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Mathematical Tripos Part III: Lent Term 2016/17

Topics in Mathematics of Information – Exercise Sheet I

1. Suppose $\hat{f}$ is piecewise smooth and continuous and $\hat{f}(\xi) = 0$ for all $|\xi| > B\pi$. Show that

$$f = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{B}\right)\varphi_k, \qquad \text{where} \qquad \varphi_k = \frac{\sin(\pi(B\cdot - k))}{\pi(B\cdot - k)}.$$

and that

$$f_N = \sum_{|k|\leqslant N} f\left(\frac{k}{B}\right)\varphi_k \to f \qquad \text{in } L^\infty.$$

(Use the fact that $\left\{\frac{1}{\sqrt{2B\pi}} e^{-ikB^{-1}\cdot} \; ; \; k \in \mathbb{Z}\right\}$ is an orthonormal basis of $L^2[-B\pi, B\pi]$.)

2. Given a sequence of closed subspace $\{V_j\}_{j\in\mathbb{Z}}$, suppose that $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$. Prove that if $\{\varphi_{0,k} \; ; \; k \in \mathbb{Z}\}$ is an orthonormal basis of V_0, then $\{\varphi_{j,k} \; ; \; k \in \mathbb{Z}\}$ is an orthonormal basis of V_j.

3. Let $\varphi = \mathbb{1}_{[0,1]}$. Since $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ forms an orthonormal basis, we know that $\sum_k |\hat{\varphi}(\xi + 2k\pi)|^2 = 1$ for a.e. $\xi \in \mathbb{R}$. Use this to show that

$$\sum_{k\in\mathbb{Z}} \frac{4}{(\xi + 2k\pi)^2} = \frac{1}{\sin^2(\xi/2)}.$$

4. Let V_j be the space of all $f \in L^2(\mathbb{R})$ that are continuous and piecewise linear, with corners only at the points $k/2^j$ for $k \in \mathbb{Z}$.

   (a) Show that $\{V_j\}_{j\in\mathbb{Z}}$ satisfies conditions (I) to (IV) in the definition of an MRA (see the definition given in the lecture notes).

   (b) Let
$$\Delta = \mathbb{1}_{[0,1]} \star \mathbb{1}_{[0,1]}.$$
   Show that $\{\Delta(\cdot - k)\}_{k\in\mathbb{Z}}$ is a (nonorthonormal) basis of V_0.

   (c) Let φ be defined via its Fourier transform:
$$\hat{\varphi} = \frac{\hat{\Delta}}{\sqrt{\sum_n \left|\hat{\Delta}(\cdot + 2\pi n)\right|^2}}.$$
   Show that
$$\hat{\varphi} = e^{-i\xi} \frac{4\sin^2(\xi/2)}{\xi^2\sqrt{1 - \frac{2}{3}\sin^2(\xi/2)}}$$
   and that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis of V_0.

   (d) Show that the low pass filter of φ is
$$m(\xi) = \frac{e^{-i\xi}\cos^2(\xi/2)\sqrt{1 - \frac{2}{3}\sin^2(\xi/2)}}{\sqrt{1 - \frac{2}{3}\sin^2(\xi)}}$$
   and has an associated wavelet ψ with Fourier transform
$$\hat{\psi}(\xi) = -\frac{e^{i\xi/2}\sin^4(\xi/4)}{\xi^2/16} \frac{\sqrt{1 - \frac{2}{3}\cos^2(\xi/4)}}{\sqrt{1 - \frac{2}{3}\sin^2(\xi/2)}\sqrt{1 - \frac{2}{3}\sin^2(\xi/4)}}.$$
   Conclude that ψ has exactly 2 vanishing moments.

5. Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be a scaling function of an MRA. Recall from the lectures that $|\hat{\varphi}(0)| = 1$ (since $\varphi \in L^1$ implies that $\hat{\varphi}$ is continuous).

   (a) Show that $\hat{\varphi}(2\pi k) = 0$ for all $k \neq 0$.

   (b) Suppose that $\hat{\varphi}(0) = 1$. Show that
$$\sum_{n \in \mathbb{Z}} \varphi(t-n) = 1, \qquad a.e.\ t \in \mathbb{R}.$$

6. Let f be a function of support $[0,1]$ and suppose that f is equal to different polynomials of degree q on the intervals $\{[t_k, t_{k+1}]\}_{k=0}^{K-1}$, with $t_0 = 0$ and $t_K = 1$. Let ψ be a compactly supported wavelet with p vanishing moments, with support in $[-p, p+1]$. If $q < p$, compute the number of nonzero coefficients $\langle f, \psi_{j,n} \rangle$ at a fixed scale 2^j for $j \in \mathbb{N}$ sufficiently large. How should we choose p to minimize this number? If $q > p$, what is the maximum number of nonzero wavelet coefficients $\langle f, \psi_{j,n} \rangle$ at a fixed scale 2^j?

For the following questions, you are given this fact:

> $\psi \in L^2(\mathbb{R})$ is a wavelet (not necessarily derived from an MRA) if and only if
> $$\sum_{j \in \mathbb{Z}} \left| \hat{\psi}(2^j \xi) \right|^2 = 1, \qquad \sum_{j \in \mathbb{Z}} \left| \hat{\psi}(\xi + 2j\pi) \right|^2 = 1, \qquad \text{for } a.e.\ \xi \in \mathbb{R}. \tag{1}$$

7. Let φ be a scaling function associated with an MRA and let ψ be its associated wavelet. Prove that $|\hat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} \left| \hat{\psi}(2^j \xi) \right|^2$ for a.e. $\xi \in \mathbb{R}$.

8. * Let
$$\hat{\psi}(\omega) = \begin{cases} 1 & |\omega| \in [4\pi/7, \pi] \cup [4\pi, 4\pi + 4\pi/7], \\ 0 & \text{otherwise.} \end{cases}$$

   Using (1) or otherwise, prove that $\{\psi_{j,n}\ ;\ j, n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. Prove that ψ is not associated to a scaling function that generates an MRA.

   Hint: to show that ψ is not associated to an MRA, use the fact that MRA wavelets necessarily satisfy the property derived in question 7.