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Mathematical Tripos Part III: Michaelmas Term 2017/18

Topic in Mathematics of Information – Exercise Sheet II

Linear and nonlinear approximation

1. Prove that for any $f \in L^2(\mathbb{R})$, if $\|f\|_V < \infty$, then $\|f\|_\infty < \infty$. Given an example of a function $f \in L^2([0,1]^2)$ such that $\|f\|_V < \infty$ and $\|f\|_\infty = +\infty$.
2. Let $a, b > 0$. Let f be defined on the interval $[0,1]$ by
$$f(x) = \begin{cases} x^a \sin(x^{-b}) & x \in (0,1] \\ 0 & x = 0 \end{cases}.$$
Show that f is bounded and continuous, but not of bounded variation when $a \leqslant b$. Is f of bounded variation when $a = 2$ and $b = 1$?
3. Let $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$. The space of distributions on Ω is the set of real continuous linear functionals on $\varphi : \mathcal{D}(\Omega) \to \mathbb{R}$.
   (a) Prove that every $u \in L^1_{loc}(\Omega)$ can be understood as a distribution via the definition
   $$T_u(\varphi) := \int u(x)\varphi(x)\, dx, \quad \text{for } \varphi \in \mathcal{C}_c^\infty(\Omega).$$
   (b) Show further that if $u, v \in L^1_{loc}$ and $\langle u, \varphi \rangle = \langle v, \varphi \rangle$ for all $\varphi \in \mathcal{D}(\Omega)$, then $u = v$ a.e.
   (c) A distribution that corresponds to a locally integrable function is called *regular*, otherwise it is called *singular*. Can you give an example for a singular distribution?
4. Let $m \in \mathbb{N}$ and let $\varepsilon \in (0,1)$. Show that f is uniformly Lipschitz-$(m+\varepsilon)$ (as defined in the lectures) if and only if f is m-times continuously differentiable and $f^{(m)}$ satisfies $\left| f^{(m)}(x) - f^{(m)}(y) \right| \leqslant C\,|x-y|^\varepsilon$.
5. Consider a wavelet basis of $L^2([0,1])$ constructed with wavelets having $q > s$ vanishing moments and that are C^q. Construct functions $f \in W^{s,2}([0,1])$ for which the linear and nonlinear approximation errors in this basis are identical, $\varepsilon_l(f,M) = \varepsilon_n(f,M)$ for all $M \geqslant 0$.
6. Let f be a piecewise polynomial of degree q defined on $[0,1]$, with K discontinuities. Given a wavelet basis of $L^2([0,1])$ with $q+1$ vanishing moments, give upper bounds as a function of K and M, the linear approximation error $\varepsilon_l(f,M)$ and the nonlinear approximation error $\varepsilon_n(f,M)$.
7. Let $f \in L^\infty([0,1]^2)$ be uniformly Lipschitz-α with $\alpha \in (1/2, q)$ except on a smooth curve of length L. Consider its approximation error with a compactly supported wavelet basis whose wavelet is q-times differentiable and has q vanishing moments. Show that $\varepsilon_n(f,M) = \mathcal{O}(M^{-1})$.

Compressed sensing

1. Show that if $A \in \mathbb{R}^{m\times n}$ and $\Lambda \subset \{1,\dots,n\}$ are such that
$$\sum_{j\in\Lambda} |v_j| < \sum_{l\in\Lambda^c} |v_l| \quad \forall v \in \mathcal{N}(A),\ \mathbb{R}^n \ni v \neq 0,$$
then
$$\sum_{j\in\Lambda} \sqrt{v_j^2 + w_j^2} < \sum_{l\in\Lambda^c} \sqrt{v_l^2 + w_l^2}, \quad \forall v, w \in \mathcal{N}(A),\ \mathbb{R}^n \ni v, w \neq 0.$$
2. Given $A \in \mathbb{R}^{m\times n}$, show that every k-sparse vector $x \in \mathbb{R}^n$ where $x \geqslant 0$ (this means all entries are non-negative) is the unique solution to
$$\min \|z\|_{l^1} \quad \text{subject to} \quad Az = Ax \quad z \geqslant 0$$
if and only if
$$v_{\Lambda^c} \geqslant 0 \Rightarrow \sum_{j=1}^n v_j > 0$$
for all $v \in \mathcal{N}(A)\setminus\{0\}$ and all $\Lambda \subset \{1,\dots,n\}$ with $|\Lambda| \leqslant k$.

3. Let $A \in \mathbb{C}^{m\times N}$. Suppose that A has ℓ^2-normalized columns, i.e. for each column a_j of A, $\|a_j\|_2 = 1$. Show that for all s-sparse vectors

$$(1 - \mu_1(s-1))\|x\|_2^2 \leqslant \|Ax\|_2^2 \leqslant (1 + \mu_1(s-1))\|x\|_2^2,$$

where $\mu_1(s) = \max_k \max \left\{ \sum_{j\in S} |\langle a_j,\, a_k\rangle| \;;\; S \subset [N], |S| = S, k \notin S \right\}$ is the ℓ^1 coherence function of A.

You may use without proof Gershgorin's disk theorem: Let λ be an eigenvalue of a square matrix $M \in \mathbb{C}^{n\times n}$. Then, there exists an index $j \in [n]$ such that

$$|\lambda - M_{j,j}| \leqslant \sum_{k\in[n]\setminus\{j\}} |M_{j,k}|\,.$$

4. * *This question discusses the converse to Theorem 19 of the notes.* For a given matrix A, consider the following condition:

$$|\sum_{j\in S} \mathrm{sgn}(x_j) v_j| < \|v_{S^c}\|_1, \qquad v \in \mathcal{N}(A) \setminus \{0\}. \tag{1}$$

   (a) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$ and let $x = (e^{-i\pi/3}, e^{i\pi/3}, 0)$. Check that (1) is false for $S = \mathrm{Supp}(x)$ and verify that x is the unique solution to

$$\min_z \|z\|_{l^1} \quad \text{subject to} \quad Az = Ax, \tag{2}$$

   (b) Let $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$ such that $\mathrm{supp}(x) = S \subset \{1, \dots, n\}$. Show that x is the unique minimiser to (2) (where we minimize only over real vectors) implies that (1) holds.