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Mathematical Tripos Part III: Michaelmas Term 2017/18

Topic in Mathematics of Information – Exercise Sheet II

Linear and nonlinear approximation

1. Prove that for any $f \in L^1(\mathbb{R})$, if $\|f\|_V < \infty$, then $\|f\|_\infty < \infty$. Given an example of a function $f \in L^1(\mathbb{R}^2)$ such that $\|f\|_V < \infty$ and $\|f\|_\infty = +\infty$.
2. Let $a, b > 0$. Let f be defined on the interval $[0, 1]$ by

$$f(x) = \begin{cases} x^a \sin(x^{-b}) & x \in (0,1] \\ 0 & x = 0 \end{cases}.$$

Show that f is bounded and continuous, but not of bounded variation when $a \leqslant b$. Is f of bounded variation when $a = 2$ and $b = 1$?

Proof. **To show that f is bounded and continuous, note that $|f(x)| \leqslant 1$ for all $x \in [0,1]$. For continuity, we simply need to check at $x = 0$:**

$$0 \leqslant \lim_{x \to 0} f(x) \leqslant \lim_{x \to 0} |x|^a = 0.$$

Finally, to show that f is not of bounded variation, let $x_n = (n\pi + \pi/2)^{-1/b}$. Then

$$\sin(x_n^{-b}) = \begin{cases} 1 & n \text{ even}, \\ -1 & n \text{ odd}, \end{cases}$$

and

$$\sum_{n=1}^{M} |f(x_n) - f(x_{n-1})| = \sum_{n=1}^{M} |(-1)^n (x_n^a + x_{n-1}^a)| = 2 \sum_{n=1}^{M-1} x_n^a + x_M^a + x_a^a,$$

but $\sum_{n=1}^{M-1} x_n^a = \sum_{n=1}^{M-1} (n\pi + \pi/2)^{-a/b} \to \infty$ as $M \to \infty$ when $a/b \leqslant 1$.

For the last part, observe that f is differentiable: again, we only need to check at $x = 0$,

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin(h^{-b})}{h} = 0.$$

So, $f'(0) = 0$. Also for $x \neq 0$, $f'(x) = 2x \sin(x^{-1}) - \cos(x^{-1})$ and

$$\int_0^1 |f'(x)| \leqslant \int_0^1 (2x + 1) \mathrm{d}x = 2.$$

□

3. Let $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$. The space of distributions on Ω is the set of real continuous linear functionals on $\varphi : \mathcal{D}(\Omega) \to \mathbb{R}$.
   (a) Prove that every $u \in L^1_{loc}(\Omega)$ can be understood as a distribution via the definition

$$T_u(\varphi) := \int u(x)\varphi(x)\, dx, \quad \text{for } \varphi \in \mathcal{C}_c^\infty(\Omega).$$

   (b) Show further that if $u, v \in L^1_{loc}$ and $\langle u, \varphi \rangle = \langle v, \varphi \rangle$ for all $\varphi \in \mathcal{D}(\Omega)$, then $u = v$ a.e.
   (c) A distribution that corresponds to a locally integrable function is called *regular*, otherwise it is called *singular*. Can you give an example for a singular distribution?

Proof. **Note that $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$ means that**

(i) **$\exists K \subset\subset \Omega$ such that $\mathrm{Supp}(\varphi_n) \subset K$, for all n,**

(ii) **$\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly on K, for all $\alpha \in \mathbb{N}_0^N$.**

When we say that the distribution T is continuous, we mean that $T(\varphi_n) \to T(\varphi)$ whenever $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$.

For (a), it is clear that $T_u = \int_{\mathrm{Supp}(\varphi)} u\varphi$ is well defined, linear (due to linearity of the integral), and continuous, since

$$|T_u(\varphi)| \leqslant \|u\|_{L^1(\mathrm{Supp}(\varphi))} \|\varphi\|_\infty.$$

For (b), we simply need to show that $\langle u, \varphi\rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega)$ implies that $u = 0$ a.e.: Let $\psi_n \in C_c^\infty$ with $\psi_n(x) = 1$ for all $|x| \leqslant n$. Then $u\psi_n \in L^1$ and for any mollifier φ_δ,

$$u\psi_n \star \varphi_\delta(x) = \int u(y) \underbrace{\psi_n(y)\varphi_\delta(x-y)}\, g(y) \in C_c^\infty \mathrm{d}y = 0.$$

But, $u\psi_n \star \varphi_\delta \to u\psi_n$ in L^1 as $\delta \to 0$. Therefore, $u\psi_n = 0$ a.e. and since this is true for all n, we have that $u = 0$ a.e..

Finally, $\delta : \varphi \mapsto \varphi(0)$ is an example of a singular distribution. □

4. Let $m \in \mathbb{N}$ and let $\varepsilon \in (0,1)$. Show that f is uniformly Lipschitz-$(m+\varepsilon)$ (as defined in the lectures) if and only if f is m-times continuously differentiable and $f^{(m)}$ satisfies $\left|f^{(m)}(x) - f^{(m)}(y)\right| \leqslant C\,|x-y|^\varepsilon$.
5. Consider a wavelet basis of $L^2([0,1])$ constructed with wavelets having $q > s$ vanishing moments and that are C^q. Construct functions $f \in W^{s,2}([0,1])$ for which the linear and nonlinear approximation errors in this basis are identical, $\varepsilon_l(f,M) = \varepsilon_n(f,M)$ for all $M \geqslant 0$.
6. Let f be a piecewise polynomial of degree q defined on $[0,1]$, with K discontinuities. Given a wavelet basis of $L^2([0,1])$ with $q+1$ vanishing moments, give upper bounds as a function of K and M, the linear approximation error $\varepsilon_l(f,M)$ and the nonlinear approximation error $\varepsilon_n(f,M)$.
7. Let $f \in L^\infty([0,1]^2)$ be uniformly Lipschitz-α with $\alpha \in (1/2, q)$ except on a smooth curve of length L. Consider its approximation error with a compactly supported wavelet basis whose wavelet is q-times differentiable and has q vanishing moments. Show that $\varepsilon_n(f,M) = \mathcal{O}(M^{-1})$.