Natural numbers ....
and then...
What are the real numbers?

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Natural numbers

What are the real numbers?

Natural numbers

Fundamental properties of the natural numbers

i) 1 is a natural number.

ii) Every natural number n has a successor $n+1$ which is also a natural number.

iii) The well-ordering property: Every non-empty set $S \subseteq \mathbb{N}$ has a least element.

iv) If a subset $\Lambda \subset \mathbb{N}$ satisfies

- $1 \in \Lambda$
- $\forall n \in \Lambda : n+1 \in \Lambda$

then $\Lambda = \mathbb{N}$.

The last property iv is the principle of mathematical induction.
You will cover this in Foundations.

Principle of Mathematical Induction

For every $n \in \mathbb{N}$, let $P(n)$ be a statement about n. Suppose that the following hold:

- $P(1)$ is true.
- If $P(k)$ is true then $P(k+1)$ is true.

Then, $P(n)$ is true for all $n \in \mathbb{N}$.

Example

Use mathematical induction to show that for all $n \in \mathbb{N}$,

i) $1 + 2 + \ldots + n = n(n+1)/2$

ii) 3 is a factor of $4^n - 1$.

iii) 5 is a factor of $6^n - 1$.

iv) $2 + 5 + 8 + \ldots + (3n-1) = \frac{n(3n+1)}{2}$

v)* $\sum_{m=0}^{n} \binom{n}{m} = 2^n$

Wang's Paradox: *'Certainly* 1 *is small. If* n *is small then also* $n+1$ *is small. Therefore by induction all natural numbers are small.'*

What is wrong?

What are the real numbers?

Axioms

One approach to defining the real numbers is using **axioms**.

Axioms are basic properties which are assumed as true and not proven.

We will see

- 9 Field axioms describing properties of addition and multiplication.
- 3 Ordering axioms.
- The completeness axiom.

The field axioms

There is an operation $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which satisfies

(A1) $\forall a, b, c \in \mathbb{R}\colon a + (b + c) = (a + b) + c$ (associative law for $+$).

(A2) $\forall a, b \in \mathbb{R}\colon a + b = b + a$ (commutative law for $+$).

(A3) $\exists 0 \in \mathbb{R}\, \forall a \in \mathbb{R}\colon a + 0 = a = 0 + a$ (additive identity).

(A4) $\forall a \in \mathbb{R}\, \exists b \in \mathbb{R}\colon a + b = 0 = b + a$ (additive inverse). For a given $a \in \mathbb{R}$, the additive inverse (referred to as b in the previous formula) is unique and is usually denoted by $-a$.

There is an operation $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which satisfies

(M1) $\forall a, b, c \in \mathbb{R}\colon a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associative law for $\times$).

(M2) $\forall a, b \in \mathbb{R}\colon a \cdot b = b \cdot a$ (commutative law for $\times$).

(M3) $\exists 1 \in \mathbb{R} \setminus \{0\}\, \forall a \in \mathbb{R}\colon a \cdot 1 = a = 1a$ (multiplicative identity).

(M4) $\forall a \in \mathbb{R} \setminus \{0\}\, \exists b \in \mathbb{R}\colon a \cdot b = 1 = b \cdot a$ (multiplicative inverse). The multiplicative inverse of a is unique and is usually denoted by $1/a$ or a^{-1}.

(D) $\forall a, b, c \in \mathbb{R}\colon a \cdot (b + c) = a \cdot b + a \cdot c$ (distributive law).

Proving properties of the reals from the axioms

Consequences of (A1) to (A4).

1. If $a + x = a$ for all a, then $x = 0$. (uniqueness of zero element).
2. If $a + x = a + y$ then $x = y$ (cancellation law for addition).
3. $-0 = 0$.
4. $-(-a) = a$.
5. $-(a + b) = (-a) + (-b)$.

Consequences of (M1) to (M4)

1. If $a \cdot x = a$ for all $a \neq 0$ then $x = 1$ (uniqueness of multiplicative identity).
2. If $a \neq 0$ and $a \cdot x = a \cdot y$, then $x = y$ (cancellation law for multiplication).
3. If $a \neq 0$ then $(a^{-1})^{-1} = a$.

Consequences from combining all the axioms.

1. $(a + b) \cdot c = a \cdot c + b \cdot c$.
2. $a \cdot 0 = 0$.
3. $a \cdot (-b) = -(a \cdot b)$. In particular, $(-1) \cdot a = -a$.
4. $(-1) \cdot (-1) = 1$.
5. If $a \cdot b = 0$ then either $a = 0$ or $b = 0$ (or both).

Powers

For $a \in \mathbb{R}$ and $n \in \mathbb{N}$ we define

- $a^0 = 1$
- $a^n = \underbrace{a \cdots a}_{n \text{ times}}$
- $a^{-n} = (a^{-1})^n$

From these definitions, we have properties

- $a^{ij} = (a^i)^j$ for all $a \in \mathbb{R} \setminus \{0\}$ and $i, j \in \mathbb{Z}$.
- $(ab)^i = a^i b^i$ for all $a, b \in \mathbb{R} \setminus \{0\}$ and $i \in \mathbb{Z}$.
- $a^{i+j} = a^i a^j$ for all $a \in \mathbb{R} \setminus \{0\}$ and $i, j \in \mathbb{Z}$.

The order axioms

There is a subset $\mathbb{P}$ (the strictly positive numbers) of $\mathbb{R}$ such that for all $a, b \in \mathbb{R}$,

(P1) $a, b \in \mathbb{P}$ implies $a + b \in \mathbb{P}$.

(P2) $a, b \in \mathbb{P}$ implies $a \cdot b \in \mathbb{P}$

(P3) exactly one of $a \in \mathbb{P}$, $a = 0$ and $-a \in \mathbb{P}$ holds.

We write

- $a < b$ (or $b > a$) if and only if $b - a \in \mathbb{P}$
- $a \leq b$ (or $b \geq a$) if and only if $b - a \in \mathbb{P} \cup \{0\}$.

Some consequences of the order axioms

1. Reflexivity: $a \leq a$.
2. Antisymmetry: $a \leq b$ and $b \leq a$ implies $a = b$.
3. Transitivity: If $a \leq b$ and $b \leq c$, then $a \leq c$. Likewise with $<$ in place of $\leq$.
4. Trichotomy: Exactly one of the following hold: $a < b$, $a = b$ and $b < a$.
5. $0 < 1$ (equivalently, $1 \in \mathbb{P}$).
6. $a < b$ if and only if $-b < -a$.
7. $a < b$ and $c \in \mathbb{R}$ implies $a + c < b + c$.
8. If $a < b$ and $c < d$, then $a + c < b + d$.
9. $a < b$ and $0 < c$ implies $ac < bc$.
10. $a^2 \geq 0$ with equality if and only if $a = 0$.
11. $a > 0$ if and only if $1/a > 0$.
12. If $a, b > 0$ and $a < b$, then $1/b < 1/a$.