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Mathematical Tripos Part III: Michaelmas Term 2017/18

Topics in Mathematics of Information – Exercise Sheet I

Sketch of solutions for most of the questions:

1. Let $f \in L^1(\mathbb{R})$ and suppose that $\mathrm{Supp}(\hat{f}) \subset [-B\pi, B\pi]$. Show that

$$f = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{B}\right)\varphi_k, \qquad \text{where} \qquad \varphi_k = \frac{\sin(\pi(B\cdot - k))}{\pi(B\cdot - k)}.$$

and that

$$f_N = \sum_{|k|\leqslant N} f\left(\frac{k}{B}\right)\varphi_k \to f \qquad \text{in } L^\infty.$$

(Use the fact that $\left\{\frac{1}{\sqrt{2B\pi}} e^{-ikB^{-1}\cdot} \; ; \; k \in \mathbb{Z}\right\}$ is an orthonormal basis of $L^2[-B\pi, B\pi]$.)

Proof. Since $\hat{f} \in L^2[-B\pi, B\pi]$, we can write

$$\hat{f}(\xi) = \frac{1}{B}\sum_{k\in\mathbb{Z}} f\left(\frac{k}{B}\right) e^{-ik\xi/B}. \tag{1}$$

Since $\hat{f}$ is continuous and piecewise smooth, its Fourier series converge uniformly to $\hat{f}$. Moreover,

$$f(x) = \frac{1}{2\pi}\int_{-B\pi}^{B\pi} \hat{f}(\xi)e^{i\xi x}\mathrm{d}\xi.$$

Applying this to (1) and exchanging the sum and the integral, we have the required representation since

$$\int \mathbb{1}_{[-B\pi,B\pi]}(\xi)e^{-ik\xi/B+ix\xi}\mathrm{d}\xi = \varphi_k(x).$$

□

2. Given a sequence of closed subspace $\{V_j\}_{j\in\mathbb{Z}}$, suppose that $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$. Prove that if $\{\varphi_{0,k} \; ; \; k \in \mathbb{Z}\}$ is an orthonormal basis of V_0, then $\{\varphi_{j,k} \; ; \; k \in \mathbb{Z}\}$ is an orthonormal basis of V_j.

3. Let $\varphi = \mathbb{1}_{[0,1]}$. Since $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ forms an orthonormal basis, we know that $\sum_k |\hat{\varphi}(\xi + 2k\pi)|^2 = 1$ for a.e. $\xi \in \mathbb{R}$. Use this to show that

$$\sum_{k\in\mathbb{Z}} \frac{4}{(\xi + 2k\pi)^2} = \frac{1}{\sin^2(\xi/2)}.$$

$$1 = \sum_k |\hat{\varphi}(\xi + 2\pi k)|^2 = \sum_k \left|\frac{2\sin(\xi/2 + \pi k)}{(\xi + 2\pi k)}\right|^2 = \sin^2(\xi/2)\sum_k \frac{4}{(\xi + 2\pi k)^2},$$

note that you have equality everywhere: the partial sum converges uniformly on compact sets. Combining this with the observation that each summand is continuous, the limit is also a continuous function.

4. Let V_j be the space of all $f \in L^2(\mathbb{R})$ that are continuous and piecewise linear, with corners only at the points $k/2^j$ for $k \in \mathbb{Z}$.

   (a) Show that $\{V_j\}_{j\in\mathbb{Z}}$ satisfies conditions (I) to (IV) in the definition of an MRA (see the definition given in the lecture notes).

(b) Let

$$\Delta = \mathbb{1}_{[0,1]} \star \mathbb{1}_{[0,1]}.$$

Show that $\{\Delta(\cdot - k)\}_{k\in\mathbb{Z}}$ is a (nonorthonormal) basis of V_0.

(c) Let φ be defined via its Fourier transform:

$$\hat{\varphi} = \frac{\hat{\Delta}}{\sqrt{\sum_n \left|\hat{\Delta}(\cdot + 2\pi n)\right|^2}}.$$

Show that

$$\hat{\varphi} = e^{-i\xi} \frac{4\sin^2(\xi/2)}{\xi^2\sqrt{1 - \frac{2}{3}\sin^2(\xi/2)}}$$

and that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis of V_0.

(d) Show that the low pass filter of φ is

$$m(\xi) = \frac{e^{-i\xi}\cos^2(\xi/2)\sqrt{1 - \frac{2}{3}\sin^2(\xi/2)}}{\sqrt{1 - \frac{2}{3}\sin^2(\xi)}}$$

and has an associated wavelet ψ with Fourier transform

$$\hat{\psi}(\xi) = -\frac{e^{i\xi}\sin^4(\xi/4)}{\xi^2/16} \frac{\sqrt{1 - \frac{2}{3}\cos^2(\xi/4)}}{\sqrt{1 - \frac{2}{3}\sin^2(\xi/2)}\sqrt{1 - \frac{2}{3}\sin^2(\xi/4)}}.$$

Conclude that ψ has 2 vanishing moments.

For (a), we simply mention that for (III), the set of continuous, compactly supported functions can be approximated to arbitrary precision by elements of V_j; and $C_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. For (b), use the fact that $f(x) = \sum_k f(k)\Delta(x-k)$ for all $f \in V_0$.
For (c), first note that $\hat{\Delta}(\xi) = e^{-i\xi}\sin^2(\xi/2)/(\xi/2)^2$. Now,

$$\sum_k \left|\hat{\Delta}(\xi + 2k\pi)\right|^2 = \sum \frac{\sin^4(\xi/2 + k\pi)}{|\xi/2 + k\pi|^4} = \sin^4(\xi/2)\sum_k \frac{16}{|\xi + 2k\pi|^4}.$$

By twice differentiating the expression from the previous question,

$$\sum_k \frac{48}{|\xi + 2k\pi|^4} = \frac{3 - 2\sin^2(\xi/2)}{\sin^4(\xi/2)}.$$

Plugging this in yields the required result for $\hat{\varphi}$. It is also easy to see from this that $\hat{\varphi} \in L^2$, and hence $\varphi \in L^2$. Also, the integer translates of φ form an orthonormal basis since $\sum_k |\hat{\varphi}(\xi + 2k\pi)|^2 = 1$ a.e.
Note that this also shows that $\sum_k \left|\hat{\Delta}(\xi + 2k\pi)\right|^2 \in [1/3, 1]$ for some $A, B > 0$ and hence, $\{\Delta(\cdot - k)\}_k$ is in fact a Riesz basis.

One can see from the expression of $\hat{\psi}$ that ψ has exactly 2 vanishing moments, since $\hat{\psi}(0) = \hat{\psi}^{(1)}(0) = 0$. We can see $\hat{\psi}(\xi) = \sin^2(\xi/4)P(\xi)$ where $P(0) \neq 0$. If we differentiate twice, the only nonzero terms occur if the derivative falls on $\sin^2(\xi)$ and we get $\hat{\psi}^{(2)}(0) = 2P(0) \neq 0$, so there are exactly 2 vanishing moments.

5. Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be a scaling function of an MRA. Recall from the lectures that $|\hat{\varphi}(0)| = 1$ (since $\varphi \in L^1$ implies that $\hat{\varphi}$ is continuous).

(a) Show that $\hat{\varphi}(2\pi k) = 0$ for all $k \neq 0$.

(b) Suppose that $\hat{\varphi}(0) = 1$. Show that

$$\sum_{n\in\mathbb{Z}} \varphi(t-n) = 1, \qquad a.e.\ t \in \mathbb{R}.$$

Proof. Note that since $\sum_k |\hat{\varphi}(\cdot - 2\pi k)|^2 = 1$ a.e. and $|\hat{\varphi}(0)| = 1$, we have that $\hat{\varphi}(2\pi k) = 0$ for all $k \neq 0$.
For the second part, observe that $P(t) = \sum_n \varphi(t-n)$ is a 1-periodic function which is $L^1[0,1]$ since $\varphi \in L^1$. The result follows because $\int_0^1 P(x)e^{-2\pi i x\omega}\mathrm{d}x = \hat{\varphi}(2\pi\omega)$. □

6. Let f be a function of support $[0,1]$ and suppose that f is equal to different polynomials of degree q on the intervals $\{[t_k, t_{k+1}]\}_{k=0}^{K-1}$, with $t_0 = 0$ and $t_K = 1$. Let ψ be a compactly supported wavelet with p vanishing moments, with support in $[-p, p-1]$. If $q < p$, compute the number of nonzero coefficients $\langle f, \psi_{j,n}\rangle$ at a fixed scale 2^j for $j \in \mathbb{N}$ sufficiently large. How should we choose p to minimize this number? If $q > p$, what is the maximum number of nonzero wavelet coefficients $\langle f, \psi_{j,n}\rangle$ at a fixed scale 2^j?

Proof. If $q < p$, then $\langle f, \psi_{j,n}\rangle = 0$ whenever $t_n \notin \text{interior}(\text{Supp}(\psi_{j,n}))$. So, $\langle f, \psi_{j,n}\rangle$ is potentially nonzero only if there exists k such that

$$\frac{-p+n}{2^j} < t_k < \frac{p+n-1}{2^j} \iff 2^j t_k - p + 1 < n < 2^j + p.$$

There are at most $2p-2$ integers in this range.
If $q \geqslant p$, then $\langle f, \psi_{j,n}\rangle$ is potentially nonzero if $|\text{Supp}(\psi_{j,n}) \cap [0,1]| \neq 0$. This occurs if

$$\frac{-p+n}{2^j} < 1 \quad \text{and} \quad \frac{p+n-1}{2^j} > 0,$$

that is, $1-p < n < 2^j + p$. There are at most $2^j + 2p - 2$ integers in this range.
So, to minimize the number of nonzero coefficients, we should choose $p = q+1$ (since choosing a smaller □

For the following questions, you are given this fact:

> $\psi \in L^2(\mathbb{R})$ is a wavelet (not necessarily derived from an MRA) if and only if
>
> $$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}(2^j\xi)\right|^2 = 1, \qquad \sum_{j\in\mathbb{Z}} \left|\hat{\psi}(\xi + 2j\pi)\right|^2 = 1, \qquad \text{for } a.e.\ \xi \in \mathbb{R}. \tag{2}$$

7. Let φ be a scaling function associated with an MRA and let ψ be its associated wavelet. Prove that $|\hat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} \left|\hat{\psi}(2^j\xi)\right|^2$ for a.e. $\xi \in \mathbb{R}$.

Proof. From lectures,

$$|\hat{\varphi}(2\xi)|^2 + \left|\hat{\psi}(2\xi)\right|^2 = |\hat{\varphi}(\xi)|^2 \left(|m(\xi)|^2 + |m(\xi+\pi)|^2\right),$$

where m is the low pass filter.
So,

$$|\hat{\varphi}(\xi)|^2 = \left|\hat{\varphi}(2^N\xi)\right|^2 + \sum_{j=1}^{N} \left|\hat{\psi}(2^j\xi)\right|^2, \qquad \forall N \geqslant 1.$$

Since $|\hat{\varphi}(\xi)| \leqslant 1$ for all $\xi \in \mathbb{R}$, $\sum_{j=1}^{N} \left|\hat{\psi}(2^j\xi)\right|^2$ is increasing in N and bounded by 1. In particular, its limit exists. Also,

$$\lim_{N\to\infty} \left|\hat{\varphi}(2^N\xi)\right|$$

exists. Note that

$$\int \left|\hat{\varphi}(2^N\xi)\right|^2 \mathrm{d}\xi = \frac{1}{2^N} \int |\hat{\varphi}(\xi)|^2 \to 0 \qquad N \to \infty.$$

So, by Fatou's lemma,

$$\int \lim_{N\to\infty} \left|\hat{\varphi}(2^N\xi)\right|^2 \leqslant \lim_{N\to\infty} \frac{1}{2^N} \int |\hat{\varphi}(\xi)|^2 = 0.$$

So, $\lim_{N\to\infty} \left|\hat{\varphi}(2^N\xi)\right| = 0$. NB: we only need to go to this trouble as we only have $\varphi \in L^2$ – if $\varphi \in L^1$, then by the Riemann Lebesgue lemma, we have that $\hat{\varphi}(2^N\xi) \to 0$ as $N \to \infty$. □

8. * Let

$$\hat{\psi}(\omega) = \begin{cases} 1 & |\omega| \in [4\pi/7, \pi] \cup [4\pi, 4\pi + 4\pi/7], \\ 0 & \text{otherwise.} \end{cases}$$

Using (2) or otherwise, prove that $\{\psi_{j,n}\ ;\ j,n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. Prove that ψ is not associated to a scaling function that generates an MRA.

Hint: to show that ψ is not associated to an MRA, use the fact that MRA wavelets necessarily satisfy the property derived in question 7.

Proof. Let $K = \{\omega\ ;\ |\omega| \in [4\pi/7, \pi] \cup [4\pi, 4\pi + 4\pi/7]\}$. Then, by (2), ψ is a wavelet if $\{2^j K\ ;\ j \in \mathbb{Z}\}$ partitions $\mathbb{R}$ and $\{K + 2k\pi\ ;\ k \in \mathbb{Z}\}$ partitions $\mathbb{R}$. One can easily check that this is true.
To show that ψ cannot come from an MRA, suppose for now that ψ does come from an MRA. Then, recall from question 6 that this implies that

$$|\hat{\varphi}(\xi)|^2 = \sum_{j\in\mathbb{N}} \left|\hat{\psi}(2^j\xi)\right|^2$$

for a.e. $\xi \in \mathbb{R}$. So,

$$|\hat{\varphi}(\xi)| = \mathbb{1}_{\tilde{K}}$$

where $\tilde{K} = \{\omega\ ;\ |\omega| \in (0, 4\pi/7] \cup [\pi, 8\pi/7] \cup [2\pi, 16\pi/7]\}$. Since the translates of φ form an orthonormal system, we also have that

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2\pi k)|^2 = 1$$

for a.e. $\xi \in \mathbb{R}$. If we now consider $I = (0, 2\pi/7)$, then $|\hat{\varphi}(\xi)| = 1$ and for $\xi \in I$, $2\pi < \xi + 2\pi \leqslant 16\pi/7$. So, $|\hat{\varphi}(\xi + 2\pi)| = 1$ for all $\xi \in I$. But this would imply that for all $\xi \in I$,

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2\pi k)|^2 \geqslant |\hat{\varphi}(\xi + 2\pi)|^2 + |\hat{\varphi}(\xi)|^2 = 2$$

which yields the required contradiction. □