Structured uncertainty and μ-analysis

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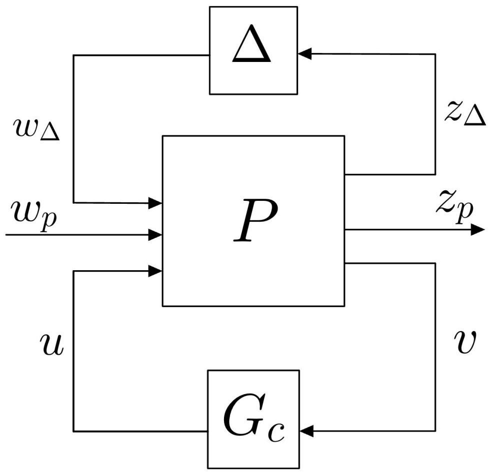
1 Structured uncertainty

Dealing with **structured uncertainty** and the analysis of feedback control systems affected by it, we are going to consider the *general control configuration* depicted in the figure below. Here z_Δ and w_Δ are the so-called *uncertainty channels*, which in turn are the signals connecting the **uncertainty sources** with the known part of the feedback control system. On the other hand the signals z_p and w_p are chosen in order to satisfy **performance requirements** from which the name **performance channel**. In the following we deal with mainly with **parametric uncertainty**.



$P\Delta G_c$ general structure

In such a general structure the sources of uncertainty Δ_i are pulled out to form a **block diagonal matrix** Δ, that is:

$$\Delta = \text{diag}\{\Delta_i\} = \begin{bmatrix} \Delta_1 & 0 & \dots & 0 \\ 0 & \Delta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \Delta_n \end{bmatrix} \tag{1}$$

We will consider different sources of uncertainty (real, complex full mzatrix, real repeated real scalar values).

1.1 Uncertainty sources

In the analysis of stability and performances by using the μ-analysis we will consider mainly:

- **Real scalar** perturbations $\Delta_i \in \mathbb{R}$ such that $|\Delta_i| \leq 1$;
- **Complex full matrix** perturbations $\Delta_i(s) \in \mathbb{C}^{m,m}$, $\|\Delta_i\|_\infty \leq 1$;
- **Repeated real scalar** perturbations $\Delta_i I_r$, where I_r is an $r \times r$ identity matrix, and $\Delta_i \in \mathbb{R}$, $\Delta_i \leq 1$

In order to perform a computer-aided μ-analysis, how we will see we use the command `mu` that among all of the parameters, take as input a matrix called `deltaset`. The main objective of such a matrix is *describing the blocks of Equation* (1) (the block diagonal matrix we mentioned) in term of: (i) type, (ii) dimensions, (iii) number of independent locations in which the uncertainty itself appears. By using a summarizing table, we mention the principal pattern we can find in analyzing our feedback control systems.

Type of Δ_i	`deltaset` **row**
Scalar real parameter	$[-1 \quad 1]$ or $[-1 \quad 0]$
f-repeated real parameter	$[-f \quad 0]$
Scalar 1×1 unmodeled dynamics	$[1 \quad 1]$ or $[1 \quad 0]$
$r \times c$ full unmodeled dynamics	$[r \quad c]$

Table 1: `deltaset` for describing Δ

Wheter in the feedback control system there are U uncertainty sources the final matrix will be such that `deltaset` $\in \mathbb{R}^{U,2}$. In the following sections we are going to do some examples which will clarify what is the structure of the `deltaset` matrix according to the structure of the uncertain plant $G_p(s)$. The models can be considered for the uncertainty are the following, given the generic parameter k:

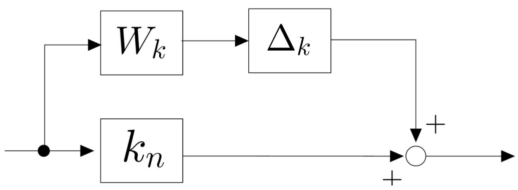
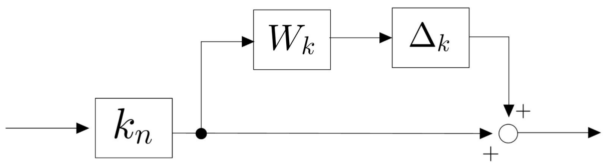
MODEL SET	**Mathematical description**
ADDITIVE UNCERTAINTY SET 	$k = k_n + W_k\Delta_k,\ \lvert\Delta_k\rvert \leq 1$
MULTIPLICATIVE UNCERTAINTY SET 	$k = k_n(1 + W_k\Delta_k),\ \lvert\Delta_k\rvert \leq 1$

Table 2: Uncertainty model sets

1.2 Fundamental bricks for structured uncertainty

Going on to analyze the structure of our uncertain system we can find in the transfer function some fundamental bricks that put all together will give us the description through block diagrams of the uncertain plant. Some aspects are presented in details for the first example while they are not repeated for the following ones which are using exactly the same steps.

1.2.1 Real Pole in zpk form

Example 1 (pole in the feedback path)

Given the following transfer function for an uncertain plant

$$G_p(s) = \frac{k}{s+p} \quad \underline{k} \leq k \leq \overline{k}, \quad \underline{p} \leq p \leq \overline{p} \tag{2}$$

we want to obtain its block diagram description. In particular for each parameter is required to obtain the central estimate k_n and p_n and the radius of uncertainty W_k and W_p. Compute

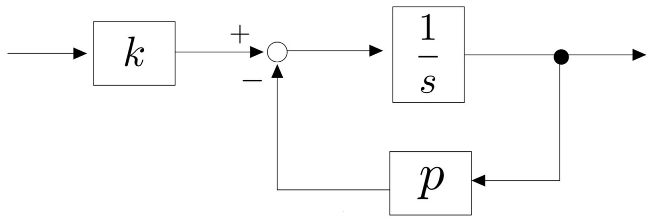
also the matrix Δ and the associated `deltaset` for the command `mu`.
The first step is for computing the value k_n, p_n, W_k and W_p:

$$k_n = \frac{\underline{k} + \overline{k}}{2} \quad W_k = \frac{\overline{k} - \underline{k}}{2}, \qquad p_n = \frac{\underline{p} + \overline{p}}{2} \quad W_p = \frac{\overline{p} - \underline{p}}{2} \tag{3}$$

At the end of the day, we are likely to build such block diagrams in SIMULINK[1], for this reason we have to manipulate the expression of Equation (2) in order to be able to represent them. If we put in evidence s at the denominator we can write:

$$G_p(s) = \frac{k}{s(1 + \frac{p}{s})} = k\frac{\frac{1}{s}}{1 + \frac{p}{s}}$$

The first term of the expression is simply a gain, while we can see the second term as a *feedback configuration* with negative feedback.



Block diagram for Equation (2)

Additive Uncertainty Set

If we use the as uncertainty set the *additive one* the block diagram we have showed changes as follows:

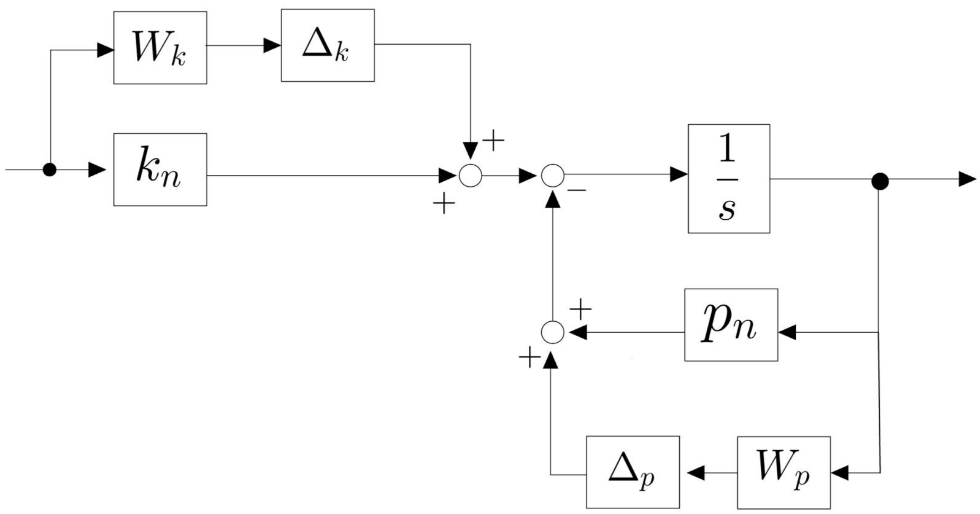


Figure 1: Block diagram of $G_p(s)$ (additive model)

Note that we have only added the for the parameters the uncertain description as indicated in Table 2. With the objective of obtaining the the $P\Delta G_c$ structure we have to: (i) put such a diagram in the Feedback control System (FCS) scheme; (ii) pull out all of the Δ_i blocks; (iii) pull out the controller G_c.
More interesting is the construction of the matrix Δ and its describing matrix `deltaset`. For this aim is crucial that all of the uncertainty channels are properly numbered. The block diagram becomes the one in Figure 2, where there are also the signals entering/coming from the (structured) uncertainty block Δ. According to the numbering we use in the block diagram we have to insert the diagonal blocks into Δ as reported here:

[1]Some algebraic manipulation are needed in order to avoid non proper blocks that – for sure– are going to raise an error during the simulation.

$$\Delta = \begin{bmatrix} \Delta_k & 0 \\ 0 & \Delta_p \end{bmatrix} \qquad \texttt{deltaset} = [\underbrace{\texttt{-1,1}}_{1};\underbrace{\texttt{-1,1}}_{2}] \tag{4}$$

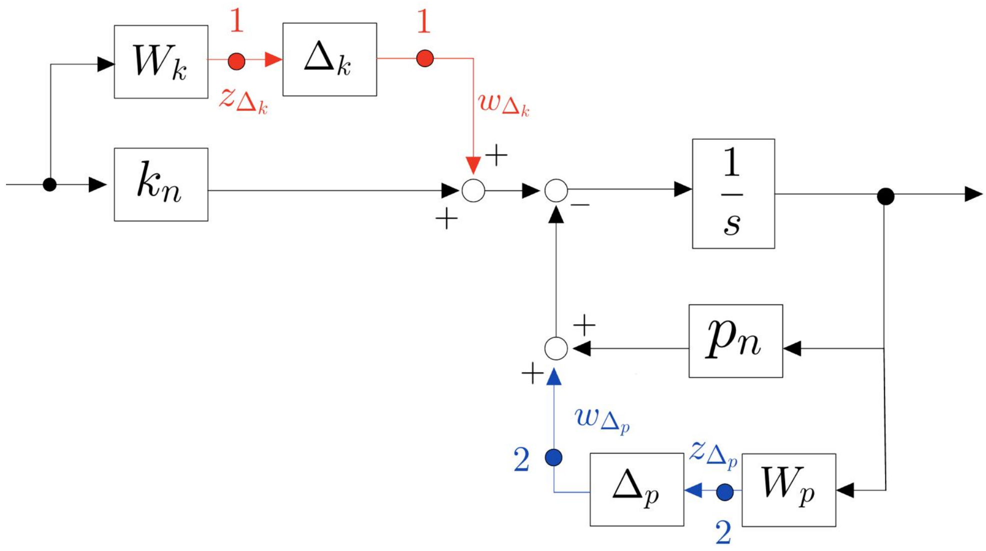


Figure 2: Block diagram of $G_p(s)$ (additive) with numbered sources

The first block of Δ is the one related to the uncertain parameter k, the second the one related to the uncertain parameter p. Let the colors guide you in the understanding the relationship existing between the block diagram and related Δ and `deltaset`.

Multiplicative Uncertainty Set

Identical reasoning can be done, if we assume to take as model of uncertainty the *multiplicative one*. Substituting in the general scheme, the structure for the uncertain parameters from the Table 2, the scheme in Figure 3 is obtained. The associated matrix Δ and the `deltaset` array are the same as before.

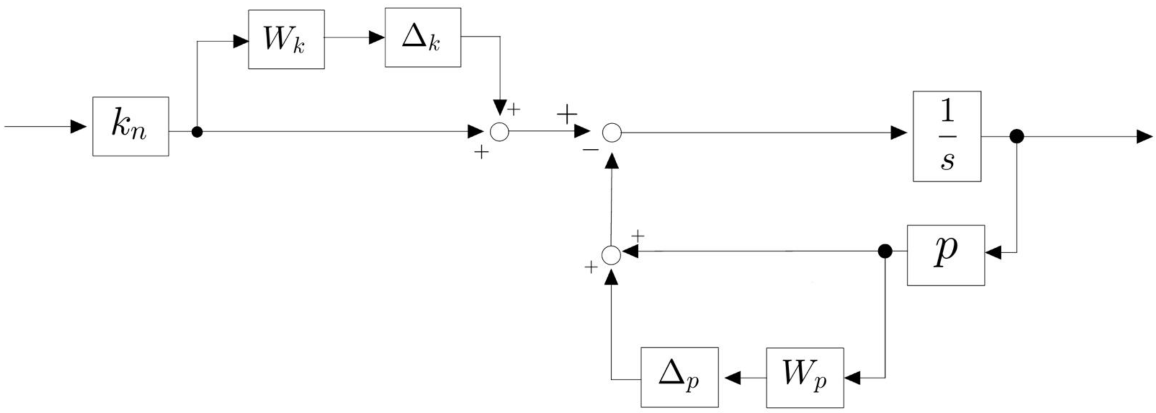


Figure 3: Block diagram for $G_p(s)$ (multiplicative set)

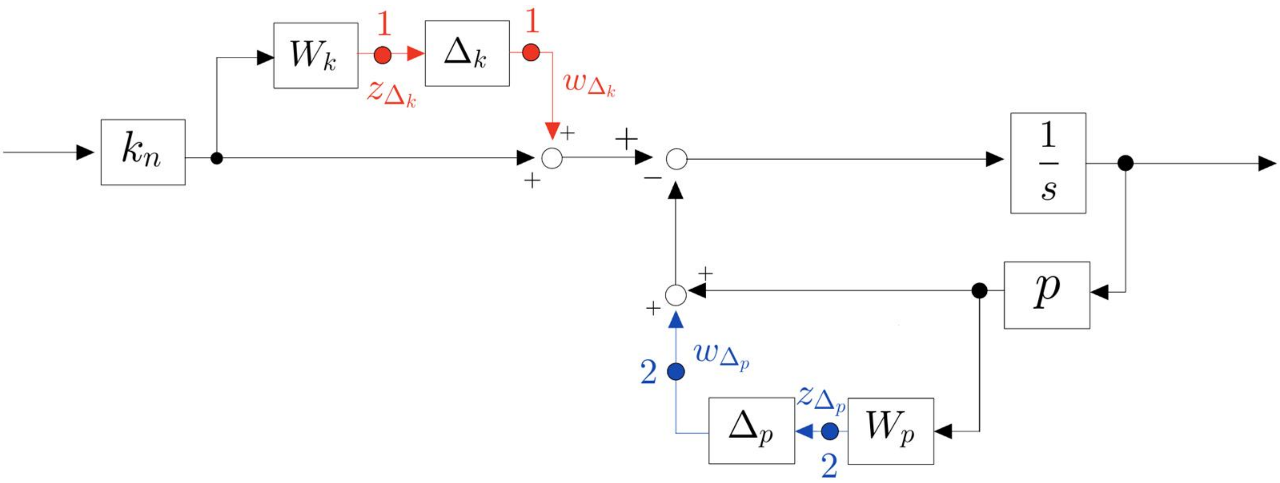


Figure 4: Block diagram for $G_p(s)$ (multiplicative set) with numbered uncertainty channels

Example 2 (model the reciprocal of an uncertain parameter)

Given the following plant

$$G_p(s) = \frac{k}{1+s\tau}, \tag{5}$$

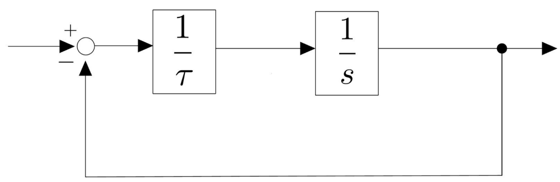
is required to find a block diagram description of it, in a way that can be implemented in a simulink scheme.

A rearragement of the terms is needed:

$$G_p(s) = k\,\frac{\frac{1}{s}\cdot\frac{1}{\tau}}{1+\frac{1}{s\tau}} \tag{6}$$

We omit the gain, to focus our attention on the second part of the transfer function.

The second term can be represented as:



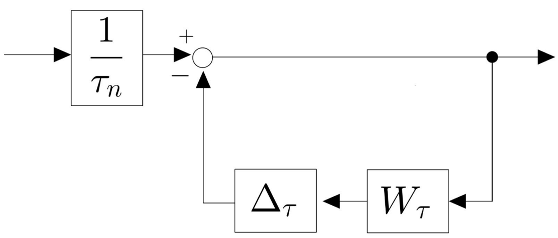
The second term can be recasted in a basic feedback structure whose direct path has the block $1/s\tau$, simply 1 in the feedback path. But how can we draw the diagram for the inverse of the uncertain parameter? It depends from the type of uncertainty set.

Multiplicative set

The term $1/\tau$ can be written as:

$$\frac{1}{\tau} = \frac{1}{\tau_n}\cdot\frac{1}{1+W_\tau\Delta_\tau} \tag{7}$$

The related block diagram representation is:

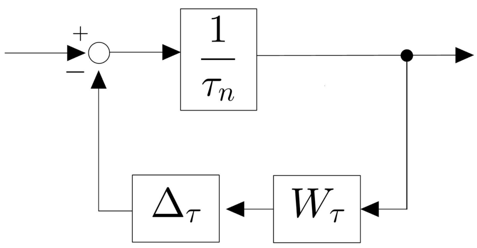


Additive set

The term $1/\tau$ can be written as:

$$\frac{1}{\tau} = \frac{1}{\tau_n + W_\tau\Delta_\tau} = \frac{\frac{1}{\tau_n}}{1+\frac{W_\tau\Delta_\tau}{\tau_n}} \tag{8}$$

The related block diagram representation is:



The remaining part of the discussion, about the port numbering and block Δ description, the way we retrieve Δ_τ and W_τ does not change. This is the reason why we omit this part.

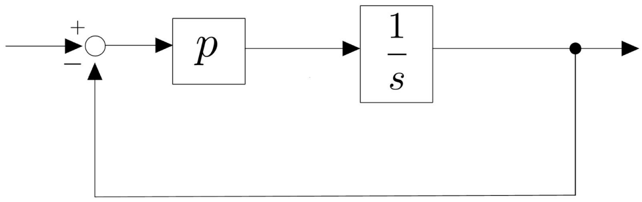
1.2.2 Real pole in DC-gain form

Here the objective is to analyze the block diagrams for the uncertain plant described by the following transfer function in *DC-gain (or time constant) form*:

$$G_p(s) = \frac{k}{1+\frac{s}{p}} \qquad \underline{k} \leq k \leq \overline{k}, \quad \underline{p} \leq p \leq \overline{p} \tag{9}$$

Following the same path we algebrically manipulate the Equation (9) in order to obtain:

$$G_p(s) = k \; \frac{\frac{p}{s}}{1+\frac{p}{s}}$$



No other comments are needed for modeling p since it is a simple uncertain parameter whose modeling can follow the Table 2.

1.2.3 Complex conjugate poles

Given the following transfer function for an uncertain plant

$$G_p(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \underline{k} \leq k \leq \overline{k}, \quad \underline{\omega}_n \leq \omega_n \leq \overline{\omega}_n \tag{10}$$

Here it is useful to apply the properties of the *Laplace transform* in order to pass into a state-space description. In particular we know, by definition, that given a certain transfer function $G_p(s)$ it is defined as

$$G_p(s) = \frac{y(s)}{u(s)} \iff y(s)(s^2 + 2\zeta\omega_n s + \omega_n^2) = u(s)\; k\omega_n^2 \iff \tag{11}$$

$$\frac{1}{\omega_n^2}\ddot{y}(t) + \frac{2\zeta}{\omega_n}\dot{y}(t) + y(t) = ku(t) \tag{12}$$

We can retrieve from the *differential equation* eq. (12) the state space description by introducing some state variables. In particular:

$$x_1(t) = y(t) \quad x_2(t) = \dot{y}(t) \tag{13}$$

Here we assume **zero initial conditions** so that we can write[2]:

$$\begin{cases} \dot{x_1}(t) = x_2(t) \\ \dot{x_2}(t) = -2\zeta\omega_n x_2(t) - \omega_n^2 x_1(t) + k\omega_n^2 u(t) \\ y = x_1 \end{cases} \implies \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

In order to draw the block diagram we individuate the equation containing the $u(t)$ that is the input till reaching the output $y(t)$. The resulting block diagram is:

[2]Here the matrices of the state space description are:

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ k\omega_n^2 \end{bmatrix}, \quad C = [1 \quad 0]$$

Since they are not strictly necessary we put them here.

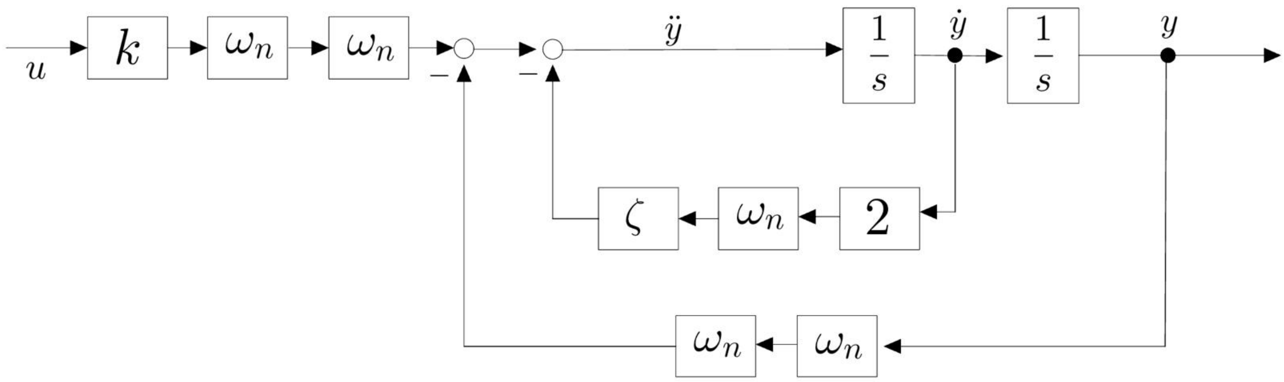


Figure 5: Block diagram of $G_p(s)$

This is the general form of the transfer function containing the complex conjugate poles, the single parameters must be substituted with their own uncertain description using one of the uncertainty sets.
This example is interesting since here there are repeated parameters, we can number with 1 the uncertainty related to the parameter k, 2 the uncertainty related to the damping factor ζ, finally all the others are the uncertainty channels related to the natural frequency ω_n. The resulting `deltaset` is

$$\texttt{deltaset} = [-1, 0;\ -1, 0;\ -5, 0] \tag{14}$$

The description obtained so far by using the state space description is quite general, so that we could consider also the case of **non-zero** initial conditions. Clearly also the previous example would have been provided us with the possibility to follow the same appproach, however when simple manipulations can be algebraically done by using directly the transfer function it is more convient and fast.

2 Robust stability with structured uncertainty

At this stage, since the controller is known, we can collapse the blocks P and G_c in a single block which we call N. The structure we obtain can be called $N-\Delta$. In this context N contains all the known parts of the feedback control system: the controller, actuators and nominal plant, the weighting functions accounting for the performance requiremens, The figure ...
This operation coincides with a *upper linear fractional transformation (upper LFT)*, where we can write

$$\begin{bmatrix} z_\Delta \\ z_p \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{12} & N_{22} \end{bmatrix} \begin{bmatrix} w_\Delta \\ w_p \end{bmatrix} \tag{15}$$

It can be prooved that ensuring robust stability for the $N-\Delta$ structure is equivalent to check that:

$$\det(I - N_{11}\Delta(j\omega)) \neq 0 \quad \forall\omega, \forall\Delta,\ \|\Delta\|_\infty < 1 \tag{16}$$

where N_{11} is the matrix tranfer function from z_Δ to w_Δ.

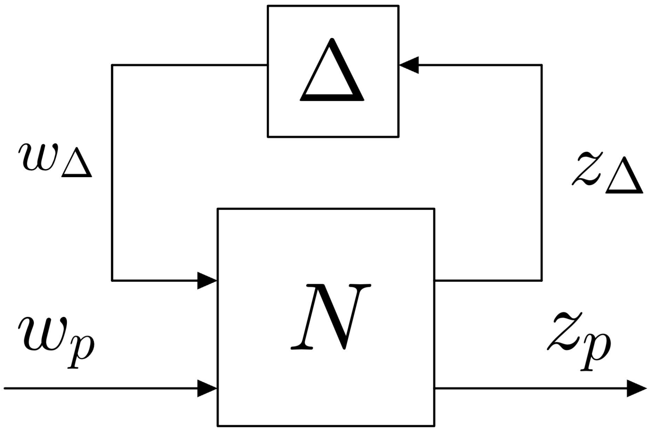


Figure 6: $N - \Delta$ structure

3 Robust performance with structured stabiliy

In the context of robust control, robust performance requirements can be expressed as:

$$\|F(s,\Delta)\|_\infty < 1, \quad \forall\Delta, \quad z_p = F(s,\Delta)w_p \tag{17}$$

where $F(s,\Delta)$ is a given matrix transfer function accounting for the performance requirements. Such a functional F can be written as:

$$F(s,\Delta) = \left\| \begin{matrix} W_S S(s,\Delta) \\ W_T T(s,\Delta) \end{matrix} \right\| \tag{18}$$

By using the properties of the LFT, can be shown that ensuring the robust performance in this case is equivalent to ensure the robust stability of the structure $N - \hat{\Delta}$ obtained by introducing a fictitious scalar complex uncertainty Δ_p, which in turn is the same to require that:

$$\det(I - N\Delta(j\omega)) \neq 0 \quad \forall\omega, \forall\Delta \tag{19}$$

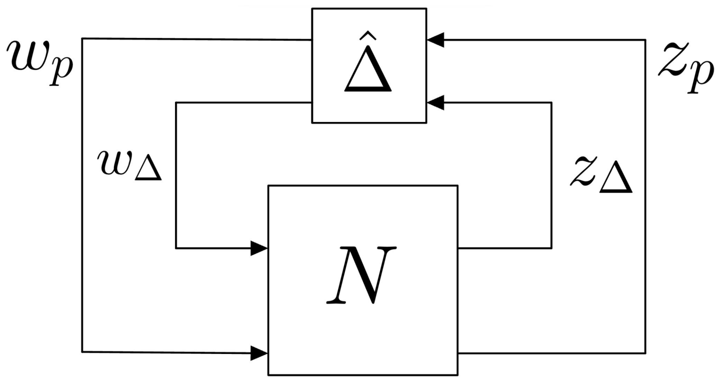


Figure 7: $N - \hat{\Delta}$ structure

Note that both conditions for robust stability and robust performance, come from the generalized Nyquist Criterion for MIMO systems.

4 Structured singular value μ

The conditions in Equation (16) and Equation (19) we have just obtained are of little practical interest, since we cannot explore all the frequencies and all the uncertainties, just to prove that the system is robust! An alternative, more practical, approach is needed. In particular, it may be useful to start by substituing in the determinant expression a certain Δ and to decrement it gradually. In particular:

- **Checking Robust Stability** is equivalent to compute the *smallest structured uncertainty* Δ such that
$$\det(I - N_{11}\Delta(j\omega)) = 0, \forall\omega \tag{20}$$
- **Checking Robust Performance** is equivalent to find the *smallest structured uncertainty* $\hat{\Delta}$ such that the stability is destroyed, that is:
$$\det(I - N\Delta(j\omega)) = 0, \forall\omega \tag{21}$$

In particular, at the end of the day:

- If the smallest uncertainty is less than one, then the robust stability/robust performance is **not fulfilled** since the stability is broken for an allowed value of Δ. In other words given our uncertain plant there are some values of the allowed uncertainty for which the robust requirement is 'broken'.
- If the smallest uncertainty is greater or equal than one, RS/RP is fulfilled, since such an uncertainty does not belong to the set of the allowed uncertainty.

Such aspects can be better formalized by introducing k_m being the singulatity margin and the definition of a novel tool which will provide us with the possibility to check in an effective way RS/RP. This leads to the definition of the **structured singular value** μ, that for the structure N, namely, is defined as:

$$\mu(N, j\omega) = \frac{1}{\min\{k_m \in \mathbb{R} : \ \det(I - N(j\omega)\Delta(j\omega)) = 0, \ \|\Delta\|_\infty < 1\}} \tag{22}$$

Note that $\mu(N, j\omega)$ is a real non-negative function of ω.

5 μ-analysis

We are are ready for give necessary and sufficient conditions for robust stability and robust performance in terms of μ. Performing the **μ-analysis** is the same to say that we are using the structured singular values in order to check Robust Stability/Robust Performance.

ROBUST STABILITY

Assume that G_c stabilizes N, the $N - \Delta$ structure is robustly stable if and only if

$$\mu(N_{11}, j\omega) < 1 \quad \forall\omega \tag{23}$$

ROBUST PERFORMANCE

Assume that G_c stabilizes N (provides internal stability), the following conditions are equivalent:

- $\|F(s, \Delta)\|_\infty < 1, \quad \|\Delta\|_\infty < 1$
- $\mu(N, j\omega) < 1 \quad \forall\omega$

5.1 μ_{ub}, μ_{lb}: bounds on the true μ

The results we have just given are very nice, except for the fact that computing the structured singular value is an hard non-convex problem, this is the reason why, practically speaking only some numerical upper and lower bounds μ_{ub}, μ_{lb} on the real μ can be computed.
Since, mostly we are not going to compute the real value for μ, you can imagine that the conditions on the *bounds of the true* μ are no more necessary and sufficient. In particular:

5.2 Sufficient conditions for robustness

$$\sup_\omega \mu_{ub}(N_{11}, \omega) < 1 \implies \textsf{Robust stability} \tag{24}$$

$$\sup_\omega \mu_{ub}(N, \omega) < 1 \implies \textsf{Robust performance} \tag{25}$$

This says us that if μ_{up} is less than one, RS/RP requirement is fulfilled, on the contrary if $\mu_{up} > 1$ and $\mu_{lb} < 1$ nothing can be said since the true μ might be greater than one, but also less than one.

5.3 Necessary conditions for robustness

$$\textsf{Robust stability} \implies \sup_\omega \mu_{lb}(N_{11}, \omega) < 1 \tag{26}$$

$$\textsf{Robust performance} \implies \sup_\omega \mu_{lb}(N, \omega) < 1 \tag{27}$$

Note that if $\mu_{lb} > 1$, then RS/RP are not fulfilled since the condition on the lower bound is a necessary one.

5.4 Robust performance - Constraining T

5.4.1 1st approach: structured description of the uncertainty

5.4.2 2nd approach: unstructured description of the uncertainty

5.5 Robust performance - Constraining S

5.5.1 1st approach: structured description of the uncertainty

5.5.2 2nd approach: unstructured description of the uncertainty

5.5.3 3rd approach: using the exact μ

5.6 Robust stability: changing the PUIs