Claire Minahan

Midterm Exam

1. Consider the following propositions:
   - $\neg [p \rightarrow (q \wedge r)]$ $\qquad$ $(p \wedge \neg q) \vee (p \wedge \neg r)$

   (a) Use a truth table to show that these propositions are logically equivalent.

p	q	r	$q \wedge r$	$\neg [p \rightarrow (q \wedge r)]$	$p \wedge \neg q$	$p \wedge \neg r$	$(p \wedge \neg q) \vee (p \wedge \neg r)$
T	T	T	T	**F**	F	F	**F**
T	T	F	F	**T**	F	T	**T**
T	F	T	F	**T**	T	F	**T**
T	F	F	F	**T**	T	T	**T**
F	T	T	T	**F**	F	F	**F**
F	T	F	F	**F**	F	F	**F**
F	F	T	F	**F**	F	F	**F**
F	F	F	F	**F**	F	F	**F**

   (b) What can we then conclude about the following proposition?

$$\neg [p \rightarrow (q \wedge r)] \leftrightarrow (p \wedge \neg q) \vee (p \wedge \neg r)$$

We can then conclude that this proposition is always true because they are logically equivalent.

2. Let the domain for x and y be Z, the set of integers.

   (a) Determine the truth value of $\forall x \exists y\ (y = x/2)$. Briefly explain your answer.
   This is equivalent to the English statement: "For all values of x, there exists some y such that y = x/2." This is not true, therefore the truth value is FALSE. For example, if x = 5, then there is no y = 5/2 that is an integer, therefore not all values of x satisfy the statement.

   (b) Determine the truth value of $\exists y \forall x\ (y = x/2)$. Briefly explain your answer.
   This is equivalent to the English statement: "There exists some y such that for all value of x, y = x/2." This is not true, therefore the truth value is FALSE. For example, if y = 3, not all x will produce 3, only the integer 6.

3. Using a proof by contraposition, prove the following statement:
   Let a, b, c be integers. If a does not divide b + c, then a does not divide b or a does not divide c.

<u>Proof</u>: Let a, b, c be integers, where a does not divide b + c and does and a | b and a | c. By definition, there exists some integer k such that b = ak and some integer m such that c = am. If this is true, then b + c = ak + am = a(k + m). Thus, by definition a | b+c.

Therefore, by contraposition, if a does not divide b + c, then a does not divide b or a does not divide c. □

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4. Consider the following statement:

$\sqrt[3]{2}$, the cube root of 2, is irrational.

(a) Using a proof by contradiction, prove that if n^3 is even, then n is even, where n is an integer.

Assume, for contradiction, that n is odd and n^3 is even. By definition with some integer k, n = 2k + 1. $n^3 = (2k + 1)^3 = (2k + 1)(2k+1)(2k+1) = (4k^2 + 4k + 1)(2k + 1) = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$. Let m be some integer such that $m = 4k^3 + 6k^2 + 3k$, therefore, the above is 2m + 1. By definition this is odd, so n^3 is both even and odd.

Therefore, by contradiction, if n^3 is even, then n is even, where n is an integer. □

(b) Using a proof by contradiction, prove that $\sqrt[3]{2}$, the cube root of 2, is irrational.

Proof: Suppose, for contradiction, that $\sqrt[3]{2}$ is rational. By definition $\sqrt[3]{2} = a/b$, where a and b are integers, $b \neq 0$ and gcd(a, b) = 1. Then $2 = (a/b)^3$; thus $2b^3 = a^3$. By definition a^3 is even, and if this is true, then a is even see above proof). Then a is even and there exists an integer k such that a = 2k. Then $2b^3 = (2k)^3 = 8k^3$, then $b^3 = 4k^3 = 2(2k^3)$. Then b^3 is also eve, thus b is even. Since a and b are both even, gcd(a, b) greater than or equal to 2. This contradicts the definition of $\sqrt[3]{2}$ being rational. Therefore, by contradiction, $\sqrt[3]{2}$ is irrational. □

5. Let the universe be Z and let:

$A = \{1, 2, 4, 8\}$, $B = \{-2, -1, 0, 1, 2\}$, and $C = \{8, 9, 10\}$

(a) Find $(\overline{A \cup C}) \cap B$.

{-2, -1, 0}

(b) Find $C \times A$

{(8, 1), (8, 2), (8, 4), (8, 8), (9, 1), (9, 2), (9, 4), (9, 8), (10, 1), (10, 2), (10, 4), (10, 8)}

6. Let A and B be sets. Prove the following statement:

$A \cap (\overline{A \cup B}) = \emptyset$

Proof: Assume, for contradiction, the set is not empty and contains an element x. This leads to $x \in A \cap (\overline{A \cup B})$. By definition, this leads to x being an element that is in both sets, A and $(\overline{A \cup B})$. This leads to $\{x \mid x \in A \land \neg (x \in B \land A)\}$. This means that x is both in A and not in A.

Therefore, by contradiction, $A \cap (\overline{A \cup B}) = \emptyset$ □

7. Let A, B, and C be sets. Prove the following statement:

If $A \subseteq B$ or $A \subseteq C$, then $A \subseteq B \cup C$

Proof: Let a be a set such that $x \in A$. By definition, if $A \subseteq B$, then $\forall x (x \in A \rightarrow x \in B)$, meaning $x \in B$. If $A \subseteq C$, then by definition $\forall x (x \in A \rightarrow x \in C)$, meaning $x \in C$; then $x \in C \lor x \in B$. By definition, A is a subset of the union of B and C.

Therefore, $A \subseteq B$ or $A \subseteq C$, then $A \subseteq B \cup C$. □

8. Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be functions.
   (a) What are the domain and codomain of $g \circ f$, the composition of g and f?

   Domain: set A. Codomain: set A. Composition of g and f: $g \circ f$

   (b) Prove or disprove the following statement:
   If $g \circ f$ is an injection, then g is also an injection.

Proof: If $g \circ f$ is an injection, then by definition each output may not have multiple inputs. Suppose there is are elements $x, z \in A$ and $y \in B$ and that $f(x) = y$ and $f(z) = y$ and $g(y) = x$. $g \circ f = g(f(x)) = g(y) = x$. $g \circ f = g(f(z)) = g(y) = w$. If both f(x) and f(z) = y, then z = x and there are not multiple inputs for one output.

Therefore, if $g \circ f$ is an injection, then g is also an injection. □

   (c) Prove or disprove the following statement:
   If $g \circ f$ is a surjection, then g is also a surjection.

Proof: if $g \circ f$ is a surjection, then all possible outputs must have some input. Suppose there are elements $x \in A$ and $y \in B$ where $f(x) = y$ and $g(y) = x$. Then $g \circ f = g(f(x)) = g(y) = x$. Since $g(y) = x$, there is some input for the output which by definition is a surjection.

Therefore, if $g \circ f$ is a surjection, then g is also a surjection. □

9. Recall the following statement:
   The set of natural numbers, N, is countably infinite.

   Briefly explain how — if at all — this statement can be proven.

Let some infinite set be a subset of N. Then this subset is countably infinite. Prove it is a bijection which then also make N countably infinite

10. Consider the following statement:
    For all positive integers n, $133 \mid (11^{n+1} + 12^{2n+1})$.

    In order to prove this by induction…

    (a) …what must be shown in the Basis Step?

Let n = 1. Then we have $11^{1+1} + 12^{2-1} = 121 + 12 = 133$, and certainly $133 \mid 133$

    (b) …what can be assumed for the Inductive Hypothesis?

Suppose, for some k, $133 \mid (11^{k+1} + 12^{2k-1})$.

    (c) …what must then, given the hypothesis, be shown in the Inductive Step?

Let n = k + 1. Then we have: $11^{k+1+1} + 12^{2(k+1)-1} = 11 \cdot 11^{k+1} + 12^2 \cdot 12^{2k-1} = 11 \cdot 11^{k+1} + (11 + 133) \cdot 12^{2k-1} = 11(11^{k+1} + 12^{2k-1}) + 133 \cdot 12^{2k-1}$. By hypothesis, the first part is divisible by 133 and 133 is divisible by 133.

Therefore, by PMI, for all positive integers n, $133 \mid (11^{n+1} + 12^{2n-1})$

(d) …what axiom allows us to complete this proof?

The axiom that $x^{a+b} = x^a \cdot x^b$

11. Suppose that the following propositions are known to be equivalent:

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Consider, then, the following generalization, for all integers $n \geq 1$:

$$p \vee (q_1 \wedge q_1 \wedge \ldots \wedge q_n) \equiv (p \vee q_1) \wedge (p \vee q_2) \wedge \ldots \wedge (p \vee q_n)$$

Using a proof by induction, prove the latter of the above statements.

Basis Step: Let n = 1. Then we have: $p \vee q_1 \equiv (p \vee q_1)$, and certainly two equivalent propositions are equal.

Inductive Hypothesis: Suppose, for some k, $p \vee (q_1 \wedge q_1 \wedge \ldots \wedge q_k) \equiv (p \vee q_1) \wedge (p \vee q_2) \wedge \ldots \wedge (p \vee q_k)$.

Inductive Step: Let n = k + 1. We then have $p \vee (q_1 \wedge q_1 \wedge \ldots \wedge q_k \wedge q_{k+1})$. Using the known proposition above, $p \vee (\wedge\, q_{k+1}) = p \vee q_{k+1}$. By hypothesis this equals $(p \vee q_1) \wedge (p \vee q_2) \wedge \ldots \wedge (p \vee q_k) \wedge (p \vee q_{k+1})$

Therefore, by PMI, $p \vee (q_1 \wedge q_1 \wedge \ldots \wedge q_n) \equiv (p \vee q_1) \wedge (p \vee q_2) \wedge \ldots \wedge (p \vee q_n)$. □