Homework 1- 5

Homework 1

1. (1.1.1) For each of the following statements, identify whether or not the statement is a proposition. If so, what is its truth value; if not, why?
   (a) Boston is the capital of Massachusetts.

   This is a proposition. Truth value: T

   (b) $5 + 7 = 10$

   This is a proposition. Truth value: F

   (c) $x + 2 = 11$

   This is not a proposition because it is neither true nor false.

   (d) Answer this question.

   This is not a proposition because it is neither true nor false.

2. (1.1.5) State the negation of each of the following propositions.
   (a) Mei has an MP3 player.

   Mei does not have an MP3 player.

   (b) There is no pollution in New Jersey.

   There is pollution in New Jersey.

   (c) $2 + 1 = 3$
   $\mathbf{2 + 1 \neq 3}$
   (d) The summer in Maine is hot and sunny.

   The summer in Maine is not hot and sunny.

3. (1.1.13) Let p and q be the propositions: ∧∨¬ ↔
   - *p:* "It is below freezing."
   - *q:* "It is snowing."

   Write the following propositions using *p*, *q*, and logical operators.

   (a) It is below freezing and snowing. $\boldsymbol{p \land q}$
   (b) It is below freezing but not snowing. $\boldsymbol{p \land \neg q}$
   (c) It is not below freezing and it is not snowing. $\boldsymbol{\neg p \land \neg q}$
   (d) It is either snowing or below freezing (or both). $\boldsymbol{p \lor q}$

4. (1.1.19) Determine whether each of the following implications is true or false.
   (a) If $1 + 1 = 2$, then $2 + 2 = 5$. **false**
   (b) If $1 + 1 = 2$, then $2 + 2 = 4$. **true**
   (c) If $1 + 1 = 3$, then $2 + 2 = 5$. **true**
   (d) If monkeys can fly, then $1 + 1 = 3$. **true**

5. (1.1.20) Determine whether each of the following implications is true or false.
   (a) If $1 + 1 = 3$, then unicorns exist. **true**
   (b) If $1 + 1 = 3$, then dogs can fly. **true**
   (c) If $1 + 1 = 2$, then dogs can fly. **false**
   (d) If $2 + 2 = 4$, then $1 + 2 = 3$. **true**

6. (1.1.18) Determine whether each of the following biconditionals is true or false.
   (a) $2 + 2 = 4$ if and only if $1 + 1 = 2$. **true**
   (b) $1 + 1 = 2$ if and only if $2 + 3 = 4$. **false**
   (c) $1 + 1 = 3$ if and only if monkeys can fly. **true**
   (d) $0 > 1$ if and only if $2 > 1$. **false**

7. Let p and q be the propositions:
   - *p:* "Swimming at the shore is allowed."
   - *q:* "Sharks have been spotted near the shore."

   Express each of the following propositions as an English sentence.

   (a) $p \wedge q$

   Swimming at the shore is allowed and sharks have been spotted near the shore.

   (b) $p \rightarrow \neg q$

   If swimming at the shore is allowed, then sharks have not been spotted near the shore.

   (c) $\neg q \rightarrow p$

   If sharks have not been spotted near the shore, then swimming at the shore is allowed.

   (d) $\neg q \vee (\neg p \wedge q)$

   Sharks have not been spotted near the shore, or swimming is not allowed at the shore and sharks have been spotted near the shore.

8. (1.1.16) Let p, q, and r be the propositions:
   - *p:* "You ace the final exam."
   - *q:* "You do every exercise in the book."
   - *r:* "You get an 'A'."

   Write the following propositions using p, q, r, and logical operators.

   (a) To get an 'A', it is necessary for you to ace the final exam.

   $\boldsymbol{r \rightarrow p}$

   (b) You ace the final exam but don't do every exercise in the book; nevertheless, you get an 'A'.

   $\boldsymbol{(p \wedge \neg q) \wedge r}$

(c) Acing the final exam and doing every exercise in the book is sufficient for getting an 'A'.

$\boldsymbol{(p \wedge q) \rightarrow r}$

(d) You will get an 'A' if and only if you either do every exercise in the book or ace the final exam.

$\boldsymbol{r \leftrightarrow (q \vee p)}$

9. (1.1.26) Rewrite each of the following propositions in the form "If…then…".
   (a) I will remember to send you the address if you send me an email address.

   If you send me an email address, then I will remember to send you the address.

   (b) That you got the job implies that you had the best credentials.

   If you got the job, then you had the best credentials.

   (c) Having a valid password is necessary in order to login to the server.

   If you can login to the server, then you have a valid password.

   (d) It is possible for you to reach the summit unless you begin your climb too late.

   If you begin your climb too late, then it is not possible for you to reach the summit.

10. (1.1.33) Construct a truth table for each of the following compound propositions.
   (a) $p \wedge \neg p$

p	$p \wedge \neg p$
T	**F**
F	**F**

(b) $(p \vee \neg q) \rightarrow q$

p	q	$(p \vee \neg q)$	$(p \vee \neg q) \rightarrow q$
T	T	T	**T**
T	F	T	**F**
F	T	F	**F**
F	F	T	**T**

(c) $(p \vee q) \rightarrow (p \wedge q)$

p	q	$(p \vee q)$	$(p \wedge q)$	$(p \vee q) \rightarrow (p \wedge q)$
T	T	T	T	**T**
T	F	T	F	**F**
F	T	T	F	**F**
F	F	F	F	**T**

(d) $(p \to q) \leftrightarrow (\neg q \to \neg p)$

p	q	$(p \to q)$	$(\neg q \to \neg p)$	$(p \to q) \leftrightarrow (\neg q \to \neg p)$
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	T	**T**
F	F	T	T	**T**

11. (1.1.38) Construct a truth table for each of the following compound propositions.

(a) $(p \vee q) \wedge r$

p	q	r	$(p \vee q)$	$(p \vee q) \wedge r$
T	T	T	T	**T**
T	T	F	T	**F**
T	F	T	T	**T**
T	F	F	T	**F**
F	T	T	T	**T**
F	T	F	T	**F**
F	F	T	F	**F**
F	F	F	F	**F**

(b) $(p \wedge q) \vee r$

p	q	r	$(p \wedge q)$	$(p \wedge q) \vee r$
T	T	T	T	**T**
T	T	F	T	**T**
T	F	T	F	**T**
T	F	F	F	**F**
F	T	T	F	**T**
F	T	F	F	**F**
F	F	T	F	**T**
F	F	F	F	**F**

Homework 2

1. (1.3.5) Using a truth table, show that $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	**T**	T	T	**T**
T	T	F	T	**T**	T	F	**T**
T	F	T	T	**T**	F	T	**T**
T	F	F	F	**F**	F	F	**F**
F	T	T	T	**F**	F	F	**F**
F	T	F	T	**F**	F	F	**F**
F	F	T	T	**F**	F	F	**F**
F	F	F	F	**F**	F	F	**F**

2. (1.3.20) Without using a truth table, show that $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$.

$$\begin{aligned}
\boldsymbol{p \leftrightarrow q} &\equiv \boldsymbol{(p \wedge q) \vee (\neg p \wedge \neg q)} \\
&\equiv \boldsymbol{(p \to q) \wedge (q \to p)} \\
&\equiv \boldsymbol{(\neg p \wedge q) \wedge (\neg q \vee \mathrm{p})} \\
&\equiv \boldsymbol{((\neg p \vee q) \wedge \neg q) \vee ((\neg p \vee q) \wedge p)} \\
&\equiv \boldsymbol{((\neg p \vee q) \vee (q \wedge \neg q)) \vee ((\neg p \vee q) \vee (p \wedge \neg p))} \\
&\equiv \boldsymbol{((\neg p \vee q) \vee \mathrm{F}) \vee ((\neg p \vee q) \vee F)} \\
&\equiv \boldsymbol{(\neg p \vee q) \vee (q \wedge p)} \\
&\equiv \boldsymbol{(p \wedge q) \vee (\neg p \wedge \neg q)} \qquad \textbf{:)}
\end{aligned}$$

3. (1.3.25) Without using a truth table, show that $\neg (p \leftrightarrow q) \equiv \neg p \leftrightarrow q$.

$$\begin{aligned}
\boldsymbol{\neg (p \leftrightarrow q)} &\equiv \boldsymbol{\neg p \leftrightarrow q} \\
&\equiv \boldsymbol{\neg ((p \to q) \wedge (q \to p))} \\
&\equiv \boldsymbol{\neg (p \to q) \vee \neg (q \to p)} \\
&\equiv \boldsymbol{\neg (\neg p \vee q) \vee \neg (\neg q \vee p)} \\
&\equiv \boldsymbol{(p \wedge \neg q) \vee (q \wedge \neg p)} \\
&\equiv \boldsymbol{((p \wedge \neg q) \vee q) \wedge ((p \wedge \neg q) \vee \neg p)} \\
&\equiv \boldsymbol{((\mathrm{p} \vee q) \wedge (\neg q \vee q)) \wedge ((p \vee \neg p) \wedge (\neg q \vee \neg p))} \\
&\equiv \boldsymbol{((\mathrm{p} \vee q) \wedge \mathrm{T}) \wedge (\mathrm{T} \wedge (\neg q \vee \neg p))} \\
&\equiv \boldsymbol{(\mathrm{p} \vee q) \wedge (\neg q \vee \neg p)} \\
&\equiv \boldsymbol{(\neg p \to q) \wedge (\neg q \to p)} \\
&\equiv \boldsymbol{\neg p \leftrightarrow q} \qquad \textbf{:)}
\end{aligned}$$

4. (1.3.35) Show that $(p \to q) \to r \neq p \to (q \to r)$.

p	q	r	$p \to q$	$(p \to q) \to r$	$q \to r$	$p \to (q \to r)$
T	T	T	T	**T**	T	**T**
T	T	F	T	**F**	F	**F**
T	F	T	F	**T**	T	**T**
T	F	F	F	**T**	T	**F**
F	T	T	T	**T**	T	**T**
F	T	F	T	**F**	F	**T**
F	F	T	T	**T**	T	**T**
F	F	F	T	**F**	T	**T**

5. (1.3.36) Show that $(p \wedge q) \rightarrow r \neq (p \rightarrow r) \wedge (q \rightarrow r)$.

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$
T	T	T	T	**T**	T	T	**T**
T	T	F	T	**F**	F	F	**F**
T	F	T	F	**T**	T	T	**T**
T	F	F	F	**T**	F	T	**F**
F	T	T	F	**T**	T	T	**T**
F	T	F	F	**T**	T	F	**F**
F	F	T	F	**T**	T	T	**T**
F	F	F	F	**T**	T	T	**F**

1. (1.4.5) Let $P(x)$ be the predicate:
   - $P(x)$: "Student x spends more than five hours every weekday in class."

   … where the domain for x consists of all students at some university. Express each of the following quantified predicates as an English sentence.

   (a) $\exists x\,(P\,(x))$
   There exists a student at some university who spends more than five hours every weekday in class.

   (b) $\forall x\,(P\,(x))$
   All students spend more than five hours every weekday in class at some university.

   (c) $\exists x\,(\neg P\,(x))$
   There exists a student at some university who doesn't spend more than five hours every weekday in class.

   (d) $\forall x\,(\neg P\,(x))$
   No students spend more than five hours every weekday in class at some university.

2. (1.4.9) Let $P(x)$ and $Q(x)$ be the predicates:
   - $P(x)$: "Person x can speak Russian."
   - $Q(x)$: "Person x knows the computer language C++."

… where the domain for x consists of all people whom you know. Write each of the following quantified predicates using $P(x)$, $Q(x)$, quantifiers, and logical operators.

(a) You know someone who can speak Russian and knows C++.
$\boldsymbol{\exists x\,(P\,(x) \wedge Q(x))}$

(b) You know someone who can speak Russian but doesn't know C++.
$\boldsymbol{\exists x\,(P\,(x) \wedge \neg\, Q(x))}$

(c) Everyone you know can speak Russian or knows C++.
$\boldsymbol{\forall x\,(P\,(x) \vee Q(x))}$

(d) Nobody you know can speak Russian or knows C++.
$\boldsymbol{\forall x\,(\neg\,(P\,(x) \vee Q(x)))}$

3. (1.4.17) Let the domain x consist of the integers 0, 1, 2, 3, and 4. Rewrite each of the following quantified statements using only $P(x)$, disjunctions, conjunctions, and negations.
   (a) $\forall x\ (P\ (x))$
   $P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4)$
   (b) $\exists x\ (\neg P\ (x))$
   $\neg P(0) \vee \neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4)$

4. (1.4.15) Let the domain for n consist of all integers. Determine whether each of the following quantified predicates is true or false:
   (a) $\forall n\ (n^2 \geq 0)$ **T**
   (b) $\forall n\ (n^2 \geq n)$ **T**
   (c) $\exists n\ (n^2 = 2)$ **F**
   (d) $\exists n\ (n^2 < 0)$ **F**

5. (1.5.27) Let the domain n consist of all integers. Determine whether each of the following quantified predicates is true or false:
   (a) $\forall\ n\ \exists\ m\ (n^2 < m)$ **T**
   (b) $\exists\ m\ \forall\ n\ (n^2 < m)$ **F**
   (c) $\exists\ n\ \forall\ m\ (nm = m)$ **T**
   (d) $\exists\ n\ \exists\ m\ (n^2 + m^2 = 5)$ **T**

6. (1.7.1) Using a direct proof, prove the following statement:
   Let a and b be integers. If a and b are odd, then $a + b$ is even.

By definition, there exists an integer k such that a = 2k + 1, and m such that b = 2m + 1. Then a + b = 2k + 1 + 2m + 1 = 2k + 2m + 2 = 2(k + m + 1). Let some integer y = k + m + 1, then a + b = 2y.

Therefore, if a and b are odd, a + b is even.

7. (1.7.13) Using a proof by contraposition, prove the following statement:
   Let x be real. If x is irrational, then $1/x$ is irrational.

Assume x is irrational and 1/x is not irrational. If 1/x is not irrational then 1/x = some integers a/b, where b ≠ 0. So, x = 1/(1/x) = 1/(a/b) = b/a, which leads to x being rational.

Therefore, if x is irrational, then 1/x is irrational.

8. (1.7.16) Consider the following statement:
   Let x, y, and z be integers. If $x + y + z$ is odd, then at least one of x, y, or z is odd.
   (a) Which proof technique should be used to prove the above statement? Briefly explain your answer. **Contradiction, because to assume x, y, or z is odd, is to assume that they are all even.**
   (b) Prove the above statement.

Assume x, y, and z are all even. By definition, if all of them are even then x = 2k, y = 2m, and z = 2w. Then x + y + z = 2k + 2m + 2w = 2(k + m + w) which, by definition, is even. This shows that if all the integers are even, then the result is cannot be odd.

Therefore, if x + y + z is odd, then at least one of them has to be odd.

Homework 3

Homework 3 – Sets

Due: Thursday, January 28th

1. (1.7.20) Consider the following statement:
Let n be an integer. If $3n + 2$ is even, then n is even.

(a) Using a proof by contraposition, prove the above statement.
Proof: Let *n* be an integer that is odd. By definition, n = 2k + 1
Then, 3(2k + 1) + 2 = 6k + 3 + 2 = 6k + 4 + 1 = 2(3k + 2) + 1. Some integer m = 3k + 2 s.t. 3n + 2 = 2m + 1, which means 3n + 2 is odd.
Therefore, by contraposition, if 3n + 2 is even, then n is even.

(b) Using a proof by contradiction, prove the above statement.
Proof: Suppose by contradiction, that 3n+2 is even, and n is odd. By definition, 3n+2 = 3(2k + 1)+2 = 6k + 3 + 2 = 6k + 4 + 1 = 2(3k+2) +1. m = 3k+2, 3n+2 = 2m+1, which means 3n+2 is odd, and even. Therefore, by contradiction, if 3n+2 is even, then n is even.

2. (1.7.24) Using the notion of "without loss of generality," prove the following statement:
If there are 3 socks, either blue or black, then there must be at least 2 of the same color.
Theorem: If there exist 3 socks, either blue or black, then there must be at least 2 socks of the same color.
Sketch:
blue, blue, blue; blue, blue, black; blue, black, blue, blue, black, black.
"Without loss of generality," suppose the first sock is blue. Using the four cases above, in all possible cases, there are at least 2 socks of the same color. Therefore, if there exists 3 socks, either black or blue, then there must be at least 2 socks of the same color.

3. (1.8.8) Using the notion of "without loss of generality", prove the following statement:
Let x and y be integers. If x and y have opposite parity, then $5x + 5y$ is odd.
"Without loss of generality", assume x is odd and y is even. Then x = 2k + 1 and y = 2m, then 5x + 5y = 5(2k + 1) + 5(2m) = 10k + 1 + 10m = 2(5k + 5m) + 1, which by definition is odd.
Therefore, is x and y have opposite parity, then 5x + 5y is odd.

4. (1.8.10) Consider the following statement:
There exists a positive integer x such that $x = E^{x-1}{}_{i=1}\ i$.

(a) Prove the above statement.
For example, 3. The sum of 2 + 1 is 3. Therefore, there exists a positive integer x such that $x = E^{x-1}{}_{i=1}$ *i*.

(b) Is your proof **constructive** or non-constructive?
constructive

5. (1.8.12) Consider the following statement:

Let a, b, and c be real numbers. There exists a pair of a, b, or c whose product is non-negative.

(a) Prove the above statement.

Cases: a+ b+ c+, a+ b+ c-, a+ b- c-. Without loss of generality, assume a is non-negative. Using the cases above, in all possible cases there is always a pair of either non-negative or negative. Every real number is either negative or non-negative.
In all possible cases, there are at least two numbers with the same sign. If they have same sign, product must be non-negative.

(b) Is your proof constructive or **non-constructive**?

Non-constructive

6. (2.1.1) List the members of the following sets:

(a) $\{x \in R \mid x^2 = 1\}$ **{-1, 1}**

(b) $\{x \in Z^+ \mid x \leq 12\}$ **{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}**

(c) $\{x \in R \mid \sqrt{x} \in Z \land x < 100\}$ **{0, 1, 4, 9, 16, 25, 36, 49, 64, 81}**

(d) $\{x \in Z \mid x^2 = 2\}$ **∅**

7. (2.1.10) Determine whether each of the following statements is true or false:

(a) $\{2\} \in \{x \in Z \mid x > 1\}$ **F**

(b) $\{2\} \in \{2, \{2\}\}$ **T**

(c) $\{2\} \in \{\{2\}, \{2, \{2\}\}\}$ **T**

(d) $\{2\} \in \{\{\{2\}\}\}$ **F**

8. (2.1.11) Determine whether each of the following statements is true or false:

(a) $0 \in \emptyset$ **F**

(b) $\emptyset \subseteq \{0\}$ **T**

(c) $\{0\} \in \{0\}$ **F**

(d) $\{\emptyset\} \subseteq \{\emptyset\}$ **T**

9. (2.2.1) Let A and B be the sets:

- A: The set of students who live within one mile of school.
- B: The set of students who walk to class.

… describe each of the following sets as an English phrase:

(a) $A \cap B$

The set of students who live within one mile of school and who walk to class.

(b) $A \cup B$

The set of students who live within one mile of school and/or walk to class.

(c) $A - B$

The set of students who live within one of school but do not walk to class.

(d) $B - A$

The set of students who walk to class but do not live within one mile of school.

10. (2.1.20) Find two sets, A and B, such that $A \in B$ and $A \subseteq B$.

A: {2, 4} **B: {2, 4, {1, 5, 6}}**

11. (2.2.14) Find two sets, A and B, such that $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

A: {1, 5, 7, 8, 3, 6, 9} **B: {2, 10, 3, 6, 9}**

12. (2.2.16) Consider the following statement:

Let A and B be sets. Then $(A \cap B) \subseteq A$.

(a) Draw a Venn diagram to illustrate the above statement.

(b) Prove the above statement.

Let $x \in A \cap B$. By definition, x is $x \in A \wedge x \in B$. So, x is an element of A, $x \in A$.

Therefore, $(A \cap B) \subseteq A$.

13. (2.2.16) Consider the following statement:

Let A and B be sets. Then $A \cap (B - A) = \emptyset$.

(a) Draw a Venn diagram to illustrate the above statement.

(b) Prove the above statement.

Let $x \in A \cap (B - A)$. By definition, $x \in A \wedge x \in B - A$. By definition of difference, $x \in A \wedge (x \in B \wedge \neg (x \in A)) = x \in A \wedge (\neg(x \in A) \wedge x \in B) = (x \in A \wedge \neg(x \in A)) \wedge x \in B$ which is always false. So, $A \cap (B - A) \subseteq \emptyset$.

Therefore, $A \cap (B - A) = \emptyset$.

Homework 4

1. (2.2.20) Prove the following statement:

   Let A, B, and C be sets. Then $(A \cup B) \subseteq (A \cup B \cup C)$.

Proof: Let A ∪ B be a set such that $x \in (A \cup B)$. By definition, $x \in A \vee x \in B$, which means that x is in A or x is in B. Because this is a disjunction, adding in that x is an element of C does not change the truth value. So, x is an element of A or x is an element of B or x is an element of C is still true.

Therefore, $(A \cup B) \subseteq (A \cup B \cup C)$.

2. (2.2.20) Prove the following statement:

   Let A, B, and C be sets. Then $(A - C) \cap (C - B) = \emptyset$

Proof: Assume, for contradiction, the set is not empty and contains an element x. This leads to $x \in (A - C) \cap (C - B)$. By definition, x is an element that is in A but not C AND an element that is in C but not B; $\{x \mid x \in A \wedge x \in \neg C \wedge x \in C \wedge x \in \neg B\}$. This means that x is both not in C and in C.

Therefore, by contradiction, $(A - C) \cap (C - B) = \emptyset$

3. (2.2.53) Let $\mathscr{A} = \{A_1, A_2, A_3, \dots, A_n\}$, where $A_1 = \{1\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2, 3\}$, and $A_n = \{1, 2, 3, \dots, n\}$. Find the following sets:
   (a) $\cup A_{A \in \mathscr{A}}$ **$\{1, 2, 3, \dots, n\}$**
   (b) $\cap A_{A \in \mathscr{A}}$ **$\{1\}$**

4. (2.1.34) Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find the following sets:
   (a) $(A \times B) \times C$ **{(a, x, 0), (a, x, 1), (a, y, 0), (a, y, 1), (b, x, 0), (b, x, 1), (b, y, 0), (b, y, 1), (c, x, 0), (c, x, 1), (c, y, 0), (c, y, 1)}**
   (b) $(C \times B) \times A$ **{(0, x, a), (0, x, b), (0, x, c), (0, y, a), (0, y, b), (0, y, c), (1, x, a), (1, x, b), (1, x, c), (1, y, a), (1, y, b), (1, y, c)**

5. (2.1.28) Prove the following statement:

   Let A, B, C, and D be sets. If $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

Proof: By definition, $A \subseteq C$ means that every element in A is in C; $\forall x\,(x \in A \rightarrow x \in C)$. And $B \subseteq D$ means that every element of B is in D; $\forall y\,(y \in B \rightarrow y \in D\}$. $A \times B$ is a subset of pairs with an element from A being the first item, and an element from B being the second; $\{(x, y) \mid x \in A \text{ and } y \in B\}$. Because every element from A is in C, and every element from B is in D, every pair that exists in A × B also exists in C × D, which, by definition, make A × B a subset of C × D. $\{(x, y) \mid x \in A \wedge x \in C \text{ and } y \in B \wedge y \in D\}$

Therefore, $A \times B \subseteq C \times D$.

6. (2.1.33) Prove the following statement:

Let A be a set. Then $\emptyset \times A = \emptyset$.

By definition, $\emptyset \times A = \{(x, y) \mid x \in \emptyset \text{ and } y \in A\}$. This means that x is an element from the empty set, however, because there are no elements in the empty set then nothing is produced. This leads to an incomplete (x, y).

Therefore, $\emptyset \times A = \emptyset$.

7. (2.3.1 and 2.3.2) Determine whether each of the following is a function:
   (a) $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1/x$ **not a function, *f* is undefined at 0**
   (b) $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \pm\sqrt{(x^2 + 1)}$ **no, *f* assigns more than one value to each value of x**
   (c) $f: \mathbb{Z} \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ **yes**
   (d) $f: \mathbb{Z} \to \mathbb{R}$ given by $f(x) = (x^2 - 2)^{-1}$ **yes**

8. (2.3.12 and 2.3.13) Determine whether each of the following is injective, surjective, both, or neither:
   (a) $f: Z \to Z$ given by $f(n) = n - 1$ **bijective**
   (b) $f: Z \to Z$ given by $f(n) = n^2 + 1$ **injective**
   (c) $f: Z \to Z$ given by $f(n) = n^3$ **surjective**
   (d) $f: Z \to Z$ given by $f(n) = [n/2]$ **injective**

9. (2.3.17 and 2.3.19) Let A and B be the sets:
   - A: The set of teachers employed by a school
   - B: The set of offices in (that same) school

   … describe each of the following situations in English:

   (a) A function $f: A \to B$ is an injection.

   No teachers employed by a school share an office in school.

   (b) A function $f: A \to B$ is a surjection.

   Every office in school is used by one or more teachers.

10. (2.3.28) Let f be the function:
    - $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$

    A function is said to be *invertible* if its inverse can be defined.

    (a) Explain why f *is not* invertible. **It is not a bijection.**
    (b) Find an alternative domain or codomain such that f *is* invertible. **f: $\mathbf{Q} \to \mathbf{R}_{>0}$**

11. (2.3.34) Let $f : B \rightarrow C$ and $g : A \rightarrow B$ be functions.
    (a) Prove or disprove the following statement:
    If $f \circ g : A \rightarrow C$ is an injection, then $g : A \rightarrow B$ is also an injection.

Proof: If $f \circ g$: $A \rightarrow C$ is an injection then $((x, y) \in f \circ g \wedge (z, y) \in f \circ g) \rightarrow x = z$. By definition, there exists some $x, z \in A$ and $y \in C$ such that $f \circ g(x) = y$ and $f \circ g(z) = y$. Because g: $A \rightarrow B$ is a function, for every input there is an output, so by definition there exists a $w \in B$ such that $g(x) = w$ and $f(w) = y$, and there exists a $v \in B$ such that $g(z) = v$ and $f(v) = y$. So $B =\{w, v\}$, $A = \{x, z\}$, and $C = \{y\}$. The outputs in B each only take one input from A, x or z.

Therefore, g: $A \rightarrow B$ is also an injection.

    (b) Prove or disprove the following statement:
    If $f \circ g : A \rightarrow C$ is an injection, then $f : B \rightarrow C$ is also an injection.

Proof: If $f \circ g$: $A \rightarrow C$ is an injection then $((x, y) \in f \circ g \wedge (z, y) \in f \circ g) \rightarrow x = z$. Using the same reasoning as above, because $f(v) = z = f(w)$, $v = w$.

Therefore, $f : B \rightarrow C$ is also an injection.

12. (2.3.72) Prove the following statement:
    Let $f : B \rightarrow C$ and $g : A \rightarrow B$ be bijections. Then $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

    [Hint: Recall that a function is itself a set; this is a proof of set equality.]

Proof: Since $f : B \rightarrow C$ and $g : A \rightarrow B$ are bijective, then there exists an $x \in A$ such that $g(x) = y$ where $y \in B$, and $f(y) =$ where $z \in C$, this then makes $f \circ g$: $A \rightarrow C$ bijective. By definition, $(f \circ g)^{-1} (z) = x$. $g^{-1} \circ f^{-1} = g^{-1}(f^{-1}(z))$. Broken down, $f^{-1}(z) = y$ because it is a bijection. Then $g^{-1}(y) = x$, and $x = x$.

Therefore, $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

Homework 5

Homework 5 – Mathematical Induction

Due: Thursday, February 11th

1. (2.3.34) Let $f : B \to C$ and $g : A \to B$ be functions.
   (a) Prove or disprove the following statement:
   If $f \circ g : A \to C$ is a surjection, then $g : A \to B$ is also a surjection.

Proof: If $f \circ g : A \to C$ is a surjection, then ∀ y ∈ C(∃ x ∈ A((x, y) ∈ f)). By definition, there exists some z ∈ C that is a result of some x ∈ A, such that $f \circ g$(x) = z. If this is a surjection, then all outputs have an input, if this is true from A → C then this must be true from A → B because there must be some input there that produces an output, hence, all outputs being produced.

Therefore, if $f \circ g : A \to C$ is a surjection, then $g : A \to B$ is also a surjection.

   (b) Prove or disprove the following statement:
   If $f \circ g : A \to C$ is a surjection, then $f : B \to C$ is also a surjection.

Proof: If $f \circ g : A \to C$ is a surjection, then ∀ y ∈ C(∃ x ∈ A((x, y) ∈ f)). By definition, there exists some z ∈ C that is a result of some x ∈ A, such that $f \circ g$(x) = z. If this is a surjection, then all outputs have an input. This does not guarantee that f is also a surjection because value y in B that doesn't have an input in C, but that does have an input when $f \circ g$.

Therefore, if $f \circ g : A \to C$ is a surjection, then $f : B \to C$ is not always a surjection.

2. (2.5.1 and 2.5.2) Determine whether each of the following sets is finite, countably infinite, or uncountable.

(a) The set of integers less than 100	**countably infinite**
(b) The set of real numbers between 0 and ½	**uncountable**
(c) The set of positive integers less than 1000000000	**finite**
(d) The set $\{2, 3\} \times Z^+$	**countably infinite**

3. (2.5.10) Find the two uncountable sets, A and B, such that $A - B$ is finite.

A = {1, 2} U Z^+	**B = {1} U Z^+**
A = {7, 8, 9} U Z^+	**B = {8} U Z^+**

4. (2.5.11) Find two uncountable sets, A and B, such that $A \cap B$ is countably infinite.
   A = {1, 2} U Z^+ and B = {3, 4} U Z^+
   A = {5, 6} U Z^+ and B = {7, 8} U Z^+

5. (5.1.3) Let $P(n)$ be the statement:
   - $P(n)$: "For all positive integers n, $1^2 + 2^2 + \ldots + n^2 = n(n + 1)(2n + 1) / 6$."

   Consider proving the above statement by induction:

   (a) Show $P(1)$, completing the basis step.

Consider n = 1. Then we have: $1^2 = (1)(2)(3)/6 = 1$

(b) Assume, for some integer $k \geq 1$, $P(k)$, completing the inductive hypothesis.
Suppose that for some $k \in N_{>0}$, $(1^2 + 2^2 + \ldots + k^2) = k(k + 1)(2k + 1) / 6$

(c) Show $P(k + 1)$, completing the inductive step (and, thus, the proof).
Let n = k + 1. Then we have: $(1^2 + 2^2 + \ldots + k^2 + (k + 1)^2)$
By hypothesis, this equals $k(k + 1)(2k + 1) / 6 + (k+1)^2 = 6(k+1)^2 + k(k+1)(2k+1)/6$
$=(k + 1)(2k\text{^}2 + 7k + 6)/6 = (k + 1)(k + 2)(2k + 3)/6$
Therefore, by the PMI, for all positive integers n, $1^2 + 2^2 + \ldots + n^2 = n(n + 1)(2n + 1) / 6$

6. (5.1.6) Using a proof by induction, prove the following statement:
For all positive integers n, $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n + 1)! - 1$.

<u>Basis Step:</u> Let n = 0. Then we have $0 \cdot 0! = (0 + 1)! - 1 = 0$.

<u>Inductive Hypothesis</u>: Suppose, for some $k \in N_{>0,}$ $1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! = (k + 1)! - 1$.

<u>Inductive Step</u>: Let n = k + 1. Then we have $1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! + (k + 1) \cdot (k + 1)!$

By hypothesis, this equal: $(k + 1)! - 1 + (k + 1) \cdot (k + 1)!$

Therefore, by PMI, for all positive integers n, $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n + 1)! - 1$.

7. (5.1.37) Using a proof by induction, prove the following statement:
For all positive integers n, $133 \mid (11^{n+1} + 12^{2n-1})$.

<u>Basis Step:</u> Let n = 1. Then we have $11^{1+1} + 12^{2-1} = 121 + 12 = 133$, and certainly $133 \mid 133$

<u>Inductive Hypothesis</u>: Suppose, for some k, $133 \mid (11^{k+1} + 12^{2k-1})$.

<u>Inductive Step:</u> Let n = k + 1. Then we have: $11^{k+1+1} + 12^{2(k+1)-1} = 11 \cdot 11^{k+1} + 12^2 \cdot 12^{2k-1}$

$= 11 \cdot 11^{k+1} + (11 + 133) \cdot 12^{2k-1} = 11(11^{k+1} + 12^{2k-1}) + 133 \cdot 12^{2k-1}$. By hypothesis, the first part is divisible by 133 and 133 is divisible by 133.

Therefore, by PMI, for all positive integers n, $133 \mid (11^{n+1} + 12^{2n-1})$

8. (5.1.52) Using a proof by induction, prove the following statement:
Let A and B be sets such that $|A| > |B|$. Then any function $f : A \to B$ is not injective.

[Hint: Note that $|B| \geq 1$, else f would not be a function. Then, since $|A| > |B|$, $|A| \geq 2$.]

<u>Basis Step:</u> Let the cardinality of A = 2. Then the cardinality of B is one. This means that the two inputs in A produce the one output in B. Certainly this is not injective.

<u>Inductive Hypothesis</u>: Suppose for A that the cardinality is k. Then B's is at most k – 1. There is then one more input than output meaning at least one output has multiple inputs.

<u>Inductive Step</u>: Let n be the cardinality of A and n = k + 1. Then the cardinality of B is at most k. This means that there is one more input than there are outputs, meaning that at least one output has multiple inputs.

Therefore, if $|A| > |B|$. Then any function $f : A \to B$ is not injective.

9. (5.1.49) Consider the following "proof":
   Theorem:
   All horses have the same color.
   Proof:
   - Consider the statement "All horses in a set of n horses have the same color."
   - *Basis Step*:
     - Consider $n = 1$.
     - Certainly, a set of one horse contains only horses of the same color.
   - *Inductive Hypothesis*:
     - Suppose that, for some $k \in N$, all horses in a set of k horses have the same color.
   - *Inductive Step*:
     - Consider $n = k + 1$. Let a set of $k + 1$ horses be denoted $H = \{h_1, h_2, \ldots, h_k, h_{k+1}\}$.
     - Two subsets of H are $H_1 = \{h_1, h_2, \ldots, h_k\}$ and $H_2 = \{h_2, \ldots, h_k, h_{k+1}\}$.
     - Then $|H_1| = k$ and $|H_2| = k$, thus, by the hypothesis, all the horses in H_1 have the same color and all the horses in H_2 have the same color.
     - Note that h_2, for instance, is an element of both H_1 and H_2.
     - Then h2 is the same color as all the horses in H1 and all those in H_2. h_2 cannot be two different colors, thus, the color of all the horses in H_1 and the color of all the horses in H_2 must be the same color
     - Thus, all the horses in H have the same color.
   - Therefore, by the Principle of Mathematical Induction, all horses have the same color.

What is the error in this "proof"?

The sets won't overlap if n + 1 = 2.

10. (5.1.78) A *triomino* is an 'L' shape formed by three 1 × 1 squares:
    (a) By example, show that a 4 × 4 checkerboard with one 1 × 1 square removed can be covered with non-overlapping triominoes.

X	X	X	X
X	X		X
X	X	X	X
X	X	X	X

    (b) By example, show that an 8 × 8 checkerboard with one 1 × 1 square removed can be covered with non-overlapping triominoes.

X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X

X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	

(c) By induction, show that, for all positive integers n, every $2^n \times 2^n$ checkerboard with one 1×1 square removed can be covered with non-overlapping triominos

<u>Basis Step:</u> Let n = 1. Then we have $2^1 \times 2^1 = 2 \times 2$, which with one square removed is by definition, a triomino.

<u>Inducive Hypothesis</u>: Suppose, for some k, $2^k \times 2^k$, with one square removed can be covered with non-overlapping triominos.

<u>Inductive Step</u>: Let n = k + 1. Then we have $2^{k+1} \times 2^{k+1} = 2*2^k \times 2*2^k$. By hypothesis, this with one square removed can be covered with non-overlapping triominos.

(d) **Therefore, by PMI, for all positive integers n, every $2^n \times 2^n$ checkerboard with one 1×1 square removed can be covered with non-overlapping triominos.**