LABORATORY OF ROBUST IDENTIFICATION AND CONTROL

Lecture notes

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Part I

Set-Membership System Identification

Chapter 1

Introduction

The *first step* of any control problem is tipically the derivatin of a **mathematical model** for the plant to be controlled that without loss of generality is a **mechatronic system**. This is the most critical step because, assuming that we use physics laws:

- We use *simplifying assumptions*;
- The value of the physical parameters involved in the equations (eg. mass, friction coefficients...) are not exactly known.

This fact is critical since *standard* approaches to controller design are **model based**, in the sense that the controller is designed by strongly relying on the mathematical model used to describe the mechatronic system under study. Clearly the neglection of some aspects, will result in a neglection of state variables! For example for certain problems the assumption of *rigid body* is satisfactory only under certain conditions. **How to solve this problem?** If we compare the two common approaches for designing a controller (state space or frequency description) the one based on frequency is much more able to face the problem of uncertainty.
In general, we can say that a controller is **robust** if it keeps good performances under the assumption of *uncertain description*. For this reason we need a **robust description** of the plant that, roughly speaking, is made up of a **nominal model** and by a model for the **uncertainty**[1]. Once such a model is derived we can apply some *robust control techniques* for designing a controller ($\mathcal{H}_\infty$, μ-synthesis...)

In order to deal with the presence of the uncertainty and to overcome the limitations related to the first principle modeling approach we will focus on **System Identification (SysId)** (Part I) and **Direct Data Driven Controller design (DDDC)** (Part II).

1.1 Mathematical modeling of dynamical systems

Since we have discussed about the importance of the mathematical model, now we can give an overview of the approaches one can track.

1.1.1 White-Box modeling *(first principles)*

The models deriving from **white-box approach** are obtained by applying the first principle of physics and all the physical phenomena involved in the equation, also all the *physical parameters*

[1] This can be modeled in an *unstructured* or *structured* way.

involved in the equation are **assumed to be exactly known**. The main idea which is useful to stress is that here **we know everything including the physical parameters**.

1.1.2 Gray-Box modeling

They are models based on equations obtained (again) by applying first principles, but this time the parameters entering the equations are not exactly known and so an estimation procedure from experimantally collected data is needed.

1.1.3 Black-Box modeling

In this case the structure of the equation is selected by the user on the basis of some "general" **a-priori information** on the system physics (eg. linearity). The parameters involved in the equation of the black-box model are then estimated/computed by using experimentally collected data. In general the parameter of a black box model do not have any *physical meaning.*

1.1.4 Some comments on the three approaches

White-box models are not very useful in practice, where is very irrealistic the fact of having the knowledge of everything! Instead, more interesting is the comparison between **Gray-box** and **Black-Box** models. In both cases we have *some information* (physical insights) and use the data in order to estimate the parameters themselves.

In **gray-box modeling** the structure of the equations is not selected by the user since it is forced by the first principle approach. For this reason in general the equations of a gray-box will depend in a **possibly complx nonlinear** way from the physical parameters to be estimated.

Example 1 (GRAY-BOX)

Let us assume that we know the plant to be modeled is an LTI one. You will remember that in the state-space representation, one can represent a dynamical system as:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

If we look inside the A matrix for example we can find:

$$A = \begin{bmatrix} \frac{m}{k^2} & \sqrt{\beta} \\ \alpha^2 \frac{k^3}{\gamma} & \gamma^2 \end{bmatrix}$$

The mathematical procedure for the estimation of the parameters will be complex since they appear in the equation in a non linear way. Then, the modeling of such a plant will become very hard. This represents the main limitation of the gray-box approach.

On the other hand, in the **black-box approach**, we have more freedom to select the structure of the equations, especially because we do it in a *more convenient way* by only exploiting some general properties derived from our physical insights.

Coming back to the example we have just seen, the matrix A will be made up of four coefficients a_{ij} which appear linearly in the equation. The main difference is that such coefficients do not have a physical meaning.

1.2 Steps for mathematical modeling

The procedure we have for obtaining the mathematical model is the following:

1. **STEP 1** Exploit available *a-priori information* on the system under study to select the structure of the mathematical equations describing the Input-Output mapping. In the most general case we do not know all the state variables, from this fact we can understand that we derive an **input-output model** (equation) for the plant like the following:

$$\underbrace{y(t)}_{\text{output}} = f(\underbrace{u(t)}_{\text{input}}, \underbrace{\theta}_{\text{parameters}}) \tag{1.1}$$

2. **STEP 2** Collect Input-Output data representing the behaviour of the system under study by performing an (open-loop) experiment. In particular, we collect the output $\tilde{y}$ from the plant using as input the sequence $\tilde{u}$.

3. **STEP 3** To formulate a suitable **mathematical problem** to estimate/compute the values of parameter $\theta = [\theta_1 ... \theta_n]^T$ in such a way that our mathematical model is going to describe the behaviour of the real system, **as well as possible (in some sense)**. A common approach is to compute the parameter by solving the following problem:

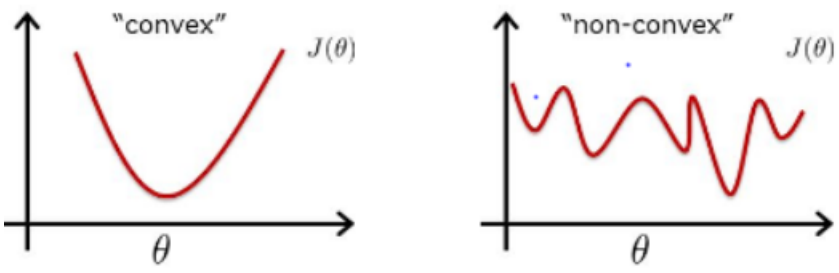
$$\hat{\theta} = \arg\min_{\theta} J(\theta) \tag{1.2}$$

where $\hat{\theta}$ is the vector of parameters to be estimated, while $J(\theta)$ is the functional to be minimized. Usually we take it as $J(\theta) = \|\tilde{y} - f(\tilde{u}, \theta)\|$. At this stage the difference between gray-box and black-box approach models comes into play, since:

- **Gray-box models** $\Longrightarrow$ $f(\tilde{u}, \theta)$ will depends by a complex nonlinear function from θ (parameters);
- **Black-box models** $\Longrightarrow$ $f(\tilde{u}, \theta)$ will be selected by the user in order to depend linearly from θ (if possible) or anyway in the *simplest possible way*.

1.3 Gray-Box vs Black-Box

We have said that in the case of **Gray-box models** in general $f(u, \theta)$ may be a *nonlinear* and *non convex* function of θ. This imply that the problem (1.2) is going to be a **non convex optimization problem**, and in this case it is not trivial to solve it, the best I can say is to find **local minima**. In some situations we could be particularly lucky, and choosing a particular initial point there is the possibility of finding the global minimum. However there is no way to certify it! Clearly a local minima, could correspond to a bad estimate of the parameter θ.



For **Black-box** models, by selecting the parametrization of f such that it could be a **convex function of** θ, the problem 1.2 becomes a **convex optimization problem**. In 1D the functional to be minimized is something similar to the one showed in the figure above. Convex functions have a unique **global minima**,there is no chance to be trapped in a local minima as in the non convex case. Clearly, waht is missed in this kind of approach is the *physical meaning* of the estimated parameters. Let us give an example to better clarify this aspect:

Example (Second-order LTI system)

Suppose that from first principles of physics we derive the following transfer function:

$$H(s) = \frac{\frac{p_1^2}{p_2}s + \frac{p_3}{\sqrt{p_4}}}{s^2 + \frac{p_1 p_4}{p_3^2}s + 1}$$

where $p_1, ..., p_4$ are physical (meaningful) parameters. What does we miss by modeling the transfer function by using the following model derived for example after a *black-box procedure*?

$$H(s) = \frac{\theta_1 s + \theta_2}{s^2 + \theta_3 s + \theta_4}$$

The physical meaning is clearly missed, but in many situation the objective is not to be grasped to the physical meaning of the parameters taken singularly, but (a) to derive an I/O model for the plant; (b) to use such a model for designing a (model-based) controller.

In order to conclude this discussion we can say that:

- The **black-box approach** is the *best choice* when we want either to simulate the I/O behaviour of the system or to design a **feedback control system**; an important remark to do is that the **structure** of a such a model must be selected by exploiting the most important a-priori information on the system (eg. **linearity**, **time invariance**...)
- The **gray-box approach** is the *best choice* when we want to estimate the values for some **physical parameters**.

It is true that – in all the approaches can be used for **System Identification** – the experimental data plays a crucial role, but also the *a-priori information* are of paramount importance. Besides, given the experimentally collected data there is an infinite number of functions which can interpolate those data. But if we apply an arbitrary input and then we compare the estimated output with the true one, we can confirm that the derived function just overfits the provided data. Conclusion: together with *a-posteriori information* (collected data), we need also the *a-priori information*, otherwise a well SysId procedure cannot be performed.

Chapter 2

System Identification: Least Squares Approach

We have introduced in the first chapter the concept of *System Identification* and we caught the importance of experimentally collected data, clearly in this more or less complex procedure one has to take into account that since that – the data are collected by performing experiments on the real plant, they can be affected by **uncertainty/measurement noise**.
Moreover, even if the system to be identified is a **continuous-time one** the most natural model for SysId is the discrete-time one, since samples of continuos time signals are collected.

2.1 Regression form for describing dynamical systems

There are evidences that – in a quite general manner – any dynamical system can be represented by using the so-called **regression form**, which stabilizes a relation between the current output, the input samples and the samples of the previous output. It is defined as follows:

$$y(k) = f(y(k-1), y(k-2), ..., y(k-n), u(k-1), ..., u(k-m), \theta) \tag{2.1}$$

For any physical system $m \leq n$ where n is the system order that is the number of *state variables*.

2.2 Error-in-variables(EIV): General setting for SysId

The most general setting describing an experiment performed on a plant to be identified is the **Error-in-variables (EIV)**, here both output $y(k)$ and input $u(k)$ are affected by measurement noise $\eta(k)$ and $\xi(k)$ respectively. Then the collected data can be represented by:

$$\tilde{u}(k) = u(k) + \xi(k) \tag{2.2}$$
$$\tilde{y}(k) = y(k) + \eta(k) \tag{2.3}$$

In some situation the sequence input $u(k)$ can be assumed to be perfectly known so $\xi(k) = 0$, because we build it in order to stimulate the system.[1] However, the **EIV** is more general and encapsulate also the situation in which the system to be identified is a subsystem from a more complex plant, then, since both $u(t)$ and $y(t)$ must be measured, both input and output are corrupted by uncertainty/measurement noise.

The following is a figure that shows schematically the setting we have just described:

[1] Later, when the concept on noise will be better formalized, we will give to such an approach the name of **output error (OE)**.

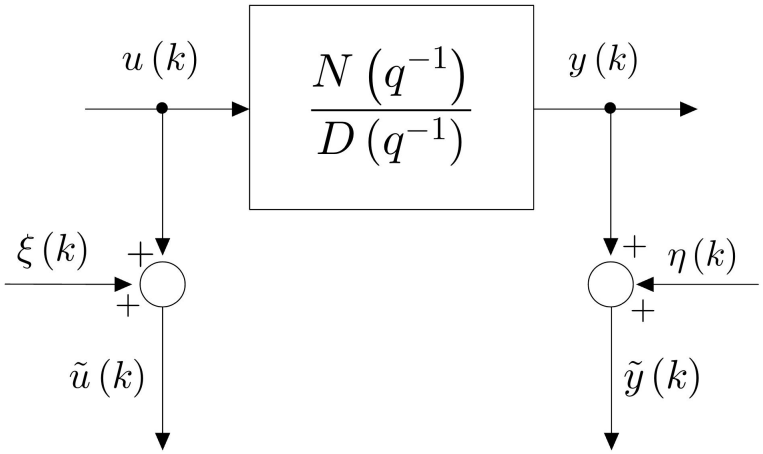


Figure 2.1: EIV SysId setting

Once we have fixed the setting, the a-priori information to provide are about:

- The **model** for example $f \in \mathcal{F}$ where $\mathcal{F}$ is associated with a certain class of systems (eg. LTI, nonlinear, stable...)
- The **noise** and in particular information reguarding the **statistical distribution** (white, gaussian...) or the **boundedness** (depending on the approach we are going to follow).

Now we are going to show an example which helps us to understand why the *regression form* is an effective model for describing in the most general manner a dynamical system.

Example (Regression form for a second order LTI system)

Let us consider a case in which we want to derive a model for a linear time-invariant system of the second order (a-priori assumption on the model: $f \in \mathcal{LTI}$, $n = 2$), furthermore suppose there is no noise in the data. The regression form $y(u(k), \theta)$ is:

$$y(k) = -\theta_1 y(k-1) - \theta_2 y(k-2) + \theta_3 u(k) + \theta_4 u(k-1) + \theta_5 u(k-2) \tag{2.4}$$

It is useful now to remind an important property (**backward-shift operator**) that tells us $s(k-r) = q^{-r}s(k)$, for this reason the 2.4 becomes:

$$\begin{aligned}
&y(k) = -\theta_1 q^{-1} y(k) - \theta_2 q^{-2} y(k) + \theta_3 u(k) + \theta_4 q^{-1} u(k) + \theta_5 q^{-2} u(k) \iff \\
&y(k)[1 + \theta_1 q^{-1} + \theta_2 q^{-2}] = u(k)[\theta_3 + \theta_4 q^{-1} + \theta_5 q^{-2}] \\
&\frac{y(k)}{u(k)} = \frac{\theta_3 + \theta_4 q^{-1} + \theta_5 q^{-2}}{1 + \theta_1 q^{-1} + \theta_2 q^{-2}} \iff H(z) = \frac{\theta_3 z^2 + \theta_4 z + \theta_5}{z^2 + \theta_1 z + \theta_2}
\end{aligned}$$

The last step comes up from the fact that can be dimostrated that it holds that $q^{-1} = z^{-1}$ and so from the regression form, passing through the backward-shift operator we can derive the transfer function of the system to be identified. Clearly a state-space description can be obtained once the parameters have been estimated by using the *realization theory* (transfer function $\rightarrow$ state space).

This example shows us in an inductive way that the regression form is the right one to use!

2.3 Least Squares estimation of the parameters θ_i

The objective of this paragraph is to show gradually – using significative examples – the problem of **parameter estimation** performed by using the **Least Squares (LS) model**, then we will analyze the pros and cons of such a method and some assumptions under which this kind of approach shows very nice properties (in a certain sense).

2.3.1 Estimation of parameters in the noise-free case

Let us consider again the case of a 2^{nd} order LTI system; we have seen it is characterized by the following regression form:

$$y(k) = -\theta_1 y(k-1) - \theta_2 y(k-2) + \theta_3 u(k) + \theta_4 u(k-1) + \theta_5 u(k-2)$$

The objective of the SysId procedure is to estimate the parameter $\theta = [\theta_1, \theta_2, \theta_3, \theta_4, \theta_5]$. We have to carry out an **open-loop experiment** on the plant by injecting the sequence $u(k)$ and collecting the output $y(k)$ for $k = 1, ..., H$. Since in the regression form we use samples till $k-2$ we have to start from $n+1 = 3$. Then:

$$\begin{aligned}
y(3) &= -\theta_1 y(2) - \theta_2 y(1) + \theta_3 u(3) + \theta_4 u(2) + \theta_5 u(1) \\
y(4) &= -\theta_1 y(3) - \theta_2 y(2) + \theta_3 u(4) + \theta_4 u(3) + \theta_5 u(2) \\
y(5) &= -\theta_1 y(4) - \theta_2 y(3) + \theta_3 u(5) + \theta_4 u(4) + \theta_5 u(3) \\
y(6) &= -\theta_1 y(5) - \theta_2 y(4) + \theta_3 u(6) + \theta_4 u(5) + \theta_5 u(4) \\
y(7) &= -\theta_1 y(6) - \theta_2 y(5) + \theta_3 u(7) + \theta_4 u(6) + \theta_5 u(5)
\end{aligned} \tag{2.5}$$

In this case $H = 3n+1 = 7$, it is quite evident we can express the equation 2.5 in matrix form as:

$$\underbrace{\begin{bmatrix} y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} -y(2) & -y(1) & u(3) & u(2) & u(1) \\ -y(3) & -y(2) & u(4) & u(3) & u(2) \\ -y(4) & -y(3) & u(5) & u(4) & u(3) \\ -y(5) & -y(4) & u(6) & u(5) & u(4) \\ -y(6) & -y(5) & u(7) & u(6) & u(5) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}}_{\theta} \tag{2.6}$$

Then the five equations can be rewritten as $y = A\theta$, and in the case in which the matrix A is invertible ($\det(A) \neq 0$), the problem of estimating the θ parameters is simply:

$$\theta = A^{-1} y \tag{2.7}$$

The fact that must be $\det(A) \neq 0$ is not so hard to guarantee since the first two columns of the matrix A containing the samples of the output are likely to be very different! In order to prove it, it is sufficient to understand that for each LTI system there is a transient in which the output is not perfectly stabilized, even when the stimula assume very simple shapes (eg. step...). The matrix A is square by construction, then since we have $h = 2n+1 = 5$ parameters, we need 5 equations to obtain a unique solution to the problem. This approach is valid even with nonlinear functions which *depends linearly on the parameters*, we are sampling input and output, for this reason it is not important that the function f (of the regression form) is nonlinear. Important remarks:

- For a system of order n we need to estimate $h = 2n+1$ parameters;
- The minimal number of samples we need is $H = 3n+1$
- We can start applying the regression form function from the instant $k = n+1$, since it depends on both preceeding output and input.

2.3.2 Estimation of parameters in the noisy case

The fact that the collected samples were noisy free was only a simplificative assumption made up to introduce the problem. In real-world applications there is always uncertainty. Let us consider an example which will make necessary the collection of more and more data.
Let us consider a static system of the type[2]

$$y(k) = \theta u(k) \tag{2.8}$$

It seems that we can correctly estimate θ by just collecting a *simple pair* $(u(k), y(k))$ in order to obtain $\theta = \frac{y(1)}{u(1)}$. Now let us assume that the input data are exact, while the output sample $y(1)$ is corrupted by a noise $\eta(1)$. What is obtained is as follows:

$$\begin{cases} \tilde{u}(k) = u(k) \\ \tilde{y}(k) = y(k) + \eta(k) \end{cases}$$

Since $\tilde{y(k)} \neq y(k)$, the estimate given by $\frac{y(1)}{u(1)}$ is completely wrong, since:

$$\hat{\theta} = \frac{\tilde{y}(1)}{u(1)} = \frac{y(1) + \eta(1)}{u(1)} = \theta + \frac{\eta(1)}{u(1)} \neq \theta$$

What to do? The idea is to collect a number of data $H \gg 2n + 1$, in this case the matrix A becomes a tall matrix, there is not a unique solution as in the case of the noise-free example, but we can get an approximation $\hat{\theta}$ such that $\tilde{y} \approx y$. The following steps can be done:

$$\tilde{y} = A\theta \rightarrow A^T\tilde{y} = (A^TA)\theta \iff \theta = \underbrace{(A^TA)^{-1}A^T}_{A^*}\tilde{y} \qquad \text{(Normal Equations)}$$

where A^* is the Moore-Penrose pseudoinverse (generalization of the inverse for non-square matrices)[3]. It can be demonstrated[4] that the (Normal Equations) is the solution of the problem:

$$\theta_{LS} = \arg\min_{\theta} \|\tilde{y} - A\theta\|_2^2 \tag{LS}$$

that is the well-known **Least-Squares problem**. This is a statistical approach to *parameter estimation* which has nice properties:

- The *computational burden* is very low! The only needed operation is the inversion of A^TA;
- There is a recursive way to solve it that reduces the work load in presence of big matrices (online computation).[5]
- The most important and 'powerful' property is the **consistency property**, which holds when two assumptions are satisfied. The next paragraph deals with the explanation of such assumptions.

[2]This could be for example a model for a resistor in which flows a certain current ($u(k)$) and we want to measure the voltage ($y(k)$).

[3]Keep in mind it is derived from the Singular Value Decomposition (SVD), which the generalization of the spectral Decomposition for non-symmetric matrices.

[4]The problem (LS) appears to be a convex quadratic unconstrained minimization problem. If the functional is explicitly written as a quadratic function, then after computing the gradient, its root raises the normal equations.

[5]See for more details: `https://en.wikipedia.org/wiki/Recursive_least_squares_filter`

2.3.3 Least Squares: consistency property

Theorem 1 (Consistency Theorem). *If the following two assumptions are satisfied:*

1. *The noise can be considered as an additive term entering the problem that is*

$$\begin{aligned} y(k) = &- \theta_1 y(k-1) - \theta_2 y(k-2) - ... - \theta_n y(k-n) \\ &+ \theta_{n+1} u(k) + \theta_{n+2} u(k-1) + ... + \theta_{n+m+1} u(k-m) + \underbrace{e(k)}_{\text{EQUATION ERROR}} \end{aligned}$$

2. *The samples $e(k)$, $k = 1, ..., H$ are indipendent and identically distributed (white) random variables which can be modeled through a* ***zero-mean Gaussian noise***

Then, it holds that[6]*:*

$$\lim_{H \to \infty} \mathbb{E}[\theta_{LS}] = \theta \tag{2.9}$$

In a simplified way such a theorem states that under the two assumptions (satisfied), if you enlarge H, $\theta_{LS} \to \theta$.

2.3.4 Analysis of the assumptions

At this point, we wonder if the just exposed result, solved all of our problem for the parameter estimation, and then if the LS approach can be used in general. This is nothing but verifying if the two hypotesis are satisfied. For the sake of clarity let us take the most general setting for an experiment on a plant to identify (EIV).
Grasping on the a-priori information that the system is an LTI second-order one, let us take the associated regression form

$$y(k) = -\theta_1 y(k-1) - \theta_2 y(k-2) + \theta_3 u(k) + \theta_4 u(k-1) + \theta_5 u(k-2)$$

Since $u(k) = \tilde{u}(k) - \eta(k)$ and $y(k) = \tilde{y}(k) - \xi(k)$, we can substitute them obtaining:

$$\begin{aligned} \tilde{y}(k) = &- \theta_1 \tilde{y}(k-1) - \theta_2 \tilde{y}(k-2) + \theta_3 \tilde{u}(k) + \theta_4 \tilde{u}(k-1) + \theta_5 \tilde{u}(k-2) + \\ &\underbrace{+\theta_1 \eta(k-1) + \theta_2 \eta(k-2) - \theta_3 \xi(k) - \theta_4 \xi(k-1) - \theta_5 \xi(k-2)}_{e(k)} \end{aligned} \tag{2.10}$$

It is evident, I can envelope all the terms associated with the noise samples in a term which I call $e(k)$. Then, **the first assumption is satisfied**. What about the second? We have to check if the sequence

$$e(k) = \theta_1 \eta(k-1) + \theta_2 \eta(k-2) - \theta_3 \xi(k) - \theta_4 \xi(k-1) - \theta_5 \xi(k-2) \tag{EE}$$

is a white one (samples iid). Let us analyze a pair of samples:

$$\begin{aligned} e(3) &= \theta_1 \eta(2) + \theta_2 \eta(1) - \theta_3 \xi(3) - \theta_4 \xi(2) - \theta_5 \xi(1) \\ e(4) &= \theta_1 \eta(3) + \theta_2 \eta(2) - \theta_3 \xi(4) - \theta_4 \xi(3) - \theta_5 \xi(2) \end{aligned}$$

How it is highlighted, only by taking two of the $e(k)$ we can note they depend from common samples, for this reason the sequence $e(k)$ itself it is not white at all! They will provide an estimate θ_{LS} which is not going to enjoy of the consistency property. Even if the setting was OE instead of EIV, the same conclusion would have been drawn.

[6] We take the expected value of the estimate since random variables are introduced in the problem by adding $e(k)$, the estimate itself becomes a random variable.

2.4 The Equation Error (EE) noise structure

We have concluded in the former paragraph that the LS estimate is not suitable in the case we have either an EIV or an OE setting. Thus, what is the case in which the LS can be used? (again: that is, the two assumptions are verified). It is necessary to better dissect the properties of (2.10). In particular, it is useful (passing through the backward-shift operator) finding what is the relation between $\tilde{y}(k)$, $u(k)$ and $e(k)$. In order to discover such properties let us assume that the setting used is the Output Error (without loss of generality). Using $s(k-r) = q^{-r}s(k)$ we can write:

$$\tilde{y}(k)[1 + \theta_1 q^{-1} + ... + \theta_n q^{-n}] = u(k)[\theta_{n+1} + \theta_{n+2} q^{-1} + ... + \theta_{n+m+1} q^{-m}] + e(k) \iff \quad (2.11)$$

$$\tilde{y}(k) = \frac{[\theta_{n+1} + \theta_{n+2} q^{-1} + ... + \theta_{n+m+1} q^{-m}]}{[1 + \theta_1 q^{-1} + ... + \theta_n q^{-n}]} u(k) + \frac{1}{[1 + \theta_1 q^{-1} + ... + \theta_n q^{-n}]} e(k) = \quad (2.12)$$

$$= \frac{N(q^{-1})}{D(q^{-1})} u(k) + \frac{1}{D(q^{-1})} e(k) = \frac{N(z)}{D(z)} u(k) + \frac{1}{D(z)} e(k) \quad (2.13)$$

The deriving setting is represented in the figure below:

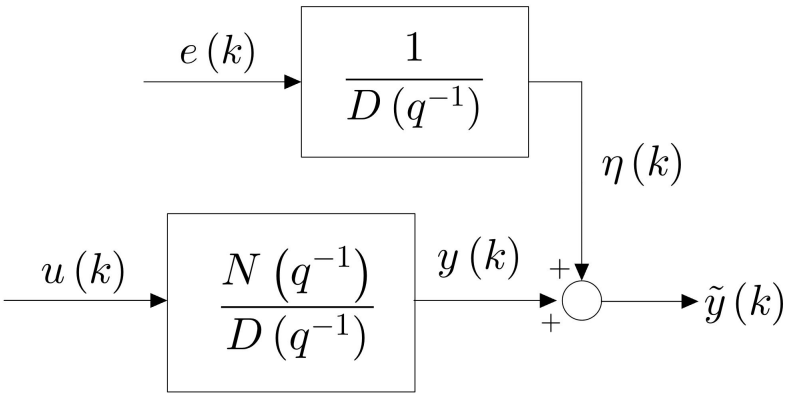


Figure 2.2: Equation Error (EE) noise structure

In this way we can conclude that the LS estimate makes sense if the date corrupted by a random sequence $e(k)$ (the noise) filtered by a system whose transfer function is the denominator of the system to be identified! No sense, since the sensor in general has nothing to share with the plant we want to identify. However there are some cases in which the LS approach can be used, we refer to the few cases in which the plant to be identified is such that $D(q^{-1}) = 1$. This occurs when I have to do:

- Identification of **FIR systems** (Finite impulse response);
- Identification of static systems;

When the denominator of the transfer function is equal to one, the Equation Error plays the role of the *output measurement error* $\eta(k)$. In this case also the **second assumption** is satisfied.

2.4.1 System Identification of Finite Impulse Response (FIR) systems

This of type of system has a transfer function which depends only on the samples of the input and not on on the previous output samples.

$$\tilde{y}(k) = \theta_1 u(k) + \theta_2 u(k-1) + ... + \theta_n u(k-n) + \underbrace{e(k)}_{\eta(k)} \tag{2.14}$$

Here also the second assumption is satisfied since the error samples are iid. In the case of FIR the property (2.9) is fulfilled.

2.4.2 System Identification of Static systems

Here the output is a function of parameters θ_i and of the input $u(k)$ at the current instant.

$$y(k) = \theta_1 g_1(u(k)) + \theta_2 g_2(u(k)) + ... + \theta_n g_n(u(k)) + e(k) \tag{2.15}$$

The functions $g_i(u(k))$ can be trigonometric functions, polynomial or anyway any other basic function.

2.5 Final remarks

Real experimets have data characterized by noise and in general the consistency property does not hold, in some situations in which the problem has a particular structure the assumptions required are perfectly fulfilled. We have understood that the most critical problem to manage is the second assumption which require the error to be white, zero mean and Gaussian (very strong assumptions!), moreover the resulting noise structure reaches a quite strange conclusion in which it is required that the system and the sensor share a part of their model.
In the following we will see another approach that, differently from LS, replaces the Assumption (2) with something that is significantly *less strong*.

Chapter 3

Set-Membership Identification: Introduction

The objective of this chapter is to introduce an approach for the parameter estimation which requires to do less strong assumption on the noise affecting the experimentally collected data. After a brief introduction with the crucial ingredients, we will go on with some instructive examples which will bring us to the complete formulation of the **Set-Membership System Identification procedure**.

3.1 Ingredients for Set-Membership System Identification

As usual in order to perform correctly the procedure of System Identification we need some crucial ingredients:

❶ **A-priori assumption on the system:**

- ✓ We use the general **regression form**

$$y(k) = f(y(k-1), y(k-2), ..., y(k-n), u(k-1), ..., u(k-m), \theta) \tag{3.1}$$

- ✓ The **class of function** $\mathcal{F}$ and the order of the system n;

❸ **A-priori information of the noise** and in particular:

- ✓ **Noise structure**: is referred to the way the uncertainty enters into the problem.
- ✓ **Characteristic of the signal**, it is remarkable that here we assume something different and weaker. We will assume that the noise sequence/sequences (depending on the noise structure) belongs to a certain bounded set $\mathcal{B}$.

3.2 Set-Membership Identification of LTI system with EE noise structure

In this paragraph we will show what is obtained in term of parameter estimation, when we have that the a-priori information on the noise are the following:

✓ The uncertainty enter in the problem as an additive term which we call $e(k)$ (the same of the first assumption of the theorem), that is:

$$\begin{aligned} y(k) = &- \theta_1 y(k-1) - \theta_2 y(k-2) - ... - \theta_n y(k-n) \\ &+ \theta_{n+1} u(k) + \theta_{n+2} u(k-1) + ... + \theta_{n+m+1} u(k-m) + \underbrace{e(k)}_{\text{EQUATION ERROR}} \end{aligned}$$

✓ We suppose on the sequence characterizing the error is **bounded** (this is the crucial difference with respect to what requires the *consistency theorem*), that is:

$$e(k) \in \mathcal{B}_e \iff |e(k)| \leq \Delta_e, \ k = 1, ..., H \tag{3.2}$$

Part II

Direct Data-Driven Control