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Mathematical Tripos Part III: Michaelmas Term 2017/18

Topic in Mathematics of Information – Exercise Sheet III

1. Show that if $A \in \mathbb{R}^{m\times n}$ and $\Lambda \subset \{1,\ldots,n\}$ are such that

$$\sum_{j\in\Lambda} |v_j| < \sum_{l\in\Lambda^c} |v_l| \quad \forall v \in \mathcal{N}(A),\ \mathbb{R}^n \ni v \neq 0,$$

then

$$\sum_{j\in\Lambda} \sqrt{v_j^2 + w_j^2} < \sum_{l\in\Lambda^c} \sqrt{v_l^2 + w_l^2}, \quad \forall v, w \in \mathcal{N}(A),\ (v,w) \neq (0,0).$$

Hint: For $u, v \in \mathcal{N}(A)$, consider $u = \cos(\theta)v + \sin(\theta)w$.

2. Given $A \in \mathbb{R}^{m\times n}$, show that every k-sparse vector $x \in \mathbb{R}^n$ where $x \geqslant 0$ (this means all entries are non-negative) is the unique solution to

$$\min \|z\|_{l^1} \quad \text{subject to} \quad Az = Ax \quad z \geqslant 0$$

if and only if

$$v_{\Lambda^c} \geqslant 0 \Rightarrow \sum_{j=1}^n v_j > 0$$

for all $v \in \mathcal{N}(A) \setminus \{0\}$ and all $\Lambda \subset \{1,\ldots,n\}$ with $|\Lambda| \leqslant k$.

3. Let $A \in \mathbb{C}^{m\times N}$. Show that $\delta_s(A) = \max_{S\subset[N],|S|\leqslant s} \|A_S^* A_S - I_S\|_{2\to 2}$. Suppose that $\delta_s(A) < 1$. Show that for any $S \subset [N]$ with $|S| \leqslant s$, $\frac{1}{1+\delta_s(A)} \leqslant \|(A_S^* A_S)^{-1}\|_{2\to 2} \leqslant \frac{1}{1-\delta_s(A)}$.

For all x with $\mathrm{Supp}(x) \subset S$,

$$\begin{aligned} &(1-\delta)\|x\|_2^2 \leqslant \|Ax\|_2^2 \leqslant (1+\delta)\|x\|_2^2 \\ \iff\ & -\delta\|x\|_2^2 \leqslant \|Ax\|_2^2 - \|x\|^2 \leqslant \delta\|x\|_2^2 \\ \iff\ & |\langle (A_S^* A_S - I_S)x,\, x\rangle| \leqslant \delta. \end{aligned}$$

So,

$$\delta_s = \sup_{S\subset[N],|S|=s} \sup_{\|x\|=1} |\langle (A_S^* A_S - I_S)x,\, x\rangle| = \sup_{S\subset[N],|S|=s} \|A_S^* A_S - I_S\|_{2\to 2}$$

since $A_S^* A_S - I_S$ is self-adjoint.

Finally, since we have for x with $\mathrm{Supp}(x) \subset S$,

$$(1-\delta)\|x\|_2^2 \underbrace{\leqslant}_{(a)} \langle A_S^* A_S x,\, x\rangle \underbrace{\leqslant}_{(b)} (1+\delta)\|x\|_2^2,$$

(a) implies that $(1-\delta_s)\|x\| \leqslant \|A_S^* A_S x\|$, so letting $x = (A_S^* A_S)^{-1} z$, we have $\|z\| \geqslant \|(A_S^* A_S)^{-1} z\|$ which implies that $\|A_S^* A_S)^{-1}\| \leqslant \frac{1}{1-\delta_s}$.

(b) implies that $\|A_S^* A_S\| \leqslant (1+\delta_2)$. So, $\|x\| = \|(A_S^* A_S)^{-1} A_S^* A_S x\| \leqslant \|(A_S^* A_S)^{-1}\|(1+\delta_s)\|x\|$.

4. Let $A \in \mathbb{C}^{m\times N}$. Suppose that A has ℓ^2-normalized columns, i.e. for each column a_j of A, $\|a_j\|_2 = 1$. Show that for all s-sparse vectors

$$(1-\mu_1(s-1))\|x\|_2^2 \leqslant \|Ax\|_2^2 \leqslant (1+\mu_1(s-1))\|x\|_2^2,$$

where $\mu_1(s) = \max_k \max \left\{ \sum_{j\in S} |\langle a_j,\, a_k\rangle| \ ;\ S \subset [N], |S| = s, k \notin S \right\}$ is the ℓ^1 coherence function of A.

You may use without proof Gershgorin's disk theorem: Let λ be an eigenvalue of a square matrix $M \in \mathbb{C}^{n\times n}$. Then, there exists an index $j \in [n]$ such that

$$|\lambda - M_{j,j}| \leqslant \sum_{k\in[n]\setminus\{j\}} |M_{j,k}|\,.$$

5. *This question discusses the converse to Theorem 19 of the notes.* For a given matrix A, consider the following condition:

$$|\sum_{j\in S} \mathrm{sgn}(x_j) v_j| < \|v_{S^c}\|_1, \qquad v \in \mathcal{N}(A) \setminus \{0\}. \tag{1}$$

(a) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$ and let $x = (e^{-i\pi/3}, e^{i\pi/3}, 0)$. Check that (1) is false for $S = \mathrm{Supp}(x)$ and verify that x is the unique solution to

$$\min_z \|z\|_{l^1} \quad \text{subject to} \quad Az = Ax, \tag{2}$$

(b) Let $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$ such that $\mathrm{supp}(x) = S \subset \{1, \dots, n\}$. Show that x is the unique minimiser to (2) (where we minimize only over real vectors) implies that (1) holds.

Hint: Show that $\langle v, \mathrm{sign}(x - tv)\rangle < \|v_{S^c}\|_1$ for all $v \in \mathcal{N}(A) \setminus \{0\}$ and all $t > 0$.

6. Let $D \in \mathbb{R}^{N\times M}$ with $M \geqslant N$ and let $A \in \mathbb{R}^{m\times N}$. The restricted isometry constant δ_s adapted to D is defined as the smallest constant such that

$$(1 - \delta_s)\|z\|_2^2 \leqslant \|Az\|_2^2 \leqslant (1 + \delta_s)\|z\|_2^2$$

for all $z \in \mathbb{R}^N$ of the form $z = Dx$ for some s-sparse $x \in \mathbb{R}^M$.

If A is an $m \times N$ subgaussian random matrix, show that the RIP constant adapted to D of $m^{-1/2}A$ satisfy $\delta_s \leqslant \delta$ with probability at least $1 - \varepsilon$ provided that

$$m \geqslant C\delta^{-2}(s \ln(eM/s) + \ln(2\varepsilon^{-1})).$$

7. Let $A \in \mathbb{C}^{m\times N}$ be a matrix with ℓ_2-normalized columns. Let $x \in \mathbb{C}^N$ and let $S \subset [N]$ be an index set of s largest absolute entries of x. Assume that

$$\|A_S^* A_S - I\|_{2\to 2} \leqslant \alpha$$

for some $\alpha \in (0, 1)$ and that there exists $u = A^*h \in \mathbb{C}^N$ with $h \in \mathbb{C}^m$ such that

$$u_S = \mathrm{sign}(x_S), \quad \|u_{S^c}\|_\infty \leqslant \beta, \quad \|h\|_2 \leqslant \gamma\sqrt{s},$$

for some constants $\beta \in (0, 1)$ and $\gamma > 0$. Suppose that we are given $y = Ax + e$ with $\|e\|_2 \leqslant \eta$. Show that any solution $\hat{x} \in \mathbb{C}^N$ of

$$\min_z \|z\|_1 \quad \text{subject to} \quad \|Az - y\|_2 \leqslant \eta$$

satisfies

$$\|x - \hat{x}\|_2 \leqslant C\sigma_s(x)_1 + D\sqrt{s}\eta$$

for appropriate constants $C, D > 0$ which depend only on α, β, γ.

Hint: Let $z = \hat{x} - x$. First use the fact that $(A_S^* A_S)$ is invertible to show that $\|z_S\|_2 \lesssim \eta + \|z_{S^c}\|_1$. Now, how does the existence of a dual certificate allow for control on $\|z_{S^c}\|_1$?