2 Description of the coursework

If you use a random number generator for any of the problems below, seed the generator so that the results are reproducible.

Problem 1. *Consider a random variable X whose cumulative distribution function (cdf) is given by*

$$F(x) = \begin{cases} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-\frac{y^2}{2}) dy, & \text{if } x < 0, \\ \frac{3}{4}, & \text{if } 0 \leq x < 3, \\ 1, & \text{if } x \geq 3. \end{cases} \tag{1}$$

a) Describe how one can generate a sample from F.

b) Write Python code to generate a sample from F and plot a histogram of a sample from F.

Problem 2. *Consider a stochastic process $(X_t)_{t\geq 0}$ that is characterised by the following stochastic differential equation*

$$dX_t = \mu X_t dt + (X_t - \gamma e^{\mu t})\sigma dW_t, \quad X_0 = 10, \tag{2}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, $\gamma \in (0, 10)$ and $(W_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion.

a) Explain how one can simulate a sample path of $(X_t)_{t\geq 0}$ on the time grid

$$0 < h < 2h < \ldots < Nh = T,$$

where $T > 0$, $h = T/N$ and $N \in \mathbb{N}$. Implement your suggested method in Python and plot sample paths of $(X_t)_{t\geq 0}$ for model parameters of your choice.

b) Does there exist an exact scheme for simulating a sample path of $(X_t)_{t\geq 0}$ on the time grid specified in part a) of this problem? Justify your answer.

c) Analyse the probability distribution of X_t, where $t = 5$, and how it depends on the model parameters. Use both mathematical arguments and output from your Python code for your analysis.

Problem 3. *Consider a financial market consisting of two risky assets and one riskless asset. Let $t \geq 0$. The riskless asset has time-t price $B_t = e^{rt}$, where $r \geq 0$ is the constant interest rate. The time-t prices of the two risky assets are denoted by $S_t^{(1)}$ and $S_t^{(2)}$ and satisfy*

$$\begin{aligned} dS_t^{(1)} &= S_t^{(1)} \left(rdt + \sigma_1 dW_t^{(1)} \right), \\ dS_t^{(2)} &= S_t^{(2)} \left(rdt + \sigma_2 \left(\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)} \right) \right), \end{aligned} \tag{3}$$

where $S_0^{(1)}, S_0^{(2)} > 0$ are the initial prices of the risky assets, $\sigma_1, \sigma_2 > 0$ are the volatilities, $\rho \in [-1, 1]$ is a correlation parameter and $(W_t^{(1)})_{t \geq 0}$ and $(W_t^{(2)})_{t \geq 0}$ are independent standard one-dimensional Brownian motions under the risk-neutral measure.

Consider a financial derivative whose payoff at maturity $T > 0$ is given by

$$\left(K - \frac{1}{2}(S_{\frac{T}{2}}^{(1)} + S_T^{(1)}) \right)^+ 1_{\{S_{\frac{T}{2}}^{(2)} < L\}}, \tag{4}$$

where $K, L \in [0, \infty)$ are constants.

a) *Write down a Monte Carlo estimator together with an asymptotic 99.5% confidence interval for the time-0 price of the financial derivative with payoff given in (4) and justify your answer. Explain how you can generate the random numbers needed for your Monte Carlo estimation. Implement this Monte Carlo estimator and an asymptotic 99.5% confidence interval for the time-0 price in Python (for general model parameters).*

b) *Provide an estimate of the time-0 price of the financial derivative with payoff (4) and a corresponding 99.5% confidence interval for the following model parameters:*

$$S_0^{(1)} = 100, S_0^{(2)} = 90, r = 0.04, \sigma_1 = 0.2, \sigma_2 = 0.5, \rho = 0.75, T = 1, K = 100, L = 80.$$

c) *Suggest three different control variate estimators that are a good choice for estimating the time-0 price of the financial derivative with payoff (4). The control variate estimators need to differ in the choice of the control variates, and not just in the choice of any multiplier (the multiplier was denoted by b in the lecture notes). Argue why you think they are good choices and compare their performance.*
Which estimator is most suitable for estimating the time-0 price of the financial derivative with payoff (4)? Justify your answer.
For this analysis, choose some informative combinations of model parameters yourself. In particular, do not restrict your analysis to the parameters considered in part b) of this problem.

d) *Consider the following function $g : [-1, 1] \to \mathbb{R}$, where*

$$g(\rho) = \mathbb{E}\left[e^{-rT} \left(K - \frac{1}{2}(S_{\frac{T}{2}}^{(1)} + S_T^{(1)}) \right)^+ 1_{\{S_{\frac{T}{2}}^{(2)} < L\}} \right],$$

and ρ is the correlation parameter that occurs in (3). Suggest two methods that are good choices for approximating the derivative $g'(\rho) = \frac{\partial}{\partial \rho} g(\rho)$ for all $\rho \in [-1, 1]$ and argue why you think they are good choices. Implement your suggested methods in Python and use them to discuss the sensitivity of the time-0 price of the financial derivative with payoff (4) with respect to ρ.
Which of the two methods for approximating the derivative g' do you prefer to use? Justify your answer.
For this analysis, choose some informative combinations of model parameters yourself. In particular, do not restrict your analysis to the parameters considered in part b) of this problem.