Inverse Problems
Variational regularisation

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February 6, 2020

Outline

Variational regularisation

Let's return to Tikhonov regularisaton: The regularised solution is u_α:

$$(A^*A + \alpha \mathrm{Id})u_\alpha = A^* f_\delta \tag{1.1}$$

One can check (do this!) that this is the first order optimality condition of

$$\min_{u \in \mathcal{U}} \|Au - f_\delta\|^2 + \frac{\alpha}{2}\|u\|^2. \tag{1.2}$$

Since this is a convex optimisaton problem, (1.1) is a necessary and sufficient condition for the minimum of the functional (1.2).

- $\|Au - f\|^2$ is called the data fidelity term.
- $\mathcal{J}(u) \stackrel{\text{def.}}{=} \frac{1}{2}\|u\|^2$ is called the regularisation term, and penalises some unwanted features of the solution (in this case, large norm).
- α is the regularisation parameter.

Variational regularisation

We will now study more general variational regularisers of the form

$$R_\alpha f \in \operatorname{argmin}_{u\in\mathcal{U}} \frac{1}{2}\|Au - f_\delta\|_{\mathcal{V}}^2 + \alpha\mathcal{J}(u). \tag{1.3}$$

where

- $A : \mathcal{U} \to \mathcal{V}$ is a bounded linear operator between a Banach spaces $\mathcal{U}$ and a Hilbert space $\mathcal{V}$.
- $\mathcal{J} : \mathcal{U} \to [0, \infty]$.
- $f_\delta \in \mathcal{V}$ satisfies $\|Au - f_\delta\|_{\mathcal{V}} \leqslant \delta$.

Example: smoothing regularisers

Let $\mathcal{J}(u) = \|Lu\|_{\mathcal{Z}}$ where $L : \mathcal{U} \to \mathcal{Z}$ is a linear (possibly unbounded) operator . Popular choices include differential operators, e.g. $L = \nabla$, $\mathcal{U} = W^{1,2}$, $\mathcal{Z} = L^2$.

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For $\mathcal{J}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$, u is a minimizer if and only if

$$A^*Au - A^*g - \lambda\Delta u = 0,$$

with Neumann boundary condition $\nabla u \cdot \eta = 0$ on $\partial\Omega$ where η is the outward unit normal to $\partial\Omega$.

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- Intuition is to encourages solutions with small gradient which best fit the observation data g, so noise is removed.
- For imaging applications, leads to oversmooth reconstructions as Δ has very strong isotropic smoothing properties.

Example: Lasso

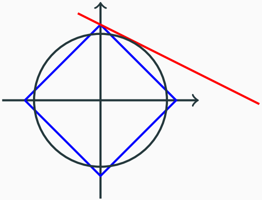
Consider $\mathcal{U} = \mathcal{V} = \ell_2(\mathbb{N})$ and $\mathcal{J}(u) = \begin{cases} \|u\|_1 & u \in \ell_1(\mathbb{N}) \\ +\infty & u \in \ell_2(\mathbb{N}) \setminus \ell_2(\mathbb{N}) \end{cases}$.

The problem

$$\min_u \|Au - f\|_2^2 + \frac{\alpha}{2} \|u\|_1$$

is called the lasso in statistics and can be shown to promote sparse solutions.

One can also consider $\mathcal{J}(u) = \|Wu\|_1$ where $W : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$. For example, W is some wavelet transform.



Consider $\langle u, a \rangle = f$ where $u \in \mathbb{R}^2$ is unknown, $a \in \mathbb{R}^2$ and $f \in \mathbb{R}$, solutions are along the red line. The solution of smallest ℓ_1 norm will be 1-sparse, whereas the solution of smallest ℓ_2 norm is 2-sparse.

Example: Total variation

Instead of $\mathcal{J}(u) = \int_\Omega |\nabla u|^2$, one could consider $\mathcal{J}(u) = \int_\Omega |\nabla u|$.

Deblurring example:

$$\min_u \mathcal{J}(u) + \|Ku - b\|_{L^2}^2 , \quad \text{where} \quad Ku = h \star u$$

$\mathcal{J}(x) = \|Dx\|_2^2$

$\mathcal{J}(x) = \|Dx\|_1$

Example: Total variation

The use of $\int_\Omega |\nabla u|^2$ leads to smooth solutions, the point of $\int_\Omega |\nabla u|$ is that this makes sense not only for $u \in W^{1,1}(\Omega)$ but also for functions of bounded variation.

Given $u \in L^1(\Omega)$ for $\Omega \subset \mathbb{R}^d$, define

$$\mathrm{TV}(u) \stackrel{\text{def.}}{=} \sup \left\{ \langle u, \mathrm{div}\varphi \rangle \; ; \; \varphi \in C_c^\infty(\Omega; \mathbb{R}^d), \; \sup_{\omega \in \Omega} \|\varphi(\omega)\|_2 \leqslant 1 \right\}.$$

Let $\|u\|_{BV} \stackrel{\text{def.}}{=} \|u\|_{L^1} + \mathrm{TV}(u)$, and the space of bounded variations $\left\{ u \in L^1 \; ; \; \mathrm{TV}(u) < \infty \right\}$ is a Banach space with norm $\|\cdot\|_{BV}$.

Contains $W^{1,1}(\Omega)$ and also discontinuous functions such as χ_C where $C \subset \Omega$ has Lipschitz boundary, in which case, $TV(\chi_C) = \mathrm{Per}(C)$.

Bayesian viewpoint of variational methods

Given $f \in \mathbb{R}^N$, there are two components to (linear) inverse problems:

1. A data model: $f = Au_0 + n$ where $u_0 \in \mathbb{R}^N$ is the underlying object to be recovered, T is some linear transform (e.g. a blurring operator, a subsampled Fourier transform, or the identity matrix), and n is the noise. Typically, the entries in n are assumed to be Gaussian distributed with mean 0 and variance σ^2.
2. An a-priori probability density: $P(u) = e^{-p(u)}$. This represents the idea that we have of the solution.

Bayesian viewpoint of variational methods

By Bayes' rule, the posteriori probability of u knowing f is

$$P(u|f)P(f) = P(f|u)P(u),$$

where $P(f|u) = \exp\left(-\frac{1}{\sigma^2}\|f - Au\|_2^2\right)$. So,

$$P(u|f) = \frac{\exp\left(-\frac{1}{\sigma^2}\|f - Au\|_2^2 - p(u)\right)}{P(f)},$$

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The maximum a posteriori (MAP) reconstruction is:

$$u^* \in \operatorname*{argmax}_u P(u|f).$$

Equivalently, $$u^* \in \operatorname*{argmin}_u p(u) + \frac{1}{\sigma^2}\|f - Au\|_2^2 .$$

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Other choices of noise distributions:

- Additive Laplace noise $e^{-\frac{1}{\sigma^2}\|f-Au\|_1}$ with corresponding data fidelity term $\|Au - f\|_1$
- Poisson noise $\prod_{i,j} \frac{u_{i,j}^{f_{i,j}}}{f_{i,j}!} e^{-u_{i,j}}$ with data fidelity term $\int u - f \log(u)$.

We now study regularisers of the form

$$R_\alpha(f) \in \operatorname*{argmin}_u \alpha \mathcal{J}(u) + \frac{1}{2} \|f - Au\|_2^2 .$$

Usual questions:

- Given $f = Au^\dagger$, do we have convergence $R_\alpha(f) \to u^\dagger$?
- Do we have convergent regularisers?
- Convergence rates?

Outline

Banach spaces are complete, normed vector spaces.

Dual spaces

For every Banach space $\mathcal{U}$, its dual space $\mathcal{U}^*$ is the space of continuous linear functionals on $\mathcal{U}$, that is, $\mathcal{U}^* = \mathcal{L}(\mathcal{U}, \mathbb{R})$. Given $u \in \mathcal{U}$ and $p \in \mathcal{U}^*$, we write the dual product $\langle p,\, u\rangle \stackrel{\text{def.}}{=} p(u)$. The dual space is a Banach space equipped with the norm

$$\|p\|_{\mathcal{U}^*} = \sup_{u \in \mathcal{U},\, \|u\|_{\mathcal{U}} \leqslant 1} \langle p,\, u\rangle.$$

Banach spaces are complete, normed vector spaces.

Bi-dual

The bi-dual space of $\mathcal{U} \stackrel{\text{def.}}{=} (\mathcal{U}^*)^*$. Every $u \in \mathcal{U}$ defines a continuous linear mapping on $\mathcal{U}^*$, by

$$\langle Eu,\, p\rangle \stackrel{\text{def.}}{=} \langle p,\, u\rangle.$$

$E : \mathcal{U} \to \mathcal{U}^{**}$ is well defined and is a continuous linear isometry. If E is injective, then $\mathcal{U}$ is called reflexive. Examples of reflexive Banach spaces include Hilbert spaces, L^q, ℓ^q for $q \in (1, \infty)$. We call $\mathcal{U}$ separable if there exists a countable dense subset of $\mathcal{U}$.

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Adjoint

For any $A \in \mathcal{L}(\mathcal{U}, \mathcal{V})$, there exists a unique operator $A^* : \mathcal{V}^* \to \mathcal{U}^*$ called the adjoint of A such that for all $u \in \mathcal{U}$ and $p \in \mathcal{V}$,

$$\langle A^* p,\, u \rangle = \langle p,\, Au \rangle.$$

In infinite dimensions, bounded sequences do not have to have convergent subsequences.

E.g. In ℓ^2, consider e_j the canonical basis. Then, $\|e_j\| = 1$ for all j but there does not exists $u \in \ell^2$ such that $\|e_j - u\| \to 0$.

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Weak and weak-* convergence

We say that $\{u_k\} \subset \mathcal{U}$ converges weakly to $u \in \mathcal{U}$ if and only if for all $p \in \mathcal{U}^*$, we have $\langle p, u_k \rangle \to \langle p, u \rangle$.

For $\{p_k\} \subset \mathcal{U}^*$, we say $\{p_k\}$ converges weak-* to $p \in \mathcal{U}^*$ if for all $u \in \mathcal{U}$, we have $\langle p^k, u \rangle \to \langle p, u \rangle$ for all $u \in \mathcal{U}$.

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- **Banach-Alaogu Theorem**: Let $\mathcal{U}$ be a separable normed vector space. Then every bounded sequence has a weak-* convergent subsequence.
- Every bounded sequence in a reflexive Banach space has a weakly convergent subsequence.

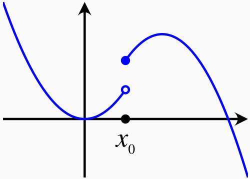
Lower semi-continuity

One useful property is the notion of sequential lower semicontinuity:

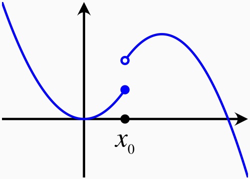
Let $\mathcal{X}$ be a Banach space with topology $\tau_{\mathcal{X}}$. The functional $E : \mathcal{X} \to [-\infty, \infty]$ is said to be sequentially lower semi-continuous with respect to $\tau_{\mathcal{X}}$ at $u \in \mathcal{X}$ if

$$E(x) \leqslant \liminf_{j\to\infty} E(x_j)$$

for all sequences $\{x_j\}_j \subset \mathcal{X}$ with $x_j \to x$ in the topology $\tau_{\mathcal{X}}$ of $\mathcal{X}$.



Not lsc



lsc

"$E(x_0)$ is a good lower bound for function values near x_0"

Example: Lower semi-continuity

Let $\mathcal{U}$ be any normed space with norm $\|\cdot\|_{\mathcal{U}}$, then $E(u) = \|u\|_{\mathcal{U}}$ is lower semicontinuous with respect to the weak topology:

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Let $u^j \to u$ weakly, by the Hahn-Banach theorem, there exists an element $f \in \mathcal{U}^*$ such that $f(u) = \|u\|_{\mathcal{U}}$ and $\|f\| = 1$. Therefore,

$$\|u\|_{\mathcal{U}} = f(u) = \lim_j f(u^j) \leqslant \liminf_j \left\| u^j \right\|_{\mathcal{U}}.$$

Example: Lower semi-continuity

The functional $\|\cdot\|_1 : \ell^2 \to [0, \infty]$ is lower semi-continuous with respect to ℓ_2 convergence.

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Given any $\{u^j\} \subset \ell_2$ with $u^j \to u$ in ℓ_2, we have

$$u_k^j = \langle e_k,\, u^j \rangle \to \langle \delta_k,\, u \rangle = u_k.$$

So, by Fatou's lemma

$$\|u\|_1 = \sum_k \lim_{j\to\infty} \left\| u_k^j \right\| \leqslant \liminf_{j\to\infty} \sum_k \left| u_k^j \right| = \liminf_j \left\| u^j \right\|_2 .$$

Background: infinity

We consider functionals $E : \mathcal{U} \to \bar{\mathbb{R}} \stackrel{\text{def.}}{=} \mathbb{R} \cup \{-\infty, +\infty\}$.

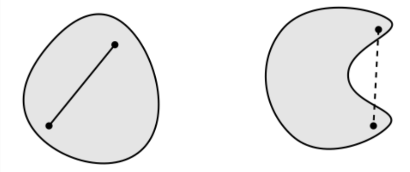
- Useful to model constraints. E.g. if $E : [-1, \infty) \to \mathbb{R}^2$ maps $x \mapsto x^2$, consider instead $\bar{E} : \mathbb{R} \to \bar{\mathbb{R}}$ defined by $\bar{E}(x) = E(x)$ for $x \in [-1, \infty)$ and $\bar{E}(x) = +\infty$ otherwise. No need to worry if $E(x + y)$ is well-defined.
- We then consider unconstrained minimisation (although the function may no longer be differentiable).
- The indicator function on a set $C \subset \mathcal{U}$ is $\iota_C \stackrel{\text{def.}}{=} \begin{cases} 1 & x \in C \\ +\infty & x \notin C \end{cases}$

  So, we can write $\min_{u \in C} E(u) = \min_{u \in \mathcal{U}} E(u) + \iota_C(u)$.

We denote $\mathrm{dom}(E) \stackrel{\text{def.}}{=} \{u \in \mathcal{U} \; ; \; E(u) < \infty\}$. We say E is proper if $\mathrm{dom}(E) \neq \emptyset$.

Background: Convexity

A subset $C \subseteq \mathcal{U}$ is called convex if $\lambda u + (1-\lambda)v \in \mathcal{C}$ for all $\lambda \in (0,1)$ and $u, v \in \mathcal{C}$



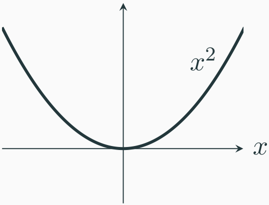
A functional $E : \mathcal{U} \to \bar{\mathbb{R}}$ is called convex if

$$E(\lambda u+(1-\lambda)v) \leqslant \lambda E(u)+(1-\lambda)E(v), \forall \lambda \in (0,1) \quad \text{and} \quad \forall u, v \in \mathrm{dom}(E), u \neq v.$$

It is called strictly convex if the inequality is strict.

Minimising functionals

A functional is called coercive if for all $u_j \in \mathcal{U}$ with $\|u_j\| \to +\infty$, we have $E(u_j) \to +\infty$. Equivalently, if $\{E(u_j)\}_j$ is bounded, then $\{u_j\}_j$ must be bounded.



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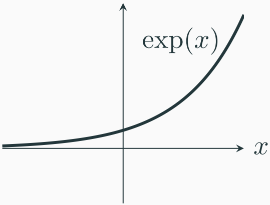

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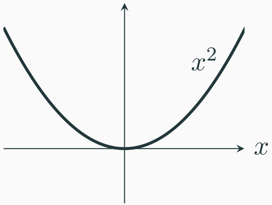

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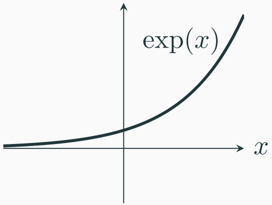

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Coercivity is sufficient to ensure boundedness of minimising sequences:

Lemma 2.1

Let $E : \mathcal{U} \to \bar{\mathbb{R}}$ be a proper coercive functional, bounded from below. Then, $\inf_{u \in \mathcal{U}} E(u)$ exists in $\mathbb{R}$ and there exists a minimising sequence $\{u_j\}$ such that $E(u_j) \to \inf_u E(u)$ and all minimising sequences are bounded.

Theorem 2.2 (The Direct method of Calculus)

Let $\mathcal{U}$ be a Banach space and $\tau_{\mathcal{U}}$ a topology (not necessarily the norm topology) on $\mathcal{U}$ such that bounded sequences have $\tau_{\mathcal{U}}$ convergent subsequences. Let $E : \mathcal{U} \to \bar{\mathbb{R}}$ be proper coercive and $\tau_{\mathcal{U}}$-l.s.c, and bounded from below. Then E has a minimiser.

Proof.

- The assumptions imply that there exists a bounded minimising sequence $\{u_j\}_j$.
- By assumption on the topology $\tau_{\mathcal{U}}$, there exists a subsequence u_{k_j} and $u_* \in \mathcal{U}$ which converges $\tau_{\mathcal{U}}$ to u_*.
- Due to $\tau_{\mathcal{U}}$-lsc, we have $E(u^*) \leqslant \liminf_{k\to\infty} E(u_{j_k}) = \inf_u E(u) > \infty$. Therefore, u_* is a minimiser.

$\square$

Minimising functionals

- Key ingredient: bounded sequences have convergent subsequences.
- If $\mathcal{U}$ is a reflexive Banach space and E is a proper, bounded from below, coercive, lsc wrt weak topology, then a minimiser exists, since reflexive Banach spaces are weakly compact.
- A convex function is lsc wrt weak topology if and only if it is lsc with respect to strong topology.
- If E has at least one minimiser and is strictly convex, then the minimiser is unique: let u, v be two minimisers of E. If $u \neq v$, then

$$E(u) \leqslant E(\frac{1}{2}u + \frac{1}{2}v) < \frac{1}{2}E(u) + \frac{1}{2}E(v) \leqslant E(u)$$

  which is a contradiction. Not however that strict convexity is not necessary for uniqueness of minimisers (e.g. think for $f(x) = |x|$).

Outline

Well-posedness and regularisation properties

We now study the properties of

$$R_\alpha f \in \operatorname{argmin}_{u \in \mathcal{U}} \Phi_{\alpha,f}(u) \stackrel{\text{def.}}{=} \frac{1}{2} \|Au - f\|_{\mathcal{V}}^2 + \alpha \mathcal{J}(u)$$

as a convergent regularisation for

$$Au = f \tag{3.1}$$

where $A : \mathcal{U} \to \mathcal{V}$ is a bounded linear operator and $\mathcal{U}$, $\mathcal{V}$ are Banach spaces.

- When do minimisers exist? (i.e. well-posedness of the regularised problem)
- Is $R_\alpha : \mathcal{V} \to \mathcal{U}$ continuous?
- Are there parameter choice rules that guarantee the convergence of the minimisers to an appropriated generalised solution? (Need equivalent notions of minimal-norm solution and least squares solution)

Theorem 1

Let $\mathcal{U}$ and $\mathcal{V}$ be Banach spaces with topologies $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{V}}$ respectively. Let $\|\cdot\|_{\mathcal{V}}$ be $\tau_{\mathcal{V}}$-lsc. Assume that

(i) $A : \mathcal{U} \to \mathcal{V}$ *is* $\tau_{\mathcal{U}} \to \tau_{\mathcal{V}}$ *continuous.*

(ii) $\mathcal{J} : \mathcal{U} \to (0, +\infty]$ *is proper,* $\tau_{\mathcal{U}}$*-lsc and its non-empty sublevel-sets* $\{u \in \mathcal{U} \; ; \; \mathcal{J}(u) \leqslant C\}$ *are* $\tau_{\mathcal{U}}$*-sequentially compact*

Then,

(i') *for any fixed* $\alpha > 0$ *and* $f \in \mathcal{V}$*, there exists a minimiser of* $u^\alpha \in \operatorname{argmin}_u \frac{1}{2} \|Au - f\|_{\mathcal{V}}^2 + \alpha \mathcal{J}(u)$.

(ii') *Suppose that* $\mathcal{V}$ *is a Hilbert space. If* A *is injective or* $\mathcal{J}$ *is strictly convex, then* u^α *is unique.*

Existence of minimisers

Since $\Phi_{\alpha,f}(u) \geqslant 0$, there exists a minimising sequence u_j so that

$$\lim_{j\to\infty} \Phi_{\alpha,f}(u_j) = \inf_{u\in\mathcal{U}} \Phi_{\alpha,f}(u) \stackrel{\text{def.}}{=} L.$$

In particular, $J(u_j)$ is uniformly bounded. Since the level sets of $\mathcal{J}$ are $\tau_{\mathcal{U}}$ sequentially compact, there exists a subsequence u_{j_k} which converges $\tau_{\mathcal{U}}$ to some $u \in \mathcal{U}$

By continuity of A, Au_{j_k} converges to Au in $\tau_{\mathcal{V}}$. By lsc properties of $\mathcal{J}$ and $\|\cdot\|_{\mathcal{V}}$, we have

$$\Phi_{\alpha,f}(u) \leqslant \liminf_{k\to\infty} \Phi_{\alpha,f}(u_{j_k}) \leqslant L.$$

Therefore, u is a minimiser.

Finally, we saw that the minimum is unique if $\Phi_{\alpha,f}$ is strictly convex. Note that $u \mapsto \|Au - f\|_{\mathcal{V}}$ is strictly convex if and only if A is injective (exercise!).

Variational regularisers are continuous

Theorem 2

Under the assumptions of Theorem 4, assume also

- $\mathcal{V}$ *is a Hilbert space and that either* A *is injective or* $\mathcal{J}$ *is strictly convex.*
- *norm convergence in* $\mathcal{V}$ *implies convergence in* $\tau_{\mathcal{V}}$.

Then, given $f_j \to f$ *in* $\mathcal{V}$, $u_j \stackrel{\text{def.}}{=} R_\alpha f_j$ *exists and is unique, and* u_j *converges to* $u \stackrel{\text{def.}}{=} R_\alpha f$ *in* $\tau_{\mathcal{U}}$.

Variational regularisers are continuous

We first show that $\Phi_{\alpha,f}(u_j)$ is bounded: Since $\mathcal{J}$ is proper, there exists $\tilde{u}$ such that $\Phi_{\alpha,f}(\tilde{u}) < \infty$

$$\Phi_{\alpha,f}(u_j) \leqslant 2\Phi_{\alpha,f_j}(u_j) + \|f - f_j\|_{\mathcal{V}}^2 \leqslant 2\Phi_{\alpha,f_j}(\tilde{u}) + \|f - f_j\|_{\mathcal{V}}^2$$

By compactness of the sublevel sets of $\mathcal{J}$, there exists a subsequence u_{j_k} which converges $\tau_{\mathcal{U}}$ to some $\hat{u} \in \mathcal{U}$. By continuity of A, lsc of $\|\cdot\|_{\mathcal{V}}$ and lsc of $\mathcal{J}$, we have

$$\Phi_{\alpha,f}(\hat{u}) \leqslant \liminf_k \Phi_{\alpha,f_{j_k}}(u_{j_k}) \leqslant \liminf \Phi_{\alpha,f_{j_k}}(u) = \Phi_{\alpha,f}(u).$$

By uniqueness of minimisers, $\hat{u} = u$

Repeat this for any subsequence of $\{u_j\}$ to see that all subsequences have a subsequence which converge to u. Therefore, the entire sequence u_j converges to u in $\tau_{\mathcal{U}}$.

Definition 3 ($\mathcal{J}$-minimising solutions)

Let

- $u^\dagger_{\mathcal{J}} \in \operatorname{argmin}_{u\in\mathcal{U}} \|Au - f\|_{\mathcal{V}}$ and
- $\mathcal{J}(u^\dagger_{\mathcal{J}}) \leqslant \mathcal{J}(\tilde{u})$ for all $\tilde{u} \in \operatorname{argmin}_{u\in\mathcal{U}} \mathcal{F}(Au, f)$.

Then, $u^\dagger_{\mathcal{J}}$ is called a $\mathcal{J}$-minimising solution of (3.1).

Remarks

- If $\mathcal{V}$ is a Hilbert space, then $\mathbb{L} \stackrel{\text{def.}}{=} \left\{v\ ;\ v \in \operatorname{argmin}_{u\in\mathcal{U}} \|Au - f\|_{\mathcal{V}}\right\}$ is non-empty if and only if $f \in \mathcal{R}(A) \oplus \mathcal{R}(A)^{\perp}$.
- In the following, we assume that (3.1) is solvable, i.e. for any f, there exists $u^\dagger$ such that $Au^\dagger = f$ and $\mathcal{J}(u^\dagger) < \infty$.
- But, even in this case the existence of a $\mathcal{J}$-minimising solution is not guaranteed. Furthermore, even when there is existence, in general, there is no uniqueness.

Existence of minimisers

Theorem 4

Let $\mathcal{U}$ and $\mathcal{V}$ be Banach spaces with topologies $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{V}}$ respectively. Let $\|\cdot\|_{\mathcal{V}}$ be $\tau_{\mathcal{V}}$-lsc. Suppose that $Au = f$ has a solution with finite $\mathcal{J}$-value. Assume that

(i) $A : \mathcal{U} \to \mathcal{V}$ *is* $\tau_{\mathcal{U}} \to \tau_{\mathcal{V}}$ *continuous.*

(ii) $\mathcal{J} : \mathcal{U} \to (0, +\infty]$ *is proper,* $\tau_{\mathcal{U}}$*-lsc and its non-empty sublevel-sets* $\{u \in \mathcal{U} \; ; \; \mathcal{J}(u) \leqslant C\}$ *are* $\tau_{\mathcal{U}}$*-sequentially compact*

Then, there exists a $\mathcal{J}$-minimising solution $u^{\dagger}_{\mathcal{J}}$.

Let $\mathbb{L} \overset{\text{def.}}{=} \{u \; ; \; Au = f\}$. Consider $\inf_{u \in \mathbb{L}} \mathcal{J}(u)$.

- $\mathbb{L}$ is nonempty by assumption and closed by continuity of A.
- Since $\mathcal{J} \geqslant 0$, there exists a minimising sequence u_n. By compactness of sublevel sets, there exists a subsequence u_{n_k} which $\tau_{\mathcal{U}}$ converges to u_*.
- u_* is a minimiser as $\mathcal{J}$ is $\tau_{\mathcal{U}}$-lsc: $\inf_{u \in \mathbb{L}} \mathcal{J}(u) = \liminf_k \mathcal{J}(u_{n_k}) \geqslant J(u_*)$.

Theorem 5

Under the assumptions of Theorem 4, if $\alpha = \alpha(\delta)$ is such that $\alpha(\delta) \to 0$ and $\delta^2/\alpha(\delta) \to 0$ as $\delta \to 0$, then $u_\delta \stackrel{\text{def.}}{=} u_\delta^{\alpha(\delta)}$ converges (up to a subsequence) $\tau_\mathcal{U}$ to $u^\dagger_\mathcal{J}$ as $\mathcal{J}$ minimising solution and $\mathcal{J}(u_\delta) \to \mathcal{J}(u^\dagger_\mathcal{J})$.

- Since u_δ is a minimiser:
  - $\|Au_\delta - f_\delta\|^2 + \alpha(\delta)\mathcal{J}(u_\delta) \leqslant \frac{1}{2}\left\|Au^\dagger_\mathcal{J} - f_\delta\right\|^2 + \alpha(\delta)\mathcal{J}(u^\dagger_\mathcal{J})$
  - $J(u_\delta) \leqslant \mathcal{J}(u^\dagger_\mathcal{J}) + \frac{\delta^2}{2\alpha(\delta)}$.
- by compactness of the sublevel sets of $\mathcal{J}$, up to a subsequence u_{δ_n} converges to u_* as $\delta_n \to 0$. By continuity of A, $Au_{\delta_n} \stackrel{\tau_\mathcal{V}}{\to} Au_*$.
- $Au_* = f$ follows by lsc of $\|\|_\mathcal{V}$ wrt $\tau_\mathcal{V}$ and by minimality of u_{δ_n}:

$$\frac{1}{2}\|Au_* - f\|^2 \leqslant \liminf \|Au_{\delta_n} - f_\delta\|^2 \leqslant \liminf \frac{1}{2}\|Au_{\delta_n} - f_\delta\| + \alpha(\delta_n)\mathcal{J}(u_{\delta_n})$$
$$\leqslant \liminf \frac{1}{2}\left\|Au^\dagger_\mathcal{J} - f_\delta\right\| + \alpha(\delta_n)\mathcal{J}(u^\dagger_\mathcal{J}) = 0$$

- Finally $\mathcal{J}(u_*) \leqslant \liminf_{n\to\infty} \mathcal{J}(u_{\delta_n}) \leqslant \liminf_{n\to\infty} \mathcal{J}(u^\dagger_\mathcal{J}) + \frac{\delta_n^2}{2\alpha(\delta_n)} = \mathcal{J}(u^\dagger_\mathcal{J})$.

Examples: Tikhonov

Let $\mathcal{U}$ be a Hilbert space and $\mathcal{J}(u) = \|u\|^2$.

- $\mathcal{J}$ is weakly-lsc and bounded sequences have weakly convergent subsequences. So, Theorem 4 holds with weak convergence.
- Hilbert spaces satisfy the Radon Riesz property: if u_k converge weakly to u and $\|u_k\| \to \|u\|$, then $\|u_k - u\| \to 0$. So, we have strong convergence as well as weak convergence of solutions.

Let $\mathcal{U} = \ell^2$ be the space of square summable sequences. Let $\mathcal{J}(u) = \|u\|_1$.

- $\mathcal{J}$ is weakly lsc in ℓ^2.
- We have $\|\cdot\|_2 \leqslant \|\cdot\|_1$, so $\mathcal{J}(u) \leqslant C$ implies $\|u\|_2 \leqslant C$ and bounded sequences have weakly convergent subsequences in ℓ^2. So, the sublevel-sets of $\mathcal{J}$ are weakly sequentially compact in ℓ^2.

Theorem 4 thus guarantees weak convergence in ℓ_2 of solutions.

Example: Bounded variation

Recall $\|u\|_{BV} = \|u\|_{L^1} + TV(u)$. Consider $A : L^2(\Omega) \to L^2(\Omega)$ and

$$\mathcal{J}(u) = \begin{cases} \|u\|_{BV} & u \in BV(\Omega) \\ +\infty & \text{otherwise} \end{cases}.$$

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TV is lower semi-continuous with respect to L^1 convergence (exercise) and we have Rellich's compactness theorem:

Theorem 6

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipscthiz boundary, and let $(u_n)_n \subset BV(\Omega)$ be such that $\sup_n \|u\|_{BV} < \infty$. Then there exists $u \in BV(\Omega)$ and a subsequence $(u_{n_k})_k$ such that $u_{n_k} \to u$ in $L^1(\Omega)$.

Therefore, Theorem 4 guarantees strong convergence in L^1.

What if we take $\mathcal{J}(u) = \|u\|_{TV}$ on domain Ω?

Compactness of sublevel sets is problematic as $\mathcal{J}(\alpha\chi_\Omega) = 0$ for all $\alpha \in \mathbb{R}$, but additional compactness can come from the data fidelity term:

Example: total variation

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Compactness of sublevel sets is problematic as $\mathcal{J}(\alpha\chi_\Omega) = 0$ for all $\alpha \in \mathbb{R}$, but additional compactness can come from the data fidelity term:

Theorem 3.1 (Poincaré inequality)

Let $\Omega \subset \mathbb{R}^N$. For $u \in BV(\Omega)$, let $m(u) = \frac{1}{|\Omega|}\int_\Omega u(x)\mathrm{d}x$. Then there exists $C > 0$ such that

$$\|u - m(u)\|_{L^p} \leqslant C\,\mathrm{TV}(u), \qquad \forall u \in BV(\Omega),$$

for all $p \in [1, N/(N-1)]$. This holds for $p = 2$ and $N = 2$.

Example: total variation

Suppose that $A\chi_\Omega \neq 0$ and $A : L^p \to L^p$ is a bounded linear operator for some $p \in [1, N/(N-1)]$.

Given u_n s.t. $\mathrm{TV}(u_n) + \frac{1}{2}\|Au_n - f_\delta\|_2^2 \leqslant C$, $m(u_n)$ is also uniformly bounded:

- let $w_n = m(u_n)$ and $v_n = u_n - m(u_n)$. Then, $\int v_n = 0$ and $J(v_n) = J(u_n)$. So, by the Poincaré inequality, $\|v_n\|_{L^p} \leqslant C'$ for $p \in [1, N/(N-1)]$ (NB: true for some $p \leqslant 2$).
- Observe now that $C \geqslant \|Au_n - f_\delta\|_2 \geqslant \|Au_n\|_2 - \|f_\delta\|_2$, so $\|Au_n\|_p \leqslant \Omega^{(2-p)/2}\|Au_n\|_2^p$ is uniformly bounded. Hence

$$C \geqslant \|Au_n\|_p = m(u_n)\|A\chi_\Omega\|_p - \|Av_n\|_p .$$

So, Poincaré inequality tells us that $\|u_n\|_1$ is uniformly bounded, and Rellich's compactness theorem allows us to extract a L^1 convergent subsequence.

Outline

Towards convergence rates

We have established convergence of a regularised solution u_δ to a $\mathcal{J}$-minimising solution $u^\dagger_{\mathcal{J}}$ as $\delta \to 0$. We now establish results on the *speed* of convergence.

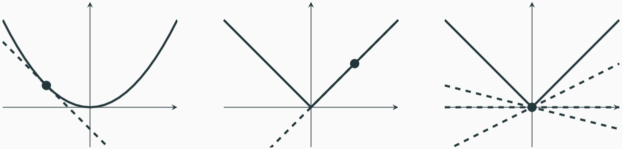
For convex functionals, we can generalise the concept of a derivative for non-differentiable functions.

Definition 7

A functional $E : \mathcal{U} \to \bar{\mathbb{R}}$ is called subdifferentiable at $u \in \mathcal{U}$ if there exists an element $p \in \mathcal{U}^*$ such that $E(v) \geqslant E(u) + \langle p, v - u\rangle$ for all $v \in \mathcal{U}$. We call p a subgradient at u. The collection of all subgradients at u

$$\partial E(u) \stackrel{\text{def.}}{=} \{p \in \mathcal{U}^* \; ; \; E(v) \geqslant E(u) + \langle p, v - u\rangle, \forall v \in \mathcal{U}\}$$

is called the subdifferential of E at u.



Let $E : \mathbb{R} \to \mathbb{R}$ be $E(u) = |u|$. Then, $\partial E(u) = \begin{cases} \text{sign}(u) & u \neq 0 \\ [-1, 1] & u = 0 \end{cases}$

The subdifferential

- If E is differentiable at u, then $\partial E(u) \stackrel{\text{def.}}{=} \{\nabla E(u)\}$.
- Let $E : \mathcal{U} \to \bar{\mathbb{R}}$ and $F : \mathcal{U} \to \bar{\mathbb{R}}$ be proper lsc convex functions and suppose that there exists $u \in \mathrm{dom}(E) \cap \mathrm{dom}(F)$ such that E is continuous at u. Then $\partial(E + F) = \partial E + \partial F$.
- Let E be convex. Then, u is a minimises E if and only if $0 \in \partial E(u)$.
- If $E : \mathcal{U} \to \bar{\mathbb{R}}$ is a proper convex function and $u \in \mathrm{dom}(E)$, then $\partial E(u)$ is a weak-* compact convex subset of $\mathcal{U}^*$.

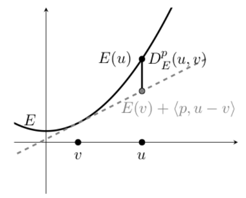
Bregman distances

Convergence rates are typically derived via analysis of the *dual problem* and studied in terms of the *Bregman distances* associated with the (convex) regularisation functional $\mathcal{J}$.

Definition 8

Given a convex functional $\mathcal{J}$, $u, v \in \mathcal{U}$ such that $\mathcal{J}(v) < \infty$ and $q \in \partial\mathcal{J}(v)$, the generalised Bregman distance is given by

$$\mathcal{D}^q_{\mathcal{J}}(u, v) = \mathcal{J}(u) - \mathcal{J}(v) - \langle q,\, u - v\rangle. \tag{4.1}$$



Example: For $\mathcal{J}(u) = \frac{1}{2}\|u\|^2$, the subgradient at v is $q = v$, so

$$\mathcal{D}^v_{\mathcal{J}}(u, v) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\|v\|^2 - \langle v,\, u - v\rangle = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - \langle v,\, u\rangle = \frac{1}{2}\|u - v\|^2.$$

We say that a $\mathcal{J}$-minimising solution $u^\dagger_{\mathcal{J}}$ satisfies the **source condition** if there exists $p^\dagger \in \mathcal{V}$ such that $A^* p^\dagger \in \partial \mathcal{J}(u^\dagger)$.

Theorem 4.1

Assume that the source condition is satisfied at a $\mathcal{J}$-minimising solution $u^\dagger_{\mathcal{J}}$ and u_δ be a regularised solution. Then, letting $v = A^* p^\dagger$,

$$D^v_{\mathcal{J}}(u_\delta, u^\dagger_{\mathcal{J}}) \leqslant \frac{1}{2\alpha} \left(\delta + \alpha \left\| p^\dagger \right\| \right)^2 .$$

Proof.

Since u_δ is a minimiser,
$\alpha\mathcal{J}(u_\delta) + \frac{1}{2}\left\|Au_\delta - f_\delta\right\|^2 \leqslant \alpha\mathcal{J}(u^\dagger_{\mathcal{J}}) + \frac{1}{2}\left\|Au^\dagger_{\mathcal{J}} - f_\delta\right\|^2.$

- $\alpha D^v_{\mathcal{J}}(u_\delta, u^\dagger_{\mathcal{J}}) + \frac{1}{2}\left\|Au_\delta - f_\delta\right\|^2 + \alpha\langle A^* p^\dagger,\, u_\delta - u^\dagger_{\mathcal{J}}\rangle \leqslant \frac{\delta^2}{2}.$
- LHS is equal to
$$\frac{1}{2}\left\|Au_\delta - f_\delta + \alpha p^\dagger\right\|^2 + \alpha D^v_{\mathcal{J}}(u_\delta, u^\dagger_{\mathcal{J}}) - \frac{\alpha^2}{2}\left\|p^\dagger\right\|^2 + \alpha\langle p^\dagger,\, f_\delta - f_\dagger\rangle.$$
- Rearranging and by Cauchy-Schwarz:
$$D^v_{\mathcal{J}}(u_\delta, u^\dagger_{\mathcal{J}}) \leqslant \frac{1}{2\alpha}\left(\delta^2 + \alpha^2\left\|p^\dagger\right\|^2 + 2\alpha\left\|p^\dagger\right\|\delta\right).$$

□

Outline

Let us return to

$$\min_{u \in L^2(\Omega)} TV(u) + \frac{1}{2} \|Au - f\|_{L^2}^2$$

When $A = \mathrm{Id}$, this is known as the ROF model (after Rudin, Osher and Fatemi who first introduced the total variation functional for image processing).

- Recall that $TV(u) = \int_\Omega |\nabla u|$ is well defined on $W^{1,1}(\Omega)$.
- For $u \in W^{1,1}([a,b])$, define continuous function $\tilde{u}(x) - \tilde{u}(a) = \int_a^x u'(t)\mathrm{d}t$ which coincides with u a.e.. So, functions in $W^{1,1}([a,b])$ cannot have discontinuities, and given $f \in W^{1,1}([a,b]^2)$, since $f(\cdot, x) \in W^{1,1}([a,b])$ for a.e. x, images cannot have jumps across vertical/horizontal boundaries.

Key point: is that J is well defined for a more general class of functions which can have discontinuities.

Motivating example

We shall see in this example that not only can $\int |\nabla u|$ be extended to a larger class of functions where edges are permitted, it is actually necessary to do so.

Consider

$$\min_{u \in W^{1,1}([0,1])} \mathcal{E}(u), \qquad \mathcal{E}(u) = \lambda \int_0^1 \left|u'(t)\right| \mathrm{d}t + \int_0^1 |u(t) - g(t)|^2 \, \mathrm{d}t,$$

where $g = \chi_{(1/2,1]}$.

We will show that this minimization problem does not have a solution in $W^{1,1}$.

Motivating example

Let u be a minimizer.

- **Maximum/minimum principles** $u \leqslant 1$ a.e.:
  Let $v \in \min\{u, 1\}$. Then,
  - $v' = u'$ on $\{u < 1\}$ and $v' = 0$ on $\{u \geqslant 1\}$. Therefore, $\int |v'| \leqslant \int |u'|$.
  - Since $g \leqslant 1$, $\|v - g\|^2 \leqslant \|u - g\|^2$.

  So, $\mathcal{E}(v) \leqslant \mathcal{E}(u)$ and this inequality is strict if $v \neq u$. Similarly, $u \geqslant 0$ a.e..
- **'Symmetry'** Note that $g(t) = 1 - g(1-t)$. Let $\tilde{u} = 1 - u(1-t)$. Then $\|\tilde{u} - g\|^2 = \|u - g\|^2$ and $\|\tilde{u}'\|_1 = \|u'\|_1$. So, $\mathcal{E}(\tilde{u}) = \mathcal{E}(u)$.
  Also,
  $$\mathcal{E}\left(\frac{\tilde{u} + u}{2}\right) \leqslant \frac{1}{2}\mathcal{E}(\tilde{u}) + \frac{1}{2}\mathcal{E}(u) = \mathcal{E}(u)$$
  and by strict convexity of $\|\cdot\|_2^2$, this inequality is strict if $\tilde{u} \neq u$.
- Let $m = \min u = u(a)$ and let $M = \max u = u(b)$. From the previous observation, $M = 1 - m$. Then, (assume $b > a$, case $a \geqslant b$ is similar)
  $$\left\|u'\right\|_1 \geqslant \int_a^b \left|u'(t)\right| \mathrm{d}t \geqslant \int_a^b u'(t) = M - m = 1 - 2m.$$
  Also, since $m \leqslant 1 - m$, we must have $m \in [0, 1/2]$.

To summarize, we have shown that $u \in [m, 1-m]$ for some $m \in [0, 1/2]$, $u(1-t) = 1 - u(t)$, and

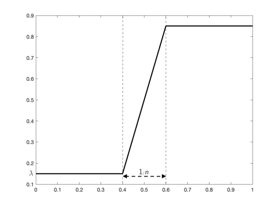
$$\mathcal{E}(u) \geqslant \lambda(M - m) + \int_0^{1/2} m^2 + \int_{1/2}^1 (1-M)^2 = \lambda(1-2m) + m^2.$$

The RHS is minimal when $m = \lambda$ if $\lambda \leqslant 1/2$ and $m = 1/2$ if $\lambda \geqslant 1/2$. In the latter case, we see that $u \equiv 1/2$ achieves the minimum and is the unique minimizer.

Motivating example

Assume now that $\lambda < 1/2$. Then for any minimizer u, $\mathcal{E}(u) \geqslant \lambda(1-\lambda)$. Let us construct a minimizing sequence: For $n \geqslant 2$, define

$$u_n(t) = \begin{cases} \lambda & t \leqslant 1/2 - 1/n, \\ \frac{1}{2} + n(t-1/2)(1/2-\lambda) & |t - 1/2| \leqslant 1/n, \\ 1-\lambda & t \geqslant 1/2 + 1/n. \end{cases}$$



- $\int_0^1 |u_n'| = \int_0^1 u_n' = 1 - 2\lambda$.
- $\mathcal{E}(u_n) \leqslant \lambda(1-2\lambda) + \left(1 - \frac{2}{n}\right)^2 \lambda^2 + \frac{2}{n} \to \lambda(1-\lambda)$ as $n \to \infty$. So $\inf_u \mathcal{E}(u) = \lambda(1-\lambda)$.

The L^1 limit of u_n is $u = \lambda\chi_{[0,1/2)} + (1-\lambda)\chi_{[1/2,1]}$, which is not in $W^{1,1}$. Note also that since $\int |u_n'| = 1 - 2\lambda$ for all n, it is natural to assume that $\int |u'|$ makes sense.

Total variation regularisation

A natural extension of the functional F is to define for $u \in L^1$:

$$F(u) = \inf \left\{ \lim_{n\to\infty} \int_0^1 \left| u_n'(t) \right| \mathrm{d}t \; ; \; u_n \to u \text{ in } L^1, \quad \lim_{n\to\infty} \int_0^1 \left| u_n' \right| < \infty \right\}.$$

This definition is consistent with the more standard definition of total variation thanks to the following result

Theorem 5.1

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary, let $u \in BV(\Omega)$. Then, there exists a sequence (u_n) of functions in $C^\infty(\Omega) \cap W^{1,1}(\Omega)$, such that

1. $u_n \to u$ *in* L^1.
2. $J(u_n) = \int_\Omega |\nabla u| \to J(u) = \int_\Omega |Du|$.

Distributional interpretation of TV

Given $u \in L^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$, define $T_u : \mathcal{D}(\Omega) \stackrel{\text{def.}}{=} \mathcal{C}_c^\infty(\Omega) \to \mathbb{R}$ by

$$T_u(\varphi) = \int \varphi(x) u(x) \mathrm{d}x$$

This is a continuous linear form on $\mathcal{D}(\Omega)$, aka a distribution. Write $T_u \in \mathcal{D}(\Omega)'$.

The derivative of T_u are defined to be, for $i = 1, \ldots, n$

$$\partial_i T_u(\varphi) = \int \partial_i \varphi(x) u(x) \mathrm{d}x.$$

Denote $Du = (\partial_i T_u)_{i=1}^n$.

If $TV(u) < \infty$, then $\langle Du, \varphi \rangle \leqslant TV(u) \|\varphi\|_\infty$, so Du is a continuous linear form on the space of continuous vector fields and by Riesz' representation theorem, it defines a Radon measure on Ω, and $|Du|(\Omega) = J(u)$.

Subdifferential of the total variation functional

Recall that for each $u \in L^1(\Omega)$, $J(u) = \sup_{p \in \mathcal{K}} \int_\Omega u(x)p(x)\mathrm{d}x$ where

$$\mathcal{K} = \left\{ -\mathrm{div}\varphi \; ; \; \varphi \in C_c^\infty(\Omega; \mathbb{R}^N), \|\varphi\|_\infty \leqslant 1 \right\}.$$

Taking the closure of $\mathcal{K}$ in L^2 gives

$$K = \left\{ -\mathrm{div}z \; ; \; z \in L^\infty(\Omega, \mathbb{R}^N), \|z\|_\infty \leqslant 1, -\mathrm{div}z \in L^2(\Omega), z \cdot \eta_\Omega = 0 \right\}.$$

and since K is the largest set for which $J(u) = \sup_{p \in K} \int_\Omega u(x)p(x)\mathrm{d}x$. we also have

$$K = \left\{ p \in L^2(\Omega) \; ; \; \int p(x)u(x)\mathrm{d}x \leqslant J(u), \; \forall u \in L^2(\Omega) \right\}.$$

In the definition of K, $-\mathrm{div}z \in L^2(\Omega)$ means that there exists $\gamma \in L^2(\Omega)$ such that

$$\int_\Omega \gamma u = \int z \cdot \nabla u, \qquad \forall u \in C_c^\infty(\Omega).$$

Theorem 9

We have $\partial J(u) = \{p \in K \; ; \; \langle p, u \rangle = J(u)\}$

Proof.

If $p \in K$ and $\langle p, u \rangle = J(u)$, then for all $v \in L^2$,

$$J(v) \geqslant \langle v, p \rangle = J(u) + \langle v - u, p \rangle.$$

For the converse, if $p \in \partial J(u)$, the for all $t > 0$ and all $v \in L^2$,

$$tJ(v) = J(tv) \geqslant J(u) + \langle tv - u, p \rangle$$

Letting $t \to 0$ yields $J(u) \leqslant \langle p, u \rangle$.

Dividing by t and letting $t \to +\infty$ yields $J(v) \geqslant \langle v, p \rangle$. $\square$

Note that $K = \partial J(0)$ and $p \in \partial J(u)$ means

- $p = -\mathrm{div}(z)$ where $\|z\|_\infty \leqslant 1$ and $-\int \mathrm{div}(z) u = \int |Du| = J(u)$.

We can think of $z \cdot Du = |Du|$ so z is the vector field which is normal to the level lines of u.

TV denoising

Let's consider the ROF model

$$\min_{u \in L^2} \alpha TV(u) + \frac{1}{2} \|u - f\|_{L^2}^2 .$$

Here, $A = \mathrm{Id}$ and the source condition asks that $\partial TV(u) \neq \emptyset$.

Source condition example 1

Let $C \subset \Omega$ have C^∞ boundary and consider $f = 1_C$. Then

$$TV(1_C) = \mathrm{Per}(C) = \int_{\partial C} 1 = \int_{\partial C} \langle \eta_{\partial C}, \eta_{\partial C} \rangle.$$

Since $\eta_{\partial C} \in C^\infty(\partial C, \mathbb{R}^2)$ and $\|\eta_{\partial C}(x)\|_2 = 1$, we can extend to $\psi \in C_0^\infty(\Omega; \mathbb{R}^2)$ with $\sup_x \|\psi(x)\|_2 \leqslant 1$. Therefore, by the divergence theorem

$$TV(1_C) = \int_{\partial C} \langle \psi, \eta_{\partial C} \rangle = \int_C \mathrm{div}(\psi) = \langle \mathrm{div}(\psi), 1_C \rangle$$

and $\mathrm{div}(\psi) \in \partial TV(0)$. So, the source condition is satisfied.

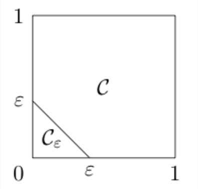
Source condition example 2

Suppose now that $C = [0,1]^2$ and suppose that $p_0 \in \partial TV(1_C) \subset L^2(\Omega)$. Then,

$$\langle p,\, 1_C\rangle = TV(1_C) = \mathrm{Per}(C) = 4.$$

Since $TV(u) \geqslant \langle p_0,\, u\rangle$ for all u,

$$TV(1_{C\setminus C_\varepsilon}) \geqslant \langle p_0,\, 1_{C\setminus C_\varepsilon}\rangle = \langle p_0,\, 1_C\rangle - \langle p_0,\, 1_{C_\varepsilon}\rangle$$



Suppose now that $C = [0,1]^2$ and suppose that $p_0 \in \partial TV(1_C) \subset L^2(\Omega)$. Then,

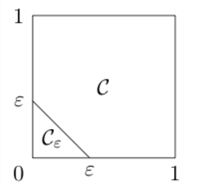
$$\langle p,\, 1_C \rangle = TV(1_C) = \mathrm{Per}(C) = 4.$$

Since $TV(u) \geqslant \langle p_0,\, u \rangle$ for all u,

$$4 - 2\varepsilon + \sqrt{2}\varepsilon = TV(1_{C \setminus C_\varepsilon}) \geqslant \langle p_0,\, 1_{C\setminus C_\varepsilon} \rangle = \langle p_0,\, 1_C \rangle - \langle p_0,\, 1_{C_\varepsilon} \rangle = 4 - \langle p_0,\, 1_{C_\varepsilon} \rangle$$

$$\frac{\varepsilon}{\sqrt{2}} \sqrt{\int_{C_\varepsilon} |p_0|^2} \geqslant |\langle p_0,\, 1_{C_\varepsilon} \rangle| \geqslant (2 - \sqrt{2})\varepsilon \implies \sqrt{\int_{C_\varepsilon} |p_0|^2} \geqslant \sqrt{2}(2 - \sqrt{2}).$$

Contradiction since $p_0 \in L^2$. Therefore $\partial J(1_C) = \emptyset$.



Consider

$$\min_{u \in L^2(\Omega)} \alpha TV(u) + \frac{1}{2} \|Au - f\|_{L^2}.$$

Under the source condition with $v = A^* p = \mathrm{div}(z)$, we have a bound not only on the Bregman divergence $d := J(u) - J(u_0) - \langle v, u - u_0 \rangle$, but also on the total variation outside the saturation points of z.

Theorem 5.2

Assume that $v = A^* p \in \partial TV(u^\dagger)$ *and* $f = Au^\dagger$*. Let* $v = -\mathrm{div} z$ *with* $\|z\|_\infty \leqslant 1$ *and*

$$U_r \overset{\text{def.}}{=} \{x \in \Omega\,;\ |z(x)| < r\}.$$

For each $r \in (0,1)$,

$$(1 - r) \int_{U_r} |Du| \leqslant \frac{\delta^2}{2\lambda} + \frac{\lambda \|p\|_{L^2}^2}{2} + \delta \|p\|_{L^2}.$$

Proof.

$$
\begin{aligned}
d &:= J(u) - J(u_0) - \langle v,\, u - u_0 \rangle \\
&= J(u) - J(u_0) + \langle \mathrm{div} z,\, u \rangle - \langle \mathrm{div} z,\, u_0 \rangle \\
&= J(u) + \langle \mathrm{div} z,\, u \rangle \qquad\qquad \text{since } J(u_0) = \langle -\mathrm{div} z,\, u_0 \rangle \\
&= J(u) - \int (z, Du) = J(u) - \int_{\Omega \setminus U_r} (z, Du) - \int_{U_r} (z, Du) \\
&\geqslant J(u) - \int_{\Omega \setminus U_r} |Du| - r \int_{U_r} |Du| \geqslant (1 - r) \int_{U_r} |Du| \,.
\end{aligned}
$$

The conclusion now follows by applying the upper bound on d. $\square$

Example

Let us consider the case of denoising. Let $B_R \subset \mathbb{R}^2$ be the ball of radius R with origin 0 and let $u^\dagger = \chi_{B_R}$. Then let $p = -\mathrm{div}(z)$ where z is defined by

$$z(x) = \frac{q\left(\Big|\,|x| - R\Big|\right)}{|x|}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad q(s) = \max\{1 - s/\varepsilon, 0\}.$$

(In polar coordinates (r, θ), we can write $z(r, \theta) = q(|r - R|)\binom{\cos(\theta)}{\sin(\theta)})$. One can show that $\|p\| = \mathcal{O}(\varepsilon^{-1/2})$. Then, by choosing $U = \{x \in \Omega\,;\, \mathrm{dist}(x, \partial B_R) \geqslant \varepsilon\}$, the minimizer u satisfies

$$\int_U |Du| \leqslant \mathcal{O}\left(\frac{\delta^2}{\lambda} + \frac{\lambda}{\varepsilon} + \frac{\delta}{\varepsilon}\right) = \mathcal{O}\left(\frac{\delta}{\sqrt{\varepsilon}}\right)$$

provided that $\lambda = \delta\sqrt{\varepsilon}$.

Therefore, most of the total variation of u is concentrated around ∂B_R and this points to the ability of TV regularization in dampening oscillations away from the true edge ∂B_R.

Outline

The dual perspective

We have so far considered

$$\min_u \mathcal{J}(u) + \frac{1}{2} \|Au - f\|^2$$

When $\mathcal{J}$ is a convex functional, it is often convenient (both from a theoretical and practical perspective) to consider the dual formulation.

The convex conjugate

Let V be a real topological vector space and let V^* be its dual.

Definition 10

Given $F : V \to (-\infty, +\infty]$, its convex conjugate is $F^* : V^* \to (-\infty, +\infty]$ defined by

$$F^*(y) \stackrel{\text{def.}}{=} \sup_{x \in V} \{\langle x, y \rangle - F(x)\}.$$

- F^* is convex regardless of whether F is convex.
- We have the Fenchel Young inequality: $\langle x, y \rangle \leqslant F(x) + F^*(y)$,
- if F is convex and lower semi-continuous, then $F^{**} = F$.
- if F is convex, then $y \in \partial F(x)$ if and only if $F(x) + F^*(y) = \langle x, y \rangle$.

(a) if $F(x) = \frac{1}{2}\|x\|^2$ and V is a Hilbert space, then $F^*(y) = \frac{1}{2}\|y\|^2$:
  - $F^*(y) = \sup_x \langle x, y\rangle - \frac{1}{2}\|x\|^2 \leqslant \frac{1}{2}\|y\|^2$.
  - Setting $x \overset{\text{def.}}{=} y$ in the supremum above yields $F^*(y) \geqslant \frac{1}{2}\|y\|^2$.

(b) If $F(x) = \|x\|$ and $\|\cdot\|_*$ is its dual norm, then

$$F^*(y) = \begin{cases} 0 & \|y\|_* \leqslant 1 \\ +\infty & \text{otherwise.} \end{cases}$$

(c) If $F = \iota_K$ (takes value 0 for $x \in K$ and $+\infty$ otherwise) with K being a convex set, then $F^*(y) = \sup_{x\in K}\langle x, y\rangle$.

Absolutely one-homogeneous functionals

A functional $E : \mathcal{U} \to \bar{\mathbb{R}}$ is absolutely one-homogeneous if $E(\lambda u) = |\lambda| \, E(u)$ for all $\lambda \in \mathbb{R}$ and $u \in \mathcal{U}$. Clearly $E(0) = 0$.

Examples: $\|\cdot\|_p$, the total variation functional.

- Let E be convex, absolutely one-homogeneous and let $p \in \partial E(u)$. Then $E(u) = \langle p, u \rangle$.
- Let E be proper, convex, lsc, absolutely one-homogeneous. Then, E^* is the characteristic function of the convex set $\partial E(0)$.
- for any $u \in \mathcal{U}$, $p \in \partial E(u)$ if and only if $p \in \partial E(0)$ and $E(u) = \langle p, u \rangle$.

Primal and dual formulations

Let V, Y be real topological vector spaces with duals V^* and Y^*. Let $y \in Y$ and $b_j \in \mathbb{R}$ for $j = 1, \ldots, M$. Consider **the primal problem**:

$$\min_{x \in V} F_0(x) \text{ subject to } Ax = y, \tag{6.1}$$

$$F_j(x) \leqslant b_j,\ j \in [M], \tag{6.2}$$

where $F_0 : V \to (-\infty, +\infty]$ is called the objective function and $F_j : V \to (-\infty, +\infty]$ for $j \in [M]$ are called the constraint functions. $A : V \to Y$ is a continuous linear functional. The set $K \overset{\text{def.}}{=} \{x \in V \ ; \ Ax = y, F_j(x) \leqslant b_j\}$ is called the admissible set.

The **Lagrange function** is defined for $x \in V$, $\xi \in Y^*$ and $\nu \in \mathbb{R}^M$ with $\nu_\ell \geqslant 0$ for all $\ell \in [M]$ by

$$L(x, \xi, \nu) \overset{\text{def.}}{=} F_0(x) + \langle \xi,\, Ax - y \rangle + \sum_{\ell=1}^{M} \nu \left(F_\ell(x) - b_\ell \right).$$

The variables ξ and ν are called the **Lagrange multipliers**.

The Lagrange dual function is defined as

$$H(\xi, \nu) \overset{\text{def.}}{=} \inf_{x \in V} L(x, \xi, \nu), \qquad \xi \in Y^*,\ \nu \in \mathbb{R}^M_{\geqslant 0}.$$

If $x \mapsto L(x, \xi, \nu)$ is unbounded from below, then we write $H(\xi, \nu) = -\infty$.

Primal and dual formulations

Properties of the dual function H:

- The dual function is always concave since it is the pointwise infimum of a family of affine functions.
- We have $H(\xi, \nu) \leqslant \inf_{x \in K} F_0(x)$ for all $\xi \in Y^*$ and $\nu \in \mathbb{R}^M_{\geqslant 0}$. Indeed, we have $H(\xi, \nu) \leqslant \inf_{x \in K} L(x, \xi, \nu)$, and note that given any $x \in K$, we have $Ax - y = 0$ and $F_\ell(x) - b_\ell \leqslant 0$, so $L(x, \xi, \nu) \leqslant F_0(x)$.

So, $H(\xi, \nu)$ serves as a lower bound for the infimum of F_0 over K, and since we want this lower bound to be as tight as possible, it makes sense to consider

$$\sup_{\xi \in Y^*, \nu \in \mathbb{R}^M} H(\xi, \nu) \text{ subject to } \nu_\ell \geqslant 0,\ \ell \in [M]. \tag{6.3}$$

This optimisation problem is called the **dual problem** and (6.1) is called the **primal problem**.

Primal and dual formulations

- If D^* is the supremum of (6.3) and P^* is the infimum of (6.1), then we have in general $D^* \leqslant P^*$ (this is called **weak duality**). and $P^* - D^*$ is called the duality gap.
- When $D^* = P^*$, then we say we have **strong duality**.

Primal and dual formulations

Consider now $\inf_{x \in V} E(Ax) + F(x)$, where $E : Y \to (-\infty, +\infty]$ and $F : V \to (-\infty, +\infty]$ are convex functionals, and $A : V \to Y$ is a continuous linear operator. This is equivalent to

$$\inf_{z \in Y, x \in V} E(z) + F(x) \text{ subj. to } Ax = z$$

the Lagrange dual is for $\xi \in Y^*$ as

$$\begin{aligned} H(\xi) &= \inf_{x,z} \{E(z) + F(x) + \langle \xi, Ax - z \rangle\} \\ &= \inf_{x,z} \{E(z) + F(x) + \langle A^* \xi, x \rangle - \langle \xi, z \rangle\} \\ &= -\sup_{z \in Y} \langle \xi, z \rangle - E(z) - \sup_{x \in V} \langle -A^* \xi, x \rangle - F(x) \\ &= -E^*(\xi) - F^*(-A^* \xi). \end{aligned}$$

So, the dual problem is

$$\sup_{\xi \in Y^*} -E^*(\xi) - F^*(-A^* \xi)$$

Theorem 6.1 (Strong duality)

Suppose that E and F are proper convex functionals, there exists $u_0 \in V$ such that $F(u_0) < \infty$, $E(Au_0) < \infty$ and E is continuous at Au_0. Then, strong duality holds and there exists at least one dual optimal solution. Moreover, if p^* is a primal optimal solution and d^* is a dual optimal solution, then

$$Ap^* \in \partial E^*(d^*) \quad \text{and} \quad A^* d^* \in -\partial F(p^*)$$

Primal and dual formulations

We are interested in the case

$$\min_u \frac{1}{2} \|Au - f_\delta\|^2 + \alpha\mathcal{J}(u)$$

So, $E(Au) = \frac{1}{2}\|Au - f_\delta\|^2$ and $F(u) = \alpha\mathcal{J}(u)$.

- $E^*(v) = -\langle v,\, f_\delta\rangle + \frac{1}{2}\|v\|^2$.
- If $\mathcal{J}$ is absolute one-homogeneous, then $\mathcal{J}^*(v) = \iota_K$ where $K = \partial J(0)$, and $(\alpha J)^*(v) = \alpha J^*(\alpha^{-1}v)$.

Therefore, the dual problem is

$$\sup_v \langle v,\, f_\delta\rangle - \frac{1}{2}\|v\|^2 + \iota_K\left(\frac{A^* v}{\alpha}\right) = \sup_{v : A^* v \in \partial\mathcal{J}(0)} \alpha\left(\langle v,\, f_\delta\rangle - \frac{\alpha}{2}\|\alpha v\|^2\right). \quad (6.4)$$

If p_δ and u_δ are dual and primal solutions, then the optimality conditions take the form

$$A^* p_\delta \in \partial\mathcal{J}(u_\delta) \quad \text{and} \quad p = \frac{f_\delta - Au_\delta}{\alpha}$$

NB: the dual solution is unique since it is the projection onto a closed convex set.

Formal limits problems as $\delta \to 0$ are

$$\inf_{u:Au=f} \mathcal{J}(u) = \inf_{u\in\mathcal{U}} \iota_{\{f\}}(Au) + \mathcal{J}(u) \tag{6.5}$$

and

$$\sup_{p:A^* p\in\partial\mathcal{J}(0)} \langle f,\, p\rangle = -\inf_{p} \langle -f,\, p\rangle + \iota_{\partial J(0)}(A^* p) \tag{6.6}$$

Lemma 11

For $J : \mathcal{U} \to [0, \infty]$ absolute one-homogeneous and coercive, we have $0 \in \mathrm{int}(\partial J(0))$.

Proof.

Indeed, if not, then there exists e_n and u_n with $\|e_n\| \to 0$ such that $J(u_n) < \langle e_n,\, u_n\rangle$. Since J is one-homogeneous, we can assume that $\|u_n\| = 1$. Therefore, $\lim_{n\to\infty} J(u_n) \leqslant \lim_{n\to\infty} \|e_n\| \, \|u_n\| = 0$. Letting $\lambda_n = 1/J(u_n)$, we have $\|\lambda_n u_n\| \to +\infty$ but $J(\lambda_n u_n) = 1$. Contradiction since J is coercive.

□

Primal-Dual relations

- We can apply Theorem 6.1 to (6.6) with $F = \iota_{\partial J(0)}$ and $E = \langle -f, \cdot \rangle$. Cleary, $E(0) = 0$, $F(A^*0) = 0$, and F is continuous at 0. In this case, we have strong duality and (6.5) has at least one solution.
- However, unlike the case where $\alpha > 0$, there is no guarantee that a dual solution to (6.6) exists, and it may not be unique if it does exist.
- If a dual solution p exists, then it is related to any primal solution u by $A^*p \in \partial J(u)$.

What is the behaviour of p_δ as $\delta \to 0$?

The source condition implies dual convergence

Theorem 6.2

Suppose that the source condition holds at a $\mathcal{J}$-minimising solution $u^\dagger_{\mathcal{J}}$. Then, p_α the solution to (6.4) *with data f is uniformly bounded in α. Moreover, $p_\alpha \to p^\dagger$ strongly in $\mathcal{V}$ as $\alpha \to 0$, where $p^\dagger$ is a solution to* (6.6) *with smallest norm.*

Proof.

- Let p_α be a solution to to (6.4) with $f_\delta = f$, we have

$$\langle f, p_\alpha \rangle - \frac{\alpha}{2} \|p_\alpha\|^2 \geqslant \langle f, p^\dagger \rangle - \frac{\alpha}{2} \left\|p^\dagger\right\|^2, \tag{6.7}$$

  and $p^\dagger$ being a solution to (6.6) implies that $\langle f, p^\dagger \rangle \geqslant \langle f, p_\delta \rangle$. So, $\left\|p^\dagger\right\| \geqslant \|p_\alpha\|$.
- We may extract a subsequence such that $p_{\alpha_{n_k}}$ weakly converges to p_* (recall that the closed unit ball of a Hilbert space is weakly sequentially compact). Taking the limit of $\lambda \to 0$ in (6.7) yields $\langle f, p_* \rangle \geqslant \langle y, p^\dagger \rangle$.
- Note that $A^* p_{\alpha_{n_k}}$ converges weakly to $A^* p_*$, and so $A^* p_* \in \partial \mathcal{J}(0)$ (since this is a weakly closed set). So, p_* is a solution to (6.6).

Proof.

- Finally, p_* is the solution of minimal norm since

$$\|p_*\| \leqslant \liminf_k \left\|p_{\alpha_{n_k}}\right\| \leqslant \left\|p^\dagger\right\|,$$

and hence, $p_* = p^\dagger$, $\left\|p_{\alpha_{n_k}}\right\| \to \left\|p^\dagger\right\|$ and $p_{\alpha_{n_k}} \to p_0$ strongly in $\mathcal{H}$. This implies $\lim_{\delta\to 0}\left\|p_\alpha - p^\dagger\right\| = 0$, since otherwise, we can extract a subsequence p_{α_k} such that $\left\|p_{\alpha_k} - p^\dagger\right\| > \varepsilon$ and by the above argument, extract a further subsequence which converges strongly to $p^\dagger$.

$\square$

Note: since the solution to (6.4) with f_δ is $P_K(f_\delta/\alpha)$ the orthogonal projection onto $\{p\ ;\ A^*p \in \partial\mathcal{J}(0)\}$, we have

$$\|p_\alpha - p_\delta\| = \|P_K(f/\alpha) - P_K(f_\delta/\alpha)\| \leqslant \delta/\alpha \leqslant C.$$

So, $\|p_\delta\|$ is also uniformly bounded in δ and converges to $p^\dagger$ as $\delta/\alpha(\delta) \to 0$.

The minimal norm certificate

The dual solutions $p_{\alpha,\delta}$ converge to the minimal norm dual solution $p^\dagger$ as $\alpha, \delta \to 0$ (with $\delta/\alpha \leqslant c$). This often means that $A^* p^\dagger$ control the structural properties of $u_{\alpha,\delta}$ for small α and δ.

Example Let $\mathcal{J} = \|\cdot\|_1$ in $\mathbb{R}^n$. Suppose that $A^* p^\dagger \in \partial J(u^\dagger)$ satisfies $\left\|(A^* p^\dagger)_{S^c}\right\|_\infty < 1$ for $S \overset{\text{def.}}{=} \mathrm{Supp}(u)$. Then $\mathrm{Supp}(u_{\alpha,\delta}) = S$.

- $A^* p \in \partial J(u)$ means that $\|A^* p\|_\infty \leqslant 1$ and $(A^* p)_S = \mathrm{sign}(u_S)$.
- If $\left\|(A^* p^\dagger)_{S^c}\right\|_\infty < 1$, then $\|(A^* p_{\alpha,\delta})_{S^c}\|_\infty < 1$ for all α, δ sufficiently small. This means $\mathrm{Supp}(u_{\alpha,\delta}) \subseteq S$.
- Since we have convergence of $u_{\alpha,\delta}$ to u, we actually have $\mathrm{Supp}(u_{\alpha,\delta}) = S$.

Similar notions of structural stability (stability of level curves) for $\mathcal{J} = TV$.

Summary

We studied variational regularisers of the form

$$R_\alpha(f) = \mathrm{argmin}_u\, \alpha \mathcal{J}(u) + \frac{1}{2} \|Au - f\|^2 .$$

which is a natural generalisation of Tikhonov regularisation.

- This is a convergent regularisation under appropriate continuity properties of A, $\mathcal{J}$ is proper, lsc with compact sublevel sets and $\delta^2/\alpha(\delta) \to 0$.
- We introduced a source condition for studying convergence rates:
  - this gives convergence rates in terms of Bregman distances under a source condition.
  - For convex regularisers, we saw how to reformulate using the dual problem. The source condition is simply saying that the limit dual problem ($\alpha \to 0$) has a solution.
  - The source condition guarantees dual convergence, and this can provide finer notions of convergence.