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Notes on free field theory and black hole evaporation

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1 Introduction

In this write-up, I do my best to give a brief introduction to free quantum field theory and the basic physics that goes into the calculation of black hole evaporation. These notes were compiled towards the end of my summer at the Center for Theoretical Physics at MIT under the MIT Summer Research Program. I have done my best to be as detailed as possible in many of the derivations, filling in steps that are not made explicit in the resources I used to compile these notes. It is in no way meant to be a comprehensive overview of the subject matter, but rather an introduction with enough mathematical detail and rigor to be satisfying. If you are reading this, I hope it can be helpful in some way, if even to just stir excitement about the field!

The notes are outlined as follows: first, we discuss free scalar field theory through the method of canonical quantization and compute some basic results of the theory, like correlation functions and vacuum energy. Next, we examine the same free field theory from the perspective of path integrals by first deriving the path integral in non-relativistic quantum mechanics then generalizing to field theory. After this, we spend a short time reviewing the basics of the theory of general relativity and the Schwarzschild solution before going on to derive the equation of motion for free field theory in an arbitrary metric as opposed to the flat spacetime considered in the first sections. Next, we discuss a useful way to decompose Minkowski spacetime that makes manifest the entangled nature of the vacuum and show that this decomposition greatly resembles the near-horizon geometry of the Schwarzschild solution. Finally, we use all the results from above to solve the free scalar field in the Schwarzschild metric and show how the effect of black hole evaporation and the possible loss of information emerges.

2 Free field theory: canonical quantization

In this section, I discuss free field theory in the context of canonical quantization and derive some basic results, including the vacuum energy of the free field and the Feynman propagator. The section is largely drawn from the lecture notes by Tong and the textbook by Srednicki [1].

2.1 Overview

One of the most instructive systems to study in quantum field theory (QFT) is that of a free massive scalar field. Here, a single degree of freedom is associated with each point in space, and the space of states is thought of as an infinite tensor product of Hilbert spaces at each point in space. The primary massive scalar field is typically denoted by $\phi(\vec{x})$ and is an operator on the space of states. In the Schrödinger formalism of quantum mechanics, operators are time-independent and states evolve in time according to the Schrödinger equation, where the Hamiltonian drives the temporal evolution. For a free massive scalar field, the Hamiltonian is given by:

$$\mathcal{H} = \frac{1}{2}\int d^3x \, \left(\pi(\vec{x})^2 + (\nabla\phi(\vec{x}))^2 + m^2\phi(\vec{x})^2\right), \tag{2.1}$$

where $\pi(\vec{x})$ is the conjugate momentum of $\phi(\vec{x})$. To move from the Schrödinger formalism to the Heisenberg picture, time dependence is given to the field operators in the following way:

$$\phi(x) \equiv \phi(\vec{x}, t) \equiv e^{iHt}\phi(\vec{x})e^{-iHt}, \tag{2.2}$$

where $x \equiv (t, \vec{x})$. It is, of course, possible to work with a purely space-dependent field. Then, using signature $(-, +, +, +)$, the field is given by:

$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right], \tag{2.3}$$

where $\vec{p}\cdot\vec{x}$ is the usual 3-vector dot product, and $a_{\vec{p}}$, $a^\dagger_{\vec{p}}$ are annihilation/creation operators satisfying the following commutation relations:

$$[a_{\vec{p}}, a_{\vec{q}}] = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] = 0, \tag{2.4}$$

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}). \tag{2.5}$$

To make the field appear explicitly Lorentz invariant, it is useful to express it in the Heisenberg picture. For a massive scalar field, the Heisenberg operator field can be expressed in the following form:

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[a_{\vec{p}}e^{ipx} + a^\dagger_{\vec{p}}e^{-ipx}\right], \tag{2.6}$$

where $p \equiv (E_{\vec{p}}, \vec{p})$ is the momentum 4-vector, $E_{\vec{p}} = +\sqrt{|\vec{p}|^2+m^2}$, and $px \equiv \eta_{\mu\nu}p^\mu x^\nu = -E_{\vec{p}}t + \vec{p}\cdot\vec{x}$, where $\eta_{\mu\nu}$ is the Minkowski metric under signature $(-,+,+,+)$, and the Einstein summation convention is used. The ground state of the system, $|\Omega\rangle$, often referred to as the vacuum, is annihilated by all $a_{\vec{p}}$, and the state $|\vec{p}\rangle \equiv a^\dagger_{\vec{p}}|\Omega\rangle$ is interpreted as an excitation of the field where a particle of momentum $\vec{p}$ is created. The action of the field $\phi(x)$ on the vacuum is interpreted as creating a particle at the point $\vec{x}$ and time t: $|x\rangle \equiv \phi(x)\,|\Omega\rangle$.

A keen reader may note that Eq. 2.6 does not appear completely Lorentz-invariant since the measure:

$$\int \frac{d^3\vec{p}}{\sqrt{2E_{\vec{p}}}} \tag{2.7}$$

is itself not Lorentz-invariant. In fact, the Lorentz-invariant measure is given by the following expression:

$$\int \frac{d^3\vec{p}}{2E_{\vec{p}}}. \tag{2.8}$$

To justify this, first note that the following measure is Lorentz-invariant:

$$\int d^4p\ \delta(p^2+m^2)\theta(p^0), \tag{2.9}$$

where θ is the unit step function. To see why, note that $\int d^4p$ is Lorentz-invariant, and $\delta(p^2+m^2)$ is as well, since the following holds for all Lorentz transformations Λ:

$$p^\mu \to \Lambda^\mu{}_\nu p^\nu, \tag{2.10}$$

$$p^2 = \eta_{\mu\nu}p^\mu p^\nu \to \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma p^\rho p^\sigma = \eta_{\rho\sigma}p^\rho p^\sigma = p^2, \tag{2.11}$$

where we use the fact that $\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$. Finally, $\theta(p^0)$ is Lorentz-invariant under the action of the restricted Lorentz group (which is the Lorentz group without time and parity reversals) since if $p^0 > 0$, then under a transformation from the restricted group this property is maintained. Now, we use the following property of the delta function:

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}, \tag{2.12}$$

where x_i are roots of $g(x)$. Applying this to our problem we get:

$$\delta\left(p^2+m^2\right)=\delta\left(-(p^0)^2+\vec{p}\cdot\vec{p}+m^2\right)=\frac{\delta\left(p^0-E_{\vec{p}}\right)}{2E_{\vec{p}}}+\frac{\delta\left(p^0+E_{\vec{p}}\right)}{2E_{\vec{p}}}. \tag{2.13}$$

Finally, integrating over p^0 gives:

$$\int_{-\infty}^{\infty}dp^0\int d^3\vec{p}\,\delta(p^2+m^2)\theta(p^0)=\int_{0}^{\infty}dp^0\int d^3\vec{p}\,\frac{\delta(p^0-E_{\vec{p}})}{2E_{\vec{p}}}=\int\frac{d^3\vec{p}}{2E_{\vec{p}}}. \tag{2.14}$$

Since the measure in Eq. 2.6 is not Lorentz-invariant, it is tempting to think that the field is itself not Lorentz-invariant. However, although it is not explicit, the definition of $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ is such that the whole expression is Lorentz-invariant. One could redefine $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ such that the Lorentz-invariant measure is explicit, but this choice is inconsequential.

2.2 Motivation

Before discussing some elementary results of free field theory, I would like to provide some motivation for the form of Eq. 2.6. We start by considering the equation of motion (EOM) governing dynamics in non-relativistic quantum mechanics with no external potential, using the Schrödinger picture:

$$i\frac{\partial}{\partial t}\psi(x)=-\frac{1}{2m}\nabla^2\psi(x). \tag{2.15}$$

In this case, the energy operator is $H=\frac{P^2}{2m}=-\frac{1}{2m}\nabla^2$, and it drives the time evolution of the state.[1] In an attempt to make the theory relativistic, one can consider replacing the energy operator with the relativistic form of the energy: $H=\sqrt{P^2+m^2}=+\sqrt{-\nabla^2+m^2}$. However, this replacement is rife with issues. For example, a formal expansion in powers of ∇^2 leads to arbitrarily high spatial derivatives of the state, implying non-locality of the theory since it would become possible to solve the state at a single point and proceed to calculate it at all other points to arbitrary accuracy. Another problem is the asymmetry between spatial and temporal derivatives, which is not consistent with the relativistic principle of placing space and time on equal footings.

One way to alleviate the aforementioned issues is to square the operators on both sides, giving rise to the following equation:

$$-\frac{\partial^2}{\partial t^2}\phi(x)=(-\nabla^2+m^2)\phi(x), \tag{2.16}$$

which can be rewritten as:

$$(\partial_\mu\partial^\mu-m^2)\phi(x)=0, \tag{2.17}$$

where $\partial_0\equiv\frac{\partial}{\partial t}$, $\partial_i\equiv\frac{\partial}{\partial x^i}$[2], and $\partial^\mu\equiv\eta^{\mu\nu}\partial_\nu$. This is the Klein-Gordon equation. A very important distinction must be made here between the interpretation of $\psi(x)$ in Eq. 2.15 and $\phi(x)$ in Eq.2.17, which is that $\psi(x)$ represents the state of the system, while $\phi(x)$ will be quantized and thus represents an operator field, acting on states in the theory.

[1] Throughout, we use capital P and X to denote the momentum and position operators.

[2] As is standard in the literature, Latin indices are reserved for spatial coordinates while Greek indices run over all spacetime coordinates.

We can find the general solution of the classical EOM in Eq. 2.17 through a change of variables to momentum space by way of a Fourier transformation. Expressing the scalar field $\phi(x)$ in terms of a Fourier transformation, we have:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \tilde{\phi}(\vec{p}, t). \tag{2.18}$$

Then, inserting this in the Klein-Gordon equation we have:

$$(\partial_\mu \partial^\mu - m^2)\phi(x) = \left(\frac{\partial^2}{\partial t^2} + \left(|\vec{p}|^2 + m^2 \right) \right) \tilde{\phi}(\vec{p}, t) = 0 \tag{2.19}$$

which is the EOM describing a simple harmonic oscillator with angular frequency $\omega_{\vec{p}} = \pm\sqrt{|\vec{p}|^2 + m^2}$. We are interested in positive-frequency solutions, so we take a positive $\omega_{\vec{p}}$. The general solution to the Klein-Gordon equation is then a linear combination of $a(\vec{p})e^{i\omega_{\vec{p}} t}$ and $b(\vec{p})e^{-i\omega_{\vec{p}} t}$. Note that for different $\vec{p}$, these solutions are decoupled and don't interact, hence the notion of a "free" field theory. Finally, to get the quantum field as in Eq. 2.3, we must quantize these infinitely many harmonic oscillators. In non-relativistic quantum mechanics, the Hamiltonian for a harmonic oscillator is:

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 Q^2, \tag{2.20}$$

where Q is the position operator. The Hilbert space is spanned by the energy eigenstates $|n\rangle$ where, up to a normalizaiton constant, $H|n\rangle = (n + \frac{1}{2})\omega |n\rangle$. These eigenstates are created by the action of the operator

$$a^\dagger = \sqrt{\frac{\omega}{2}} Q - \frac{i}{\sqrt{2\omega}} P, \tag{2.21}$$

on the ground state $|0\rangle$, which satisfies $a|0\rangle = 0$. In terms of a and $a^\dagger$, our Hamiltonian[3] is:

$$H = \omega \left(a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2} \right). \tag{2.22}$$

So, to quantize the real scalar field, we associate a creation operator $a^\dagger_{\vec{p}}$ to each momentum $\vec{p}$ and a corresponding annihilation operator $a_{\vec{p}}$ satisfying $a_{\vec{p}}|\Omega\rangle = 0$ for all momenta $\vec{p}$. Our general solution is then a sum over all momenta of these annihilation and creation operators as given in Eq. 2.3.

2.3 Vacuum energy

In this section, we will derive the vacuum energy for the free scalar field and discuss the implications of this non-trivial result. The Hamiltonian for a free, massive scalar field is given by Eq. 2.1, which we rewrite here for convenience:

$$\mathcal{H} = \frac{1}{2} \int d^3x \, \left(\pi(\vec{x})^2 + (\nabla\phi(\vec{x}))^2 + m^2\phi(\vec{x})^2 \right). \tag{2.23}$$

[3] There is an interesting subtlety to do with operator ordering that leads to a non-zero ground state energy when we define our Hamiltonian as in Eq. 2.20. We will encounter a divergence when we compute the energy of the free field vacuum in Section 2.3 since we will have infinitely many oscillators, each with non-zero ground state energy. We will show how we may relieve this tension by being more careful in quantizing our Hamiltonian as we account for the non-abelian nature of operators in quantum mechanics.

Our first task is to substitute the fields into the Hamiltonian. Doing so gives:

$$\begin{aligned}\mathcal{H} = \frac{1}{2}\int d^3x \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\Bigg[&-\frac{\sqrt{E_{\vec{p}}E_{\vec{q}}}}{2}\left(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right)\left(a_{\vec{q}}e^{i\vec{q}\cdot\vec{x}} - a^\dagger_{\vec{q}}e^{-i\vec{q}\cdot\vec{x}}\right) \\ &-\frac{\vec{p}\cdot\vec{q}}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}}\left(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right)\left(a_{\vec{q}}e^{i\vec{q}\cdot\vec{x}} - a^\dagger_{\vec{q}}e^{-i\vec{q}\cdot\vec{x}}\right) \\ &+\frac{m^2}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}}\left(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right)\left(a_{\vec{q}}e^{i\vec{q}\cdot\vec{x}} + a^\dagger_{\vec{q}}e^{-i\vec{q}\cdot\vec{x}}\right)\Bigg].\end{aligned} \tag{2.24}$$

Next, we leverage the following identity:

$$\delta^{(3)}(\vec{p}\pm\vec{q}) = \frac{1}{(2\pi)^3}\int d^3x\ e^{i\vec{x}\cdot(\vec{p}\pm\vec{q})}, \tag{2.25}$$

and then integrate over $\vec{q}$. After collecting terms and massaging a bit, we have:

$$\mathcal{H} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}E_{\vec{p}}\left(a_{\vec{p}}a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}}a_{\vec{p}}\right). \tag{2.26}$$

Finally, we use the commutation relation in Eq. 2.5 to get the following result:

$$\mathcal{H} = \int \frac{d^3p}{(2\pi)^3}E_{\vec{p}}\left(a^\dagger_{\vec{p}}a_{\vec{p}} + \frac{1}{2}(2\pi)^3\delta^{(3)}(0)\right). \tag{2.27}$$

Now, to compute the vacuum energy, we act on the vacuum state with the Hamiltonian, which immediately gives rise to the following striking result:

$$\mathcal{H}\left|\Omega\right\rangle = \infty\left|\Omega\right\rangle. \tag{2.28}$$

This divergence is a result of two well-known issues with quantum field theories, the so-called "infrared" divergence, evidenced by the delta function, and the "ultraviolet" divergence, evidenced by the integral of $E_{\vec{p}}$ over all momenta. Note that both of these arise from the second term in the integral. The infrared divergence arises from the infinite size of space, and can be regulated by computing energy densities, or working within a finite volume. The ultraviolet divergence is a result of the infinite degrees of freedom of the theory within a finite volume and can be regulated by working on a lattice, or, equivalently, introducing an energy cutoff.

A practical approach can be taken to deal with these issues if we are only interested in energy differences from the vacuum state. The idea is to subtract off the infinite energy term and define the ground state to have zero energy. Doing so, our Hamiltonian becomes:

$$\mathcal{H} = \int \frac{d^3p}{(2\pi)^3}E_{\vec{p}}a^\dagger_{\vec{p}}a_{\vec{p}}. \tag{2.29}$$

It turns out that we can actually arrive at this answer in a natural way if we are careful about the ordering of the Q and P operators when we quantize the harmonic oscillator. Specifically, if we adopt the following Hamiltonian:

$$H = \frac{1}{2}\left(\omega Q - iP\right)\left(\omega Q + iP\right), \tag{2.30}$$

which is the same as the classical Hamiltonian, we find our quantum Hamiltonian is given by $H = \omega a^\dagger_{\vec{p}}a_{\vec{p}}$, which annihilates the ground state, in contrast to $H = \omega\left(a^\dagger_{\vec{p}}a_{\vec{p}} + \frac{1}{2}\right)$ which gives a non-zero ground state energy.

2.4 The Feynman propagator

An important quantity in quantum field theory is the Feynman propagator, also known as the time-ordered two-point correlation function. This is given by:

$$\Delta_F(x-y) \equiv \langle\Omega| T\phi(x)\phi(y)|\Omega\rangle \,, \tag{2.31}$$

where T is the time-ordering operator defined by:

$$T\phi(x)\phi(y) = \begin{cases} \phi(x)\phi(y), & \text{if } x^0 > y^0, \\ \phi(y)\phi(x), & \text{if } x^0 < y^0. \end{cases} \tag{2.32}$$

To understand the meaning of $\Delta_F(x-y)$, lets assume for simplicity that $x^0 > y^0$. Then:

$$\Delta_F(x-y) = \langle x|y\rangle \,, \tag{2.33}$$

so the Feynman propagator gives the amplitude for a particle starting at position y to be found at position x. As we will see in Section 3.4 when we formulate free field theory with the path integral, the Feynman propagator emerges naturally in the computation of the path integral. This quantity is important for calculating observables like scattering amplitudes.

We will now work out the propagator in the cases $x^0 > y^0$ and $x^0 < y^0$, then give an equivalent way to express Δ_F as a single four-momentum integral. Suppose $x^0 > y^0$. Then:

$$\phi(x)\phi(y) = \int \frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}_1}E_{\vec{p}_2}}}\left[a_{\vec{p}_1}e^{ip_1x} + a^\dagger_{\vec{p}_1}e^{-ip_1x}\right]\left[a_{\vec{p}_2}e^{ip_2y} + a^\dagger_{\vec{p}_2}e^{-ip_2y}\right]. \tag{2.34}$$

Now, since $a_{\vec{p}}|\Omega\rangle = 0$ (and, equivalently, $\langle\Omega|\, a^\dagger_{\vec{p}} = 0$), the only term surviving from the expressions in the brackets in Eq. 2.34 is $a_{\vec{p}_1}a^\dagger_{\vec{p}_2}e^{ip_1\cdot x - ip_2\cdot y}$. Now, we invoke the commutation relation in Eq. 2.5 and integrate over $\vec{p}_2$ to get:

$$\Delta_F(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{e^{ip(x-y)}}{2E_{\vec{p}}}. \tag{2.35}$$

Clearly, if we exchange x and y in Eq. 2.34, we simply swap the sign in the exponent in Eq. 2.35. So, we have for $x^0 < y^0$:

$$\Delta_F(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{e^{-ip(x-y)}}{2E_{\vec{p}}}. \tag{2.36}$$

These expressions are useful, however there exists a more elegant way to express the Feynman propagator as a single 4-momentum integral:

$$\Delta_F(x-y) = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2+m^2}. \tag{2.37}$$

This looks very similar to the expressions above in the special cases, however note that here we treat p^0 itself as an integration variable. The equivalence of this expression to the ones above can be shown by picking a particular integral contour in the complex plane, depending on the time-ordering of x and y, around the poles $\pm E_{\vec{p}}$ of the p^0 integral. One can show that by doing this, Eq. 2.35 and Eq. 2.36 emerge naturally from the residue theorem.

3 Free field theory: path integral formalism

In this section, we will derive the path-integral formulation of non-relativistic quantum mechanics, apply it to the harmonic oscillator, and then show how this can be naturally extended to free field theory. Our final task will be to show how the Feynman propagator of Section 2.4 emerges naturally in the path integral formalism. The structure of this section is directly inspired by Chapters 6-8 of Srednicki's Textbook [1], and further details can be found there.

3.1 The path integral in non-relativistic quantum mechanics

The path integral approach to quantum mechanics was first proposed by Richard Feynman. The general idea is that the probability amplitude for a particle starting at a point q' at time t' to be found at a point q'' at time t'' is given by the following expression:

$$K \propto \int \mathcal{D}q \; e^{iS}, \tag{3.1}$$

where:

$$S = \int_{t'}^{t''} dt \; L(q, \dot{q}, t) \tag{3.2}$$

is the action, $L(q, \dot{q}, t)$ is the Lagrangian of the system, and $\mathcal{D}q$ is an appropriate measure, where the path integral is understood as a sum over all spacetime paths connecting (q', t') and (q'', t''). Throughout this section I will use q to denote position in the non-relativistic theory so as to not confuse with the 4-vector x discussed in previous sections.

We shall start by deriving the path integral formalism from the familiar Schrödinger picture. In the Schrödinger formalism, time evolution of an initial quantum state $|\psi_0\rangle$ is governed by the unitary operator $U(t) \equiv e^{-iHt}$, i.e., $|\psi(t)\rangle = U(t)\,|\psi_0\rangle$. If we start, then, with a position eigenstate $|q'\rangle$, and evolve by a time $t'' - t'$, we find ourselves in the state $e^{-iH(t''-t')}\,|q'\rangle$. So, the probability amplitude of finding the particle in the position eigenstate $|q''\rangle$ at time t'' is $\langle q''|\, e^{-iH(t''-t')}\,|q'\rangle$. For notational simplicity, we define the state $|q, t\rangle \equiv e^{iHt}\,|q\rangle$, which is an eigenstate of the Heisenberg position operator, $Q(t) = e^{iHt} Q e^{-iHt}$. Then, our quantity of interest is neatly expressed as $\langle q'', t''|q', t'\rangle$.

We start by dividing the total time interval $T \equiv t'' - t'$ into $N+1$ equal time intervals of length $\delta t = \frac{T}{N+1}$. Then, for $i = 1$ to N, pick some position eigenstate $|q_i\rangle$ and recall the following identity:

$$\mathbb{1} = \int dq_i \,|q_i\rangle\,\langle q_i|\,. \tag{3.3}$$

We can then insert these complete sets of position eigenstates for each time step into the inner product $\langle q'', t''|q', t'\rangle$:

$$\langle q'', t''|q', t'\rangle = \int dq_1 \cdots dq_N \,\langle q''|e^{-iH\delta t}|q_N\rangle\,\langle q_N|e^{-iH\delta t}|q_{N-1}\rangle \cdots \langle q_1|e^{-iH\delta t}|q'\rangle\,. \tag{3.4}$$

To make progress, let us first examine $\langle q_{i+1}|e^{-iH\delta t}|q_i\rangle$. Consider a Hamiltonian of the form:

$$H = \frac{P^2}{2m} + V(Q). \tag{3.5}$$

Then:

$$e^{-iH\delta t} = e^{-i\delta t\left(\frac{P^2}{2m}+V(Q)\right)}, \tag{3.6}$$

which is formally given by the following infinite series:

$$e^{-iH\delta t} = \sum_{n=0}^{\infty} (-i\delta t)^n \left[\frac{P^2}{2m} + V(Q)\right]^n. \tag{3.7}$$

This is non-trivial since we are working with operators that do not commute. To actually compute this, we invoke the Baker-Campbell-Hausdorff formula, which says that:

$$\exp\left(X + Y + \frac{1}{2}[X,Y] + \cdots\right) = \exp(X)\exp(Y), \tag{3.8}$$

for operators X and Y. Since we will eventually take $\delta t \to 0$, we keep only those terms that are linear in δt:

$$e^{-iH\delta t} = e^{-i\delta t\frac{P^2}{2m}} e^{-i\delta t V(Q)}. \tag{3.9}$$

Next, we will insert a complete basis of momentum eigenstates. Recall that, for a complete momentum basis, as was the case for the position basis, we have the following identity:

$$\mathbb{1} = \int dp_i \, |p_i\rangle \langle p_i| \, . \tag{3.10}$$

Inserting this complete basis into $\langle q_{i+1}|e^{-iH\delta t}|q_i\rangle$, we have:

$$\begin{aligned}
\langle q_{i+1}|e^{-iH\delta t}|q_i\rangle &= \int dp_i \, \langle q_{i+1}|e^{-i\delta t\frac{P^2}{2m}}|p_i\rangle \langle p_i|e^{-i\delta t V(Q)}|q_i\rangle \, , \\
&= \int dp_i \; e^{-i\delta t\left(\frac{p_i^2}{2m}+V(q_i)\right)} \langle q_{i+1}|p_i\rangle \langle p_i|q_i\rangle \, .
\end{aligned} \tag{3.11}$$

Now, the wavefunction $\langle q|p\rangle$ for a momentum eigenstate is of the form $\langle q|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipq}$, so substituting this, we have:

$$\langle q_{i+1}|e^{-iH\delta t}|q_i\rangle = \int \frac{dp_i}{2\pi} e^{-i\delta t H(p_i,q_i)} e^{ip_i(q_{i+1}-q_i)}. \tag{3.12}$$

Now, if we insert the above equation into the full integral for $\langle q'', t''|q', t'\rangle$, we have:

$$\langle q'', t''|q', t'\rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} \exp\left\{ i\delta t \left(p_j \frac{q_{j+1}-q_j}{\delta t} - H(p_j, q_j) \right) \right\}. \tag{3.13}$$

Finally, we take the limit $\delta t \to 0$, giving:

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q\mathcal{D}p \exp\left\{ i \int_{t'}^{t''} dt \left[p(t)\dot{q}(t) - H(p(t), q(t)) \right] \right\}, \tag{3.14}$$

where the measure $\mathcal{D}q\mathcal{D}p$ is understood as taking all paths through phase space. To get to the schematic form of the path integral given in Eq. 3.1, where the integral is done only over paths in position space, we can explicitly compute the integral over momenta by computing

Gaussian integrals. This is relatively straightforward due to the simple Hamiltonians we are considering. We will perform this computation below.

First, for simplicity, we define the following: $A \equiv \frac{i\delta t}{2m}$ and $B_j \equiv i\left(q_{j+1} - q_j\right)$. Then, taking only the momentum integrals from Eq. 3.13:

$$\begin{aligned}
&\int \prod_{j=0}^{N} \frac{dp_j}{2\pi} \exp\left\{i\left[p_j\left(q_{j+1} - q_j\right) - \delta t H\left(p_j, q_j\right)\right]\right\} \\
&= \int \prod_{j=0}^{N} \frac{dp_j}{2\pi} \exp\left\{-Ap_j^2 + B_j p_j - i\delta t V(q_j)\right\}, \\
&= \sqrt{\frac{\pi^{N+1}}{A^{N+1}}} \frac{1}{(2\pi)^{N+1}} \exp\left\{\sum_{j=0}^{N}\left(-i\delta t V(q_j) + \frac{B_j^2}{4A}\right)\right\}, \\
&= \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \exp\left\{i\delta t \sum_{j=0}^{N}\left[\frac{1}{2}m\left(\frac{q_{j+1} - q_j}{\delta t}\right)^2 - V(q_j)\right]\right\}.
\end{aligned} \tag{3.15}$$

Finally, taking the limit $\delta t \to 0$ (and equivalently $N \to \infty$) and plugging into Eq. 3.13, we get:

$$\begin{aligned}
\langle q'', t'' | q', t' \rangle &= \lim_{N\to\infty} \int \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \prod_{k=1}^{N} dq_k \exp\left\{i\delta t \sum_{j=0}^{N}\left[\frac{1}{2}m\left(\frac{q_{j+1} - q_j}{\delta t}\right)^2 - V(q_j)\right]\right\}, \\
&= \int \mathcal{D}q \exp\left\{i\int_{t'}^{t''} L\left(\dot{q}(t), q(t)\right)\right\},
\end{aligned} \tag{3.16}$$

where the measure $\mathcal{D}q$ is therefore explicitly given by:

$$\mathcal{D}q = \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \prod_{k=1}^{N} dq_k. \tag{3.17}$$

3.2 Time-ordered correlation functions

In Section 2.4, we used the canonical quantization formulation of QFT to calculate the Feynman propagator, which is just the amplitude for some initial position eigenstate onto some final position eigenstate that obeys time-ordering. We will be interested in computing this quantity in the path integral formalism of QFT. However, we can actually make good progress by first analyzing the equivalent question in the non-relativistic theory. To do so, we will employ a trick that involves adding some force term $f(t)q(t)$ in the Hamiltonian, then taking functional derivatives of the resulting *functional* to "bring down" position functions at a given time, which is equivalent to acting on the initial state with time-ordered position operators. After taking as many derivatives as desired, $f(t)$ is set to 0. Finally, we make a substitution $H \to (1 - i\epsilon)H$, where ϵ is some small quantity, then take $t' \to -\infty$, and $t'' \to \infty$, which has the effect of selecting the initial and final states as the ground state. This is desired since it allows us to calculate correlation functions effectively with the ground state as our initial and final states. Below we explicitly outline the procedure described in this paragraph.

As mentioned above, we are interested in calculating $\langle q'', t''|TQ(t_1)\cdots Q(t_n)|q', t'\rangle$, where T is the time-ordering operator, defined in Eq. 2.32. If we look back at the derivation of the path integral in Eq. 3.14, we see that inserting a position operator at some time t_1 will simply mean that the operator acts on the position eigenstate at that time, which introduces a factor $q(t_1)$ into the integral. In other words:

$$\langle q'', t''|Q(t_1)|q', t'\rangle = \int \mathcal{D}q\mathcal{D}p\; q(t_1)\exp\left\{i\int_{t'}^{t''} dt\,[p(t)\dot{q}\,(t) - H\,(p(t), q(t))]\right\}. \tag{3.18}$$

Now, we will compute the RHS of the above equation by first making the substitution $H \to H - fq$, then taking functional derivatives with respect to f. Inserting the modified Hamiltonian into Eq. 3.14, we have:

$$\begin{aligned}\langle q'', t''|q', t'\rangle_f &\equiv \int \mathcal{D}q\mathcal{D}p \exp\left\{i\int_{t'}^{t''} dt\,[p(t)\dot{q}\,(t) - H\,(p(t), q(t)) + f(t)q(t)]\right\}, \\ &= \int \mathcal{D}q\mathcal{D}p\; e^{iS_f},\end{aligned} \tag{3.19}$$

where we define $S_f \equiv \int_{t'}^{t''} dt\,[p(t)\dot{q}\,(t) - H\,(p(t), q(t)) + f(t)q(t)]$ for convenience. Now, let us take a functional derivative with respect to f. To do so, we introduce the functional derivative $\frac{\delta}{\delta f(t)}$ which is defined by:

$$\frac{\delta}{\delta f(t_1)} f(t_2) = \delta(t_1 - t_2), \tag{3.20}$$

where the RHS is the Dirac delta function. It is easy to verify that the usual properties of derivatives, such as linearity and the chain rule, follow from this definition. Applying to Eq. 3.19:

$$\begin{aligned}\frac{\delta}{\delta f(t_1)}\langle q'', t''|q', t'\rangle_f &= i\int \mathcal{D}q\mathcal{D}p\; e^{iS_f} \times \frac{\delta}{\delta f(t_1)} S_f \\ &= i\int \mathcal{D}q\mathcal{D}p\; e^{iS_f} \times \int_{t'}^{t''} dt\;\delta(t_1 - t)q(t) \\ &= i\int \mathcal{D}q\mathcal{D}p\; q(t_1)e^{iS_f}, \\ &= i\,\langle q'', t''|Q(t_1)|q', t'\rangle_f\,.\end{aligned} \tag{3.21}$$

Evaluating at $f = 0$ afterwards, we have:

$$\frac{1}{i}\frac{\delta}{\delta f(t_1)}\langle q'', t''|q', t'\rangle_f\Big|_{f=0} = \langle q'', t''|Q(t_1)|q', t'\rangle\,. \tag{3.22}$$

We can repeat this procedure n times to get the following general result:

$$\langle q'', t''|TQ(t_1)\cdots Q(t_n)|q', t'\rangle = \frac{1}{i^n}\frac{\delta}{\delta f(t_1)}\cdots\frac{\delta}{\delta f(t_n)}\langle q'', t''|q', t'\rangle_f\Big|_{f=0}. \tag{3.23}$$

Since we will be interested in correlation functions, we would like our initial and final states to be asymptotic vacuum states as opposed to position eigenstates. We are therefore interested in the following quantity:

$$\langle 0|0\rangle_f \equiv \lim_{\substack{t'\to-\infty,\\ t''\to\infty}} \langle 0|e^{-i(H-fQ)(t''-t')}|0\rangle\,. \tag{3.24}$$

By inserting a complete position basis two times we get the following:

$$\begin{aligned}
\langle 0|0\rangle_f &= \lim_{\substack{t'\to-\infty,\\ t''\to\infty}} \langle 0|\left(\int dq''\,|q''\rangle\,\langle q''|\right) e^{-i(H-fQ)(t''-t')}\left(\int dq'\,|q'\rangle\,\langle q'|\right)|0\rangle\,,\\
\langle 0|0\rangle_f &= \lim_{\substack{t'\to-\infty,\\ t''\to\infty}} \int dq''dq'\,\langle 0|q''\rangle\,\langle q'',t''|q',t'\rangle_f\,\langle q'|0\rangle\,,\\
\langle 0|0\rangle_f &= \lim_{\substack{t'\to-\infty,\\ t''\to\infty}} \int dq''dq'\psi_0^*(q'')\,\langle q'',t''|q',t'\rangle_f\,\psi_0(q'),
\end{aligned} \tag{3.25}$$

where $\psi_0(q)$ is the ground-state wave function. We can find a way to compute this, up to some constant factors, as a path integral with arbitrary initial and final positions. In order to do this, we must make the substitution $H \to (1-i\epsilon)H$, where ϵ is taken to be small. To see why this will help, let us first express $|q',t'\rangle$ in terms of energy eigenstates:

$$\begin{aligned}
|q',t'\rangle &= e^{iHt'}\,|q'\rangle\,,\\
&= \sum_n e^{iHt'}\,|n\rangle\,\langle n|q'\rangle\,,\\
&= \sum_n \psi_n^*(q')e^{iE_nt'}\,|n\rangle\,,
\end{aligned} \tag{3.26}$$

where ψ_n is the nth energy eigenstate. Then, taking $H \to (1-i\epsilon)H$ and $t' \to -\infty$, we see that, assuming $E_0 = 0$,[4] the ground state is selected since all other states become exponentially suppressed. In other words:

$$\lim_{t'\to-\infty} |q',t'\rangle = \psi_0^*(q')\,|0\rangle\,. \tag{3.27}$$

A similar argument shows that for $t'' \to \infty$:

$$\lim_{t''\to\infty} \langle q'',t''| = \langle 0|\,\psi_0(q''). \tag{3.28}$$

In Eq. 3.25, we are integrating over all possible initial and final positions, q' and q''. However, by making the substitution $H \to (1-i\epsilon)H$, we have shown that in the limits $t' \to -\infty$ and $t'' \to \infty$, the initial and final choices of position are, up to normalization, inconsequential to the evaluation of the path integral. We therefore get the following clean result:

$$\langle 0|0\rangle_f = \int \mathcal{D}q\mathcal{D}p \exp\left\{i\int_{-\infty}^{\infty} dt\,(p\dot{q} - (1-i\epsilon)H + fq)\right\}. \tag{3.29}$$

[4]If this is not the case, simply redefine the Hamiltonian to enforce this condition.

3.3 The harmonic oscillator in the path integral formalism

In this section, we will analyze the harmonic oscillator in the context of the path integral for non-relativistic quantum theory. Our ultimate goal will be to compute the time-ordered two-point correlation function, $\langle 0|TQ(t_1)Q(t_2)|0\rangle$. We will do this by first solving for $\langle 0|0\rangle_f$, then applying the functional derivative method discussed in Section 3.2. This will also help us study free fields in QFT as the situation is analogous to the harmonic oscillator in the non-relativistic theory. In fact, the derivations are practically identical.

We start with the Hamiltonian for the harmonic oscillator:

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2. \tag{3.30}$$

Then, making the substitution $H \to (1-i\epsilon)H$, and keeping only linear terms in ϵ, we observe the following:

$$\begin{aligned}(1-i\epsilon)H &= \frac{1}{2}P^2\frac{(1-i\epsilon)}{m} + \frac{1}{2}\omega^2Q^2(1-i\epsilon)m,\\ &= \frac{1}{2}P^2\frac{(1-i\epsilon)(1+i\epsilon)}{(1+i\epsilon)m} + \frac{1}{2}\omega^2Q^2(1-i\epsilon)m,\\ &= \frac{1}{2}P^2\frac{1+\epsilon^2}{(1+i\epsilon)m} + \frac{1}{2}\omega^2Q^2(1-i\epsilon)m,\\ &= \frac{1}{2}P^2\frac{1}{(1+i\epsilon)m} + \frac{1}{2}\omega^2Q^2(1-i\epsilon)m.\end{aligned} \tag{3.31}$$

We are pedantic so as to make explicit that $m \to (1+i\epsilon)m$ in the kinetic term and $m \to (1-i\epsilon)m$ in the potential term. This makes expressing the Lagrangian of the system under the substitution $H \to (1-i\epsilon)H$ simple:

$$L = \frac{1}{2}(1+i\epsilon)m\dot{q}^2 - \frac{1}{2}(1-i\epsilon)m\omega^2q^2. \tag{3.32}$$

Since the potential of our Hamiltonian is of the simple form discussed in Section 3.1, we can jump straight to the path integral over position as in Eq. 3.16. So, the first quantity we are interested in computing is given by:

$$\langle 0|0\rangle_f = \int \mathcal{D}q \exp\left\{i\int_{-\infty}^{\infty} dt(L+fq)\right\}. \tag{3.33}$$

Let us start by changing variables using the Fourier transform:

$$\tilde{q}(E) = \int dt\ e^{iEt}q(t), \tag{3.34}$$

$$\tilde{f}(E) = \int dt\ e^{iEt}f(t). \tag{3.35}$$

Then, for the inverse transform we have:

$$q(t) = \int \frac{dE}{2\pi}e^{-iEt}\tilde{q}(E), \tag{3.36}$$

$$f(t) = \int \frac{dE}{2\pi} e^{-iEt} \tilde{f}(E). \tag{3.37}$$

The purpose of this change of variables is to allow us to explicitly extract the time-dependence in the action integrand. Then, we can express the action in terms of an integral over energy. Before we insert, note the following:

$$\dot{q}(t) = -i \int \frac{dE}{2\pi} E e^{-iEt} \tilde{q}(E), \tag{3.38}$$

which is found by taking the derivative of the inverse transform. Following Srednicki [1], we set $m = 1$ from now on for simplicity. Then, carrying out our change of variables in the action integrand:

$$\begin{aligned} L + fq =& \frac{1}{2}(1+i\epsilon)\dot{q}^2 - \frac{1}{2}(1-i\epsilon)\omega^2 q^2 + fq, \\ =& -\frac{1}{2}(1+i\epsilon) \int \frac{dE}{2\pi} \frac{dE'}{2\pi} EE' e^{-i(E+E')t} \tilde{q}(E)\tilde{q}(E') \\ & -\frac{1}{2}(1-i\epsilon)\omega^2 \int \frac{dE}{2\pi} \frac{dE'}{2\pi} e^{-i(E+E')t} \tilde{q}(E)\tilde{q}(E') \\ & + \int \frac{dE}{2\pi} \frac{dE'}{2\pi} e^{-i(E+E')t} \tilde{f}(E)\tilde{q}(E'). \end{aligned} \tag{3.39}$$

Collecting terms, we have:

$$\begin{aligned} L + fq = \frac{1}{2} \int \frac{dE}{2\pi} \frac{dE'}{2\pi} e^{-i(E+E')t} \Big(& - \tilde{q}(E)\tilde{q}(E') \left[(1+i\epsilon)EE' + (1-i\epsilon)\omega^2\right] \\ & + \tilde{f}(E)\tilde{q}(E') + \tilde{f}(E')\tilde{q}(E) \Big). \end{aligned} \tag{3.40}$$

Note the breaking apart of the $2f(E)q(E')$ term into terms where the integration variables are swapped. This will help us in the future to simplify when we change the path integral variable. It initially looks like we have not made much progress, however note that the only time-dependence is in the exponential factor, so we can compute the action integral easily using the following identity:

$$\frac{1}{2\pi} \int dt\ e^{-i(E+E')t} = \delta(E+E'). \tag{3.41}$$

This means that integrating over E' will select $-E$ in place of all E':

$$S = \frac{1}{2} \int \frac{dE}{2\pi} \Big(\tilde{q}(E)\tilde{q}(-E) \left[(1+i\epsilon)E^2 - (1-i\epsilon)\omega^2\right] + \tilde{f}(E)\tilde{q}(-E) + \tilde{f}(-E)\tilde{q}(E) \Big). \tag{3.42}$$

Now, again for notational simplicity, we will redefine ϵ such that $\epsilon \to \frac{\epsilon}{E^2+\omega^2}$, which means that:

$$(1+i\epsilon)E^2 - (1-i\epsilon)\omega^2 \to E^2 - \omega^2 + i\epsilon. \tag{3.43}$$

This is of course valid since $E^2 + \omega^2 > 0$. The next step is to make a change of variables for the path integral itself. Define $\tilde{z}(E) = \tilde{q}(E) + \frac{\tilde{f}(E)}{E^2-\omega^2+i\epsilon}$. Importantly, since $\tilde{q}(E)$ is shifted by a constant with respect to $\tilde{q}(E)$, the measure change is simply $\mathcal{D}q = \mathcal{D}z$. In terms of this new variable, the action becomes:

$$S = \frac{1}{2}\int \frac{dE}{2\pi}\left(\tilde{z}(E)\left(E^2-\omega^2+i\epsilon\right)\tilde{z}(-E) - \frac{\tilde{f}(E)\tilde{f}(-E)}{E^2-\omega^2+i\epsilon} + \tilde{f}(E)\tilde{z}(-E) - \tilde{f}(-E)\tilde{z}(E)\right). \tag{3.44}$$

Now, the usefulness of splitting the $2\tilde{f}(E)\tilde{q}(E')$ term becomes apparent as the last two terms in the integral will cancel in an integral over all energies. Our expression for the action is thus:

$$S = \frac{1}{2}\int \frac{dE}{2\pi}\left(\tilde{z}(E)\left(E^2-\omega^2+i\epsilon\right)\tilde{z}(-E) - \frac{\tilde{f}(E)\tilde{f}(-E)}{E^2-\omega^2+i\epsilon}\right). \tag{3.45}$$

Plugging into the path integral we have:

$$\langle 0|0\rangle_f = \int \mathcal{D}z \exp\left\{\frac{i}{2}\int \frac{dE}{2\pi}\left(\tilde{z}(E)\left(E^2-\omega^2+i\epsilon\right)\tilde{z}(-E) - \frac{\tilde{f}(E)\tilde{f}(-E)}{E^2-\omega^2+i\epsilon}\right)\right\}. \tag{3.46}$$

The cleverness in this derivation is now evident: the second term in the action has no z-dependence and can thus be pulled out of the path integral. Also, note the following:

$$\langle 0|0\rangle_{f=0} = \int \mathcal{D}z \exp\left\{\frac{i}{2}\int \frac{dE}{2\pi}\tilde{z}(E)\left(E^2-\omega^2+i\epsilon\right)\tilde{z}(-E)\right\}, \tag{3.47}$$

which is the first term in the exponent. If there is no disturbing force, i.e. $f = 0$, then we reasonably expect the inner product $\langle 0|0\rangle_{f=0}$ to be unity. We have thus managed to express $\langle 0|0\rangle_f$ purely in terms of an integral of f:

$$\langle 0|0\rangle_f = \exp\left\{-\frac{i}{2}\int \frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(-E)}{E^2-\omega^2+i\epsilon}\right\}. \tag{3.48}$$

We can, of course, express this in terms of time-domain integrals by reversing the Fourier transform:

$$\langle 0|0\rangle_f = \exp\left\{-\frac{i}{2}\int \frac{dE}{2\pi}dtdt' f(t)\frac{e^{-iE(t-t')}}{E^2-\omega^2+i\epsilon}f(t')\right\}. \tag{3.49}$$

We then define the following function:

$$G(t-t') = -\int \frac{dE}{2\pi}\frac{e^{-iE(t-t')}}{E^2-\omega^2+i\epsilon}, \tag{3.50}$$

which is a Green's function for the EOM of a Harmonic oscillator. Showing this is just a matter of inserting $G(t-t')$ into the oscillator EOM and noting the result is $\delta(t-t')$. Note the striking similarity between $G(t-t')$ and the Feynman propagato in Eq.r 2.36. In fact, when we generalize this result to free field theory, the analagous function will, up to a factor of i, turn out to be the Feynman propagator! With this, we then have:

$$\langle 0|0\rangle_f = \exp\left\{\frac{i}{2}\int dtdt' f(t)G(t-t')f(t')\right\}. \tag{3.51}$$

Finally, we can compute $\langle 0|TQ(t_2)Q(t_1)|0\rangle$ by the method outlined in Section 3.1. First, let us compute the functional derivative of $\langle 0|0\rangle_f$:

$$\begin{aligned}\frac{\delta}{\delta f(t_1)}\langle 0|0\rangle_f &= \langle 0|0\rangle_f \frac{\delta}{\delta f(t_1)}\left(\frac{i}{2}\int dtdt' f(t)G(t-t')f(t')\right),\\ &= \frac{i}{2}\langle 0|0\rangle_f\left(\int dtdt'\delta(t_1-t)G(t-t')f(t') + \int dtdt' f(t)G(t-t')\delta(t_1-t')\right),\\ &= \frac{i}{2}\langle 0|0\rangle_f\left(\int dt\ G(t_1-t)f(t) + \int dt\ f(t)G(t-t_1)\right),\\ &= i\langle 0|0\rangle_f\int dt\ G(t-t_1)f(t).\end{aligned} \tag{3.52}$$

Thus for our two-point correlation function, we have:

$$\begin{aligned}&\langle 0|TQ(t_2)Q(t_1)|0\rangle\\ &= \frac{1}{i^2}\frac{\delta}{\delta f(t_2)}\frac{\delta}{\delta f(t_1)}\langle 0|0\rangle_f\bigg|_{f=0},\\ &= -\frac{\delta}{\delta f(t_2)}\left(i\langle 0|0\rangle_f\int dt\ G(t-t_1)f(t)\right)\bigg|_{f=0},\\ &= -i\langle 0|0\rangle_f\left(\int dt\ G(t-t_1)\delta(t-t_2) + \int dtdt' G(t-t_1)f(t)G(t'-t_2)f(t')\right)\bigg|_{f=0}.\end{aligned} \tag{3.53}$$

Finally, setting $f=0$, we find $\langle 0|0\rangle_{f=0} = 1$ and the second term disappears, leaving the following neat result for the time-ordered two-point correlation function:

$$\langle 0|TQ(t_2)Q(t_1)|0\rangle = \frac{1}{i}G(t_2-t_1). \tag{3.54}$$

3.4 Free field theory

In this section, we will present the results of applying the path integral formalism to a free scalar field. As mentioned in Section 3.3, computing the path integral in free field theory is almost identical to computing the path integral for the harmonic oscillator in the non-relativistic theory, so we will skip most of the calculation details and merely outline the general procedure.

The Hamiltonian for a free massive scalar field is given by:

$$\mathcal{H} = \frac{1}{2}\int d^3x\ \left(\pi(\vec{x})^2 + (\nabla\phi(\vec{x}))^2 + m^2\phi(\vec{x})^2\right), \tag{3.55}$$

which has a corresponding Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2. \tag{3.56}$$

Generalizing from the path integral in the non-relativistic theory, we have the following expression for the ground state to ground state transition amplitude in the presence of some source J:

$$\langle\Omega|\Omega\rangle_J = \int\mathcal{D}\phi\exp\left\{i\int d^4x\,(\mathcal{L}+J\phi)\right\}, \tag{3.57}$$

where $|\Omega\rangle$ is the vacuum state, J is some classical source (equivalent to the force f in the harmonic oscillator), and the path integral is understood as being done over all field configurations. By changing variables in an identical fashion to the harmonic oscillator (see Section 3.3 and Srednicki, Chapter 8 [1] for more details), we arrive at the following result:

$$\langle\Omega|\Omega\rangle_J = \exp\left\{\frac{1}{2}\int d^4x d^4x' J(x)\Delta_F(x-x')J(x')\right\}, \tag{3.58}$$

where:

$$\Delta_F(x-x') \equiv i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-x')}}{p^2+m^2-i\epsilon}, \tag{3.59}$$

is the Feynman propagator. Recall that the use of ϵ is a trick that allows us to compute the path integral more easily. Thus, taking $\epsilon \to 0$, we recover Eq. 2.37. Just as $G(t-t')$ is a Green's function of the oscillator equation, the Feynman propagator Δ_F is a Green's function of the Klein-Gordon equation, which one can verify by confirming the following equation holds:

$$\left(-\partial_\mu\partial^\mu + m^2\right)\Delta_F(x-x') = \delta^{(4)}\left(x-x'\right). \tag{3.60}$$

Finally, by taking functional derivatives with respect to J, then setting $J=0$, as we did with the force f previously, we find the following:

$$\langle\Omega|T\phi(x)\phi(x')|\Omega\rangle = \Delta_F(x-x'), \tag{3.61}$$

which is consistent with our definition of the Feynman propagator in Section 2.4.

4 QFT in curved spacetime

In this section, our goal is to review general relativity, the Schwarzschild solution, and derive the equation of motion for a free scalar field in some arbitrary background metric. We will start with a *very* brief review of general relativity, the Schwarzschild solution to Einstein's equations, and some useful coordinates in the Schwarzschild geometry. We then go on to derive the EOM for a free scalar field in an arbitrary metric.

4.1 Review of general relativity and the Schwarzschild solution

In general relativity, spacetime is a 4-dimensional manifold where the "force" of gravity is encoded in the geometry of the manifold. The important dynamical variable is the spacetime metric, $g_{\mu\nu}$, which is a $(0,2)$ tensor field. The metric can be thought of as acting locally on vectors associated with a point to give a notion of directions and angles between them. Specifically, the metric is used to define an inner product between vectors X and Y in the following way:

$$X \cdot Y \equiv g_{\mu\nu}X^\mu Y^\nu, \tag{4.1}$$

where we remind the reader that the Einstein summation convention is in effect. It is standard to express the metric in a way that emphasizes the invariance of the spacetime line element, ds^2:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \tag{4.2}$$

For example, in Minkowski spacetime, we call the following expression the metric:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{4.3}$$

The metric is solved from Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{4.4}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor, which encodes the presence of energy in spacetime. Since it will be useful in our later calculation of black hole evaporation, we will define the energy-momentum tensor:

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g_{\mu\nu}}, \tag{4.5}$$

where g is reserved for the determinant of the metric. Later, we will show how we can calculate this explicitly when S_{matter} is the action for a free scalar field. Given a metric, the paths $x^\mu(\tau)$ taken through spacetime for test particles can be found from the geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}, \tag{4.6}$$

where $\Gamma^\mu{}_{\rho\sigma}$ is the Christoffel symbol and is related to the metric by:

$$\Gamma^\mu{}_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}\left(\partial_\rho g_{\nu\sigma} + \partial_\sigma g_{\nu\rho} - \partial_\nu g_{\rho\sigma}\right). \tag{4.7}$$

From a differential geometry perspective, the Christoffel symbols are the unique connection coefficients associated with the Levi-Civita connection, ${}^{LC}\nabla$. Roughly speaking, this connection identifies autoparallel curves in spacetime with geodesic curves. The choice of the Levi-Civita connection is equivalent to enforcing that the covariant derivative annihilate the metric, and that the manifold be torsion-free, i.e. the $(1,2)$-tensor field T is zero, where $T(\omega, X, Y) \equiv \omega\left(\nabla_X Y - \nabla_Y X - [X,Y]\right)$, and $[X,Y] \equiv XY - YX$ is the usual commutator.

In the presence of no sources, the Einstein equations have a unique solution for a spherically symmetric spacetime that is asymptotically Minkowski spacetime. This is known as the Schwarzschild solution, and in the standard coordinates it is given by:

$$ds^2 = -\frac{r-2GM}{r}dt^2 + \frac{r}{r-2GM}dr^2 + r^2 d\Omega_2^2, \tag{4.8}$$

where $d\Omega_2^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$, and M is usually interpreted as the mass of some point mass at $r = 0$. I emphasize here that the Schwarzschild solution does not represent the exact geometry of black holes in an astrophysical sense. However, there is a very neat way to approximate an astrophysical black hole geometry by "gluing" together a piece of the Schwarzschild geometry and the Minkowski geometry. More details and very useful diagrams can be found in Harlow [2].

There are two apparent divergences in the spacetime interval ds^2 when it is expressed in these coordinates: $r = 2GM$ and $r = 0$. However, only the $r = 0$ divergence is physical, which can be shown in a coordinate invariant way by contracting the Riemann tensor. In doing so, one finds the contraction diverges, and since this physically encodes the strength of tidal forces and is coordinate-independent, it verifies the physical nature of the singularity at $r = 0$. The apparent divergence at $r = r_s \equiv 2GM$, known as the Schwarzschild radius, is an artifact of our coordinate choice. While the metric does not diverge at r_s, it is still a very important surface in the geometry since below the Schwarzschild radius, the time and radial component of the line element change signs. This means that below this point, a decreasing radius is as inevitable as time marching forward in our familiar Minkowski spacetime.

Now, we will discuss some useful coordinate transformations for studying the geometry of the Schwarzschild solution. First, we introduce the tortoise radial coordinate, whose name is inspired from the story of Achilles and the tortoise:

$$r_* \equiv r + r_s \log\left(\frac{r}{r_s} - 1\right). \tag{4.9}$$

Taking $r_s = 1$, we then define the Kruskal-Szekeres coordinates by:

$$U \equiv -e^{\frac{r_* - t}{2}}, \tag{4.10}$$

$$V \equiv e^{\frac{r_* + t}{2}}. \tag{4.11}$$

To justify this coordinate choice, let us solve for the form of radial null geodesics in the Schwarzschild coordinates. If we set $d\Omega_2^2 = 0$ (radial), and $ds^2 = 0$ (null), we are left with the following equation:

$$\frac{r - r_s}{r} dt^2 = \frac{r}{r - r_s} dr^2, \tag{4.12}$$

which becomes:

$$\begin{aligned} \int dt &= \int dr \frac{r}{r - r_s}, \\ t = \pm\left(r + r_s\left(\frac{r - r_s}{r_s}\right)\right) &+ C, \\ t &= \pm r_* + C, \end{aligned} \tag{4.13}$$

for some constant C. So the tortoise coordinate is a natural choice for radial, null geodesics. Furthermore, if we take U to be equal to some negative constant, we find:

$$\begin{aligned} U &= -C, \\ e^{\frac{r_* - t}{2}} &= C, \\ \frac{r_* - t}{2} &= C, \\ t &= r_* + C, \end{aligned} \tag{4.14}$$

where at each step we redefined C. Thus, the contours of U are radial null geodesics. A similar thing holds for the coordinate V. Another nice feature of these coordinates becomes apparent when you multiply them together. We do so below to get:

$$\begin{aligned} UV &= -e^{r_*}, \\ UV &= (1 - r)e^r. \end{aligned} \tag{4.15}$$

When $r = r_s = 1$, $UV = 0$, so the event horizon occurs when either U or V is zero. When $r = 0$, $UV = 1$, so the singularity occurs on the rotated hyperbola. Finally, we can rewrite the Schwarzschild metric in our new coordinates. First, let us compute dU and dV:

$$\begin{aligned} dU &= \frac{1}{2}U dr_* - \frac{1}{2}U dt, \\ &= \frac{1}{2}U \left[\frac{r}{r-1}dr - dt\right], \end{aligned} \tag{4.16}$$

$$\begin{aligned} dV &= \frac{1}{2}V dr_* + \frac{1}{2}V dt, \\ &= \frac{1}{2}V \left[\frac{r}{r-1}dr + dt\right]. \end{aligned} \tag{4.17}$$

Then, note the following:

$$\begin{aligned} dU dV &= \frac{1}{4}UV\frac{r}{r-1}\left[\frac{r}{r-1}dr^2 - \frac{r-1}{r}dt^2\right], \\ &= -\frac{re^r}{4}\left[\frac{r}{r-1}dr^2 - \frac{r-1}{r}dt^2\right], \end{aligned} \tag{4.18}$$

where we used the fact that $UV = (1-r)e^r$. Thus, we have the following result:

$$dU dV + dV dU = -\frac{re^r}{2}\left[\frac{r}{r-1}dr^2 - \frac{r-1}{r}dt^2\right], \tag{4.19}$$

and so our metric becomes:

$$ds^2 = -\frac{2e^{-r}}{r}\left(dU dV + dV dU\right) + r^2 d\Omega_2^2. \tag{4.20}$$

If one wishes to have a metric that is diagonal, they can further define new coordinates:

$$U = T - X, \tag{4.21}$$

$$V = T + X. \tag{4.22}$$

Then, we have the following simple relation:

$$dU dV = dV dU = (dT - dX)(dT + dX) = dT^2 - dX^2, \tag{4.23}$$

so the metric becomes:

$$ds^2 = \frac{4e^{-r}}{r}\left(-dT^2 + dX^2\right) + r^2 d\Omega_2^2. \tag{4.24}$$

Changing to the $(X,\ T)$ coordinates from the $(U,\ V)$ coordinates is effectively a 45 degree rotation. Previously, we showed that the event horizon occurs when U or V is zero, i.e. the vertical and horizontal axis lines in $U-V$ space. This corresponds to the lines $T = X$ and $T = -X$, which is just a 45 degree rotation from the horizontal and vertical axes. Similarly, we showed that the rotated hyperbola $UV = 1$ corresponds to the singularity. This becomes the hyperbola $T^2 - X^2 = 1$, which is, again, a 45 degree rotation from the hyperbola in $U-V$ space.

4.2 Free scalar field in an arbitrary metric

In this section, we derive the equation of motion for a scalar field in an arbitrary metric $g_{\mu\nu}$. Recall the Klein-Gordon equation, which is the EOM for a free scalar field in Minkowski spacetime:

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = \partial_\mu\partial^\mu\phi = m^2\phi. \tag{4.25}$$

At first pass, if one is looking to generalize this equation, they may think to replace $\eta_{\mu\nu}$ with the metric $g_{\mu\nu}$ and replace the partial derivatives with the covariant derivative ∇_μ to get the following equation:

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = \nabla_\mu\nabla^\mu\phi = m^2\phi. \tag{4.26}$$

This equation turns out to be correct, although we can arrive to it in a more 'rigorous' way by varying the action of the free scalar field and applying the principle of least action. The Lagrangian of the free scalar field, generalized to arbitrary metric $g_{\mu\nu}$, is:

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2. \tag{4.27}$$

The generalized action is thus:

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right), \tag{4.28}$$

where the factor $\sqrt{-g}$ is necessary for well-defined integration on a manifold.

Before varying the action to derive Eq. 4.26, we will state two results that will help us but we will prove only one of them. The first result, which we will only state, is the following:

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(A^\mu\sqrt{-g}\right). \tag{4.29}$$

The next result, which is proved through integration by parts, is:

$$\int d^4x\sqrt{-g}\partial_\mu\phi\partial^\mu\psi = -\int d^4x\ \phi\partial_\mu\left(\sqrt{-g}\partial^\mu\psi\right), \tag{4.30}$$

for scalars ϕ and ψ. To demonstrate this, first note the following:

$$\sqrt{-g}\partial_\mu\phi\partial^\mu\psi = \partial_\mu\left(\phi\sqrt{-g}\partial^\mu\psi\right) - \phi\partial_\mu\left(\sqrt{-g}\partial^\mu\psi\right), \tag{4.31}$$

which is verified by the product rule. Then:

$$\int d^4x\sqrt{-g}\partial_\mu\phi\partial^\mu\psi = \int d^4x\ \partial_\mu\left(\phi\sqrt{-g}\partial^\mu\psi\right) - \int d^4x\ \phi\partial_\mu\left(\sqrt{-g}\partial^\mu\psi\right). \tag{4.32}$$

The first term is an overall derivative and, given any reasonable boundary conditions, we can neglect it after integration over all of spacetime, so our final result is:

$$\int d^4x\sqrt{-g}\partial_\mu\phi\partial^\mu\psi = -\int d^4x\ \phi\partial_\mu\left(\sqrt{-g}\partial^\mu\psi\right). \tag{4.33}$$

Equipped with the results above, we will go on to vary the action by making the substitutions $\phi \to \phi + \delta\phi$ and $\partial_\mu \phi \to \partial_\mu \phi + \partial_\mu(\delta\phi)$. Up to first order in $\delta\phi$, then, we have the following:

$$\begin{aligned} S' &= -\frac{1}{2} \int d^4x \sqrt{-g} \left[g^{\mu\nu} \left(\partial_\mu \phi + \partial_\mu(\delta\phi) \right) \left(\partial_\nu \phi + \partial_\nu(\delta\phi) \right) + m^2 \left(\phi + \delta\phi \right)^2 \right], \\ S' &= -\frac{1}{2} \int d^4x \sqrt{-g} \left[g^{\mu\nu} \left(\partial_\mu \phi \partial_\nu \phi + \partial_\mu \phi \partial_\nu(\delta\phi) + \partial_\mu(\delta\phi) \partial_\nu \phi \right) + m^2 \left(\phi^2 + 2\phi\delta\phi \right) \right], \\ S' - S &= -\frac{1}{2} \int d^4x \sqrt{-g} \left[g^{\mu\nu} \left(\partial_\mu \phi \partial_\nu(\delta\phi) + \partial_\mu(\delta\phi) \partial_\nu \phi \right) + 2m^2 \phi\delta\phi \right]. \end{aligned} \tag{4.34}$$

Now, since $g^{\mu\nu}$ is a symmetric tensor, $g^{\mu\nu} \partial_\mu \phi \partial_\nu(\delta\phi) = g^{\mu\nu} \partial_\mu(\delta\phi) \partial_\nu \phi$. Defining $\delta S \equiv S' - S$ then, we have:

$$\delta S = -\int d^4x \sqrt{-g} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu(\delta\phi) + m^2 \phi\delta\phi \right]. \tag{4.35}$$

Now we use the result in Eq. 4.33, where we associate $\delta\phi$ with ψ, to get:

$$\begin{aligned} \delta S &= \int d^4x \left[\delta\phi \partial_\mu \left(\sqrt{-g} \partial^\mu \phi \right) - \sqrt{-g} m^2 \phi\delta\phi \right], \\ &= \int d^4x \sqrt{-g} \delta\phi \left[\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \partial^\mu \phi \right) - m^2 \phi \right]. \end{aligned} \tag{4.36}$$

Finally, we apply the principle of least action, which states that $\delta S = 0$, to get the following EOM:

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \partial^\mu \phi \right) = m^2 \phi. \tag{4.37}$$

This form of the equation is certainly very useful, however to get to the ansatz in Eq. 4.26, we must use the identity in Eq. 4.29 where we identify A^μ with $\partial^\mu \phi$. Then, we have:

$$\nabla_\mu (\partial^\mu \phi) = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \partial^\mu \phi \right) = m^2 \phi. \tag{4.38}$$

Since ϕ is a scalar, $\nabla^\mu \phi = \partial^\mu \phi$, so then:

$$\nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \partial^\mu \phi \right) = m^2 \phi, \tag{4.39}$$

so we have recovered our naive guess in Eq. 4.26.

5 Spacetime entanglement

In this section, we will discuss an expression of the vacuum state in free field theory as an entangled state between two separate regions of spacetime known as the Rindler wedges. To do this, we will divide spacetime into two regions and study the states of observers that are experiencing a constant acceleration with respect to the vacuum. We will find that these accelerating observers perceive a thermal distribution of particles. The benefit of studying this is that the region near the horizon of a black hole greatly resembles this system, and understanding the vacuum as an entangled state across two regions of spacetime will help us to understand the quantum state near the black hole's event horizon.

To start, we will briefly review what entanglement means in non-relativistic quantum theory. We will then go on to demonstrate the vacuum-entangled state via the Rindler decomposition of Minkowski spacetime.

5.1 Entanglement in non-relativistic quantum mechanics

A quantum state is most generally represented as a non-negative, Hermitian operator ρ of unit trace acting on some Hilbert space, $\mathcal{H}$. A state is called pure if there exists an element of the Hilbert space $|\psi\rangle \in \mathcal{H}$ such that:

$$\rho = |\psi\rangle\langle\psi|. \tag{5.1}$$

A state that is not pure is referred to as mixed. The Von Neumann entropy is a very useful way to quantify the "pureness" of a quantum state. It is defined in the following way:

$$S(\rho) \equiv -\mathrm{Tr}\left(\rho \log \rho\right). \tag{5.2}$$

Among many other things, the following important quality is true of the Von Neumann entropy: $S(\rho) \geq 0$, and $S(\rho) = 0$ if and only if ρ is pure. Furthermore, if $d = \dim(\mathcal{H})$, then $S(\rho) \leq \log d$, and $S(\rho) = \log d$ if and only if ρ is maximally mixed. Thus, we see that $S(\rho)$ gives a nice quantitative description of the "pureness" of a quantum state.

In order to discuss entanglement, we will talk about bipartite quantum systems, which is a quantum system where the Hilbert space can be tensor-factored in the following way:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{5.3}$$

If we have an operator M_A acting on space $\mathcal{H}_A$ and some other operator N_B acting on $\mathcal{H}_B$, we can naturally define the operator $M_A \otimes N_B$ on $\mathcal{H}$ by:

$$\left(M_A \otimes N_b\right)\left(v \otimes w\right) \equiv \left(M_A v\right) \otimes \left(N_B w\right), \tag{5.4}$$

where $v \in \mathcal{H}_A$ and $w \in \mathcal{H}_B$. We can then consider the effect of making a measurement on the $\mathcal{H}_A$ factor of the state $\rho_{AB} \in \mathcal{H}$. For any quantum state ρ, the probability of measuring some outcome i is given by:

$$P(\rho, i) = \mathrm{Tr}\left(\rho \Pi_i\right), \tag{5.5}$$

where Π_i is the projection operator for that outcome. In our bipartite system then, if we make a measurement only on the $\mathcal{H}_A$ factor, we are interested in the following quantity:

$$P\left(\rho_{AB}, i\right) = \mathrm{Tr}_{AB}\left(\rho_{AB}\left(\Pi_i \otimes I_B\right)\right), \tag{5.6}$$

where Π_i is the projection operator for outcome i in the $\mathcal{H}_A$ factor and I_B is the identity operator in the $\mathcal{H}_B$ factor. Now, the idea is to define an operator ρ_A on the $\mathcal{H}_A$ factor by the following:

$$\langle a_j|\rho_A|a_k\rangle \equiv \sum_n \langle a_j b_n|\rho_{AB}|a_k b_n\rangle, \tag{5.7}$$

where $\{a_j\}$ and $\{b_j\}$ are bases of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Then, if we choose c_{jk} such that $\Pi_i |a_j\rangle = \sum_k c_{jk} |a_k\rangle$, then:

$$\begin{aligned}
\mathrm{Tr}\left(\rho_A \Pi_i\right) &= \sum_j \langle a_j|\rho_A \Pi_i|a_j\rangle\,, \\
&= \sum_{j,k} c_{jk} \langle a_j|\rho_A|a_k\rangle\,, \\
&= \sum_{j,k,n} c_{jk} \langle a_j b_n|\rho_{AB}|a_k b_n\rangle\,, \\
&= \sum_{j,n} \langle a_j b_n|\, \rho_{AB} \left(\sum_k c_{jk} |a_k\rangle \otimes |b_n\rangle\right), \\
&= \sum_{j,n} \langle a_j b_n|\, \rho_{AB} \left(\Pi_i |a_j\rangle \otimes |b_n\rangle\right), \\
&= \sum_{j,n} \langle a_j b_n|\rho_{AB} \left(\Pi_i \otimes I_B\right) |a_j b_n\rangle\,, \\
&= \mathrm{Tr}\left(\rho_{AB} \left(\Pi_i \otimes I_B\right)\right).
\end{aligned} \tag{5.8}$$

Thus, we can understand the probability of this measurement as the exact same as what would be made if we had the state ρ_A in the Hilbert space $\mathcal{H}_A$. The operation done in Eq. 5.7 is called the partial trace.

We now make use of a way to write pure states in a bipartite system known as the Schmidt decomposition. We will not prove the statement, but refer the reader to section C.2.2 of Harlow [2] for details on its derivation. The Schmidt decomposition says:

- If $|\psi\rangle$ is a pure state of $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $|A| \leq |B|$, there are constants $p_a \in [0, 1]$ with $\sum_a p_a = 1$, an orthonormal basis $|a, \psi\rangle_A$ of $\mathcal{H}_A$, and an orthonormal basis $|a, \psi\rangle_B$ of a subspace of $\mathcal{H}_B$ such that:

$$|\psi\rangle = \sum_a \sqrt{p_a}\, |a, \psi\rangle_A\, |a, \psi\rangle_B\,. \tag{5.9}$$

Finally, with this expression, we say that a state $\rho_{AB} = |\psi\rangle\langle\psi|$ is entangled if and only if more than one p_a is non-zero. If there is only non-zero p_a, then $\psi = |\psi_a\rangle_A \otimes |\psi_b\rangle_B$, in other words our state is a product of pure states. In words, then, to be entangled means the bipartite state ρ_{AB} is pure while the reduced states ρ_A and ρ_B are mixed. In fact, one can prove that, if $S_A \equiv -\mathrm{Tr}\left(\rho_A \log \rho_A\right)$ and $S_B \equiv -\mathrm{Tr}\left(\rho_B \log \rho_B\right)$, then $S_A = S_B$, and this is often referred to as the entanglement entropy.

Now that we have a rigorous way to define entanglement in non-relativistic quantum mechanics, we will turn to the vacuum state in the free scalar field theory and show that it can be decomposed according to physical regions in space in a fashion that makes the entanglement of these two regions apparent.

5.2 The Rindler decomposition of spacetime

Our goal in this section will be to express the vacuum state as an entangled state between two regions of spacetime. The general idea will be to pick a spatial coordinate x, then divide the total Hilbert space into two pieces across some point and find basis states on either side of the point such that the vacuum can be expressed as an entangled state for the bipartite system. To accomplish this, our basis states will be eigenstates of the boost operator that acts along the x-direction.

To start, we will state a result from Section 3.3 of Harlow [2] where there is an excellent explanation of how one can generally express the vacuum state in the following way:

$$|\Omega\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\pi\omega_i} |i^*\rangle_L |i\rangle_R . \tag{5.10}$$

In words, this equation says that the vacuum can be expressed as a sum of boost eigenstates from the left and the right wedges. It is reminiscent of Eq. 5.9, and is thus a manifestly entangled state with its components coming from Hilbert spaces in different regions of spacetime. In fact we can take the partial trace of $\rho \equiv |\Omega\rangle \langle\Omega|$ to get the reduced density matrix. The total density matrix is:

$$\rho = \frac{1}{Z} \sum_{i,j} e^{-\pi(\omega_i+\omega_j)} |i^*\rangle_L |i\rangle_R \langle j^*|_L \langle j|_R . \tag{5.11}$$

Taking the partial trace to get the reduced density matrix for the right side, we have:

$$\begin{aligned} \rho_R &= \sum_k {}_L\langle k^*|\rho|k^*\rangle_L , \\ &= \frac{1}{Z} \sum_{i,j,k} e^{-\pi(\omega_i+\omega_j)} \langle k^*|i^*\rangle_L |i\rangle_R \langle j^*|k^*\rangle_L \langle j|_R , \\ &= \frac{1}{Z} \sum_{i,j} e^{-\pi(\omega_i+\omega_j)} |i\rangle_R \langle j^*|i^*\rangle_L \langle j|_R , \\ &= \frac{1}{Z} \sum_i e^{-2\pi\omega_i} |i\rangle_R \langle i| , \end{aligned} \tag{5.12}$$

where from line 2 to 3 we sum over k, and line 3 to 4 over j, using the orthonormality condition on the eigenstates. Thus, the reduced density matrix ρ_R is a thermal distribution of boost eigenstates with temperature $T = \frac{1}{2\pi}$. As we discussed in Section 5.1, the reduced density matrix can be interpreted as the "effective" state that an observer only acting on that part of the Hilbert space sees. Thus, the boosted observer on one side is exposed to a thermal state.

In order to get a more concrete version of Eq. 5.10, we will introduce the Rindler coordinates and solve for the field in this coordinate system. The Rindler coordinates for the left and right Rindler wedges are implicitly defined by the following:

$$x = e^{\xi_R} \cosh \tau_R = -e^{-\xi_L} \cosh \tau_L, \tag{5.13}$$

$$t = e^{\xi_R} \sinh \tau_R = e^{-\xi_L} \sinh \tau_L. \tag{5.14}$$

The picture to have in mind is shown in Fig. 1. The coordinates are chosen so that acting with the boost operator is equivalent to translating τ_R (τ_L) forwards (backwards) in time,

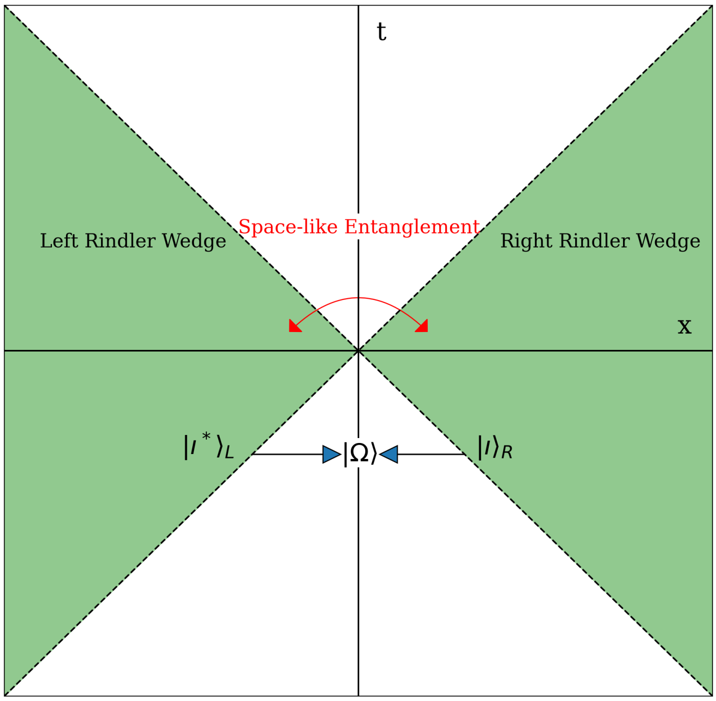

Figure 1. Rindler decomposition of Minkowski spacetime.

and constant ξ_R (ξ_L) corresponds to constant proper acceleration. Our first task will be to solve the Klein-Gordon equation in Rindler coordinates to gain an intuition for the modes in these coordinates. Starting from Eqs. 5.13 and 5.14 we have:

$$dx = x\ d\xi_R + t\ d\tau_R = -x\ d\xi_L - t\ d\tau_L, \tag{5.15}$$

$$dt = t\ d\xi_R + x\ d\tau_R = -t\ d\xi_L - x\ d\tau_L. \tag{5.16}$$

To get the metric in Rindler coordinates we then compute $-dt^2 + dx^2$:

$$\begin{aligned} -dt^2 + dx^2 &= \left(x^2 - t^2\right)\left(-d\tau_{R/L}^2 + d\xi_{R/L}^2\right), \\ &= e^{2\xi_R}\left(-d\tau_R^2 + d\xi_R^2\right), \\ &= e^{-2\xi_L}\left(-d\tau_L^2 + d\xi_L^2\right). \end{aligned} \tag{5.17}$$

So, our metric becomes:

$$\begin{aligned} ds^2 &= e^{2\xi_R}\left(-d\tau_R^2 + d\xi_R^2\right) + dy^2 + dz^2, \\ &= e^{-2\xi_L}\left(-d\tau_L^2 + d\xi_L^2\right) + dy^2 + dz^2. \end{aligned} \tag{5.18}$$

In the right Rindler wedge, the Klein-Gordon equation is:

$$\begin{aligned} &\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = m^2\phi, \\ &\left(-e^{-2\xi_R}\partial_{\tau_R}^2 + e^{-2\xi_R}\partial_{\xi_R}^2 + \partial_y^2 + \partial_z^2\right)\phi = m^2\phi, \end{aligned} \tag{5.19}$$

while in the left Rindler wedge it is:

$$\left(-e^{2\xi_L}\partial_{\tau_L}^2 + e^{2\xi_L}\partial_{\xi_L}^2 + \partial_y^2 + \partial_z^2\right)\phi = m^2\phi. \tag{5.20}$$

To solve this equation generally, we will use separation of variables and look for mode solutions. In the right wedge this looks like:

$$\begin{aligned} f_{R,\omega\vec{k}} &= e^{-i\omega\tau_R}e^{i(k_y y + k_z z)}\psi_{R,\omega\vec{k}}(\xi_R), \\ &= e^{-i\omega\tau_R}e^{i\vec{k}\cdot\vec{y}}\psi_{R,\omega\vec{k}}(\xi_R), \end{aligned} \tag{5.21}$$

where we define the two-vectors $\vec{y} \equiv (y, z)$ and $\vec{k} \equiv (k_y, k_z)$. To figure out what form $\psi_{R,\omega\vec{k}}(\xi_R)$ must take, we will insert our mode solutions into Eq. 5.19. Doing so, we get the following differential equation for $\psi_{R,\omega\vec{k}}(\xi_R)$:

$$\left(-\frac{\partial^2}{\partial\xi_R^2} + \left(m^2 + \vec{k}\cdot\vec{k}\right)e^{2\xi_R} - \omega^2\right)\psi_{R,\omega\vec{k}}(\xi_R) = 0, \tag{5.22}$$

which one may identify as an effective Schrödinger's equation for a particle in an exponential potential. In identical fashion, we assume mode solutions for the left wedge of the form:

$$f_{L,\omega\vec{k}} = e^{-i\omega\tau_L}e^{i\vec{k}\cdot\vec{y}}\psi_{L,\omega\vec{k}}(\xi_L), \tag{5.23}$$

which means that $\psi_{L,\omega\vec{k}}(\xi_L)$ satisfies the following:

$$\left(-\frac{\partial^2}{\partial\xi_L^2} + \left(m^2 + \vec{k}\cdot\vec{k}\right)e^{-2\xi_L} - \omega^2\right)\psi_{L,\omega\vec{k}}(\xi_L) = 0. \tag{5.24}$$

These equations can be solved exactly using Bessel functions. However, it is useful to interpret them by recalling some basic facts about potentials in the Schrödinger equation, assuming our solutions are in some sense normalizable. Specifically, in the right (left) wedge, if ξ_R (ξ_L) goes to negative (positive) infinity, all modes will oscillate, while taking ξ_R (ξ_L) to positive (negative) infinity exponentially suppresses those low energy modes while some higher energy modes can oscillate. There is of course also the transverse momentum term, which, when higher, has the effect of suppressing modes. Put another way, modes are generally confined towards the Rindler horizon, with the lower energy/higher transverse momentum modes confined more strongly.

Now that we have an idea of the classical solutions in the Rindler coordinates, we can quantize the field as in Section 2 by writing it as a sum over all modes where each mode is associated with a creation and annihilation operator. The annihilation operators, $a_{R/L,\omega\vec{k}}$, will annihilate the Rindler vacuum, which I will denote as $|0\rangle$, while the creation operators $a^\dagger_{R/L,\omega\vec{k}}$ will create states of definite boost energy in the respective Rindler wedge when acting on the Rindler vacuum. I want to emphasize here that the Rindler vacuum is *not* the same state as the vacuum $|\Omega\rangle$, which is annihilated by a different set of annihilation operators, namely the ones we defined in Section 2. With this in mind, we can now expand the field in terms of the Rindler modes:

$$\phi = \sum_{\omega,\vec{k}} \left(f_{R,\omega\vec{k}} a_{R,\omega\vec{k}} + f_{L,\omega\vec{k}} a_{L,\omega\vec{k}} + f^*_{R,\omega\vec{k}} a^\dagger_{R,\omega\vec{k}} + f^*_{L,\omega\vec{k}} a^\dagger_{L,\omega\vec{k}} \right). \tag{5.25}$$

We now appeal to Eq. 5.10, the schematic way to express the vacuum state in terms of boost eigenstates in the left and right regions. First, let us consider what a boost eigenstate looks like in one of the wedges. The most general state will have n_1 particles with boost energy ω_1 and momentum $\vec{k}_1$, n_2 particles with boost energy ω_2 and momentum $\vec{k}_2$, and so on. In other words, we can make the following identification with Eq. 5.10:

$$|i\rangle_R = |n_1\rangle_{R,\omega_1\vec{k}_1} \otimes |n_2\rangle_{R,\omega_2\vec{k}_2} \otimes \cdots \otimes |n_N\rangle_{R,\omega_N\vec{k}_N}, \tag{5.26}$$

where:

$$|n\rangle_{R,\omega\vec{k}} \equiv \underbrace{a^\dagger_{R,\omega\vec{k}} a^\dagger_{R,\omega\vec{k}} \cdots a^\dagger_{R,\omega\vec{k}}}_{n \text{ times}} |0\rangle, \tag{5.27}$$

and the state $|i\rangle_R$ has boost energy $\sum_j n_j \omega_j$. To get the sum over all boost eigenstates in Eq. 5.10, we need every combination of Eq. 5.26. To achieve this, we can first fix ω and $\vec{k}$, then sum over all possible n, which will give us all the states where each particle has boost energy ω and transverse momentum $\vec{k}$. This looks like: $\sum_{n=0}^{\infty} |n\rangle_{R,\omega\vec{k}}$. Then, we can vary ω and $\vec{k}$ by taking a tensor product over all possible values. In other words:

$$\sum_i |i\rangle_R = \bigotimes_{\omega,\vec{k}} \sum_{n=0}^{\infty} |n\rangle_{R,\omega\vec{k}}. \tag{5.28}$$

It is possible to verify that every term of the form of Eq. 5.26 appears exactly once in the sum in Eq. 5.28. We are now in a position to write the final form of Eq. 5.10 for the Rindler modes:

$$|\Omega\rangle = \bigotimes_{\omega\vec{k}} \left(\sqrt{1 - e^{-2\pi\omega}} \sum_n e^{-\pi\omega n} |n\rangle_{L,\omega(-\vec{k})} |n\rangle_{R,\omega\vec{k}} \right). \tag{5.29}$$

The negative $\vec{k}$ for the left state is a subtlety to do with the path integral derivation of Eq. 5.10 and more details on this can be found in Harlow [2].

What we have demonstrated in this section is that the vacuum state can be expressed explicitly as an entangled state between distinct regions of spacetime. Furthermore, for a Rindler observer with a constant proper acceleration, the effective state she has access to is a thermal distribution of boost eigenstates with a "temperature" $T = \frac{1}{2\pi}$. The hesitation in calling this quantity a temperature stems from its dimensionless nature. This may seem strange at first, but it is actually a consequence of not identifying an appropriate length scale in defining the Rindler coordinates, which are themselves dimensionless. For an observer with constant proper acceleration a, the natural length scale is $1/a$. If we restore $\hbar$'s, c's, and k_B's, the temperature, known as the *Unruh temperature*, becomes:

$$T_{Unruh} = \frac{\hbar a}{2\pi k_B c}. \tag{5.30}$$

5.3 Rindler vs. Schwarzschild

In this section we justify the advantage of studying the Unruh effect. Namely, we show that the near-horizon metric of the Schwarzschild spacetime in appropriate coordinates very closely resembles the metric of the Rindler observer. This allows us to understand the near-horizon quantum states very well through understanding the thermally entangled states of the Rindler decomposition. Recall that the Schwarzschild metric is given by Eq. 4.8, which we reproduce here with $r_s = 1$:

$$ds^2 = -\frac{r-1}{r}dt^2 + \frac{r}{r-1}dr^2 + r^2 d\Omega_2^2. \tag{5.31}$$

If we change to the tortoise coordinate $r_* = r + \log(r-1)$, the metric becomes:

$$ds^2 = \frac{r-1}{r}\left(-dt^2 + dr_*^2\right) + r^2 d\Omega_2^2. \tag{5.32}$$

Now, if we restrict ourselves to the near-horizon region, i.e. $r \approx 1$, then $r^2 d\Omega_2^2 \approx d\Omega_2^2$, and:

$$\begin{aligned}
\frac{r-1}{r} &= 1 - \frac{1}{r}, \\
&= 1 - \frac{1}{1+(r-1)}, \\
&\approx 1 - (1 - r + 1), \\
&= r - 1, \\
&= e^{\log(r-1)} = e^{r_* - r} \approx e^{r_* - 1},
\end{aligned} \tag{5.33}$$

so our metric becomes:

$$ds^2 \approx e^{r_* - 1}\left(-dt^2 + dr_*^2\right) + d\Omega_2^2. \tag{5.34}$$

This is strikingly similar to the Rindler metric in Eq. 5.18. Indeed, if we identify $dy^2 + dz^2$ with $d\Omega_2^2$, and let $r_* = 2\xi_R + 1 - \log 4$ and $t = 2\tau_R$, they are exactly equivalent. The power of the Rindler spacetime is thus apparent: by associating the tortoise coordinate with the $\xi_{R/L}$ coordinate (which is related to acceleration), and Schwarzschild time with $\tau_{R/L}$, we find spacetime metrics that are effectively the same. This means that the lessons we learned about decomposing the spacetime vacuum according to entangled boost eigenstates in the Rindler metric will prove very fruitful in studying the near-horizon region of a black hole.

6 Black hole evaporation, thermodynamics and the information paradox

In this section, we will compute black hole evaporation and discuss some consequences of the effect, including the thermodynamics of a black hole and the emergence of the information paradox. To achieve this, we will start by solving for the free field in the Schwarzschild metric, then study the effective Schrödinger equation which emerges for the radial component, just as was done in Section 5.2 for the Rindler metric. Equipped with this, we will deduce the associated quantum state outside the black hole horizon then go on to compute the rate of black hole evaporation.

6.1 Free scalar field in the Schwarzschild metric

In this section, we will solve the free scalar field EOM in the Schwarzschild metric. Our starting point is Eq. 4.37, which we derived in Section 4.2 by applying the least action principle to the free scalar Lagrangian. We rewrite here for convenience:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) = m^2\phi. \tag{6.1}$$

Note that we will use φ to represent the azimuthal angle while ϕ represents the field. I will start by expressing the Schwarzschild metric and its inverse in matrix form:

$$g_{\mu\nu} = \begin{pmatrix} -\frac{r-r_s}{r} & & & \\ & \frac{r}{r-r_s} & & \\ & & r^2 & \\ & & & r^2\sin\theta \end{pmatrix}, \tag{6.2}$$

$$g^{\mu\nu} = \begin{pmatrix} -\frac{r}{r-r_s} & & & \\ & \frac{r-r_s}{r} & & \\ & & \frac{1}{r^2} & \\ & & & \frac{1}{r^2\sin\theta} \end{pmatrix}. \tag{6.3}$$

From here, computing the determinant of $g_{\mu\nu}$ is straightforward:

$$g \equiv \det\left(g_{\mu\nu}\right) = -r^4\sin^2\theta, \tag{6.4}$$

and we can immediately compute the $\sqrt{-g}$ factor:

$$\sqrt{-g} = r^2\sin\theta. \tag{6.5}$$

Plugging this all into Eq. 4.37 we have:

$$g^{00}\partial_t^2\phi + \frac{1}{r^2}\partial_r\left(r^2 g^{11}\partial_r\phi\right) + \frac{g^{22}}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\phi\right) + g^{33}\partial_\varphi^2\phi = m^2\phi, \tag{6.6}$$

which becomes:

$$\begin{aligned} -\frac{r}{r-r_s}\partial_t^2\phi + \frac{1}{r^2}&\left[\left(2r-r_s\right)\partial_r\phi + r\left(r-r_s\right)\partial_r^2\phi\right] \\ &+ \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\phi\right) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\phi = m^2\phi. \end{aligned} \tag{6.7}$$

Following [2], we use separation of variables and assume a solution of the following form:

$$\phi(t,r,\theta,\varphi) = \frac{1}{r}Y_{lm}(\theta,\varphi)e^{-i\omega t}\psi_{\omega l}(r), \tag{6.8}$$

where Y_{lm} are the usual spherical harmonics, and the subscript m denotes the quantum number for the complex exponential factor $e^{im\varphi}$ of Y_{lm} and is not to be confused with the mass of the scalar field, m. Recall that the spherical harmonics satisfy the following:

$$\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta Y_{lm}\right) + \frac{1}{\sin^2\theta}\partial_\varphi^2 Y_{lm} = -l(l+1)Y_{lm}. \tag{6.9}$$

We will eventually like to write an effective Schrödinger equation for the radial component, $\psi_{\omega l}$ as a function of the tortoise coordinate, r_*. Recall that this is defined as:

$$r_* \equiv r + r_s \log\left(\frac{r - r_s}{r_s}\right). \tag{6.10}$$

To help us later, let us compute ∂_r in terms of this tortoise coordinate:

$$\begin{aligned}\frac{\partial}{\partial r} &= \frac{\partial r_*}{\partial r}\frac{\partial}{\partial r_*},\\ &= \frac{r}{r - r_s}\frac{\partial}{\partial r_*},\end{aligned} \tag{6.11}$$

and ∂_r^2 as well:

$$\begin{aligned}\frac{\partial^2}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial r_*}{\partial r}\frac{\partial}{\partial r_*}\right),\\ &= \frac{\partial^2 r_*}{\partial r^2}\frac{\partial}{\partial r_*} + \frac{\partial r_*}{\partial r}\frac{\partial}{\partial r}\frac{\partial}{\partial r_*},\\ &= \frac{\partial^2 r_*}{\partial r^2}\frac{\partial}{\partial r_*} + \left(\frac{\partial r_*}{\partial r}\right)^2\frac{\partial^2}{\partial r_*^2},\\ &= -\frac{r_s}{(r - r_s)^2}\frac{\partial}{\partial r_*} + \frac{r^2}{(r - r_s)^2}\frac{\partial^2}{\partial r_*^2}.\end{aligned} \tag{6.12}$$

Now, we compute the radial derivatives that appear in Eq. 6.7. First, we have:

$$\begin{aligned}\partial_r\left(\frac{\psi}{r}\right) &= -\frac{\psi}{r^2} + \frac{1}{r}\partial_r\psi,\\ &= -\frac{\psi}{r^2} + \frac{1}{r - r_s}\partial_{r_*}\psi.\end{aligned} \tag{6.13}$$

Then:

$$\begin{aligned}\partial_r^2\left(\frac{\psi}{r}\right) &= \partial_r\left(-\frac{\psi}{r^2} + \frac{1}{r}\partial_r\psi\right)\\ &= \frac{2\psi}{r^3} - \frac{2}{r^2}\partial_r\psi + \frac{1}{r}\partial_r^2\psi,\\ &= \frac{2\psi}{r^3} - \frac{2}{r\,(r - r_s)}\partial_{r_*}\psi + \frac{1}{(r - r_s)^2}\left[r\partial_{r_*}^2\psi - \frac{r_s}{r}\partial_{r_*}\psi\right].\end{aligned} \tag{6.14}$$

If we plug this into the radial part of Eq. 6.7, we have:

$$\frac{1}{r^2}\left[(2r - r_s)\,\partial_r\phi + r\,(r - r_s)\,\partial_r^2\phi\right] = \frac{Y_{lm}e^{-i\omega t}}{r}\left[\frac{r}{r - r_s}\partial_{r_*}^2\psi - \frac{r_s}{r^3}\psi\right]. \tag{6.15}$$

For the angular part of Eq. 6.7, we have the following after inserting ϕ from Eq. 6.8 and utilizing the property of spherical harmonics in Eq. 6.9:

$$\begin{aligned}\frac{1}{r^2}\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\phi\right) + \frac{1}{\sin^2\theta}\partial_\varphi^2\phi\right] &= \frac{e^{-i\omega t}\psi}{r^3}\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta Y_{lm}\right) + \frac{1}{\sin^2\theta}\partial_\varphi^2 Y_{lm}\right],\\ &= -\frac{e^{-i\omega t}\psi}{r^3}l(l+1)Y_{lm}.\end{aligned} \tag{6.16}$$

Finally, the temporal part of Eq. 6.7, after plugging in Eq. 6.8, is:

$$-\frac{r}{r-r_s}\partial_t^2\phi = \omega^2\frac{r}{r-r_s}Y_{lm}e^{-i\omega t}\frac{\psi}{r}. \tag{6.17}$$

Putting all these pieces together, we get the following equation for the radial component:

$$\begin{aligned} -\frac{d^2}{dr_*^2}\psi + \frac{r-r_s}{r^3}\left(m^2r^2 + l(l+1) + \frac{r_s}{r}\right)\psi = \omega^2\psi, \\ \frac{d^2\psi}{dr_*^2} = \left(V(r_*) - \omega^2\right)\psi, \end{aligned} \tag{6.18}$$

where we define the potential $V(r_*)$ as:

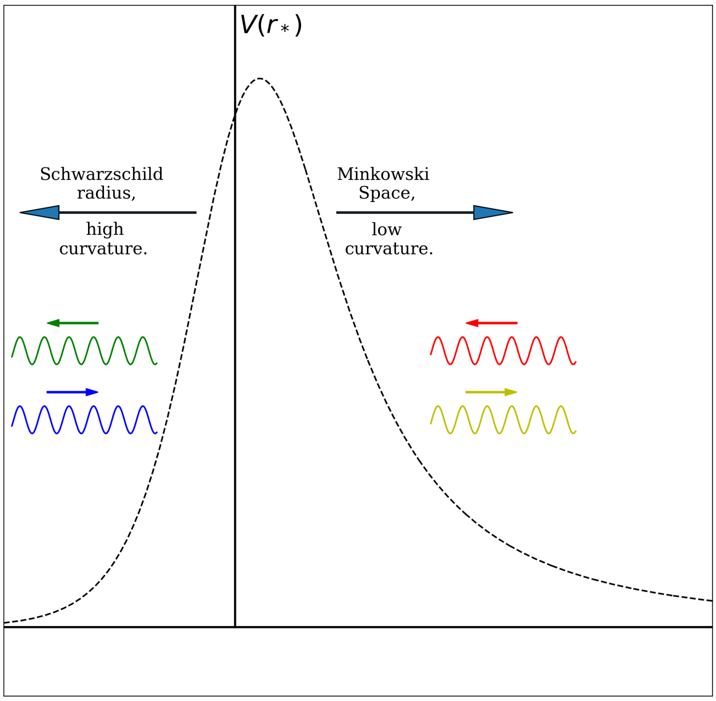

Figure 2. Effective potential for the radial component of the free scalar field in the Schwarzschild metric, where we set $m = 0$ and choose $l = 1$.

$$V(r_*) \equiv \frac{r-r_s}{r^3}\left(m^2r^2 + l(l+1) + \frac{r_s}{r}\right) \tag{6.19}$$

Just as with the free field in Rindler coordinates, we have found an effective Schrödinger equation for the radial component ψ. In Fig. 2 we plot this potential.

The problem that emerges is a scattering problem between the highly-curved near-horizon region, and the asymptotically flat region. By observing Eq. 6.18, we can glean information about the relevant modes. As expected, we see that freely propagating modes only occur when $\omega^2 > V(r_*)$. We will assume that we are studying the state after black hole formation has stabilized and there is no infalling matter. We are then interested in the modes that escape the near horizon region towards the essentially flat region outside. Since we care about those modes that are actually able to get through the barrier, we will study the case where the mass disappears, i.e. $m \to 0$. Additionally, notice that for higher angular momentum modes, i.e. when l is large, the potential barrier becomes much higher, so the modes most likely to tunnel through are those with lower angular momentum. An argument based on this sort of logic makes clear that Hawking radiation will be dominated by the lowest-spin field present, which would be photons in our universe based on the standard model.

To make progress from here, we need an understanding of the quantum state that exists outside of the black hole, and for this we ideally require suitable initial conditions. The question of the state is a very interesting one and details can be found in Hawking's paper [3]. The upshot is that we can effectively "guess" what the correct state should look like after matter stops falling in, and go from there. We will do this in the next section and use our result to compute Hawking radiation and black hole evaporation.

6.2 Black hole evaporation and thermodynamics

In this section we first demonstrate Hawking radiation and compute the rate at which energy is lost by a black hole before going on to discuss the thermodynamic properties of black holes.

As we mentioned in Section 6.1, to make progress we need to understand the state of the black hole exterior. The state we will write down can be justified heuristically by assuming the black hole has stabilized, that no matter is falling into it, and that the near-horizon region is thermally entangled with the interior, analogous to the entangled vacuum state in the Unruh effect. With this in mind, we write the state as a product of the Minkowski vacuum and a thermally entangled state, completely analogous to Eq. 5.29:

$$\rho_{\text{exterior}} = |\Omega\rangle\langle\Omega| \otimes \left(\bigotimes_{\omega,l,m} \left[(1 - e^{-\beta\omega}) \sum_n e^{-\beta\omega n} |n\rangle_{\omega lm} \langle n| \right] \right), \tag{6.20}$$

where:

$$|n\rangle_{\omega lm} \equiv \underbrace{a^\dagger_{\omega lm} \cdots a^\dagger_{\omega lm}}_{\text{n times}} |\Omega\rangle_S \,, \tag{6.21}$$

where $|\Omega\rangle_S$ is the vacuum for the Schwarzschild region, $\beta = \frac{1}{T_{Hawking}}$, $T_{Hawking} = \frac{T_{Unruh}}{2}$, and $a^\dagger_{\omega lm}$ is the creation operator for the ωlm mode on the Schwarzschild vacuum. The reason for the factor of 2 difference between the Unruh temperature and Hawking temperature is the relationship between the Unruh "time" coordinate, τ, and Schwarzschild time, t. Recall that when we expressed the Unruh temperature dimensionally in Eq. 5.30, the natural length scale $\frac{1}{a}$ was chosen, where a was the proper acceleration. For $T_{Hawking}$ we do a similar thing, taking the scale to be r_S, the Schwarzschild radius. Thus, the dimensionful Hawking temperature is:

$$T_{Hawking} = \frac{\hbar c^3}{8\pi k_B G M}. \tag{6.22}$$

Now that we have an idea of the black hole exterior state, we can go on to calculate the rate of energy loss from the black hole at far distances. To do this, we integrate the average energy flux over a sphere of radius r then take $r \to \infty$. Recalling that T^{0r} is the energy flux through the radial direction in the energy-momentum tensor $T_{\mu\nu}$, we have the following:

$$\frac{dE}{dt} = -\lim_{r\to\infty} \int d\theta d\varphi \; r^2 \sin\theta \left\langle T^{0r} \right\rangle, \tag{6.23}$$

where $\langle T^{0r}\rangle$ is the expectation value for the given state of the tensor component (recall that $T_{\mu\nu}$ is expressed in terms of the fields, which are themselves operators, hence the expectation values). To find the energy-momentum tensor, we invoke Eq. 4.5. First, though we prove the following result:

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu}. \tag{6.24}$$

To see why this is true, first note that for a matrix M, $\log(\det M) = \mathrm{Tr}(\log M)$. Then, taking differentials on either side we have the following:

$$\frac{\delta(\det M)}{\det M} = \mathrm{Tr}\left(M^{-1}\delta M\right). \tag{6.25}$$

If we apply this to $g_{\mu\nu}$, we have:

$$\frac{\delta g}{g} = g^{\mu\nu}\delta g_{\mu\nu}, \tag{6.26}$$

and so:

$$\frac{\delta}{\delta g^{\mu\nu}(x)} g(x') = g(x') g_{\mu\nu}(x') \delta^{(4)}(x-x'). \tag{6.27}$$

With this in hand, we turn to the functional derivative in Eq. 4.5:

$$\begin{aligned}
\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}(x)} &= \frac{\delta}{\delta g^{\mu\nu}(x)} \int d^4x' \left(-\frac{1}{2} g^{\alpha\beta}(x')\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2\right), \\
&= \int d^4x' \sqrt{-g}\left[-\frac{1}{2}\delta^{(4)}(x-x')\delta^\alpha_\mu\delta^\beta_\nu\partial_\alpha\phi\partial_\beta\phi\right] \\
&+ \int d^4x' \left[-\frac{1}{2}g^{\alpha\beta}(x')\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2\right] \frac{\delta}{\delta g^{\mu\nu}(x)}\sqrt{-g}, \\
&= -\frac{1}{2}\sqrt{-g}\partial_\mu\phi\partial_\nu\phi + \int d^4x' \left[-\frac{1}{2}g^{\alpha\beta}(x')\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2\right] \frac{-g}{2\sqrt{-g}} g^{\mu\nu}\delta^{(4)}(x-x'), \\
&= -\frac{1}{2}\sqrt{-g}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(g^{\alpha\beta}(x)\partial_\alpha\phi\partial_\beta\phi + m^2\phi^2\right)\right].
\end{aligned} \tag{6.28}$$

Thus, for a free scalar field, the stress-energy tensor is:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(g^{\alpha\beta}(x)\partial_\alpha\phi\partial_\beta\phi + m^2\phi^2\right). \tag{6.29}$$

As mentioned above, we are interested in the T^{0r} component. Recalling that $T^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}T_{\alpha\beta}$, we have:

$$\begin{aligned} T^{0r} &= g^{\mu 0}g^{r\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g^{0r}\left(g^{\alpha\beta}(x)\partial_\alpha\phi\partial_\beta\phi + m^2\phi^2\right), \\ &= g^{00}g^{rr}\partial_0\phi\partial_r\phi, \\ &= -\dot{\phi}\partial_r\phi. \end{aligned} \tag{6.30}$$

We will start by examining the expectation value for the state with exactly one mode. Thus:

$$\langle T^{0r}\rangle = \langle\Omega|a_{\omega lm}\dot{\phi}\partial_r\phi a^\dagger_{\omega lm}|\Omega\rangle\,. \tag{6.31}$$

As before, we expand the field in terms of the ωlm modes, giving:

$$\phi = \sum_{\omega,l,m}\left[f_{\omega lm}a_{\omega lm} + f^*_{\omega lm}a^\dagger_{\omega lm}\right]. \tag{6.32}$$

It is not too difficult, but rather tedious, to show that after utilizing the commutation relations for the annihilation and creation operators, and normal ordering, only two terms survive, leaving us with:

$$\langle T^{0r}\rangle = \dot{f}^*_{\omega lm}\partial_r f_{\omega lm} + \dot{f_{\omega lm}}\partial_r f^*_{\omega lm}. \tag{6.33}$$

Since we will be taking $r \to \infty$, it is now useful to re-express our modes to reflect this approximation. Specifically, we know that the radial function $\psi_{\omega l}(r)$ will look like a free particle at large distances, and thus the function is just the wavefunction for a free particle that is weighted by the probability amplitude of transmission:

$$\psi_{\omega l} = \frac{T_{\omega l}}{\sqrt{2\omega}}e^{i\omega r}, \tag{6.34}$$

where $T_{\omega l}$ is the transmission probability amplitude, and $\sqrt{2\omega}$ is a normalization factor. Thus, we have for our expectation value:

$$\begin{aligned} \langle T^{0r}\rangle &= i\omega f^*_{\omega lm}f_{\omega lm}\left(i\omega - \frac{1}{r}\right) + i\omega f_{\omega lm}f^*_{\omega lm}\left(i\omega + \frac{1}{r}\right), \\ &= -2\omega^2\frac{Y_{lm}}{Y^*_{lm}}\frac{|T_{\omega l}|^2}{2\omega}, \\ &= -\frac{\omega}{r^2}Y_{lm}Y^*_{lm}|T_{\omega l}|^2. \end{aligned} \tag{6.35}$$

So for one particle in mode ωlm we have:

$$\begin{aligned} \frac{dE}{dt} &= \lim_{r\to\infty}\int d\theta d\varphi r^2\sin\theta\,\langle\dot{\phi}\partial_r\phi\rangle\,, \\ &= -\omega|T_{\omega l}|^2\int d\theta d\varphi\sin\theta Y^*_{lm}Y_{lm}, \\ &= -\omega|T_{\omega l}|^2. \end{aligned} \tag{6.36}$$

At this point it is interesting to reinterpret the meaning of the transmission probability, $|T_{\omega l}|^2$. By time-reversal symmetry, one can verify that the transmission probability of some mode has to equal the absorption probability. So, we can think of $|T_{\omega l}|^2$ as the absorption probability of some mode, $P_{abs}(\omega l)$. Unfortunately, there is no closed-form description of $P_{abs}(\omega l)$, however it can be solved numerically from the potential $V(r_*)$.

We showed that the energy lost by a black hole due to a ωlm mode is $\omega P_{abs}(\omega, l)$. To then get the total rate of energy loss from the black hole, we must compute the average number of particles found in each mode and add the contributions from each mode. From Eq. 6.20, we see that the expectation value of n in the mode ωlm is:

$$\begin{aligned}
\langle n\rangle &= \mathrm{Tr}\left(\rho_{\text{exterior}} n_{op}^{\omega lm}\right), \\
\langle n\rangle &= \mathrm{Tr}\left(|\Omega\rangle\langle\Omega|\right)\mathrm{Tr}\left((1-e^{-\beta\omega})\sum_{n=0}^{\infty} e^{-\beta\omega n}\,|n\rangle_{\omega lm}\langle n|\, n_{op}^{\omega lm}\right), \\
\langle n\rangle &= \mathrm{Tr}\left((1-e^{-\beta\omega})\sum_{n=0}^{\infty} n e^{-\beta\omega n}\,|n\rangle_{\omega lm}\langle n|\right), \\
\langle n\rangle &= \left(1-e^{-\beta\omega}\right)\sum_{n=0}^{\infty} n e^{-\beta\omega n}\mathrm{Tr}\left(|n\rangle_{\omega lm}\langle n|\right), \\
\langle n\rangle &= \left(1-e^{-\beta\omega}\right)\sum_{n=0}^{\infty} n e^{-\beta\omega n}, \\
\langle n\rangle &= \frac{1}{e^{\beta\omega}-1}.
\end{aligned} \tag{6.37}$$

where $n_{op}^{\omega lm}$ is the number operator for the ωlm state which acts trivially on other mode states. Thus, the total energy loss from the black hole is given by:

$$\frac{dE}{dt} = -\sum_{l,m}\int_0^{\infty}\frac{d\omega}{2\pi}\frac{\omega P_{abs}(\omega, l)}{e^{\beta\omega}-1}. \tag{6.38}$$

If it were not for the factor $P_{abs}(\omega, l)$, this would be the energy loss from a perfectly radiating blackbody. The factor $P_{abs}(\omega, l)$ is thus called a greybody factor, since it causes a deviation from blackbody radiation.

Now that we have established the mechanism for black hole evaporation and have associated the notion of temperature with a black hole, it is interesting to study the thermodynamic properties of the black hole. The hawking temperature, after setting $\hbar = c = k_B = 1$, is given by:

$$T_{Hawking} = \frac{1}{8\pi GM}. \tag{6.39}$$

Thus, if we identify the black hole's mass with energy, we can compute the entropy of the black hole by recalling the canonical definition in statistical mechanics and assuming that

$S(E=0)=0$:

$$\begin{aligned}
&\frac{dS}{dE}=\frac{1}{T},\\
&\int dS=\int 8\pi GMdE,\\
&S=4\pi GM^2,\\
&S=4\pi r_s^2\times\frac{1}{4G},\\
&S=\frac{A}{4G},\\
&S=2\pi\frac{A}{l_p^2},
\end{aligned}\tag{6.40}$$

where $l_p\equiv\sqrt{8\pi G}$ is the Planck length. This remarkable expression is known as the Bekenstein-Hawking formula. It is particularly interesting since it is purely thermodynamic in nature and reveals no information about the specific form of the microstates of the black hole. Apparently, all we need to characterize the number of microstates of the black hole is its mass. Additionally, as the last part of the expression makes explicit by comparing the black hole surface area to the Planck scale, this entropy is enormous. For a solar mass black hole, it is of order 10^{78}!

6.3 The information problem

Now that we have covered the basics of black hole evaporation, we are in a position to discuss the problem that emerges relating to the conservation of information. In classical physics, time evolution is generated by a Hamiltonian, and by changing the sign of the Hamiltonian, one can always "run back" time to infer the initial state. A similar thing holds in quantum mechanics, where the time-evolution operator is unitary, and by changing the sign of the Hamiltonian, we can evolve a state backwards in time to infer its initial state. Quite often, we may not think of quantum mechanics in this deterministic way because when we make measurements in the theory with some external apparatus, often times we are only guaranteed the probability of given outcomes. However, it can be shown that by including the apparatus in the system itself that the time-evolution is in fact unitary.

Now, turning to black holes, let us consider starting with some pure state $|\psi\rangle$ that collapses and forms a black hole. After forming, we are left with the state given by Eq. 6.20 outside the black hole. As time goes on, this state becomes more and more mixed, so its entanglement entropy increases, until the black holes reaches the size of order the Planck scale, at which point one of the following three possibilities occurs:

1. Evaporation halts at the Planck scale, leaving behind a *remnant*. In order for unitary evaporation to hold, the final state must be pure, which means the remnant must have extremely high entanglement entropy, far beyond the entropy expected from the Bekenstein-Hawking entropy in Eq. 6.40.

2. Evaporation continues normally and the final burst of evaporation, on grounds of energy conservation, cannot purify the entanglement entropy of the previous radiation.

This means there is no unitary map from the initial state to the final state, so we have *information loss.*

3. The state given by Eq. 6.20 is only approximately correct, and information is carried through very subtle correlations between the evaporating photons which keeps the total state pure. In this case, the thermal nature of the radiation is merely a consequence of looking at small chunks of radiation at a time.

The issues with each of these interpretations is what's known as *the black hole information problem.*

7 Conclusions

The unification of quantum mechanics and general relativity remains an elusive open problem in theoretical physics. In these notes, we showed that progress can be made in certain limiting regimes. However, their combination into some unified theory will probably require a completely new way to look at the world, in much the same way that quantum theory and general relativity drastically altered our descriptions of physical phenomena.

It is curious to compare to the historical development of thermodynamics and statistical mechanics to the progression of black hole thermodynamics. Classical thermodynamics deals only with those bulk features of a system which are readily measured, such as temperature, pressure, volume, entropy, etc. It wasn't until the likes of Boltzmann, Maxwell, Gibbs, and others that we were able to probe deeper on the meaning of these quantities. This led to the birth of statistical mechanics, were it became clear that these macroscopic phenomena could be naturally explained in terms of microscopic configurations. Of course, statistical mechanics has since developed into a discipline in its own right with very powerful applications. The similarities of this historical development to the progress of black hole thermodynamics is striking. We saw in these notes that the entropy of a black hole can be characterized by the macroscopic quantity of surface area through the Bekenstein-Hawking formula. If we thus put faith in the general principles of thermodynamics and statistical mechanics, which have proved accurate in a wide spectrum of physical situations, then we naturally expect the next stage to be the birth of the statistical mechanics of black hole states. There has been tons of work on this problem, with many solutions emerging in string theory. For this reason, it is thus an exciting time to be studying black holes and quantum gravity! Indeed, if we can get a strong grip on this problem, it would go a long way in developing a consistent theory of quantum gravity.

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