

Coursework 1 for Inverse Problems (MA505250)

This coursework accounts for 20% of your final grade. Submit by 28th February.

Email cmhsp20@bath.ac.uk typed up solutions and your Matlab scripts.

Exercise 1 (Integral operators). For $\Omega = [0, 1]^2$ and $\mathcal{U} = L^2(\Omega)$, we consider the integral operator $A : \mathcal{U} \rightarrow \mathcal{U}$ with

$$(Au)(y) \stackrel{\text{def.}}{=} \int_{\Omega} k(x, y)u(x) dx, \quad y \in \Omega$$

for $k \in L^2(\Omega \times \Omega)$. Show that

- (a) A is linear with respect to u ,
- (b) A is a bounded linear operator, i.e. $\|Au\|_{\mathcal{U}} \leq \|A\|_{\mathcal{L}(\mathcal{U}, \mathcal{U})} \|u\|_{\mathcal{U}}$. Give also an estimate for $\|A\|_{\mathcal{L}(\mathcal{U}, \mathcal{U})}$,
- (c) the adjoint A^* is given via

$$(A^*v)(y) = \int_{\Omega} k(y, x)v(x) dx.$$

[3 Marks]

Exercise 2 (Inverse problem of differentiation). We consider the problem of differentiation, formulated as the inverse problem of finding u from $Au = f$ with the integral operator $A : L^2([0, 1]) \rightarrow L^2([0, 1])$ defined as

$$(Au)(y) \stackrel{\text{def.}}{=} \int_0^y u(x) dx.$$

- (a) Let f be given by

$$f(x) := \begin{cases} 0 & x < \frac{1}{2}, \\ 1 & x > \frac{1}{2}. \end{cases}$$

Show that $f \in \overline{\mathcal{R}(A)}$.

- (b) Let f be given as in a). Show that $f \in \overline{\mathcal{R}(A)} \setminus \mathcal{R}(A)$.

Hint. Consider the Picard criterion.

- (c) Prove or falsify: *The Moore-Penrose inverse of A is continuous.*

[3 Marks]

Exercise 3 (Generalised inverse). 1. Let $m, n \in \mathbb{N}$ with $m \geq n \geq 2$. Compute the Moore-Penrose inverses of the following matrices:

- (a) $A = (1, 1, \dots, 1) \in \mathbb{R}^{1 \times n}$
- (b) $A = \text{diag}(a_1, \dots, a_n) \in \mathbb{R}^{n \times n}$ with $a_j \in \mathbb{R}$ for $j \in \{1, \dots, n\}$
- (c) $A \in \mathbb{R}^{m \times n}$ with $A^T A = I_n$

2. Let $a, b \in \mathbb{R}$ with $a < b$. Compute the Moore-Penrose inverse of the operator $A : L^2([a, b]) \rightarrow \mathbb{R}$ with

$$Au = \int_a^b u(x) dx.$$

[3 Marks]

Exercise 4 (Convolution). Many forward problems are either modelled as convolutions or they are modelled as the composition of several components one of which is a convolution. Therefore convolutions play an important role in inverse problems. The inverse problem is to recover $U \in L^2(\Omega)$ from $AU = H \star U$ for some kernel $H \in L^2(\Omega)$ where $\Omega = [0, 1]^2$. Here, $A : L^2(\Omega) \rightarrow L^2(\Omega)$.

In this exercise, we implement a discrete version of this problem. From now on, discretize the domain Ω into an $n \times n$ evenly spaced grid, let $A : u \in \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ be defined by $Au = h \star u$ where $h \in \mathbb{R}^{n \times n}$ is a discretised version of H and \star denotes the circular convolution operation.

By the convolution theorem, we know that given $f = Au$,

$$u = (2\pi)^{-n/2} \mathcal{F}^{-1} \left(\frac{\mathcal{F}(f)}{\mathcal{F}(h)} \right) \quad (1)$$

where \mathcal{F} denotes the discrete Fourier transform and the division is a pointwise division.

- (a) Implement this inversion formula in Matlab, taking u to be the Matlab phantom `phantom(256)` and h to be the discretized Gaussian filter with $n = 256$ and $s = 3$:

```
x = [0 : n/2 - 1, -n/2 : -1];
[Y, X] = meshgrid(x, x);
h = exp((-X.^2 - Y.^2)/(2 * s^2));
h = h/sum(h(:));
```

Demonstrate empirically that this is highly unstable to noise.

- (b) Consider the Tikhonov regularised solution

$$\operatorname{argmin}_u \frac{1}{2} \|Au - f\|_2^2 + \frac{\alpha}{2} \|u\|_2^2.$$

Using the convolution theorem, give a closed form expression for the regularised solution u (similar to (1)), implement this reconstruction in Matlab for different values of α and empirically show that it is stable.

- (c) Demonstrate the use of Morozov's discrepancy principle for the selection of α , given noisy measurements $f = Au + e$ and an estimate of $\|e\|_2$.

[5 Marks]

Exercise 5 (The Radon transform). (a) The Matlab command `f = radon(u, phi)`; computes a discretised two-dimensional Radon transform of a discrete image u for a vector of angles ϕ . Use this command to set up a matrix R that maps the column-vector representation of u into the column-vector representation of the sinogram f for an arbitrary image $u \in \mathbb{R}_{\geq 0}^{64 \times 64}$ and angles ϕ with $\phi(j) = j$ for $j \in \{0, 2, \dots, 178\}$.

- (b) Create a noisy sinogram by applying R to a down-sampled version of the Shepp-Logan phantom (built-in in Matlab; use the command `phantom`) and subsequently adding non-negative, random numbers to the sinogram. Create multiple versions with different noise levels.

- (c) Compute a singular value decomposition of \mathbf{R} via the Matlab command `svd` and visualise selected singular vectors of your choice.
- (d) Create a 'pseudo'-inverse of \mathbf{R} by constructing an appropriate matrix with inverted singular values and apply this matrix to the column-vector representations of your noisy sinograms. Regularise the Moore-Penrose inverse using
 - (i) Truncated singular value decomposition;
 - (ii) Tikhonov regularisation.

[5 Marks]