Math tools for ROTATIONS (Formulary)

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Direction Cosine Matrices (DCM)

Position of a particle
$\mathbf{R} = X\mathbf{I} + Y\mathbf{J} + Z\mathbf{K}$ Position of the particle in F1
$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ Position of the particle in F2
$\mathbf{R_O} = X_O\mathbf{I} + Y_O\mathbf{J} + Z_O\mathbf{K}$ Position in F1 of the O of F2

F1 vs F2 $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix} + \begin{bmatrix} \mathbf{I}\cdot\mathbf{i} & \mathbf{I}\cdot\mathbf{j} & \mathbf{I}\cdot\mathbf{k} \\ \mathbf{J}\cdot\mathbf{i} & \mathbf{J}\cdot\mathbf{j} & \mathbf{J}\cdot\mathbf{k} \\ \mathbf{K}\cdot\mathbf{i} & \mathbf{K}\cdot\mathbf{j} & \mathbf{K}\cdot\mathbf{k} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{T} \doteq \begin{bmatrix} \mathbf{I}\cdot\mathbf{i} & \mathbf{I}\cdot\mathbf{j} & \mathbf{I}\cdot\mathbf{k} \\ \mathbf{J}\cdot\mathbf{i} & \mathbf{J}\cdot\mathbf{j} & \mathbf{J}\cdot\mathbf{k} \\ \mathbf{K}\cdot\mathbf{i} & \mathbf{K}\cdot\mathbf{j} & \mathbf{K}\cdot\mathbf{k} \end{bmatrix}$

Interpretations of DCM $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ can be seen as: $\begin{cases} \text{Coordinate transformation} & \text{F2} \to \text{F1} \\ \text{Rotation} & \text{F1} \to \text{F2} \end{cases}$

Euler angles

Elementary 3D-rotation matrices

$$\mathbf{T}_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & cos\phi \end{bmatrix} \quad \text{Rotation about } X \text{ (or x) of } \phi$$

$$\mathbf{T}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad \text{Rotation about } Y \text{ (or y) of } \theta$$

$$\mathbf{T}_3(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Rotation about } Z \text{ (or z) of } \psi$$

Rotations as product of the matrices $\mathbf{T}_\diamond$

- **6 Tait-Bryan rotations** the most used are 123 and 321
- **6 proper Euler rotations** the most common is 313

Angle-axis representation $\mathbf{T} \equiv \mathbf{T}(\beta, \mathbf{u})$

Theorem (angle-axis representation)

(i) Any rotation of a rigid body where a point is fixed is **equivalent** to a rotation in which the rotation axis passes through the fixed point;

(ii) The rotation axis is the **eigenvector u** corresponding to the eigenvalues 1 of the rotation matrix.

Quaternions

Basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$

Properties
$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\otimes\mathbf{j}\otimes\mathbf{k} = -1$
$\mathbf{i}\otimes\mathbf{j} = -\mathbf{j}\otimes\mathbf{i} = \mathbf{k}$
$\mathbf{j}\otimes\mathbf{k} = -\mathbf{k}\otimes\mathbf{j} = \mathbf{i}$
$\mathbf{k}\otimes\mathbf{i} = -\mathbf{i}\otimes\mathbf{k} = \mathbf{j}$

Notations
$$\begin{aligned} \mathfrak{q} &= q_0 + q_1\mathbf{i} + q_1\mathbf{j} + q_2\mathbf{k} = \\ &= q_0 + \mathbf{q} = \\ &= (q_0, q_1, q_2, q_3) = \\ &= (q_0, \mathbf{q}) = \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix} \end{aligned}$$

Null element $\mathfrak{O} = (0, \mathbf{0})$ **Complex conjugate** $\mathfrak{q}^* = q_0 - q = \begin{bmatrix} q_0 \\ -\mathbf{q} \end{bmatrix} = (q_0, -\mathbf{q})$ **Identity** $\mathfrak{I} = (1, \mathbf{0})$

Quaternion norm $|\mathfrak{q}| = \|\mathfrak{q}\| = |\mathfrak{q}^*| = \sqrt(q \cdots q^*) = \sum_{i=0}^{3} q_i{}^2$ **Reciprocal quaternion** $\mathfrak{q}^{-1} = \frac{\mathfrak{q}^*}{\|\mathfrak{q}\|}$

Sum $\mathfrak{q} + \mathfrak{p} = q_0 + p_0 + \mathbf{q} + \mathbf{p}$ **Dot product** $\mathfrak{q}\cdot\mathfrak{p} = \sum_{i=0}^{3} q_i p_i$

Hamilton product $\mathfrak{q} \otimes \mathfrak{p} = (q_0 + \mathbf{q}) \otimes (p_0 + \mathbf{p}) = (...) = (q_0 p_0 - \mathbf{q} \cdot \mathbf{p}) + (q_0 \mathbf{p} + p_0 \mathbf{q} + \mathbf{q} \times \mathbf{p})$

Quaternion associated with rotations $\mathfrak{q} \doteq \left(\cos\frac{\beta}{2},\ \mathbf{u} \sin\frac{\beta}{2} \right) = \left(\cos\frac{\beta}{2},\ u_1 \sin\frac{\beta}{2},\ u_2 \sin\frac{\beta}{2},\ u_3 \sin\frac{\beta}{2} \right)$

Theorem $(0, p) = \mathfrak{q} \otimes (0, \mathbf{r}) \otimes \mathfrak{q}^*$

$$\mathbf{T}_1(\phi) \longleftrightarrow \mathfrak{q}_1 = \left(\cos\frac{\phi}{2}, \sin\frac{\phi}{2}, 0, 0 \right)$$

Elementary transformations $\mathbf{T}_2(\theta) \longleftrightarrow \mathfrak{q}_2 = \left(\cos\frac{\theta}{2}, 0, \sin\frac{\theta}{2}, 0 \right)$

$$\mathbf{T}_3(\psi) \longleftrightarrow \mathfrak{q}_3 = \left(\cos\frac{\psi}{2}, 0, 0, \sin\frac{\psi}{2} \right)$$

Any rotation can be expressed as: $\mathfrak{q} = \mathfrak{q}_1 \otimes \mathfrak{q}_2 \otimes \cdots \otimes \mathfrak{q}_n$

Inverse rotation $\mathfrak{q}^{-1} = \mathfrak{q}^*$