CONVEX OPTIMIZATION AND ENGINEERING APPLICATIONS (Formulary)

1. Introduction

Standard form for optimization problems $p^* = \min_x f_0(x)$ s.t. $f_i(x) \le 0, i = 1, ..., m$
where: $f_0 : \mathbb{R}^n \to \mathbb{R}$ (**objective function**), $f_i : \mathbb{R}^n \to \mathbb{R}$, (inequality constraints)
Feasible set $\mathcal{X} = \{x \in \mathbb{R}^n : \ f_i(x) \le 0\}$ — **Optimal solution** $x^* \in \mathcal{X} : f(x^*) = p^*$ — **Optimal value** p^*
Equality constraints $h_i(x) = 0 \ i = 1, ..., p \iff h_i(x) \le 0, -h_i(x) \le 0$ – **Optimal Set** $\mathcal{X}_{opt} = \{x \in \mathcal{X} : f(x) = p^*\}$
Optimization problems in maximization form $p^* = \max_x f_0 \iff -p^* = min_x \ -f_0(x) = min_x \ g_0(x)$
ε-suboptimality of a solution $x \in \mathbb{R}^n \iff p^* \le f_0(x) \le p^* + \varepsilon$

2. Vector, Projections, Functions, Gradients

Vector $x \in \mathbb{R}^n = [x_1 \quad x_2 \quad ... \quad x_n]^T$, $x_i \in \mathbb{R}$ – **Sum** $x, y \in \mathbb{R}^n, x + y \iff x_i + y_i, \ i = 1, ..., n$
Scalar multiplication $\alpha \in \mathbb{R}, x \in \mathbb{R}^n, \ \alpha x \iff \alpha x_i \ i = 1, ..., n$
Subspace $\mathcal{V} \subseteq \mathcal{X}$ is a **subspace** $\iff \forall x, y \in \mathcal{V}, \ \alpha x + \beta y \in \mathcal{V}$
$S = \{x^{(1)}, ..., x^{(n)}\}$, $\text{span}(S) = \{\alpha_1 x^{(1)} + ... + \alpha_n x^{(n)}, \ \alpha_i \in \mathbb{R}, \ i = 1, ..., n\}$
Basis of a vector space $B = \{b^{(1)}, ..., b^{(n)}\} \iff \text{span}(B) = \mathcal{X}, \forall x \in \mathcal{X}, x = \alpha_1 b^{(1)} + ... + \alpha_n b^{(n)}, \alpha_i \in \mathbb{R}$
Direct sum of subspaces $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$, $\mathcal{X} \oplus \mathcal{Y} = \{x + y : \ x \in \mathcal{X}, y \in \mathcal{Y}\}$
Affine set $\mathcal{X}$ vector space, $\mathcal{V} \subseteq \mathcal{X} \to$ affine set: $\mathcal{A} = \{x \in \mathcal{X} : x = x_0 + v, \ v \in \mathcal{V}\}$, note that: $\dim_{\mathbb{R}} \mathcal{A} = \dim_{\mathbb{R}} \mathcal{V}$
Line (1D-(affine set)) $L = \{x \in \mathbb{R}^n : \ x = x_0 + v, \ v \in \text{span}(u), \|u\|_2 = 1\}$
Norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R} \to \|x\| \ge 0, \ \|x\| = 0 \iff x = 0, \ \|xy\| = \|x\|\|y\|, \ \|\alpha x\| = |\alpha| \cdot \|x\|, \ \|x + y\| \le \|x\| + \|y\|$
ℓ_p-norms $\|x\|_p = \left(\sum_{i=0}^n |x_i|^p\right)^{1/p}$, $p = 2 \to$ Euclidean Distance, $p = 1 \to$ Manahattan Distance, $p = \infty \to \max x_i$
Inner Product $\langle x, x\rangle \ge 0$, $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$, $\langle x, y\rangle = \langle y, x\rangle$, $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$, $\cos\theta = \frac{\langle x, y\rangle}{\|x\|_2\|y\|_2}$
Standard inner product $x, y \in \mathcal{X}$, $\langle x, y\rangle = x^T y = \sum_{i=1}^n x_i y_i$, **Schwarz inequality** $\langle x, y\rangle \le \|x\|_2\|y\|_2$,
$\|x\|_2 = \sqrt{x^T x} = \sqrt{\langle x, x\rangle}$, $x \perp y \iff \langle x, y\rangle = 0 \iff \cos\theta = 0$

$S = \{x^{(1)}, ..., x^{(n)}\}$ **mutually orthogonal** $\iff \langle x, y\rangle = \begin{cases} 0 & i \neq j \\ \neq 0 & i = j \end{cases}$ **orthonormal** $\iff \langle x, y\rangle = \begin{cases} 1 & i \neq j \\ 0 & i = j \end{cases}$

Vector $\perp$ Subspace $\mathcal{S} \subseteq \mathcal{X}$, $x \in \mathcal{X}$, $x \perp \mathcal{S} \iff \langle x, s\rangle = 0, \forall s \in \mathcal{S}$, $\mathcal{S}^\perp = \{x \in \mathcal{X} : \langle x, s\rangle = 0, \forall s \in \mathcal{S}\}$
Orthogonal decomposition of a vector $\forall x \in \mathcal{X}, x = x_1 + x_2 : \ x_1 \in \mathcal{S}, x_2 \in \mathcal{S}^\perp \iff \mathcal{X} = \mathcal{S} \oplus \mathcal{S}^\perp$

Projections $\Pi_{\mathcal{S}}(x) = \min_{y \in \mathcal{S}} \|y - s\|$ – (*Projection Theorem*) $\begin{cases} x^* \in \mathcal{S} \text{ is unique} \\ (x - x^*) \perp \mathcal{S} \iff (x - x^*) \in \mathcal{S}^\perp \end{cases}$

Functions $f : \mathbb{R}^n \to \mathbb{R}$ (function), $f : \mathbb{R}^n \to \mathbb{R}^m$ (map), **Domain of** f: $\text{dom}(f) = \{x \in \mathbb{R}^n : \ \|f(x)\| < \infty\}$
$\text{graph}(f) = \{(x, f(x)) \in \mathbb{R}^{n+1} : \ x \in \mathbb{R}^n\}$, $\text{epi}(f) = \{(x, t) \in \mathbb{R}^{n+1} : \ x \in \mathbb{R}^n, t >= f(x)\}$
Contour curves $C_f(t) = \{x \in \mathbb{R}^n : f(x) = t, \ t \in \mathbb{R}\}$ – **α-sublevel set** $S_\alpha = \{x \in \mathbb{R}^n : \ f(x) \le t, \ t \in \mathbb{R}\}$
f is **linear** $\iff f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$, $\alpha, \beta \in \mathbb{R}$, $\tilde{f}$ is **affine** $\iff \tilde{f} - x_0$ is linear
any affine function can be written as: $f(x) = a^T x + b$ where $a \in \mathbb{R}^n, \ b \in \mathbb{R}$, $b = f(0)$, $a_i = f(e_i) - b$,
Hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n : \ a^T x = b\}$, $a \in \mathbb{R}^n$ (normal direction), $\mathcal{H}_- = \{x \in \mathbb{R}^n : \ a^T x \le b\}$, (closed half-space)
$\mathcal{H}_{++} = \{x \in \mathbb{R}^n : \ a^T x > b\}$ (open half-space)

Gradient Given $f : \mathbb{R}^n \to \mathbb{R}$ differentiable $\nabla f = \left[\frac{\partial f}{\partial x_1} \ \frac{\partial f}{\partial x_2} \ ... \ \frac{\partial f}{\partial x_n}\right]^T$, $\nabla f(x_0)^T \cdot v$ (directional derivatives)
the rate of variation is maximal when $\nabla f(x_0) \parallel v$ minimal when $\nabla f(x_0) \perp v$, moreover $\nabla f(x_0) \perp C_f(t) \ \forall x_0 \in \text{dom}(f)$

3. Matrices

Matrix $A \in \mathbb{R}^{m,n}$, $a_{ij} \in \mathbb{R}$ — $AB_{ij} = R_i(A) \cdot C_j(B)$ (rows by column) — $(AB)^T = B^T A^T$
An **affine map** $f = Ax + b$, $A \in \mathbb{R}^{m,n}, \ b \in \mathbb{R}^n$ (generalization of an **affine function**)

Subspaces associated with a matrix $\begin{cases} \mathcal{R}(A) = \{Ax : x \in \mathbb{R}^n\} & \text{range of A} \\ \mathcal{N}(A) = \{x \in \mathbb{R}^n : \ Ax = 0\} & \text{nullspace of A} \\ \dim_{\mathbb{R}} \mathcal{R}(A) = \text{rank}(A), \ 1 \le \text{rank}(A) \le \min(m, n) & \text{rank of A} \end{cases}$

Fundamental thm. of linear algebra For any $A \in \mathbb{R}^{m,n}$ $\begin{cases} \mathbb{R}^n \supseteq \mathcal{N}(A) \perp \mathcal{R}(A^T) \equiv \mathcal{N}(A)^\perp \iff \mathbb{R}^n = \mathcal{N}(A) \oplus \mathcal{R}(A^T) \\ \mathbb{R}^m \supseteq \mathcal{R}(A) \perp \mathcal{N}(A^T) \equiv \mathcal{R}(A)^\perp \iff \mathbb{R}^m = \mathcal{R}(A) \oplus \mathcal{N}(A^T) \\ \forall x \in \mathbb{R}^n, \ x = A^T x + \zeta, \ \zeta \in \mathcal{N}(A) \\ \forall w \in \mathbb{R}^m, \ w = Ax + \xi, \ \xi \in \mathcal{N}(A^T) \end{cases}$

Singular matrix $A \in \mathbb{R}^{n,n}$ is **singular** $\iff \det(A) = 0 \iff \mathcal{N}(A) \neq \{0\} \iff \text{rank}(A) \neq \min(m, n)$
Inverse matrix $A \in \mathbb{R}^{n,n}$, $\det(A) \neq 0 \ \exists A^{-1} \in \mathbb{R}^{n,n} : AA^{-1} = I_n$, $(AB)^{-1} = B^{-1}A^{-1}$
Similar matrices $A, B \in \mathbb{R}^{n,n}$ are similar $\iff \exists P : B = P^{-1}AP$, P is **non-singular** (columns: basis for $\mathbb{R}^{n,n}$)
Eigenvalues/Eigenvector $u \in \mathbb{R}^{n,n}$ **eigenvector** for $A \iff Au = \lambda u, \ u \neq 0$,
$\lambda \to$ **eigenvalue associated with** u, **Eigenvalues** $\to_{\text{roots of}}$ $p(\lambda) = \det(A - \lambda I_n) = 0$, **Eigenspace** $\to \phi_i = \mathcal{N}(A - \lambda_i I_n)$
<u>**Algebraic Multiplicity**</u> ν_i (multiplicity of the root of $p(\lambda)$) <u>**Geometric Multiplicity**</u> $\mu_i = \dim_{\mathbb{R}}(\phi_i) \longrightarrow \nu_i \le \mu_i$
Diagonalizable matrices (thm.) $\lambda_i, \ i = 1, ..., k \le n$ (distinct eigenvalues of A), $U^{(i)} = [u_1^{(i)} \quad \dots \quad u_{\nu_i}^{(i)}]$ the eigenvector wrt λ_i, **if** $\nu_i = \mu_i, \ i = 1, ..., k$, $U = [U^{(1)} \quad \dots \quad U^k]$ is **invertible** and $A = U\Lambda U^{-1}$, $\Lambda = diag(\lambda_1 I_{\mu_1}, ..., \lambda_k I_{\mu_k})$
Matrix norms ...is a function $f : \mathbb{R}^{m,n} \to \mathbb{R}$ — additional property <u>$f(AB) \le f(A)f(B)$</u> **sub-multiplicativity**
Frobenius norm (extension of ℓ_2-norm) $\|A\|_F = \sqrt{\text{trace}(A^T A)} = \sqrt{\sum_{j=1}^m \sum_{i=1}^n a_{ij}^2} = \sqrt{\sum_{i=1}^n \lambda_i(A^T A)}$

Operator norms (max input-output gain of $y = Au$) $\|A\|_p \doteq \max_{u\neq 0} \frac{\|Au\|_p}{\|u\|_p} = \max_{\|u\|_p=1} \|Au\|_p$ (ℓ_p-induced norms)
$\|A\|_1 = \max_{j=1,...,m} \sum_{i=1}^n |a_{ij}|$, $\|A\|_\infty = \max_{i=1,...,m} \sum_{j=1}^n |a_{ij}|$, $\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$ (see variational characterization)
Spectral radius $A \in \mathbb{R}^{n,n}$ $\rho(A) \doteq \max_{i=1,...,n} |\lambda_i(A)|$ — $\rho(A) \leq \min(\|A\|_1, \|A\|_\infty)$

4. Symmetric Matrices and SVD

$A \in \mathbb{R}^{n,n}$ is **symmetric** if $A = A^T$, $\mathbb{S}^n$ is the subspace of **symmetric matrices** $n \times n$

Examples: (Hessian matrix given $f : \mathbb{R}^2 \to \mathbb{R}$) $H = \begin{bmatrix} \frac{\partial f}{\partial x_1^2} & \frac{\partial f}{\partial x_1 x_2} \\ \frac{\partial f}{\partial x_2 x_1} & \frac{\partial f}{\partial x_2^2} \end{bmatrix}$, (quadratic function) $q(x) = \frac{1}{2}x^T Hx + c^T x + d$

Quadratic form Given $A \in \mathbb{S}^n$, $\forall x \neq 0$ the function $q(x) = x^T Ax$ is a **quadratic form associated with A**
Spectral theorem Given $A \in \mathbb{S}^n$ it holds that: $\lambda_i \in \mathbb{R}$, $i = 1, ..., n$, exists $U = [u_1 \quad \dots \quad u_n]$ orthogonal and $A = U\Lambda U^T = \sum_{i=1}^n \lambda_i u_i u_i^T$, $\Lambda = diag(\lambda_1, ..., \lambda_n)$. I can sort them $\lambda_{max}(A) = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = \lambda_{min}(A)$
Rayleigh quotient given $A \in \mathbb{S}^n$, we can define $\forall x \in \mathbb{R}^n$ the quantity $r_A(x) = \frac{x^T Ax}{x^T x}$ (Rayleigh quotient)

Theorem For any $A \in \mathbb{S}^n$, $\forall x \neq 0$, $\lambda_{max} \leq r_A(x) \leq \lambda_{min}$, where $\begin{cases} \lambda_{max}(A) = \max_{\|x\|_2=1} x^T Ax & (x^* = u_1) \\ \lambda_{min}(A) = \min_{\|x\|_2=1} x^T Ax & (x^* = u_n) \end{cases}$

...on the ℓ_2-induced norm (by using the definition of operator norm and ℓ_2-norm) $\frac{\|Ax\|_2^2}{\|x\|_2^2} = \frac{(Ax)^T(Ax)}{x^T x} = \frac{x^T(A^T A)x}{x^T x}$
that is the Rayleigh quotient of $A^T A \in \mathbb{S}^n$ from the theorem follows that $\|A\|_2 = \max_{\|x\|\neq 0} r_{A^T A}(x) = \sqrt{\lambda_{max}(A^T A)}$ $\square$

Given a symmetric matrix, it is said to be $\begin{cases} \textbf{Positive definite}(A \succ 0) & \forall x \in \mathbb{R}^n, x^T Ax > 0 \iff \lambda_i > 0,\ i = 1, ..., n \\ \textbf{Positive semidefinite}(A \succeq 0) & \forall x \in \mathbb{R}^n, x^T Ax \geq 0 \iff \lambda_i \geq 0,\ i = 1, ..., n \\ \textbf{Negative definite}(A \prec 0) & \forall x \in \mathbb{R}^n, x^T Ax < 0 \iff \lambda_i < 0,\ i = 1, ..., n \\ \textbf{Negative semidefiite}(A \preceq 0) & \forall x \in \mathbb{R}^n, x^T Ax \leq 0 \iff \lambda_i \leq 0,\ i = 1, ..., n \end{cases}$

Matrix square-root $A \succeq 0 \Rightarrow \exists B :\ A = B^2$, $B = A^{1/2}$ is the **matrix square-root**
Cholesky decomposition $A \succ 0 \iff \exists B : A = B^T B$, such a B can be computed as: $B = U\Lambda^{1/2}U^T$, where U comes from the spectral decomposition and $\Lambda^{1/2} = diag(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})$
SINGULAR VALUE DECOMPOSITION(SVD) (It holds for any matrix $A \in \mathbb{R}^{m,n}$, $A^T A$ is very important!)
Theorem Given $A \in \mathbb{R}^{m,n}$, it can be written as: $A = U\tilde{\Sigma}V^T$, $U \in \mathbb{R}^{m,m}$, $V \in \mathbb{R}^{n,n}$ are *orthogonal matrices* while $\tilde{\Sigma} = \begin{bmatrix} \Sigma & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}$, $\Sigma = diag(\sigma_1, ..., \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq 0$ are the **singular values**, $U = [u_1, ..., u_r]$ are the **left singular vector**, $V = [v_1, ..., v_r]$ are the **right singular vector**.

SVD (compact form) $A = U_r \Sigma V_r = \sum_{i=1}^r (\sigma_i u_i v_i^T)$, moreover $\begin{cases} \sigma_i^2 = \lambda_i(A^T A), \sigma_1^2 = \lambda_{max}(A^T A) \\ u_i \textbf{ eigenvectors of } AA^T \\ v_i \textbf{ eigenvectors of } A^T A \end{cases}$

Properties of SVD We can redefine some properties by using SVD $\begin{cases} rank(A) = r \\ \mathcal{N}(A) = \mathcal{R}([v_{r+1} \dots v_n]),\ \mathcal{R}(A) = \mathcal{R}([u_1 \dots u_r]) \\ \|A\|_F^2 = \sum_{i=1}^r \sigma_i^2, \|A\|_2^2 = \lambda_{max}(A^T A) = \sigma_1^2 \\ \|A\|_* = \sum_{i=1}^r \sigma_i \textbf{ nuclear norm} \end{cases}$

Moore-Penrose pseudo-inverse $A^\dagger = V\tilde{\Sigma}^\dagger U^T$ (or in compact form) $A^\dagger = V_r \Sigma^{-1} U_r^T$, (full column) $A^\dagger = (A^T A)^{-1} A$
Condition number How close is A to be singular? $\kappa(A) = \frac{\sigma_1}{\sigma_n}$ the larger κ the closest A to be singular.
Low rank approximation $A_k = \arg\min_{A_k \in \mathbb{R}^{m,n}, rank(A_k)=k} \|A - A_k\|_F^2 = \sum_{i=1}^{\mathbf{k}} (\sigma_i u_i v_i^T)$
Ratio of the total variance in A_k is the quantity $\eta_k = \frac{\|A_k\|_F^2}{\|A\|_F^2} = \frac{\sigma_1^2 + ... + \sigma_k^2}{\sigma_1^2 + ... + \sigma_n^2}$ is useful a graph k vs η_k to choose properly k — **Norm approximation error** $e_k = \frac{\|A - A_k\|_F^2}{\|A\|_F^2} = 1 - \eta_k$
Application: Principal component Analysis (PCA) I collect in a matrix $X \in \mathbb{R}^{n,m}$ by columns m samples characterized by n features. $\tilde{X} = [x_1 - \bar{x} \ ...\ x_m - \bar{x}]$, $\bar{x} = \frac{1}{m} \sum_{i=1}^m x_i$ (baricenter). $x_i \in \mathbb{R}^n$
I want a direction $z \in \mathbb{R}^n$ along which the projections of the centered data have the maximal variance.
Ingredients: $\begin{cases} \text{projection on } z & \alpha_i = \tilde{x}_i^T z,\ i = 1, ..., m \\ \text{variance} & \frac{1}{m} \sum_{i=1}^m {\alpha_i}^2 = \sum_{i=1}^m z^T \tilde{x}_i \tilde{x}_i^T z = z^T \tilde{X}\tilde{X}^T z \end{cases}$
Problem to solve: $\max_{z \in \mathbb{R}^n} z^T \tilde{X}\tilde{X}^T z$, $\tilde{X} \underset{\text{SVD}}{=} U_r \Sigma {V_r}^T$, $\tilde{X}\tilde{X}^T = U_r \Sigma^2 {U_r}^T$ (spectral decomposition)
from the theorem: $z = u_1$ (**first principal component**) $\to$ the i-th principal component $\iff$ i-th row of U from SVD. $U_k \in \mathbb{R}^{n,k}$ (first k principal components, columns of U) $\longrightarrow$ I choose k according to η_k, then: $x = \bar{x} + U_k z$, $z \in \mathbb{R}^k$ (zero-mean random factors). **Result:** we are using $z \in \mathbb{R}^k$, instead of $x \in \mathbb{R}^n$, $k < n$ to represent our data. $\square$

5. Least Squares (LS)

Applications: $\begin{cases} \text{approximation for a "fat" system of equations} \\ \text{polynomial approximation/regression} \end{cases}$ $y = Ax \underset{\text{relax}}{\longrightarrow} y \simeq Ax$, **residuals** $r(x) = Ax - y$
Objective: find an x such that the squared-residuals are "small" $f_0(x) = \sum_i r_i^2(x) = \sum_i (a^T x - y_i)^2 = \|Ax - y\|_2^2$
Least Squqres problem: $\min_{x \in \mathbb{R}^n} \|Ax - y\|_2^2$ (unconstrained)— **Solution** we have to find the roots of $\nabla f_0(x) = 0$
Rewriting of the f_0 $\|Ax - y\|_2^2 = x^T Qx - 2b^T x + c$, $Q = A^T A \succ 0$, $b = A^T y$, $c = \|y\|_2^2$

$\nabla f_0(x)=0 \iff Qx=b \iff \mathbf{A^T A x}=\mathbf{A^T y}$ (normal equations) $x=(A^TA)^{-1}A^Ty$ (for full column A)
Geometric interpretation (projections) LS problem can be recasted as: $\min_{\tilde{y}\in\mathcal{R}(A)}\|\tilde{y}-y\|_2^2$
You know from the projection theorem that: (i) $\tilde{y}\in\mathcal{R}(A)$, (ii) $(\tilde{y}-y)\perp\mathcal{R}(A) \iff (\tilde{y}-y)\in\mathcal{N}(A^T) \iff A^T(\tilde{y}-y)=0 \iff A^T(Ax-y)=0 \iff A^TAx=A^Ty$ (normal equations)

Variants for the LS problem
- LS+**Equality constraints**: $\min_x f_0$ s.t. $Cx=d\to x=\bar{x}+Nz$, span(N)=$\mathcal{N}(C)$; $\min_x\|\bar{A}x-\bar{y}\|_2^2,\ \bar{A}=AN,\ \bar{y}=y-A\bar{x}$
- LS+**weighted residuals**: $f_0=\sum w_ir_i^2(x)=\|W(Ax-y)\|_2^2,\ W=diag(w_1,...,w_n)$; $\min_x\|A_wx-y_w\|_2^2,\ A_w=WA,\ y_w=Wy$
- LS+ℓ_2-**regularization**: $\min_x\|Ax-y\|_2^2+\gamma\|x\|_2^2,\ \left\|\begin{bmatrix}a\\b\end{bmatrix}\right\|_2^2=\|a\|_2^2+\|b\|_2^2$; $\min_x\|\tilde{A}x-\tilde{y}\|,\tilde{y}=[y\quad 0_n]^T,\ \tilde{A}=[A\quad\sqrt{\lambda}I_n]^T$

6. Convexity: sets and functions

Given $P=\{x^{(1)},x^{(2)},...,x^{(m)}\}$,
- **linear combination** $\lambda_1x^{(1)}+\lambda_2x^{(2)}+...+\lambda_mx^{(m)},\lambda_i\in\mathbb{R}$
- **convex combination** $\lambda_i\in\mathbb{R},\lambda_i\geq 0,\sum\lambda_i=1$

Well-known subspaces
- linear hull: **subspace with all linear combinations**
- affine hull: aff(P), $\lambda_i\in\mathbb{R},\ \sum\lambda_i=1$
- convex hull: co(P) **subspace with all convex combinations**
- conic hull: conic(P) $\lambda_i\in\mathbb{R},\lambda_i\geq 0$

CONVEX SETS $\mathcal{C}\subseteq\mathbb{R}^n$ **convex** $\iff \forall x,y\in\mathcal{C}\to$ the segment joining x and y lies in $\mathcal{C}$. More formally, a convex combination of any two points falls within $\mathcal{C}$...
$\underbrace{\forall x,y,\ \lambda\in[0,1],\ x,y\in\mathcal{C}\to\ \lambda x+(1-\lambda y)\in\mathcal{C}}_{\text{convex set}},\quad \underbrace{x\neq y,\lambda\in(0,1),\lambda x+(1-\lambda)y\in\text{relint}(C)}_{\text{strictly convex set}}$

Cone and convex cone Given $\mathcal{C}\subseteq\mathbb{R}^n$ is a **cone** $\iff x\in\mathcal{C}\to\alpha x\in\mathcal{C},\ \alpha>0$ — $\mathcal{C}$ also convex $\to$ **convex cone**

Operations preserving convexity
- Intersections with convex sets: $\mathcal{C}_1,...,\mathcal{C}_n$ convex $\to\mathcal{C}=\bigcap_{i=1}^n\mathcal{C}_i$ **is convex**
- Affine transformations: $f(x)=Ax+b,\mathcal{C}$ cvx $\to f(C)=\{f(x):\ x\in\mathcal{C}\}$ cvx

Supporting Hyperplane $\underset{\textbf{Thm.: exists separating } \mathcal{H} \text{ for } \mathcal{C} \text{ at } z}{\underline{\text{Given } \mathcal{C} \text{ convex set}, z\in\partial\mathcal{C}}}$, $\mathcal{H}$ **supporting hyperplane** $\iff z\in\mathcal{H}$ and $\mathcal{C}\subseteq\mathcal{H}_-$

Separating Hyperplane $\underset{\textbf{Thm.: } \mathcal{C}_1\cap\mathcal{C}_2=\emptyset \text{ exists sep. } \mathcal{H}}{\underline{\text{Given } \mathcal{C}_1,\mathcal{C}_2 \text{ convex sets}}}$ the hyperplane $\mathcal{H}$:
- **separates** the sets $\mathcal{C}_1\subseteq\mathcal{H}_-,\mathcal{C}_2\subseteq\mathcal{H}_+$
- **strictly separates** the sets $\mathcal{C}_1\subseteq\mathcal{H}_{--},\mathcal{C}_2\subseteq\mathcal{H}_{++}$

CONVEX FUNCTIONS

$f:\mathbb{R}^n\to\mathbb{R}$ is
- **convex** $\iff$ **(i)** dom(f) convex, **(ii)** $x,y\in$ dom(f), $\lambda\in[0,1]\to f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$
- **strictly convex** $\iff x,y\in$ dom(f), $\lambda\in[0,1]\to f(\lambda x+(1-\lambda)y)<\lambda f(x)+(1-\lambda)f(y)$
- **strongly convex** $\iff\exists m>0:\ \tilde{f}(x)=f(x)-\frac{m}{2}\|x\|_2^2$ is convex $\underset{\text{imply}}{\to}$ strictly convex

Properties f convex(cvx) $\Leftrightarrow$ epi(f) is cvx — f cvx $\Rightarrow S_\alpha$ (α-sublevel set) is convex $\forall\alpha$ — f str. cvx $\overset{\Rightarrow}{\not\Leftarrow} S_\alpha$ str. cvx
f,g strictly convex $\Rightarrow f+g$ strictly convex — f convex, g strongly convex $\Rightarrow f+g$ strongly convex
h,g convex, g non decreasing $\Rightarrow h\circ g$ cvx — g concave, h convex non-increasing $\Rightarrow h\circ g$ cvx
Non-negative linear combinations $f_i:\mathbb{R}^n\to\mathbb{R},\ i=1,...,m$ convex $\Rightarrow f=\sum_{i=1}^m\alpha_if_i,\ \alpha_i\geq 0$ is convex over $\cap_i$dom(f_i)
Affine variable transformation $f:\mathbb{R}^n\to\mathbb{R},\ g(x)=f(Ax+b)$ is convex

Conditions for convexity
- **First-order condition** (f differentiable): f convex $\iff\forall x,y\in$ dom(f), $f(y)\geq f(x)+\nabla f(x)^T(y-x)$
- **Second-order condition** (f twice differentiable):
  - f convex $\iff\nabla^2f(x)\succeq 0\ \forall x\in$ dom(f)
  - f strictly convex $\overset{\Leftarrow}{\not\Rightarrow}\nabla^2f(x)\succ 0$
  - f strongly convex $\exists m:\ \nabla^2f(x)\succ mI,\ \forall x\in$ dom(f)
- **Restriction to a line**: f convex $\iff\underbrace{g(t)=f(x_0+tv)}_{\text{restriction of } f \text{ to a line}},\ x_0,v\in\mathbb{R}^n,\ t\in\mathbb{R}$ is convex

Pointwise maximum $f_\alpha(x)$ convex and $\alpha\in\mathcal{A}$, $\mathcal{A}$ compact set(closed and bounded), $f(x)=\max_{\alpha\in\mathcal{A}}f_\alpha(x)$ is convex
Jensen's inequality Given $f:\mathbb{R}^n\to\mathbb{R}$ convex and $z\in\mathbb{R}^n$ a random variable such that $p\{z\in\text{int dom}(f)\}=1$ it holds that: $f(\mathbb{E}[z])\leq\mathbb{E}[f(z)]$. **Case** z **discrete R.V.** $p(z=x^{(i)})=\theta_i,\ i=1,...,m,\ \sum_i\theta_i=1,\ \theta_i\geq 0\Rightarrow f\big(\sum_i\theta_ix^{(i)}\big)\leq\sum_i\theta_if\big(x^{(i)}\big)$

7. Convex problems

$$p^*=\min_{x\in\mathbb{R}^n}\overbrace{f_0(x)}^{\text{convex}}\qquad(1)$$
$$\text{s.t.}\quad\underbrace{f_i(x)\leq 0}_{\text{convex}\to 0-\text{sublevel set}},\ i=1,...,m\qquad(2)$$
$$\underbrace{h_i(x)=0}_{\text{affine}\to\text{ flat}},i=1,...,p\qquad(3)$$

(1)-(3) $\to$ **CONVEX OPTIMIZATION PROBLEM**

Feasible set $\mathcal{X}$: convex since intersectio n between convex sets. $\mathcal{X}=\mathbb{R}^n$ (**unconstrained problem**).

Active/inactive constraint
- $f_i(x^*)<0$ inactive at x^*
- $f_i(x)=0$ active at x^*

In some situations... $\mathcal{X}\neq\varnothing$ **but** $\mathcal{X}_{opt}=\varnothing$

Local and global optima (thm) Given $\min_{x\in\mathcal{X}} f_0(x)$ ($f_0, \mathcal{X}$ convex) it holds that: (i) if $x \in \mathcal{X}$ is a local minimizer $\rightarrow$ is a **global minimizer**, (ii) the optimal set $\mathcal{X}_{opt}$ is a convex set.

PROBLEM TRANSFORMATIONS: how to formulate the problem into an "equivalent" way?

(i) Affine variable transformation

Original problem

$$\min_{x\in\mathcal{X}} f_0(x)$$

Transformed problem

$$\min_{x\in\mathcal{X}} \varphi(f_0(x))$$

$\varphi : \mathbb{R} \rightarrow \mathbb{R}$ must be continuous and stricltly increasing over $\mathcal{X}$

(ii) Addition of **slack variables**

Original Problem

$$\min_{x\in\mathbb{R}^n} \sum_{i=1}^{r} \varphi_i(x)$$
$$\text{s.t. } f_i(x) \le 0,\ h_i(x) = 0$$

Transformed Problem

$$\min_{x,t} \sum_{i=1}^{r} t_i$$
$$\text{s.t } f_i(x) \le 0,\ h_i(x) = 0$$
$$\varphi_i(x) \le t_i$$

This transformation is effective when the objective is the sum of functions φ_i.

(iii) Epigraphic formulation

Original problem

$$p^* = \min_{x\in\mathbb{R}^n} f_0(x)$$
$$\text{s.t } f_i(x) \le 0,\ h_i(x) = 0$$

Transformed problem

$$g^* = \min_{x\in\mathbb{R}^n, t\in\mathbb{R}} t$$
$$\text{s.t } f_i(x) \le 0,\ h_i(x) = 0$$
$$f_0(x) \le t$$

This type of transformation can be applied for all problems in a way that the objective function becomes linear, pushing the original objective into the constraints.

(iv) Replacement $h_i(x) = 0 \leftrightarrow h_i(x) \le 0$

Original problem

$$p^* = \min_{x\in\mathbb{R}^n} f_0(x)$$
$$\text{s.t } f_i(x) \le 0,\ h_i(x) = 0$$

Transformed problem

$$g^* = \min_{x\in\mathbb{R}^n} f_0(x)$$
$$\text{s.t } f_i(x) \le 0,\ h_i(x) \le 0$$

I can do this substitution if...
f_0 strictly decreasing, $x : h(x) < 0 \in \text{relint}(\mathcal{X}) \leftarrow$ (minimization)
f_0 strictly increasing, $x : h(x) < 0 \in \text{relint}(\mathcal{X}) \leftarrow$ (maximization)

Optimality (Prop) Consider $\min_{x\in\mathcal{X}} f_0(x)$ with $f_0, \mathcal{X}$ convex. A feasible solution $x \in \mathcal{X}$ is **optimal** $\iff \forall y \in \mathcal{X}$ $\nabla f(x)^T (y - x) \ge 0$. **Geometric interpretation**: there is no direction for which the objective function can decrease.

Categories of directions There are directions $v_\pm$ for which: $\begin{cases} \nabla f_0(x)^T \cdot v_+ > 0 & f_0 \text{ increase} \\ \nabla f_0(x)^T \cdot v_+ < 0 & f_0 \text{ decrease} \rightarrow \text{descent direction} \end{cases}$

Optimality (unconstrained) $x \in \mathcal{X}$ is optimal $\iff \nabla f_0(x) = 0$

Optimality (equality constrained) $x \in \mathcal{X}_{opt} \iff Ax = b,\ \exists \nu \in \mathcal{R}^m :\ \nabla f_0(x) + A^T\nu = 0$

Optimality (inequality constrained) Given x feasible, let $\mathcal{A}(x) = \{i :\ f_i(x) = 0\}$, $x \in \mathcal{X}_{opt} \iff \exists \lambda_i \ge 0,\ i \in \mathcal{A}(x)$ $\nabla f_0(x) + \sum_{i\in\mathcal{A}(x)} \lambda_i \nabla f_i(x) = 0$.

8. Quadratic programs (QP)

Quadratic function $f_0(x) = \frac{1}{2}x^T Hx + c^T x + d$ (if $H = 0$, f_0 is linear) The matrix $\begin{cases} H \succeq 0 & f_0 \text{ convex, elliptical paraboloid} \\ H \succ 0 & f_0 \text{ strongly convex} \\ H \preceq 0 & f_0 \text{ concave} \end{cases}$

Unconstrained minimization of f_0

Linear case ($H = 0$)

$f_0 = c^T x + d$, two cases: $\begin{cases} p^* = -\infty & c \ne 0 \\ p^* = d & c = 0 \end{cases}$

Quadratic case ($H \ne 0$) $\rightarrow \nabla f_0(x) = 0$ is applied
$\nabla f_0(x) = Hx + c = 0 \iff Hx = -c$. 2 cases:
(i) $c \notin \mathcal{R}(H) \rightarrow p^* = -\infty$ (unbounded below)
(ii) $c \in \mathcal{R}(H) \rightarrow \begin{cases} x^* = -H^\dagger c + \zeta,\ \zeta \in \mathcal{N}(H) & \text{in general} \\ x^* = -H^{-1}c & \lambda_i > 0,\ \forall i \end{cases}$

Polyhedron $\mathcal{P} = \{x : a_i^T x \le b_i,\ i = 1, ..., m\} = \{x \in \mathbb{R}^n :\ Ax \le b, A \in \mathbb{R}^{m,n},\ b \in \mathbb{R}^m\}$ (cvx: intersect. of halfspaces)
...properties: The image of $\mathcal{P}$ through an affine map is still a polyhedron — The set $\mathcal{P} = \{x \in \mathbb{R}^n : Ax \le b, Cx = d\}$ is a polyhedron. It can be obtained by properly parametrizing x. Polyhedron+Bounded $\rightarrow$ **Polytope**

QP (standard form)
$$\min_{x\in\mathbb{R}^n} \frac{1}{2}x^T Hx + c^T x + d$$
$$\text{subject to: } Ax \le b,\ Fx = g$$

The problem is tractable $\iff H \succeq 0$

Example(1): Markowitz's Porftolio
$r_i, x_i \rightarrow$ return and investment for asset a_i
$\hat{r} = \mathbb{E}\{r\}$, $\Sigma \doteq \mathbb{E}\{(r - \hat{r})(r - \hat{r})^T\}$
$f_0(x) = \underbrace{x^T\Sigma x}_{\text{risk}} - \underbrace{\gamma\ \hat{r}^T x}_{\text{return}}$, $\gamma > 0$
constraints: $x \ge 0$, $\sum_i x_i = 1$

Example(2): LASSO Problem (see also coord. descent method)
ℓ_1-penalty: $\lambda\|x\|_1 = \lambda\sum_i |x_i|$ (use of slack variables...)
Resulting problem:

$$\min_{x,u} \|Ax - y\|_2^2 + \lambda \sum_{i=1}^{n} u_i$$
$$\text{subject to: } x_i \le u_i,\ x_i \ge -u_i,\ i = 1, ..., n$$

9. Linear programs (LP)

Inequality form

$$\min_x c^T x$$
$$\text{s.t. } Ax \le b,\ A_{eq}x = b_{eq}$$

without loss of generality we can assume that $d = 0$: being a constant term, nothing change.

$\longrightarrow$ *problem transformation* $\longrightarrow$

(slack variables) $e \ge 0, x_+ \ge 0, x_- \ge 0$

$\tilde{x} = (x_+, x_-, e)$, $\forall x\ x = x_+ - x_-$

objective $c^T x_+ - c^T x_-$

constraints $Ax_+ - Ax_- + e = b$

$A_{eq}x_+ - A_{eq}x_- = b_{eq}$

$\tilde{A} = \begin{bmatrix} A & -A & I \\ A_{eq} & -A_{eq} & 0 \end{bmatrix}, \tilde{b} = \begin{bmatrix} b \\ b_{eq} \end{bmatrix}$

$\tilde{c} = [c^T \quad -c^T \quad 0]$

Standard form

$$\min_x \tilde{c}^T \tilde{x}$$
$$\text{s.t. } \tilde{A}\tilde{x} = \tilde{b},\ \tilde{x} \ge 0$$

Can be solved by using:
(i) Simplex algorithm
(ii) Interior point algorithm

Geometric interpretation The halfspace $\mathcal{H}(x_f) = \{x \in \mathbb{R}^n :\ c^T(x - x_f) = 0,\ c \in \mathbb{R}^n\}$ is important $x_f \in \mathcal{X}$ is optimal $\iff$ $c^T x \ge c^T x_f$, $\forall x \in \mathcal{X}$ $\iff$ $c^T(x - x_f) \ge 0, \forall x \in \mathcal{X}$ – if x_f is not optimal $\iff$ there exists at least one x for which $c^T(x - x_f) < 0$ (feasible descent direction) $\rightarrow$ the optimal solution is on vertex/edge/facet of the polyhedron representing the feasible set which is totally contained in the halfspace $\mathcal{H}_+(x_f)$

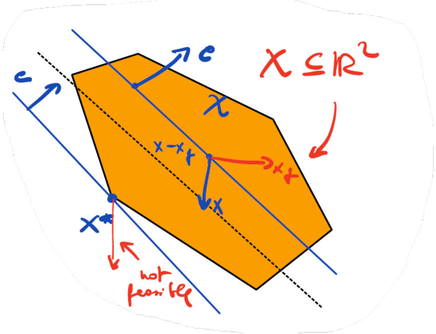
Feasible set and solutions The following situations can arise: $\begin{cases} \mathcal{X} = \varnothing & p^* = +\infty \\ \mathcal{X} \text{ bounded} & x^* \text{ is on a vertex/edge/facet} \\ \mathcal{X} \text{ unbounded} & \exists c :\ p^* = -\infty \text{ unbounded below} \end{cases}$

Example(1): Diet problem

Most economical diet (total cost: $c^T x$) satisfying the nutritional requirements ($Ax \ge b$).

n foods, their cost is c_i, m nutritional element, their recommended quantity b_i. a_{ij} quantity of the nutrient i in the j-th food. x_i quantity in the diet of the i-th food.

$$\min_{x \in \mathbb{R}^n} c^T x$$
$$\text{s.t. } Ax \ge b,\ x \ge 0$$



Example(2): Max flow problem

Maximize a certain quantity f, sent from a source to a sink, through a network represented by a digraph.

- C (matrix of capacities), c_{ij} (link capacity from i to j)
- X (matrix of effective flow) – clearly $x_{ij} \le c_{ij}$
- $\phi_{in} = \sum_{i=1}^n x_{ik}$, $\phi_{out} = \sum_{j=1}^n x_{kj}$

Constraints $\begin{cases} \phi_{in} - \phi_{out} = -f & \text{at source (node 1)} \\ \phi_{in} - \phi_{out} = f & \text{at sink} \\ \phi_{in} - \phi_{out} = 0 & \text{intermediate (node } n\text{)} \end{cases}$

Objective $\max_{x \in \mathbb{R}^n, f \in \mathbb{R}} f \iff -\min_{x \in \mathbb{R}^n, f \in \mathbb{R}} -f$

$$-p^* = \min_{x \in \mathbb{R}^n, f \in \mathbb{R}} -f$$
$$\text{s.t. } (X^T - X)\mathbf{1} = \begin{bmatrix} -f \\ 0 \\ f \end{bmatrix},\ 0_{n \times n} \le X \le C$$

10. Second-order cone programs (SOCP)

Second order cone $\mathcal{K}_p = \{(u, t) :\ u \in \mathbb{R}^{p-1},\ t \in \mathbb{R}, \|u\|_2 \le t\}$ extension of $\mathcal{K}_3 = \{(x, y, z) :\ \sqrt{x^2 + y^2} \le z\}$

$\mathbb{R}^n$**-cone** $\mathcal{K}_n = \{x \in \mathbb{R}^n : \sqrt{x_2^2 + x_3^2 + ... + x_n^2} \le x_1\}$

Rotated cone $\mathcal{K}_n^r \doteq \{x \in \mathbb{R}^n :\ x_3^2 + x_4^2 + ... + x_n^2 \le 2x_1x_2,\ x_1, x_2 > 0\}$

Useful equivalence $\|u\|_2^2 \le yz\ y \ge 0,\ z \ge 0 \iff \|\binom{2u}{y-z}\|_2 \le y + z$

Standard form for SOCP

$$\min_x c^T x$$
$$\text{s.t. } Ax = b, x^i \in \mathcal{K}_{n_i}$$

x^i is the i-th block of the decision variable, n_i its dimension.

SOC constraints
Given $u = Ax + b$, $t = c^T x + d$

$$\mathcal{K} = \{x : \|Ax + b\|_2 \le c^T x + d\}$$

I can rewrite the problem as:

$$\min_x c^T x$$
$$\text{s.t. } \|A_i x + b_i\|_2 \le c_i^T x + d_i$$

SOCP class encapsulate also the Linear and Quadratic programs.

LP as SOCP $\Longrightarrow$

$$\min_x c^T x$$
$$\text{s.t. } a_i^T x \le b_i,\ i = 1, ..., m$$

problem transformation

$$\min_x c^T x$$
$$\text{s.t. } \|C_i x + d_i\|_2 \le b_i - a^T x$$
$$C_i = 0, d_i = 0,\ i = 1, ..., m$$

QP as SOCP $\Longrightarrow$

$$\min_x x^T Q x + c^T x + d$$
$$\text{s.t. } a_i^T x \le b_i,\ i = 1, ..., m$$

problem transformation

$$\min_{x,y} c^T x + y$$
$$\text{s.t. } \left\| \begin{pmatrix} 2Q^{1/2} \\ y - 1 \end{pmatrix} \right\|_2 \le y + 1$$
$$\text{s.t. } a_i^T x \le b_i,\ i = 1, ..., m$$

Sum of Norms (SON) $\min_x \sum_{i=1}^p \|A_i x - b_i\|_2 \longrightarrow \min_{x,y} \sum_{i=1}^p y_i$ s.t. $\|A_i x - b_i\|_2 \le y_i$, $i = 1, ..., p$, y_i slack variables

Max of Norms $\min_x \max_{i=1,...,p} \|A_i x - b\|_2 \longrightarrow \min_{x,y} y$ s.t. $\|A_i x - b\|_2 \le y,\ i = 1, ..., p$

Example(1): Fermat-Weber Point

> Where to locate a warehouse in order to serve in the best way some location services?

$x \in \mathbb{R}^2$: position of the warehouse
$y_i \in \mathbb{R}^2,\ i = 1, ..., m$: position of the location services
I have to solve: $\min_x \frac{1}{m} \sum_{i=1}^m \|x - y_i\|_2$ (use SON)

$$\underbrace{\begin{aligned} &\min_{x,t} \frac{1}{m} \sum_{i=1}^m t_i \\ &\text{s.t. } \|x - y_i\|_2 \le t_i,\ i = 1, ..., m \end{aligned}}_{\text{SOC Program}}$$

Example(2): LP with probability constraints

> How to solve an LP problem when one or more data are random or uncertain?

In particular:

-a_i random normally distributed vectors $\begin{cases} \mathbb{E}\{a_i\} = \bar{a}_i \\ \text{var}(a_i) = \Sigma_i \succ 0 \end{cases}$

-$a_i^T x$ is a random variable $\to$ ($\mu = \bar{a}_i^T x,\ \sigma^2 = x^T \Sigma_i x$),
-We know a-priori $P\{a_i^T x \le b_i\} \ge p_i, i = 1, ..., m$
If $p_i > 0.5$ the LP problem can be rewritten as:

$$\underbrace{\begin{aligned} &\min_x c^T x \\ &\text{s.t. } \bar{a}_i^T x \le b_i - \Phi^{-1}(p_i) \|\Sigma_i^{1/2} x\|_2 \end{aligned}}_{\text{SOC Program}}$$

11. CVX modeling

Example (LP Program)

```
cvx_begin [quiet]
    [cvx_solver [mosek|sedumi]]
    variable(s) x(2)    % name(dimension)
    minimize (c'*x)     %objective function
    subject to
        A*x <= b;       %constraints
cvx_end
```

You can write the problem also in a non-standard form, `cvx` will recast the problem in a standard form.

Variables: $\begin{cases} \texttt{x} & \text{the minimizer} \\ \texttt{cvx\_optval} & \text{the value } p^* \\ \texttt{cvz\_status} & \texttt{Solved, Unfeasible, Unbounded} \end{cases}$

useful function: `quad_form(x,Q)`, returns $x^T Q x$ if $Q \succ 0$

12. General optimization and Lagrangian Duality

Langrangian $\mathcal{L}(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x)$ – $\lambda_i,\ \nu_i$ **Lagrange multipliers** – $f_0(x) \ge \mathcal{L}(x, \lambda, \nu)$

Lagrangian dual function $g(\lambda, \nu) = \inf_{x \in \mathcal{D}} \mathcal{L}(x, \lambda, \nu),\ \lambda > 0,\ g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ – **Properties** $\begin{cases} \text{jointly concave in}(\lambda, \nu) \\ g(\lambda, \nu) \le p* \ \ \forall \lambda > 0, \forall \nu \end{cases}$

Dual problem (What is the best $g(\lambda, \nu)$)? $d^* = \max_{\lambda,\nu} g(\lambda, \nu)$ subject to : $\lambda > 0$ – $d^* \le p^*$ (**...is always convex**)

Duality gap $\delta^* = p^* - d^* = \begin{cases} = 0 & \text{strong duality (further conditions are needed)} \\ > 0 & \text{weak duality} \end{cases}$

Slater's condition for STRONG DUALITY Let $f_i(x)$ convex functions, $h_i(x)$ affine functions, $f_i(x), i = 1, ..., k \le m$ affine, if $\exists x \in \text{relint } \mathcal{D}$: $f_i(x) \le 0,\ i = 1, ..., k,\ f_{k+1}(x) < 0, ..., f_m(x) < 0,\ h_i(x) = 0,\ i = 1, ..., p$ then: $p^* = d^*$ (there is no gap), moreover if the problem is not unbounded below $\exists(\lambda^*, \nu^*) :\ g(\lambda^*, \nu^*) = d^* = p^*$.

Primal problem

$$p^* = \min_x \max_{\lambda,\nu} \mathcal{L}(x, \lambda, \nu)$$

Dual problem

$$d^* = \max_{\lambda,\nu} \min_x \mathcal{L}(x, \lambda, \nu)$$

Primal solution from dual solution (Required: strong duality) $d^* = p^* = f_0(x^*)$, $d^* = g(\lambda^*, \nu^*) = \mathcal{L}(x, \lambda^*, \nu^*)$

$$f_0(x^*) = \mathcal{L}(x^*, \lambda^*, \nu^*) = f_0(x^*) + \sum_{i=1}^m \lambda_i{}^* f_i(x^*) + \sum_{i=1}^p \nu_i^* h_i(x^*) \to \begin{cases} \sum_{i=1}^m \lambda_i{}^* f_i(x^*) = 0 & \text{complementary slackness} \\ x^* & \text{minimizer (wrt } x\text{) of } \mathcal{L}(x, \lambda^*, \nu^*) \end{cases}$$

Complementary slackness For the i-th constraint: either $\lambda_i = 0, f_i(x) \le 0$, or $\lambda_i > 0, f_i(x) = 0$

Solution recovery If f_0 convex, $\nabla_x\ \mathcal{L}(x, \lambda^*, \nu^*) = 0$ gives a **global minimizer** (unique if f_0 strictly convex).

KKT conditions for optimality(optimal $\iff$ $\delta^* = 0$)
f_0 differentiable, strong duality holds.

$$\begin{cases} \text{1. primal feasibility} & f_i(x^*) \le 0,\ h_i(x^*) = 0 \\ \text{2. dual feasibility} & \lambda_i^* \ge 0,\ i = 1, ..., m \\ \text{3. complementary slackness} & \sum_{i=1}^m \lambda_i^* f_i(x^*) = 0 \\ \text{4. Lagrangian stationarity} & \nabla_x\ \mathcal{L}(x, \lambda^*, \nu^*)\big|_{x=x^*} = 0 \end{cases}$$

Sensitivity of the optimal solution p^*

Perturbed Problem

$$\begin{aligned} p^*(u, v) = &\min_{x \in \mathbb{R}^n} f_0(x) \\ &\text{s.t. } f_i(x) \le u_i,\ h_i(x) = v_i \end{aligned}$$

Remark: $p^*(0, 0) = p^*$ – $u_i > 0 \to$ relaxing, $u_i < 0$ tightning. When the primal problem is convex and $p^*(u, v)$ differentiable:

$$\lambda_i^* = -\frac{\partial p^*(u, v)}{\partial u_i}\Big|_{(u,v)=(0,0)},\ \ \nu_i^* = -\frac{\partial p^*(u, v)}{\partial v_i}\Big|_{(u,v)=(0,0)}$$

Interpretation: (here I use the complementary slackness)
$\lambda_i^* = 0 \Rightarrow f_i(x^*) < 0$ the constraint is inactive, I don't care about perturbations $\to$ not resource critical!
$\lambda_i^* > 0 \Rightarrow f_i(x^*) = 0$ constraint active (resource critical) $\to p^*_{new} = p^* - \lambda_i^* u_i$.

In particular: $\begin{cases} u_i < 0 & p^* \text{ increase (worse solution)} \\ u_i > 0 & p^* \textbf{ can} \text{ decrease (better solution)} \end{cases}$

Remember that: the Lagrange multipliers are given by the solution of the dual problem.

13. Gradient Algorithm (GA) (unconstrained case)

General structure: $x_{k+1} = x_k + s_k v_k$

$s_k > 0,\ s_k \in \mathbb{R}$ (step-size)

$v_k \in \mathbb{R}^n$ (update/search direction)

For Gradient Algorithm: $\begin{cases} f_0 \text{ differentiable} \\ x_k \in \text{dom}(f_0) \\ v_k \in \mathbb{R}^n (\text{chosen wrt } \nabla f_0(x)) \end{cases}$

$f_0 : \mathbb{R}^n \to \mathbb{R},\ f_0(x_k + s v_k) \simeq f_0(x_k) + s\nabla f_0(x_k)^T v_k$

$t \to \infty,\ \delta_k = \lim_{s\to\infty} \frac{f_0(x_k+sv_k)-f_0(x_k)}{s} = \nabla f_0(x_k)^T v_k$

I want a direction for which f_0 decrease (descent). $\forall k$ moving toward $v_k \to$ go to min.

Steepest descent direction: $v_k = -\frac{\nabla f_0(x)}{\|f_0(x)\|_2}$

Algorithm 1 Gradient Algorithm

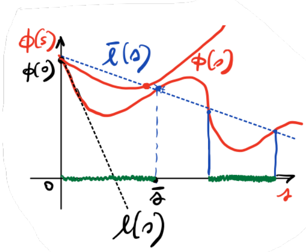
1: $k = 0$, choose a descent v_k ($v_k = -\nabla f_0(x_k)$)
2: Determine the step-size s_k
3: $x_{k+1} = x_k + s_k v_k$
4: **if** Stop criterion **then**
5: **return** x_k
6: **else**
7: back to 2
8: **end if**

STEPSIZE SELECTION $\overbrace{\phi(s) = f_0(x_k + s v_k)}^{\text{restriction of } f_0 \text{ along } v_k}, s \geq 0$

$\phi(0) = f_0(x_k)$, I want $s > 0:\ f(x_{k+1}) = \phi(s) < \phi(0)$

Exact line-search (best s possible) (non-convex)

$$s^* = \arg\min_{s\geq 0} \phi(s)$$



$\ell(s) = \phi(0) + s\delta_k$ (tan at 0)

$\bar{\ell}(s) = \phi(0) + \alpha s \delta_k$

$\bar{\ell}(s) \geq \ell(s) \geq \phi(s)$

$s:\ \phi(s) \leq \bar{\ell}(s)$ are OK

Armijo condition

$\phi(s) \leq \phi(0) + s\alpha\delta_k$

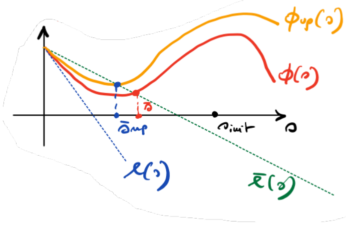
Algorithm 2 Backtracking line-search

1: $\alpha \in (0,1),\ \beta \in (0,1), s_{init} = 1,\ v_k$ descent
2: **if** $f_0(x_k + sv_k) \leq f(x_k) + s\alpha\nabla f_0(x_k)^T v_k$ **then**
3: **return** $s_k \leftarrow s$
4: **else**
5: $s = \beta s$, back to 2 ▷ update until not Armijo's meet
6: **end if**

CONVERGENCE OF GA: STOPPING CRITERION

$\nabla f_0(x), \exists L : \|\nabla f_0(y) - \nabla f_0(x)\|_2 \leq L\|y-x\|_2, \forall x,y$ (Lipschitz)

$\Rightarrow f_0(x) \leq \underbrace{f_0(y) + \nabla f_0(x)^T(x-y) + \frac{L}{2}\|x-y\|_2}_{\text{strongly cvx, quadratic approximant}}$



I can introduce $\phi_{up}(s)$ since ∇f_0 is Lipschitz. $\bar{s}_{up} \to$ intersection $\phi_{up}(s)$-$\bar{\ell}(s)$

$\bar{s}_{up} = \frac{2}{L}(1-\alpha)$

Backtracking ensures:

$\forall k, s_k \geq \min(s_{init}, \beta\bar{s}_{up}) \doteq s_{\text{lb}}$

It holds that (GA converges to a stationary point):

$\alpha s_{\text{lb}} \sum_{i=0}^k \|\nabla f_0(x_i)\|_2^2 \leq f_0(x_0) - f_0^*,\ \lim_{k\to\infty} \|\nabla f_0(x_i)\|_2 = 0$

Stopping criterion $\to$ $\|\nabla f_0(x)\|_2 < \varepsilon, \varepsilon > 0$

f_0 convex $\to x^*$ is a global minimizer.

Convergence $\underset{\|\nabla f_0(x_k)\|_2 \to 0}{}$ $\begin{cases} O(1/\sqrt{k}) & f_0 \text{ generic} \\ O(1/\varepsilon) & f_0 \text{ convex (sublinear)} \\ O(1/\log(1/\varepsilon)) & f_0 \text{ strongly convex (linear)} \end{cases}$

Grad. Algorithm as minimization of $q(x)$

GA can be interpreted also as minimization of the quadratic approximant when $\|x - x_k\|_2$ is small $\to f_0(x) \simeq q(x) = f_0(x_k) + \nabla f_0(x_k)^T(x-x_k) + \frac{1}{2s}\|x-x_k\|_2$

$\nabla q(x) = \nabla f_0(x_k) + \frac{1}{s}(x - x_k) = 0 \Leftrightarrow \underbrace{x_{k+1} = x_k - s\nabla f_0(x_k)}_{\text{Gradient Algorithm}}$

Conclusion: x is updated $\forall k$ by minimizing $q(x_k)$. □

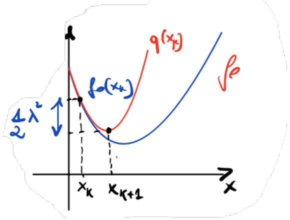
14. Newton Algorithm (NA) (unconstrained case)

Key concept: Finding (starting from x_k) the roots of a non linear function $g : \mathbb{R} \to \mathbb{R}$. We want the roots of $\nabla f_0(x) = 0$.

$\tilde{g}(x) = g(x_k) + g'(x_k)(x - x_k)$. x_{k+1} is given by $\tilde{g}(x) = 0$

$$\overbrace{x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)} = x_k - (\nabla^2 f_0(x_k))^{-1}\nabla f_0(x_k)}^{\text{NEWTON METHOD}} \quad (4)$$

$$\underbrace{q_0(x) = f(x_k) + \nabla f_0(x)^T(x - x_k) + \frac{1}{2}(x-x_k)^T\nabla^2 f_0(x_k)(x-x_k)}_{\text{(better) quadratic approximant of } f_0}$$



$\arg\min_x q_0(x) = x_{k+1}$

$\min_x q_0(x) = f(x_{k+1})$

$\min_x q_0(x) - f(x) = \frac{1}{2}\lambda^2(x)$

$\underbrace{\lambda^2 = \nabla f_0{}^T \nabla^2 f_0{}^{-1} \nabla f_0}_{\text{Newton decrement}}$

where $\nabla f_0 \leftrightarrow \nabla f_0(x_k)$

Damped Newton Method $x_{k+1} = x_k + tv$

$v = -\nabla^2 f_0(x_k)^{-1}\nabla f_0(x_k), \nabla f_0(x_k)^T \cdot v = -\lambda^2 < 0$ (**descent**)

Algorithm 3 Choice of t (step-size)

1: $\alpha \in (0, \frac{1}{2}],\ \beta \in (0,1), t = 1$
2: **while** $f_0(x_k + tv) > f(x_k) + \alpha t \nabla f_0(x_k)^T v$ **do** $t \leftarrow t\beta$
3: **end while**
4: $x_{k+1} = x_k + tv$

Convergence properties f_0 strongly convex, $\nabla f_0, \nabla^2 f_0$ Lipschitz continuous, $\eta \in (0, m^2/L)$.

2 phases: $\begin{cases} \|\nabla f_0(x_k)\|_2 \geq \eta & \text{Damped phase} \\ \|\nabla f_0(x_k)\|_2 < \eta & \text{Quadratically convergent phase} \end{cases}$

Stopping criterion $f_0(x_k) - f_0^* \leq \lambda_k^2$

OTHER ASPECTS (1): EQUALITY CONSTRAINTS

I choose $x_0 : Ax_0 = b$, and $x_{k+1} = x_k + tv$, where

$$v = \arg\min_{z\in\mathcal{N}(A)} \nabla f_0(x)^T(z-x) + \frac{1}{2}(z-x)^T\nabla^2 f_0(x_k)(z-x)$$

In this way, since $Ax_{k+1} = Ax_k + tAv = b$, x_k remains in the feasible set, $Av = 0$.

OTHER ASPECTS(2): QUASI-NEWTON METHODS

Secant condition $H(\nabla f(x) - \nabla f(y)) = (x - y) \Rightarrow$ a matrix H satisfying such a condition can approximate the Hessian.

Algorithm 4 Quasi-Newton methods

1: $H_k = I_n$
2: Update s.t.: $H_{k+1}\nabla f_0(x_{k+1}) - \nabla f_0(x_k) = x_{k+1} - x_k$

15. Approaches for constrained optimization

When there are constraints solving $\nabla f_0(x) = 0$ it is not sufficient, x^* must be feasible and $\mathcal{X} \neq \mathbb{R}^n$

1^{st} approach: *Projected Gradient Method*

$\mathcal{P}_{\mathcal{X}}(x) \leftarrow$ projection of x on the feasible set $\mathcal{X}$ (convex, non-empty).

Algorithm 5 Projected Gradient Method

```
k = 0
w_{k+1} = x_k − s_k ∇f_0(x_k)        ▷ Gradient Step
x_{k+1} = P_X(w_{k+1})               ▷ ensure feasibility
```

(used only for set for which is simple to compute projections, otherwise other more general methods are used).

The s_k (step-size) is chosen as follows: $s_k = \bar{s}2^{-t(k)}$

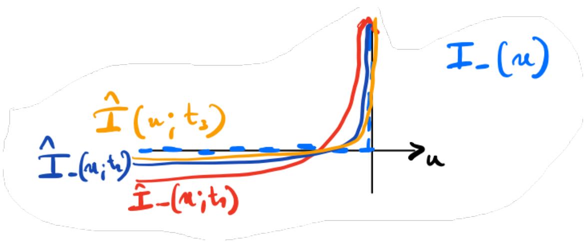
Algorithm 6 Stepsize selection

```
1: j = 0
2: z_j = P_X(x_k − s̄ 2^{−j} ∇f_0(x_k))
3: if f(z_j) ≤ f(x_k) − α∇f_0(x_k)^T (x_k − z_j) then
4:     t(k) = j
5: else
6:     goto 2
7: end if
```

2^{nd} approach: *Barrier Method*

Here we want to solve ($f_0, ..., f_m$ convex and smooth):

$$p^* = \min_x f_0(x)$$
$$\text{s.t. } f_i(x) \leq 0,\ i = 1, ..., m$$
$$Ax = b$$



The problem can be rewritten as:

$$p^* = \min_x f_0(x) + \sum_{i=1}^{m} I_-(f_i(x)) \tag{5}$$

$$I_-(u) = \begin{cases} 0 & u \leq 0 \\ \infty & u > 0 \end{cases} \quad \text{(Indicator function)} \tag{6}$$

$I_-(u)$ non-differentiable $\rightarrow \hat{I}_-(u;t) = -\frac{1}{t}\log(-u)$

$$p^*(t) = \min_x f_0(x) \overbrace{-\frac{1}{t}\sum_{i=1}^{m} log(-f_i(x))}^{\phi(t) \rightarrow \textbf{logarithmic barrier}} = \tag{7}$$

$$= \min_x t f_0(x) + \phi(t) \tag{8}$$

$$\text{s.t. } Ax = b \tag{9}$$

Parametrizing the x, (8) becomes unconstrained (GA, NM can be used).
For $t \rightarrow \infty$ (8)$\rightarrow$(5) that is $p^*(t) \rightarrow p^*,\ x^*(t) \rightarrow x^*$
Central path$\{x^*(t) : t > 0\}$, $p^*(t)$ ε-suboptimal $\varepsilon = m/t$

Algorithm 7 Sequential Barrier Method

```
x_0 (strictly feasible), t←t_0, μ > 1, ε > 0
loop
    Solve min_{Ax=b} t f_0(t) + φ(t)        ▷ Centering step
    x←x*(t)
    if m/t < ε then                         ▷ Stopping criterion
        break
    else
        t ← μt          ▷ Increase t (better approx of I_−(u))
    end if
end loop
```

<u>Phase I Problem</u>: starting from a feasible x_0

$$\tilde{s} = \min_{x,s} s$$
$$\text{s.t. } f_i(x) \leq s,$$
$$Ax = b$$

Solved by using the barrier method, starting from:
$x_0 :\ Ax_0 = b,$
$s_0 = \eta + \max_i f_i(x),\ \eta > 0$

$\rightarrow \tilde{s} < 0$. Since $f_i(\tilde{x}) \leq \tilde{s} \leq 0 \longrightarrow x_0 = \tilde{x}$
$\rightarrow \tilde{s} > 0$. $\nexists x :\ f_i(x) \leq 0 \longrightarrow$ problem infeasible
$\rightarrow \tilde{s} = 0$. Only of theoretical interest

Remark: For carrying out the *centering step* the NEWTON METHOD tailored for equality constraints is used.

16. Geometric Programs(GP)

For GP: **variables** $\rightarrow$ positive (physical quantities), **objective/constraints** $\rightarrow$ non-negative linear combination of positive monomials.
Positive monomial Given $x \in \mathbb{R}^n,\ a \in \mathbb{R}^n,\ c > 0, x > 0\ cx^a = cx_1^{a_1}x_2^{a_2}...x_n^{a_n} \in \mathbb{R}^+$

Posynomial $f : \mathbb{R}^n_{++} \rightarrow \mathbb{R}$, $f(x) = \sum_{i=1}^k c_i x^{a_{(i)}}$, $c_i, x > 0, a_{(i)} \in \mathbb{R}^n$ – **Generalized posynomial** $\begin{cases} \text{pointwise maximum} \\ \text{fractional power} \\ \text{addition/multiplication} \end{cases}$
composite fun. of posynomials

Monomials, Posynomials and Generalized posynomials **are not convex!** $\rightarrow$ <u>Problem transformation is needed</u>
Convex form for monomials $y_i = \log x_i \rightarrow \tilde{g}(x) = cx_1^{a_1}...x_n^{a_n} = e^{\log c}e^{\log x_1^{a_1}}...e^{\log x_n^{a_n}} = e^{a_1y_1+...+a_ny_n+\log c} = e^{a^Ty+b}$. I can take the log (f convex $\circ$ f increasing) obtaining $g(x) = \log \tilde{g}(x) = a^Ty + b \rightarrow$ Linear Program
Convex form for posynomials $\tilde{f}(x) = \sum_{i=1}^k c_i x^{a_{(i)}} = \sum_{i=1}^k e^{a_{(i)}^T y + b_i} \rightarrow \tilde{g}(x) = \log\left(\sum_{i=1}^k e^{a_{(i)}^T y + b_i}\right) = \text{lse}(Ax + b)$
log-sum-exp (lse) function is convex – $A \in \mathbb{R}^{k,n},\ b = [b_1 \quad \dots \quad b_n]^T$

Standard form for GP Here $f_0(x), f_i(x)$ are posynomials, $h_i(x)$ are (possibly) monomials

$$\min_x \sum_{k=1}^{K_0} c_k x^{a_{(k)}}$$
$$\text{s.t. } \sum_{k=1}^{K_i} c_k x^{a_{(k)}} \leq 1,\ i = 1, ..., m$$
$$g_i x^{r_{(i)}} = 0,\ i = 1, ..., p$$

Geometric programs in standard form

$$\min_y \operatorname{lse}(A_0 y + b_0)$$
$$\text{s.t. } \operatorname{lse}(A_i y + b_i) \leq 0$$
$$Ry + h = 0$$
$$A_0 \in \mathbb{R}^{K_0, n}, b_0 \in \mathbb{R}^{K_0},\ A_i \in \mathbb{R}^{K_i, n}, b_0 \in \mathbb{R}^{K_i}$$

Generalized GP $f_0, ..., f_m$ are generalized posynomials $\to$ GGP (Generalized GP)
Fractional power $f_1(x)^\alpha + f_2(x)^\beta \leq 1 \Rightarrow \underbrace{f_1(x) \leq t_1,\ f_2(x) \leq t_2,\ t_1^\alpha + t_2^\beta \leq 1}_{\text{GP constraints}}$

Pointwise maximum power $\max(f_1(x), f_2(x)) + f_3(x) \leq 1 \Rightarrow \underbrace{f_1(x) \leq t, f_2(x) \leq t, t + f_3(x) \leq 1}_{\text{GP constraints}}$

APPENDIX

Schur complement $M = \begin{bmatrix} A & X^T \\ X & B \end{bmatrix}, A, B \in \mathbb{S}^n, X \in \mathbb{R}^{n,m}. \quad M \succeq 0 \iff S = A - XB^{-1}X^T \succeq 0$

Young inequality If $a \geq 0$, $b \geq 0$, $p > 1$, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ then $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$ (Jensen's inequality can be used in order to demonstrate it)

Maxima of inner product over a ball $\max_{\|x\|_p \leq 1} x^T y$ $\begin{cases} \max_{\|x\|_2 \leq 1} x^T y = \|y\|_2,\ x^* = \frac{y}{\|y\|_2} & p = 2 \\ \max_{\|x\|_\infty \leq 1} x^T y = \sum_{i=1}^n |y_i| = \|y\|_1,\ x_\infty^* = \operatorname{sgn}(y) & p = \infty \\ \max_{\|x\|_1 \leq 1} x^T y = \max_i |y_i| = \|y\|_\infty & \\ [x_1^*]_i = \begin{cases} \operatorname{sign}(y_i) & \text{if } i = m \\ 0 & \text{otherwise} \end{cases},\ i = 1, ..., n & p = 1 \end{cases}$