MA10174 - Semester 1, 2021/22

Problem Sheet 2

1. Using the principle of induction, prove that:

   a. 5 is a factor of $6^n - 1$ for all $n \in \mathbb{N}$

   b. **Homework** $2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n+1)}{2}$, for all $n \in \mathbb{N}$.

2. **Homework** Solve

$$a. \quad -1 \leq \frac{x-1}{2x+1} \leq 1, \qquad b. \quad x^2 - 5x + 6 \leq 0$$

3. Solve the inequalities

$$a. \quad \frac{2x}{|x-1|} \geq 1, \quad b. \quad \frac{x-1}{|2x|} \leq 1.$$

4. Solve

$$a. \quad |-3x+7| \leq 1, \quad b. \quad |x^2 - x| > 1, \quad c. \quad |x-1| - 3 > -1.$$

5. **Homework**

   a. Find all values of x satisfying $|x-3| + |x-1| = 3$.

   b. Find all values of x satisfying $|3x - 17| \geq |3x - 17| + 4$.

6. a. Show that for any $a, b \in \mathbb{R}$

$$2ab \leq a^2 + b^2$$

   and that equality holds only if $a = b$.

   b. Show that for any $a, b \geq 0$

$$\sqrt{\frac{a}{2}} + \sqrt{\frac{b}{2}} \leq \sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$$

   c. Show that for all $x, y \geq 0$

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}$$

7. **Homework** Find c and r so that $|x-c| = r$ if and only if $|x-1| = 2|x-2|$.

8. Let $S \subset \mathbb{R}$ be a non-empty set and let $s_0 \in \mathbb{R}$. We say that s_0 is the maximum of S if $s_0 \in S$ and $s \leq s_0$ for all $s \in S$. We say that s_0 is the minimum of S if $s_0 \in S$ and $s \geq s_0$ for all $s \in S$. For each of the following sets in $\mathbb{R}$, decide which has i) a maximum, ii) a supremum, iii) a minimum, iv) an infimum.

   a. $\mathbb{Q} \cap [0, \sqrt{2}]$.

   b. $\{3^n | n \in \mathbb{Z}\}$.

   c. $\{1/n | n = 1, 2, \ldots\}$.

9. **Homework** Let $A \subseteq B$ be non-empty subsets of $\mathbb{R}$.

   a. Prove that if B has a supremum, then A has a supremum and $\sup(A) \leq \sup(B)$.

   b. Prove that if B has an infimum, then A has an infimum and $\inf(B) \leq \inf(A)$.

10. **Homework** Let $A = \{\frac{2n}{2n+1} : n \in \mathbb{N}\}$. Show that A is bounded. Find $\sup(A)$ and $\inf(A)$.