Coursework 1 for Inverse Problems (MA505250)

This coursework accounts for 20% of your final grade. Submit by 28th February.

Email cmhsp20@bath.ac.uk typed up solutions and your Matlab scripts.

Exercise 1 (Integral operators). For $\Omega = [0,1]^2$ and $\mathcal{U} = L^2(\Omega)$, we consider the integral operator $A : \mathcal{U} \to \mathcal{U}$ with

$$(Au)(y) \stackrel{\text{def.}}{=} \int_\Omega k(x,y)u(x)\,dx, \qquad y \in \Omega$$

for $k \in L^2(\Omega \times \Omega)$. Show that

(a) A is linear with respect to u,

(b) A is a bounded linear operator, i.e. $\|Au\|_{\mathcal{U}} \leqslant \|A\|_{\mathcal{L}(\mathcal{U},\mathcal{U})}\|u\|_{\mathcal{U}}$. Give also an estimate for $\|A\|_{\mathcal{L}(\mathcal{U},\mathcal{U})}$,

(c) the adjoint A^* is given via

$$(A^*v)(y) = \int_\Omega k(y,x)v(x)\,dx.$$

[3 Marks]

Exercise 2 (Inverse problem of differentiation). We consider the problem of differentiation, formulated as the inverse problem of finding u from $Au = f$ with the integral operator $A : L^2([0,1]) \to L^2([0,1])$ defined as

$$(Au)(y) \stackrel{\text{def.}}{=} \int_0^y u(x)\,dx\,.$$

(a) Let f be given by

$$f(x) := \begin{cases} 0 & x < \frac{1}{2}, \\ 1 & x > \frac{1}{2}. \end{cases}$$

Show that $f \in \overline{\mathcal{R}(A)}$.

(b) Let f be given as in a). Show that $f \in \overline{\mathcal{R}(A)} \setminus \mathcal{R}(A)$.

Hint. Consider the Picard criterion.

(c) Prove or falsify: *The Moore-Penrose inverse of A is continuous.*

[3 Marks]

Exercise 3 (Generalised inverse). 1. Let $m, n \in \mathbb{N}$ with $m \geqslant n \geqslant 2$. Compute the Moore-Penrose inverses of the following matrices:

(a) $A = (1, 1, \ldots, 1) \in \mathbb{R}^{1\times n}$

(b) $A = \operatorname{diag}(a_1, \ldots, a_n) \in \mathbb{R}^{n\times n}$ with $a_j \in \mathbb{R}$ for $j \in \{1, \ldots, n\}$

(c) $A \in \mathbb{R}^{m\times n}$ with $A^TA = I_n$

2. Let $a, b \in \mathbb{R}$ with $a < b$. Compute the Moore-Penrose inverse of the operator $A : L^2([a,b]) \to \mathbb{R}$ with

$$Au = \int_a^b u(x)\,dx.$$

[3 Marks]

Exercise 4 (Convolution). Many forward problems are either modelled as convolutions or they are modelled as the composition of several components one of which is a convolution. Therefore convolutions play an important role in inverse problems. The inverse problem is to recover $U \in L^2(\Omega)$ from $\mathcal{A}U = H \star U$ for some kernel $H \in L^2(\Omega)$ where $\Omega = [0,1]^2$. Here, $\mathcal{A} : L^2(\Omega) \to L^2(\Omega)$.

In this exercise, we implement a discrete version of this problem. From now on, discretize the domain Ω into an $n \times n$ evenly spaced grid, let $A : u \in \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be defined by $Au = h \star u$ where $h \in \mathbb{R}^{n\times n}$ is a discretised version of H and $\star$ denotes the circular convolution operation.

By the convolution theorem, we know that given $f = Au$,

$$u = (2\pi)^{-n/2}\mathcal{F}^{-1}\left(\frac{\mathcal{F}(f)}{\mathcal{F}(h)}\right) \tag{1}$$

where $\mathcal{F}$ denotes the discrete Fourier transform and the division is a pointwise division.

(a) Implement this inversion formula in Matlab, taking u to be the Matlab phantom `phantom(256)` and h to be the discretized Gaussian filter with $n = 256$ and $s = 3$:

$$\begin{aligned}
&\mathtt{x} = [0 : \mathtt{n}/2 - 1, -\mathtt{n}/2 : -1];\\
&[\mathtt{Y}, \mathtt{X}] = \mathtt{meshgrid}(\mathtt{x}, \mathtt{x});\\
&\mathtt{h} = \mathtt{exp}((-\mathtt{X}.^2 - \mathtt{Y}.^2)/(2 * \mathtt{s}^2));\\
&\mathtt{h} = \mathtt{h}/\mathtt{sum}(\mathtt{h}(:));
\end{aligned}$$

Demonstrate empirically that this is highly unstable to noise.

(b) Consider the Tikhonov regularised solution

$$\mathrm{argmin}_u\ \frac{1}{2}\left\|Au - f\right\|_2^2 + \frac{\alpha}{2}\left\|u\right\|^2.$$

Using the convolution theorem, give a closed form expression for the regularised solution u (similar to (1)), implement this reconstruction in Matlab for different values of α and empirically show that it is stable.

(c) Demonstrate the use of Mozorov's discrepancy principle for the selection of α, given noisy measurements $f = Au + e$ and an estimate of $\|e\|_2$.

[5 Marks]

Exercise 5 (The Radon transform). (a) The Matlab command `f = radon(u, phi);` computes a discretised two-dimensional radon transform of a discrete image `u` for a vector of angles `phi`. Use this command to set up a matrix `R` that maps the column-vector representation of `u` into the column-vector representation of the sinogram `f` for an arbitrary image $\mathtt{u} \in \mathbb{R}^{64\times 64}_{\geqslant 0}$ and angles `phi` with `phi(j) = j` for $j \in \{0, 2, \ldots, 178\}$.

(b) Create a noisy sinogram by applying `R` to a down-sampled version of the Shepp-Logan phantom (built-in in Matlab; use the command `phantom`) and subsequently adding non-negative, random numbers to the sinogram. Create multiple versions with different noise levels.

(c) Compute a singular value decomposition of `R` via the Matlab command `svd` and visualise selected singular vectors of your choice.

(d) Create a 'pseudo'-inverse of `R` by constructing an appropriate matrix with inverted singular values and apply this matrix to the column-vector representations of your noisy sinograms. Regularise the Moore-Penrose inverse using

  (i) Truncated singular value decomposition;

  (ii) Tikhonov regularisation.

[5 Marks]