What are the real numbers continued...

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Semester 1 (2021)

Useful Inequalities

Max, min, sup and inf

The completeness axiom

Useful Inequalities

The binomial inequality

Proposition 2.3

Let $x \in \mathbb{R}$ be such that $x > -1$ and $n \in \mathbb{N}$. Then,

$$(1 + x)^n \geq 1 + nx.$$

The absolute value

Definition 2.4

The absolute value of a real number $x \in \mathbb{R}$ is

$$|x| := \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

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e.g. $|-2| = 2$ *and* $|3| = 3$.

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Proposition 2.5

i) for all $x \in \mathbb{R}$, $x \leq |x|$.

ii) for all $x \in \mathbb{R}$, $-x \leq |x|$.

iii) for all $x \in \mathbb{R}$, $|-x| = |x|$.

iv) for all $x, y \in \mathbb{R}$, $|xy| = |x||y|$.

The triangle inequality

Proposition 2.6

For all $x, y \in \mathbb{R}$,

1. $|x+y| \leq |x| + |y|$ (triangle inequality),
2. $\big||x| - |y|\big| \leq |x-y|$ (reverse triangle inequality).

Intervals

Given $a, b \in \mathbb{R}$, we set

$$\begin{aligned}
(a, b) &= \{x \in \mathbb{R} : \; a < x < b\}, \\
[a, b] &= \{x \in \mathbb{R} : \; a \leq x \leq b\}, \\
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\end{aligned}$$

Furthermore, for any $a \in \mathbb{R}$,

$$\begin{aligned}
(a, \infty) &= \{x \in \mathbb{R} : \; x > a\}, \\
[a, \infty) &= \{x \in \mathbb{R} : \; x \geq a\}, \\
(-\infty, a) &= \{x \in \mathbb{R} : \; x < a\}, \\
(-\infty, a] &= \{x \in \mathbb{R} : \; x \leq a\}.
\end{aligned}$$

Max, min, sup and inf

Maximum and minimum of sets

Definition 2.10

Consider a set $S \subseteq \mathbb{R}$. Assume that $S \neq \emptyset$.

(i) Let $s_0 \in \mathbb{R}$. We say s_0 is the maximum of S and write $s_0 = \max S$ if $s_0 \in S$ and $s \leq s_0$ for all $s \in S$.

(ii) Let $s_0 \in \mathbb{R}$. We say s_0 is the minimum of S and write $s_0 = \min S$ if $s_0 \in S$ and $s \geq s_0$ for all $s \in S$.

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Not all subsets of reals have maximums and minimums!

- $(-\infty, 0)$ has no maximum and no minimum.
- $(-2, 0]$ has maximum of 0 and no minimum.
- $\{\frac{1}{n} : \ n \in \mathbb{N}\}$ has maximum 1 but no minimum.

Upper and lower bounds

Consider a set $S \subseteq \mathbb{R}$.

1. A number $M \in \mathbb{R}$ is said to be an **upper bound** of S if $s \leq M$ for all $s \in S$.
2. A number $m \in \mathbb{R}$ is said to be a **lower bound** of S if $s \geq m$ for all $s \in S$.
3. The set S is called **bounded above** if it has an upper bound and **bounded below** if it has a lower bound. We say that S is **bounded** if it is bounded above and below.

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3. The set S is called **bounded above** if it has an upper bound and **bounded below** if it has a lower bound. We say that S is **bounded** if it is bounded above and below.

- $(-\infty, 0)$ has 3 has an upper bound but has no lower bound.
- $(-2, 0]$ has 1 has an upper bound and -2 as a lower bound.
- $\{\frac{1}{n} :\ n \in \mathbb{N}\}$ has 1 as an upper bound and -8 as a lower bound.

Supremum and infimum

Supremum and infimum

Consider a set $S \subseteq \mathbb{R}$.

1. A number $T \in \mathbb{R}$ is called **supremum** or **least upper bound** of S if T is an upper bound of S and any other upper bound M of S satisfies $T \leq M$. We write $T = \sup S$.
2. A number $t \in \mathbb{R}$ is called **infimum** or **greatest lower bound** of S if t is a lower bound of S and any other lower bound m of S satisfies $t \geq m$. We write $t = \inf S$.

The completeness axiom

$\mathbb{Q}$ and $\mathbb{R}$ are different!

For example, there is no element $r \in \mathbb{Q}$ such that $r^2 = 2$.

Consider $S = \{x \in \mathbb{Q} : x^2 < 2\}$.

S is clearly bounded. Does S has a least upper bound in $\mathbb{Q}$?

The completeness axiom

(C) Every **non-empty** set of real numbers that is **bounded above** has a least upper bound.

This captures our intuition that $\mathbb{Q}$ is riddled with holes while $\mathbb{R}$ is complete.

Using the completeness axiom, we can prove

- the existence of $\sqrt{2}$ in $\mathbb{R}$.
- for all $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n \geq x$. (Archimedean property)
- $\mathbb{Q}$ is dense in $\mathbb{R}$.