Additional exercises 2 for MA505250

Exercise 1. Suppose that $E : \mathcal{U} \to \bar{\mathbb{R}}$ is absolute one-homogeneous. Then

(a) $E(u) = \langle p,\, u\rangle$ for all $p \in \partial E(u)$.

(b) $E^*(p) = \iota_{\partial E(0)}$

(c) $u \in \mathcal{U}$ and $p \in \partial E(u)$ if and only if $p \in \partial E(0)$ and $E(u) = \langle p,\, u\rangle$.

Exercise 2 (Moreau's identity). Let $F : \mathbb{R}^N \to (-\infty, \infty]$ be convex, proper, l.s.c. with bounded sublevel sets and let $P_F(x) = (I + \partial F)^{-1}(x)$. Show that for

$$P_F(x) + P_{F^*}(x) = x, \qquad \forall x,$$

and hence deduce that for all $\delta > 0$,

$$P_{\delta F}(x) + \delta P_{\delta^{-1} F^*}(x/\delta) = x.$$

Exercise 3 (Dual of the ROF model). Recall that the ROF model for image denoising is as follows

$$\min_{u \in L^2(\Omega)} \frac{1}{2} \|u - f\|_{L^2}^2 + \alpha \mathrm{TV}(u).$$

Consider it in the finite-dimensional setting (i.e. after discretisation). Let $u \in \mathbb{R}^n$ be a discrete signal. Then the ROF model can be re-written as follows

$$\min_{u \in \mathbb{R}^n} \frac{1}{2} \|u - f\|_2^2 + \alpha \|Du\|_1,$$

where D denotes the (discrete) gradient (you can think of it being just a matrix). See the appendix for a discussion on this discretization.

1. Show that the dual is

$$\sup_{\xi \in \mathbb{R}^n} -\frac{1}{2} \|D^*\xi - f\|_2^2 + \frac{1}{2} \|f\|_2^2 \text{ subject to } \|\xi\|_\infty \leqslant \alpha$$

2. Describe how the primal and dual solutions are related

3. Explain how to solve the dual using Forward-Backward.