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Quiz 4 – Functions

1. Prove the following statement:

   Let A be a set. Then $\emptyset \times A = \emptyset$.

Proof: Assume $\emptyset \times A$ is not an empty set and $(x, y) \in (\emptyset \times A)$. By definition, $\emptyset \times A = \{(x, y) \mid x \in \emptyset$ and $y \in A\}$. This means that x is an element from the empty set, however, because there are no elements in the empty set then nothing is produced. This leads to an incomplete (x, y).

Therefore, by contradiction, $\emptyset \times A = \emptyset$.

2. Determine whether each of the following is a function:
   (a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \pm \sqrt{x^2 + 1}$ **not a function**
   (b) $f : \mathbb{Z} \to \mathbb{R}$ given by $f(x) = (x^2 - 2)^{-1}$ **yes, it is a function**

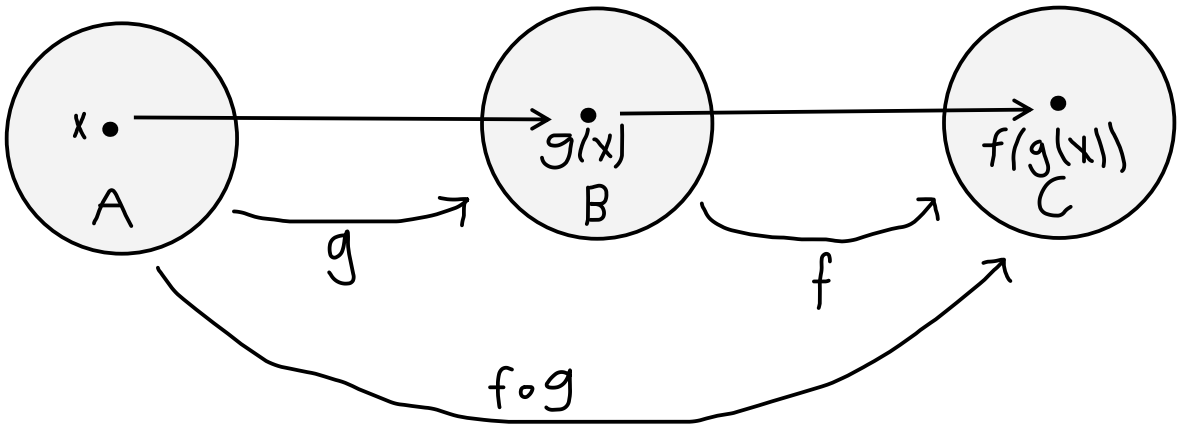
3. Determine whether each of the following is injective, surjective, both, or neither:
   (a) $f : Z \to Z$ given by $f(n) = n^3$ **injective**
   (b) $f : Z \to Z$ given by $f(n) = [n/2]$ **surjective**

4. Let A and B be the sets:
   - A: The set of teachers employed by a school
   - B: The set of offices in (that same) school

   … describe each of the following situations in English:

   (a) A function $f : A \to B$ is an injection.
   No teachers employed by a school share an office in school.
   (b) A function $f : A \to B$ is a surjection.
   Every office in school is used by one or more teachers.

5. Let $g : A \to B$, $f : B \to C$, and $f \circ g : A \to C$ be functions.
   (a) Draw and label a Venn diagram to illustrate this situation.



(b) Let $f \circ g : A \to C$ be an injection. Give a counterexample to show that $f : B \to C$ need not also be an injection.

f : R → R with $f(x) = x^2$ and g: R → $R_{>0}$ with $g(x) = 2x$.

$f(x)$ is not an injection because some outputs have multiple inputs (e.g. $f(2) = f(-2) = 4$).
$f \circ g : A \to C$ is an injection because of the domain on g(x), so no output has multiple inputs.
$f(g(x)) = (2x)^2$