Structured uncertainty and μ-analysis

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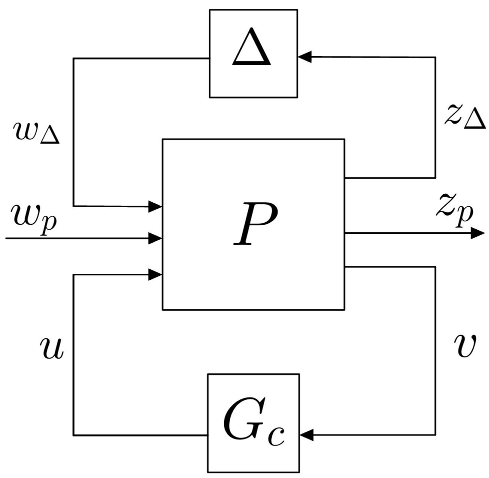
1 Structured uncertainty

Dealing with **structured uncertainty** and the analysis of feedback control systems affected by it, we are going to consider the *general control configuration* depicted in the figure below. Here z_Δ and w_Δ are the so-called *uncertainty channels*, which in turn are the signals connecting the **uncertainty sources** with the known part of the feedback control system. On the other hand the signals z_p and w_p are chosen in order to satisfy **performance requirements** from which the name **performance channel**. In the following we deal with mainly with **parametric uncertainty**.



$P\Delta G_c$ general structure

In such a general structure the sources of uncertainty Δ_i are pulled out to form a **block diagonal matrix** Δ, that is:

$$\Delta = \mathrm{diag}\{\Delta_i\} = \begin{bmatrix} \Delta_1 & 0 & \dots & 0 \\ 0 & \Delta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \Delta_n \end{bmatrix} \tag{1}$$

We will consider different sources of uncertainty (real, complex full mzatrix, real repeated real scalar values).

1.1 Uncertainty sources

In the analysis of stability and performances by using the μ-analysis we will consider mainly:

- **Real scalar** perturbations $\Delta_i \in \mathbb{R}$ such that $|\Delta_i| \leq 1$;
- **Complex full matrix** perturbations $\Delta_i(s) \in \mathbb{C}^{m,m}$, $\|\Delta_i\|_\infty \leq 1$;
- **Repeated real scalar** perturbations $\Delta_i I_r$, where I_r is an $r \times r$ identity matrix, and $\Delta_i \in \mathbb{R}$, $\Delta_i \leq 1$

In order to perform a computer-aided μ-analysis, how we will see we use the command `mu` that among all of the parameters, take as input a matrix called `deltaset`. The main objective of such a matrix is *describing the blocks of Equation* (1) (the block diagonal matrix we mentioned) in term of: (i) type, (ii) dimensions, (iii) number of independent locations in which the uncertainty itself appears. By using a summarizing table, we mention the principal pattern we can find in analyzing our feedback control systems.

Type of Δ_i	**`deltaset` row**
Scalar real parameter	$[-1 \quad 1]$ or $[-1 \quad 0]$
f-repeated real parameter	$[-f \quad 0]$
Scalar 1×1 unmodeled dynamics	$[1 \quad 1]$ or $[1 \quad 0]$
$r \times c$ full unmodeled dynamics	$[r \quad c]$

Table 1: `deltaset` for describing Δ

Wheter in the feedback control system there are U uncertainty sources the final matrix will be such that `deltaset` $\in \mathbb{R}^{U,2}$. In the following sections we are going to do some examples which will clarify what is the structure of the `deltaset` matrix according to the structure of the uncertain plant $G_p(s)$. The models can be considered for the uncertainty are the following, given the generic parameter k:

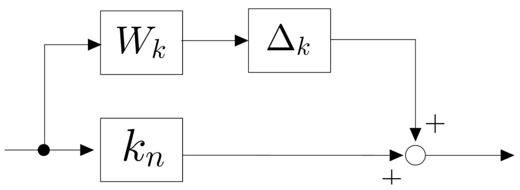
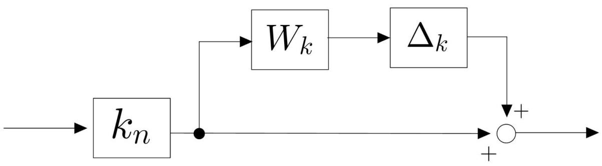
MODEL SET	**Mathematical description**
ADDITIVE UNCERTAINTY SET 	$k = k_n + W_k\Delta_k,\ \lvert\Delta_k\rvert \leq 1$
MULTIPLICATIVE UNCERTAINTY SET 	$k = k_n(1 + W_k\Delta_k),\ \lvert\Delta_k\rvert \leq 1$

Table 2: Uncertainty model sets

1.2 Fundamental bricks for structured uncertainty

Going on to analyze the structure of our uncertain system we can find in the transfer function some fundamental bricks that put all together will give us the description through block diagrams of the uncertain plant. Some aspects are presented in details for the first example while they are not repeated for the following ones which are using exactly the same steps.

1.2.1 Real Pole in zpk form

Example 1 (pole in the feedback path)

Given the following transfer function for an uncertain plant

$$G_p(s) = \frac{k}{s+p} \quad \underline{k} \leq k \leq \overline{k}, \quad \underline{p} \leq p \leq \overline{p} \tag{2}$$

we want to obtain its block diagram description. In particular for each parameter is required to obtain the central estimate k_n and p_n and the radius of uncertainty W_k and W_p. Compute

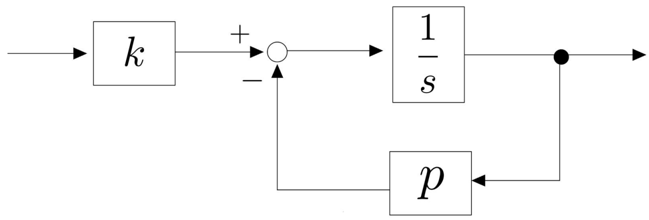
also the matrix Δ and the associated `deltaset` for the command `mu`.
The first step is for computing the value k_n, p_n, W_k and W_p:

$$k_n = \frac{\underline{k}+\overline{k}}{2} \quad W_k = \frac{\overline{k}-\underline{k}}{2}, \qquad p_n = \frac{\underline{p}+\overline{p}}{2} \quad W_p = \frac{\overline{p}-\underline{p}}{2} \tag{3}$$

At the end of the day, we are likely to build such block diagrams in SIMULINK[1], for this reason we have to manipulate the expression of Equation (2) in order to be able to represent them. If we put in evidence s at the denominator we can write:

$$G_p(s) = \frac{k}{s(1+\frac{p}{s})} = k\frac{\frac{1}{s}}{1+\frac{p}{s}}$$

The first term of the expression is simply a gain, while we can see the second term as a *feedback configuration* with negative feedback.



Block diagram for Equation (2)

Additive Uncertainty Set

If we use the as uncertainty set the *additive one* the block diagram we have showed changes as follows:

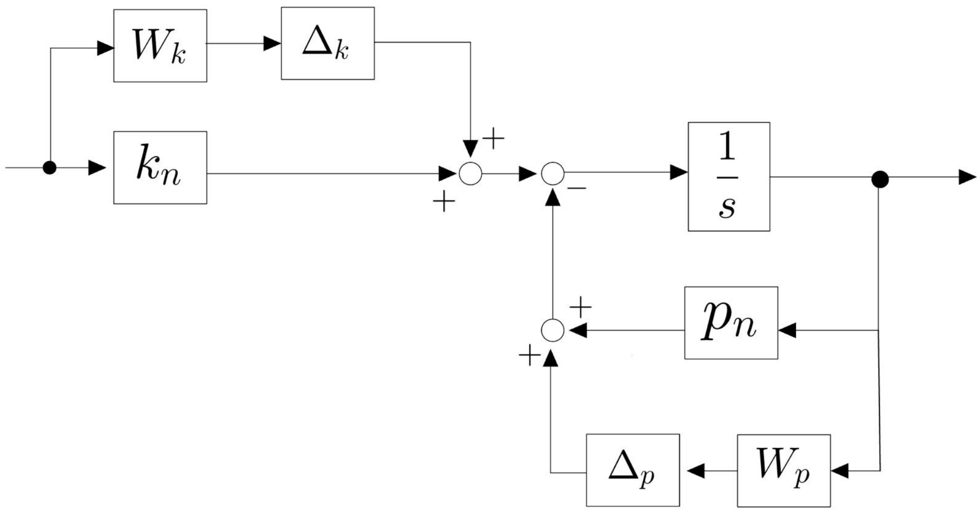


Figure 1: Block diagram of $G_p(s)$ (additive model)

Note that we have only added the for the parameters the uncertain description as indicated in Table 2. With the objective of obtaining the the $P\Delta G_c$ structure we have to: (i) put such a diagram in the Feedback control System (FCS) scheme; (ii) pull out all of the Δ_i blocks; (iii) pull out the controller G_c.
More interesting is the construction of the matrix Δ and its describing matrix `deltaset`. For this aim is crucial that all of the uncertainty channels are properly numbered. The block diagram becomes the one in Figure 2, where there are also the signals entering/coming from the (structured) uncertainty block Δ. According to the numbering we use in the block diagram we have to insert the diagonal blocks into Δ as reported here:

[1]Some algebraic manipulation are needed in order to avoid non proper blocks that – for sure– are going to raise an error during the simulation.

$$\Delta = \begin{bmatrix} \Delta_k & 0 \\ 0 & \Delta_p \end{bmatrix} \qquad \texttt{deltaset} = [\underbrace{\texttt{-1,1}}_{1};\underbrace{\texttt{-1,1}}_{2}] \tag{4}$$

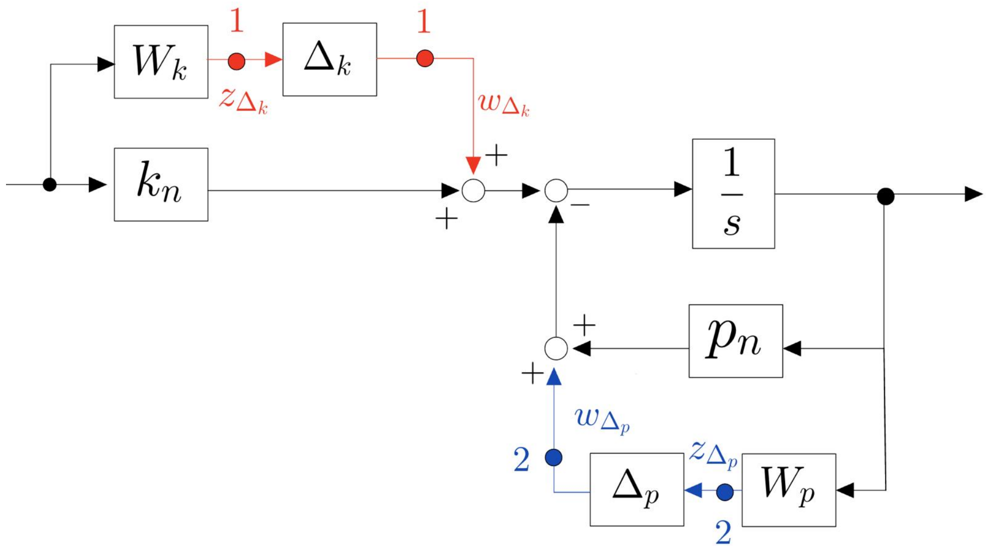


Figure 2: Block diagram of $G_p(s)$ (additive) with numbered sources

The first block of Δ is the one related to the uncertain parameter k, the second the one related to the uncertain parameter p. Let the colors guide you in the understanding the relationship existing between the block diagram and related Δ and `deltaset`.

Multiplicative Uncertainty Set

Identical reasoning can be done, if we assume to take as model of uncertainty the *multiplicative one*. Substituting in the general scheme, the structure for the uncertain parameters from the Table 2, the scheme in Figure 3 is obtained. The associated matrix Δ and the `deltaset` array are the same as before.

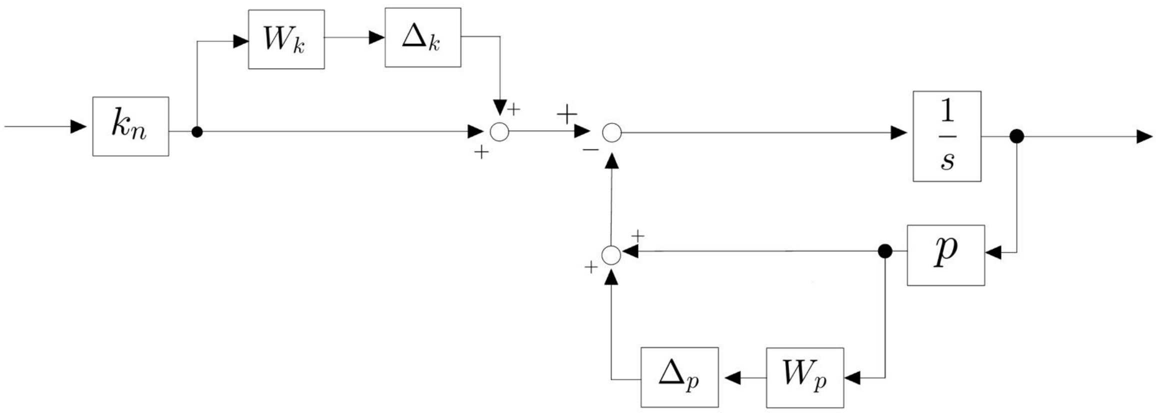


Figure 3: Block diagram for $G_p(s)$ (multiplicative set)

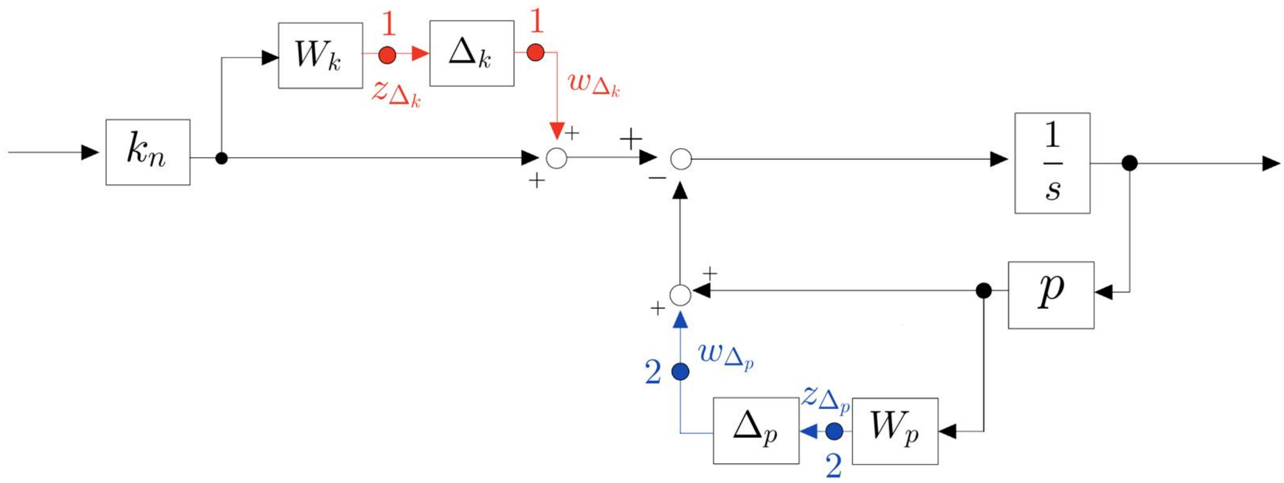


Figure 4: Block diagram for $G_p(s)$ (multiplicative set) with numbered uncertainty channels

Example 2 (model the reciprocal of an uncertain parameter)

Given the following plant

$$G_p(s) = \frac{k}{1 + s\tau}, \tag{5}$$

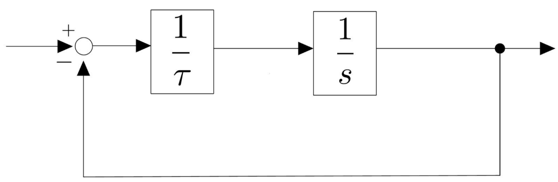
is required to find a block diagram description of it, in a way that can be implemented in a simulink scheme.

A rearragement of the terms is needed:

$$G_p(s) = k\,\frac{\frac{1}{s}\cdot\frac{1}{\tau}}{1 + \frac{1}{s\tau}} \tag{6}$$

We omit the gain, to focus our attention on the second part of the transfer function.

The second term can be represented as:



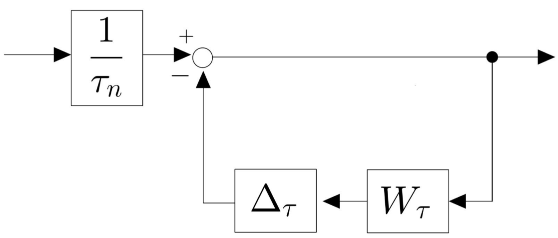
The second term can be recasted in a basic feedback structure whose direct path has the block $1/s\tau$, simply 1 in the feedback path. But how can we draw the diagram for the inverse of the uncertain parameter? It depends from the type of uncertainty set.

Multiplicative set

The term $1/\tau$ can be written as:

$$\frac{1}{\tau} = \frac{1}{\tau_n}\cdot\frac{1}{1 + W_\tau \Delta_\tau} \tag{7}$$

The related block diagram representation is:

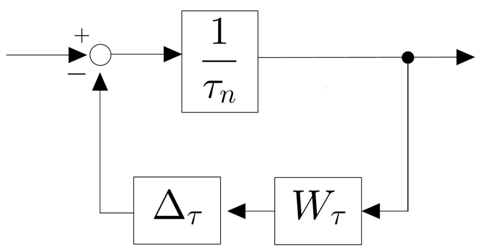


Additive set

The term $1/\tau$ can be written as:

$$\frac{1}{\tau} = \frac{1}{\tau_n + W_\tau \Delta_\tau} = \frac{\frac{1}{\tau_n}}{1 + \frac{W_\tau \Delta_\tau}{\tau_n}} \tag{8}$$

The related block diagram representation is:



The remaining part of the discussion, about the port numbering and block Δ description, the way we retrieve Δ_τ and W_τ does not change. This is the reason why we omit this part.

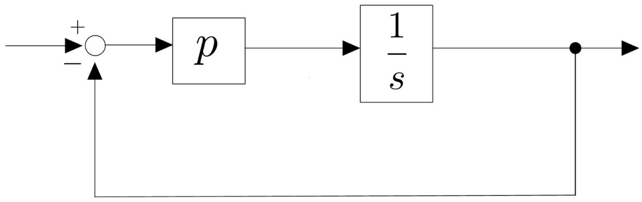
1.2.2 Real pole in DC-gain form

Here the objective is to analyze the block diagrams for the uncertain plant described by the following transfer function in *DC-gain (or time constant) form*:

$$G_p(s) = \frac{k}{1+\frac{s}{p}} \qquad \underline{k} \leq k \leq \overline{k}, \quad \underline{p} \leq p \leq \overline{p} \tag{9}$$

Following the same path we algebrically manipulate the Equation (9) in order to obtain:

$$G_p(s) = k \; \frac{\frac{p}{s}}{1+\frac{p}{s}}$$



No other comments are needed for modeling p since it is a simple uncertain parameter whose modeling can follow the Table 2.

1.2.3 Complex conjugate poles

Given the following transfer function for an uncertain plant

$$G_p(s) = \frac{k\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \quad \underline{k} \leq k \leq \overline{k}, \quad \underline{\omega}_n \leq \omega_n \leq \overline{\omega}_n \tag{10}$$

Here it is useful to apply the properties of the *Laplace transform* in order to pass into a state-space description. In particular we know, by definition, that given a certain transfer function $G_p(s)$ it is defined as

$$G_p(s) = \frac{y(s)}{u(s)} \iff y(s)(s^2+2\zeta\omega_n s+\omega_n^2) = u(s)\; k\omega_n^2 \iff \tag{11}$$

$$\frac{1}{\omega_n^2}\ddot{y}(t) + \frac{2\zeta}{\omega_n}\dot{y}(t) + y(t) = ku(t) \tag{12}$$

We can retrieve from the *differential equation* eq. (12) the state space description by introducing some state variables. In particular:

$$x_1(t) = y(t) \quad x_2(t) = \dot{y}(t) \tag{13}$$

Here we assume **zero initial conditions** so that we can write[2]:

$$\begin{cases} \dot{x_1}(t) = x_2(t) \\ \dot{x_2}(t) = -2\zeta\omega_n x_2(t) - \omega_n^2 x_1(t) + k\omega_n^2 u(t) \\ y = x_1 \end{cases} \implies \begin{cases} \dot{x}(t) = Ax(t)+Bu(t) \\ y(t) = Cx(t) \end{cases}$$

In order to draw the block diagram we individuate the equation containing the $u(t)$ that is the input till reaching the output $y(t)$. The resulting block diagram is:

[2]Here the matrices of the state space description are:

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ k\omega_n^2 \end{bmatrix}, \quad C = [1 \quad 0]$$

Since they are not strictly necessary we put them here.

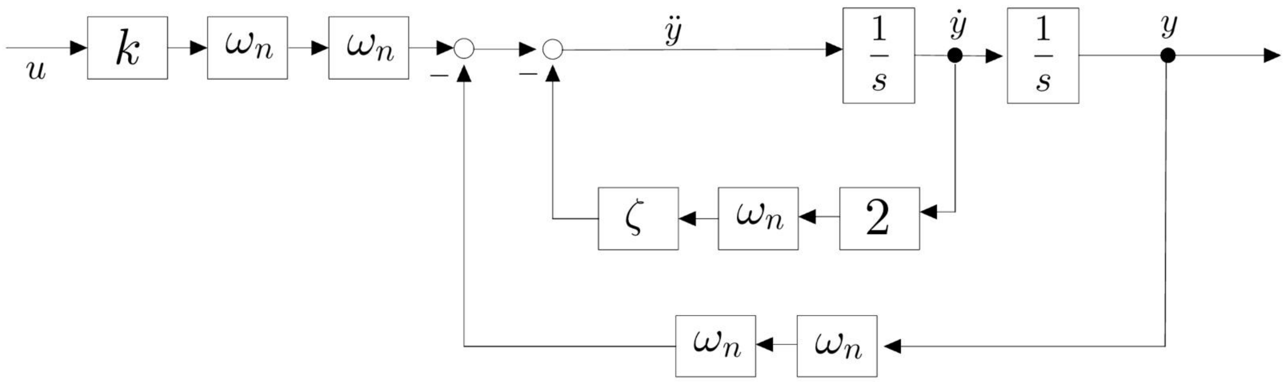


Figure 5: Block diagram of $G_p(s)$

This is the general form of the transfer function containing the complex conjugate poles, the single parameters must be substituted with their own uncertain description using one of the uncertainty sets.
This example is interesting since here there are repeated parameters, we can number with 1 the uncertainty related to the parameter k, 2 the uncertainty related to the damping factor ζ, finally all the others are the uncertainty channels related to the natural frequency ω_n. The resulting `deltaset` is

$$\texttt{deltaset} = [-1, 0;\ -1, 0;\ -5, 0] \tag{14}$$

The description obtained so far by using the state space description is quite general, so that we could consider also the case of **non-zero** initial conditions. Clearly also the previous example would have been provided us with the possibility to follow the same appproach, however when simple manipulations can be algebraically done by using directly the transfer function it is more convient and fast.

2 Robust stability with structured uncertainty

3 Robust performance with structured stabiliy

4 Structured singular value μ

5 μ-analysis