MA10274 Sequences and Functions [1]

Lecture Notes

Semester 1, 2021/22

[1] These notes will be updated as we progress through the semester. Please send any corrections to Clarice Poon (cmhsp20@bath.ac.uk).

Why study analysis?

Analysis is the theory of limits and the concepts dependent on limits, including derivatives and integrals. Here are some examples of sequences:

- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$ converges to 0.
- $1, -1, 2, -2, 3, -3, 4, -4, \cdots$ doesn't have a limit.

We can also define sequences by adding up numbers:

- $a_n := \sum_{j=1}^{n} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ goes to infinity.
- $b_n := \sum_{j=1}^{n} \frac{1}{j^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$ converges to $\frac{\pi}{6}$.
- $c_n := c_{n-1} + c_{n-2}$, where $c_0 = 0,\ c_1 = 1$.

The last sequence c_n is the Fibbonacci sequence – it diverges but the ratio sequence $d_n := c_n / c_{n-1}$ converges to the golden ratio $\frac{1}{2}(\sqrt{5} + 1)$. In this unit, we will be developing theoretical tools to understand when sequences converge or diverge. So why study analysis?

It's pretty

Fractals such as the Mandelbrot fractal or Newton's fractal are generated by looking at the convergence and divergence of sequences.

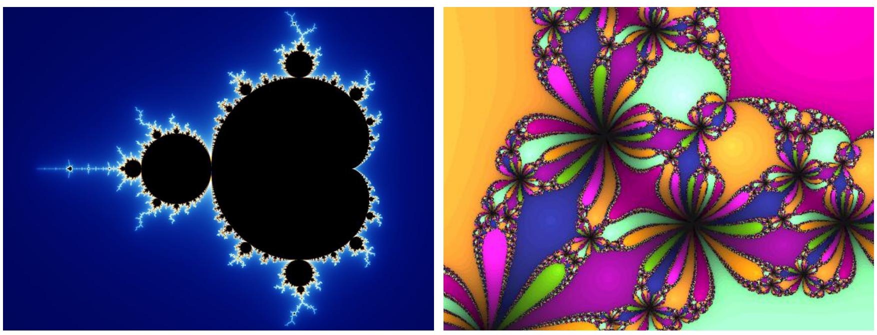


Figure 1: Left: the Mandelbrot set. Right: a Newton fractal

Analysis is crucial for the design and study of algorithms

Example 1: Computing integrals Consider the problem of computing an integral $\int_a^b f(x)\mathrm{d}x$. We know how to evaluate simple integrals such as

$$\int_0^1 e^x \mathrm{d}x \quad \text{or} \quad \int_0^\pi \cos(x)\mathrm{d}x.$$

But what about

$$\int_0^1 e^{x^2}\mathrm{d}x \quad \text{or} \quad \int_1^{2000} \exp\left(\sin\left(\cos\left(\sinh\left(\cosh\left(\tan^{-1}\left(\log(x)\right)\right)\right)\right)\right)\right)\mathrm{d}x?$$

Being able to evaluate integrals is an important part of scientific computing, but often, there is no closed form solution and we need to design an algorithm to *approximate* these integrals. One would typically construct I_N from samples $f(x_1), \ldots, f(x_N)$ for some $N \in \mathbb{N}$ with I_N converging to $\int_a^b f(x)\mathrm{d}x$ as $N \to \infty$.

Example 2: Solving linear systems Suppose we want to recover an unknown x given observations $\hat{f}$ with the knowledge that $\hat{f}$ is generated by a system of linear equations, i.e.

$$\text{Find} \quad x \in \mathbb{R}^n \quad \text{such that} \quad Ax = \hat{f}.$$

This situation arises in many practical problems from medical imaging to weather forecasting, however, these systems are often too large to fit into computer memory and we cannot directly invert the matrix A from the observations $\hat{f}$. We therefore need to construct a sequence x_n which converges to the solution x.

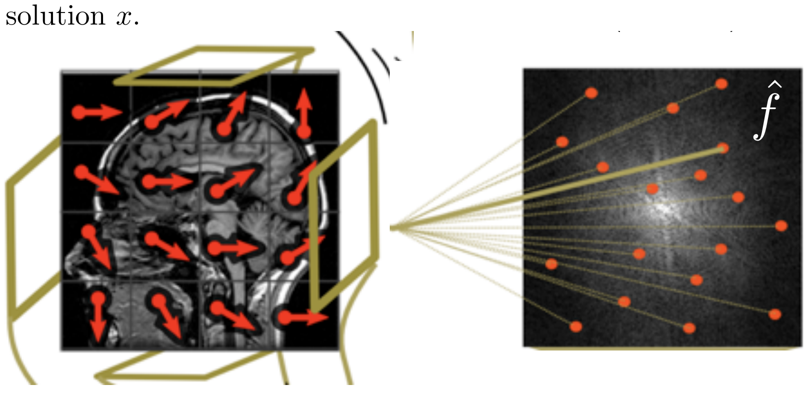
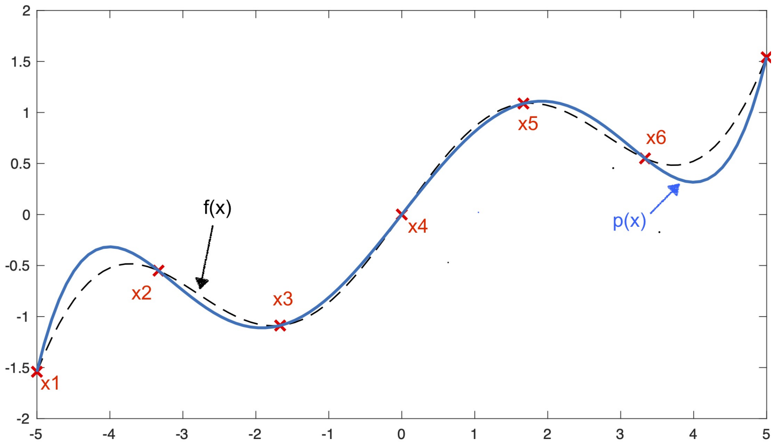


Figure 2: Solving linear systems is important in magnetic resonance imaging (MRI): the MRI scanner outputs a sequence of numbers $\hat{f}$ from which you want to reconstruct an image x (representing the cross section of the head). The physics of the MRI scanner is modelled using a linear system.

Example 3: Interpolation Suppose we want to approximate some function f from given N data points x_i and $y_i := f(x_i)$ for $i = 1, \dots, N$. That is, compute a function p_N such that

$$p_N(x_i) = y_i \quad i = 1, \dots, N.$$

This problem is called interpolation, and once this function is computed, you can then evaluate this function at a previously unseen point x.



This problem is of paramount importance in machine learning. To give a concrete example, you may want to automatically classify a handwritten digit.

In this setting, the x_i's represent the images and the y_i's take values in $\{0, 1, \dots, 9\}$. The understanding of sequences (and functions) is important here: we want to construct approximations p_N that will converge to the true (unknown) function f as the number of data points N increases.

Chapter 1

Some basic concepts

This semester, we will see a mathematically rigorous development of the theory of sequences. This means that all statements will be accompanied by proofs, based on logical arguments. For this reason, we first discuss the basic elements of logic, followed by other notions that the theory depends on.

1.1 Key logical concepts

But let's first clarify a key expression in the preceding paragraph.

Definition 1.1. A *statement* (or *proposition*) is a sentence that is either true or false but not both.

Example 1.2. The following are statements.

- 7 is an odd integer. (True)
- $2 < 7$. (True)
- All integers are odd. (False)

But the following is *not* a statement.

- Welcome to the University of Bath!

If we have two statements, we can form other statements (compound statements) from them using the logical connectives 'and' and 'or'. In the following, let P and Q denote two statements. (Here P and Q are place holders that may be replaced by *any* statement.)

Conjunction The expression $P \wedge Q$ stands for 'P and Q'. This statement is true if both P and Q are true; otherwise it is false.

Disjunction The expression $P \vee Q$ stands for 'P or Q'. This statement is true if either P or Q or both are true; otherwise it is false.

Remark. We always use the inclusive or; so $P \vee Q$ means 'either P or Q *or both*'. This is still the case if we use words instead of symbols; so 'P or Q' means the same. Even the phrase 'either P or Q' should be interpreted this way. The word 'either' is used merely as an aid to parse a sentence.

Example 1.3. Consider the following statement.

- All integers are odd, and $2 < 7$.

This is false, as not all integers are odd. But consider the following.

- All integers are odd, or $2 < 7$.

This statement is true, as $2 < 7$.

We can use a *truth table* to demonstrate the truth values of a compound statement.

P	Q	$P \wedge Q$	$P \vee Q$
true	true	true	true
true	false	false	true
false	true	false	true
false	false	false	false

Negation Given a statement P, the expression $\neg P$ stands for 'not P'. This statement is true if P is false and is false if P is true.

Here is the corresponding truth table.

P	$\neg P$
true	false
false	true

Implication Given two statements P and Q, the expression $P \Rightarrow Q$ stands for 'if P, then Q'. The truth table for $P \Rightarrow Q$ is

P	Q	$P \Rightarrow Q$
true	true	true
true	false	false
false	true	true
false	false	true

You can check that this has the same truth values as $\neg P \vee Q$.

Equivalence The expression $P \Leftrightarrow Q$ stands for '$P \Rightarrow Q$ and $Q \Rightarrow P$'. It means that P is true when Q is true and vice versa.

Instead of 'if P, then Q', we sometimes say 'P implies Q'. The statement $P \Leftrightarrow Q$ can be expressed in words by saying 'P is equivalent to Q' or 'P if, and only if, Q'. A common abbreviation for the latter is 'P iff Q'.

Example 1.4. Consider a number x. Then the following are true statements.

- $x > 0 \Rightarrow x \geq 0$.
- $x > 0 \Leftrightarrow -x < 0$.

The above concepts have the following truth tables.

P	Q	$P \Rightarrow Q$	$P \Leftrightarrow Q$
true	true	true	true
true	false	false	false
false	true	true	false
false	false	true	true

1.2 Transformation of logical expressions

These are laws about how to manipulate complex chains of conjunctions and disjunctions.

Proposition 1.5 (Distributive laws)**.** *Given any three statements P, Q, R, the following equivalences hold true.*

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$$
$$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

Proof. It suffices to compare the truth tables. Here is the truth table (with an intermediate step) for $P \wedge (Q \vee R)$.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$
true	true	true	true	true
true	true	false	true	true
true	false	true	true	true
true	false	false	false	false
false	true	true	true	false
false	true	false	true	false
false	false	true	true	false
false	false	false	false	false

Here is the truth table (with two intermediate steps) for $(P \wedge Q) \vee (P \wedge R)$.

P	Q	R	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
true	true	true	true	true	true
true	true	false	true	false	true
true	false	true	false	true	true
true	false	false	false	false	false
false	true	true	false	false	false
false	true	false	false	false	false
false	false	true	false	false	false
false	false	false	false	false	false

Comparing the last columns, we see that the first equivalence holds true. The second one is proved with the same method, but we omit the details here.

□

Proposition 1.6 (De Morgan's laws)**.** *Given two statements P and Q, the following equivalences hold true.*

$$\begin{aligned} \neg(P \vee Q) &\Leftrightarrow \neg P \wedge \neg Q \\ \neg(P \wedge Q) &\Leftrightarrow \neg P \vee \neg Q \end{aligned}$$

Proof. This is proved with the same method as Proposition 1.5. We omit the details. □

Example 1.7. What is the negation of the statement $P \Rightarrow Q$?

We first recall that $\neg P \vee Q$ is another way to write $P \Rightarrow Q$. Then by De Morgan's law,

$$\neg(\neg P \vee Q) \;\Leftrightarrow\; \neg(\neg P) \wedge \neg Q.$$

But $\neg(\neg P) \Leftrightarrow P$. Hence

$$\neg(P \Rightarrow Q) \;\Leftrightarrow\; P \wedge \neg Q.$$

That is, $P \wedge \neg Q$ is another way to express $\neg(P \Rightarrow Q)$.

Proposition 1.8 (Contrapositive)**.** *Given two statements P and Q, the implication*

$$P \Rightarrow Q$$

is equivalent to

$$\neg Q \Rightarrow \neg P$$

Proof. We show two ways of proving this.
The **first proof** is via truth tables:

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Note that the columns of $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ are the same.

The **second proof** exploits the equivalence between $P \Rightarrow Q$ and $(\neg P) \vee Q$: We can write $(\neg P) \vee Q$ in the form $Q \vee (\neg P)$. Now we observe that $Q \Leftrightarrow \neg(\neg Q)$, and therefore, we conclude that $Q \vee (\neg P)$ is equivalent to $\neg(\neg Q) \vee (\neg P)$. Finally, the last statement can be written in the form $\neg Q \Rightarrow \neg P$.

$\square$

1.3 The notation of set theory

Again we begin with a definition of the sort of object we consider here.

Definition 1.9. A *set* is a collection of distinct objects. These objects are called the *elements* of the set.

The order of the elements doesn't matter. For example, the set $\{1, 2, 3\}$ comprises the numbers 1, 2, and 3, and is exactly the same as $\{3, 2, 1\}$. That is,

$$\{1, 2, 3\} = \{3, 2, 1\}.$$

The following is some notation that we use in connection with sets. First we have some statements related to sets.

Notation	**Meaning**
$x \in A$	x is an element of the set A.
$x \notin A$	x is not an element of the set A.
$A \subseteq B$	A is a *subset* of another set B; i.e., every element of A is also an element of B.

Here is some notation related to forming new sets from two given sets A and B.

Notation	**Meaning**
$A \cap B$	The *intersection* of A and B (i.e., the set containing all objects that are elements of A and of B).
$A \cup B$	The *union* of A and B (i.e., the set containing all objects that are elements of A or of B).
$A \setminus B$	The *set difference* (i.e., the set of all elements of A that are not elements of B).
$A \times B$	The *Cartesian product*, comprising all pairs (a, b) with one element $a \in A$ and one element $b \in B$.

Finally, here are some specific sets that we will use often.

Notation	**Meaning**
$\emptyset$	The *empty set* (i.e., the set containing no objects at all).
$\mathbb{N} = \{1, 2, 3, \dots\}$	The set of *natural numbers*.
$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$	The set of *integers*.
$\mathbb{N}_0 = \{0, 1, 2, \dots\}$	The set of non-negative integers.
$\mathbb{Q} = \{p/q \colon p \in \mathbb{Z} \text{ and } q \in \mathbb{N}\}$	The set of *rational numbers* (numbers given as fractions).
$\mathbb{R}$	The set of *real numbers* (to be discussed later).
$\mathbb{C}$	The set of *complex numbers* (to be discussed in Algebra 1A).

Example 1.10.

- $2 \in \mathbb{N}$, $-2 \notin \mathbb{N}$, $2/3 \notin \mathbb{N}$
- $\mathbb{N} \cap \mathbb{Z} = \mathbb{N}$ and $\mathbb{N} \cup \mathbb{Z} = \mathbb{Z}$
- $\mathbb{Z} \subset \mathbb{Q}$
- $\mathbb{Z} \setminus \mathbb{N}_0 = \{-1, -2, -3, \dots\}$
- $\{1, 2\} \times \{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$.

1.4 Quantifiers

Quantifiers are expressions that indicate how a variable is to be interpreted in a logical statement. Sometimes they are expressed by symbols, so that a logical statement can be represented in terms of a formula. We have two such symbols: $\forall$ stands for the phrase 'for all', and $\exists$ stands for 'there exists'. For example, the formula

$$\forall x \colon x \in \mathbb{R} \Rightarrow x^2 \geq 0$$

means

'For all x, if x is a real number, then $x^2 \geq 0$.'

The formula

$$\exists x \colon x \in \mathbb{R} \wedge x^2 \geq 0$$

means

'There exists x such that x is a real number and $x^2 \geq 0$.'

Both of these are of course true statements. Since most of the time, the variables in such a statement are taken from a specific set, there following usage of the quantifiers is also common:

$$\forall x \in \mathbb{R} \colon x^2 \geq 0$$

(meaning 'for all real numbers x, the inequality $x^2 \geq 0$ holds true'); and

$$\exists x \in \mathbb{R}\colon x^2 \geq 0$$

(meaning 'there exists a real number x such that $x^2 \geq 0$).

Example 1.11. Is this a true statement:

$$\exists x \in \mathbb{Z}\, \forall y \in \mathbb{N}\colon x < y \ ?$$

Yes, this is a true statement, because there does a exist an integer x that is smaller than any natural number y (e.g. $x = -1$ will do).

We have seen De Morgan's laws in Proposition 1.6, which tell us how conjunction and disjunction behave under negation. We have a similar law for quantifiers. In the following, we write $P(x)$ for a statement depending on a variable x.

Proposition 1.12. *(i) The statements*

$$\neg(\forall x\colon P(x)) \qquad \textit{and} \qquad \exists x\colon \neg P(x)$$

are equivalent.

(ii) The statements

$$\neg(\exists x\colon P(x)) \qquad \textit{and} \qquad \forall x\colon \neg P(x)$$

are equivalent.

This proposition should be intuitively clear. It is not proved here, because it belongs to the theory of logic rather than analysis.

Example 1.13. Recall the statement from Example 1.11

$$\exists x \in \mathbb{Z}\, \forall y \in \mathbb{N}\colon x < y. \tag{1}$$

Show that its negation is

$$\forall x \in \mathbb{Z}\, \exists y \in \mathbb{N}\colon x \geq y. \tag{2}$$

Solution Using (ii) of Proposition 1.12, the negation of statement (1) is

$$\forall x \in \mathbb{Z} : \neg\left(\forall y \in \mathbb{N} : x < y\right).$$

Now apply (i) of Proposition 1.12 to the inner statement is obtain the equivalent statement

$$\forall x \in \mathbb{Z} : \exists y \in \mathbb{N} : \neg\left(x < y\right)$$

which is equivalent to (2).

1.5 Functions

A question that we study a lot in analysis is how one varying quantity depends on another. The following concepts formalises the idea of quantities depending on one another.

Definition 1.14. Let A and B be sets. A *function* from A to B is a rule that assigns to each element of A a unique element of B. We write $f\colon A \to B$ to indicate that the symbol f denotes a function from A to B. Given any $a \in A$, we then write $f(a)$ for the element from B assigned to a by the function. The set A is called the *domain* of f and B is called the *codomain* or *target set* of f.

So in order to specify a function, we need three things: a domain, a codomain, and a rule.

Example 1.15. We may define a function from $\mathbb{R}$ to $\mathbb{R}$ by assigning to every real number x its square. This gives a function $f\colon \mathbb{R} \to \mathbb{R}$ with the property that $f(x) = x^2$ for all $x \in \mathbb{R}$.

As the example shows, an equation such as $f(x) = x^2$ can be used to describe a specific function once the domain and codomain are known. Another way to refer to the same rule is $x \mapsto x^2$. The function should not be confused with the expression x^2, however. In order to obtain a well-defined function, we need to say what the domain and the codomain are and we need to specify the rule. Even so, you will sometimes see phrases such as 'the function $f(x) = x^2$'. This wording is acceptable when the context leaves no doubt about the domain and codomain, but should not be used otherwise.

Not all functions are given by such a convenient formula.

Example 1.16. We may define a function $f\colon \mathbb{R} \to \mathbb{Q}$ as follows: if $x \in \mathbb{R}$ is a rational number, then $f(x) = \frac{1}{2}$. Otherwise $f(x) = 0$. Thus

$$f(x) = \begin{cases} 1/2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Terminology: For a function $f : A \to B$, we say that f is

- *injective* if $f(a) = f(b)$ implies that $a = b$.
- *surjective* if for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.
- *bijective* if it is both injective and surjective.

1.6 The natural numbers

These are the numbers $1, 2, 3, \ldots$, comprising the set $\mathbb{N}$ that we have already seen. It is assumed here that you have at least an intuitive understanding of the natural numbers. Therefore, this discussion is very brief and serves mostly to highlight one aspect of $\mathbb{N}$ that will be important later.

What characterises the natural numbers are the following fundamental properties:

(i) 1 is a natural number (i.e., $1 \in \mathbb{N}$).

(ii) Every natural number n has a successor $n+1$, which is also a natural number (so $\forall n \in \mathbb{N} \colon n+1 \in \mathbb{N}$).

(iii) Every non-empty subset S of $\mathbb{N}$ has a least element. that is, there exists $s_0 \in S$ such that $s_0 \leq s$ for all $s \in S$. This is called the well-ordering property.

(iv) If a subset $\Lambda \subseteq \mathbb{N}$ satisfies $1 \in \Lambda$ and $\forall n \in \Lambda \colon n+1 \in \Lambda$, then $\Lambda = \mathbb{N}$.

Remark. Note that the integers $\mathbb{Z}$ is not well ordered, since $\mathbb{Z}$ has no least element. However, every non-empty subset of $\mathbb{Z}$ which is bounded from below has a least element.

The last property (iv) is the principle of induction and is the basis for various proofs of statements about the natural numbers. In particular, if $P(n)$ denotes a statement about $n \in \mathbb{N}$, then $P(n)$ may be true for some values of n and false for others. Here are some examples of statements

- $P(n)$ is the statement $n = n^2$. Then, $P(1)$ is true, but $P(n)$ is false for all $n \neq 1$, $n \in \mathbb{N}$.
- $P(n)$ is the statement $n = n + 5$. Then $P(n)$ is false for all $n \in \mathbb{N}$.
- $P(n)$ is the statement $n < 10$. Then $P(n)$ is true for $n = 1, \ldots, 9$ and false for all other elements of $\mathbb{N}$.

The principle of mathematical induction can be formulated as follows: For each $n \in \mathbb{N}$, let $P(n)$ be a statement about n. Suppose that

- $P(1)$ is true.
- If $P(k)$ is true, then $P(k+1)$ is true.

Then, $P(n)$ is true for all $n \in \mathbb{N}$.

Example 1.17. Prove that $1 + 2 + \cdots + n = n(n+1)/2$ for all $n \in \mathbb{N}$.

Solution. Let $\Lambda \subseteq \mathbb{N}$ be the set of all $n \in \mathbb{N}$ such that the formula holds true. Then clearly $1 \in \Lambda$, as $1 = 1 \cdot 2/2$.

Moreover, if $n \in \Lambda$, then

$$\begin{aligned} 1+2+\cdots+(n+1) &= (1+\cdots+n)+(n+1) \\ &= \frac{n(n+1)}{2}+(n+1) \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

So if $n \in \Lambda$, then $n+1 \in \Lambda$. By induction, it follows that $\Lambda = \mathbb{N}$. That is, the formula is true for all $n \in \mathbb{N}$.

Chapter 2

The real numbers

We take an axiomatic approach to the real numbers here. That is, we first state their basic properties, or axioms. These may be taken as self-evident and are not proved. Later on, however, we will prove any further statements by reducing them to these axioms. The axioms are presented in several groups, each of which correspond to a specific aspect of the theory.

2.1 Field axioms

The first group comprises the *field axioms*, which describe the algebraic properties under addition and multiplication.

(A1) $\forall a, b, c \in \mathbb{R}\colon a + (b + c) = (a + b) + c$ (associative law for $+$).

(A2) $\forall a, b \in \mathbb{R}\colon a + b = b + a$ (commutative law for $+$).

(A3) $\exists 0 \in \mathbb{R}\, \forall a \in \mathbb{R}\colon a + 0 = a = 0 + a$ (additive identity).

(A4) $\forall a \in \mathbb{R}\, \exists b \in \mathbb{R}\colon a + b = 0 = b + a$ (additive inverse). For a given $a \in \mathbb{R}$, the additive inverse (referred to as b in the previous formula) is unique and is usually denoted by $-a$.

(M1) $\forall a, b, c \in \mathbb{R}\colon a(bc) = (ab)c$ (associative law for $\times$).

(M2) $\forall a, b \in \mathbb{R}\colon ab = ba$ (commutative law for $\times$).

(M3) $\exists 1 \in \mathbb{R} \setminus \{0\}\, \forall a \in \mathbb{R}\colon a1 = a = 1a$ (multiplicative identity).

(M4) $\forall a \in \mathbb{R} \setminus \{0\}\, \exists b \in \mathbb{R}\colon ab = 1 = ba$ (multiplicative inverse). The multiplicative inverse of a is unique and is usually denoted by $1/a$ or a^{-1}.

(D) $\forall a, b, c \in \mathbb{R}\colon a(b + c) = ab + ac$ (distributive law).

Here are a few consequences of the field axioms.

Proposition 2.1. *Consequences of (A1) to (A4).*

(i) If $a + x = a$ for all a, then $x = 0$. *(uniqueness of zero element).*

(ii) If $a + x = a + y$ then $x = y$ *(cancellation law for addition, implying uniqueness of additive inverse of a).*

(iii) $-0 = 0$.

(iv) $-(-a) = a$.

(v) $-(a + b) = (-a) + (-b)$.

Consequences of (M1) to (M4)

(vi) If $a \cdot x = a$ for all $a \neq 0$ then $x = 1$ *(uniqueness of multiplicative identity).*

(vii) If $a \neq 0$ and $a \cdot x = a \cdot y$, then $x = y$ *(cancellation law for multiplication, implying uniqueness of multiplicative inverse of a).*

(viii) If $a \neq 0$ then $(a^{-1})^{-1} = a$.

Consequences from combining all the axioms.

(ix) $(a + b) \cdot c = a \cdot c + b \cdot c$.

(x) $a \cdot 0 = 0$.

(xi) $a \cdot (-b) = -(a \cdot b)$. *In particular,* $(-1) \cdot a = -a$.

(xii) $(-1) \cdot (-1) = 1$.

(xiii) If $a \cdot b = 0$ then either $a = 0$ or $b = 0$ *(or both). Moreover, if* $a \neq 0$ and $b \neq 0$, then $1/(a \cdot b) = (1/a) \cdot (1/b)$.

Proof. i) By A2, there exists $(-a)$ such that $(-a)+a = 0$, so, $(-a)+(a+x) = (-a)+a = 0$. By the associative law A1, $0+x = ((-a)+a)+x = 0$. So, $x = 0$.

ii) We add the additive inverse of a, $(-a)$, to both sides: $(-a)+(a+x) = (-a) + (a + y)$. By A1, this is $(-a + a) + x = (-a + a) + y$. By A3 and A4, we conclude $x = y$.

iii) By A4, $0 + (-0) = 0$. By i) (uniqueness of zero element), we have $(-0) = 0$.

iv) By A4, a is the additive inverse of $-a$, so since the additive inverse is unique, $-(-a) = a$.

v) By A1 (associativity) and A2 (commutivity)

$$\begin{aligned}(a+b)+((-a)+(-b)) &= (a+b)+((-b)+(-a)) && A2\\ &= a+(b+-b+(-a)) && A1\\ &= a+(0+(-a)) && A4\\ &= a+(-a) && A3\\ &= 0 && A4\end{aligned}$$

It follows that $(-a)+(-b) = -(a+b)$ is the additive inverse of $(a+b)$.

vi) We multiply both sides by a^{-1}, then apply M1, followed by M3 and M4.

vii) Recall that by M4, $a^{-1}a = 1$, so $a^{-1}(ab) = a^{-1}(ac)$. By M1, $(a^{-1}a)b = (a^{-1}a)c$. By M4, $1 \cdot b = 1 \cdot c$. Therefore, by M3, $b = c$.

viii) From M4, we see that a is an multiplicative inverse of a^{-1}, by uniqueness of multiplicative inverse, the result follows.

ix) By M2, $(a+b)c = c(a+b)$. We then apply D to deduce $(a+b)c = ca+cb$. We conclude by again applying M2.

x) By A3, M3 and ix, $a + a \cdot 0 = a \cdot 1 + a \cdot 0 = a \cdot (1+0) = a$. It follows by adding $(-a)$ to both sides and applying A1 and A4, that $a \cdot 0 = 0$.

xi) $(ab) + (a \cdot (-b)) = a(b + (-b)) = a \cdot 0$ first applying D, followed by A4 and x).

xii) $(-1) \cdot (-1) = -(-1 \cdot 1) = -(-1) = 1$, where we apply xi, M3 and iv.

xiii) The statement is clearly true when $a = 0$ and $b = 0$ (by x). Suppose $a \neq 0$, then $a^{-1} \cdot (a \cdot b) = (a^{-1} \cdot a)b = 1 \cdot b = b$. So, $b = 0$. Similarly, $b \neq 0$ implies $a = 0$.

If $a \neq 0$ and $b \neq 0$, then $(a \cdot b)((1/a) \cdot (1/b)) = (a \cdot 1/a) \cdot (b \cdot 1/b)$ by M1 and M2. So, $(a \cdot b)((1/a) \cdot (a/b)) = 1$, and $(1/a) \cdot (1/b) = 1/(a \cdot b)$. □

Remark (Rational numbers)**.** Elements of $\mathbb{R}$ which can be written in the form $a \cdot b^{-1}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$ are called rational numbers. The set of rational numbers is denoted by $\mathbb{Q}$.

Remark. If you replace $\mathbb{R}$ with a set F and define operations $+$ and $\cdot$ over F satisfying **A1** to **A4**, **M1** to **M4** and **D**, then it is called a field. The set of rational numbers $\mathbb{Q}$ and complex numbers $\mathbb{C}$ are fields. However, the natural numbers $\mathbb{N}$ and integers $\mathbb{Z}$ are not fields (why?).

2.2 Powers

In addition to the arithmetic operations $+$ and $\times$, we have powers of real numbers. These are defined as follows. For $a \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$a^n = \underbrace{a \cdots a}_{n \text{ times}}.$$

Earlier we have seen the notation a^{-1} for the multiplicative inverse. We further define

$$a^{-n} = (a^{-1})^n$$

for $n \in \mathbb{N}$, as well as $a^0 = 1$. Then we have powers a^i for every integer $i \in \mathbb{Z}$, provided that $a \neq 0$. With these definitions, we have the following properties of powers.

(i) $a^{ij} = (a^i)^j$ for all $a \in \mathbb{R} \setminus \{0\}$ and $i, j \in \mathbb{Z}$.

(ii) $(ab)^i = a^i b^i$ for all $a, b \in \mathbb{R} \setminus \{0\}$ and $i \in \mathbb{Z}$.

(iii) $a^{i+j} = a^i a^j$ for all $a \in \mathbb{R} \setminus \{0\}$ and $i, j \in \mathbb{Z}$.

You are probably familiar with powers with rational (and perhaps real) exponents, too. Later on, we will rigorously develop the theory of powers of the form a^x for $a > 0$ and for every real number x.

2.3 Order axioms

The next few axioms describe how the real numbers are ordered. There is a subset $\mathbb{P}$ (the strictly positive numbers) of $\mathbb{R}$ such that for all $a, b \in \mathbb{R}$,

(P1) $a, b \in \mathbb{P}$ implies $a + b \in \mathbb{P}$.

(P2) $a, b \in \mathbb{P}$ implies $a \cdot b \in \mathbb{P}$

(P3) exactly one of $a \in \mathbb{P}$, $a = 0$ and $-a \in \mathbb{P}$ holds.

We write $a < b$ (or $b > a$) if and only if $b - a \in \mathbb{P}$ and $a \leq b$ (or $b \geq a$) if and only if $b - a \in \mathbb{P} \cup \{0\}$.

Proposition 2.2. *(i) Reflexivity:* $a \leq a$.

(ii) Antisymmetry: $a \leq b$ *and* $b \leq a$ *implies* $a = b$.

(iii) Transitivity: If $a \leq b$ *and* $b \leq c$*, then* $a \leq c$*. Likewise with* $<$ *in place of* $\leq$.

(iv) Trichotomy: Exactly one of the following hold: $a < b$, $a = b$ *and* $b < a$.

(v) $0 < 1$ *(equivalently,* $1 \in \mathbb{P}$*).*

(vi) $a < b$ *if and only if* $-b < -a$.

(vii) $a < b$ *and* $c \in \mathbb{R}$ *implies* $a + c < b + c$.

(viii) *If* $a < b$ *and* $c < d$, *then* $a + c < b + d$.

(ix) $a < b$ *and* $0 < c$ *implies* $ac < bc$.

(x) $a^2 \geq 0$ *with equality if and only if* $a = 0$.

(xi) $a > 0$ *if and only if* $1/a > 0$.

(xii) *If* $a, b > 0$ *and* $a < b$, *then* $1/b < 1/a$.

Proof. (i) $a - a \in \mathbb{P} \cup \{0\}$ by A4.

(ii) By definition, $b-a, a-b \in \mathbb{P}\cup\{0\}$. If $b-a = 0$ or $a-b = 0$, then $a = b$ by properties of addition. Otherwise, $b-a \in \mathbb{P}$ and $-(b-a) = a-b \in \mathbb{P}$ which is impossible by P3.

(iii) By definition of $<$, $b - a \in \mathbb{P}$ and $c - b \in \mathbb{P}$, so $(c - b) + (b - a) = c + (-b) + b + (-a) = c + 0 + (-a) = c - a \in \mathbb{P}$. For $\leq$, consider also the cases of $a = b$ and or $b = c$.

(iv) By P3, exactly one of $b - a \in \mathbb{P}$ or $b - a = 0$ or $-(b - a) = a - b \in \mathbb{P}$ holds. Now apply the definition of $<$.

(v) By P3, exactly one of $1 \in \mathbb{P}$, $1 = 0$ and $-1 \in \mathbb{P}$ holds. We know from M3 that $1 \neq 0$. If $-1 \in \mathbb{P}$, then $(-1) \cdot (-1) = 1 \in \mathbb{P}$ by Proposition 17 (v), and this contradicts P3. So, $1 \in \mathbb{P}$.

(vi) $a < b$ iff $b - a \in \mathbb{P}$, but $b - a = -a + -(-b)$ so $-a > -b$ by definition of $>$.

(vii) $a + c < b + c$ iff $b - a = (b + c) - (a + c) \in \mathbb{P}$ iff $b > a$.

(viii) $(b + d) - (a + c) = (b - a) + (d - c)$ by properties of addition. Since $(b - a), (d - c) \in \mathbb{P}$, it follows by P1 that $(b - a) + (d - c) \in \mathbb{P}$. Hence by definition of $>$, $b + d > a + c$.

(ix) $a < b$ and $0 < c$ implies $b - a \in \mathbb{P}$ and $c \in \mathbb{P}$. By P2, $c \cdot (b - a) = bc - ac \in \mathbb{P}$ which is equivalent to $bc > ac$.

(x) First, $a^2 = 0$ iff $a = 0$ by Proposition 13 i. If $a \in \mathbb{P}$, then $a^2 \in \mathbb{P}$ by P2. If $-a \in \mathbb{P}$, then again by P2, $\mathbb{P} \ni (-a) \cdot (-a) = a \cdot a = a^2$. Therefore, $a^2 \geq 0$.

(xi) Assume $a \in \mathbb{P}$. Then, $a^{-1} \cdot a = 1 \in \mathbb{P}$ by (v), which implies $a^{-1} = 1 \cdot a^{-1} = a^{-1} \cdot a \cdot a^{-1} \in \mathbb{P}$. The implication that $a^{-1} \in \mathbb{P}$ implies $a \in \mathbb{P}$ is similar.

(xii) We use (x) and (viii). If $a, b > 0$ and $a < b$, then multiplying both sides by a^{-1} and applying (x) and (viii), $1 < b \cdot a^{-1}$. Now multiplying both sides by b^{-1} yields $b^{-1} < a^{-1}$.

□

2.4 Some useful inequalities

Proposition 2.3 (Binomial inequality)**.** *Let x be a real number with $x > -1$ and let n be a positive integer. Then,*

$$(1+x)^n \geq 1 + nx$$

Proof. We proceed by induction. The inequality is trivially true when $n = 1$. Assume this is true for $k \in \mathbb{N}$:

$$(1+x)^k \geq 1 + kx$$

holds for all $x > -1$. Then $0 < x - (-1) = x + 1$.

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)^k(1+x) && \text{by definition} \\
&\geq (1+kx)(1+x) && \text{by hypothesis and Propositon 2.2(ix)} \\
&= 1 + x + kx + kx^2 && \text{by A1-A4} \\
&\geq 1 + (1+k)x.
\end{aligned}$$

The result follows by induction. □

Definition 2.4. The *absolute value* of a number $x \in \mathbb{R}$ is

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Here are some of the basic properties of the absolute value.

Proposition 2.5. *For all $x \in \mathbb{R}$,*

(i) $x \leq |x|$,

(ii) $-x \leq |x|$,

(iii) $|-x| = |x|$.

(iv) For all $x, y \in \mathbb{R}$, the identity $|xy| = |x||y|$ holds true.

Proof. (i) If $x \geq 0$, then $|x| = x \geq x$. If $x < 0$, then $|x| = -x > 0 > x$, using (vi) and (iii) of Proposition 2.2.

(ii) The proof is similar to (i).

(iii) This follows by definition of $|x|$ and $x > 0$ iff $-x < 0$.

(iv) The idea is to use P3 to split into cases and apply the definition of $|x|$ with $(-x)(-y) = xy$. If $x, y \geq 0$, then $xy \geq 0$ by P2. Hence, $|xy| = xy = |x||y|$. If $x, y < 0$, then $-x, -y > 0$ and by P2 $0 < (-x)(-y) = xy$. So, $|x \cdot y| = xy = (-x) \cdot (-y) = |x| \cdot |y|$. If $x \geq 0$ and $y < 0$, then $-y > 0$ and $-(xy) = x \cdot (-y) \geq 0$. So, $|xy| = -xy = x \cdot (-y) = |x||y|$. □

Proposition 2.6 (Triangle inequality)**.** *For all $x, y \in \mathbb{R}$,*

(i) $|x + y| \leq |x| + |y|$ *(triangle inequality),*

(ii) $\big||x| - |y|\big| \leq |x - y|$ *(reverse triangle inequality).*

Proof. From Proposition 2.5,

$$-|x| \leq x \leq |x| \quad \text{and} \quad -|y| \leq y \leq |y|.$$

Adding these inequalities yields

$$-(|x| + |y|) = -|x| - |y| \leq x + y \leq |x| + |y|.$$

In particular, $x + y \leq |x| + |y|$ and $-(x + y) \leq |x| + |y|$. So, if $x + y \geq 0$, then $|x + y| = x + y \leq |x| + |y|$. On the other hand, if $x + y < 0$, then $|x + y| = -(x + y) \leq |x| + |y|$.

For the reverse triangle inequality, by applying the triangle inequality,

$$|x| = |x + y + (-y)| \leq |x - y| + |y|.$$

So, $|x| - |y| \leq |x - y|$. By reversing the roles of x and y, we obtain $|y| - |x| \leq |y - x| = |x - y|$. Since $||x| - |y||$ is either $|x| - |y|$ or $|y| - |x|$, we have $||x| - |y|| \leq |x - y|$. □

Now we apply these tools to some examples.

Example 2.7. Assuming that $|x - y| \leq 1$, show that

$$\left|x^2 - y^2\right| \leq (2|x| + 1)|x - y|.$$

Solution. We first factorise $x^2 - y^2$ into $x^2 - y^2 = (x + y)(x - y)$. Thus

$$\left|x^2 - y^2\right| = |x + y||x - y|$$

by Proposition 2.5. Furthermore,

$$|x + y| = |2x + (y - x)| \leq |2x| + |y - x| = 2|x| + |x - y| \leq 2|x| + 1.$$

Here we have used the triangle inequality, Proposition 2.5, and the hypothesis of the problem. Combining these two facts, we obtain

$$\left|x^2 - y^2\right| \leq (2|x| + 1)|x - y|.$$

Example 2.8. For what values of $x \in \mathbb{R}$ do we have the inequality

$$\frac{1}{|x-1|} \geq 1\,?$$

Solution. We can rule out $x = 1$ immediately, because the fraction makes no sense for this value.

Multiplying with $|x-1|$ (which is always positive), we obtain

$$1 \geq |x-1|.$$

If $x > 1$, then this means that $1 \geq x - 1$, so $x \leq 2$. If $x < 1$, then the inequality becomes $1 \geq 1 - x$, which we can transform into $0 \geq -x$ and then into $x \geq 0$.

To summarise, the inequality holds true when $1 < x \leq 2$ or $0 \leq x < 1$.

2.5 Intervals

The following notation is often useful when we deal with inequalities. Given $a, b \in \mathbb{R}$, we set

$$\begin{aligned}
(a,b) &= \{x \in \mathbb{R} \colon a < x < b\}\,, \\
[a,b] &= \{x \in \mathbb{R} \colon a \leq x \leq b\}\,, \\
(a,b] &= \{x \in \mathbb{R} \colon a < x \leq b\}\,, \\
[a,b) &= \{x \in \mathbb{R} \colon a \leq x < b\}\,.
\end{aligned}$$

Furthermore, for any $a \in \mathbb{R}$,

$$\begin{aligned}
(a,\infty) &= \{x \in \mathbb{R} \colon x > a\}\,, \\
[a,\infty) &= \{x \in \mathbb{R} \colon x \geq a\}\,, \\
(-\infty,a) &= \{x \in \mathbb{R} \colon x < a\}\,, \\
(-\infty,a] &= \{x \in \mathbb{R} \colon x \leq a\}\,.
\end{aligned}$$

Example 2.9. The set of solutions in Example 2.8 is $(1,2] \cup [0,1)$.

2.6 Maximum, minimum, supremum and infimum

You are probably familiar with the notion of maximum/minimum (greatest/least element):

Definition 2.10 (Maximum and minimum)**.** Consider a set $S \subseteq \mathbb{R}$. Assume that $S \neq \emptyset$.

(i) Let $s_0 \in \mathbb{R}$. We say s_0 is the maximum of S and write $s_0 = \max S$ if $s_0 \in S$ and $s \leq s_0$ for all $s \in S$.

(ii) Let $s_0 \in \mathbb{R}$. We say s_0 is the minimum of S and write $s_0 = \min S$ if $s_0 \in S$ and $s \geq s_0$ for all $s \in S$.

Note that not all sets have maximum or minimums. For example $S := (-\infty, 0]$ has maximum of 0 and no minimum. Now consider the set $S := (-\infty, 0)$: note that $0 \notin S$ and actually, no maximum exists. We need some more definitions about (least) upper bounds and (greatest) lower bounds.

Definition 2.11. Consider a set $S \subseteq \mathbb{R}$.

(i) A number $M \in \mathbb{R}$ is said to be an *upper bound* of S if $s \leq M$ for all $s \in S$.

(ii) A number $m \in \mathbb{R}$ is said to be a *lower bound* of S if $s \geq m$ for all $s \in S$.

(iii) The set S is called *bounded above* if it has an upper bound and *bounded below* if it has a lower bound. We say that S is *bounded* if it is bounded above and below.

Definition 2.12 (Supremum and infimum)**.** Consider a set $S \subseteq \mathbb{R}$.

(i) A number $T \in \mathbb{R}$ is called *supremum* or *least upper bound* of S if T is an upper bound of S and any other upper bound M of S satisfies $T \leq M$. We write $T = \sup S$.

(ii) A number $t \in \mathbb{R}$ is called *infimum* or *greatest lower bound* of S if t is a lower bound of S and any other lower bound m of S satisfies $t \geq m$. We write $t = \inf S$.

Remark (Relationship between infimum/minimum and supremum/maximum)**.** Let $\emptyset \neq S \subset \mathbb{R}$ be such that $\sup S$ exists. Then, S has a maximum if and only if $\sup S \in S$. In this case, $\sup S = \max S$. Likewise, if $\inf S$ exists, then S has a minimum if and only if $\inf S$ exists and then $\inf S = \min S$.

Example 2.13.

S	$\inf(S)$	$\min(S)$	$\sup(S)$	$\max(S)$
$(0,1]$	0	None	1	1
$[0,1] \cup [2,3)$	0	0	3	None
$(0,\infty)$	0	None	None	None
$\mathbb{N}$	1	1	None	None
$\{-\frac{1}{4}, 2, 3, \pi\}$	$-\frac{1}{4}$	$-\frac{1}{4}$	π	π

2.7 Completeness axiom

So far, we have the axioms on arithmetic and ordering hold for both $\mathbb{R}$ and $\mathbb{Q}$, but we know they are different, for example, there is no rational number q for which $q^2 = 2$ but there is a real number r for which $r^2 = 2$. Here's another difference: consider the following sets

$$S_1 = \{x \in \mathbb{R} : x^2 < 2\} \quad \text{and} \quad S_2 = \{x \in \mathbb{Q} : x^2 < 2\}.$$

Both these sets are upper bounded, e.g. $x \leq 4$ for all $x \in S_1$ and all $x \in S_2$. Let's make some observations

(i) the set S_1 has no maximum element but, it has $\sqrt{2}$ as the least upper bound.

(ii) the set S_2 does not have a maximum element and it also has no least upper bound in $\mathbb{Q}$: Suppose for a contradiction that $s \in \mathbb{Q}$ is the least upper bound. Define

$$q = s + \frac{2 - s^2}{s + 2} = \frac{2s + 2}{s + 2}$$

Now, either $s^2 < 2$ or $s^2 > 2$:

(a) $s^2 < 2$ implies that $q > s$ and

$$q^2 - 2 = \frac{(2s + 2)^2 - 2(s + 2)^2}{(s + 2)^2} = \frac{2(s^2 - 2)}{(s + 2)^2} < 0.$$

So, $q \in S_2$ and $q > s$, so s is not an upper bound. Contradiction.

(b) $s^2 > 2$ implies that $q < s$ and $q^2 - 2 > 0$. So, q is an upper bound to S_2 and is less than s which contradicts s being the least upper bound.

The example above shows us a bounded set in $\mathbb{R}$ which has a least upper bound, but when considered in $\mathbb{Q}$, no least upper bound exists. One of the intuitive difference between $\mathbb{R}$ and $\mathbb{Q}$ is that $\mathbb{Q}$ is riddled with holes while $\mathbb{R}$ is complete – this intuition is capture by the completeness axiom:

(C) Every **non-empty** set of real numbers that is **bounded above** has a least upper bound.

Equipped with the completeness axiom, it is possible to prove additional properties about $\mathbb{R}$, such as

- the existence of $\sqrt{2}$ in $\mathbb{R}$
- the natural numbers is not upper bounded by any number in $\mathbb{R}$: for all $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n \geq x$.

- the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ (given any $x \in \mathbb{R}$ and any $\epsilon > 0$, there is $q \in \mathbb{Q}$ such that $|x - q| < \epsilon$, i.e. you can approximate any real number to arbitrary precision using rational numbers).

- The set of real numbers $\mathbb{R}$ is uncountable, i.e. you cannot assign a bijection from $\mathbb{N}$ to $\mathbb{R}$.

We end our discussion of the real numbers here, interested readers can refer to Chapter 2 of "Introduction to real analysis" by Bartle and Sherbert for more information about the completeness axioms.

Chapter 3

Sequences

Example 3.1 (Example of sequences)**.**

- $\frac{3}{10}, \frac{33}{100}, \frac{333}{1000}, \frac{3333}{10000}, \cdots$ converges to $\frac{1}{3}$.
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$ converges to 0.
- $1, -1, 2, -2, 3, -3, 4, -4, \cdots$ doesn't have a limit.

We formally define a sequence as follows:

Definition 3.2. A *sequence* of real numbers is a function from $\mathbb{N}$ to $\mathbb{R}$.

We will also use the expressions 'real sequence' and 'sequence in $\mathbb{R}$' for the same concept.

Although this means that a sequence is a rule assigning a real number to every element of $\mathbb{N}$, the structure of $\mathbb{N}$ also allows the interpretation implied by the word 'sequence'. We use the notation $(a_1, a_2, a_3, \dots)$ for the rule that assigns the number a_1 to 1, the number a_2 to 2, etc. Often this is abbreviated as $(a_n)_{n\in\mathbb{N}}$ or $(a_n)_{n=1}^{\infty}$.

Example 3.3. The sequence $(1, 2, 4, 8, 16, \ldots)$ may be written as $(2^{n-1})_{n\in\mathbb{N}}$, and the sequence $(1, 1/2, 1/3, \dots)$ may be written as $(1/n)_{n\in\mathbb{N}}$.

3.1 Convergence

When studying sequences, we are often interested in their behaviour for very large values of n. Of particular importance is the question whether for a given sequence $(a_n)_{n\in\mathbb{N}}$, the values a_n approach a certain number when n tends to infinity.

Definition 3.4. A real sequence $(a_n)_{n\in\mathbb{N}}$ *converges* to a real number L if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. If so, then L is called the *limit* of the sequence. We denote convergence of $(a_n)_{n\in\mathbb{N}}$ to L by $a_n \to L$ as $n \to \infty$, or alternatively by $L = \lim_{n\to\infty} a_n$.

If a sequence does not converge, then we say that it *diverges*.

The condition in this definition is perhaps a bit complicated, so we discuss it briefly. We may write it with the help of quantifiers as follows: convergence of $(a_n)_{n\in\mathbb{N}}$ to L means that

$$\forall \epsilon > 0\, \exists N \in \mathbb{N}\, \forall n \geq N \colon |a_n - L| < \epsilon.$$

The quantity $|a_n - L|$ can be thought of as the distance between a_n and L. So the inequality $|a_n - L| < \epsilon$ means that a_n is less than ϵ away from L. The whole statement means that no matter what positive number ϵ you choose, you can always find a corresponding natural number N such that from N onward (for all $n \geq N$), the members of the sequence are less than ϵ away from L. In other words, the sequence eventually approximates L arbitrarily well.

One more subtlety: the number N can (and will in general) depend on ϵ. Choosing ϵ smaller may force you to increase the value of N in order to still satisfy the condition.

Example 3.5. Does the sequence $(1/n)_{n\in\mathbb{N}}$ converge?

Solution. If we suspect convergence, then we have to identify a limit. Inspection of the sequence suggests the limit 0. Now to check if indeed $\lim_{n\to\infty} 1/n = 0$, we need to verify the criterion. This is what we do next.

Suppose we're given some $\epsilon > 0$. We examine the inequality $|1/n - 0| < \epsilon$, which can be rewritten in the form $1/n < \epsilon$ or $n > 1/\epsilon$. Now if we choose $N \in \mathbb{N}$ with $N > 1/\epsilon$, then any $n \geq N$ will also satisfy $n > 1/\epsilon$. Hence $|1/n - 0| < \epsilon$ for all $n \geq N$.

The conclusion is that the sequence does converge and the limit is 0.

As this simple example shows, verifying convergence directly with the criterion can be laborious. Sometimes there is no other way, but often it is possible to deduce convergence or divergence from general facts such as the following.

Proposition 3.6. *(i) For any $c \in \mathbb{R}$, the constant sequence $(c, c, c, \ldots)$ converges to c.*

(ii) The sequence $(1/n)_{n\in\mathbb{N}}$ converges to 0.

(iii) For any $a \in \mathbb{R}$, if $|a| < 1$, then the sequence $(a^n)_{n\in\mathbb{N}}$ converges to 0.

Proof. (i) For any $\epsilon > 0$, the inequality $|c - c| < \epsilon$ holds automatically, so we may choose any $N \in \mathbb{N}$ to satisfy the criterion.

(ii) The proof has been given in Example 3.5.

(iii) Suppose that $|a| < 1$. Then $1/|a| > 1$ and therefore $1/|a| - 1 > 0$. Define $b = 1/|a| - 1$. Then

$$|a^n - 0| = |a^n| = |a|^n = \left(\frac{1}{1+b}\right)^n = \frac{1}{(1+b)^n} \leq \frac{1}{1+nb}$$

by the binomial inequality (Proposition 2.3).

Fix $\epsilon > 0$. Choose $N \in \mathbb{N}$ with

$$N > \frac{1/\epsilon - 1}{b}.$$

Then for $n \geq N$, we conclude that

$$1 + nb \geq 1 + Nb > 1 + \frac{1/\epsilon - 1}{b} b = \frac{1}{\epsilon}.$$

Therefore,

$$|a^n - 0| \leq \frac{1}{1 + nb} < \frac{1}{1/\epsilon} = \epsilon.$$

This proves that $\lim_{n\to\infty} a^n = 0$.

□

Proposition 3.7. *Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are real sequences with $a_n \leq b_n$ for all $n \in \mathbb{N}$. If $L = \lim_{n\to\infty} a_n$ and $M = \lim_{n\to\infty} b_n$ exist, then $L \leq M$.*

Proof. We argue by contradiction.

Suppose (seeking contradiction) that $M < L$. Let $\epsilon = (L - M)/2$. Since $a_n \to L$ as $n \to \infty$, there exists $N_1 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N_1$. Similarly, there exists $N_2 \in \mathbb{N}$ such that $|b_n - M| < \epsilon$ for all $n \geq N_2$. Now set $N = \max\{N_1, N_2\}$. Then for all $n \geq N$,

$$a_n > L - \epsilon = L - \frac{L - M}{2} = \frac{L + M}{2}$$

and

$$b_n < M + \epsilon = M + \frac{L - M}{2} = \frac{L + M}{2}.$$

Therefore, we see that $a_n > b_n$ for $n \geq N$, which contradicts the hypothesis.

□

Corollary 3.8. *Suppose that $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence.*

(i) If $b \in \mathbb{R}$ is a number with $a_n \leq b$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n \leq b$.

(ii) If $c \in \mathbb{R}$ is a number with $a_n \geq c$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n \geq c$.

Theorem 3.9 (Uniqueness of limits)**.** *A sequence of real numbers has at most one limit.*

In other words, if a limit exists at all (if the sequence converges), then that limit is unique.

Proof. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a real sequence that converges simultaneously to $L \in \mathbb{R}$ and to $M \in \mathbb{R}$. Then, as $a_n \leq a_n$ for all n, Proposition 3.7 implies that $L \leq M$. But it also implies that $M \leq L$, and according to Proposition 2.2, this means that $L = M$. □

Definition 3.10. A real sequence $(a_n)_{n\in\mathbb{N}}$ is *bounded* if the set $\{a_n : n \in \mathbb{N}\}$ (the set of all its members) is bounded.

Proposition 3.11. *Any convergent sequence is bounded.*

Proof. Suppose that $(a_n)_{n\in\mathbb{N}}$ converges and let $L = \lim_{n\to\infty} a_n$. Apply the convergence criterion with $\epsilon = 1$. This yields a number $N \in \mathbb{N}$ such that $|a_n - L| < 1$ for all $n \geq N$. Hence

$$|a_n| < |L| + 1$$

for all $n \geq N$. Now set

$$M = \max\{|a_1|, \dots, |a_{N-1}|, |L| + 1\}.$$

Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. Hence M serves as an upper bound of $\{a_n : n \in \mathbb{N}\}$ while $-M$ serves as a lower bound. □

Theorem 3.12 (Algebra of limits)**.** *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be convergent sequences with limits $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$. Then*

(i) $\lim_{n\to\infty}(a_n + b_n) = A + B$,

(ii) $\lim_{n\to\infty}(a_n b_n) = AB$,

(iii) $\lim_{n\to\infty} Ca_n = CA$ *for any constant* $C \in \mathbb{R}$,

(iv) $\lim_{n\to\infty}(a_n - b_n) = A - B$, *and*

(v) if $b_n \neq 0$ for all $n \in \mathbb{N}$ and if $B \neq 0$, then $\lim_{n\to\infty}(a_n/b_n) = A/B$.

Proof. (i) Let $\epsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that $|a_n - A| < \epsilon/2$ for all $n \geq N_1$ and there exists $N_2 \in \mathbb{N}$ such that $|b_n - B| < \epsilon/2$ for all $n \geq N_2$.

Define $N = \max\{N_1, N_2\}$. If $n \geq N$, then

$$|(a_n+b_n)-(A+B)| = |(a_n-A)+(b_n-B)| \leq |a_n-A|+|b_n-B| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

(Here we have used the triangle inequality.) Hence $a_n + b_n \to A + B$ as $n \to \infty$.

(ii) First note that

$$\begin{aligned} |a_n b_n - AB| &= |(a_n - A)b_n + A(b_n - B)| \\ &\leq |(a_n - A)b_n| + |A(b_n - B)| \\ &= |a_n - A||b_n| + |A||b_n - B| \end{aligned} \tag{1}$$

by the triangle inequality and Proposition 2.5. According to Proposition 3.11, there exists $M > 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$.

Now let $\epsilon > 0$. Choose $N_1 \in \mathbb{N}$ such that

$$|a_n - A| < \frac{\epsilon}{2M + 1}$$

for all $n \geq N_1$ and choose $N_2 \in \mathbb{N}$ such that

$$|b_n - B| < \frac{\epsilon}{2|A| + 1}$$

for all $n \geq N_2$. Set $N = \max\{N_1, N_2\}$. Then for $n \geq N$, we have the inequalities

$$|a_n - A||b_n| < \frac{\epsilon}{2M + 1} M \leq \frac{\epsilon}{2}, \quad \text{and} \quad |A||b_n - B| < |A|\frac{\epsilon}{2|A| + 1} \leq \frac{\epsilon}{2}.$$

Combine these estimates with (1). This yields

$$|a_n b_n - AB| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(iii) Consider the sequence $(c_n)_{n \in \mathbb{N}}$ with $c_n = C$ for every $n \in \mathbb{N}$. Then $\lim_{n \to \infty} c_n = C$ by Proposition 3.6. Now we simply apply (ii), which gives

$$\lim_{n \to \infty} (C a_n) = \lim_{n \to \infty} (c_n a_n) = CA.$$

(iv) Note that

$$a_n - b_n = a_n + (-1)b_n.$$

By (iii), we have the limit $\lim_{n \to \infty}((-1)b_n) = -B$. Now (i) implies that

$$\lim_{n \to \infty} (a_n - b_n) = A + (-B) = A - B.$$

(v) Here we first prove that

$$\lim_{n \to \infty} \frac{1}{b_n} = \frac{1}{B}. \tag{2}$$

To this end, first note that

$$\left| \frac{1}{b_n} - \frac{1}{B} \right| = \left| \frac{B - b_n}{b_n B} \right| = \frac{|B - b_n|}{|b_n||B|}.$$

Since $B \neq 0$, we know that $|B| > 0$. Therefore, there exists $N_1 \in \mathbb{N}$ such that $|b_n - B| < |B|/2$ for all $n \geq N_1$. Then it also follows that

$$|b_n| \geq |B| - |b_n - B| > \frac{|B|}{2}$$

by the reverse triangle inequality.

Now fix $\epsilon > 0$ arbitrarily. Choose $N_2 \in \mathbb{N}$ such that

$$|b_n - B| < \frac{\epsilon |B|^2}{2}$$

for all $n \geq N_2$. Set $N = \max\{N_1, N_2\}$. Then for $n \geq N$, we find that

$$\left| \frac{1}{b_n} - \frac{1}{B} \right| = \frac{|B - b_n|}{|b_n||B|} < \frac{\epsilon |B|^2/2}{|B|^2/2} = \epsilon.$$

This proves (2).

Finally, we prove the full statement with the help of (ii). We now know that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(a_n \frac{1}{b_n} \right) = \frac{A}{B}.$$

□

Example 3.13. Find

$$\lim_{n\to\infty} \frac{2^{-n} + n^3 - 7n}{5n^3 + 2n^2 - 8n + 9}.$$

Solution. We need to rewrite this expression a bit before we can apply the algebra of limits theorem. Note that

$$\frac{2^{-n} + n^3 - 7n}{5n^3 + 2n^2 - 8n + 9} = \frac{\frac{(1/2)^n}{n^3} + 1 - \frac{7}{n^2}}{5 + \frac{2}{n} - \frac{8}{n^2} + \frac{9}{n^3}}.$$

Now

$$\frac{(1/2)^n}{n^3} = \left(\frac{1}{2}\right)^n \left(\frac{1}{n}\right) \left(\frac{1}{n}\right) \left(\frac{1}{n}\right) \to 0 \cdot 0 \cdot 0 \cdot 0 = 0 \quad \text{as } N \to \infty$$

by Theorem 3.12. Similarly,

$$\frac{7}{n^2} \to 0, \quad \frac{2}{n} \to 0, \quad \frac{8}{n^2} \to 0, \quad \frac{9}{n^3} \to 0 \quad \text{as } N \to \infty.$$

Using Theorem 3.12 again, we finally find that

$$\lim_{n\to\infty} \frac{2^{-n} + n^3 - 7n}{5n^3 + 2n^2 - 8n + 9} = \frac{0 + 1 - 0}{5 + 0 - 0 + 0} = \frac{1}{5}.$$

Proposition 3.14. *Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence.*

(i) If $a_n \to L$ as $n \to \infty$ for some $L \in \mathbb{R}$, then $|a_n| \to |L|$ as $n \to \infty$.

(ii) If $|a_n| \to 0$ as $n \to \infty$, then $a_n \to 0$ as $n \to \infty$ as well.

But caution: the second statement becomes false if 0 is replaced by any other value. For example, consider the sequence with $a_n = (-1)^n$. Then $|a_n| = 1 \to 1$ as $n \to \infty$, but $(a_n)_{n\in\mathbb{N}}$ oscillates between -1 and 1 and does not converge.

Proof. (i) Suppose that $a_n \to L$ as $n \to \infty$. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. Then by the reverse triangle inequality,

$$||a_n| - |L|| \leq |a_n - L| < \epsilon$$

as well.

(ii) This is an exercise.

□

In addition to convergent sequences, the following types of sequences are sometimes useful, too.

Definition 3.15. (i) We say that a real sequence $(a_n)_{n\in\mathbb{N}}$ *diverges to* ∞ if for all $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $a_n > M$ for all $n \geq \mathbb{N}$. In this case, we write $a_n \to \infty$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = \infty$.

(ii) We say that $(a_n)_{n\in\mathbb{N}}$ *diverges to* $-\infty$ if for all $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $a_n < M$ for all $n \in \mathbb{N}$. In this case, we write $a_n \to -\infty$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = -\infty$.

Some, but not all, of the statements from Theorem 3.12 extend to sequences diverging to ∞ or $-\infty$. You can explore the corresponding statements in an exercise.

3.2 Monotone sequences

Definition 3.16. A real sequence $(a_n)_{n\in\mathbb{N}}$ is called

(i) *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$,

(ii) *strictly increasing* if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$,

(iii) *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$,

(iv) *strictly decreasing* if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.

We say that the sequence is *monotone* if it has any of the above properties.

Warning: Some books use the expressions 'nondecreasing' in place of 'increasing' and 'nonincreasing' in place of 'decreasing'.

The following is one of the reasons why monotone sequences are useful.

Theorem 3.17. *If a real sequence is monotone and bounded, then it converges.*

Proof. We only give the proof for increasing sequences, as the arguments for the other cases are essentially the same.

Suppose that $(a_n)_{n\in\mathbb{N}}$ is a bounded, increasing sequence. Then the set $\{a_n : n \in \mathbb{N}\}$ is bounded (by definition of bounded sequences) and non-empty (as, e.g., a_1 belongs to the set). The completeness axiom (C) says that

$$L := \sup\{a_n : n \in \mathbb{N}\}$$

exists. We claim that $L = \lim_{n\to\infty} a_n$.

In order to verify this claim, choose $\epsilon > 0$. Since L is the least upper bound, there exists $N \in \mathbb{N}$ such that $a_N > L - \epsilon$. Now for all $n \geq N$, we have the inequality

$$a_n \geq a_N > L - \epsilon.$$

By the definition of L, we also know that $a_n \leq L$, and both inequalities together imply that

$$|a_n - L| < \epsilon$$

for all $n \geq N$. □

The theorem tells you that a limit exists, but it does not tell you what it is. Nevertheless, this is useful information and we can often find out what the limit is with other means once we know it exists.

Example 3.18. Show that there exists a number $x \in \mathbb{R}$ with $x^2 = 2$.

Solution. Let $a_1 = 2$ and define a sequence $(a_n)_{n\in\mathbb{N}}$ recursively by the formula[1]

$$a_{n+1} = a_n - \frac{a_n^2 - 2}{2a_n}, \quad n = 1, 2, \dots .$$

First we claim that $a_n > 0$ for all $n \in \mathbb{N}$ and hence the definition makes sense. We prove this by induction. It is obvious that $a_1 > 0$. Assuming that $a_n > 0$, we obtain the inequality

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} > 0$$

as well. This is the induction step, so we have indeed proved that $a_n > 0$ for all $n \in \mathbb{N}$.

[1] This sequence is generated by an application of Newton's method (which you may have seen in school) for finding a zero of the function $x \mapsto x^2 - 2$. This information is not necessary to understand the arguments, but may explain the background of the underlying ideas.

Now note that for all $n \in \mathbb{N}$,

$$a_{n+1}^2 - 2 = \left(\frac{a_n}{2} + \frac{1}{a_n}\right)^2 - 2 = \frac{a_n^2}{4} - 1 + \frac{1}{a_n^2} = \frac{a_n^4 - 4a_n^2 + 4}{4a_n^2} = \frac{(a_n^2 - 2)^2}{4a_n^2} \geq 0. \tag{3}$$

Therefore,

$$a_{n+1} = a_n - \frac{a_n^2 - 2}{2a_n} \leq a_n$$

for all $n \in \mathbb{N}$. That is, we have a decreasing sequence. Because we have already seen that $a_n > 0$ for all $n \in \mathbb{N}$, it is bounded.

Inequality (3) even implies that $a_n > 1$ for all $n \in \mathbb{N}$, because we know that $a_n > 0$ and any number $a \in (0, 1]$ will satisfy $a^2 - 2 < 0$.

Theorem 3.17 implies that $x = \lim_{n\to\infty} a_n$ exists. As we have seen that $a_n > 1$ for all $n \in \mathbb{N}$, the limit will satisfy $x \geq 1$. By the algebra of limits theorem (Theorem 3.12), it also satisfies

$$x = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(\frac{a_n}{2} + \frac{1}{a_n}\right) = \frac{x}{2} + \frac{1}{x}.$$

Hence

$$\frac{x}{2} = \frac{1}{x},$$

which is equivalent to the equation $x^2 = 2$.

Example 3.19. Show that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

exists.

Solution. According to Theorem 3.17, it suffices to show that the sequence with members $a_n = (1 + 1/n)^n$ is bounded and increasing.

We first show that $(a_n)_{n\in\mathbb{N}}$ is increasing. To this end, we write

$$a_n = \left(\frac{n+1}{n}\right)^n.$$

Thus

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^n} = \frac{(n+2)^{n+1} n^n}{(n+1)^{n+1}(n+1)^n} = \frac{n+2}{n+1}\left(\frac{(n+2)n}{(n+1)^2}\right)^n.$$

Now

$$\frac{(n+2)n}{(n+1)^2} = \frac{n^2 + 2n}{(n+1)^2} = \frac{(n+1)^2 - 1}{(n+1)^2} = 1 - \frac{1}{(n+1)^2}.$$

Therefore,

$$\frac{a_{n+1}}{a_n} = \frac{n+2}{n+1}\left(1 - \frac{1}{(n+1)^2}\right)^n.$$

In the next step, we want to apply the binomial inequality (Proposition 2.3) to the power. In order to do so, we need to check that

$$-\frac{1}{(n+1)^2} \geq -1.$$

But this is evidently true, as $n+1 > 1$ for all $n \in \mathbb{N}$. Hence the binomial inequality implies that

$$\left(1 - \frac{1}{(n+1)^2}\right)^n \geq 1 - \frac{n}{(n+1)^2}.$$

We now obtain the inequality

$$\frac{a_{n+1}}{a_n} \geq \frac{n+2}{n+1}\left(1 - \frac{n}{(n+1)^2}\right) = \frac{n^3+3n^2+3n+2}{n^3+3n^2+3n+1} > 1.$$

Therefore, we have proved that $a_{n+1} > a_n$, and the sequence is in fact strictly increasing.

Next we need to show that it is bounded. We consider even indices (of the form $n = 2k$ for some $k \in \mathbb{N}$) first. We compute

$$\frac{1}{a_{2k}} = \left(\frac{2k}{2k+1}\right)^{2k} = \left[\left(1 - \frac{1}{2k+1}\right)^k\right]^2 \geq \left[1 - \frac{k}{2k+1}\right]^2 = \left[\frac{k+1}{2k+1}\right]^2.$$

(We have used the binomial inequality again in the third step.) Since $2(k+1) > 2k+1$ for every $k \in \mathbb{N}$, we conclude that

$$\frac{k+1}{2k+1} > \frac{1}{2}.$$

Hence

$$\frac{1}{a_{2k}} > \frac{1}{4},$$

which means that $a_{2k} < 4$ for all $k \in \mathbb{N}$. For odd indices n, we use the monotonicity that we have already proved. If n is odd, then $a_n < a_{n+1} < 4$ as well. So the sequence is bounded. Therefore, there exists a limit.

We will come back to the limit of this sequence later. It is common to use the symbol e for it. That is,

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

3.3 Tests for convergence

In this section we discuss ways to find out if a given sequence is convergent. We already know one criterion: if a sequence is bounded and monotone, then it is convergent by Theorem 3.17. But this does not apply to all interesting sequences, which is why we want other tests.

Theorem 3.20 (Pinching theorem/sandwich theorem)**.** *Suppose that* $(a_n)_{n\in\mathbb{N}}$*,* $(b_n)_{n\in\mathbb{N}}$*, and* $(c_n)_{n\in\mathbb{N}}$ *are real sequences with* $a_n \le b_n \le c_n$ *for all* $n \in \mathbb{N}$*. If* $(a_n)_{n\in\mathbb{N}}$ *and* $(c_n)_{n\in\mathbb{N}}$ *converge and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n,$$

then $(b_n)_{n\in\mathbb{N}}$ *converges as well and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

Proof. Assume that $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ do indeed converge with $a_n \to L$ and $c_n \to L$ as $n \to \infty$ for some $L \in \mathbb{R}$. We need to show that $b_n \to L$ as well as $n \to \infty$.

To this end, fix $\epsilon > 0$. Choose $N \in \mathbb{N}$ so large that

$$|a_n - L| < \epsilon \quad \text{and} \quad |c_n - L| < \epsilon$$

for all $n \in \mathbb{N}$. Then

$$L - \epsilon < a_n \le b_n \le c_n < L + \epsilon$$

for $n \ge N$, which implies that $|b_n - L| < \epsilon$. □

Example 3.21. Consider the sequence $(b_n)_{n\in\mathbb{N}}$ with

$$b_n = \frac{\sin n}{n}, \quad n \in \mathbb{N}.$$

This permits an application of Theorem 3.20 with $a_n = -1/n$ and $c_n = 1/n$ for $n \in \mathbb{N}$. Then $a_n \le b_n \le c_n$, because $|\sin n| \le 1$, for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty}(-1/n) = \lim_{n\to\infty}(1/n) = 0$, it follows that

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

Whenever we deal with convergence, we examine only the behaviour of a sequence for very large indices. Therefore, we can always ignore any finite number of members. This observation gives rise to the following version of the sandwich theorem.

Corollary 3.22 (Bitten sandwich theorem)**.** *Suppose that $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, and $(c_n)_{n\in\mathbb{N}}$ are real sequences and there exists $N \in \mathbb{N}$ such that $a_n \leq b_n \leq c_n$ for all $n \geq N$. If $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ converge with*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n,$$

then $(b_n)_{n\in\mathbb{N}}$ converges as well and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

Proposition 3.23. *If $(a_n)_{n\in\mathbb{N}}$ is a real sequence such that $\lim_{n\to\infty} a_n = 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $(1/a_n)_{n\in\mathbb{N}}$ is divergent.*

Proof. We first prove that $1/|a_n| \to \infty$ as $n \to \infty$. To this end, let $M > 0$. Set $\epsilon = 1/M$ and choose $N \in \mathbb{N}$ such that $|a_n| < \epsilon$ for all $n \geq N$. Then $1/|a_n| > 1/\epsilon = M$ for all $n \geq N$. So indeed, we have shown that $1/|a_n| \to \infty$ as $n \to \infty$.

But this means that $(1/a_n)_{n\in\mathbb{N}}$ cannot be convergent, because if it were, then Proposition 3.14 would imply that $1/|a_n|$ is convergent as well, and we have just seen that this is not the case. □

For the next theorem we will need the following auxiliary result.

Lemma 3.24. *Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers such that*

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists. Then for any $s > r$ there exists a number $C > 0$ such that

$$a_n \leq Cs^n$$

for all $n \in \mathbb{N}$.

Proof. Fix $s > r$ and let $\epsilon = s - r$. Then there exists $N \in \mathbb{N}$ such that

$$\left|\frac{a_{n+1}}{a_n} - r\right| < \epsilon$$

for all $n \geq N$. But then

$$\frac{a_{n+1}}{a_n} \leq r + \epsilon = s$$

for $n \geq N$. Therefore,

$$a_{N+k} = \frac{a_{N+k}}{a_{N+k-1}} \frac{a_{N+k-1}}{a_{k+k-2}} \cdots \frac{a_{N+1}}{a_N} a_N \leq s^k a_N$$

for all $k \in \mathbb{N}$.

Define

$$C = \max\left\{\frac{a_1}{s}, \frac{a_2}{s^2}, \ldots, \frac{a_N}{s^N}\right\}.$$

Then it is clear that $a_n \leq Cs^n$ for $n \leq N$, and for $n > N$, we know that

$$a_n \leq a_N s^{n-N} \leq Cs^N s^{n-N} = Cs^n.$$

□

Theorem 3.25 (Growth factor test)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers. Suppose that*

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists (allowing divergence to infinity, so $0 \leq r \leq \infty$).

(i) If $0 \leq r < 1$, then $a_n \to 0$ as $n \to \infty$.

(ii) If $r > 1$, then $a_n \to \infty$ as $n \to \infty$.

If $r = 1$, then the test is inconclusive.

Proof. (i) If $r < 1$, we choose $s \in (r, 1)$ and use Lemma 3.24. We find $C > 0$ such that

$$0 < a_n \leq Cs^n$$

for all $n \in \mathbb{N}$. Then the result follows from the pinching theorem.

(ii) If $r > 1$, then we can apply statement (i) to the sequence $(1/a_n)_{n\in\mathbb{N}}$. If $r < \infty$, we find that

$$\frac{1/a_{n+1}}{1/a_n} = \frac{1}{\frac{a_{n+1}}{a_n}} \to \frac{1}{r} < 1$$

as $n \to \infty$ by Theorem 3.12. If $r = \infty$, then

$$\frac{1/a_{n+1}}{1/a_n} \to 0$$

as $n \to \infty$. (That's because for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $a_{n+1}/a_n > 1/\epsilon$ for all $n \geq N$, and then

$$\frac{1/a_{n+1}}{1/a_n} = \frac{1}{a_{n+1}/a_n} < \epsilon$$

for $n \geq N$.) In either case, we conclude that $(1/a_n)_{n\to\infty}$ converges to 0. Then $a_n \to \infty$ as $n \to \infty$, as shown in an exercise.

□

Example 3.26. Does the sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n = \frac{n}{2^n}$ converge?

Solution. We compute

$$\frac{a_{n+1}}{a_n} = \frac{2^n(n+1)}{2^{n+1}n} = \frac{1}{2}\left(1 + \frac{1}{n}\right) \to \frac{1}{2}$$

as $n \to \infty$. Hence the growth factor test says that $a_n \to 0$ as $n \to \infty$.

This example can be generalised.

Corollary 3.27. *Let $k \in \mathbb{N}$ and $a > 1$. Then*

$$\frac{n^k}{a^n} \to 0 \quad \textit{and} \quad \frac{a^n}{n^k} \to \infty$$

as $n \to \infty$.

Roughly speaking, this result says that exponential growth (as in a^n) beats polynomial growth (as in n^k) as $n \to \infty$.

Proof. We use the growth factor test:

$$\frac{\frac{(n+1)^k}{a^{n+1}}}{\frac{n^k}{a^n}} = \frac{1}{a}\left(1 + \frac{1}{n}\right)^k \to \frac{1}{a}$$

as $n \to \infty$. This proves the convergence of the first sequence, and if we take the reciprocal value, it also proves the divergence of the second sequence. □

3.4 Subsequences

Here we study sequences obtained from other sequences by selecting only some of the members.

Definition 3.28. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers and $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers. Then $(a_{n_k})_{k\in\mathbb{N}}$ is called a *subsequence* of $(a_n)_{n\in\mathbb{N}}$.

Example 3.29. Consider the alternating sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n = (-1)^n$ for all $n \in \mathbb{N}$. It is divergent, but contains two convergent subsequences. If we take all even indices, i.e., $n_k = 2k$ for $k \in \mathbb{N}$, then $a_{n_k} = (-1)^{2k} = 1$ for all k, so $(a_{n_k})_{k\in\mathbb{N}} = (1, 1, 1, \dots)$. If we take all odd indices, i.e., $n_k = 2k - 1$ for $k \in \mathbb{N}$, then $(a_{n_k})_{k\in\mathbb{N}} = (-1, -1, -1, \dots)$.

Proposition 3.30. *Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence converging to $L \in \mathbb{R}$. Then any subsequence of $(a_n)_{n\in\mathbb{N}}$ also converges to L.*

Proof. Consider a subsequence $(a_{n_k})_{k\in\mathbb{N}}$. Given $\epsilon > 0$, we can choose $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. Now choose $K \in \mathbb{N}$ such that $n_K \geq N$. Then for every $k \geq K$, we know that $n_k \geq N$ as well, since $(n_k)_{k\in\mathbb{N}}$ is increasing. Therefore,

$$|a_{n_k} - L| < \epsilon$$

for all $k \geq K$, and we conclude that $a_{n_k} \to L$ as $k \to \infty$. □

Corollary 3.31. *If a sequence $(a_n)_{n\in\mathbb{N}}$ has two subsequences converging to different limits, then $(a_n)_{n\in\mathbb{N}}$ is divergent.*

Example 3.32. The sequence from Example 3.29 has two subsequences that converge to 1 and -1, respectively.

The following is one of the key theorems concerning subsequences.

Theorem 3.33 (Bolzano-Weierstrass)**.** *Every bounded real sequence contains a convergent subsequence.*

Proof. We first want to find a monotone subsequence. To this end, define the set

$$S = \{n \in \mathbb{N} \colon a_m \leq a_n \text{ for all } m > n\}.$$

We distinguish two possible cases.

If S is infinite, we choose $n_1, n_2, \ldots \in S$ such that $n_1 < n_2 < \cdots$. Then it follows that $a_{n_1} \geq a_{n_2} \geq \cdots$, so the subsequence $(a_{n_k})_{k\in\mathbb{N}}$ is decreasing.

If S is finite, choose $n_1 = \max S + 1$. Since $n_1 \notin S$, there exists $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$. But then $n_2 > \max S$ as well, so $n_2 \notin S$. Therefore, there exists $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$. We may continue this construction indefinitely. Hence we obtain a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ that is strictly increasing.

In either case, the subsequence is bounded again. So Theorem 3.17 applies. The conclusion is that $(a_{n_k})_{k\in\mathbb{N}}$ converges. □

3.5 Cauchy sequences

The following notion can be used to characterise convergence.

Definition 3.34. A real sequence $(a_n)_{n\in\mathbb{N}}$ is called a *Cauchy sequence* if

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall m, n \geq N \colon |a_m - a_n| < \epsilon.$$

Theorem 3.35. *A sequence $(a_n)_{n\in\mathbb{N}}$ converges if, and only if, it is a Cauchy sequence.*

Proof. Suppose that $(a_n)_{n\in\mathbb{N}}$ is convergent. Then there exists a limit

$$L = \lim_{n\to\infty} a_n.$$

Now fix $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$|a_n - L| < \frac{\epsilon}{2}$$

for all $n \geq N$. Then for any pair of indices $m, n \geq N$,

$$|a_m - a_n| \leq |a_m - L| + |L - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by the triangle inequality. So the Cauchy condition is satisfied.

Now suppose that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Then there exists $N_1 \in \mathbb{N}$ such that $|a_m - a_n| < 1$ for all $m, n \geq N_1$. In particular,

$$|a_n| \leq |a_n - a_N| + |a_N| < |a_N| + 1$$

for all $n \geq N_1$. It follows that

$$|a_n| \leq \max\{|a_1|, \ldots, |a_N|\} + 1$$

for all $n \in \mathbb{N}$, and the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded.

By Theorem 3.33, there exists a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$. Let $L = \lim_{k\to\infty} a_{n_k}$. We claim that $a_n \to L$ as $n \to \infty$ as well.

In order to prove this statement, fix $\epsilon > 0$. Choose $N_0 \in \mathbb{N}$ such that $|a_m - a_n| < \epsilon/2$ for $m, n \geq N_0$ and choose $K \in \mathbb{N}$ such that $n_K \geq N_0$ and $|a_{n_k} - L| < \epsilon/2$ for all $k \geq K$. Define $N = \max\{N_0, n_K\}$. Then for any $n \geq N$,

$$|a_n - L| \leq |a_n - a_{n_K}| + |a_{n_K} - L| < \epsilon.$$

Hence we have proved that $a_n \to L$ as $n \to \infty$. $\square$

It may not be immediately obvious why the concept of a Cauchy sequence is useful. After all, the condition in the definition seems more complicated than the definition of convergence. But the key difference is that in order to verify convergence directly, you need to have an idea of what the limit is. For Cauchy sequences, you can simply check the condition and decide whether or not it is satisfied.