Key logical concepts

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Common constructs for creating new statements

Manipulating statements

We'll be *rigorously* developing the theory of sequences. This means all our claims will be accompanied by *proofs*.

So, we first need to introduce the rules of the game

Notation and key logical concepts.

Statements

Definition

A ‘statement’ or ‘proposition’ is a sentence which is either true or false.

Examples

- ‘$2 < 7$’ is true.
- ‘all integers are odd’ is false.
- ‘2 is a natural number’ is true.

Not a statement:

- ‘Welcome to Bath’
- ‘You should eat an apple a day’

Common constructs for creating new statements

Common constructs

Conjunction	'and'	$\wedge$
Disjunction	'or'	$\vee$
Negation	'not'	$\neg$
Implication	'if ... then ...'	$\Rightarrow$
Equivalence	'if and only if'	$\Leftrightarrow$

Conjunction and disjunction

Let P and Q be statements.

Conjunction

$P \wedge Q$ means 'P and Q'.

True if both P and Q are true. False otherwise.

Disjunction

$P \vee Q$ means 'P or Q'.

True if either P or Q is true. False if both false.

Example Let P be '$2 < 7$'. Let Q be 'all integers are odd'

- $P \wedge Q$ is false.
- $P \vee Q$ is true.

Truth tables

Truth tables are convenient ways of expressing truth.

P	Q	$P \wedge Q$
true	true	true
true	false	false
false	true	false
false	false	false

P	Q	$P \vee Q$
true	true	true
true	false	true
false	true	true
false	false	false

Negation

Negation

Given a statement P, the expression $\neg P$ stands for 'not P'.

The truth table is

P	$\neg P$
true	false
false	true

Implication

Given two statements P and Q, the expression $P \Rightarrow Q$ stands for 'if P, then Q' or 'P implies Q.

If 'you insert a coin into this machine' then 'you receive a coke'.

The only way to contradict 'if P then Q' is if P is true but Q is false.

The truth table for $P \Rightarrow Q$ is

P	Q	$P \Rightarrow Q$
true	true	true
true	false	false
false	true	true
false	false	true

This is the same as $\neg P \vee Q$.

Equivalence

Equivalence

The expression $P \Leftrightarrow Q$ stands for '$P \Rightarrow Q$ and $Q \Rightarrow P$'.

It means that P is true when Q is true and vice versa.

Also say 'P if and only if Q'. Write iff for short.

P	Q	$P \Rightarrow Q$	$P \Leftrightarrow Q$
true	true	true	true
true	false	false	false
false	true	true	false
false	false	true	true

P and Q have the same truth values. So they are interchangeable.

Manipulating statements

Distributive laws for conjunction and disjunction

Proposition 1.5 (Distributive laws)

Given any three statements P, Q, R, the following hold:

i) $P \wedge (Q \vee R) \;\Leftrightarrow\; (P \wedge Q) \vee (P \wedge R)$

ii) $P \vee (Q \wedge R) \;\Leftrightarrow\; (P \vee Q) \wedge (P \vee R)$

Prove this using truth tables.

De Morgan's law

Proposition 1.6 (De Morgan's laws)

Given two statements P and Q, the following hold true.

i) $\neg(P \vee Q) \;\Leftrightarrow\; \neg P \wedge \neg Q$

ii) $\neg(P \wedge Q) \;\Leftrightarrow\; \neg P \vee \neg Q$

Figure 1: Augustus De Morgan (1806–1871)

Again, prove this using truth tables.

Contrapositive

Proposition 1.8 (Contrapositive)

Given statements P and Q, $P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$.

Example:

- 'If it is raining, then the ground is wet'.
- If the ground is not wet, then it is not raining' .

Example: Let x be an integer.

i) If $x^2 - 6x + 5$ is even, then x is odd.

ii) If x is even, then $x^2 - 6x + 5$ is odd.

This proposition tells us we can simply prove ii) to show that i) is true.