MA10174 - Semester 1, 2021/22

Problem Sheet 3

1. Prove that

$$a. \quad \lim_{n\to\infty} \frac{2n+3}{3n-7} = \frac{2}{3}, \qquad b. \quad \textbf{Homework} \quad \lim_{n\to\infty} \frac{n^4+3}{3n^5+2} = 0.$$

   by first principles.

2. Find the limit of the following sequences

$$(i) \quad a_n = \sqrt{n^2+1} - n, \quad (ii) \quad a_n = 1 + (\frac{1}{3})^n \quad (iii) \quad \textbf{Homework} \quad a_n = \frac{7^n(1-n)}{(1+n^2)9^n}.$$

   [Hint for (i): calculate first $(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)$.]

3. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers.

   a. **Homework** Prove that if $(a_n)_{n\in\mathbb{N}}$ converges to $L > 0$, then $(\sqrt{a_n})_{n\in\mathbb{N}}$ converges to $\sqrt{L}$.

   b. Prove that $(\sqrt{1+a_n^2})_{n\in\mathbb{N}}$ converges to $\sqrt{1+L^2}$.

4. **Homework** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence and $L \in \mathbb{R}$. Prove that $\lim_{n\to\infty} a_n = L$ if and only if for any $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains a_n for all but finitely many n.

5. a. Let $a \in (0,1)$. Show that $\lim_{n\to\infty}(1+a^n)^{\frac{1}{n}} = 1$.

   [*Hint: write* $(1+a^n)^{\frac{1}{n}} = 1 + x_n$, *with* $x_n > 0$ *and show that* $x_n \to 0$ *as* $n \to \infty$]

   b. Let $c > d > 0$. Show that $\lim_{n\to\infty}(c^n + d^n)^{\frac{1}{n}} = c$

   c. Compute $\lim_{n\to\infty}(3^{2n} + n^{17}3^n)^{\frac{1}{n}}$.

6. **Homework** Determine whether the following sequences are increasing or decreasing. Find their limits, if they exist.

$$a. \quad a_n = \left(\frac{n}{n^2+1}\right)_{n\in\mathbb{N}}, \quad b. \quad a_n = \left(\frac{5^{n+1}}{2^n 3^n}\right)_{n\in\mathbb{N}}, \quad c. \quad a_n = \left(\frac{n+1}{n+10}\right)_{n\in\mathbb{N}}$$

7. Determine whether the following sequences are increasing or decreasing.

$$a.\quad n^2 \sin\left(\frac{\pi}{2}n\right), \qquad b.\quad \frac{n^n}{n!}$$

8. Show that the following sequences are convergent by showing that they are monotone and bounded. Find their limits.

   a. $a_1 = \frac{1}{4}$, $a_{n+1} = \frac{a_n}{2} + a_n^2$, for all $n \geq 1$

   b. **Homework** $a_1 = 1$, $a_{n+1} = -1 + \frac{a_n}{2}$, for all $n \geq 1$