LABORATORY OF ROBUST IDENTIFICATION AND CONTROL

Lab#2 Solution

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1. Problem 1: FPS mathematical formulation

2. Problem 2: PUI computation and Linear Programming

Set-Membership Identification with EE noise structure

Problem 1: description

In this problem we assume that the plant to be identified is exactly described as a discrete-time LTI model as follows:

$$G_p(q^{-1}) = \frac{\theta_2}{1 + \theta_1 q^{-1}} \tag{1}$$

where $\theta = [\theta_1 \ \ \theta_2] = [-0.5 \ \ 2]$ Assume that the uncertainty affecting the input-output data enters the problem according to an **Equation-Error (EE)** structure. Furthermore, to perform the simulation assume that:

$$\tilde{u} = [4, -3, 2, 1]^T \quad e = [0.05, -0.25, 0.3, -0.5] \tag{2}$$

Perform a simulation of model (1) in order to simulate the experiment. The equation to be used is the following:

$$\tilde{y}(k) = -\theta_1 \tilde{y}(k-1) + \theta_2 \tilde{u}(k) + e(k), \quad \tilde{y}(1) = 0 \tag{3}$$

where e is such that $|e(k)| \leq \Delta_e$, $\Delta_e = 0.5$

Problem 1: FPS mathematical formulation (task 1)

Task 1

Assuming that the bound Δ_e is the only available information on the error e, provide the implicit mathematical definition of the feasible parameter set (FPS) $\mathcal{D}_\theta$ for the considered Set-Membership System Identification problem.

$$
\begin{aligned}
\mathcal{D}_\theta = \{\theta \in \mathbb{R}^2 : \tilde{y}(k) + \theta_1 \tilde{y}(k-1) - \theta_2 \tilde{u}(k) = e(k), k = 2, 3, 4 \\
|e(k)| \leq \Delta_e \ k = 1, ..., 4\} = \\
\{\theta \in \mathbb{R}^2 : |\tilde{y}(k) + \theta_1 \tilde{y}(k-1) - \theta_2 \tilde{u}(k)| \leq \Delta_e, \ k = 2, 3, 4\} = \\
\{\theta \in \mathbb{R}^2 : -\Delta_e \leq \tilde{y}(k) + \theta_1 \tilde{y}(k-1) - \theta_2 \tilde{u}(k) \leq \Delta_e, \\
k = 2, 3, 4\} = \{\theta \in \mathbb{R}^2 : \theta_1 \tilde{y}(k-1) - \theta_2 \tilde{u}(k) \leq \Delta_e - \tilde{y}(k), \\
-\theta_1 \tilde{y}(k-1) + \theta_2 \tilde{u}(k) \leq \Delta_e + \tilde{y}(k)\} \implies \text{(see next slide)}
\end{aligned}
\tag{4}
$$

Problem 1: FPS mathematical formulation (task 1)

$$\mathcal{D}_\theta = \{\theta \in \mathbb{R}^2 : A\theta \leq b\} \quad \text{bounded polyhedron} \rightarrow \textbf{POLYTOPE}$$

where A, θ, b are such that:

$$\underbrace{\begin{bmatrix} \tilde{y}(k-1) & -\tilde{u}(k) \\ -\tilde{y}(k-1) & \tilde{u}(k) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}}_{\theta} \leq \underbrace{\begin{bmatrix} \Delta_e - \tilde{y}(k) \\ \Delta_e + \tilde{y}(k) \end{bmatrix}}_{b} \qquad k = 2, 3, 4 \tag{5}$$

In particular:

$$A = \begin{bmatrix} 0 & 3 \\ -6.25 & -2 \\ 1.175 & -1 \\ 0 & -3 \\ 6.25 & 2 \\ -1.175 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 6.75 \\ -0.675 \\ -1.5875 \\ -5.75 \\ 1.675 \\ 2.5875 \end{bmatrix} \tag{6}$$

Problem 1: Graphical representation of the FPS (task 2)

Task 2

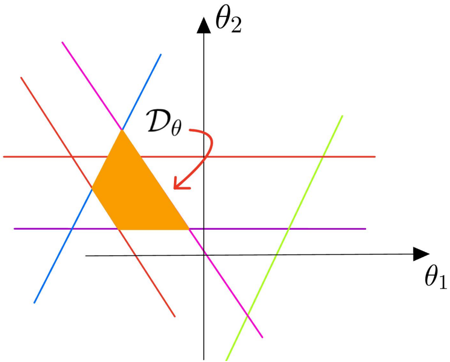
Provide a graphical representation of obtained FPS and analyze its geometrical features.

By using the (5) and substituting the data in (6) we obtain the following set of inequalities:

$$\begin{cases} 3\theta_2 \leq 6.75 \\ -6.25\theta_1 - 2\theta_2 \leq -0.675 \\ 1.175\theta_1 - \theta_2 \leq -1.5875 \\ -3\theta_2 \leq -5.75 \\ 6.25\theta_1 + 2\theta_2 \leq 1.675 \\ -1.175\theta_1 + \theta_2 \leq 2.5875 \end{cases} \tag{7}$$

Problem 1: Graphical representation of the FPS (task 2)

In the $\mathbb{R}^2$ plane, each one of the inequalities is an halfplane. If we take their intersection we obtain an $\mathbb{R}^2$-polytope which represents the FPS $\mathcal{D}_\theta$, the one indicated in orange.



At this point the following **optimization problems** have to be solved.

For the PUI_{θ_1}

$$\underline{\theta}_1 = \min_{\theta \in \mathcal{D}_\theta} \theta_1 \tag{8}$$

$$\overline{\theta}_1 = -\min_{\theta \in \mathcal{D}_\theta} -\theta_1 \tag{9}$$

For the PUI_{θ_2}

$$\underline{\theta}_2 = \min_{\theta \in \mathcal{D}_\theta} \theta_2 \tag{10}$$

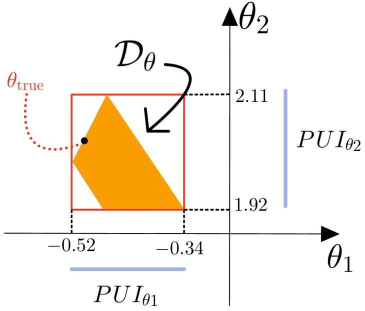
$$\overline{\theta}_2 = -\min_{\theta \in \mathcal{D}_\theta} -\theta_2 \tag{11}$$

Problem 1: PUI computation (task 3)

Task 3

Exploiting the obtained geometrical description, compute the parameter uncertainty interval PUI_{θ_1} and PUI_{θ_2}

By using MATLAB, and outerbounding the polytope with a rectangle, we can say that approximately the PUIs are those shown in the following figure:



The PUIs are then approximately:

$$PUI_{\theta_1} = [-0.52, -0.34] \tag{12}$$

$$PUI_{\theta_2} = [1.92, 2.11] \tag{13}$$

Note that $\theta_{\text{true}} = [-0.5, 2]$ is on the border of the computed $\mathcal{D}_\theta$. This is important for the SM Identification approach.

Set-Membership Identification with EE noise structure

Problem 2: description

In this problem we assume that the plant to be identified is exactly described as a DT LTI model whose transfer function is:

$$G_p(q^{-1}) = \frac{N(q^{-1})}{D(q^{-1})} = \frac{\theta_3 + \theta_4 q^{-1} + \theta_5 q^{-2}}{1 + \theta_1 q^{-1} + \theta_2 q^{-2}} \tag{14}$$

where $\theta = [\theta_1 \ \theta_2 \ \theta_3 \ \theta_4 \theta_5] = [-0.7647, 0.3012, 0, 32.24, 21.41]$. The objective here is to compute a Set-Membership estimation of the model parameters. The data are collected by simulating the system (14). In particular, as far as the input is concerned, it is a random sequence uniformly distributed in $[0, 1]$, while the error is a random sequence uniformly distributed in $[-\Delta_e, \Delta_e]$[1]. The output sequence must be computed as follows:

$$y(k) = -\theta_1 y(k-1) - \theta_2 y(k-2) + \theta_3 u(k) + \theta_4 u(k-1) + \theta_5 u(k-2) + e(k) \tag{15}$$

[1] Different Δ_e must be used from 0.1 to 100

Problem 2: Feasible Parameter Set $\mathcal{D}_\theta$ definition

Main ingredients

- *A-priori information on the system*: **second order** (n=2), **linear time-invariant (LTI)**;
- *A-priori information on the noise*:
  1. Noise structure: the uncertainty enters into the identification problem as an additive term (equation-error), then the same as before;
  2. Noise property: the noise samples are bounded, that is $|e(k)| \leq \Delta_e$, $k = 1, ..., N$, the bound Δ_e is known
- *A-posteriori information*: data generated by simulating the system with randomly generated $u(k)$ and $e(k)$.

Problem 2: Feasible Parameter Set $\mathcal{D}_\theta$ definition

Implicit mathematical formulation

Task 1

Define the feasible parameter set (FPS) $\mathcal{D}_\theta$ for the considered set-membership identification problem.

$$
\begin{aligned}
\mathcal{D}_\theta = \{\theta \in \mathbb{R}^5 :\ & y(k) = -\theta_1 y(k-1) - \theta_2 y(k-2) + \\
& + \theta_3 u(k) + \theta_4 u(k-1) + \theta_5 u(k-2) + e(k),\ k = 3, \ldots, N \\
& |e(k)| \leq \Delta_e,\ k = 1, \ldots, N\} = \\
& \{\theta \in \mathbb{R}^5 :\ y(k) + \theta_1 y(k-1) + \theta_2 y(k-2) + \\
& - \theta_3 u(k) - \theta_4 u(k-1) - \theta_5 u(k-2) = e(k),\ k = 3, \ldots, N \\
& |e(k)| \leq \Delta_e,\ k = 1, \ldots, N\} \Longrightarrow \text{(see next slide)}
\end{aligned} \tag{16}
$$

Problem 2: Feasible Parameter Set $\mathcal{D}_\theta$ definition

Implicit mathematical formulation

By eliminating the dependence on $e(k)$ samples, the set is rewritten as:

$$
\begin{aligned}
\mathcal{D}_\theta = \{\theta \in \mathbb{R}^5 :\ & |y(k) + \theta_1 y(k-1) + \theta_2 y(k-2) + \\
& - \theta_3 u(k) - \theta_4 u(k-1) - \theta_5 u(k-2)| \leq \Delta_e,\ k = 3, \ldots, N\} = \\
\{\theta \in \mathbb{R}^5 :\ & \theta_1 y(k-1) + \theta_2 y(k-2) + \\
& -\theta_3 u(k) - \theta_4 u(k-1) - \theta_5 u(k-2) \leq \Delta_e - y(k) \\
& -\theta_1 y(k-1) - \theta_2 y(k-2) + \\
& +\theta_3 u(k) + \theta_4 u(k-1) + \theta_5 u(k-2) \leq \Delta_e + y(k),\ k = 3, \ldots, N\}
\end{aligned}
\tag{17}
$$

This describes implicitly all the possible solutions for our identification problem, since for each choice of θ coherent with $\mathcal{D}_\theta$, I obtain a different model.

Problem 2: Feasible Parameter Set $\mathcal{D}_\theta$ definition

Computing the Parameter Uncertainty Intervals (PUIs)

Provided that the Feasible Parameter Set has been formulated as in (17), the problem of computing the PUI can be written as follows:

Definition (Parameter Uncertainty Interval (PUI))

For each parameter θ_j the PUI_{θ_j} is defined as follows:

$$PUI_{\theta_j} = [\underline{\theta}_j, \overline{\theta}_j] \tag{18}$$

where:

$$\underline{\theta}_j = \min_{\theta \in \mathcal{D}_\theta} \theta_j, \quad \overline{\theta}_j = \max_{\theta \in \mathcal{D}_\theta} \theta_j \Rightarrow \overline{\theta}_j = -\min_{\theta \in \mathcal{D}_\theta} -\theta_j \tag{19}$$

Problem 2: Feasible Parameter Set $\mathcal{D}_\theta$ definition

PUI computation by means of Linear Programs (LP) solution

The set in (17) can be rewritten in a more-compact matrix form as:

$$\mathcal{D}_\theta = \{\theta \in \mathbb{R}^5 : A\theta \leq b\} \tag{20}$$

where:

$$A = \begin{bmatrix} y(k-1) & y(k-2) & -u(k) & -u(k-1) & -u(k-2) \\ -y(k-1) & -y(k-2) & u(k) & u(k-1) & u(k-2) \end{bmatrix},$$
$$b = \begin{bmatrix} \Delta_e - y(k) \\ \Delta_e + y(k) \end{bmatrix}, \quad k = 3, ..., N$$

We have seen that the derived set is a polytope that is a convex one, moreover the objective of the problems in (19) is also convex (in particular is linear). The minimization of a **linear objective** over a **polytope** raises a problem of **Linear Programming (LP)**. Well known algorithms (eg. *Barrier method*) have been developed in order to solve such problems.

Problem 2: Feasible Parameter Set $\mathcal{D}_\theta$ definition

PUI computation by means of Linear Programs (LP) solution

The problems in (19) can be rewritten as:

$$\underline{\theta}_j = \min_{\theta} c^T \theta \qquad \text{subject to: } A\theta \leq b \tag{21}$$

$$\overline{\theta}_j = \max_{\theta} c^T \theta \qquad \text{subject to: } A\theta \leq b \tag{22}$$

where $c_i = 1 \iff i = j$. Linear Programming solvers can be used in order to find a global optimal solution. In MATLAB the `linprog()` command can be used.

MATLAB® Code

In the following there is an example of MATLAB code for the computation of PUI_{θ_1}

```
A=[y(2:N-1) y(1:N-2) -u(3:N) -u(2:N-1) -u(1:N-2);
-y(2:N-1) -y(1:N-2) u(3:N) u(2:N-1) u(1:N-2)];
b=[dE*ones(N-3+1)-y(3:N); dE*ones(N-3+1)+y(3:N)];
c=zeros(5,1); c(1)=1;
[~,th1_inf] = linprog(c,A,b);
[~,th1_sup] = linprog(-c,A,b); th1_sup=-th1_sup;
```

Problem 2: Feasible Parameter Set $\mathcal{D}_\theta$ definition

PUI computation by means of Linear Programs (LP) solution

Choosing $\Delta_e = 10$ the following PUIs have been computed:

	$\underline{\theta}_j$	$\overline{\theta}_j$	θ_j^c	θ_{true}
PUI_{θ_1}	-0.7984	-0.7303	-0.76438	-0.7647
PUI_{θ_2}	0.2690	0.3354	0.3022	0.3012
PUI_{θ_3}	-1.104	1.1064	0.0012	0
PUI_{θ_4}	31.187	33.31	32.249	32.24
PUI_{θ_5}	19.645	23.017	21.331	21.41

Also the **central estimates** have been provided. Note that all of the θ_j^{true} are inside the associated PUI.