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Quiz 5 – Recursive Definitions

1. Solve each of the following recurrence relations:
   [*Note*: You *must* use the iterative method and show all of your work.]
   (a) $a_n = 2a_{n-1} - 3$, where $a_0 = -1$
   $a_n = 2a_{n-1} - 3$
   $= 2*2a_{n-2} - 3 - 3$
   $= 2*2*2a_{n-2} - 3 - 3 - 3$
   …
   $= 2^n a_0 - 3n$
   $= 2^n(-1) - 3n$

   (b) $a_n = (n + 1)\, a_{n-1}$, where $a_0 = 2$
   $a_n = (n + 1)\, a_{n-1}$
   $= (n+1)*(n)*a_{n-2}$
   $= (n+1)*(n)*(n-1)*a_{n-2}$
   …
   $= (n+1)!\, a_0$
   $= 2*(n + 1)!$

2. Recall that every recursive definition consists of a *Basis Step* and a *Recursive Step*.
   Give a recursive definition for each of the following sets.
   (a) The set of even integers.
   ***Basis Step*: $0 \in A$**
   ***Recursive Step*: if $x \in A$ then $(x + 2) \in A$ and $(x - 2) \in A$**

   (b) The set of positive integers not divisible by 5.
   ***Basis Step*: 1, 2, 3, 4 $\in$ A**
   ***Recursive Step*: if $x \in A$ then $(x + 5) \in A$**

3. Suppose that a rectangular chocolate bar consists of *n* individual squares.
Further suppose that the bar — or any smaller, rectangular, piece of the bar — can *only* be broken into two pieces along the horizontal or vertical lines separating its squares.
For example, one possible rectangular bar of 8 squares is that with dimensions 4×2. One possible way to break this bar is as follows:

… producing one bar of 2 squares and one bar of 6. Prove the following generalization:

A bar of *n* squares requires $n - 1$ breaks to be broken down into its constituent squares.

Proof:

***Basis Step*:**

Let n = 1. Then 1-1 = 0 and certainly if there is only one square then it will take 0 breaks to get it into 1 square.

***Inductive Hypothesis*:**

Suppose for all integers, $1 \le n \le k$, a bar of n squares requires n – 1 breaks to be broken down into its constituent squares.

***Inductive Step*:**

Let n = k + 1. Let x and y be some integers such that k +1 = x*y, where x and y are the width and height of the bar. Breaking off the edge of the bar will then lead to two rectangles: (x – 1) × y and 1 × y.

By hypothesis, the first rectangle, (x – 1) × y, requires (x – 1)*y – 1 breaks to break it down to its constituent squares. Similarly, the 1 × y rectangle is proven by the hypothesis to take y – 1 breaks.

This leads to a bar with k + 1 squares taking (x-1)*y – 1 + (y – 1) + 1 breaks. Then we have: x*y – y – 1 + y – 1 + 1 = x*y – 1 = (k + 1) – 1.

Therefore, by PMI, a bar of *n* squares requires *n* – 1 breaks to be broken down into is constituent squares.