Chapter 1

Equations of Fluid Motion

Introduction

In this chapter we provide an introduction to fluid dynamics that will serve as a starting point for understanding how organisms move through fluids. We first present a very general set of equations that model incompressible, constant viscosity fluid flow. A discussion of non-dimensional scaling parameters in the model will help describe similar mechanisms of swimming and flight over a range of scales. The analytical solution for steady flow through a pipe provides one of the simplest examples of flow relevant to biological systems. We end the chapter with an introduction to non-Newtonian fluids such as mucus and saliva.

The Navier-Stokes Equations

We derive the Navier-Stokes equations of motion for an incompressible fluid using Newton's second law, $\Sigma F = ma$. At the scale relevant to most forms of animal locomotion, it makes sense to use this model because we treat the fluid as a continuum. Consider a fluid of constant density ρ with velocity components $\mathbf{u} = [u, v, w]$ at Cartesian coordinates $\mathbf{x} = [x, y, z]$. Then $\mathbf{u}(\mathbf{x}, t)$ describes the fluid velocity at any point $\mathbf{x}$ and time t. Now let's allow $g(\mathbf{x}, t)$ to denote some quantity of interest in the flow, such as a component of the velocity.

To begin the derivation, we determine an expression for the acceleration of a fluid parcel. A fluid parcel is a very small volume of fluid with constant mass that moves with the flow. There are two standard ways to describe the motion of a fluid. You can sit at a fixed point and describe the motion of a fluid parcel as it passes by, which is called the Eulerian or lab frame. Alternatively, you can imagine you are moving with the fluid parcel watching the flow around you, which is called the Lagrangian frame. The time rate of change of the flow velocity at a fixed position in the lab frame of reference would be given by the partial derivative $\frac{\partial u}{\partial t}$. Another quantity is needed to describe the acceleration of a particular fluid parcel in the Lagrangian frame. This quantity is typically called the material derivative, $\frac{D}{Dt}$.

In general we can use the material derivative to describe the time rate of change of any quantity g as one follows the fluid,

$$\frac{Dg}{Dt} = \frac{d}{dt} g[x(t), y(t), z(t), t], \tag{1}$$

where $x(t)$, $y(t)$, and $z(t)$ change with time at the local fluid velocity $\mathbf{u}$. Using the fact that $\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$, and $\frac{dz}{dt} = w$ and applying the chain rules yields the following result:

$$\frac{Dg}{Dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt} + \frac{\partial g}{\partial t} \tag{2}$$

$$\frac{Dg}{Dt} = \frac{\partial g}{\partial t} + (\mathbf{u} \cdot \nabla) g. \tag{3}$$

If g is set to a component of the velocity, this result implies that the acceleration of a fluid parcel is given by the equation

$$\frac{D\mathbf{u}}{Dt} = \frac{d\mathbf{u}}{dt} + (\mathbf{u} \cdot \nabla)\mathbf{u}. \tag{4}$$

If δV is the volume of the fluid parcel, then $\rho \delta V \frac{D\mathbf{u}}{Dt}$ is equal to the mass times the acceleration of the fluid parcel.

The next step in the derivation of the Navier-Stokes equations is to determine all of the forces acting upon the fluid parcel. Let S be the surface enclosing the volume of the fluid parcel, δV. Now consider a geometric surface element on the surface of S denoted by δS. The forces acting on δS in the direction normal to the surface are given by $p\mathbf{n}\delta S$, where $p = p(x, y, z, t)$ is a scalar function describing the pressure acting on the surface of the fluid and $\mathbf{n}$ is the normal vector. Using an identify derived from the Divergence Theorem, the net force on the fluid parcel is given by

$$\int_S p\mathbf{n} dS = -\int_V \nabla p dV \tag{5}$$

where the negative sign is included to account for the fact that **n** points in the outward direction. If we assume that ∇p is continuous, then it will be almost constant over the fluid parcel of volume δV. The net force on the parcel due to the pressure of the surrounding fluid is then equal to $-\nabla p \delta V$. The force of gravity acting on the fluid parcel is given by $mg = \rho \delta V g$, where g is the gravitational acceleration. For Newtonian fluids such as air and water, the viscous force due to the fluid's resistance to shear is linearly proportional to the dynamic viscosity of the fluid, μ. For Newtonian fluids like air and water, the viscous force acting on the fluid parcel is usually described by $\mu \Delta \mathbf{u} \delta V$.

Setting the mass times acceleration of the fluid parcel equal to the forces acting on the parcel and dividing by $\rho \delta V$ results in the following three equations that represent the conservation of linear momentum in each coordinate direction:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{1}{\rho} f_{B,x} \tag{6}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \frac{1}{\rho} f_{B,y} \tag{7}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \frac{1}{\rho} f_{B,z} \tag{8}$$

where ν is the kinematic viscosity of the fluid, which is the ratio of the coefficient of dynamic viscosity to the density of the fluid ($\nu = \mu/\rho$), while f_B indicates the body force acting on the fluid flow. This body force may be due to gravity (as discussed above) or to some other external force. This set of equations requires that the inertial force must be equal and opposite to the sum of the pressure force, viscous force, and body force.

The final step in the derivation of the Navier-Stokes equations will be to determine the condition for incompressibility. Consider an arbitrary volume of fluid V. Let S be the fixed closed surface enclosing the region V and let **n** be defined as the outward normal. The fluid enters this volume at some points on the surface and exits at other points. The volume of fluid leaving through a small surface element δS per unit time is given by $\mathbf{u} \cdot \mathbf{n} \delta \mathbf{S}$. The net rate of fluid leaving this volume may be found by integrating over the surface. For an incompressible fluid, the net rate of fluid leaving the volume (the difference between the amount entering and the amount leaving) should be zero, and an application of the Divergence theorem yields the following result:

$$\int_S \mathbf{u} \cdot \mathbf{n} dS = \int_V \nabla \cdot \mathbf{u} dV = 0 \tag{9}$$

Since this must be true for all regions of the fluid, it follows that

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{10}$$

The collective set of governing conservation equations 6 – 8 and 10 are the Navier–Stokes equations. The nonlinear nature of these partial differential equations render them difficult to solve, and only a few exact analytical solutions for specific problems are known. [LAURA FIX ME! Say we will see some of the simplifications such as disregarding/omitting viscosity or inertia, that make the equations analytically tractable, as well as some numerical approaches to approximate solutions.]

0.1 Fluid dynamic scaling

To compare the flows of different fluids over a range of length scales and velocities, it is a useful exercise to non-dimensionalize the terms in the above governing conservation equations as follows,

$$x' = \frac{x}{L}, y' = \frac{y}{L}, z' = \frac{z}{L}$$
$$u' = \frac{u}{U}, v' = \frac{v}{U}, w' = \frac{w}{U}$$
$$t' = \omega t, p' = \frac{p}{\rho U^2}$$

. Here L, U, and $1/\omega$ are characteristic flow length, velocity, and time scales respectively. The values for the characteristic scales are often chosen particular to the application. In the case of blood flow through an artery, the diameter of the vessel may be chosen as the characteristic length scale, the maximum velocity may be chosen as the characteristic velocity, and the period of the heartbeat may be chosen as the characteristic time scale. For the case of insect flight, the chord length of the wing may be chosen as the characteristic length scale, the average wing tip velocity may be chosen as the characteristic velocity, and the period of the wing beat may be chosen as the characteristic time scale.

The application of these terms to equations 6 – 8 and 10 results in the following set of equations if one neglects the body force contribution,

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'} = 0 \tag{11}$$

$$\left(\frac{\omega L}{U}\right)\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}+w'\frac{\partial u'}{\partial z'}=-\frac{\partial p'}{\partial x'}+\left(\frac{UL}{\nu}\right)\left(\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 u'}{\partial y'^2}+\frac{\partial^2 u'}{\partial z'^2}\right) \tag{12}$$

$$\left(\frac{\omega L}{U}\right)\frac{\partial v'}{\partial t'}+u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}+w'\frac{\partial v'}{\partial z'}=-\frac{\partial p'}{\partial y'}+\left(\frac{UL}{\nu}\right)\left(\frac{\partial^2 v'}{\partial x'^2}+\frac{\partial^2 v'}{\partial y'^2}+\frac{\partial^2 w'}{\partial z'^2}\right) \tag{13}$$

$$\left(\frac{\omega L}{U}\right)\frac{\partial w'}{\partial t'}+u'\frac{\partial w'}{\partial x'}+v'\frac{\partial w'}{\partial y'}+w'\frac{\partial w'}{\partial z'}=-\frac{\partial p'}{\partial z'}+\left(\frac{UL}{\nu}\right)\left(\frac{\partial^2 w'}{\partial x'^2}+\frac{\partial^2 w'}{\partial y'^2}+\frac{\partial^2 w'}{\partial z'^2}\right). \tag{14}$$

It can be observed that there are two dimensionless ratios that are common in the momentum conservation equations, and these are the only two free parameters in the Navier–Stokes equations governing the dynamics of fluid flow. They are defined as the Reynolds number *Re* and Stokes number *S*,

$$Re=\frac{\rho UL}{\mu}=\frac{UL}{\nu}, S=\frac{\omega L^2}{\nu}.$$

Of particular relevance to hemodynamics are the Reynolds number and a modified form of the Stokes number, known as the Womersley number (or Womersley–Witzig parameter) denoted by α,

$$\alpha=\sqrt{S}=L\sqrt{\frac{\omega}{\nu}}.$$

The Reynolds number can be thought of as the ratio of inertial forces to viscous forces acting in the fluid flow; flows of the same *Re* are dynamically similar. An example of biological flows where $Re >> 1$ can be seen in the case of fish swimming, while the movement of bacteria by flagella occurs at $Re << 1$. Low *Re* flows such as those experienced by bacteria are reversible, which means that if you drive motion with a boundary then move that boundary back through the same path, then no matter what the speed, the fluid will return to its initial state. A consequence of reversibility is that net fluid transport and locomotion do not occur by reciprocal motions, such as a fin that flaps in a symmetric manner. In high *Re* flows , which are dominated by pressure forces, locomotion and fluid transport are possible using reciprocal motions. Similar comparisons analogies can also be seen in pumping fluids. In the adult human heart, $Re \approx 1000$ and inertial effects dominate. Unidirectional flow is generated by the valves. The blood flow through capillaries occurs at $Re <<1$, and viscous effects dominate. Very low *Re* pumps such as the fallopian tube generate unidirectional flow through peristaltic motions.

The *Re* is also used to predict whether fluid flow may be characterized as laminar, transitional, or turbulent. For the internal, steady, incompressible flow of a fluid of constant viscosity through a circular pipe, *laminar* flow is observed to occur at fairly low speeds where $Re < 1000$. In laminar flow the fluid motion is unidirectional and slowed near stationary boundaries. For the same flow scenario, when the Reynolds number is greater than 2300, the flow is classified as being *turbulent*, where enhanced mixing is found to occur at much faster time scales as compared to pure molecular diffusion that is characteristic of laminar flows. The flow for this problem is said to be in a *transitional* state for $2000 \leq Re \leq 2300$, where it fluctuates in time from being laminar to turbulent.

The Womersley number is important to quantify the unsteady effects of fluid flow, and is naturally important in pulsatile systems common in cardiovascular flows. Within the context of a blood vessel, when the value of α is high, then the velocity is flat over most of the cross-section and there is a small region near the vessel wall known as the *boundary layer* wherein viscous effects are important (see Fig. 1). Also, the flow at the center of the tube is inertial and pulsatile. A typical value of α for the adult canine heart is roughly 13 [1]. At the other low extreme end of α, the velocity profile over the vessel cross-section is parabolic in nature, and the flow is quasi-steady and viscous dominated. The transient effects can be ignored when α is sufficiently small, and this is common in the case of microcirculation such as in capillaries and arterioles where the effect of the the heart pulsation is no longer felt [1].

When the *Re* is sufficiently small, the inertial terms in the momentum equations can be ignored, resulting in the equation below

$$\nabla p = \mu \Delta \mathbf{u}. \tag{15}$$

The above relation, also known as Stokes equation, shows that in the limit of very low *Re* and α, the flow is entirely driven through an intricate balance between the pressure gradient and viscous diffusion. To illustrate this, consider the flow in embryonic heart with typical flow passage length of 50 μm, flow velocity of 1 mm/s, and blood viscosity of 0.02 poise, and density roughly 1 g/cm^3, with a heart beat frequency of 2 Hz. The *Re* value of this flow is roughly 0.03 and α value is 0.05, both much less than 1, and hence the Stokes equation applies to this case. Figure 2 shows the distribution of the Reynolds number as a function of the characteristic flow velocity imparted by various biological pumps found in nature.

0.2 HW 1, Q1: Nondimensionalizing Navier Stokes

Using the nondimensionalization given able, please show that equations 12 - 14 hold.

1 Application: Pipe flow

Pipe flow is relevant to organisms swimming through tubes as well as to organisms that locomote by forcing fluids through cylindrical structures. Some examples of each include certain bacteria (and nanobots) that swim through blood vessels and squid and salps that swim by ejecting water through a siphon. To analyze these flows it is often useful to start with a simplified model of internal flow through a pipe.

Consider the steady, incompressible, two dimensional (radial r and axial x, see definitions in Fig. 1), incompressible, internal flow of a fluid of density ρ and uniform dynamic viscosity μ through a cylinder of radius R. The flow velocity is assumed to have no rotational component ($u_\theta = 0$), and the flow is considered to be axisymmetric about the central axis of the pipe, such that $\frac{\partial}{\partial\theta} = 0$.

$$\frac{\partial u_r}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r u_r) = 0 \tag{16}$$

$$u_x \frac{\partial u_x}{\partial x} + u_r \frac{\partial u_x}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial r^2} + \frac{1}{r}\frac{\partial u_x}{\partial r} \right) \tag{17}$$

$$u_x \frac{\partial u_r}{\partial x} + u_r \frac{\partial u_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \left(\frac{\partial^2 u_r}{\partial x^2} + \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} \right) \tag{18}$$

This problem can be solved analytically by considering a few simplifying assumptions. In the case of microscale fluid dynamics, the flow reaches a fully developed state at a short distance from its entrance to the pipe such that there is no variation in the flow velocity along the primary axial direction ($\frac{\partial}{\partial x} = 0$). This reduces the above equation set 16–18 to the following

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) = 0 \tag{19}$$

$$u_r \frac{\partial u_x}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u_x}{\partial r^2} + \frac{1}{r}\frac{\partial u_x}{\partial r} \right) \tag{20}$$

$$u_r \frac{\partial u_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} \right) \tag{21}$$

The above equations of mass and momentum are subject to the following conditions at specific boundaries in the problem domain:

$$u_r(r=0) = 0 \tag{22}$$

$$u_r(r=R) = 0 \tag{23}$$

$$u_x(r=R) = 0 \tag{24}$$

$$\left.\frac{\partial u_x}{\partial r}\right|_{r=0} = 0 \tag{25}$$

The boundary conditions 22 and 25 are obtained by imposing symmetry about the centerline. The boundary condition 23 ensures that there is no normal flow through the pipe, and condition 24 means that the layer of fluid that is in contact with the wall remains at rest (“no slip” of fluid on the solid surface). From the continuity equation 19, we obtain

$$ru_r = constant$$

. Applying the boundary conditions 22 and 23, it can be seen that there is no radial flow throughout the pipe ($u_r = 0$). This simplifies the r-momentum equation 21 to the form given below,

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = 0$$

. As the flow is incompressible, this means that the dynamic pressure is invariant in the radial direction and is only a function of the axial location ($p = p(x)$). The axial momentum conservation equation 20 now becomes

$$\frac{1}{\rho}\frac{dp}{dx} = \nu\left(\frac{\partial^2 u_x}{\partial r^2} + \frac{1}{r}\frac{\partial u_x}{\partial r}\right)$$

which can be written as,

$$\frac{r}{\mu}\frac{dp}{dx} = \frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right)$$

. Integrating both sides of the above equation in terms of r, we obtain

$$\frac{r^2}{2\mu}\frac{dp}{dx} = r\frac{\partial u_x}{\partial r} + A$$

where A is the constant of integration, the value of which is determined by applying 25 to the above equation. The resultant equation can be integrated once again in terms of r to solve for the axial velocity profile $u_x(r)$,

$$\frac{r^2}{4\mu}\frac{dp}{dx} = u_x + B$$

. The constant of integration is determined by using boundary condition 24. The solution for the axial velocity is thus given by

$$u_x(r) = -\left(\frac{R^2}{4\mu}\frac{dp}{dx}\right)\left(1 - \frac{r^2}{R^2}\right) = u_{max}\left(1 - \frac{r^2}{R^2}\right). \tag{26}$$

The internal flow through a pipe under the previously stated assumptions has a parabolic velocity profile with the peak located along the centerline, the magnitude of which depends on the pressure gradient at the particular axial location of interest, the dynamic viscosity of the fluid, and the pipe radius. The pressure gradient is referred to be adverse when $\frac{dp}{dx} > 0$ resulting in a decelerating flow, and is favorable when $\frac{dp}{dx} < 0$ and the flow accelerates.

The viscous fluid flow exerts a tangential shear stress which can be determined as the gradient of the axial velocity as given below:

$$\tau_{xr} = \tau_{rx} = \mu \frac{\partial u_x}{\partial r} = -\frac{2\mu u_{max} r}{R^2} = \frac{r}{2}\frac{dp}{dx}. \tag{27}$$

Of potential importance to locomotory control is the shear stress imposed by the fluid on the walls of the tube, which is given by,

$$\tau_w = \tau_{xr}\big|_{r=R} = -\frac{2\mu u_{max}}{R} = \frac{R}{2}\frac{dp}{dx}. \tag{28}$$

The mean flow velocity through the tube can be calculated by integrating the axial velocity profile over the cross section,

$$\overline{u}_x = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R u_x(r)\, r\, dr\, d\theta = \frac{u_{max}}{2}. \tag{29}$$

The shear stress can be redefined in terms of the volumetric flow rate Q based on mean flow velocity as

$$\tau_w = -\frac{4\mu \overline{u}_x}{R} = -\frac{4\mu Q}{\pi R^3} = -\frac{32\mu Q}{\pi D^3}. \tag{30}$$

As an example, the shear stress in conjunctival capillaries of diameter 10 μm in the adult human circulatory system is approximately 15 dynes/cm^2 [2].

This set of derived equations provides physical intuition about the flows. Equation 26 tells us that for a given pressure jump across a tube, the maximum velocity in the tube will be proportional to the square of the radius. In other words, it is difficult to have high velocities in a small tube. You can also think of this as a consequence of the no slip boundary condition and the resistance to shear. Since there is little distance between the walls where velocity is zero, there's no way for velocity to increase to a high value unless the fluid has high shear. This does not happen with viscous fluids, which resist shear. We can interpret equation 28 for shear stress at the wall in a couple of ways. For a given velocity u_{max}, the sheer stress will be proportional to the viscosity and inversely proportional to radius. Also, for a constant pressure drop $\frac{dp}{dx}$, the shear stress will be proportional to radius.

1.1 HW 1, Q2

Convert the Navier-Stokes equations to cylindrical coordinates to show that equations 16 - **??** hold.

Tables and Figures

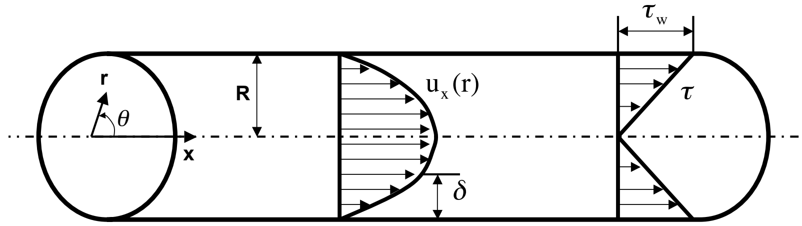

Fig. 1 Internal flow of a fluid through a circular cylinder of radius R (also known as Hagen–Poiseuille flow). Under the simplifications of two-dimensional, time-invariant, axisymmetric (no fluid rotation, u_θ=0), incompressible flow of a fluid with uniform density and viscosity, the axial velocity u_x remains unchanged along the longitudinal direction x after a certain length (typically 2–4 multiples of the radius) from the entrance, and this condition is also known as a *fully developed* flow. However, the radial variation of the axial velocity $u_x(r)$ is parabolic about the centerline axis as shown. The shear stress imposed by the flow τ in the radial direction varies linearly from the centerline, and the maximum value τ_w occurs at the walls. Note that δ indicates the thickness of the boundary layer, which is the region near the solid boundaries where the deceleration of the flow on account of fluid viscosity is non-negligible. The coordinate system used for the analysis of this problem (r–radial, x–axial, θ–rotational) is also shown.

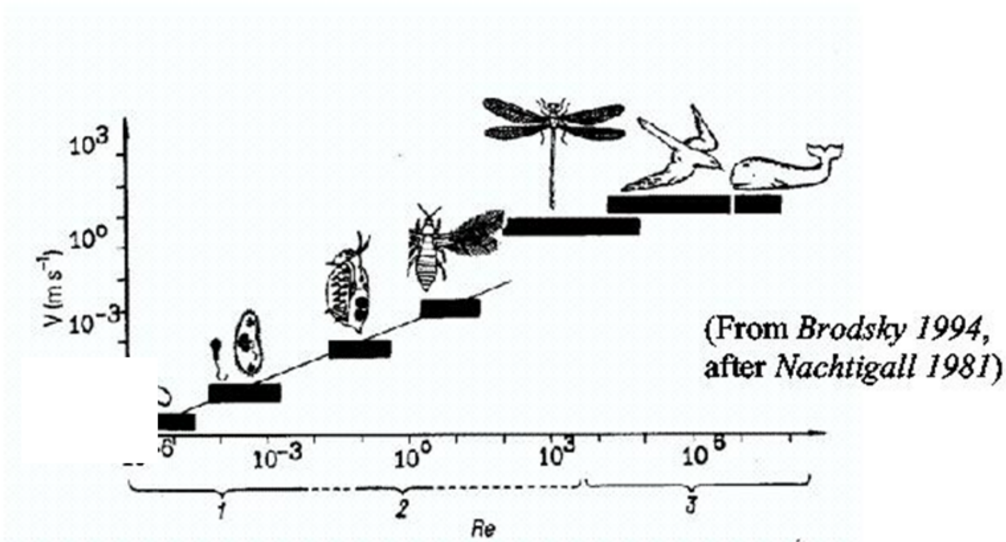

Fig. 2 Graph showing the Reynolds number (*Re*) verses the forward swimming or flying velocity for a variety of organisms. The *Re* is calculated using the kinematic viscosity of the fluid and the average speed and length of the organism.

Chapter 2

Porous Environments

Introduction

In this chapter we provide an introduction to flow through porous layers, which are relevant to many biological processes. For example, bristled wings and fins used in small-scale swimming and flight can be modeled as infinitely thin porous sheets. Flows through structures over a large range of scales can also be modeled as porous layers. For example, blood flows through an extracellular protein layer near the vessel wall called the endothelial surface layer. On a larger scale, flow through vegetation relevant to the locomotion and dispersal of organisms is also modeled with equations describing porous media. We introduce the analytical solution for steady flow in a channel lined with porous layers as a relatively simple example. We end the chapter with an introduction to flow through infinitely thin layers that is relevant to bristled wings and fins.

Darcy's Law

Henry Darcy [**?**] originally derived a constitutive equation describing the fluid flow through a porous medium based upon experimental measurements of flow rates through sand beds. Darcy's law simply states that there is a linearly proportional relationship between the volumetric flow rate through a porous medium, the viscosity of the fluid, and the pressure drop over a given distance. This law is written as follows:

$$Q = -\frac{kA}{\mu}\frac{(P_b - P_a)}{L} \tag{31}$$

where Q is the volumetric flow rate, k is the permeability of the porous medium with units of $length^2$, A is the cross-sectional area normal to the direction of flow, $P_b - P_a$ is the pressure drop across the porous region, and L is the length of the porous region. This law essentially states that if there is a pressure gradient, there will be flow from regions of high pressure to regions of low pressure that is proportional to the permeability of the medium. The model is typically only valid for Reynolds numbers on the order of 1 or below, where the pore diameter is the characteristic length scale.

Flow through porous layers

Brinkman model

The Brinkman equation was developed as a generalization of equation 31 that interpolates between the Navier-Stokes equations and Darcy's law [**?**]. The full two-dimensional Brinkman equation is given as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \alpha^2 \mu u \tag{32}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \alpha^2 \mu v \tag{33}$$

where α^2 is the hydraulic resistivity (the inverse of the hydraulic permeability). These velocities and pressures should be viewed as averages over an ensemble of different realizations of the porous medium. Since its first development, it has been rigorously shown that this equation is valid at low volume fractions of solids [**?**], and empirical work suggests that it is also reasonable at high volume fractions.

The major limitation of this model is that the relationship between the hydraulic resistivity and the structure of the layer is not well understood. For a dense porous medium (with highly packed arrays so that the layer closely resembles a solid clump), the flow within the layer is obstructed to a larger extent than it would be for a less dense porous medium (loosely packed arrays). Hence, a more tightly packed porous medium has a lower hydraulic permeability or a greater value of α^2 than a less tightly packed medium.

HW 1, Q3: Channel flow between porous layers

Derive a simplified two-dimensional representation of flow between porous layers. Consider a channel consisting of a central main region with unobstructed parallel shear flow (Region I with $\alpha^2 = 0$) that is bounded on the top and the bottom by identical porous layers (Region II with $\alpha^2 > 0$) as shown in Fig. 3.The flow field is assumed to be symmetric about the centerline $y = b$, and the analysis should be restricted to the bottom half of the system. Similar to the case of pipe flow, the flow is considered to be steady ($\partial u/\partial t = 0$), fully developed ($\partial u/\partial x = 0$), and zero in the cross-stream direction ($v = 0$).

Region I: Using the above assumptions, the x-momentum equation (32) within this free shear flow region ($\alpha^2 = 0$) can be simplified. Write down appropriate boundary conditions and simplify the equations of motion.

Region II: To distinguish between the velocity of the flow in region I (equation (**??**)) and II, u^* can be used to denote the streamwise x-directional velocity in the porous layer. Write down appropriate boundary conditions and simplify the equations of motion.

Now solve a system of coupled ODEs.

Tables and Figures

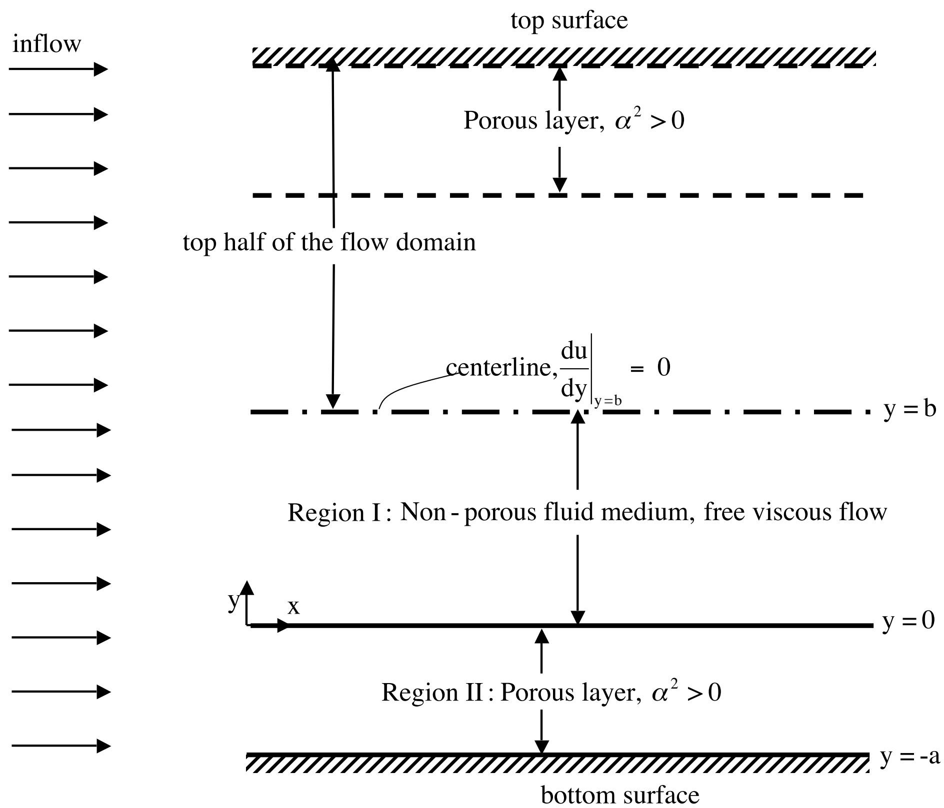

Fig. 3 Schematic of the system considered in the derivation of the simplified one-dimensional mathematical model for examining flow dynamics within a channel that partially consists of a porous layer. The interaction of the layer with the fluid is modeled as a Brinkman term by introducing a frictional force that is added in the incompressible, constant viscosity Navier–Stokes equations, where α^2 is the inverse of the hydraulic permeability of the medium. u is the velocity component in the x (streamwise) direction, and v is the velocity component in the y (cross-stream) direction. Note that this model assumes the walls in the x direction are at infinite distance (the layer is as long in x as the free shear flow central region), flow is steady ($\partial/\partial t = 0$), no cross-stream component of velocity ($v = 0$), and fully developed streamwise profile that is invariant in x ($\partial u/\partial x = 0$). Region I refers to the domain in the system where the fluid behaves as a classical Newtonian fluid. Region II refers to the domain where $\alpha > 0$, and with increasing α the hydraulic permeability decreases, so that the resistance offered by the material to the flow increases. $y = b$ represents the free surface along the centerline of the parallel flow, with the bottom wall at $y = -a$ (the model assumes centerline symmetry about $y = b$).

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