Hw#1

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Problem 1: Nondimensionalizing Navier Stokes
Using the nondimensionalization given, show that eq 12-14 hold.
Solution. Let

$$\vec{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{1}$$

so that eqs (6)-(8) in the notes become the single vector valued equation:

$$\partial_t \vec{u} + (\vec{u} \cdot \nabla)\,\vec{u} = -\frac{1}{\rho}\nabla p + \nu \Delta \vec{u} + \frac{1}{\rho}\vec{f_b} \tag{2}$$

I will omit the nondimensionalization of the incompressibility condition as it is trivial.

As in eqs (12-14) we neglect body forces $\vec{f_b} = \vec{0}$. Let L, U, T be characteristic length, velocity, and time scales respectively, and note that $\rho =$ constant is the constant density of the incompressible fluid. Then we can define:

$$x', y', z' = \frac{x}{L}, \frac{y}{L}, \frac{z}{L} \tag{3}$$

$$\vec{u}' = \frac{\vec{u}}{U} \tag{4}$$

$$t' = \frac{t}{T} \tag{5}$$

$$p' = \frac{p}{\rho U^2} \tag{6}$$

Then

$$\partial_t = \frac{1}{T}\partial_{t'} \tag{7}$$

$$\nabla = \frac{1}{L}\nabla' \tag{8}$$

$$\Delta = \frac{1}{L^2}\Delta' \tag{9}$$

Subsituting these definitions into eq(2), we obtain

$$\left(\frac{U}{T}\right)\partial_{t'}\vec{u}' + \left(\frac{U^2}{L}\right)\left(\vec{u}'\cdot\nabla'\right)\vec{u}' = \frac{\rho U^2}{\rho L}\nabla' p' + \frac{U}{L^2}\nu\Delta'\vec{u}' \tag{10}$$

Finally, multiplying both sides by $\frac{L}{U^2}$ yields:

$$\left(\frac{L}{TU}\right)\partial_{t'}\vec{u}' + \left(\vec{u}'\cdot\nabla'\right)\vec{u}' = \nabla' p' + \left(\frac{\nu}{LU}\right)\Delta'\vec{u}' \tag{11}$$

And we define the two dimensionless numbers as in the text: $Re = \frac{LU}{\nu}$, $S = \frac{L}{TU}$. Note that if $T = L/U$, then $S = 1$.

Problem 2: Convert the Navier-Stokes equations to cylindrical coordinates to show that stated equations hold.

Solution. We begin with the incompressibility condition and Navier-Stokes (for notational simplicity I will drop the vector symbol and allow context to differentiate scalar and vector):

$$\nabla \cdot u = 0 \tag{12}$$

$$\partial_t u + (u \cdot \nabla)u = -\frac{1}{\rho}\nabla p + \nu \Delta u \tag{13}$$

We use the simplifying assumptions that the flow is steady ($\partial_t u = 0$), rotationless ($\partial_\theta = 0$, $u_\theta = 0$), so that eq(13) becomes

$$(u \cdot \nabla)u = -\frac{1}{\rho}\nabla p + \nu \Delta u \tag{14}$$

I now write the ∇ and Δ operators in cylindrical coordinates, using the simplifications above. Let f be a scalar-valued function and A be a vector-valued function, and let a vector in cylindrical coordinates be written in component form as $A = (A_r, A_\theta, A_x)$ where r is the radial direction, θ the rotation, and x the lengthwise direction.

$$\nabla f = (\partial_r f, 0, \partial_x f) \tag{15}$$

$$\nabla \cdot A = \frac{1}{r}\partial_r(rA_r) + \partial_x A_x \tag{16}$$

$$\nabla^2 f = \frac{1}{r}\partial_r\left(r\partial_r f\right) + \partial_{xx} f \tag{17}$$

$$\nabla^2 A = \left(\frac{1}{r}\partial_r(r\partial_r A_r) + \partial_{xx}A_r - \frac{A_r}{r^2}, 0, \frac{1}{r}\partial_r(r\partial_r A_x) + \partial_{xx}A_x\right) \tag{18}$$

Incompressibility equation becomes

$$\nabla \cdot u = 0 \implies \frac{1}{r}\partial_r(ru_r) + \partial_x(u_x) = 0 \tag{19}$$

And Navier-Stokes becomes, (component by component)

$$r \text{ component :} \tag{20}$$

$$u_r\partial_r u_r + u_x\partial_x u_r = -\frac{1}{\rho}\partial_r p + \nu\left(\frac{1}{r}\partial_r(r\partial_r u_r) + \partial_{xx}u_r - \frac{u_r}{r^2}\right) \tag{21}$$

$$\theta \text{ component :} \tag{22}$$

$$0 \tag{23}$$

$$x \text{ component :} \tag{24}$$

$$u_r\partial_r u_x + u_x\partial_x u_x = -\frac{1}{\rho}\partial_x p + \nu\left(\frac{1}{r}\partial_r(r\partial_r u_x) + \partial_{xx}u_x\right) \tag{25}$$

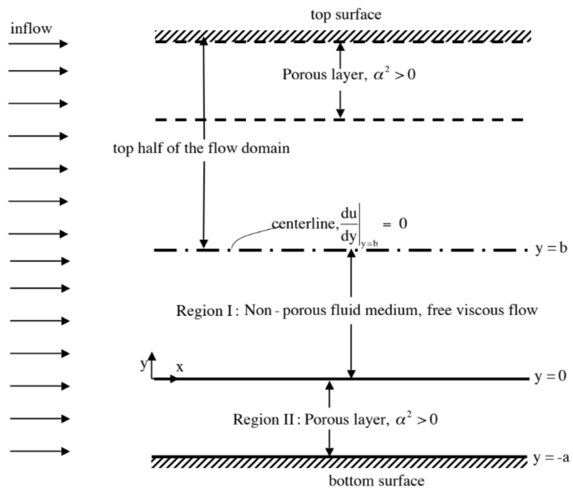

Figure 1: Problem 3 setup

Problem 3: Channel flow between porous layers

Derive a simplified two-dimensional representation of flow between porous layers. Consider a channel as shown in figure 1. The flow field is assumed symmetric about the centerline $y = b$, and the analysis should be restricted to the bottom half of the system. The flow is assumed to be steady ($u_t = 0$) and fully developed $u_x = 0$, and zero in cross-stream direction ($v = 0$).

Region I: Write the simplified x-momentum equation within the free shear flow region $\alpha^2 = 0$ along with appropriate boundary condtions.

Region II: Write down appropriate boundary conditions and simplified equations of motion.

Solve coupled system of ODEs.

Solution. Beginning from the 2D Brinkman equations:

$$\partial_t u + u\partial_x u + v\partial_y u = -\frac{1}{\rho}\partial_x p + \nu\left(\partial_{xx}u + \partial_{yy}u\right) + \alpha^2 \mu u \tag{26}$$

$$\partial_t v + u\partial_x v + v\partial_y v = -\frac{1}{\rho}\partial_y p + \nu\left(\partial_{xx}v + \partial_{yy}v\right) + \alpha^2 \mu v \tag{27}$$

We apply the simplifications: zero in cross-stream ($v = 0$), steady ($\partial_t u = 0$), fully developed ($\partial_x u = 0$), to obtain the simplified momentum equations:

$$0 = -\frac{1}{\rho}\partial_x p + \nu\partial_{yy}u + \alpha^2 \mu u \tag{28}$$

$$0 = -\frac{1}{\rho}\partial_y p \tag{29}$$

From the above we have $p(x, y) = p(x)$.

In region I, $\alpha = 0$ so that eq(28) becomes

$$0 = -\frac{1}{\rho}\partial_x p + \nu \partial_{yy} u \tag{30}$$

with boundary conditions

$$\partial_y u(y = b) = 0 \tag{31}$$
$$\lim_{y \to 0^+} u = \lim_{y \to 0^-} u \tag{32}$$

Then, solving eq(30) by direct integration:

$$u(y) = \left(\frac{y^2}{2\nu\rho}\right)\partial_x p + Ay + B \tag{33}$$

Applying BC eq(31):

$$\partial_y u(y = b) = 0 \implies -\left(\frac{b}{\nu\rho}\right)\partial_x p = A \tag{34}$$

So that

$$u(y) = \left(\frac{\partial_x p}{\nu\rho}\right)\left(\frac{1}{2}y^2 - by + C\right) \tag{35}$$

where C is some yet-to-be-determined constant. Finally, note that

$$\lim_{y \to 0^+} u(y) = \left(\frac{\partial_x p}{\nu\rho}\right) C \tag{36}$$

Turning to region II, eq(28) becomes (denoting $u \to u^*$ to help differentiate)

$$0 = -\frac{1}{\rho}\partial_x p + \nu \partial_{yy} u^* + \alpha^2 \mu u^* \tag{37}$$

with boundary conditions

$$u^*(y = -a) = 0 \tag{38}$$
$$\lim_{y \to 0^+} u = \lim_{y \to 0^-} u^* \tag{39}$$

To solve eq(37), one would normally add the general solution to the homogenous equation to a particular solution for the inhomogenous equation. However, the general solution to the homogenous equation is simply the trivial zero function, so we only need to find a particular solution. I take the anzatz $u^*(y) = Ay^2 + By + D$. Differentiating we find that

$$u^*(y) = \left(-\frac{\partial_x p}{\rho\alpha^2\mu}\right) y + D \tag{40}$$

After applying eq(38), and enforcing eq(32):

$$\boxed{\begin{aligned} u(y) &= \left(\frac{\partial_x p}{\nu\rho}\right)\left(\frac{1}{2}y^2 - by - \frac{a\nu}{\alpha^2\mu}\right) \\ u^*(y) &= \left(-\frac{\partial_x p}{\rho\alpha^2\mu}\right)(y+a) \end{aligned}} \tag{41}$$