# Additional exercises for MA505250

*Exercise* 1. Let $X$ be a Banach space and $\mathcal{J}_1 : X \to \mathbb{R}$, $\mathcal{J}_2 : X \to \mathbb{R}$ two functionals. Prove that then

(a) If $\mathcal{J}_1$ and $\mathcal{J}_2$ are weak l.s.c., $\alpha \geqslant 0$, then $\alpha \mathcal{J}_i$ and $\mathcal{J}_1 + \mathcal{J}_2$ are weak l.s.c.

(b) If $\mathcal{J}_1$ is weak l.s.c. and $\varphi : \mathbb{R} \to \mathbb{R}$ is monotonically increasing and l.s.c., then $\varphi \circ \mathcal{J}_1$ is weak l.s.c.

(c) If $\mathcal{J}_1$ is weak l.s.c. then for a Banach space $Y$ and weak sequentially continuous $\Phi : \mathcal{Y} \to X$ it follows that $\mathcal{J}_1 \circ \Phi$ is weak l.s.c.

(d) For any (non-empty) family of weak l.s.c. functionals $\mathcal{J}_i : X \to \mathbb{R}$, $i \in I$ we have that $\sup_{i \in I} \mathcal{J}_i$ is weak l.s.c.

(e) Let $\varphi : K \to \mathbb{R}$ be l.s.c., where $K$ is either $\mathbb{R}$ or $\mathbb{C}$, and $x^* \in X^*$. Then the functional

$$L_{x^*,\varphi} = \varphi \circ \langle x^*, \cdot \rangle_{X^* \times X},$$

is weak l.s.c. in $X$.

Collect what you have proven (as appropriate) and show that $\varphi(\|u\|_X)$ is weak l.s.c. for any $\varphi : [0, \infty) \to \mathbb{R}$ that is monotonically increasing and l.s.c.

*Exercise* 2. For a given noisy image $g \in L^2(\Omega)$ and rectangular image domain $\Omega \subset \mathbb{R}^2$ we consider the following variational problem

$$u_\alpha = \operatorname{argmin}_{u \in L^2(\Omega)} \left\{ \alpha \|Du\|_2^2 + \|u - g\|_2^2 \right\},$$

for $\alpha > 0$. Prove that there exists a unique minimiser $u_\alpha$ to the above problem. Show further, that under the additional assumption that $L \leqslant g \leqslant R$ a.e. in $\Omega$ the minimiser $u_\alpha$ of the above problem also fulfils $L \leqslant u_\alpha \leqslant R$ a.e. in $\Omega$.