Estimation of Frequency and Phase of Sinusoidal Signal

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1 Introduction

Estimating the frequency and phase of a single noisy Sinusoidal signal has been a popular topic in communications and signal processing.
A sinusoidal model was described by

$$y(t) = \alpha \sin(2\pi\omega t + \gamma)$$

where:

$$\begin{aligned} \alpha &= \text{amplitude}, \\ \omega &= \text{angular frequency, in radian/s}, \\ \gamma &= \text{phase shift, in radian.} \end{aligned}$$

With amplitude, $\alpha = 4$, the height of the wave was increased by four times compared to the reference wave. Angular frequency, ω represented the quantity of a complete wave in one second. With amplitude, $\alpha = 4$, there were four complete waves exist in one second while the height and position of the wave remained stationery. The phase shift, γ represented the horizontal displacement of the wave relative to the initial point. If phase shift=2, the first point of the wave shifted by two units horizontally to the left.

This report aimed to estimate the value of amplitude, α, angular frequency, ω and phase shift, γ using Bayesian inference. The expected values of these parameters, along with the assumed model, were then evaluated to assess their ability to estimate the sinusoidal model effectively.

2 Data

The provided data, y, consisted of 100 points of a sinusoidal wave collected over 1 second, plotted against time.

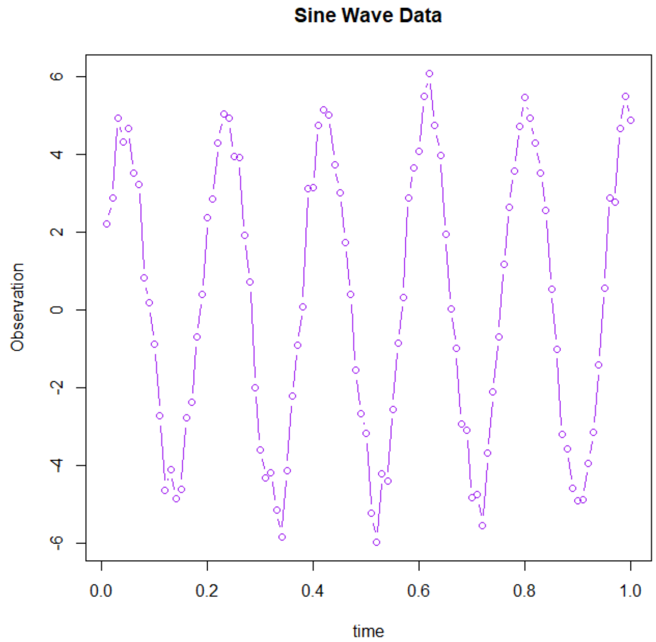

Figure 1: Sinusoidal Wave Data, y(t) Collected at time, t

3 Methods

3.1 Priors

The model for signal data collected at time t, denoted as $y(t)$, was assumed to be a sinusoidal wave model with Gaussian white noise, ϵ_t.

$$y(t) = \alpha sin(2\pi\omega t + \gamma) + \epsilon_t, \quad \epsilon_t \overset{ind}{\sim} N(0, \sigma^2)$$

The initial variance, σ^2, was assumed to be a standard normal distribution.

$$\sigma^2 \overset{\text{ind}}{\sim} \mathcal{N}(0, 1),$$

The amplitude of the wave was assumed to be uniformly distributed between [0,10].

$$\alpha \overset{\text{ind}}{\sim} \text{Uniform}(0, 10),$$

Angular frequency, ω was assumed to follow a uniform distribution

$$\omega \overset{\text{ind}}{\sim} \text{Uniform}(0, 6),$$

In order to set a reasonable bound to the prior models, y should range from [-∞,+∞]. Since the sine function is periodic with period 2π, the reasonable range of phase shift, γ should exhibit a span length of 2π and follow a uniform distribution between [$-\pi$,$+\pi$].

$$\gamma \overset{\text{ind}}{\sim} \text{Uniform}(-\pi, +\pi).$$

The posterior distribution:

$$p(\alpha, \omega | y_t) = p(y|\alpha, \omega) \cdot p(\alpha, \omega)$$

3.2 Model

Stan was used with rstan package to run MCMC on the data. Three chains were run for 15,000 iterations with a burn-in period of 5,000 iterations. The traceplot was used to indicate whether the chains mixed perfectly. Meanwhile, the posterior mean of each parameter has been plotted against the iterations to see if the values converged within burn-in period.
Meanwhile, I was interested in the relationship between parameters. Correlation plot in Figure 3 was used to see if the parameters were independent from each other.

4 Result

4.1 Tables

The posterior distributions of the parameters were summarized. The largest Monte Carlo error recorded was 0.014.

Table 1: Summary of Parameters and Posterior Distribution

Parameter	Mean	SD	2.5%	97.5%	MC Error	n_{eff}	Rhat
Amplitude, α	5.04	0.07	4.90	5.17	0.001	23577	1
Phase shift, γ	0.15	0.03	0.09	0.21	0	19002	1
Initial Variance, σ^2	0.81	0.46	0.24	1.99	0.014	21157	1
Angular frequency, ω	5.25	0.01	5.24	5.27	0	18934	1

Table 2: Correlation values of parameters

	Amplitude, α	Angular frequency, ω	Phase shift, γ	Initial Variance, σ^2
Amplitude, α	1.00	0.00	-0.01	0
Angular frequency, ω	0.00	1.00	-0.88	0
Phase shift, γ	-0.01	-0.88	1.00	0
Initial Variance, σ^2	0.00	0.00	0.00	1

4.2 Figures

The correlation plots of parameters are plotted to observe their relationship. As mentioned in the previous section, the angular frequency, γ and phase shift, ω are correlated, again proved in Figure 3.

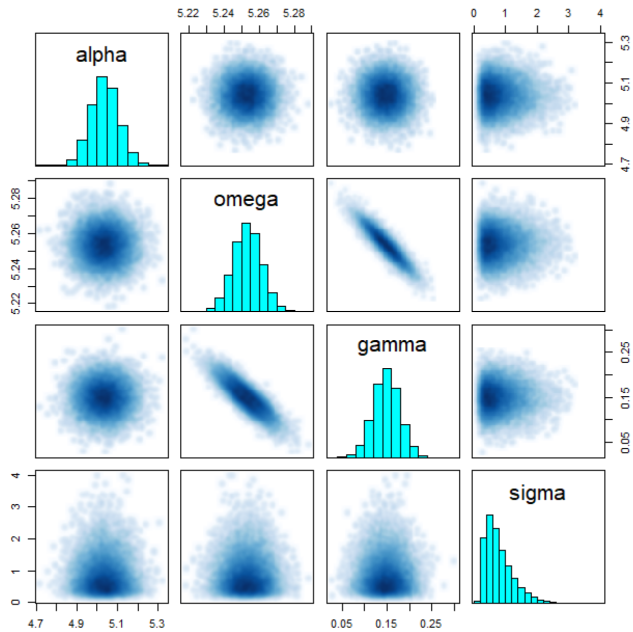

Figure 2: Correlation Plot of Parameters(Amplitude, α, Phase Shift, γ, Variance, σ^2, Angular Frequency, ω)

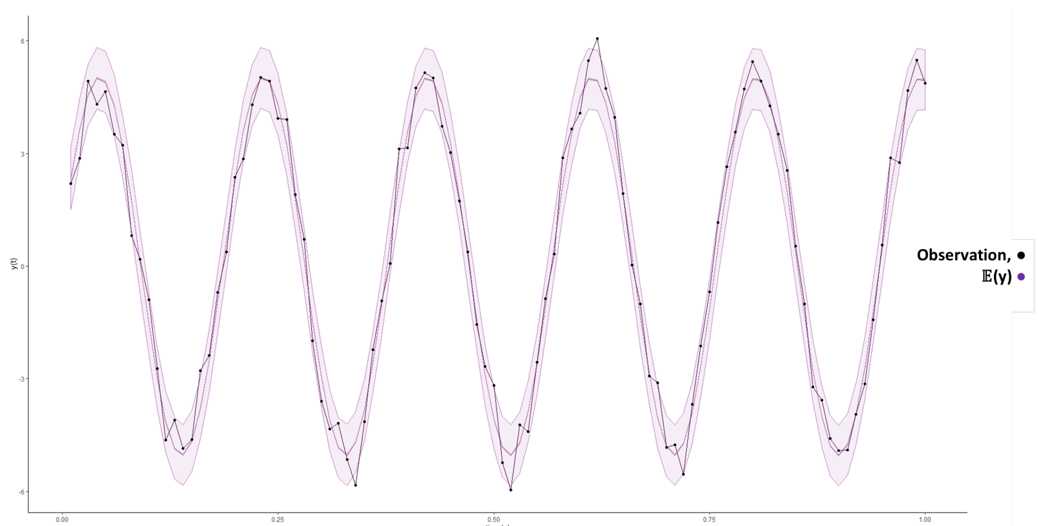

Figure 3: Posterior mean of y, $\hat{y}$(t) and observation,y plotted against time(s)

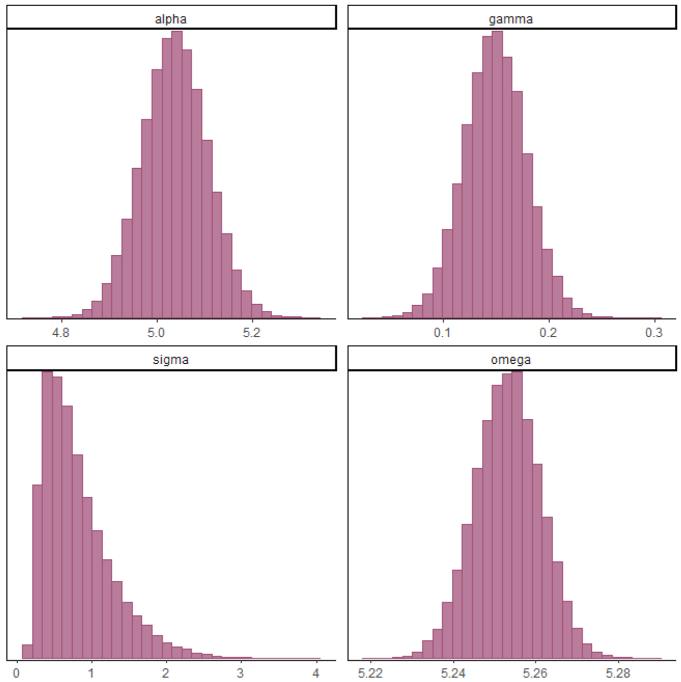

Figure 4: Posterior Histogram of Parameters based on posterior samples generated using MCMC

5 Conclusion and Discussion

This report aims to estimate the important parameters α, ω and γ of a sinusoidal model using Bayesian inference. The values of these parameters are estimated using posterior mean and 95% credible interval, as shown in Table 1.

As shown in Figure 2, the posterior mean of $y(t)$ is plotted against observation time and the 95% credible intervals fitted the observation data closely with several outliners. This model was shown to be sufficient to estimate simple periodic sine functions.

Table 2 showed that the the correlation between ω and γ is -0.88. The sine function is periodic with period 2π, hence $2\pi\omega t + \gamma$ ranges with a length of $2/pi$. At the initial point, $t = 0$, if $2\pi\omega t + \gamma = 1$, then γ needs to be euqal to $\pi/2$. Hence, these two parameters are strongly correlated.

5.1 Future Work

To simplify the model and avoid the identifiability problem, the model for y was transformed so that the priors can be described using uniform distribution instead of other distributions. However, another common technique used in control theory was building a non-linear dynamic model by rewriting $y = sin(2\pi\omega t + \gamma)$ to $y = sin(\phi)$ where $\phi = 2\pi\omega t + \gamma$. And ϕ was defined as the instantaneous phase and ranges from $[0, 2\pi]$ or $[-\pi, +\pi]$. The potential of von Mises distribution which was a circular distribution, might be advantageous compared to the uniform distribution when modeling periodic or angular variables to capture the cyclic nature of the variable.