Additional exercises for MA505250

Exercise 1. Let X be a Banach space and $\mathcal{J}_1 : X \to \mathbb{R}$, $\mathcal{J}_2 : X \to \mathbb{R}$ two functionals. Prove that then

(a) If $\mathcal{J}_1$ and $\mathcal{J}_2$ are weak l.s.c., $\alpha \geqslant 0$, then $\alpha \mathcal{J}_i$ and $\mathcal{J}_1 + \mathcal{J}_2$ are weak l.s.c.

(b) If $\mathcal{J}_1$ is weak l.s.c. and $\varphi : \mathbb{R} \to \mathbb{R}$ is monotonically increasing and l.s.c., then $\varphi \circ \mathcal{J}_1$ is weak l.s.c.

(c) If $\mathcal{J}_1$ is weak l.s.c. then for a Banach space Y and weak sequentially continuous $\Phi : \mathcal{Y} \to X$ it follows that $\mathcal{J}_1 \circ \Phi$ is weak l.s.c.

(d) For any (non-empty) family of weak l.s.c. functionals $\mathcal{J}_i : X \to \mathbb{R}$, $i \in I$ we have that $\sup_{i \in I} \mathcal{J}_i$ is weak l.s.c.

(e) Let $\varphi : K \to \mathbb{R}$ be l.s.c., where K is either $\mathbb{R}$ or $\mathbb{C}$, and $x^* \in X^*$. Then the functional

$$L_{x^*,\varphi} = \varphi \circ \langle x^*, \cdot \rangle_{X^* \times X},$$

is weak l.s.c. in X.

Collect what you have proven (as appropriate) and show that $\varphi(\|u\|_X)$ is weak l.s.c. for any $\varphi : [0, \infty) \to \mathbb{R}$ that is monotonically increasing and l.s.c.

Exercise 2. For a given noisy image $g \in L^2(\Omega)$ and rectangular image domain $\Omega \subset \mathbb{R}^2$ we consider the following variational problem

$$u_\alpha = \operatorname{argmin}_{u \in L^2(\Omega)} \left\{ \alpha \|Du\|_2^2 + \|u - g\|_2^2 \right\},$$

for $\alpha > 0$. Prove that there exists a unique minimiser u_α to the above problem. Show further, that under the additional assumption that $L \leqslant g \leqslant R$ a.e. in Ω the minimiser u_α of the above problem also fulfils $L \leqslant u_\alpha \leqslant R$ a.e. in Ω.

Exercise 3. Let the functional $\mathcal{J} : X \to \mathbb{R}$ be convex on a real Banach space X. Prove that:

(a) For $p \in \partial \mathcal{J}(u)$ and $q \in \partial J(u)$ we have $tp + (1 - t)q \in \partial \mathcal{J}(u)$ for all $t \in [0, 1]$.

(b) Let $\mathcal{J}$ be l.s.c. and consider a sequence $((u_n, p_n))$ in $X \times X^*$ with $p_n \in \partial \mathcal{J}(u_n)$, $u_n \to u$ and $p_n \overset{*}{\rightharpoonup} p$. Then $p \in \partial \mathcal{J}(u)$.

(c) For $u \in X$ the set $\partial \mathcal{J}(u)$ is weak* sequentially closed, that is:

$$\text{For } p_n \overset{*}{\rightharpoonup} p, \text{ we have that } p \in \partial \mathcal{J}(u).$$

Exercise 4. Let $\mathcal{U}$ be a Banach space and let $\mathcal{V}$ be a Hilbert space. Let $A : \mathcal{U} \to \mathcal{V}$ be a bounded linear operator. Let $J : \mathcal{U} \to [0, \infty]$ absolute one-homogeneous and coercive. Consider the problems

$$\sup_{v : A^* v \in \partial \mathcal{J}(0)} \langle f, v \rangle = -\inf_v \langle -f, v \rangle + \iota_{\partial J(0)}(A^* v) \tag{$\mathcal{P}_0$}$$

and

$$\inf_{u : Au = f} \mathcal{J}(u) = \inf_{u \in \mathcal{U}} \iota_{\{f\}}(Au) + \mathcal{J}(u) \tag{$\mathcal{D}_0$}$$

Prove that $0 \in \operatorname{int}(\partial J(0))$ and hence deduce strong duality between $(\mathcal{P}_0)$ and $(\mathcal{D}_0)$.