Additional exercises for MA505250

Exercise 1. Let X be a Banach space and $\mathcal{J}_1 : X \to \mathbb{R}$, $\mathcal{J}_2 : X \to \mathbb{R}$ two functionals. Prove that then

(a) If $\mathcal{J}_1$ and $\mathcal{J}_2$ are weak l.s.c., $\alpha \geqslant 0$, then $\alpha \mathcal{J}_i$ and $\mathcal{J}_1 + \mathcal{J}_2$ are weak l.s.c.

(b) If $\mathcal{J}_1$ is weak l.s.c. and $\varphi : \mathbb{R} \to \mathbb{R}$ is monotonically increasing and l.s.c., then $\varphi \circ \mathcal{J}_1$ is weak l.s.c.

(c) If $\mathcal{J}_1$ is weak l.s.c. then for a Banach space Y and weak sequentially continuous $\Phi : \mathcal{Y} \to X$ it follows that $\mathcal{J}_1 \circ \Phi$ is weak l.s.c.

(d) For any (non-empty) family of weak l.s.c. functionals $\mathcal{J}_i : X \to \mathbb{R}$, $i \in I$ we have that $\sup_{i \in I} \mathcal{J}_i$ is weak l.s.c.

(e) Let $\varphi : K \to \mathbb{R}$ be l.s.c., where K is either $\mathbb{R}$ or $\mathbb{C}$, and $x^* \in X^*$. Then the functional

$$L_{x^*,\varphi} = \varphi \circ \langle x^*, \cdot \rangle_{X^* \times X},$$

is weak l.s.c. in X.

Collect what you have proven (as appropriate) and show that $\varphi(\|u\|_X)$ is weak l.s.c. for any $\varphi : [0, \infty) \to \mathbb{R}$ that is monotonically increasing and l.s.c.

Solution.

(a) $\mathcal{J}_1$ and $\mathcal{J}_2$ are weak l.s.c., then

$$\begin{aligned} \alpha \mathcal{J}_1(u) &\leqslant \alpha \liminf_{n\to\infty} \mathcal{J}_1(u_n) \\ &= \liminf_{n\to\infty} (\alpha \mathcal{J}_1)(u_n), \end{aligned}$$

and

$$\begin{aligned} \left(\mathcal{J}_1 + \mathcal{J}_2\right)(u) &\leqslant \liminf_{n\to\infty} \mathcal{J}_1(u_n) + \liminf_{n\to\infty} \mathcal{J}_2(u_n) \\ &\leqslant \liminf_{n\to\infty} \left(\mathcal{J}_1 + \mathcal{J}_2\right)(u_n). \end{aligned}$$

(b)

$$\begin{aligned} \varphi(\mathcal{J}_1(u)) &\underbrace{\leqslant}_{\text{monotonicity}} \varphi\left(\liminf_{n\to\infty} \mathcal{J}_1(u_n)\right) \\ &\underbrace{\leqslant}_{l.s.c.} \liminf_{n\to\infty} \varphi(\mathcal{J}_1(u_n)). \end{aligned}$$

(c) Since Φ is weak sequentially continuous we have that if $u_n \rightharpoonup u$ in Y that $\Phi(u_n) \rightharpoonup \Phi(u)$ in X and hence

$$\mathcal{J}_1\left(\Phi(u)\right) \leqslant \liminf_{n\to\infty} \mathcal{J}_1\left(\Phi(u_n)\right).$$

(d) For all n and $i \in I$ we have that

$$\mathcal{J}_i(u_n) \leqslant \sup_{i \in I} \mathcal{J}_i(u_n),$$

and hence

$$\mathcal{J}_i(u) \leqslant \liminf_{n \to \infty} \mathcal{J}_i(u_n) \leqslant \liminf_{n \to \infty} \sup_{i \in I} \mathcal{J}_i(u_n).$$

Taking the sup on the left hand side in the above inequality we get

$$\sup_{i \in I} \mathcal{J}_i(u) \leqslant \liminf_{n \to \infty} \sup_{i \in I} \mathcal{J}_i(u_n).$$

(e) We consider $u_n \rightharpoonup u$ in X, that is

$$\lim_{n \to \infty} \langle u_n, u^* \rangle = \langle u, u^* \rangle \quad \forall u^* \in X^*.$$

Hence, $\langle u, u^* \rangle$ is continuous and hence the assertion follows from (c).

To prove the last claim we are going to use what we have proven so far. Since X is a Banach space we have

$$\|u\|_X = \sup_{\|u^*\|_{X^*} \leqslant 1} |\langle u^*, u \rangle| = \sup_{\|u^*\|_{X^*} \leqslant 1} L_{x^*, |\cdot|}(u).$$

Hence, by (d) and (e) the norm is weak l.s.c. and by (b) so is $\varphi(\|u\|_X)$.

Exercise 2. For a given noisy image $g \in L^2(\Omega)$ and rectangular image domain $\Omega \subset \mathbb{R}^2$ we consider the following variational problem

$$u_\alpha = \operatorname{argmin}_{u \in L^2(\Omega)} \left\{ \alpha \|Du\|_2^2 + \|u - g\|_2^2 \right\},$$

for $\alpha > 0$. Prove that there exists a unique minimiser u_α to the above problem. Show further, that under the additional assumption that $L \leqslant g \leqslant R$ a.e. in Ω the minimiser u_α of the above problem also fulfils $L \leqslant u_\alpha \leqslant R$ a.e. in Ω.

Solution. We consider a minimising sequence of $\mathcal{J}(u) = \alpha \|Du\|_2^2 + \|u - g\|_2^2$ (assuming that $g \in L^2(\Omega)$ the functional is bounded from below by 0). Then, we have

$$\|Du_n\|_2 < \infty$$
$$\|u_n\|_2 < \infty,$$

and hence there exists a subsequence (still denoted by u_n) such that $u_n \rightharpoonup u$ in H^1 and in particular in L^2. (As a side remark, we even get strong convergence in L^2 because of the compact embedding of H^1 in L^2). Because the norm $\|\cdot\|_2$ is l.s.c. with respect to weak convergence in L^2 and the l.s.c. of $\|D \cdot\|_2$ w.r.t. weak convergence in H^1, we are done.

Uniqueness follows from strict convexity of $\mathcal{J}$.

Now, to prove the second claim we consider a minimiser u_α and the function $u = \min(R, \max(L, u_\alpha))$. If $u_\alpha(x) \geqslant R$ for all $x \in \Omega$, then

$$|u(x) - g(x)| = |R - g(x)| \leqslant |u_\alpha(x) - g(x)|.$$

Analogously, if $u_\alpha(x) \leqslant L$ then

$$|u(x) - g(x)| \leqslant |u_\alpha(x) - g(x)|,$$

and hence

$$\frac{1}{2}\|u - g\|_2^2 \leqslant \frac{1}{2}\|u_\alpha - g\|_2^2.$$

Moreover, $u \in H^1(\Omega)$ with $Du = Du_\alpha$ a.e. on $\{L \leqslant u_\alpha \leqslant R\}$ and $Du = 0$ everywhere else (this follows from the chain rule for Sobolev functions: Let $\varphi : \mathbb{R} \to \mathbb{R}$ continuously differentiable with $\|\varphi'\|_\infty \leqslant C$ for a $C > 0$. Then the mapping $u \mapsto \varphi \circ u$ maps $H^1 \to H^1$ and $D(\varphi \circ u) = \varphi'(u) \cdot Du$.). Hence we have

$$\frac{1}{2}\int_\Omega |Du|^2 \, dx \leqslant \frac{1}{2}\int_\Omega |Du_\alpha| \, dx,$$

and hence $\mathcal{J}(u) \leqslant \mathcal{J}(u_\alpha)$ and since u_α is the unique minimiser we have that $u = u_\alpha$.

Exercise 3. Let the functional $\mathcal{J} : X \to \mathbb{R}$ be convex on a real Banach space X. Prove that:

(a) For $p \in \partial\mathcal{J}(u)$ and $q \in \partial J(u)$ we have $tp + (1-t)q \in \partial\mathcal{J}(u)$ for all $t \in [0,1]$.

(b) Let $\mathcal{J}$ be l.s.c. and consider a sequence $((u_n, p_n))$ in $X \times X^*$ with $p_n \in \partial\mathcal{J}(u_n)$, $u_n \to u$ and $p_n \overset{*}{\rightharpoonup} p$. Then $p \in \partial\mathcal{J}(u)$.

(c) For $u \in X$ the set $\partial\mathcal{J}(u)$ is weak* sequentially closed, that is:

$$\text{For } p_n \overset{*}{\rightharpoonup} p, \text{ we have that } p \in \partial\mathcal{J}(u).$$

Solution. (a) If $p, q \in \partial J(u)$, then $J(v) - J(u) \geqslant \langle p,\, v-u\rangle$ and $J(v) - J(u) \geqslant \langle q,\, v-u\rangle$ for all $v \in X$. The result follows by linearity of the inner product.
(b) For all v, $J(v) - J(u_n) \geqslant \langle p_n,\, v-u_n\rangle$. Since J is lsc, $\lim_{n\to\infty}(J(v) - J(u_n)) \leqslant J(v) - J(u)$. On the other hand,

$$|\langle p_n,\, v-u_n\rangle - \langle p,\, v-u\rangle| = |\langle p_n - p,\, v\rangle + \langle p - p_n,\, u\rangle + \langle p_n,\, u-u_n\rangle| \to 0, \qquad n \to \infty.$$

For the first 2 terms, this convergence to zero is clear by assumption, For the last term, we know that $\|p_n\|$ is uniformly bounded (by the uniform boundedness principle).
(c) For all v, $J(v) - J(u) \geqslant \langle p_n,\, v-u\rangle$. Since $p_n \overset{*}{\rightharpoonup} p$, we have that $J(v) - J(u) \geqslant \langle p,\, v-u\rangle$.

Exercise 4. Let $\mathcal{U}$ be a Banach space and let $\mathcal{V}$ be a Hilbert space. Let $A : \mathcal{U} \to \mathcal{V}$ be a bounded linear operator. Let $J : \mathcal{U} \to [0, \infty]$ absolute one-homogeneous and coercive. Consider the problems

$$\sup_{v : A^* v \in \partial \mathcal{J}(0)} \langle f,\, v\rangle = -\inf_{v} \langle -f,\, v\rangle + \iota_{\partial J(0)}(A^* v) \tag{$\mathcal{P}_0$}$$

and

$$\inf_{u : Au = f} \mathcal{J}(u) = \inf_{u \in \mathcal{U}} \iota_{\{f\}}(Au) + \mathcal{J}(u) \tag{$\mathcal{D}_0$}$$

Prove that $0 \in \operatorname{int}(\partial J(0))$ and hence deduce strong duality between ($\mathcal{P}_0$) and ($\mathcal{D}_0$).

Solution.

Claim: For $J : \mathcal{U} \to [0, \infty]$ absolute one-homogeneous and coercive, we have $0 \in \operatorname{int}(\partial J(0))$.

Proof. Since J is absolute one-homogeneous, $J(0) = 0$. So, $p \in \partial J(0)$ means that $J(u) \geqslant \langle u,\, p\rangle$ for all $u \in \mathcal{U}$. Note that clearly, $0 \in \partial J(0)$. Suppose that $0 \notin \operatorname{int}(\partial J(0))$. Then, there exist $p_n \notin \partial J(0)$ such that $\|p_n\| \to 0$. So, there exists u_n such that $J(u_n) < \langle p_n,\, u_n\rangle$. Since J is one-homogeneous, we can assume that $\|u_n\| = 1$. Therefore, $\lim_{n\to\infty} J(u_n) \leqslant \lim_{n\to\infty} \|p_n\|\, \|u_n\| = 0$. Letting $\lambda_n = 1/J(u_n)$, we have $\|\lambda_n u_n\| \to +\infty$ but $J(\lambda_n u_n) = 1$. This contradicts the assumption that J is coercive. □

The result now comes from applying Fechel-Duality theorem to ($\mathcal{P}_0$), since $\iota_{\partial J(0)}$ is continuous at zero.