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Mathematical Tripos Part III: Michaelmas Term 2017/18

Topics in Mathematics of Information - Sheet 4

1. Show that $|Du|(\Omega)$ is a semi-norm in $BV(\Omega)$ and that $BV(\Omega)$ is a Banach space with norm $\|u\|_{BV} = \|u\|_1 + |Du|(\Omega)$.
2. Some special forms of the total variation:
   (a) Let $u \in W^{1,1}(\Omega)$. Prove that in this case $|Du|(\Omega) = \|Du\|_{L^1(\Omega)}$.
   (b) Let $\Omega \subset \mathbb{R}^2$, and let $B_r = B(0,r) \subset \Omega$ denote the disc centred at the origin with radius $r > 0$. Define χ_{B_r} the indicator function of B_r and compute its total variation.
   (c) Let $u \in L^q(\Omega)$ be a piecewise smooth function. More precisely, we assume that there exists $\Omega_1, \dots, \Omega_K$ pairwise disjoint and bounded domains with Lipschitz boundary, such that $\mathrm{cl}(\Omega) = \bigcup_k \mathrm{cl}(\Omega_k)$ and $u^k := u|_{\Omega_k} \in C^1(\mathrm{cl}(\Omega_k))$. Prove that the total variation of u is given by

   $$TV(u) = \|(Du)_{L^1}\|_1 + \sum_{l<k} \int_{\Gamma_{l,k}} |u^l - u^k| \, d\mathcal{H}^1,$$

   where $(Du)_{L^1}$ is the *absolutely integrable* (aka *regular*) part of Du and $\Gamma_{l,k} = \mathrm{cl}(\Omega_l) \cap \mathrm{cl}(\Omega_k) \cap \Omega$.
3. Let the functional $\mathcal{J} : X \to \mathbb{R}$ be convex on a real Banach space X. Prove that:
   (a) For $p \in \partial\mathcal{J}(u)$ and $q \in \partial\mathcal{J}(u)$ we have $tp + (1-t)q \in \partial\mathcal{J}(u)$ for all $t \in [0,1]$.
   (b) Let $\mathcal{J}$ be l.s.c. and consider a sequence $((u_n, p_n))$ in $X \times X^*$ with $p_n \in \partial\mathcal{J}(u_n)$, $u_n \to u$ and $p_n \overset{*}{\rightharpoonup} p$. Then $p \in \partial\mathcal{J}(u)$.
   (c) For $u \in X$ the set $\partial\mathcal{J}(u)$ is weak* sequentially closed, that is:

   $$\text{For } p_n \overset{*}{\rightharpoonup} p, \quad p_n \in \partial\mathcal{J}(u), \text{ we have that } p \in \partial\mathcal{J}(u).$$

4. Prove that for every $\alpha > 0$ and a ROF-minimiser $u = u_\alpha$, i.e.

   $$u = \mathrm{argmin}_{u \in L^2(\Omega)} \left\{ \alpha |Du|(\Omega) + \frac{1}{2}\|u - g\|_2^2 \right\},$$

   we have

   $$\int_\Omega g \, dx = \int_\Omega u_\alpha \, dx.$$

   Show further that for $\alpha \to +\infty$ the minimisers u_α converge in L^1 to the average of g.
5. Let $\Omega \subset \mathbb{R}^d$. Let $g \in L^q(\Omega')$, $\infty > q > 1$, be a noisy and blurry image with blurring kernel $k \in L^1(\Omega_0)$ and $\int_\Omega k \, dx$=1. We want to reconstruct a denoised and deblurred image u on a rectangular domain Ω for which $\Omega' - \Omega_0 \subset \Omega$ under the assumption that

   $$\text{for } x \in \Omega' : \quad (u * k)(x) = \int_\Omega u(x-y)k(y) \, dy.$$

   Prove that the variational problem

   $$\min_{u \in L^q(\Omega)} \frac{\alpha}{p}\|Du\|_p^p + \frac{1}{q}\int_{\Omega'} |u * k - g|^q \, dx,$$

   attains a unique minimiser for $1 < p \leqslant q \leqslant \frac{pd}{d-p} =: p^*$, $g \in L^q(\Omega')$ and $\alpha > 0$. Drive the Euler-Lagrange equation with appropriate boundary conditions that corresponds to the above minimisation problem.
   Useful facts: 1) RellichKondrachov theorem states that for $p \in [1,d)$, $W^{1,p}$ embeds continuously into L^{p^*}. 2) The Poincaré inequality for $W^{1,p}(\Omega)$ is $\|u - m(u)\|_{L^p} \leqslant C\|\nabla u\|_{L^p}$ where $m(u) = \frac{1}{|\Omega|}\int_\Omega u(y)\mathrm{d}y$.
6. Let $u_0 \in BV(\Omega) \cap L^2(\Omega)$ and let $g \in L^2$ be such that $\|g - u_0\|_{L^2} \leqslant \delta$. Consider the denoising problem

   $$u = \mathrm{argmin}_{u \in L^2(\Omega)} \lambda J(u) + \frac{1}{2}\|u - g\|_2^2,$$

   where $J : L^2(\Omega) \to (-\infty, +\infty]$ is a convex functional. Suppose that there exists $p \in \partial J(u_0)$. Show that

   $$\int_\Omega (u - u_0)^2 + |J(u) - |J(u_0)| \leqslant C\delta(\|p\|_{L^2} + 1)$$

   provided that $\lambda \sim \delta$.

7. (Moreau's identity) Let $F : \mathbb{R}^N \to (-\infty, \infty]$ be convex, proper, l.s.c. with bounded sublevel sets and let $P_F(x) = (I + \partial F)^{-1}(x)$. Show that for

$$P_F(x) + P_{F^*}(x) = x, \qquad \forall x,$$

and hence deduce that for all $\delta > 0$,

$$P_{\delta F}(x) + \delta P_{\delta^{-1} F^*}(x/\delta) = x.$$

8. Consider the minimization problem

$$\min_{x \in \mathbb{C}^N} \|x\|_1 \text{ subject to } \|Ax - y\|_2 \leqslant \eta,$$

where $A \in \mathbb{C}^{N \times m}$. Derive the primal-dual iterates for computing solutions to this minimization problem.

Hint: Consider the problem as $\min_x F(Ax) + G(x)$ where $G(x) = \|x\|_1$ and $F(x) = \begin{cases} 0 & \|x - y\|_2 \leqslant \eta, \\ +\infty & \text{otherwise.} \end{cases}$