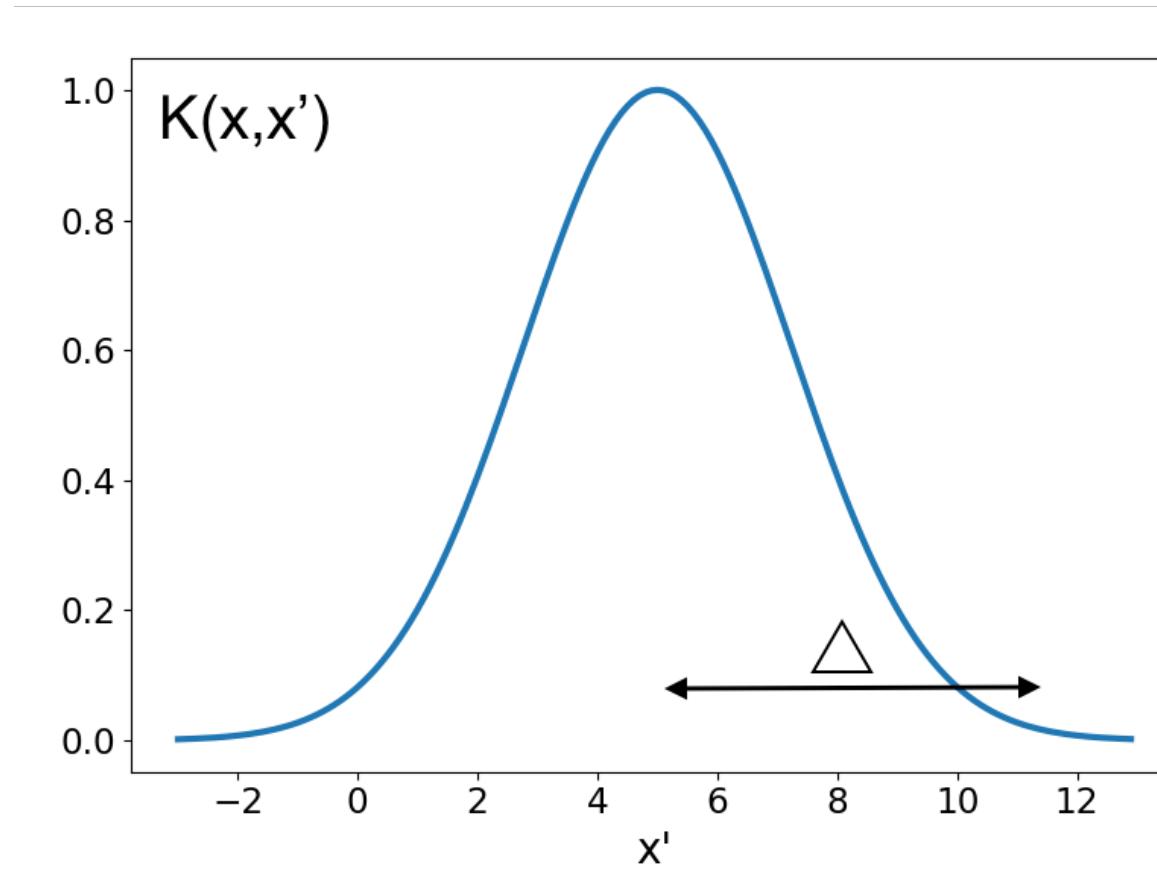
[Necessary cond] η_V must satisfy $\|\eta_V\|_\infty \leq 1$ for support stability.

[Sufficient cond] If η_V is non-degenerate, then support stability is guaranteed

Recovery under minimal separation

Typical analysis strategy to understand sparse identifiability properties of Φ :

Compute η_V and check if it is non-degenerate.



Candès and Fernandez-Granda (2012): Let $\phi(x) = (\exp(2\pi\sqrt{-1}kx))_{|k| \leq f_c}$,

if $\min_{i \neq j} |x_i - x_j| \geq \frac{C}{f_c}$, then η_V is non-degenerate. So, we have stable recovery.

Super-resolution

No super-resolution for opposite sign spikes:

If $|x - x'| < 1/f_c$, then $\mu := \delta_x - \delta_{x'}$ cannot be recovered from $P_0(\Phi\mu)$

De Castro & Fabrice (2012):

To recover N spikes with positive amplitudes, we need $f_c \geq N$ when there is no noise.

Q: Given N spikes at distance t apart, how small does the noise level $\|w\|$ need to be to identify N spikes?

Hint: Look at the certificate η_{tx} corresponding to positions $tx = (tx_i)_{i=1,\dots,N}$,
When is it non-degenerate?

Asymptotic vanishing derivatives precertificate in 1D

Theorem (Denoyelle et al, 2015):

As $t \rightarrow 0$, $\eta_{V,tx} \rightarrow \eta_w$ where

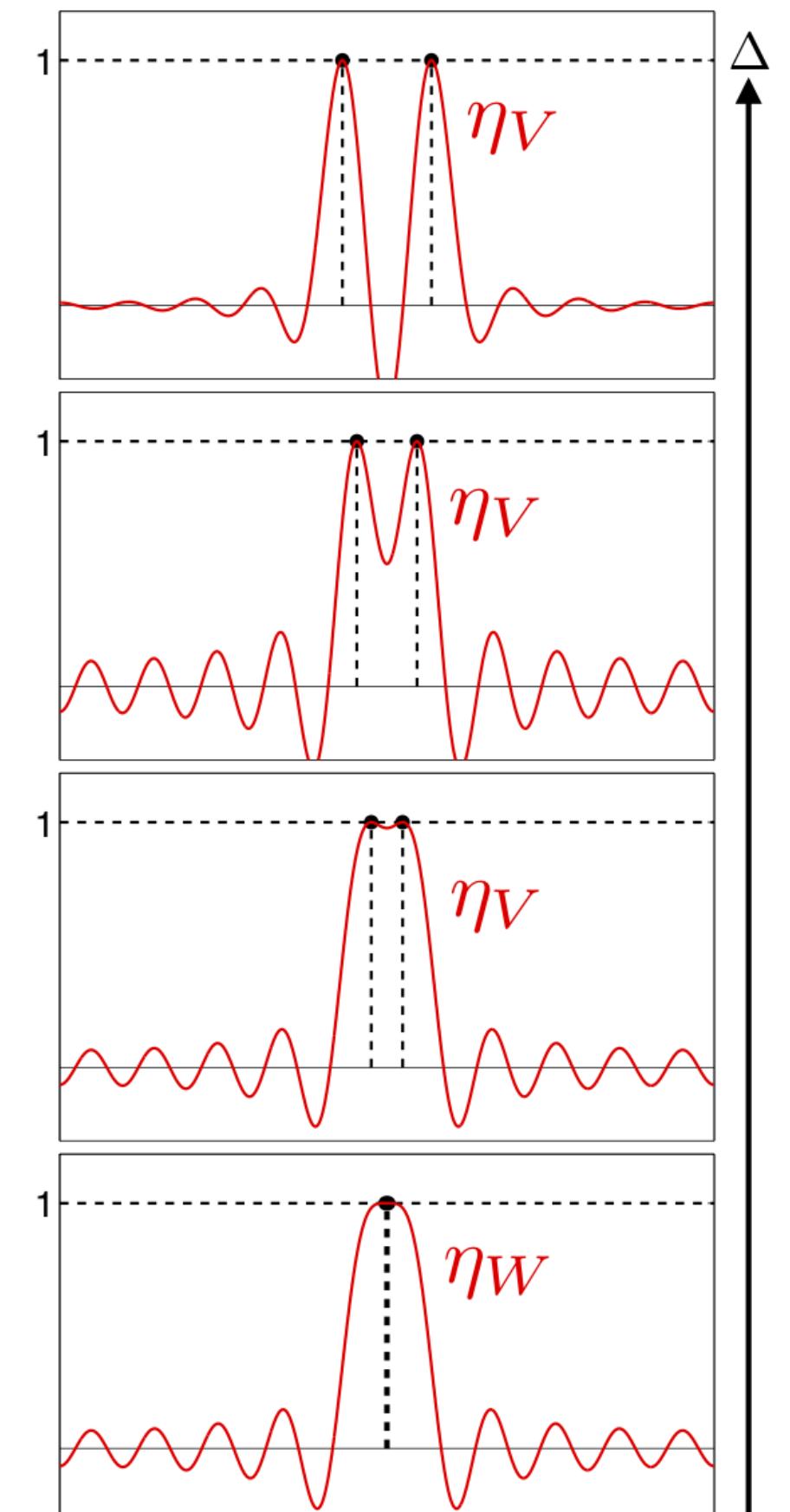
$$\eta_w = \operatorname{argmin}_{\eta=\Phi^*p} \|p\| \quad \text{s.t.} \quad \begin{cases} \eta(0) = 1 \\ \eta^{(1)}(0) = \dots = \eta^{(2N-1)}(0) = 0 \end{cases}$$

This is called non-degenerate if

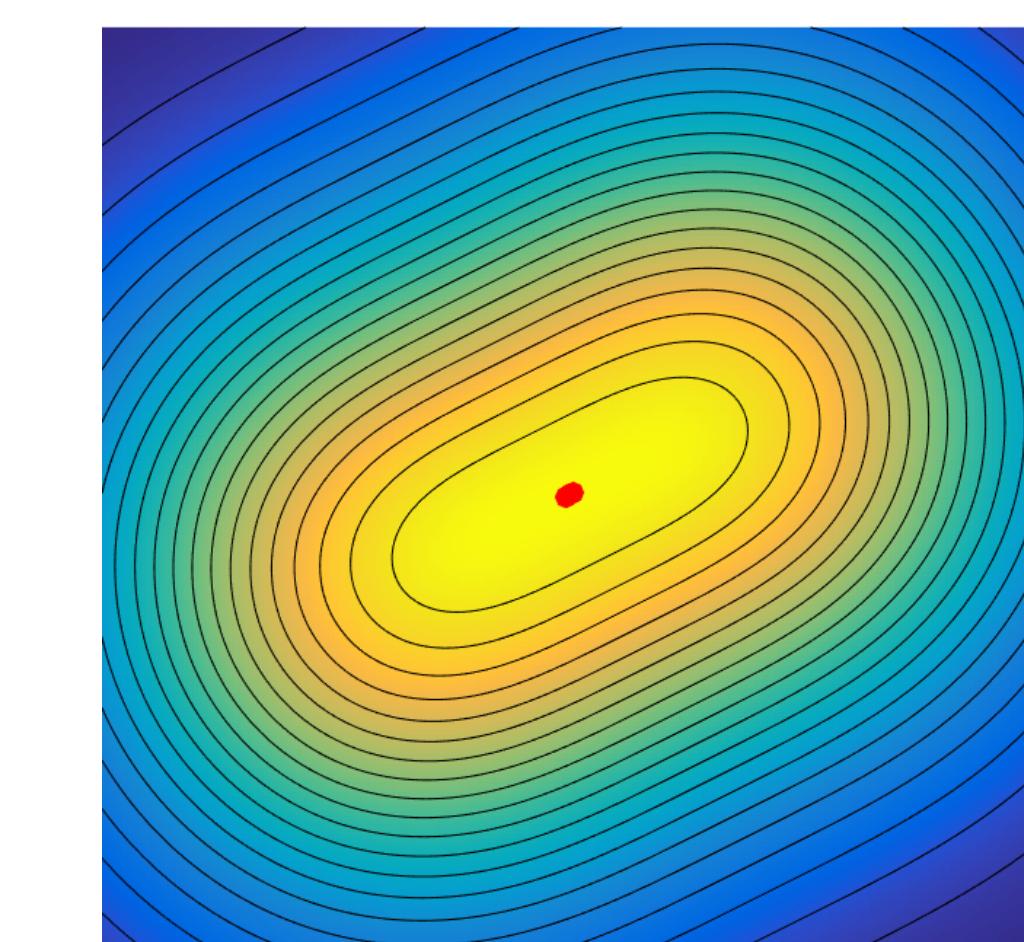
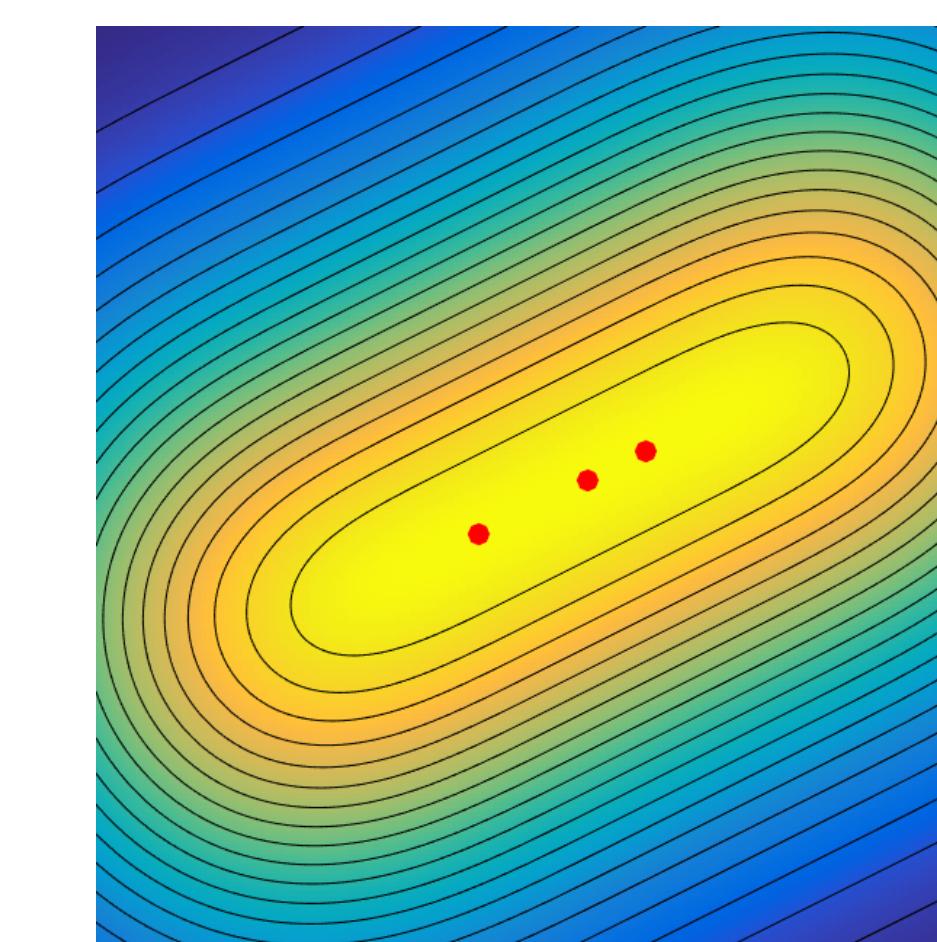
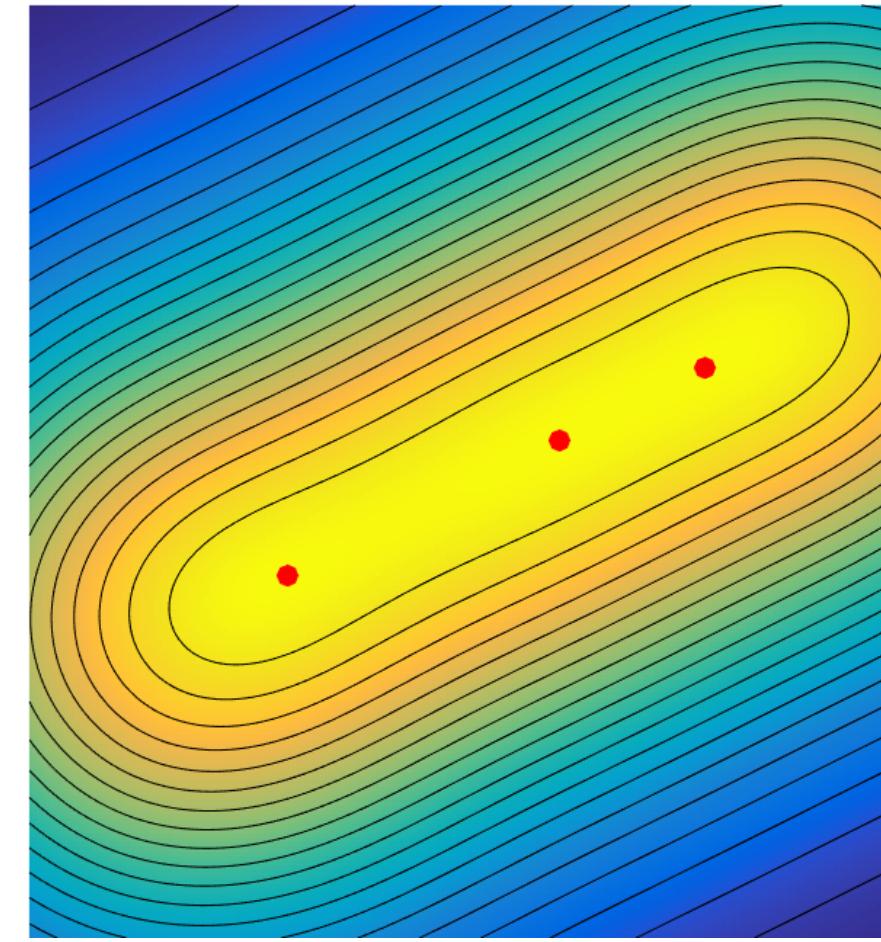
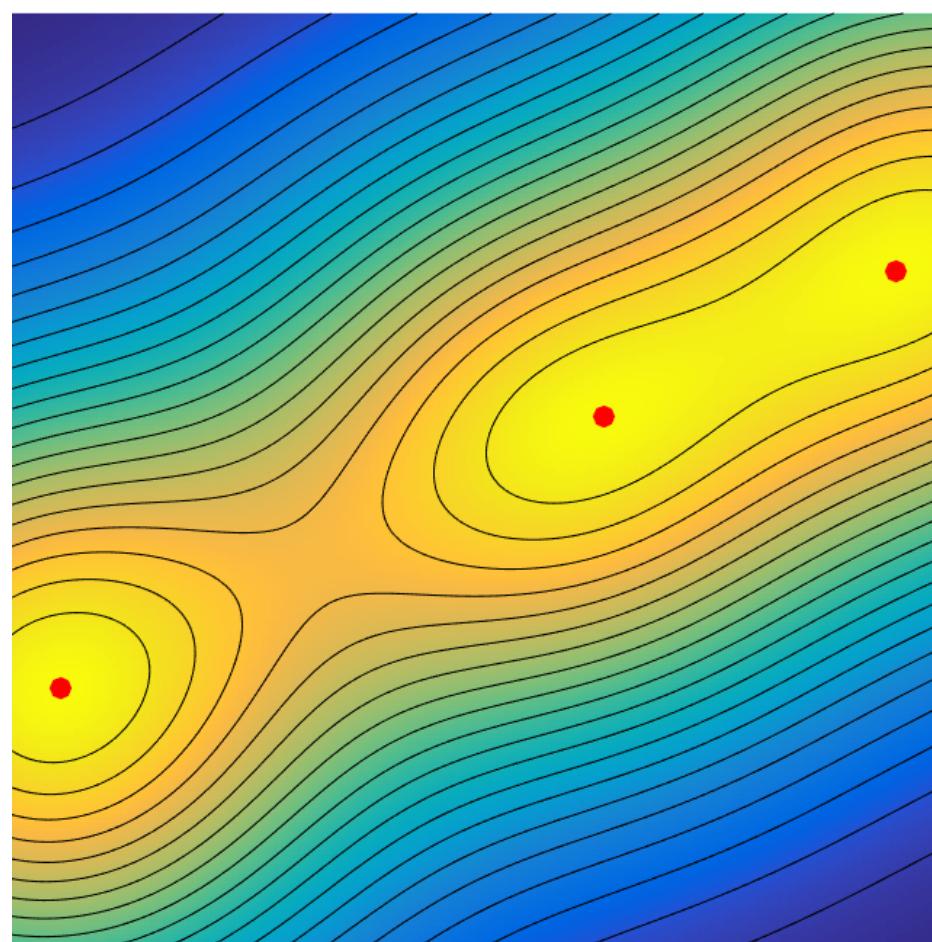
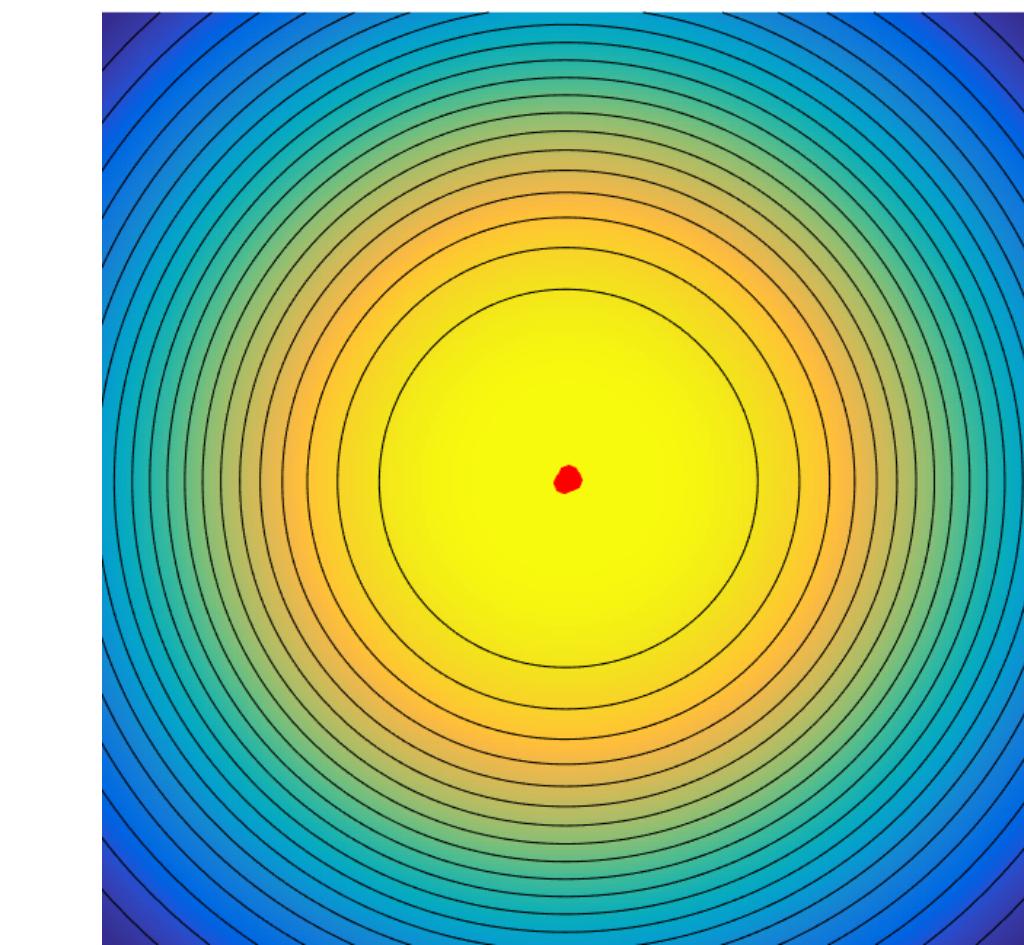
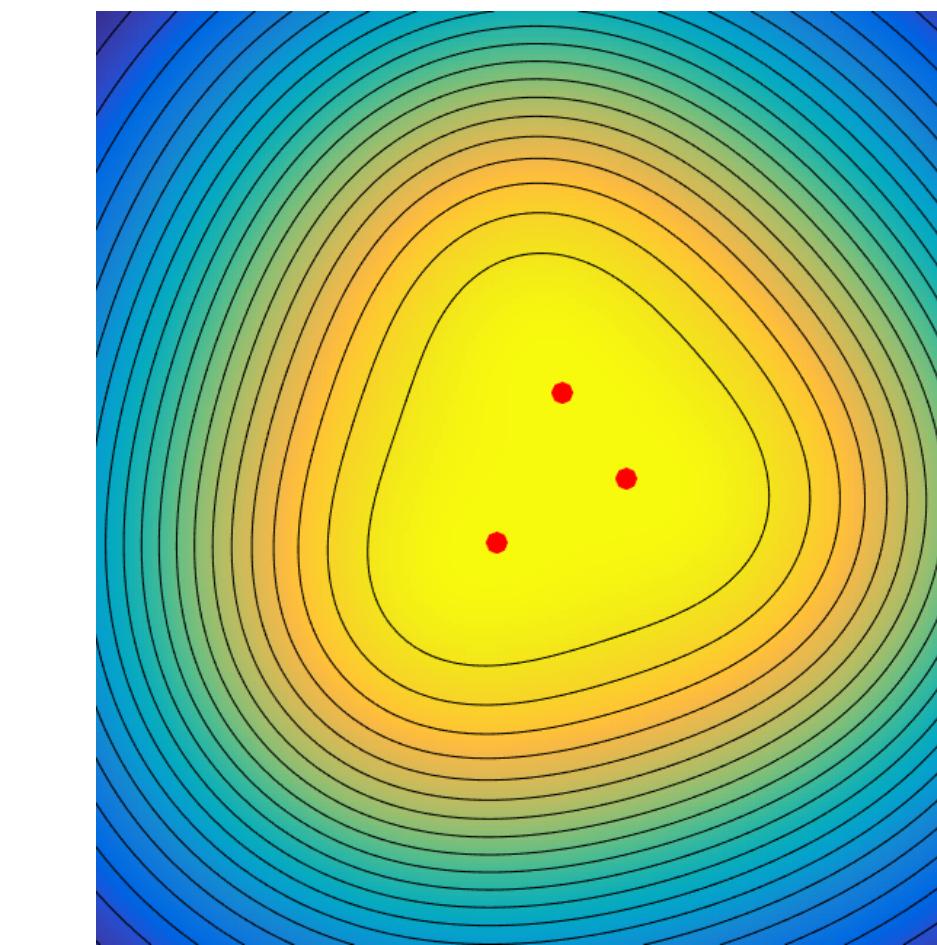
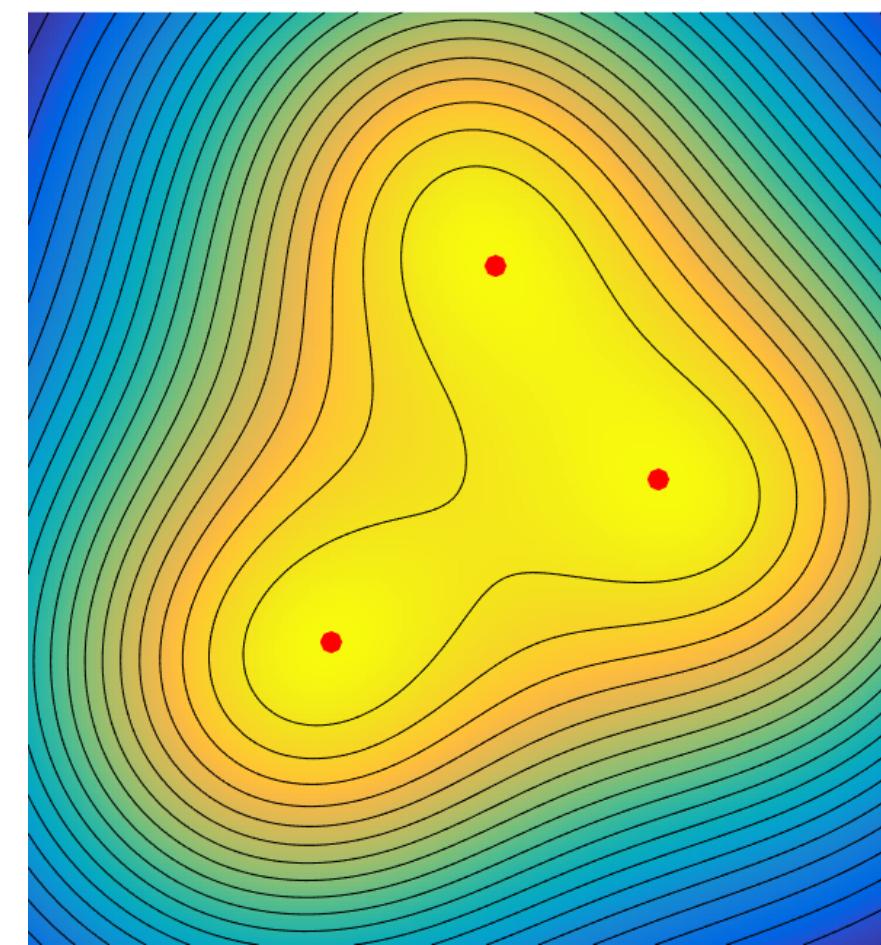
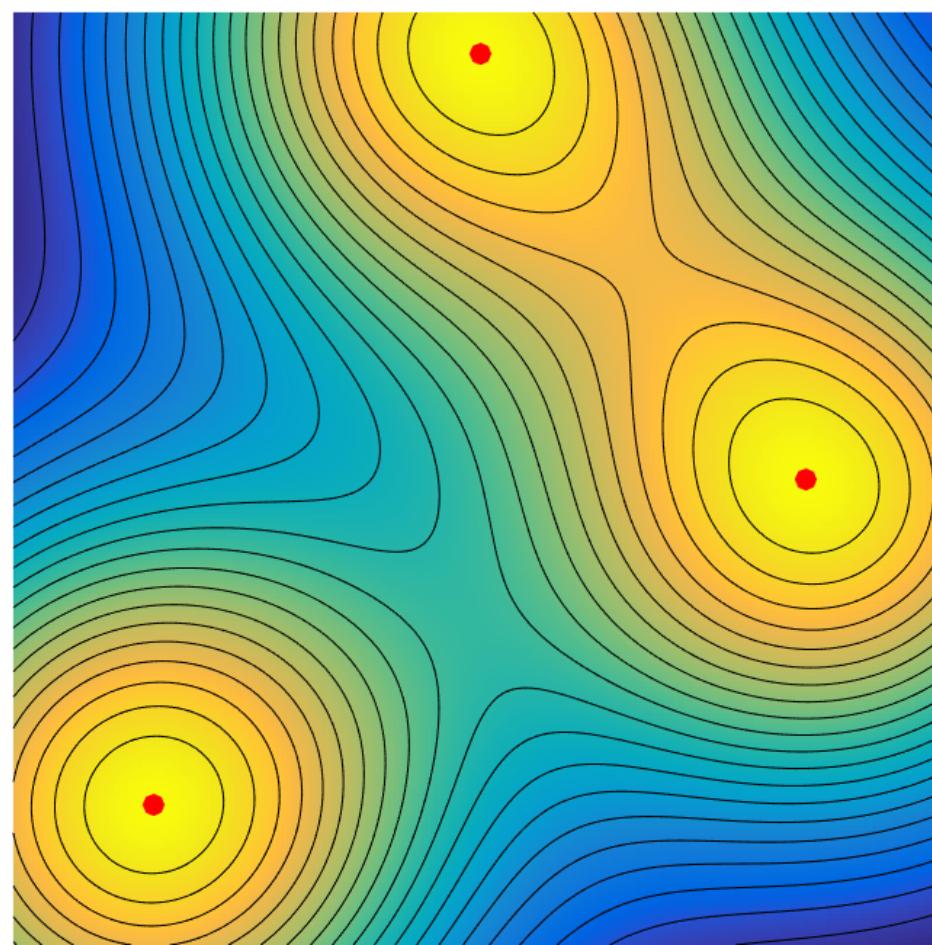
$$\eta_w^{(2N)}(0) < 0 \quad \text{and} \quad \forall z \neq 0, \quad |\eta_w(z)| < 1$$

$\eta_{V,tx}$ is non-degenerate for all t sufficiently small.

For $\|w\|/\lambda = \mathcal{O}(1)$, $\lambda = \mathcal{O}(t^{2N-1})$, $P_\lambda(\Phi\mu_{a,tx} + w)$ recovers exactly N spikes.



Asymptotic vanishing derivatives precertificate in higher dimensions



$t = 1$

$t = 0.5$

$t = 0.2$

$t = 0.1$

The limit of η_V depends on the spikes configuration!

The multivariate limiting certificate

Theorem (Poon and Peyré, 2019):

Let $p_{V,tz}$ be the precertificate associated to support $tz := (tz_i)_{i=1}^N$, then $\|p_{V,tz} - p_{w,z}\| = \mathcal{O}(t)$ where $p_{w,z} = \operatorname{argmin} \left\{ \|p\| : (\Phi^* p)(0) = 1, P(\partial)(\Phi^* p)(0) = 0, P \in \mathcal{S}_z \right\}$

The polynomial space \mathcal{S}_z is the *least interpolant polynomial space associated to z*.

Hermite interpolation problem : Given c_i, d_i ,

find $P \in \mathcal{S}$ such that $\begin{cases} P(z_i) = c_i \\ \nabla P(z_i) = d_i \end{cases}$

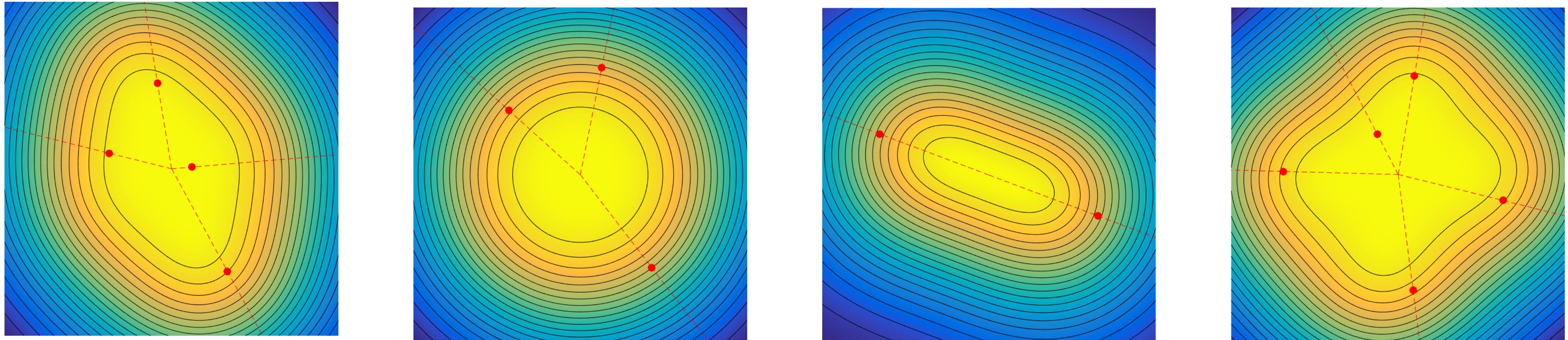
[De Boor and Ron (1990)]:

The least interpolant space is the polynomial space of least degree for which there is a unique solution.

Useful check: For support stability, it is necessary that $\|\eta_{W,z}\|_\infty \leq 1$

Gaussian convolution

$$\phi(x) = \exp(-\|x - \cdot\|^2/(2\sigma^2)) \in L^2(\mathbb{R}^2)$$

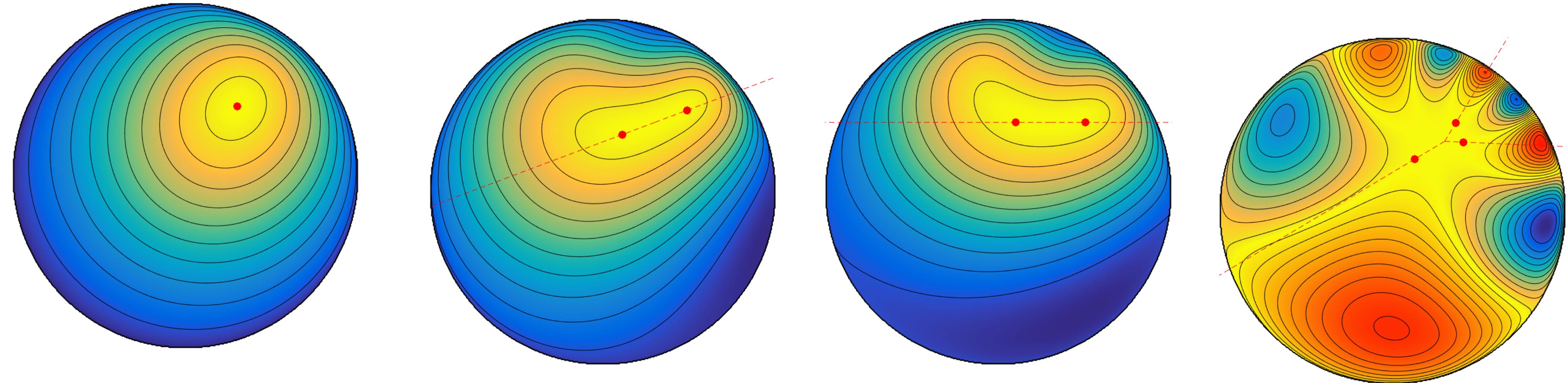


Numerical observation: $\eta_{W,z}$ is always uniformly bounded by 1.

So, we can expect super-resolution when SNR is large enough.

Neuro-imaging

Let $\mathcal{X} = \{x \in \mathbb{R}^2; \|x\| \leq 1\}$. To model MEG/EEG, $\phi(x) = u \mapsto \|x - u\|^{-2} \in L^2(\partial\mathcal{X})$



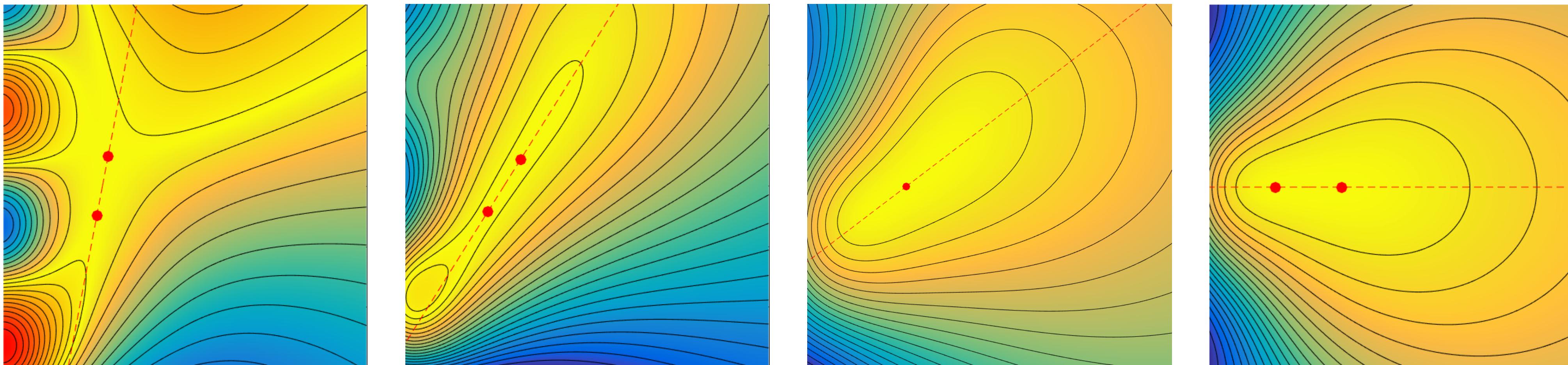
Numerical observation:

- $\eta_{W,z}$ always valid when z consists of aligned spikes
- It is not valid when the spikes are not aligned.

In general, cannot super-resolve 3 close spikes under noise.

Gaussian mixture

For $x = (m, s) \in \mathcal{X} = \mathbb{R} \times \mathbb{R}_+$, $\phi(x) = \frac{1}{s} \exp\left(-\frac{(\cdot - m)^2}{2s^2}\right) \in L^2(\mathbb{R})$



Y-axis = mean, X-axis = standard deviation

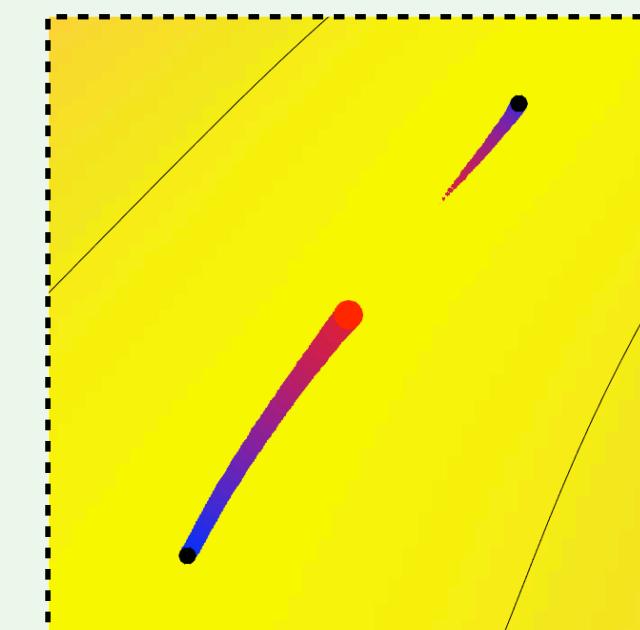
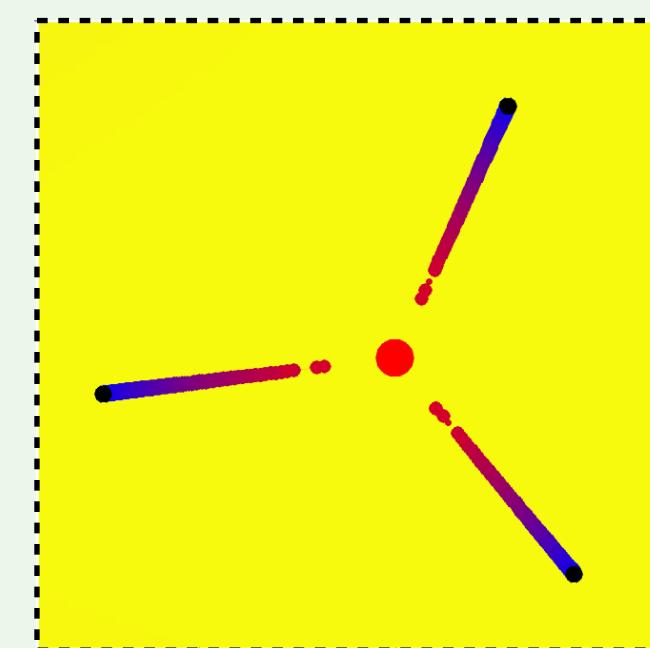
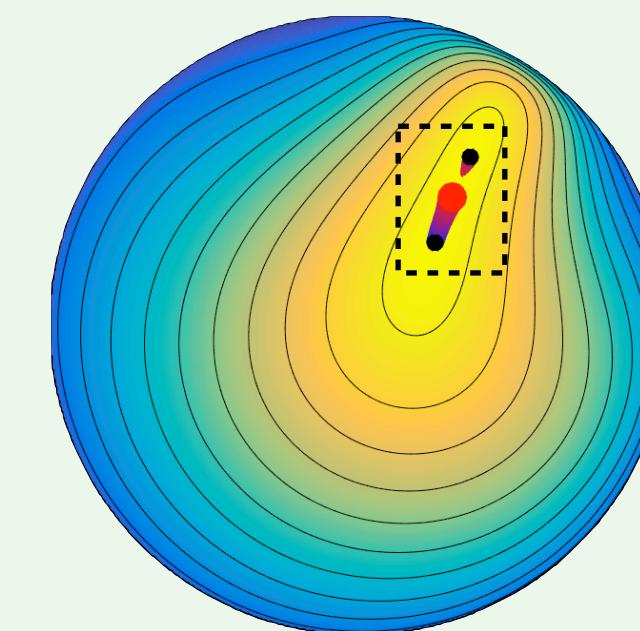
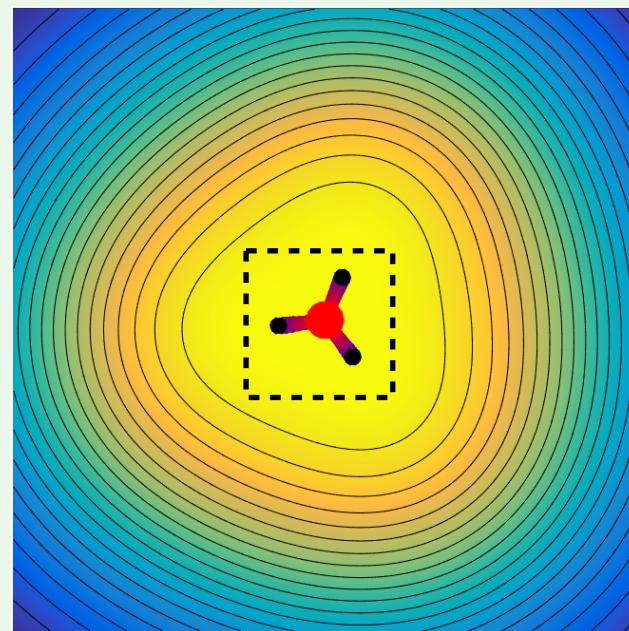
Observation: $\eta_{W,z}$ is a valid certificate if $|m_1 - m_2| \leq |s_1 - s_2|$

One cannot expect to super-resolve a mixture of 2 Gaussians when the variation in means is too large wrt variation in standard deviations.

Evolution of solutions

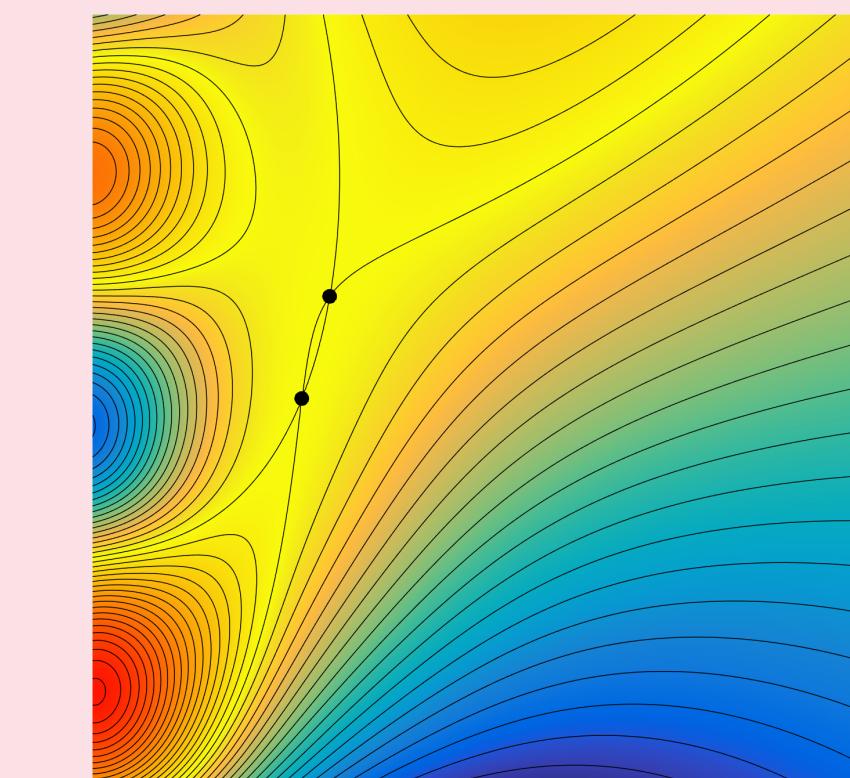
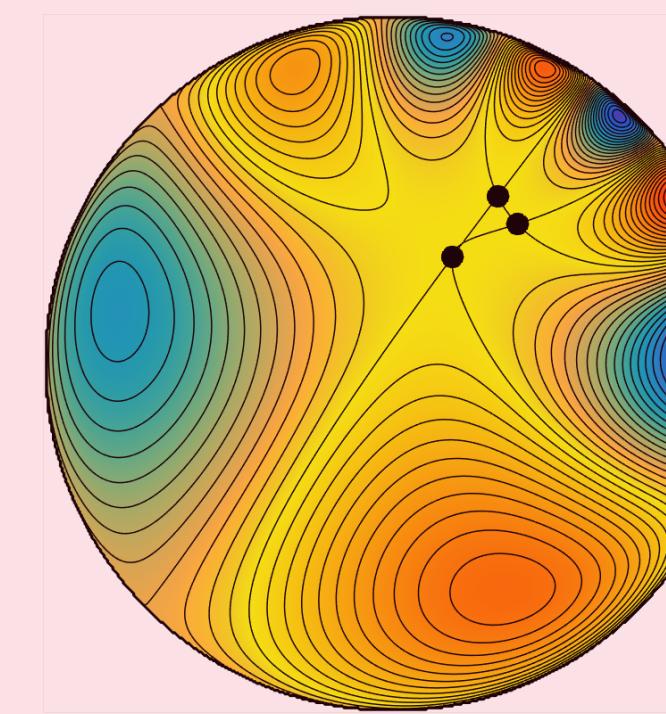
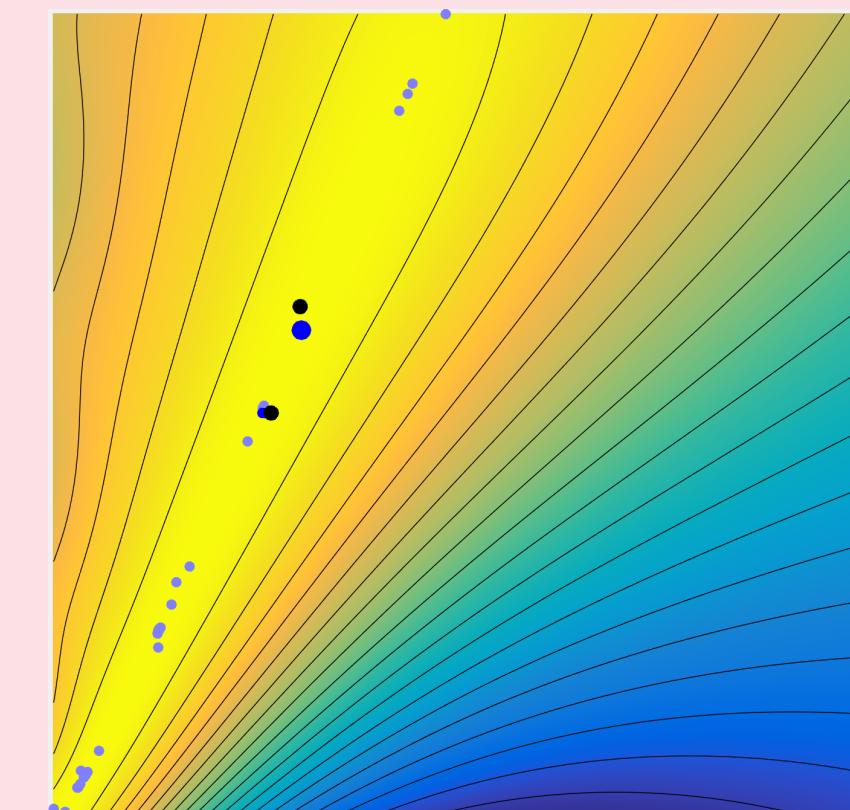
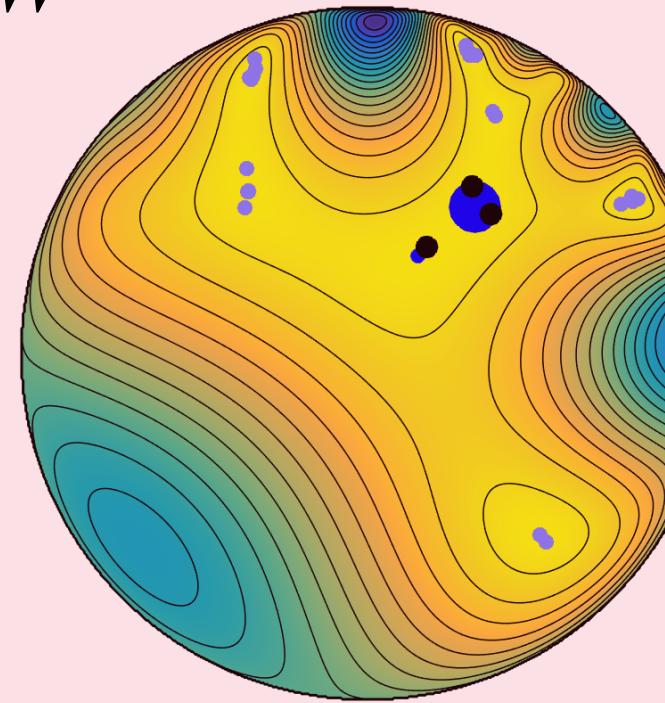
Measurements: $y = \Phi\mu_0 + \lambda w$ where $w = \Phi\hat{\mu}$ with $\hat{\mu} = \sum_{j=1}^{20} b_j \delta_{u_j}$

When $\eta_{W,z}$ is non-degenerate



Displaying evolution of solutions
from λ_{\max} (blue) to 0 (red)

When $\eta_{W,z}$ is degenerate



Solution unstable when $\eta_{W,z}$ is degenerate.
Many tiny spikes (light blue) are added!

Compressed sensing for the Blasso

Off-the-grid Compressed sensing

Problem:

- Let $\phi_\omega(x) \in \mathcal{C}(\mathcal{X})$ where $\omega \in \Omega$.
- Suppose we observe $\Phi\mu = \left(\langle \phi_{\omega_k}, \mu \rangle \right)_{k=1}^m$ where $\omega_1, \dots, \omega_m$ are drawn iid from Ω

Example:

- Random Fourier sampling :
$$\phi_\omega(x) = \exp(\sqrt{-1}2\pi\omega x)$$
 and $\omega \in \{-N, \dots, N\}$

Question:

If $\mu = \sum_{j=1}^s a_j \delta_{x_j}$, how many random samples n do we need to reconstruct m ?

Recovery results (random Fourier)

Theorem (Tang et al 2013): in the case of random Fourier samples.

If $\min_{i \neq j} |x_i - x_j| \geq C/f_c$, and $\text{sign}(a)$ is **distributed uniformly iid** on the complex unit circle, then exact recovery is guaranteed with probability at least $1 - \delta$ provided that

$$m = \mathcal{O}(s \log(s/\delta) \log(f_c/\delta))$$

Recovery results (general)

Theorem (Poon et al 2019):

If $\min_{i \neq j} d_g(x_i, x_j) \geq \Delta$, exact recovery is guaranteed with probability at least $1 - \rho$

provided that

$$m = \mathcal{O}(s \log(s/\rho)^2 + \log(L/\rho))$$

where Δ depends on s and the kernel and L depends on the bounds on the derivatives of ϕ_ω and the diameter $\sup_{x, x' \in \mathcal{X}} d_g(x, x')$.

Stable recovery: $\lambda = \epsilon/\sqrt{s}$ where ϵ is the noise level. Then,

$$W_2^2\left(\sum_j \hat{A}_j \delta_{x_j}, |\hat{\mu}| \right) \lesssim \epsilon \sqrt{s} \quad \text{and} \quad \max_j |a_j - \hat{a}_j| \lesssim \epsilon \sqrt{s}$$

In practice the bound is:
 $s \times \log \text{factors} \times \text{poly}(d)$

Sketching Gaussian mixtures

- Data samples $z_1, \dots, z_n \in \mathbb{R}^d$ drawn iid from Gaussian mixture $\xi = \sum_{i=1}^s a_i \mathcal{N}(x_i, \Sigma)$.
- Need to find: $a_1, \dots, a_s > 0$ and $x_1, \dots, x_s \in \mathbb{R}^d$
- Sketch: Draw $\omega_1, \dots, \omega_n$ iid from $\mathcal{N}(0, \Sigma^{-1}/d)$, $y := \frac{C}{n} \sum_{i=1}^n (\exp(-\sqrt{-1}\omega_k^\top z_i))_{k=1}^m$

$$y \approx \mathbb{E}_z [C \exp(-\sqrt{-1}\omega_k^\top z_i)] = \Phi \mu_0$$

$$\text{with } \mu_0 = \sum_{i=1}^s a_i \delta_{x_i} \text{ and } \phi_\omega(x) = \mathbb{E}_{z \sim \mathcal{N}(x, \Sigma)} [C \exp(\sqrt{-1}\omega^\top z)]$$

Provided that $\min_{i \neq j} \|\Sigma^{-1/2}(x_i - x_j)\| \gtrsim \sqrt{d \log(s)}$, stable recovery is guaranteed with
 $m \gtrsim s (d \log(s) \log(s/\rho) + d^2 \log(sdR)^d / \rho)$, $\epsilon = \mathcal{O}(n^{-1/2})$

Summary

- p_λ converges to p_0 the minimal solution to $D_0(y)$
- Support stability is determined by the minimal norm certificate.

One can compute a pre-certificate η_V in closed form and check its properties.

- $\|\eta_V\|_\infty > 1$ implies stability is impossible.
- $|\eta_V(x)| < 1$ outside the support $\{x_i\}_i$ and a pos-def/neg Hessian implies stability

Analysis of η_V has led to theoretical understanding of super-resolution and compressed sensing.

References

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Super resolution:

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