

An introduction to nonsmooth optimisation

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Descent methods

Gradient descent

Subgradient descent

Projected gradient descent

Douglas-Rachford splitting

Alternating Direction Method of Multipliers (ADMM)

Primal-Dual splitting

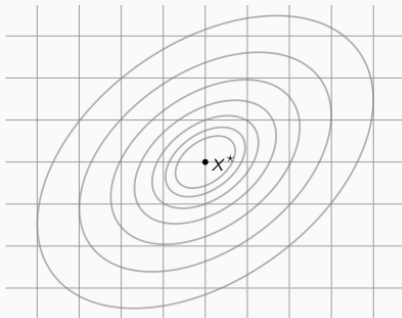
Descent methods for smooth problems

Unconstrained smooth optimisation

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$$\min_{x \in \mathbb{R}^n} F(x)$$

where $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is a proper, convex and differentiable function.



Unconstrained smooth optimisation

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where $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is a proper, convex and differentiable function.

- Assume that the set of minimisers is nonempty,

$$\operatorname{argmin}(F) = \left\{ x \in \mathbb{R}^n ; F(x) = \min_{x \in \mathbb{R}^n} F(x) \right\} \neq \emptyset$$

- $x_* \in \operatorname{argmin}(F)$ has no closed form expression in general.
- We will consider iterative algorithms which start at a point x_0 and build a sequence x_k which converge to a minimiser.

Descent methods for smooth problems

General iterative method

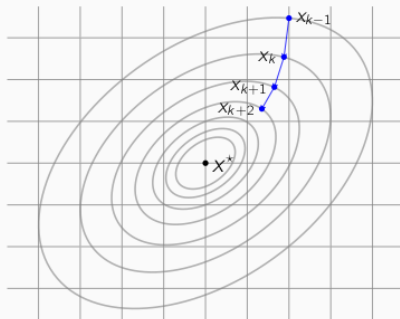
Initialization $x_0 \in \text{dom}(F)$

while *stopping criterion not satisfied* **do**

 choose step-size $\gamma_k > 0$ and a search direction d_k

 Update $x_{k+1} = x_k - \gamma_k d_k$

end

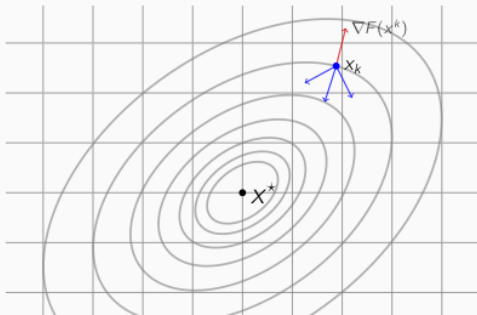


Choice of d_k to enforce descent

Descent methods

An algorithm is called a descent method if $F(x_{k+1}) < F(x_k)$.

- $\varphi_k(\gamma) \stackrel{\text{def.}}{=} F(x_k + \gamma d_k)$ is a decreasing function
- So, for d_k to be a descent direction, we need $\varphi'_k(0) = \langle \nabla F(x_k), d_k \rangle < 0$.



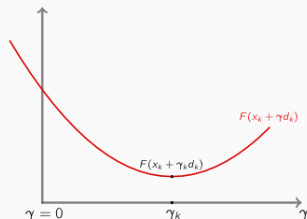
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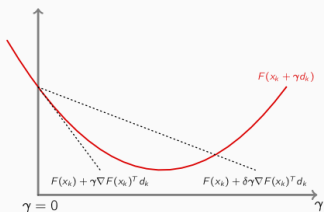
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3. Backtracking: Given direction d_k , choose $\delta \in (0, 1)$ and $\beta \in (0, 1)$ and let $\gamma = 1$.

while $F(x_k + \gamma d_k) > F(x_k) + \delta \gamma \langle \nabla F(x_k), d_k \rangle$: $\gamma = \beta \gamma$.



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- Function value: $F(x_{k+1}) - F(x_k) \leq \varepsilon$.
- Sequence residue: $\|x_k - x_{k+1}\| \leq \varepsilon$.
- Optimality condition: $\|\nabla F(x_k)\| \leq \varepsilon$.

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The method of steepest descent or gradient descent chooses $d_k = -\nabla F(x_k)$. If x_k is not a stationary point, then $\nabla F(x_k) \neq 0$ and

$$\langle \nabla F(x_k), d_k \rangle = -\|\nabla F(x_k)\|^2 < 0$$

Gradient descent

Initialization $x_0 \in \text{dom}(F)$

while *stopping criterion not satisfied* **do**

 choose step-size $\gamma_k > 0$

 Update $x_{k+1} = x_k - \gamma_k \nabla F(x_k)$

end

NB: method may not converge if γ_k is too large.

Convergence of gradient descent

Let $F \in \mathcal{C}^1$, ∇F is L -Lipschitz and $\operatorname{argmin}(F) \neq \emptyset$. Let $\gamma_k \equiv \gamma$.

General case

Choosing fixed $\gamma \in (0, 2/L)$, $\nabla F(x_k) \rightarrow 0$.

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Convex case

Suppose that F is **convex** and let x_* be a minimiser. For $\gamma \in (0, 2/L)$,

$$F(x_k) - F(x_*) \leq \frac{\|x_0 - x_*\|^2}{\theta(k+1)}, \quad \text{where } \theta = \gamma(1 - \gamma L/2). \quad (2.1)$$

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Strongly-convex case

If F is **α -strongly convex**, that is, $F(x) - \alpha \|x\|^2/2$ is convex (if $F \in \mathcal{C}^2$, equivalent to $\nabla^2 F(x) \geq \alpha \operatorname{Id}$), then

$$\|x_{k+1} - x_*\| \leq \max(1 - \gamma\alpha, \gamma L - 1) \|x_k - x_*\|.$$

Best constant is $\gamma = 2/(L + \alpha)$:

$$\|x_k - x_*\| \leq q^k \|x_0 - x_*\| \quad \text{where } q = (L - \alpha)/(L + \alpha) \in (0, 1).$$

Suppose that x_k is an element of

$$x_0 + \text{Span}\{\nabla F(x_0), \nabla F(x_1), \dots, \nabla F(x_{k-1})\}. \quad (2.2)$$

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Theorem 2.1 (Nesterov's lower complexity bound)

For any $n \geq 2$ and $x_0 \in \mathbb{R}^n$, $L > 0$ and $k < n$, there exists a convex one-time continuously differentiable function F with L -Lipschitz continuous gradient, such that for any algorithm satisfying (2.2), we have

$$F(x_k) - F(x_*) \geq \frac{L \|x_0 - x_*\|^2}{8(k+1)^2}$$

where x_ denotes a minimiser of F .*

- This result is valid only when the number of iterations is smaller than the problem size.

Suppose that x_k is an element of

$$x_0 + \text{Span}\{\nabla F(x_0), \nabla F(x_1), \dots, \nabla F(x_{k-1})\}. \quad (2.2)$$

Theorem 2.1 (Lower complexity bound for strongly convex functions)

For any $x_0 \in \ell_2(\mathbb{N})$ and $\gamma, L > 0$, there exists a γ -strongly convex one times continuously differentiable function f with L -Lipschitz gradient such that for any algorithm satisfying (2.2), we have for all k

$$f(x_k) - f(x_*) \geq \frac{\gamma}{2} \left(\frac{\sqrt{Q} - 1}{\sqrt{Q} + 1} \right)^{2k} \|x_0 - x_*\|^2.$$

where $Q = L/\gamma \geq 1$ is the condition number and x_ is a minimiser of f .*

Accelerated gradient descent

Accelerated gradient descent

Initialization $x_0 = \bar{x}_0 \in \text{dom}(F)$ and $\lambda_0 = 0$

while *stopping criterion not satisfied* **do**

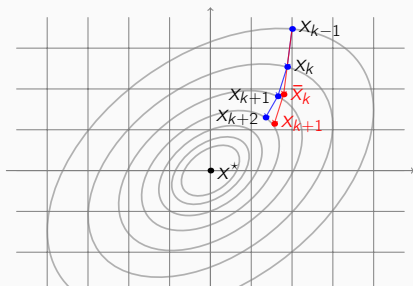
 choose step-size $\gamma_k > 0$

 Update $x_{k+1} = \bar{x}_k - \gamma_k \nabla F(\bar{x}_k)$

$$\lambda_k = \frac{1 + \sqrt{1 + 4\lambda_{k-1}^2}}{2} \text{ and } a_k = \frac{1 - \lambda_k}{\lambda_{k+1}}$$

$$\bar{x}_{k+1} = x_{k+1} + a_k(x_{k+1} - x_k)$$

end



If F is L -Lipschitz gradient, then by choosing $\gamma_k = 1/L$, AGD achieves $\mathcal{O}(1/k^2)$ convergence rate

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Let $R : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ be proper, convex and lower semi-continuous, but non-differentiable. Assume that $\operatorname{argmin}(R) \neq \emptyset$ and consider

$$\min_{x \in \mathbb{R}^n} R(x). \tag{3.1}$$

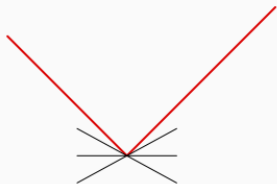
Nonsmooth optimisation

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Recall: the subdifferential of R at x is the set

$$\partial R(x) = \{p \in \mathbb{R}^n ; R(y) - R(x) \geq \langle p, y - x \rangle, \quad \forall y \in \mathbb{R}^n\}.$$



$$\partial \|x\|_1 = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}.$$

Note that $\partial R(x) = \{\nabla R(x)\}$ when R is differentiable at x .

Subgradient descent

Subgradient descent

Initialization $x_0 \in \text{dom}(R)$

while *stopping criterion not satisfied* **do**

 choose step-size $\gamma_k > 0$ and a subgradient $g_k \in \partial R(x_k)$

 Update $x_{k+1} = x_k - \gamma_k g_k$

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In general, if x_* is a solution

$$\begin{aligned}\|x_{i+1} - x_*\|^2 &= \|x_i - x_*\|^2 + \|\gamma_i g_i\|^2 - 2\gamma_i \langle g_i, x_i - x_* \rangle \\ &\leq \|x_i - x_*\|^2 + \|\gamma_i g_i\|^2 - 2\gamma_i (R(x_i) - R(x_*))\end{aligned}$$

So, summing from $i = 0, \dots, k$ and rearranging, we have

$$\sum_{i=0}^k \gamma_i (R(x_i) - R(x_*)) \leq \|x_0 - x_*\|^2 + \sum_{i=0}^k \gamma_i^2 \|g_i\|^2 - \|x_{k+1} - x_*\|^2$$

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If R is L -Lipschitz,

$$2 \min_{i=0}^k (R(x_i) - R(x_*)) \leq \frac{\|x_0 - x_*\|^2 + L \sum_{i=0}^k \gamma_i^2}{2 \sum_{i=0}^k \gamma_i}$$

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- In general choose $\sum_i \gamma_i^2 < \infty$ and $\sum_i \gamma_i = +\infty$, so γ_i converges to 0.
- Choosing $\gamma_i \equiv C/\sqrt{k+1}$ for k iterations, then

$$\min_{i=0}^k (R(x_i) - R(x_*)) \leq \frac{\|x_0 - x_*\|^2 + LC^2}{2C\sqrt{k+1}}.$$

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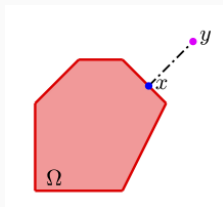
Douglas-Rachford splitting

Alternating Direction Method of Multipliers (ADMM)

Primal-Dual splitting

Indicator function: Let $\Omega \subseteq \mathbb{R}^n$

$$\iota_{\Omega}(x) = \begin{cases} +\infty & x \notin \Omega \\ 0 & x \in \Omega. \end{cases}$$



Projection onto Ω :

$$\mathcal{P}_{\Omega}(y) \stackrel{\text{def.}}{=} \operatorname{argmin}_{x \in \Omega} \|x - y\|.$$

Constrained smooth optimisation

Let $F \in \mathcal{C}^1$ with L -Lipschitz gradient and let $\Omega \subset \mathbb{R}^n$ be a closed convex set, and consider

$$\min_{x \in \Omega} F(x). \quad (4.1)$$

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Projected gradient descent method

Initialization $x_0 \in \Omega$

while *stopping criterion not satisfied* **do**

 choose step-size $\gamma_k \in (0, 2/L)$

 Update $x_{k+1/2} = x_k - \gamma_k \nabla F(x_k)$

 Project $x_{k+1} = \mathcal{P}_\Omega(x_{k+1/2})$

end

- Hyperplane: $\Omega = \{x : a^T x = b\}$, $a \neq 0$

$$\mathcal{P}_\Omega = x + \frac{b - a^T x}{\|a\|^2} a.$$

- Affine subspace: $\Omega = \{x : Ax = b\}$ with $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = m < n$

$$\mathcal{P}_\Omega = x + A^T(AA^T)^{-1}(b - Ax).$$

- Half space: $\Omega = \{x : a^T x \leq b\}$, $a \neq 0$

$$\mathcal{P}_\Omega = x + \frac{b - a^T x}{\|a\|^2} a \quad \text{if } a^T x > b \quad \text{and} \quad x \quad \text{if } a^T x \leq b.$$

- Nonnegative orthant: $\Omega = \mathbb{R}_+^n$

$$\mathcal{P}_\Omega = (\max\{0, x_i\})_i.$$

Composite optimisation

$$\min_{x \in \mathbb{R}^n} \Phi(x) \stackrel{\text{def.}}{=} F(x) + R(x). \quad (4.2)$$

where $F \in \mathcal{C}^1$ has L -Lipschitz gradient and R is proper, convex, lower semi-continuous but nonsmooth.

Note that the constrained smooth problem can be rewritten as

$$\min_{x \in \mathbb{R}^n} F(x) + \iota_{\Omega}(x).$$

Proximal mapping

Let R be proper, convex, lower semicontinuous and bounded from below. Its proximal mapping is defined by

$$\text{prox}_{\gamma R}(y) \stackrel{\text{def.}}{=} \operatorname{argmin}_{x \in \mathbb{R}^n} \gamma R(x) + \frac{1}{2} \|x - y\|^2.$$

Note that this is precisely the projection operator \mathcal{P}_Ω when $R = \iota_\Omega$.

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Note that this is precisely the projection operator \mathcal{P}_Ω when $R = \iota_\Omega$.

Optimality condition denote $y_+ \stackrel{\text{def.}}{=} \text{prox}_{\gamma R}(y)$,

$$\begin{aligned} 0 \in \gamma \partial R(y_+) + y_+ - y &\iff y \in (\text{Id} + \gamma \partial R)(y_+) \\ &\iff y_+ = (\text{Id} + \gamma \partial R)^{-1}(y). \end{aligned}$$

Proximal gradient descent

Initialization $x_0 \in \text{dom}(\Phi)$

while *stopping criterion not satisfied* **do**

 choose step-size $\gamma_k \in (0, 2/L)$

 Update $x_{k+1/2} = x_k - \gamma_k \nabla F(x_k)$

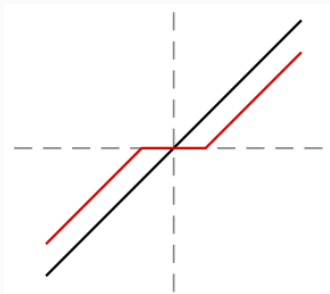
 Project $x_{k+1} = \text{prox}_{\gamma_k R}(x_{k+1/2})$

end

Example

Soft-threshold: $R(x) = |x|$,

$$\text{prox}_{\gamma R}(y) = \mathcal{T}_{\gamma}(y) = \begin{cases} y - \gamma : y > \gamma, \\ 0 : y \in [-\gamma, \gamma], \\ y + \gamma : y < -\gamma. \end{cases}$$



Consider the Lasso problem where for $y \in \mathbb{R}^m$ and a matrix $K \in \mathbb{R}^{n \times m}$,

$$R(x) = \lambda \|x\|_1 \quad \text{and} \quad F(x) = \frac{1}{2} \|Kx - y\|^2$$

- $\nabla F(x) = K^*(Kx - y)$
- $L = \|K^*K\|$
- The proximal mapping of γR is the soft-thresholding operator \mathcal{T}_γ

The iterates are therefore, for $\gamma_k \in (0, 1/\|K^*K\|]$:

$$x_{k+1} = \mathcal{T}_{\lambda\gamma_k}(x_k - \gamma_k(K^*(Kx_k - y)))$$

Specialised to the ℓ_1 case, this is sometimes known as the iterative soft thresholding algorithm (ISTA).

This is also known as Forward-Backward splitting:

$$x_{k+1} = \text{prox}_{\gamma R}(x_k - \gamma \nabla F(x_k))$$

- forward: gradient descent set in F
- backward: implicit step in R

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- forward: gradient descent set in F
- backward: implicit step in R

By definition of prox,

$$\begin{aligned}x_{k+1} &= \operatorname{argmin}_x \left\{ \frac{1}{2} \|x - (x_k - \gamma \nabla F(x_k))\|^2 + \gamma R(x) \right\} \\&= \operatorname{argmin}_x \left\{ \frac{1}{2} \|x - x_k\|^2 + \gamma \langle x - x_k, \nabla F(x_k) \rangle + \gamma R(x) \right\} \\&= \operatorname{argmin}_x \left\{ \color{red}{F(x_k)} + \langle x - x_k, \nabla F(x_k) \rangle + \frac{1}{2\gamma} \|x - x_k\|^2 + R(x) \right\}\end{aligned}$$

So, x_{k+1} minimises $R(x)$ plus majorisation of $F(x)$ at x_k if $\gamma \leq \frac{1}{L}$

Convergence properties

FB is a descent method.

- For $\gamma_k \equiv \gamma \in (0, 1/L]$, x_k converges to a minimiser and

$$\Phi(x_k) - \Phi(x_*) \leq \frac{1}{2\gamma k} \|x_0 - x_*\|^2.$$

- If F and R are strongly convex with parameters μ_F , μ_R and $\mu \stackrel{\text{def.}}{=} \mu_F + \mu_R > 0$, then

$$\Phi(x_k) - \Phi(x_*) \leq \omega^k \frac{(1 + \gamma\mu_R)}{2\gamma} \|x_0 - x_*\|^2,$$

where $\omega = (1 - \gamma\mu_F)/(1 + \gamma\mu_R)$.

The convergence rate matches that of gradient descent, although faster convergence rates can be obtained by

- incorporating Nesterov acceleration. This is called FISTA (fast iterative soft thresholding).
- Another (very effective) acceleration technique for FB is the restarted-FISTA scheme.

Other examples of proximal operators

Quadratic function $R(x) = \frac{1}{2}x^T A x + b^T x + c$, $A \succeq 0$

$$\text{prox}_{\gamma R}(y) = (\text{Id} + \gamma A)^{-1}(y - \gamma b).$$

Euclidean norm $R(x) = \|x\|$

$$\text{prox}_{\gamma R}(y) = \begin{cases} (1 - \frac{\gamma}{\|y\|})y : \|y\| > \gamma, \\ 0 : \text{o.w.} \end{cases}$$

Nuclear norm $R(x) = \sum_i \sigma_i$

$$\text{prox}_{\gamma R}(y) = U \mathcal{T}_{\gamma}(\Sigma) V^T.$$

Calculus rules for proximal operators

Quadratic perturbation $H(x) = R(x) + \frac{\alpha}{2} \|x\|^2 + \langle x, u \rangle + \beta, \alpha \geq 0$

$$\text{prox}_H = \text{prox}_{R/(\alpha+1)} \left(\frac{x - u}{\alpha + 1} \right).$$

Translation $H(x) = R(x - z)$

$$\text{prox}_H = z + \text{prox}_R(x - z).$$

Scaling $H(x) = R(x/\rho)$

$$\text{prox}_H = \rho \text{prox}_{R/\rho^2} \left(\frac{x}{\rho} \right).$$

Reflection $H(x) = R(-x)$

$$\text{prox}_H = -\text{prox}_R(-x).$$

Composition $H = R \circ L$ with L being bijective bounded linear mapping such that $L^{-1} = L^*$,

$$\text{prox}_H = L^* \circ \text{prox}_R \circ L.$$

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Two nonsmooth terms

We now let $F, G \in \Gamma_0(\mathbb{R}^n)$ and consider

$$\min_x F(x) + G(x) \tag{5.1}$$

Define the reflection operator:

$$\text{rprox}_F \stackrel{\text{def.}}{=} 2\text{prox}_F - \text{Id}.$$

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Initialization $x_0 \in \text{dom}(\Phi)$, $\gamma > 0$, $\mu \in (0, 2)$

while *stopping criterion not satisfied* **do**

$x_k = \text{prox}_{\gamma F}(z_k)$
 $z_{k+1} = (1 - \frac{1}{2}\mu)z_k + \frac{1}{2}\mu \text{rprox}_{\gamma G}(\text{rprox}_{\gamma F}(z_k))$

end

Convergence

- There is guaranteed **convergence** to a fixed point: $z_k \rightarrow z_*$ if $\gamma > 0$ and $\mu \in (0, 2)$.

Convergence

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(i) $\text{prox}_{\gamma F}$ is the soft thresholding operator.

$$\begin{aligned} \text{(ii) } \text{prox}_{\gamma G}(x) &= \operatorname{argmin}_z \left\{ \|z - x\|_2^2 ; Az = b \right\} \\ &= x + \operatorname{argmin}_y \left\{ \|y\|_2^2 ; A(y + x) = b \right\} \\ &= x + A^\dagger(b - Ax). \end{aligned}$$

Let x_* be a solution, then $0 \in \partial F(x_*) + \partial G(x_*)$.

Intuition for the Douglas-Rachford iterates

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The fixed point formulation

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leads to the fixed point iterations

$$\begin{aligned} x_k &= \text{prox}_{\gamma F}(z_k) \\ z_{k+1} &= z_k + \mu (\text{prox}_{\gamma G}(2x_k - z_k) - x_k) \end{aligned}$$

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Substituting the first line into the second line gives:

$$\begin{aligned} z_{k+1} &= z_k + \mu (\text{prox}_{\gamma G}(\text{rprox}_{\gamma F}(z_k)) - \text{prox}_{\gamma F}(z_k)) \\ &= (1 - \frac{\mu}{2})z_k + \frac{\mu}{2} \text{rprox}_{\gamma G}(\text{rprox}_{\gamma F}(z_k)) \end{aligned}$$

Descent methods

Gradient descent

Subgradient descent

Projected gradient descent

Douglas-Rachford splitting

Alternating Direction Method of Multipliers (ADMM)

Primal-Dual splitting

Alternating Direction Method of Multipliers (ADMM)

We now consider the following constrained optimisation problem:

$$\min_{x,y} F(x) + G(y) \quad \text{such that} \quad Ax + By = \zeta, \quad (6.1)$$

where $F, G \in \Gamma_0(\mathbb{R}^n)$.

The **augmented Lagrangian** formulation is

$$\min_{x,y} \sup_{\psi} F(x) + G(y) - \langle \psi, Ax + By - \zeta \rangle + \frac{\gamma}{2} \|Ax + By - z\|^2 \quad (6.2)$$

Note that the two formulations are equivalent since the supremum is $+\infty$ if $Ax + By \neq z$.

The ADMM iterations: alternate between descents on x , y and ascent on ψ

$$\min_{x,y} \sup_{\psi} F(x) + G(y) - \langle \psi, Ax + By - \zeta \rangle + \frac{\gamma}{2} \|Ax + By - z\|^2 \quad (6.3)$$

ADMM

Initialization $y_0, \psi_0, \gamma > 0$

while *stopping criterion not satisfied* **do**

$$x_{k+1} \in \operatorname{argmin}_x F(x) + G(y_k) - \langle \psi_k, Ax + By_k - \zeta \rangle + \frac{\gamma}{2} \|Ax + By_k - \zeta\|^2$$

$$y_{k+1} \in \operatorname{argmin}_y F(x_{k+1}) + G(y) - \langle \psi_k, Ax_{k+1} + By - \zeta \rangle + \frac{\gamma}{2} \|Ax_{k+1} + By - \zeta\|^2$$

$$\psi_{k+1} = \psi_k + \gamma(\zeta - Ax_{k+1} - By_{k+1})$$

end

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ADMM

Initialization $y_0, \psi_0, \gamma > 0$

while *stopping criterion not satisfied* **do**

$$x_{k+1} \in \operatorname{argmin}_x F(x) - \langle \psi_k, Ax \rangle + \frac{\gamma}{2} \|Ax + By_k - \zeta\|^2$$

$$y_{k+1} \in \operatorname{argmin}_y G(y) - \langle \psi_k, By \rangle + \frac{\gamma}{2} \|Ax_{k+1} + By - \zeta\|^2$$

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end

Consider the Lasso example again:

$$\min_{x,y} \lambda \|y\|_1 + \frac{1}{2} \|Kx - b\|^2 \text{ such that } x - y = 0.$$

Let

- $F(x) = \frac{1}{2} \|Kx - b\|^2$, $G(y) = \lambda \|y\|_1$
- $A = \text{Id}$, $B = -\text{Id}$ and $\zeta = 0$.

The ADMM iterates are:

$$x_{k+1} = (\gamma \text{Id} + K^* K)^{-1} (K^* b + \gamma y_k + \psi_k)$$

$$y_{k+1} = \mathcal{T}_{\lambda/\gamma}(x_{k+1} - \psi_k/\gamma)$$

$$\psi_{k+1} = \psi_k + \gamma(y_{k+1} - x_{k+1}).$$

Recall that ADMM solves the problem:

$$\min_{x,y} F(x) + G(y) \text{ s.t. } Ax + By = \zeta.$$

Equivalence to Douglas-Rachford

Recall that ADMM solves the problem:

$$\min_{x,y} F(x) + G(y) \text{ s.t. } Ax + By = \zeta.$$

The dual problem is

$$\begin{aligned} & \max_p \min_{x,y} F(x) + G(y) + \langle p, \zeta - Ax - By \rangle \\ &= \max_p \min_x (-\langle A^* p, x \rangle + F(x)) + \min_y (-\langle B^* p, y \rangle + G(y)) + \langle p, \zeta \rangle \\ &= \max_p -F^*(A^* p) - G^*(B^* p) + \langle p, \zeta \rangle \end{aligned}$$

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Equivalence

ADMM is equivalent to applying DR on the dual problem

$$\min_p \tilde{F}_\zeta^*(p) + \tilde{G}^*(p)$$

where $\tilde{G}^*(p) \stackrel{\text{def.}}{=} G^*(B^* p)$ and $\tilde{F}_\zeta^*(p) \stackrel{\text{def.}}{=} F^*(A^* p) - \langle p, \zeta \rangle$.

Equivalence to Douglas-Rachford

To prove the equivalence between ADMM and DR, we will rewrite the ADMM iterates in terms of $\xi_k \stackrel{\text{def.}}{=} Ax_k$, $\eta_k \stackrel{\text{def.}}{=} By_k$ and the functions \tilde{G}^* and \tilde{F}_ζ^* .

We will make use of the following lemma:

Lemma 1

Suppose that F^ is continuous at A^*p and G^* is continuous at B^*q . Define*

$$\tilde{F}(\xi) = \min_{Ax=\xi} F(x) \quad \text{and} \quad \tilde{G}(\eta) = \min_{By=\eta} G(y)$$

Then,

- *the minimums are achieved (thanks to Fenchel Rockafellar duality).*
- *$\tilde{G}^*(q) = G^*(B^*q)$ and $\tilde{F}^*(p) = F^*(A^*p)$.*

Moreover, defining $\tilde{F}_\zeta^(p) = \tilde{F}^*(p) - \langle \zeta, p \rangle$, we have*

$$\text{prox}_{\gamma \tilde{F}_\zeta^*}(x) = \text{prox}_{\gamma \tilde{F}^*}(x + \gamma \zeta).$$

Step 1: rewrite the iterates in terms of $\xi_k \stackrel{\text{def.}}{=} Ax_k$ and $\eta_k \stackrel{\text{def.}}{=} By_k$

On ξ_k

Recall the iteration on x_k is

$$x_{k+1} \in \operatorname{argmin}_x F(x) - \langle \psi_k, \xi \rangle + \frac{\gamma}{2} \|\xi + \eta_k - \zeta\|^2 \text{ s.t. } Ax = \xi$$

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So, $\xi_{k+1} = Ax_{k+1}$ satisfies

$$\xi_{k+1} \in \operatorname{argmin}_{\xi} \tilde{F}(\xi) - \langle \psi_k, \xi \rangle + \frac{\gamma}{2} \|\xi + \eta_k - \zeta\|^2$$

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Therefore,

$$0 \in \partial \tilde{F}(\xi_{k+1}) - \psi_k + \gamma(\xi_{k+1} + \eta_k - \zeta)$$

which implies

$$\xi_k = \operatorname{prox}_{\tilde{F}/\gamma}(\psi_k/\gamma + \zeta - \eta_k).$$

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$$\xi_k = \operatorname{prox}_{\tilde{F}/\gamma}(\psi_k/\gamma + \zeta - \eta_k).$$

Similarly,

$$\eta_{k+1} = \operatorname{prox}_{\tilde{G}/\gamma}(\psi_k/\gamma + \zeta - \xi_{k+1})$$

Step 2: rewrite in terms of \tilde{F}^* and \tilde{G}^*

Moreau's identity

Let f be proper, convex and lower semicontinuous and let f^* be its convex conjugate. Then for $\gamma > 0$, $x = \text{prox}_{f/\gamma}(x) + \frac{1}{\gamma} \text{prox}_{\gamma f^*}(\gamma x)$

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Define

$$u_k \stackrel{\text{def.}}{=} \psi_k - \gamma\eta_k, \quad v_{k+1} \stackrel{\text{def.}}{=} \psi_k + \gamma(\zeta - \xi_{k+1}) \quad \text{and} \quad \tilde{F}_\zeta^*(p) \stackrel{\text{def.}}{=} \tilde{F}^*(p) - \langle \zeta, p \rangle.$$

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Then

(i) $v_{k+1} = \gamma\eta_k + \text{prox}_{\gamma \tilde{F}_\zeta^*}(u_k)$

(ii) From $\text{prox}_{\gamma \tilde{G}^*}(v_{k+1}) = v_{k+1} - \gamma\eta_{k+1} = \psi_{k+1}$,

$$u_{k+1} = \text{prox}_{\gamma \tilde{G}^*}(v_{k+1}) - \gamma\eta_{k+1} = 2\text{prox}_{\gamma \tilde{G}^*}(v_{k+1}) - v_{k+1} = \text{rprox}_{\gamma \tilde{G}^*}(v_{k+1})$$

The iterates are therefore

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The 3rd line of the ADMM iteration is

$$\psi_{k+1} = \psi_k + \gamma(\zeta - \xi_{k+1} - \eta_{k+1}) \stackrel{\text{def. of } v_k}{=} v_{k+1} - \gamma\eta_{k+1}$$

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By definition of u_k , we have $u_k = \psi_k - \gamma\eta_k = v_k - 2\gamma\eta_k$.

Equivalence to Douglas-Rachford

The iterates are therefore

$$\begin{bmatrix} v_{k+1} = \gamma\eta_k + \text{prox}_{\gamma\tilde{F}_\zeta^*}(u_k) \\ u_{k+1} = \text{rprox}_{\gamma\tilde{G}^*}(v_{k+1}) \end{bmatrix} \iff \begin{bmatrix} v_{k+1} = \gamma\eta_k + \frac{1}{2}u_k + \frac{1}{2}\text{rprox}_{\gamma\tilde{F}_\zeta^*}(u_k) \\ u_{k+1} = \text{rprox}_{\gamma\tilde{G}^*}(v_{k+1}) \end{bmatrix}$$

The 3rd line of the ADMM iteration is

$$\psi_{k+1} = \psi_k + \gamma(\zeta - \xi_{k+1} - \eta_{k+1}) \stackrel{\text{def. of } v_k}{=} v_{k+1} - \gamma\eta_{k+1}$$

By definition of u_k , we have $u_k = \psi_k - \gamma\eta_k = v_k - 2\gamma\eta_k$.

This is precisely the Douglas-Rachford iterations

$$v_{k+1} = \frac{1}{2}v_k + \frac{1}{2}\text{rprox}_{\gamma\tilde{F}_\zeta^*}(\text{rprox}_{\gamma\tilde{G}^*}(v_k))$$

for the minimisation of

$$\begin{aligned} \Phi(p) &\stackrel{\text{def.}}{=} \tilde{F}_\zeta^*(p) + \tilde{G}^*(p) \\ &= F^*(A^*p) + G^*(B^*p) - \langle \zeta, p \rangle. \end{aligned}$$

Descent methods

Gradient descent

Subgradient descent

Projected gradient descent

Douglas-Rachford splitting

Alternating Direction Method of Multipliers (ADMM)

Primal-Dual splitting

Consider the problem

$$\min_x F(Kx) + G(x).$$

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Then, by considering the Fenchel conjugate of

$$F(Kx) = \sup_y \langle Kx, y \rangle - F^*(y),$$

we have the saddle point problem

$$\min_x \sup_y \langle Kx, y \rangle - F^*(y) + G(x)$$

The primal dual splitting scheme essentially alternating between

- an implicit descent on x an implicit ascent on y

The optimality conditions of $\min_x \sup_y \langle Kx, y \rangle - F^*(y) + G(x)$ are

$$x_* - \tau K^* y_* \in x_* + \tau \partial G(x_*) \quad \text{and} \quad y_* + \sigma K x_* \in y_* + \sigma \partial F^*(y_*)$$

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Primal-Dual splitting

Initialization $x_0, y_0, \sigma, \tau > 0$

while *stopping criterion not satisfied* **do**

$$x_{k+1} = \text{prox}_{\tau G}(x_k - \tau K^* y_*)$$

$$y_{k+1} = \text{prox}_{\sigma F^*}(y_k + \sigma K(2x_{k+1} - x_k))$$

end

If $\sigma\tau \|K\|^2 < 1$, (x_k, y_k) converges to a fixed point (x_*, y_*) which is a solution to the saddle point problem if a solution exists.

The primal-dual gap

For all x and y ,

$$\begin{aligned}\min_{x'} \langle Kx', y \rangle - F^*(y) + G(x') &\leq \min_{x'} \max_{y'} \langle Kx', y' \rangle - F^*(y') + G(x') \\ &\leq \max_{y'} \langle Kx, y' \rangle - F^*(y') + G(x)\end{aligned}$$

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Defining the primal-dual gap as

$$\mathcal{G}(x, y) = \max_{y'} (\langle y', Kx \rangle - F^*(y') + G(x)) - \min_{x'} (\langle y, Kx' \rangle - F^*(y) + G(x')),$$

we have

$$\mathcal{G}(\bar{x}_n, \bar{y}_n) \leq \frac{C}{k}$$

where $\bar{x}_n = \frac{1}{n} \sum_{k=1}^n x_k$ and $\bar{y}_n \stackrel{\text{def.}}{=} \frac{1}{n} \sum_{k=1}^n y_k$.

Example: Let D be the finite differences operator and $A \in \mathbb{R}^{m \times n}$.

$$\min_{x \in \mathbb{R}^n} \|Dx\|_1 + \frac{1}{2} \|Ax - b\|_2^2$$

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- For $\text{prox}_{\tau G}$, note that $z = \text{prox}_{\tau G}(x)$ satisfies

$$z \in \operatorname{argmin}_z \frac{1}{2} \|z - x\|^2 + \frac{\tau}{2} \|Az - b\|^2 \iff z = (\text{Id} + \tau A^* A)^{-1}(\gamma A^* b + x)$$

There are two key ingredients to dealing with nonsmooth optimisation of the form

$$\min_x F(x) + \sum_i R_i(K_i x)$$

where F is smooth, R_i are non-smooth and K_i are linear operators.

1. Gradient descent
2. Proximal mapping

Key splitting methods:

- $F + R$ Forward-Backward splitting.
- $R_1 + R_2$ Douglas-Rachford splitting
- $R_1 + R_2(K \cdot)$ Primal-dual splitting, ADMM.