Sequences and Functions

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Practical things

Lectures

- Odd weeks: 2 lectures. Thursday lecture on Zoom. Friday lecture in ~~CB 5.6~~ **Wolfson** 4W 1.7.
- Even weeks: 1 lecture.Thursday lecture on Zoom.

Tutorials:

- Even weeks on Tuesdays. Your tutor will arrange homework submissions in the first tutorial in week 2.

Assessment

- 2 hour exam in May covering material from both semesters.

Analysis

Analysis is a field of mathematics dealing with limits, including derivatives and integrals.

In this unit, we make a first step in the study of this subject by developing tools to analyse the behaviour of sequences.

A sequence is a map from $\mathbb{N}$ to $\mathbb{R}$. Here are some examples:

- $a_n = 1/n$: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
- $a_n = (-1)^n n$: $-1, 2, -3, 4, -5, \ldots$
- $a_n = \sum_{j=1}^{n} \frac{1}{j}$: $1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \ldots$
- $a_n = \sum_{j=1}^{n} \frac{1}{j^2}$: $1, 1 + \frac{1}{4}, 1 + \frac{1}{4} + \frac{1}{9}, \ldots$

The need for approximation

Our world is continuous but computers can only handle finite objects.

Basic example: we like to deal with π and $\sqrt{2}$, but these are irrational numbers, requiring infinitely many digits to store them.

Problems in continuous mathematics:

- Evaluate $\int_0^1 f(x)\mathrm{d}x$
- Find x such that $f(x) = 0$.
- Find a vector x such that $\sum_{j=1}^{n} a_{ij} x_j = b_i$ for all $i = 1, \ldots, N$.

Often, we cannot solve these problems exactly and to solve them on a computer, we need to approximate them.

Study of sequences is a vital part of algorithm design!

Most algorithms in use today produce a sequence which will converge to the desired solution.

Analysis is needed to understand and develop algorithms.

Example: Newton's method

Find x such that $f(x) = 0$.

Newton's method

Given initial guess a_0 define

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} = a_n - \frac{a_n^2 - 2}{2a_n}$$

Example: $f(x) = x^2 - 2$. The solution is $x = \sqrt{2}$. Starting at $a_0 = 2$, the sequence is (to 4 d.p.)

$$2, 1.5, 1.4167, 1.4142, 1.4142, \ldots$$

When does this method succeed at solving $f(x) = 0$?

How quickly does the sequence converge to the solution?

Example: Evaluating integrals

We know how to evaluate simple integrals such as

$$\int_0^1 e^x \mathrm{d}x \quad \text{or} \quad \int_0^\pi \cos(x)\mathrm{d}x.$$

But what about

$$\int_0^1 e^{x^2} \mathrm{d}x \quad \text{or} \quad \int_1^{2000} \exp\left(\sin\left(\cos\left(\sinh\left(\cosh\left(\tan^{-1}\left(\log(x)\right)\right)\right)\right)\right)\right) \mathrm{d}x?$$

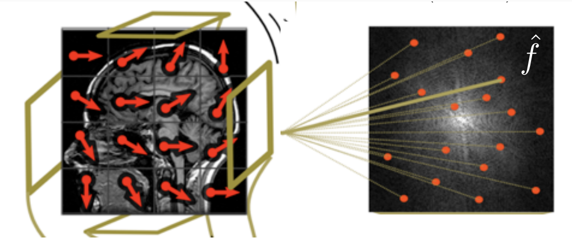
One would typically construct I_N from samples $f(x_1), \ldots, f(x_N)$ for some $N \in \mathbb{N}$ with I_N converging to $\int_a^b f(x)\mathrm{d}x$ as $N \to \infty$.

Solving linear systems

In many practical problems from medical imaging to weather forecasting, we need to solve linear systems

$$\text{Find} \quad x \in \mathbb{R}^n \quad \text{such that} \quad Ax = \hat{f}.$$

But these systems are often too large to fit into computer memory. So, we need to construct a sequence x_n which converges to the solution x.

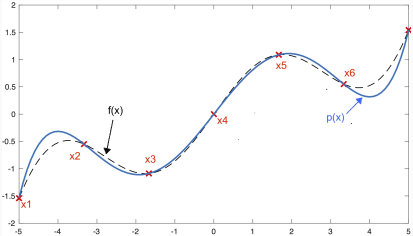


Example 3: Interpolation

Suppose we want to approximate some function f from given N data points x_i and $y_i := f(x_i)$ for $i = 1, \ldots, N$.

Compute a function p_N such that

$$p_N(x_i) = y_i \quad i = 1, \ldots, N.$$



This problem is called interpolation, and once p_N is constructed, you can then evaluate this function at a previously unseen point x.

Example 3: Interpolation

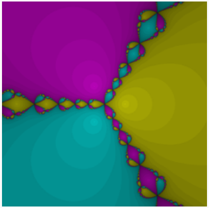
This problem is of paramount importance in machine learning. To give a concrete example, you may want to automatically classify a handwritten digit.

- x_i represent an image. y_i is a value in $\{0, 1, \ldots, 9\}$.
- Study of sequences (and functions) is important here: we want to construct p_N that will converge to the true (unknown) function f as the number of data points N increases.

Analysis is beautiful

We won't be studying algorithms in this unit, but the theoretical tools we build up will be useful throughout your mathematical careers.

I hope you find beauty in the arguments and concepts we encounter this semester!



Newton fractal for $z^3 - 1$

Unit outline

- Getting started: logic.
- What are the real numbers?
- Sequences: tools for checking convergence and divergence.
- Series (sequences constructed by adding up numbers).