Notation of Set Theory and Quantifiers

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Notation of Set theory

Quantifiers

Functions

Notation of Set theory

Sets

Definition 1.9

A **set** is a collection of distinct objects. These objects are called **elements** of the set.

Ordering doesn't matter. $\{1, 2, 3\} = \{3, 2, 1\}$.

Example of sets

- the empty set: $\emptyset$
- the natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots, \}$
- the integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots, \}$
- the rational numbers: $\mathbb{Q} = \{\frac{p}{q} : \ p \in \mathbb{Z}, \ q \in \mathbb{N}\}$.
- the real numbers: $\mathbb{R}$
- the complex numbers: $\mathbb{C}$

Notation in set theory

- $x \in A$ means x is an element of A.
  Example $1 \in \mathbb{N}$.
- $x \notin A$ means x is not an element of A.
  Example $-1 \notin \mathbb{N}$.
- $A \subseteq B$ means every element of A is an element of B.
  Examples: $\emptyset \subseteq \mathbb{N}$, $\mathbb{N} \subset \mathbb{Z}$.
- $A \cap B = \{x : \ x \in A \text{ and } x \in B\}$.
  Example: $\mathbb{N} \cap \{-1, -2, 0, 1, 3\} = \{1, 3\}$.
- $A \cup B = \{x : \ x \in A \text{ or } x \in B\}$
  Example: $\mathbb{N} \cup \{-1, -2, 1, 3\} = \{-2, -1, 0, 1, 2, 3, \ldots\}$.
- $A \setminus B = \{x : \ x \in A \text{ and } x \notin B\}$
  Example $\mathbb{Z} \setminus \mathbb{N} = \{\ldots, -2, -1, 0\}$.
- $A \times B = \{(x, y) : \ x \in A \text{ and } y \in B\}$.
  Example $\{1, 2\} \times \{3, 2\} = \{(1, 3), (1, 2), (2, 3), (2, 2)\}$.

Quantifiers

Quantifiers

Quantifiers like *for all* or *there exists* tells us how to interpret a variable in a statement.

- for every real number x, $x^2 \geq 0$.
- for every natural number, either n is a perfect square or $\sqrt{n}$ is irrational.
- there exists a real number such that $x^2 - 1 = 0$.

We use the notation

- $\exists$ to mean 'there exists'
- $\forall$ to mean 'for all'.

$\forall$ 'means For all' and $\exists$ mean 'there exists'

Examples

i) $\forall x : x \in \mathbb{R} \Rightarrow x^2 \geq 0.$

'for all x, if x is a real number, then $x^2 \geq 0$.

ii) $\exists x : x \in \mathbb{R} \wedge x^2 \geq 0$

'there exists x such that x is a real number and $x^2 \geq 0$'.

We write '$\exists x \in A : P(x)$' to mean
there exists an element x of A such that $P(x)$ holds.

Examples

- $\forall x \in \mathbb{R} : x^2 \geq 0.$
- $\exists x \in \mathbb{R} : x^2 \geq 0.$

Example

Translate the following statement into English.

$$\exists x \in \mathbb{Z} : \forall y \in \mathbb{N} : x < y.$$

There exists an integer x such that for every natural number y, $x < y$.

This is true. Take $x = -1$.

Warning: the ordering matters!

$$\forall m \in \mathbb{N} : \exists n \in \mathbb{N} : n - m = 1.$$

For every natural number m, there exists a natural number n such that $n = m + 1$*.*

True.

$$\exists n \in \mathbb{N} : \forall m \in \mathbb{N} : n - m = 1.$$

There exists a natural number n such that for all natural numbers m, $n = m + 1$*.*

False.

Using negation with quantifiers

Proposition 1.12

- The statements $\neg(\forall x : P(x))$ and $\exists x : \neg P(x)$ are equivalent.
- The statements $\neg(\exists x : P(x))$ and $\forall x : \neg P(x)$.

These are intuitively clear. We will not prove this.

It is not the case that all men are called Bob.
There exists a man who is not called Bob.

There does not exist an integer that is irrational.
Every integer x is not irrational.

Example

What is the negation of $\exists x \in \mathbb{Z}, \forall y \in \mathbb{N} : x < y$?

Functions

Functions

Definition 1.14

Let A and B be sets. A function from A to B is a rule that assigns to each element of A a unique element of B.

- Write $f : A \to B$.
- given $a \in A$, $f(a)$ is the element from B assigned to a.
- A is called the **domain** of f.
- B is called the **codomain** of f.

Examples

Two functions

i) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ for all $x \in \mathbb{R}$.

ii) $f : \mathbb{R} \to \mathbb{Q}$ defined as

$$f(x) = \begin{cases} \frac{1}{2} & x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = b$ where $b \in \mathbb{R}$ is such that $b^2 = x$. Is this a function?

No, f can assign 1 to either -1 or 1. But ok if we restrict the domain and codomain to $[0, \infty)$.