CONVEX OPTIMIZATION AND ENGINEERING APPLICATIONS (Formulary)

1. Introduction

Standard form for optimization problems $p^* = \min_x f_0(x)$ s.t. $\quad f_i(x) \leq 0, i = 1, ..., m$
where: $f_0 : \mathbb{R}^n \to \mathbb{R}$ (**objective function**), $f_i : \mathbb{R}^n \to \mathbb{R}$, (inequality constraints)
Feasible set $\mathcal{X} = \{x \in \mathbb{R}^n : \ f_i(x) \leq 0\}$ — **Optimal solution** $x^* \in \mathcal{X} : f(x^*) = p^*$ — **Optimal value** p^*
Equality constraints $h_i(x) = 0 \ i = 1, ..., p \iff h_i(x) \leq 0, -h_i(x) \leq 0$ – **Optimal Set** $\mathcal{X}_{opt} = \{x \in \mathcal{X} : f(x) = p^*\}$
Optimization problems in maximization form $p^* = \max_x f_0 \iff -p^* = min_x \ -f_0(x) = min_x \ g_0(x)$
ε-suboptimality of a solution $x \in \mathbb{R}^n \iff p^* \leq f_0(x) \leq p^* + \varepsilon$

2. Vector, Projections, Functions, Gradients

Vector $x \in \mathbb{R}^n = [x_1 \quad x_2 \quad ... \quad x_n]^T$, $x_i \in \mathbb{R}$ – **Sum** $x, y \in \mathbb{R}^n, x + y \iff x_i + y_i, \ i = 1, ..., n$
Scalar multiplication $\alpha \in \mathbb{R}, x \in \mathbb{R}^n, \ \alpha x \iff \alpha x_i \ i = 1, ..., n$
Subspace $\mathcal{V} \subseteq \mathcal{X}$ is a **subspace** $\iff \forall x, y \in \mathcal{V}, \ \alpha x + \beta y \in \mathcal{V}$
$S = \{x^{(1)}, ..., x^{(n)}\}$, $\text{span}(S) = \{\alpha_1 x^{(1)} + ... + \alpha_n x^{(n)}, \ \alpha_i \in \mathbb{R}, \ i = 1, ..., n\}$
Basis of a vector space $B = \{b^{(1)}, ..., b^{(n)}\} \iff \text{span}(B) = \mathcal{X}, \forall x \in \mathcal{X}, x = \alpha_1 b^{(1)} + ... + \alpha_n b^{(n)}, \ \alpha_i \in \mathbb{R}$
Direct sum of subspaces $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$, $\mathcal{X} \oplus \mathcal{Y} = \{x + y : \ x \in \mathcal{X}, y \in \mathcal{Y}\}$
Affine set $\mathcal{X}$ vector space, $\mathcal{V} \subseteq \mathcal{X} \to$ affine set: $\mathcal{A} = \{x \in \mathcal{X} : x = x_0 + v, \ v \in \mathcal{V}\}$, note that: $\dim_{\mathbb{R}} \mathcal{A} = \dim_{\mathbb{R}} \mathcal{V}$
Line (1D-(affine set)) $L = \{x \in \mathbb{R}^n : \ x = x_0 + v, \ v \in \text{span}(u), \|u\|_2 = 1\}$
Norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R} \to \|x\| \geq 0, \ \|x\| = 0 \iff x = 0, \ \|xy\| = \|x\|\|y\|, \ \|\alpha x\| = |\alpha| \cdot \|x\|, \ \|x + y\| \leq \|x\| + \|y\|$
ℓ_p-norms $\|x\|_p = \left(\sum_{i=0}^n |x_i|^p\right)^{1/p}$, $p = 2 \to$ Euclidean Distance, $p = 1 \to$ Manahattan Distance, $p = \infty \to \max x_i$
Inner Product $\langle x, y\rangle \geq 0$, $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$, $\langle x, y\rangle = \langle y, x\rangle$, $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$, $\cos\theta = \frac{\langle x, y\rangle}{\|x\|_2\|y\|_2}$
Standard inner product $x, y \in \mathcal{X}$, $\langle x, y\rangle = x^T y = \sum_{i=1}^n x_i y_i$, **Schwarz inequality** $\langle x, y\rangle \leq \|x\|_2\|y\|_2$,
$\|x\|_2 = \sqrt{x^T x} = \sqrt{\langle x, x\rangle}$, $x \perp y \iff \langle x, y\rangle = 0 \iff \cos\theta = 0$
$S = \{x^{(1)}, ..., x^{(n)}\}$ **mutually orthogonal** $\iff \langle x, y\rangle = \begin{cases} 0 & i \neq j \\ \neq 0 & i = j \end{cases}$, **orthonormal** $\iff \langle x, y\rangle = \begin{cases} 1 & i \neq j \\ 0 & i = j \end{cases}$
Vector $\perp$ Subspace $\mathcal{S} \subseteq \mathcal{X}$, $x \in \mathcal{X}$, $x \perp \mathcal{S} \iff \langle x, s\rangle = 0, \forall s \in \mathcal{S}$, $\mathcal{S}^\perp = \{x \in \mathcal{X} : \langle x, s\rangle = 0, \forall s \in \mathcal{S}\}$
Orthogonal decomposition of a vector $\forall x \in \mathcal{X}, x = x_1 + x_2 : \ x_1 \in \mathcal{S}, x_2 \in \mathcal{S}^\perp \iff \mathcal{X} = \mathcal{S} \oplus \mathcal{S}^\perp$
Projections $\Pi_{\mathcal{S}}(x) = \min_{y \in \mathcal{S}} \|y - s\|$ – (*Projection Theorem*) $\begin{cases} x^* \in \mathcal{S} \text{ is unique} \\ (x - x^*) \perp \mathcal{S} \iff (x - x^*) \in \mathcal{S}^\perp \end{cases}$
Functions $f : \mathbb{R}^n \to \mathbb{R}$ (function), $f : \mathbb{R}^n \to \mathbb{R}^m$ (map), **Domain of** f: $\text{dom}(f) = \{x \in \mathbb{R}^n : \ \|f(x)\| < \infty\}$
$\text{graph}(f) = \{(x, f(x)) \in \mathbb{R}^{n+1} : \ x \in \mathbb{R}^n\}$, $\text{epi}(f) = \{(x, t) \in \mathbb{R}^{n+1} : \ x \in \mathbb{R}^n, t >= f(x)\}$
Contour curves $C_f(t) = \{x \in \mathbb{R}^n : f(x) = t, \ t \in \mathbb{R}\}$ – **α-sublevel set** $S_\alpha = \{x \in \mathbb{R}^n : \ f(x) \leq t, \ t \in \mathbb{R}\}$
f is **linear** $\iff f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$, $\alpha, \beta \in \mathbb{R}$, $\tilde{f}$ is **affine** $\iff \tilde{f} - x_0$ is linear
any affine function can be written as: $f(x) = a^T x + b$ where $a \in \mathbb{R}^n, \ b \in \mathbb{R}, \ b = f(0), \ a_i = f(e_i) - b$,
Hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n : \ a^T x = b\}$, $a \in \mathbb{R}^n$ (normal direction), $\mathcal{H}_- = \{x \in \mathbb{R}^n : \ a^T x \leq b\}$, (open half-space)
$\mathcal{H}_{++} = \{x \in \mathbb{R}^n : \ a^T x > b\}$ (closed half-space)
Gradient Given $f : \mathbb{R}^n \to \mathbb{R}$ differentiable $\nabla f = \left[\frac{\partial f}{\partial x_1} \ \frac{\partial f}{\partial x_2} \ ... \ \frac{\partial f}{\partial x_n}\right]^T$, $\nabla f(x_0)^T \cdot v$ (directional derivatives)
the rate of variation is maximal when $\nabla f(x_0) \parallel v$ minimal when $\nabla f(x_0) \perp v$, moreover $\nabla f(x_0) \perp C_f(t) \ \forall x_0 \in \text{dom}(f)$

3. Matrices

Matrix $A \in \mathbb{R}^{m,n}$, $a_{ij} \in \mathbb{R}$ — $AB_{ij} = R_i(A) \cdot C_j(B)$ (rows by column) — $(AB)^T = B^T A^T$
An **affine map** $f = Ax + b$, $A \in \mathbb{R}^{m,n}$, $b \in \mathbb{R}^n$ (generalization of an **affine function**)
Subspaces associated with a matrix $\begin{cases} \mathcal{R}(A) = \{Ax : x \in \mathbb{R}^n\} & \text{range of A} \\ \mathcal{N}(A) = \{x \in \mathbb{R}^n : \ Ax = 0\} & \text{nullspace of A} \\ \dim_{\mathbb{R}} \mathcal{R}(A) = \text{rank}(A), \ 1 \leq \text{rank}(A) \leq \min(m, n) & \text{rank of A} \end{cases}$
Fundamental thm. of linear algebra For any $A \in \mathbb{R}^{m,n}$ $\begin{cases} \mathbb{R}^n \supseteq \mathcal{N}(A) \perp \mathcal{R}(A^T) \equiv \mathcal{N}(A)^\perp \iff \mathbb{R}^n = \mathcal{N}(A) \oplus \mathcal{R}(A^T) \\ \mathbb{R}^m \supseteq \mathcal{R}(A) \perp \mathcal{N}(A^T) \equiv \mathcal{R}(A)^\perp \iff \mathbb{R}^m = \mathcal{R}(A) \oplus \mathcal{N}(A^T) \\ \forall x \in \mathbb{R}^n, \ x = A^T x + \zeta, \ \zeta \in \mathcal{N}(A) \\ \forall w \in \mathbb{R}^m, \ w = Ax + \xi, \ \xi \in \mathcal{N}(A^T) \end{cases}$
Singular matrix $A \in \mathbb{R}^{n,n}$ is **singular** $\iff \det(A) = 0 \iff \mathcal{N}(A) \neq \{0\} \iff \text{rank}(A) \neq \min(m, n)$
Inverse matrix $A \in \mathbb{R}^{n,n}$, $\det(A) \neq 0 \ \exists A^{-1} \in \mathbb{R}^{n,n} : AA^{-1} = I_n$, $(AB)^{-1} = B^{-1}A^{-1}$
Similar matrices $A, B \in \mathbb{R}^{n,n}$ are similar $\iff \exists P : B = P^{-1}AP$, P is **non-singular** (columns: basis for $\mathbb{R}^{n,n}$)
Eigenvalues/Eigenvector $u \in \mathbb{R}^{n,n}$ **eigenvector** for $A \iff Au = \lambda u, \ u \neq 0$,
$\lambda \to$ **eigenvalue associated with** u, **Eigenvalues** $\to_{\text{roots of}} p(\lambda) = \det(A - \lambda I_n) = 0$, **Eigenspace** $\to \phi_i = \mathcal{N}(A - \lambda_i I_n)$
Algebraic Multiplicity ν_i (multiplicity of the root of $p(\lambda)$) **Geometric Multiplicity** $\mu_i = \dim_{\mathbb{R}}(\phi_i) \longrightarrow \nu_i \leq \mu_i$
Diagonalizable matrices (thm.) $\lambda_i, \ i = 1, ..., k \leq n$ (distinct eigenvalues of A), $U^{(i)} = [u_1^{(i)} \quad ... \quad u_{\nu_i}^{(i)}]$ the eigenvector wrt λ_i, **if** $\nu_i = \mu_i, \ i = 1, ..., k$, $U = [U^{(1)} \quad ... \quad U^k]$ is **invertible** and $A = U\Lambda U^{-1}$, $\Lambda = diag(\lambda_1 I_{\mu_1}, ..., \lambda_k I_{\mu_k})$
Matrix norms ...is a function $f : \mathbb{R}^{m,n} \to \mathbb{R}$ — additional property $f(AB) \leq f(A)f(B)$ **sub-multiplicativity**
Frobenius norm (extension of ℓ_2-norm) $\|A\|_F = \sqrt{\text{trace}(A^T A)} = \sqrt{\sum_{j=1}^m \sum_{i=1}^n a_{ij}^2} = \sqrt{\sum_{i=1}^n \lambda_i(A^T A)}$

Operator norms (max input-output gain of $y = Au$) $\|A\|_p \doteq \max_{u\neq 0} \frac{\|Au\|_p}{\|u\|_p} = \max_{\|u\|_p=1} \|Au\|_p$ (ℓ_p-induced norms)
$\|A\|_1 = \max_{j=1,...,m} \sum_{i=1}^{n} |a_{ij}|$, $\|A\|_\infty = \max_{i=1,...,m} \sum_{j=1}^{n} |a_{ij}|$, $\|A\|_2 = \sqrt{\lambda_{\max}(A^TA)}$ (see variational characterization)
Spectral radius $A \in \mathbb{R}^{n,n}$ $\rho(A) \doteq \max_{i=1,...,n} |\lambda_i(A)|$ — $\rho(A) \leq \min(\|A\|_1, \|A\|_\infty)$

4. Symmetric Matrices and SVD

$A \in \mathbb{R}^{n,n}$ is **symmetric** if $AA^T = I_n \iff A^{-1} = A^T$, $\mathbb{S}^n$ is the subspace of **symmetric matrices** $n \times n$

Examples: (Hessian matrix given $f : \mathbb{R}^2 \to \mathbb{R}$) $H = \begin{bmatrix} \frac{\partial f}{\partial x_1^2} & \frac{\partial f}{\partial x_1 x_2} \\ \frac{\partial f}{\partial x_2 x_1} & \frac{\partial f}{\partial x_2^2} \end{bmatrix}$, (quadratic function) $q(x) = \frac{1}{2}x^THx + c^Tx + d$

Quadratic form Given $A \in \mathbb{S}^n$, $\forall x \neq 0$ the function $q(x) = x^TAx$ is a **quadratic form associated with A**
Spectral theorem Given $A \in \mathbb{S}^n$ it holds that: $\lambda_i \in \mathbb{R}$, $i = 1, ..., n$, exists $U = [u_1 \quad \dots \quad u_n]$ orthogonal and $A = U\Lambda U^T = \sum_{i=1}^{n} \lambda_i u_i u_i^T$, $\Lambda = diag(\lambda_1, ..., \lambda_n)$. I can sort them $\lambda_{max}(A) = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n = \lambda_{min}(A)$
Rayleigh quotient given $A \in \mathbb{S}^n$, we can define $\forall x \in \mathbb{R}^n$ the quantity $r(x) = \frac{x^TAx}{x^Tx}$ (Rayleigh quotient)

Theorem For any $A \in \mathbb{S}^n$, $\forall x \neq 0$, $\lambda_{max} \leq r(x) \leq \lambda_{min}$, where $\begin{cases} \lambda_{max}(A) = \max_{\|x\|_2=1} x^TAx & (x^* = u_1) \\ \lambda_{min}(A) = \min_{\|x\|_2=1} x^TAx & (x^* = u_n) \end{cases}$

...on the ℓ_2-induced norm (by using the definition of operator norm and ℓ_2-norm) $\frac{\|Ax\|_2^2}{\|x\|_2^2} = \frac{(Ax)^T(Ax)}{x^Tx} = \frac{x^T(A^TA)x}{x^Tx}$
that is the Rayleigh quotient of $A^TA \in \mathbb{S}^n$ from the theorem follows that $\|A\|_2 = \max_{\|x\|\neq 0} r_{A^TA}(x) = \sqrt{\lambda_{max}(A^TA)}$ $\square$

Given a symmetric matrix, it is said to be $\begin{cases} \textbf{Positive definite}(A \succ 0) & \forall x \in \mathbb{R}^n, x^TAx > 0 \iff \lambda_i > 0,\ i = 1, ..., n \\ \textbf{Positive semidefinite}(A \succeq 0) & \forall x \in \mathbb{R}^n, x^TAx \geq 0 \iff \lambda_i \geq 0,\ i = 1, ..., n \\ \textbf{Negative definite}(A \prec 0) & \forall x \in \mathbb{R}^n, x^TAx < 0 \iff \lambda_i < 0,\ i = 1, ..., n \\ \textbf{Negative semidefiite}(A \preceq 0) & \forall x \in \mathbb{R}^n, x^TAx \leq 0 \iff \lambda_i \leq 0,\ i = 1, ..., n \end{cases}$

Matrix square-root $A \succeq 0 \Rightarrow \exists B :\ A = B^2$, $B = A^{1/2}$ is the **matrix square-root**
Cholesky decomposition $A \succ 0 \iff \exists B : A = B^TB$, such a B can be computed as: $B = U\Lambda^{1/2}U^T$, where U comes from the spectral decomposition and $\Lambda^{1/2} = diag(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})$
SINGULAR VALUE DECOMPOSITION(SVD) (It holds for any matrix $A \in \mathbb{R}^{m,n}$, A^TA is very important!)
Theorem Given $A \in \mathbb{R}^{m,n}$, it can be written as: $A = U\tilde{\Sigma}V^T$, $U \in \mathbb{R}^{m,m}$, $V \in \mathbb{R}^{n,n}$ are *orthogonal matrices* while $\tilde{\Sigma} = \begin{bmatrix} \Sigma & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}$, $\Sigma = diag(\sigma_1, ..., \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq 0$ are the **singular values**, $U = [u_1, ..., u_r]$ are the **left singular vector**, $V = [v_1, ..., v_r]$ are the **right singular vector**.

SVD (compact form) $A = U_r\Sigma V_r = \sum_{i=1}^{r} (\sigma_i u_i v_i^T)$, moreover $\begin{cases} \sigma_i^2 = \lambda_i(A^TA), \sigma_1^2 = \lambda_{max}(A^TA) \\ u_i \textbf{ eigenvectors of } AA^T \\ v_i \textbf{ eigenvectors of } A^TA \end{cases}$

Properties of SVD We can redefine some properties by using SVD $\begin{cases} rank(A) = r \\ \mathcal{N}(A) = \mathcal{R}([v_{r+1} \dots v_n]),\ \mathcal{R}(A) = \mathcal{R}([u_1 \dots u_r]) \\ \|A\|_F^2 = \sum_{i=1}^{r} \sigma_i^2, \|A\|_2^2 = \lambda_{max}(A^TA) = \sigma_1^2 \\ \|A\|_* = \sum_{i=1}^{r} \sigma_i \textbf{ nuclear norm} \end{cases}$

Moore-Penrose pseudo-inverse $A^\dagger = V\tilde{\Sigma}^\dagger U^T$ (or in compact form) $A^\dagger = V_r\Sigma^{-1}U_r^T$, (full column) $A^\dagger = (A^TA)^{-1}A$
Condition number How close is A to be singular? $\kappa(A) = \frac{\sigma_1}{\sigma_n}$ the larger κ the closest A to be singular.
Low rank approximation $A_k = \arg\min_{A_k \in \mathbb{R}^{m,n}, rank(A_k)=k} \|A - A_k\|_F^2 = \sum_{i=1}^{\mathbf{k}} (\sigma_i u_i v_i^T)$
Ratio of the total variance in A_k is the quantity $\eta_k = \frac{\|A_k\|_F^2}{\|A\|_F^2} = \frac{\sigma_1^2 + ... + \sigma_k^2}{\sigma_1^2 + ... + \sigma_n^2}$ is useful a graph k vs η_k to choose properly k — **Norm approximation error** $e_k = \frac{\|A - A_k\|_F^2}{\|A\|_F^2} = 1 - \eta_k$
Application: Principal component Analysis (PCA) I collect in a matrix $X \in \mathbb{R}^{n,m}$ by columns m samples characterized by n features. $\tilde{X} = [x_1 - \bar{x} \ ...\ x_m - \bar{x}]$, $\bar{x} = \frac{1}{m}\sum_{i=1}^{m} x_i$ (baricenter). $x_i \in \mathbb{R}^n$
I want a direction $z \in \mathbb{R}^n$ along which the projections of the centered data have the maximal variance.

Ingredients: $\begin{cases} \text{projection on } z & \alpha_i = \tilde{x}_i^T z,\ i = 1, ..., m \\ \text{variance} & \frac{1}{m}\sum_{i=1}^{m} {\alpha_i}^2 = \sum_{i=1}^{m} z^T\tilde{x}_i\tilde{x}_i^T z = z^T\tilde{X}\tilde{X}^T z \end{cases}$

Problem to solve: $\max_{z \in \mathbb{R}^n} z^T\tilde{X}\tilde{X}^Tz$, $\tilde{X} \underset{\text{SVD}}{=} U_r\Sigma {V_r}^T$, $\tilde{X}\tilde{X}^T = U_r\Sigma^2 {U_r}^T$ (spectral decomposition)
from the theorem: $z = u_1$ (**first principal component**) $\to$ the i-th principal component $\iff$ i-th row of U from SVD. $U_k \in \mathbb{R}^{n,k}$ (first k principal components, columns of U) $\longrightarrow$ I choose k according to η_k, then: $x = \bar{x} + U_k z$, $z \in \mathbb{R}^k$ (zero-mean random factors). **Result:** we are using $z \in \mathbb{R}^k$, instead of $x \in \mathbb{R}^n$, $k < n$ to represent our data. $\square$

5. Least Squares (LS)

Applications: $\begin{cases} \text{approximation for a "fat" system of equations} \\ \text{polynomial approximation/regression} \end{cases}$ $y = Ax \underset{\text{relax}}{\longrightarrow} y \simeq Ax$, **residuals** $r(x) = Ax - y$
Objective: find an x such that the squared-residuals are "small" $f_0(x) = \sum_i r_i^2(x) = \sum_i (a^Tx - y_i)^2 = \|Ax - y\|_2^2$
Least Squqres problem: $\min_{x \in \mathbb{R}^n} \|Ax - y\|_2^2$ (unconstrained)— **Solution** we have to find the roots of $\nabla f_0(x) = 0$
Rewriting of the f_0 $\|Ax - y\|_2^2 = x^TQx - 2b^Tx + c$, $Q = A^TA \succ 0,\ b = A^Ty,\ c = \|y\|_2^2$

$\nabla f_0(x) = 0 \iff Qx = b \iff \mathbf{A^T A x} = \mathbf{A^T y}$ (normal equations) $x = (A^TA)^{-1}A^Ty$ (for full column A)
Geometric interpretation (projections) LS problem can be recasted as: $\min_{\tilde{y}\in\mathcal{R}(A)} \|\tilde{y} - y\|_2^2$
You know from the projection theorem that: (i) $\tilde{y} \in \mathcal{R}(A)$, (ii) $(\tilde{y} - y) \perp \mathcal{R}(A) \iff (\tilde{y} - y) \in \mathcal{N}(A^T) \iff A^T(\tilde{y} - y) = 0 \iff A^T(Ax - y) = 0 \iff A^TAx = A^Ty$ (normal equations)

Variants for the LS problem

- LS+**Equality constraints**: $\min_x f_0$ s.t. $Cx = d \to x = \bar{x} + Nz$, span$(N)$=$\mathcal{N}(C)$
  $\min_x \|\bar{A}x - \bar{y}\|_2^2,\ \bar{A} = AN,\ \bar{y} = y - A\bar{x}$
- LS+**weighted residuals**: $f_0 = \sum w_i r_i^2(x) = \|W(Ax - y)\|_2^2,\ W = diag(w_1, ..., w_n)$
  $\min_x \|A_w x - y_w\|_2^2,\ A_w = WA,\ y_w = Wy$
- LS+ℓ_2-**regularization**: $\min_x \|Ax - y\|_2^2 + \gamma\|x\|_2^2,\ \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\|_2^2 = \|a\|_2^2 + \|b\|_2^2$
  $\min_x \|\tilde{A}x - \tilde{y}\|, \tilde{y} = [y \quad 0_n]^T,\ \tilde{A} = [A \quad \sqrt{\lambda}I_n]^T$

6. Convexity: sets and functions

Given $P = \{x^{(1)}, x^{(2)}, ..., x^{(m)}\}$, $\begin{cases} \textbf{linear combination} & \lambda_1x^{(1)} + \lambda_2x^{(2)} + ... + \lambda_m x^{(m)}, \lambda_i \in \mathbb{R} \\ \textbf{convex combination} & \lambda_i \in \mathbb{R}, \lambda_i \geq 0, \sum \lambda_i = 1 \end{cases}$

Well-known subspaces

- linear hull: **subspace with all linear combinations**
- affine hull: aff(P), $\lambda_i \in \mathbb{R},\ \sum \lambda_i = 1$
- convex hull: co(P) **subspace with all convex combinations**
- conic hull: conic(P) $\lambda_i \in \mathbb{R}, \lambda_i \geq 0$

CONVEX SETS $\mathcal{C} \subseteq \mathbb{R}^n$ **convex** $\iff \forall x, y \in \mathcal{C} \to$ the segment joining x and y lies in $\mathcal{C}$. More formally, a convex combination of any two points falls within $\mathcal{C}$...
$\underbrace{\forall x, y,\ \lambda \in [0,1],\ x, y \in \mathcal{C} \to\ \lambda x + (1 - \lambda y) \in \mathcal{C},}_{\text{convex set}}\quad \underbrace{x \neq y, \lambda \in (0,1), \lambda x + (1 - \lambda)y \in \text{relint}(C)}_{\text{strictly convex set}}$
Cone and convex cone Given $\mathcal{C} \subseteq \mathbb{R}^n$ is a **cone** $\iff x \in \mathcal{C} \to \alpha x \in \mathcal{C},\ \alpha > 0$ — $\mathcal{C}$ also convex $\to$ **convex cone**

Operations preserving convexity

- Intersections with convex sets: $\mathcal{C}_1, ..., \mathcal{C}_n$ convex $\to \mathcal{C} = \bigcap_{i=1}^n \mathcal{C}_i$ **is convex**
- Affine transformations: $f(x) = Ax + b, \mathcal{C}$ cvx $\to f(C) = \{f(x) :\ x \in \mathcal{C}\}$ cvx

Supporting Hyperplane $\underset{\textbf{Thm.}\text{: exists separating } \mathcal{H} \text{ for } \mathcal{C} \text{ at } z}{\underline{\text{Given } \mathcal{C} \text{ convex set}, z \in \partial\mathcal{C}}}$, $\mathcal{H}$ **supporting hyperplane** $\iff z \in \mathcal{H}$ and $\mathcal{C} \subseteq \mathcal{H}_-$

Separating Hyperplane $\underset{\textbf{Thm.}: \mathcal{C}_1\cap\mathcal{C}_2=\emptyset \text{ exists sep. } \mathcal{H}}{\underline{\text{Given } \mathcal{C}_1, \mathcal{C}_2 \text{ convex sets}}}$ the hyperplane $\mathcal{H}$: $\begin{cases} \textbf{separates} \text{ the sets} & \mathcal{C}_1 \subseteq \mathcal{H}_-, \mathcal{C}_2 \subseteq \mathcal{H}_+ \\ \textbf{strictly separates} \text{ the sets} & \mathcal{C}_1 \subseteq \mathcal{H}_{--}, \mathcal{C}_2 \subseteq \mathcal{H}_{++} \end{cases}$

CONVEX FUNCTIONS

$f : \mathbb{R}^n \to \mathbb{R}$ is

- **convex** $\iff$ **(i)** dom(f) convex, **(ii)** $x, y \in \text{dom}(f),\ \lambda \in [0,1] \to f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$
- **strictly convex** $\iff x, y \in \text{dom}(f),\ \lambda \in [0,1] \to f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$
- **strongly convex** $\iff \exists m > 0 :\ \tilde{f}(x) = f(x) - \frac{m}{2}\|x\|_2^2$ is convex $\underset{\text{imply}}{\to}$ strictly convex

Properties f convex(cvx) $\Leftrightarrow$ epi(f) is cvx — f cvx $\Rightarrow S_\alpha$ (α-sublevel set) is convex $\forall\alpha$ — f str. cvx $\overset{\Rightarrow}{\not\Leftarrow} S_\alpha$ str. cvx
f, g strictly convex $\Rightarrow f + g$ strictly convex — f convex, g strongly convex $\Rightarrow f + g$ strongly convex
h, g convex, g non decreasing $\Rightarrow h \circ g$ cvx — g concave, h convex non-increasing $\Rightarrow h \circ g$ cvx
Non-negative linear combinations $f_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, ..., m$ convex $\Rightarrow f = \sum_{i=1}^m \alpha_i f_i,\ \alpha_i \geq 0$ is convex over $\cap_i \text{dom}(f_i)$
Affine variable transformation $f : \mathbb{R}^n \to \mathbb{R},\ g(x) = f(Ax + b)$ is convex

Conditions for convexity

- **First-order condition** (f differentiable): f convex $\iff \forall x, y \in \text{dom}(f),\ f(y) \geq f(x) + \nabla f(x)^T(y - x)$
- **Second-order condition** (f twice differentiable): $\begin{cases} f \text{ convex} \iff \nabla^2 f(x) \succeq 0\ \forall x \in \text{dom}(f) \\ f \text{ strictly convex} \overset{\Leftarrow}{\not\Rightarrow} \nabla^2 f(x) \succ 0 \\ f \text{ strongly convex } \exists m :\ \nabla^2 f(x) \succ mI,\ \forall x \in \text{dom}(f) \end{cases}$
- **Restriction to a line**: f convex $\iff \underbrace{g(t) = f(x_0 + tv)}_{\text{restriction of } f \text{ to a line}},\ x_0, v \in \mathbb{R}^n,\ t \in \mathbb{R}$ is convex

Pointwise maximum $f_\alpha(x)$ convex and $\alpha \in \mathcal{A}$, $\mathcal{A}$ compact set(closed and bounded), $f(x) = \max_{\alpha\in\mathcal{A}} f_\alpha(x)$ is convex
Jensen's inequality Given $f : \mathbb{R}^n \to \mathbb{R}$ convex and $z \in \mathbb{R}^n$ a random variable such that $p\{z \in \text{int dom}(f)\} = 1$ it holds that: $f(\mathbb{E}[z]) \leq \mathbb{E}[f(z)]$. **Case** z **discrete R.V.** $p(z = x^{(i)}) = \theta_i,\ i = 1, ..., m,\ \sum_i \theta_i = 1,\ \theta_i \geq 0 \Rightarrow f\big(\sum_i \theta_i x^{(i)}\big) \leq \sum_i \theta_i f\big(x^{(i)}\big)$

7. Convex problems

$$p^* = \min_{x\in\mathbb{R}^n} \overbrace{f_0(x)}^{\text{convex}} \qquad (1)$$
$$\text{s.t.} \quad \underbrace{f_i(x) \leq 0}_{\text{convex}\to 0-\text{sublevel set}},\ i = 1, ..., m \qquad (2)$$
$$\underbrace{h_i(x) = 0}_{\text{affine}\to\text{ flat}}, i = 1, ..., p \qquad (3)$$

(1)-(3) $\to$ **CONVEX OPTIMIZATION PROBLEM**

Feasible set $\mathcal{X}$: convex since intersectio n between convex sets. $\mathcal{X} = \mathbb{R}^n$ (**unconstrained problem**).

Active/inactive constraint $\begin{cases} f_i(x^*) < 0 & \text{inactive at } x^* \\ f_i(x) = 0 & \text{active at } x^* \end{cases}$
In some situations... $\mathcal{X} \neq \varnothing$ **but** $\mathcal{X}_{opt} = \varnothing$

Local and global optima (thm) Given $\min_{x\in\mathcal{X}} f_0(x)$ ($f_0, \mathcal{X}$ convex) it holds that: (i) if $x \in \mathcal{X}$ is a local minimizer $\rightarrow$ is a **global minimizer**, (ii) the optimal set $\mathcal{X}_{opt}$ is a convex set.
PROBLEM TRANSFORMATIONS: how to formulate the problem into an "equivalent" way?

(i) Affine variable transformation

Original problem

$$\min_{x\in\mathcal{X}} \; f_0(x)$$

Transformed problem

$$\min_{x\in\mathcal{X}} \varphi(f_0(x))$$

$\varphi : \mathbb{R} \rightarrow \mathbb{R}$ must be continuous and stricltly increasing over $\mathcal{X}$

(ii) Addition of **slack variables**

Original Problem

$$\min_{x\in\mathbb{R}^n} \sum_{i=1}^{r} \varphi_i(x)$$
$$\text{s.t. } f_i(x) \leq 0, \; h_i(x) = 0$$

Transformed Problem

$$\min_{x,t} \sum_{i=1}^{r} t_i$$
$$\text{s.t } f_i(x) \leq 0, \; h_i(x) = 0$$
$$\varphi_i(x) \leq t_i$$

This transformation is effective when the objective is the sum of functions φ_i.

(iii) Epigraphic formulation

Original problem

$$p^* = \min_{x\in\mathbb{R}^n} f_0(x)$$
$$\text{s.t } f_i(x) \leq 0, \; h_i(x) = 0$$

Transformed problem

$$g^* = \min_{x\in\mathbb{R}^n, t\in\mathbb{R}} t$$
$$\text{s.t } f_i(x) \leq 0, \; h_i(x) = 0$$
$$f_0(x) \leq t$$

This type of transformation can be applied for all problems in a way that the objective function becomes linear, pushing the original objective into the constraints.

(iv) Replacement $h_i(x) = 0 \leftrightarrow h_i(x) \leq 0$

Original problem

$$p^* = \min_{x\in\mathbb{R}^n} f_0(x)$$
$$\text{s.t } f_i(x) \leq 0, \; h_i(x) = 0$$

Transformed problem

$$g^* = \min_{x\in\mathbb{R}^n} f_0(x)$$
$$\text{s.t } f_i(x) \leq 0, \; h_i(x) = 0$$

I can do this substitution if...
f_0 strictly decreasing, $x : h(x) < 0 \in \text{relint}(\mathcal{X}) \leftarrow$ (minimization)
f_0 strictly increasing, $x : h(x) < 0 \in \text{relint}(\mathcal{X}) \leftarrow$ (minimization)

Optimality (Prop) Consider $\min_{x\in\mathcal{X}} f_0(x)$ with $f_0, \mathcal{X}$ convex. A feasible solution $x \in \mathcal{X}$ is **optimal** $\iff \forall y \in \mathcal{X}$ $\nabla f(x)^T(y-x) \geq 0$. **Geometric interpretation**: there is no direction for which the objective function can decrease.
Categories of directions There are directions $v_\pm$ for which: $\begin{cases} \nabla f_0(x)^T \cdot v_+ > 0 & f_0 \text{ increase} \\ \nabla f_0(x)^T \cdot v_+ < 0 & f_0 \text{ decrease} \rightarrow \text{descent direction} \end{cases}$
Optimality (unconstrained) $x \in \mathcal{X}$ is optimal $\iff \nabla f_0(x) = 0$
Optimality (equality constrained) $x \in \mathcal{X}_{opt} \iff Ax = b, \; \exists \nu \in \mathcal{R}^m : \; \nabla f_0(x) + A^T\nu = 0$
Optimality (inequality constrained) Given x feasible, let $\mathcal{A}(x) = \{i : \; f_i(x) = 0\}$, $x \in \mathcal{X}_{opt} \iff \exists \lambda_i \geq 0, \; i \in \mathcal{A}(x)$ $\nabla f_0(x) + \sum_{i\in\mathcal{A}(x)} \lambda_i \nabla f_i(x) = 0$.

8. Quadratic programs (QP)

Quadratic function $f_0(x) = \frac{1}{2}x^T Hx + c^T x + d$ (if $H = 0$, f_0 is linear) The matrix $\begin{cases} H \succeq 0 & f_0 \text{ convex, elliptical paraboloid} \\ H \succ 0 & f_0 \text{ strongly convex} \\ H \preceq 0 & f_0 \text{ concave} \end{cases}$

Unconstrained minimization of f_0

Linear case ($H = 0$)

$f_0 = c^T x + d$, two cases: $\begin{cases} p^* = -\infty & c \neq 0 \\ p^* = d & c = 0 \end{cases}$

Quadratic case ($H \neq 0$) $\rightarrow \nabla f_0(x) = 0$ is applied
$\nabla f_0(x) = Hx + c = 0 \iff Hx = -c$. 2 cases:
(i) $c \notin \mathcal{R}(H) \rightarrow p^* = -\infty$ (unbounded below)
(ii) $c \in \mathcal{R}(H) \rightarrow \begin{cases} x^* = -H^\dagger c + \zeta, \; \zeta \in \mathcal{N}(H) & \text{in general} \\ x^* = -H^{-1}c & \lambda_i > 0, \; \forall i \end{cases}$

Polyhedron $\mathcal{P} = \{x : a_i^T x \leq b_i, \; i = 1, ..., m\} = \{x \in \mathbb{R}^n : \; Ax \leq b, A \in \mathbb{R}^{m,n}, \; b \in \mathbb{R}^m\}$ (cvx: intersect. of halfspaces)
...properties: The image of $\mathcal{P}$ through an affine map is still a polyhedron — The set $\mathcal{P} = \{x \in \mathbb{R}^n : Ax \leq b, Cx = d\}$ is a polyhedron. It can be obtained by properly parametrizing x. Polyhedron+Bounded $\rightarrow$ **Polytope**

QP (standard form)
$$\min_{x\in\mathbb{R}^n} \frac{1}{2}x^T Hx + c^T x + d$$
$$\text{subject to: } Ax \leq b, \; Fx = g$$

The problem is tractable $\iff H \succeq 0$

Example(1): Markowitz's Porftolio
$r_i, x_i \rightarrow$ return and investment for asset a_i
$\hat{r} = \mathbb{E}\{r\}$, $\Sigma \doteq \mathbb{E}\{(r-\hat{r})(r-\hat{r})^T\}$
$f_0(x) = \underbrace{x^T\Sigma x}_{\text{risk}} - \underbrace{\gamma \; \hat{r}^T x}_{\text{return}}$, $\gamma > 0$
constraints: $x \geq 0$, $\sum_i x_i = 1$

Example(2): LASSO Problem
ℓ_1-penalty: $\lambda\|x\|_1 = \lambda \sum_i |x_i|$ (use of slack variables...)
Resulting problem:

$$\min_{x,u} \|Ax - y\|_2^2 + \lambda \sum_{i=1}^{n} u_i$$
$$\text{subject to: } x_i \leq u_i, \; x_i \geq -u_i, \; i = 1, ..., n$$

9. Linear programs (LP)

Inequality form

$$\min_x c^T x$$
$$\text{s.t. } Ax \le b,\ A_{eq}x = b_{eq}$$

without loss of generality we can assume that $d = 0$: being a constant term, nothing change.

$\longrightarrow$ *problem transformation* $\longrightarrow$

(slack variables) $e \ge 0, x_+ \ge 0, x_- \ge 0$

$\tilde{x} = (x_+, x_-, e),\ \forall x\ x = x_+ - x_-$

objective $c^T x_+ - c^T x_-$

constraints $Ax_+ - Ax_- + e = b$

$A_{eq}x_+ - A_{eq}x_- = b_{eq}$

$\tilde{A} = \begin{bmatrix} A & -A & I \\ A_{eq} & -A_{eq} & 0 \end{bmatrix},\ \tilde{b} = \begin{bmatrix} b \\ b_{eq} \end{bmatrix}$

$\tilde{c} = [c^T \quad -c^T \quad 0]$

Standard form

$$\min_x \tilde{c}^T \tilde{x}$$
$$\text{s.t. } \tilde{A}\tilde{x} = \tilde{b},\ \tilde{x} \ge 0$$

Can be solved by using:
(i) Simplex algorithm
(ii) Interior point algorithm

Geometric interpretation The halfspace $\mathcal{H}(x_f) = \{x \in \mathbb{R}^n :\ c^T(x - x_f) = 0,\ c \in \mathbb{R}^n\}$ is important $x_f \in \mathcal{X}$ is optimal $\iff$ $c^T x \ge c^T x_f,\ \forall x \in \mathcal{X}$ $\iff$ $c^T(x - x_f) \ge 0, \forall x \in \mathcal{X}$ – if x_f is not optimal $\iff$ there exists at least one x for which $c^T(x - x_f) < 0$ (feasible descent direction) $\rightarrow$ the optimal solution is on vertex/edge/facet of the polyhedron representing the feasible set which is totally contained in the halfspace $\mathcal{H}_+(x_f)$

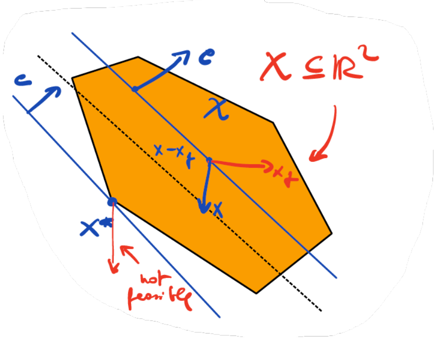
Feasible set and solutions The following situations can arise: $\begin{cases} \mathcal{X} = \varnothing & p^* = +\infty \\ \mathcal{X} \text{ bounded} & x^* \text{ is on a vertex/edge/facet} \\ \mathcal{X} \text{ unbounded} & \exists c:\ p^* = -\infty \text{ unbounded below} \end{cases}$

Example(1): Diet problem

Most economical diet (total cost: $c^T x$) satisfying the nutritional requirements ($Ax \ge b$).

n foods, their cost is c_i, m nutritional element, their recommended quantity b_i. a_{ij} quantity of the nutrient i in the j-th food. x_i quantity in the diet of the i-th food.

$$\min_{x \in \mathbb{R}^n} c^T x$$
$$\text{s.t. } Ax \ge b,\ x \ge 0$$



Example(2): Max flow problem

Maximize a certain quantity f, sent from a source to a sink, through a network represented by a digraph.

- C (matrix of capacities), c_{ij} (link capacity from i to j)
- X (matrix of effective flow) – clearly $x_{ij} \le c_{ij}$
- $\phi_{in} = \sum_{i=1}^n x_{ik}$, $\phi_{out} = \sum_{j=1}^n x_{kj}$

Constraints $\begin{cases} \phi_{in} - \phi_{out} = -f & \text{at source (node 1)} \\ \phi_{in} - \phi_{out} = f & \text{at sink} \\ \phi_{in} - \phi_{out} = 0 & \text{intermediate (node } n) \end{cases}$

Objective $\max_{x \in \mathbb{R}^n, f \in \mathbb{R}} f \iff -\min_{x \in \mathbb{R}^n, f \in \mathbb{R}} -f$

$$-p^* = \min_{x \in \mathbb{R}^n, f \in \mathbb{R}} -f$$
$$\text{s.t. } (X^T - X)\mathbf{1} = \begin{bmatrix} -f \\ 0 \\ f \end{bmatrix},\ 0_{n \times n} \le X \le C$$

10. Second-order cone programs (SOCP)

Second order cone $\mathcal{K}_p = \{(u,t) :\ u \in \mathbb{R}^{p-1},\ t \in \mathbb{R}, \|u\|_2 \le t\}$ extension of $\mathcal{K} = \{(x,y,z) :\ \sqrt{x^2 + y^2} \le z\}$
$\mathbb{R}^n$-cone $\mathcal{K}_n = \{x \in \mathbb{R}^n : \sqrt{x_2^2 + x_3^2 + ... + x_n^2} \le x_1\}$
Rotated cone $\mathcal{K}_n^r \doteq \{x \in \mathbb{R}^n :\ x_3^2 + x_4^2 + ... + x_n^2 \le 2x_1x_2,\ x_1, x_2 > 0\}$
Useful equivalence $\|u\|_2^2 \le yz\ y \ge 0,\ z \ge 0 \iff \|\binom{2u}{y-z}\|_2 \le y + z$

Standard form for SOCP

$$\min_x c^T x$$
$$\text{s.t. } Ax = b, x^i \in \mathcal{K}_{n_i}$$

x^i is the i-th block of the decision variable, n_i its dimension.

SOC constraints
Given $u = \|Ax + b\|_2$, $t = c^T x + b$

$$\mathcal{K} = \{x : \|Ax + b\|_2 \le c^T x + b\}$$

I can rewrite the problem as:

$$\min_x c^T x$$
$$\text{s.t. } \|A_i x + b_i\|_2 \le c_i^T x + b_i$$

SOCP class encapsulate also the Linear and Quadratic programs.

LP as SOCP $\Longrightarrow$

$$\min_x c^T x$$
$$\text{s.t. } a_i^T x \le b_i,\ i = 1, ..., m$$

problem transformation

$$\min_x c^T x$$
$$\text{s.t. } \|C_i x + d_i\|_2 \le b_i - a^T x$$
$$C_i = 0, d_i = 0,\ i = 1, ..., m$$

QP as SOCP $\Longrightarrow$

$$\min_x x^T Q x + c^T x + d$$
$$\text{s.t. } a_i^T x \le b_i,\ i = 1, ..., m$$

problem transformation

$$\min_{x,y} c^T x + y$$
$$\text{s.t. } \left\| \begin{pmatrix} 2Q^{1/2} \\ y - 1 \end{pmatrix} \right\|_2 \le y + 1$$
$$\text{s.t. } a_i^T x \le b_i,\ i = 1, ..., m$$

Sum of Norms (SON) $\min_x \sum_{i=1}^p \|A_i x - b_i\|_2 \longrightarrow \min_{x,y} \sum_{i=1}^p y_i$ s.t. $\|A_i x - b_i\|_2 \le y_i,\ i = 1, ..., p$, y_i slack variables
Max of Norms $\min_x \max_{i=1,...,p} \|A_i x - b\|_2 \longrightarrow \min_{x,y} y$ s.t. $\|A_i x - b\|_2 \le y,\ i = 1, ..., p$

Example(1): Fermat-Weber Point

Where to locate a warehouse in order to serve in the best way some location services?

$x \in \mathbb{R}^2$: position of the warehouse
$y_i \in \mathbb{R}^2,\ i = 1, ..., m$: position of the location services
I have to solve: $\min_x \frac{1}{m} \sum_{i=1}^m \|x - y_i\|_2$ (use SON)

$$\underbrace{\begin{aligned} &\min_{x,t} \frac{1}{m} \sum_{i=1}^m t_i \\ &\text{s.t. } \|x - y_i\|_2 \le t_i,\ i = 1, ..., m \end{aligned}}_{\text{SOC Program}}$$

Example(2): LP with probability constraints

How to solve an LP problem when one or more data are random or uncertain?

In particular:
-a_i random normally distributed vectors $\begin{cases} \mathbb{E}\{a_i\} = \bar{a}_i \\ \text{var}(a_i) = \Sigma_i \succ 0 \end{cases}$
-$a_i^T x$ is a random variable $\rightarrow$ ($\mu = \bar{a}_i^T x,\ \sigma^2 = x^T \Sigma_i x$),
-We know a-priori $P\{a_i^T x \le b_i\} \ge p_i, i = 1, ..., m$
If $p_i > 0.5$ the LP problem can be rewritten as:

$$\underbrace{\begin{aligned} &\min_x c^T x \\ &\text{s.t. } \bar{a}_i^T x \le b_i - \Phi^{-1}(p_i)\|\Sigma_i^{1/2} x\|_2 \end{aligned}}_{\text{SOC Program}}$$

11. CVX modeling

Example (LP Program)

```
cvx_begin
    cvx_solver mosek
    variable x(2)          % name(dimension)
    minimize (c'*x)        %objective function
    subject to
        A*x <= b           %constraints
cvx_end
```

You can write the problem also in a non-standard form, `cvx` will recast the problem in a standard form.

Variables: $\begin{cases} \texttt{x} & \text{the minimizer} \\ \texttt{cvx\_optval} & \text{the value } p^* \\ \texttt{cvz\_status} & \texttt{Solved, Unfeasible, Unbounded} \end{cases}$

useful function: `quad_form(x,Q)`, returns $x^T Q x$ if $Q \succ 0$

12. General optimization and Lagrangian Duality

Langrangian $\mathcal{L}(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x)$ – $\lambda_i,\ \nu_i$ **Lagrange multipliers** – $f_0(x) \ge \mathcal{L}(x, \lambda, \nu)$

Lagrangian dual function $g(\lambda, \nu) = \inf_{x \in \mathcal{D}} \mathcal{L}(x, \lambda, \nu),\ \lambda > 0,\ g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$ – **Properties** $\begin{cases} \text{jointly concave in}(\lambda, \nu) \\ g(\lambda, \nu) \le p* \ \ \forall \lambda > 0, \forall \nu \end{cases}$

Dual problem (What is the best $g(\lambda, \nu)$)? $d^* = \max_{\lambda,\nu} g(\lambda, \nu)$ subject to : $\lambda > 0$ – $d^* \le p^*$ (**...is always convex**)

Duality gap $\delta^* = p^* - d^* = \begin{cases} = 0 & \text{strong duality (further conditions are needed)} \\ > 0 & \text{weak duality} \end{cases}$

Slater's condition for STRONG DUALITY Let $f_i(x)$ convex functions, $h_i(x)$ affine functions, $f_i(x), i = 1, ..., k \le m$ affine, if $\exists x \in$ relint $\mathcal{D}$: $f_i(x) \le 0,\ i = 1, ..., k,\ f_{k+1}(x) < 0, ..., f_m(x) < 0,\ h_i(x) = 0,\ i = 1, ..., p$ then: $p^* = d^*$ (there is no gap), moreover if the problem is not unbounded below $\exists(\lambda^*, \nu^*) :\ g(\lambda^*, \nu^*) = d^* = p^*$.

Primal problem

$$p^* = \min_x \max_{\lambda,\nu} \mathcal{L}(x, \lambda, \nu)$$

Dual problem

$$d^* = \max_{\lambda,\nu} \min_x \mathcal{L}(x, \lambda, \nu)$$

Primal solution from dual solution (Required: strong duality) $d^* = p^* = f_0(x^*)$, $d^* = g(\lambda^*, \nu^*) = \mathcal{L}(x, \lambda^*, \nu^*)$

$$f_0(x^*) = \mathcal{L}(x^*, \lambda^*, \nu^*) = f_0(x^*) + \sum_{i=1}^m {\lambda_i}^* f_i(x^*) + \sum_{i=1}^p \nu_i^* h_i(x^*) \rightarrow \begin{cases} \sum_{i=1}^m {\lambda_i}^* f_i(x^*) = 0 & \text{complementary slackness} \\ x^* & \text{minimizer (wrt } x\text{) of } \mathcal{L}(x, \lambda^*, \nu^*) \end{cases}$$

Complementary slackness For the i-th constraint: either $\lambda_i = 0, f_i(x) \le 0$, or $\lambda_i > 0, f_i(x) = 0$
Solution recovery If f_0 convex, $\nabla_x\ \mathcal{L}(x, \lambda^*, \nu^*) = 0$ gives a **global minimizer** (unique if f_0 strictly convex).

KKT conditions for optimality(optimal $\iff$ $\delta^* = 0$)
f_0 differentiable, strong duality holds.

$$\begin{cases} \text{1. primal feasibility} & f_i(x^*) \le 0,\ h_i(x^*) = 0 \\ \text{2. dual feasibility} & \lambda_i^* \ge 0,\ i = 1, ..., m \\ \text{3. complementary slackness} & \sum_{i=1}^m \lambda_i^* f_i(x^*) = 0 \\ \text{4. Lagrangian stationarity} & \nabla_x\ \mathcal{L}(x, \lambda^*, \nu^*)\big|_{x=x^*} = 0 \end{cases}$$

Sensitivity of the optimal solution p^*

Perturbed Problem

$$\begin{aligned} p^*(u, v) = &\min_{x \in \mathbb{R}^n} f_0(x) \\ &\text{s.t. } f_i(x) \le u_i,\ h_i(x) = v_i \end{aligned}$$

Remark: $p^*(0, 0) = p^*$ – $u_i > 0 \rightarrow$ relaxing, $u_i < 0$ tightning. When the primal problem is convex and $p^*(u, v)$ differentiable:

$$\lambda_i^* = -\frac{\partial p^*(u, v)}{\partial u_i}\Big|_{(u,v)=(0,0)},\ \ \nu_i^* = -\frac{\partial p^*(u, v)}{\partial v_i}\Big|_{(u,v)=(0,0)}$$

Interpretation: (here I use the complementary slackness)
$\lambda_i^* = 0 \Rightarrow f_i(x^*) < 0$ the constraint is inactive, I don't care about perturbations $\rightarrow$ not resource critical!
$\lambda_i^* > 0 \Rightarrow f_i(x^*) = 0$ constraint active (resource critical) $\rightarrow p^*_{new} = p^* - \lambda_i^* u_i$.

In particular: $\begin{cases} u_i < 0 & p^* \text{ increase (worse solution)} \\ u_i > 0 & p^* \textbf{ can} \text{ decrease (better solution)} \end{cases}$

Remember that: the Lagrange multipliers are given by the solution of the dual problem.

13. Gradient Algorithm (GA) (unconstrained case)

General structure: $x_{k+1} = x_k + s_k v_k$
$s_k > 0,\ s_k \in \mathbb{R}$ (step-size)
$v_k \in \mathbb{R}^n$ (update/search direction)

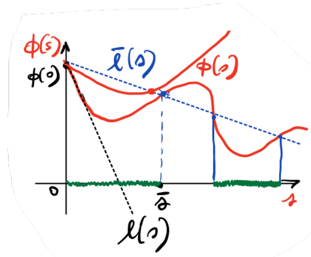
For Gradient Algorithm: $\begin{cases} f_0 \text{ differentiable} \\ x_k \in \mathrm{dom}(f_0) \\ v_k \in \mathbb{R}^n (\text{chosen wrt } \nabla f_0(x)) \end{cases}$

$f_0 : \mathbb{R}^n \to \mathbb{R},\ f_0(x_k + s v_k) \simeq f_0(x_k) + s \nabla f_0(x_k)^T v_k$
$t \to \infty,\ \delta_k = \lim_{s\to\infty} \frac{f_0(x_k+sv_k)-f_0(x_k)}{s} = \nabla f_0(x_k)^T v_k$
I want a direction for which f_0 decrease (descent). $\forall k$ moving toward $v_k \to$ go to min.
Steepest descent direction: $v_k = -\frac{\nabla f_0(x)}{\|f_0(x)\|_2}$

Algorithm 1 Gradient Algorithm
1: $k = 0$, choose a descent v_k $(v_k = -\nabla f_0(x_k))$
2: Determine the step-size s_k
3: $x_{k+1} = x_k + s_k v_k$
4: **if** Stop criterion **then**
5: **return** x_k
6: **else**
7: back to 2
8: **end if**

STEPSIZE SELECTION $\overbrace{\phi(s) = f_0(x_k + s v_k), s \geq 0}^{\text{restriction of } f_0 \text{ along } v_k}$
$\phi(0) = f_0(x_k)$, I want $s > 0:\ f(x_{k+1}) = \phi(s) < \phi(0)$
Exact line-search (best s possible) (non-convex)

$$s^* = \arg\min_{s\geq 0} \phi(s)$$



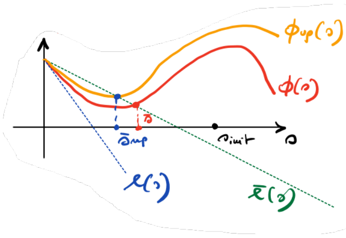
$\ell(s) = \phi(0) + s\delta_k$ (tan at 0)
$\bar{\ell}(s) = \phi(0) + \alpha s \delta_k$
$\bar{\ell}(s) \geq \ell(s) \geq \phi(s)$
$s:\ \phi(s) \leq \bar{\ell}(s)$ are OK

Armijo condition
$\phi(s) \leq \phi(0) + s\alpha\delta_k$

Algorithm 2 Backtracking line-search
1: $\alpha \in (0,1),\ \beta \in (0,1), s_{init} = 1,\ v_k$ descent
2: **if** $f_0(x_k + s v_k) \leq f(x_k) + s\alpha \nabla f_0(x_k)^T v_k$ **then**
3: **return** $s_k \leftarrow s$
4: **else**
5: $s = \beta s$, back to 2 ▷ update until not Armijo's meet
6: **end if**

CONVERGENCE OF GA: STOPPING CRITERION
$\nabla f_0(x), \exists L : \|\nabla f_0(y) - \nabla f_0(x)\|_2 \leq L\|y-x\|_2, \forall x, y$ (Lipschitz)

$$\Rightarrow f_0(x) \leq \underbrace{f_0(y) + \nabla f_0(x)^T (x-y) + \frac{L}{2}\|x-y\|_2}_{\text{strongly cvx, quadratic approximant}}$$



I can introduce $\phi_{up}(s)$ since ∇f_0 is Lipschitz. $\bar{s}_{up} \to$ intersection $\phi_{up}(s)$-$\ell(s)$
$\bar{s}_{up} = \frac{2}{L}(1-\alpha)$

Backtracking ensures:
$\forall k, s_k \geq \min(s_{init}, \beta \bar{s}_{up}) \doteq s_{\text{lb}}$

It holds that (GA converges to a stationary point):
$\alpha s_{\text{lb}} \sum_{i=0}^{k} \|\nabla f_0(x_i)\|_2^2 \leq f_0(x_0) - f_0^*,\ \lim_{k\to\infty} \|\nabla f_0(x_i)\|_2 = 0$
Stopping criterion $\to$ $\|\nabla f_0(x)\nabla\|_2 < \varepsilon, \varepsilon > 0$
f_0 convex $\to$ x^* is a global minimizer.

Convergence $_{\|\nabla f_0(x_k)\|_2 \to 0}$ $\begin{cases} O(1/\sqrt{k}) & f_0 \text{ generic} \\ O(1/\varepsilon) & f_0 \text{ convex (sublinear)} \\ O(1/\log(1/\varepsilon)) & f_0 \text{ strongly convex (linear)} \end{cases}$

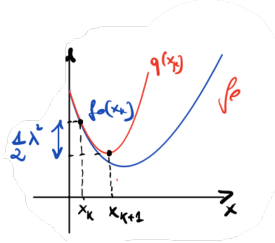
Grad. Algorithm as minimization of $q(x)$
GA can be interpreted also as minimization of the quadratic approximant when $\|x-x_k\|_2$ is small $\to f_0(x) \simeq q(x) = f_0(x_k) + \nabla f_0(x_k)^T (x - x_k)$
$\nabla q(x) = \nabla f_0(x_k) + \frac{1}{s}(x-x_k) = 0 \Leftrightarrow \underbrace{x_{k+1} = x_k - s\nabla f_0(x_k)}_{\text{Gradient Algorithm}}$
Conclusion: x is updated $\forall k$ by minimizing $q(x_k)$. □

14. Newton Algorithm (NA) (unconstrained case)

Key concept: Finding (starting from x_k) the roots of a non linear function $g : \mathbb{R} \to \mathbb{R}$. We want the roots of $\nabla f_0(x) = 0$.
$\tilde{g}(x) = g(x_k) + g'(x_k)(x - x_k)$. x_{k+1} is given by $\tilde{g}(x) = 0$

$$\overbrace{x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)} = x_k - (\nabla^2 f_0(x_k))^{-1} \nabla f_0(x_k)}^{\text{NEWTON METHOD}} \tag{4}$$

$$\underbrace{q_0(x) = f(x_k) + \nabla f_0(x)^T (x - x_k) + \frac{1}{2}(x-x_k)^T \nabla^2 f_0(x_k)(x - x_k)}_{\text{quadratic approximant of } f_0}$$



$\arg\min_x q_0(x) = x_{k+1}$
$\min_x q_0(x) = f(x_{k+1})$
$\min_x q_0(x) - f(x) = \frac{1}{2}\lambda^2(x)$
$\underbrace{\lambda^2 = \nabla f_0{}^T \nabla^2 f_0{}^{-1} \nabla f_0}_{\text{Newton decrement}}$
where $\nabla f_0 \leftrightarrow \nabla f_0(x_k)$

Damped Newton Method $x_{k+1} = x_k + tv$
$v = -\nabla^2 f_0(x_k)^{-1} \nabla f_0(x_k), \nabla f_0(x_k)^T \cdot v = -\lambda^2 < 0$ **(descent)**

Algorithm 3 Choice of t (step-size)
1: $\alpha \in (0, \frac{1}{2}],\ \beta \in (0,1), t = 1$
2: **while** $f_0(x_k + tv) > f(x_k) + \alpha t \nabla f_0(x_k)^T v$ **do** $t \leftarrow t\beta$
3: **end while**
4: $x_{k+1} = x_k + tv$

Convergence properties f_0 strongly convex, $\nabla f_0, \nabla^2 f_0$ Lipschitz continuous, $\eta \in (0, m^2/L)$.

2 phases: $\begin{cases} \|\nabla f_0(x_k)\| \geq \eta & \text{Damped phase} \\ \|\nabla f_0(x_k)\| < \eta & \text{Quadratically convergent phase} \end{cases}$

Stopping criterion $f_0(x_k) - f_0^* \leq \lambda_k^2$

OTHER ASPECTS (1): EQUALITY CONSTRAINTS
I choose $x_0 : Ax_0 = b$, and $x_{k+1} = x_k + tv$, where

$$v = \arg\min_{z\in\mathcal{N}(A)} \nabla f_0(x)^T (z - x) + \frac{1}{2}(z-x)^T \nabla^2 f_0(x_k)(z - x)$$

In this way, since $Ax_{k+1} = Ax_k + tAv = b$, x_k remains in the feasible set, $Av = 0$.

OTHER ASPECTS(2): QUASI-NEWTON METHODS
Secant condition $H(\nabla f(x) - \nabla f(y)) = (x - y) \Rightarrow$ a matrix H satisfying such a condition can approximate the Hessian.

Algorithm 4 Quasi-Newton methods
1: $H_k = I_n$
2: Update s.t.: $H_{k+1} \nabla f_0(x_{k+1}) - \nabla f_0(x_k) = x_{k+1} - x_k$

15. Approaches for constrained optimization

When there are constraints solving $\nabla f_0(x) = 0$ it is not sufficient, x^* must be feasible and $\mathcal{X} \neq \mathbb{R}^n$

1^{st} approach: *Projected Gradient Method*

$\mathcal{P}_{\mathcal{X}}(x) \leftarrow$ projection of x on the feasible set $\mathcal{X}$ (convex, non-empty).

Algorithm 5 Projected Gradient Method

$k = 0$
$w_{k+1} = x_k - s_k \nabla f_0(x_k)$ ▷ Gradient Step
$x_{k+1} = \mathcal{P}_{\mathcal{X}}(w_{k+1})$ ▷ ensure feasibility

(used only for set for which is simple to compute projections, otherwise other more general methods are used).

The s_k (step-size) is chosen as follows: $s_k = \bar{s}2^{-t(k)}$

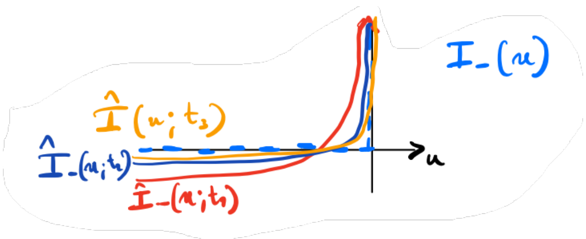
Algorithm 6 Stepsize selection

1: $j = 0$
2: $z_j = \mathcal{P}_{\mathcal{X}}(x_k - \bar{s}2^{-j}\nabla f_0(x_k))$
3: **if** $f(z_j) \leq f(x_k) - \alpha \nabla f_0(x_k)^T(x_k - z_j)$ **then**
4: $\quad t(k) = j$
5: **else**
6: $\quad$ goto 2
7: **end if**

2^{nd} approach: *Barrier Method*

Here we want to solve ($f_0, ..., f_m$ convex and smooth):

$$p^* = \min_x f_0(x)$$
$$\text{s.t. } f_i(x) \leq 0, \; i = 1, ..., m$$
$$Ax = b$$



The problem can be rewritten as:

$$p^* = \min_x f_0(x) + \sum_{i=1}^{m} I_-(f_i(x)) \tag{5}$$

$$I_-(u) = \begin{cases} 0 & u \leq 0 \\ \infty & u > 0 \end{cases} \quad \text{(Indicator function)} \tag{6}$$

$I_-(u)$ non-differentiable $\rightarrow \hat{I}_-(u;t) = -\frac{1}{t}\log(-u)$

$$p^*(t) = \min_x f_0(x) \quad \overbrace{-\frac{1}{t}\sum_{i=1}^{m} log(-f_i(x))}^{\phi(t) \rightarrow \textbf{logarithmic barrier}} \quad = \tag{7}$$
$$= \min_x t f_0(x) + \phi(t) \tag{8}$$
$$\text{s.t. } Ax = b \tag{9}$$

Parametrizing the x, (8) becomes unconstrained (GA, NM can be used).
For $t \rightarrow \infty$ (8)$\rightarrow$(5) that is $p^*(t) \rightarrow p^*$, $x^*(t) \rightarrow x^*$
Central path$\{x^*(t) : t > 0\}$, $p^*(t)$ ε-suboptimal $\varepsilon = m/t$

Algorithm 7 Sequential Barrier Method

x_0 (strictly feasible), $t \leftarrow t_0$, $\mu > 1$, $\varepsilon > 0$
loop
$\quad$ Solve $\min_{Ax=b} t f_0(t) + \phi(t)$ ▷ Centering step
$\quad x \leftarrow x^*(t)$
$\quad$ **if** $m/t < \varepsilon$ **then** ▷ Stopping criterion
$\quad\quad$ **break**
$\quad$ **else**
$\quad\quad t \leftarrow \mu t$ ▷ Increase t (better approx of $I_-(u)$)
$\quad$ **end if**
end loop

<u>Phase I Problem</u>: starting from a feasible x_0

$$\tilde{s} = \min_{x,s} s$$
$$\text{s.t. } f_i(x) \leq s,$$
$$Ax = b$$

Solved by using the barrier method, starting from:
$x_0 : \; Ax_0 = b,$
$s_0 = \eta + \max_i f_i(x), \; \eta > 0$

$\rightarrow \tilde{s} < 0$. Since $f_i(\tilde{x}) \leq \tilde{s} \leq 0 \longrightarrow x_0 = \tilde{x}$
$\rightarrow \tilde{s} > 0$. $\nexists x : \; f_i(x) \leq 0 \longrightarrow$ problem infeasible
$\rightarrow \tilde{s} = 0$. Only of theoretical interest

Remark: For carrying out the *centering step* the Newton Method tailored for equality constraints is used.

16. Geometric Programs(GP)

For GP: **variables** $\rightarrow$ positive (physical quantities), **objective/constraints** $\rightarrow$ non-negative linear combination of positive monomials.
Positive monomial Given $x \in \mathbb{R}^n$, $a \in \mathbb{R}^n$, $c > 0, x > 0$ $cx^a = cx_1^{a_1}x_2^{a_2}...x_n^{a_n} \in \mathbb{R}^+$

Posynomial $f : \mathbb{R}^n_{++} \rightarrow \mathbb{R}$, $f(x) = \sum_{i=1}^k c_i x^{a_{(i)}}$, $c_i, x > 0, a_{(i)} \in \mathbb{R}^n$ – **Generalized posynomial** (composite fun. of posynomials) $\begin{cases} \text{pointwise maximum} \\ \text{fractional power} \\ \text{addition/multiplication} \end{cases}$

Monomials, Posynomials and Generalized posynomials **are not convex!** $\rightarrow$ <u>Problem transformation is needed</u>
Convex form for monomials $y_i = \log x_i \rightarrow \tilde{g}(x) = cx_1^{a_1}...x_n^{a_n} = e^{\log c}e^{\log x_1^{a_1}}...e^{\log x_n^{a_n}} = e^{a_1 y_1 + ... + a_n y_n + \log c} = e^{a^T y + b}$. I can take the log (f convex $\circ$ f increasing) obtaining $g(x) = \log \tilde{g}(x) = a^T y + b \rightarrow$ Linear Program
Convex form for posynomials $\tilde{f}(x) = \sum_{i=1}^k c_i x^{a_{(i)}} = \sum_{i=1}^k e^{a_{(i)}^T y + b_i} \rightarrow \tilde{g}(x) = \log\left(\sum_{i=1}^k e^{a_{(i)}^T y + b_i}\right) = \text{lse}(Ax + b)$
log-sum-exp (lse) function is convex – $A \in \mathbb{R}^{k,n}$, $b = [b_1 \quad \dots \quad b_n]^T$

Standard form for GP Here $f_0(x), f_i(x)$ are posynomials, $h_i(x)$ are (possibly) monomials

Geometric programs in standard form

$$\begin{aligned}&\min_x \sum_{k=1}^{K_0} c_k x^{a_{(k)}}\\ &\text{s.t. } \sum_{k=1}^{K_i} c_k x^{a_{(k)}} \leq 1,\ i=1,...,m\\ &g_i x^{r_{(i)}} = 0,\ i=1,...,p\end{aligned}$$

$$\begin{aligned}&\min_y \operatorname{lse}(A_0 y + b_0)\\ &\text{s.t. } \operatorname{lse}(A_i y + b_i) \leq 0\\ &Ry + h = 0\\ &A_0 \in \mathbb{R}^{K_0,n}, b_0 \in \mathbb{R}^{K_0},\ A_i \in \mathbb{R}^{K_i,n}, b_0 \in \mathbb{R}^{K_i}\end{aligned}$$

Generalized GP $f_0, ..., f_m$ are generalized posynomials $\to$ GGP (Generalized GP)
Fractional power $f_1(x)^\alpha + f_2(x)^\beta \leq 1 \Rightarrow \underbrace{f_1(x) \leq t_1,\ f_2(x) \leq t_2,\ t_1^\alpha + t_2^\beta \leq 1}_{\text{GP constraints}}$

Pointwise maximum power $\max(f_1(x), f_2(x)) + f_3(x) \leq 1 \Rightarrow \underbrace{f_1(x) \leq t, f_2(x) \leq t, t + f_3(x) \leq 1}_{\text{GP constraints}}$