

# SI Model Stability

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## 1 Model

The SI (Susceptible-Infected) model is a discrete-time, deterministic model.

$$\begin{aligned}\frac{dS}{dt} &= -\frac{\alpha}{N}SI \\ \frac{dI}{dt} &= \frac{\alpha}{N}SI\end{aligned}$$

### 1.1 Parameters

- $\alpha$ : the contact rate, i.e. the average number of individuals successfully infected

$SI$  is the number of potential contacts with an infected individual. Then  $\frac{\alpha}{N}SI$  is the average number of successful infections per population times the number of potentially successful contacts.

### 1.2 Initial Conditions

$$\begin{aligned}S_0 + I_0 &= N \\ S_0 > 0, I_0 > 0\end{aligned}$$

### 1.3 Assumptions

1. The total population size,  $N$ , remains constant:  $N = S_n + I_n$ .
2. Each individual is equally likely to contract the disease.

## 2 Stability Analysis

Reduce to one equation. Since  $S = N - I$ :

$$\frac{dI}{dt} = \frac{\alpha}{N}(N - I)I$$

The equilibria / fixed points are when  $\frac{dI}{dt} = 0$ .

$$I^* = 0, I^* = N$$

Corresponding to  $S^* = N$  and  $S^* = 0$ , respectively.

This gives two  $(S, I)$  pairs: the DFE pair  $(N, 0)$  and the endemic equilibrium pair  $(0, N)$ .

The Jacobian for  $x = \frac{dS}{dt}$  and  $y = \frac{dI}{dt}$  is

$$J = \begin{vmatrix} \frac{\partial x}{\partial S} & \frac{\partial x}{\partial I} \\ \frac{\partial y}{\partial S} & \frac{\partial y}{\partial I} \end{vmatrix} = \begin{vmatrix} -\frac{\alpha}{N}I & -\frac{\alpha}{N}S \\ \frac{\alpha}{N}I & \frac{\alpha}{N}S \end{vmatrix} = \frac{\alpha}{N} \begin{vmatrix} -I & -S \\ I & S \end{vmatrix}$$

### 2.1 Disease-Free Equilibrium (DFE)

DFE:  $(S, I) = (N, 0)$

Evaluate the Jacobian.

$$J = \begin{vmatrix} 0 & -\alpha \\ 0 & \alpha \end{vmatrix}$$

Evaluate the characteristic polynomial:

$$J - \lambda I = \begin{vmatrix} 0 & -\alpha \\ 0 & \alpha \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = \begin{vmatrix} -\lambda & -\alpha \\ 0 & \alpha - \lambda \end{vmatrix}$$

Evaluate the determinant:

$$\det = -\lambda(\alpha - \lambda) - 0(-\alpha) = -\lambda(\alpha - \lambda)$$

Eigenvalues:  $\lambda_1 = 0, \lambda_2 = \alpha > 0$

### 2.2 Endemic Equilibrium

Endemic equilibrium:  $(S, I) = (0, N)$

Evaluate the Jacobian.

$$J = \begin{vmatrix} -\alpha & 0 \\ \alpha & 0 \end{vmatrix}$$

Evaluate the characteristic polynomial:

$$J - \lambda I = \begin{vmatrix} -\alpha & 0 \\ \alpha & 0 \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = \begin{vmatrix} -\alpha - \lambda & 0 \\ \alpha & -\lambda \end{vmatrix}$$

Evaluate the determinant:

$$\det = -\lambda(-\alpha - \lambda) - 0(\alpha) = \lambda(\alpha + \lambda)$$

Eigenvalues:  $\lambda_1 = 0$ ,  $\lambda_2 = -\alpha < 0$ .