SI Model Stability

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1 Model

The SI (Susceptible-Infected) model is a discrete-time, deterministic model.

$$\frac{dS}{dt} = -\frac{\alpha}{N}SI$$
$$\frac{dI}{dt} = \frac{\alpha}{N}SI$$

1.1 Parameters

- α: the contact rate, i.e. the average number of individuals successfully infected

SI is the number of potential contacts with an infected individual. Then $\frac{\alpha}{N}SI$ is the average number of successful infections per population times the number of potentially successful contacts.

1.2 Initial Conditions

$$S_0 + I_0 = N$$
$$S_0 > 0, I_0 > 0$$

1.3 Assumptions

1. The total population size, N, remains constant: $N = S_n + I_n$.
2. Each individual is equally likely to contract the disease.

2 Stability Analysis

Reduce to one equation. Since $S = N - I$:

$$\frac{dI}{dt} = \frac{\alpha}{N}(N - I)I$$

The equilibria / fixed points are when $\frac{dI}{dt} = 0$.

$$I^* = 0, I^* = N$$

Corresponding to $S^* = N$ and $S^* = 0$, respectively.

This gives two (S, I) pairs: the DFE pair $(N, 0)$ and the endemic equilbrium pair $(0, N)$.

The Jacobian for $x = \dfrac{dS}{dt}$ and $y = \dfrac{dI}{dt}$ is

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial S} & \dfrac{\partial x}{\partial I} \\ \dfrac{\partial y}{\partial S} & \dfrac{\partial y}{\partial I} \end{vmatrix} = \begin{vmatrix} -\frac{\alpha}{N}I & -\frac{\alpha}{N}S \\ \frac{\alpha}{N}I & \frac{\alpha}{N}S \end{vmatrix} = \frac{\alpha}{N}\begin{vmatrix} -I & -S \\ I & S \end{vmatrix}$$

2.1 Disease-Free Equilibrium (DFE)

DFE: $(S, I) = (N, 0)$

Evaluate the Jacobian.

$$J = \begin{vmatrix} 0 & -\alpha \\ 0 & \alpha \end{vmatrix}$$

Evaluate the characteristic polynomial:

$$J - \lambda I = \begin{vmatrix} 0 & -\alpha \\ 0 & \alpha \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = \begin{vmatrix} -\lambda & -\alpha \\ 0 & \alpha - \lambda \end{vmatrix}$$

Evaluate the determinant:

$$det = -\lambda(\alpha - \lambda) - 0(-\alpha) = -\lambda(\alpha - \lambda)$$

Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = \alpha > 0$

2.2 Endemic Equilibrium

Endemic equilibrium: $(S, I) = (0, N)$

Evaluate the Jacobian.

$$J = \begin{vmatrix} -\alpha & 0 \\ \alpha & 0 \end{vmatrix}$$

Evaluate the characteristic polynomial:

$$J - \lambda I = \begin{vmatrix} -\alpha & 0 \\ \alpha & 0 \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = \begin{vmatrix} -\alpha - \lambda & 0 \\ \alpha & -\lambda \end{vmatrix}$$

Evaluate the determinant:

$$det = -\lambda(-\alpha - \lambda) - 0(\alpha) = \lambda(\alpha + \lambda)$$

Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = -\alpha < 0$.

References

[1] Allen, L.J.S. (1994). Some discrete-time SI, SIR, and SIS epidemic models. Mathematical Biosciences 124(1), 83-105. https://doi.org/10.1016/0025-5564(94)90025-6

[2] Jacquez, J.A. & Simon, C.P. (1993). The stochastic SI model with recruitment and deaths I. comparison with the closed SIS model. Mathematical Biosciences, 117(1-2), 77-125. https://doi.org/10.1016/0025-5564(93)90018-6