

## 1 Recap: Pseudorandom Generators

1. It is possible to construct a hard-core predicate for any one-way function. Let  $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$  be any one-way function (or permutation). We define  $f'(r, x) = (r, f(x))$  for  $|r| = |x|$ . Then  $f'$  is also a one way function (or permutation, respectively), and  $h(r, x) = \langle r, x \rangle \bmod 2$  is a hard core predicate for  $f'$ .
2. A pseudorandom generator exists under the assumption that a one-way permutation exists. Formally, if  $f$  is a OWP and  $h$  is a hard-core predicate for  $f$ , then  $G(s) = (f(s), h(s))$  is a PRG with output length  $\ell(n) = n + 1$ .
3. If there exists a PRG  $G(s) = (f(s), h(s))$  with output length  $\ell(n) = n + 1$ , then

$$G'(s) = (h(s), h(f(s)), h(f^{(2)}(s)), \dots, h(f^{(m-1)}(s)))$$

is a PRG of output length  $m$  for any  $m = \text{poly}(|s|)$ .

While it is well-known that the existence of a PRG implies that a OWF must exist, is the converse also true? That is, does the existence of a one-way function also imply that a pseudorandom generator must exist? In fact, this turns out to be true as well. Hastad, Impagliazzo, Levin, and Luby established in 1989 that a PRG can be constructed from any OWF. Their construction is much more complicated than the one for OWPs, because it must address the issue that the OWF  $f$  may be very “unstructured,” and thus the distribution of  $f(x)$  may be very different from uniform, even when  $x$  is uniform. The HILL PRG construction utilizes a random seed of length  $n^{10}$  for a OWF of input length  $n$ . The details of the construction are beyond the scope of this class.

## 2 Pseudorandom Functions

### 2.1 Preliminary Concepts

Having already developed a precise definition for a pseudorandom string of bits, a natural extension is, what would a random function look like?

A function from  $\{0, 1\}^n$  to  $\{0, 1\}$  is given by specifying an output bit for every one of its inputs, of which there are  $2^n$ . Therefore, the set of all functions from  $\{0, 1\}^n$  to  $\{0, 1\}$  contains exactly  $2^{2^n}$  functions; a “random function” (with this domain and range) is a uniformly choice from this set. Such a function can also be viewed as a uniformly random  $2^n$ -bit string, which simply lists all the function’s outputs. However, stated this way, it is impossible to even look at the entire string efficiently (in  $\text{poly}(n)$  time). Therefore, we define a model in which we give oracle access to a function.

Writing  $\mathcal{A}^f$  signifies that  $\mathcal{A}$  has query access to  $f$ , i.e.,  $\mathcal{A}$  can (adaptively) query the oracle on any input  $x$  and receive the output  $f(x)$ . However,  $\mathcal{A}$  only has a “black-box” (input/output) view of  $f$ , without any knowledge of how the function  $f$  is evaluated.

**Definition 2.1** (Oracle indistinguishability). Let  $\mathcal{O} = \{O_n\}$  and  $\mathcal{O}' = \{O'_n\}$  be ensembles of probability distributions over functions from  $\{0, 1\}^{\ell_1(n)}$  to  $\{0, 1\}^{\ell_2(n)}$ , for some  $\ell_1(n), \ell_2(n) = \text{poly}(n)$ . We say that  $\mathcal{O} \stackrel{c}{\approx} \mathcal{O}'$  if, for all nuppt distinguishers  $\mathcal{D}$ ,

$$\text{Adv}_{\mathcal{O}, \mathcal{O}'}(\mathcal{D}) := \left| \Pr_{f \leftarrow O_n} [\mathcal{D}^f(1^n) = 1] - \Pr_{f \leftarrow O'_n} [\mathcal{D}^f(1^n) = 1] \right| = \text{negl}(n).$$

Naturally, we say that  $\mathcal{O} = \{O_n\}$  is pseudorandom if

$$\mathcal{O} \stackrel{c}{\approx} \left\{ U\left( \{0,1\}^{\ell_1(n)} \rightarrow \{0,1\}^{\ell_2(n)} \right) \right\},$$

i.e., if no efficient adversary can distinguish (given only oracle access) between a function sampled according to  $O_n$ , and a uniformly random function, with more than negligible advantage.

**Definition 2.2 (PRF Family).** A family  $\{f_s : \{0,1\}^{\ell_1(n)} \rightarrow \{0,1\}^{\ell_2(n)}\}_{s \in \{0,1\}^n}$  is a pseudorandom function family if it is:

- Efficiently computable: there exists a deterministic polynomial-time algorithm  $F$  such that  $F(s, x) = f_s(x)$  for all  $s \in \{0,1\}^n$  and  $x \in \{0,1\}^{\ell_1(n)}$ .
- Pseudorandom:  $\{U(\{f_s\})\}$  is pseudorandom.

Having developed a precise definition of a pseudorandom family of functions, the natural question arises: Does such a primitive even exist? And under what assumptions?

Notice that if  $\ell_1(n) = O(\log n)$ , all the outputs values of a function  $f : \{0,1\}^{\ell_1(n)} \rightarrow \{0,1\}^{\ell_2(n)}$  can be written down as a string of exactly  $2^{\ell_1(n)} \cdot \ell_2(n) = \text{poly}(n)$  bits. Moreover, all the function values can be queried in polynomial time, given oracle access. Therefore, a PRF family with  $O(\log n)$ -length input may be seen as a PRG, and vice-versa. But do there exist PRF families with longer input lengths — say,  $n$ ?

## 2.2 Constructing PRFs

**Theorem 2.3.** If a pseudorandom generator exists (i.e., if a one-way function exists), then a pseudorandom function family exists for any  $\ell_1(n), \ell_2(n) = \text{poly}(n)$ .

At first glance, this theorem may seem completely absurd. The number of functions in the family  $\{f_s\}$  with a seed length  $|s| = n$  is at most  $2^n$ , whereas the total number of functions overall (even with just one-bit outputs) is at least  $2^{2^n}$ . Therefore, our function family is  $\approx 2^{-2^n}$ -sparse, i.e., the family  $\{f_s\}$  makes up only a doubly exponentially small subset of the entire space of functions.

**Proof of Theorem 2.3.** For simplicity, we prove the theorem for  $\ell_1(n) = \ell_2(n) = n$ ; extending to other values is straightforward.

Our objective is to “stretch” an  $n$ -bit uniformly random string to produce an exponential (at least  $2^n$ ) number of “random-looking” strings. Assume without loss of generality that  $G$  is a PRG with output length  $\ell(n) = 2n$ . The basic idea is to view the output of  $G$  as two length- $n$  pseudorandom strings, which can be used recursively as inputs to  $G$  to generate an exponential number of strings.

Formally, view  $G$  as a pair of length-preserving functions  $G_0, G_1$  (i.e.,  $|G_0(s)| = |G_1(s)| = |s|$ ), where

$$G(s) = G_0(s) \parallel G_1(s).$$

The idea behind the PRF construction is that the function  $f_s(x)$  computes a path, specified by the bits of  $x$  starting from the root seed  $s$ , as shown below:



Formally, we define the function  $f_s(\cdot)$  as

$$f_s(x) = G_{x_n}(\cdots G_{x_2}(G_{x_1}(s)) \cdots).$$

Why might we expect  $f_s$  to “look random,” for a uniformly random (secret) seed  $s$ ? Intuitively,  $G_0(s)$  and  $G_1(s)$  “look like” independent uniform  $n$ -bit strings, and we might expect this pseudorandomness to propagate downward through the layers of the tree. Let us try to prove this rigorously.

Attempt 1: Design a sequence of hybrid experiments where each leaf of the tree is successively replaced by its “ideal” form, i.e., with a uniform  $n$ -bit string. Clearly, the 0th hybrid corresponds to the “real” tree construction, and the  $2^n$ th corresponds to a truly random function. However, this approach is flawed, as it requires  $2^n$  hybrid steps. (As an exercise, show that the hybrid lemma is false, in general, for an exponential number of hybrid steps.)

Attempt 2: Successively replace each layer of the tree with ideal uniform, independent entries (all at once). Thus,  $H_0$  corresponds to the real tree construction, and  $H_n$  corresponds to a truly random function. Note that we now have only  $n$  hybrid steps.

More formally, we describe hybrid distributions defining (a distribution over functions)  $f$  as follows:

- $H_0$  is the real tree construction, with a uniformly random root  $s$ , and  $f(x) = G_{x_n}(\cdots G_{x_1}(s) \cdots)$ .
- For  $i \in [n]$ ,  $H_i$  is the tree construction, but using uniformly random seeds across the  $i$ th layer of the tree. Formally,  $f(x) = G_{x_n}(\cdots G_{x_{i+1}}(s_{x_i \cdots x_1}) \cdots)$ , where the seeds  $s_y$  are uniformly random and independent for each  $y \in \{0, 1\}^i$ .

As a warm-up, we first show that  $H_0 \stackrel{c}{\approx} H_1$  (in the sense of oracle indistinguishability) assuming that  $G$  is a PRG. To prove this, we need to construct a simulator  $\mathcal{S}$  that emulates one of  $H_0$  or  $H_1$  (as oracles), depending on whether its input is  $G(U_n)$  or  $U_{2n}$ . That is, the simulator should use its

input to answer arbitrary queries. The simulator  $\mathcal{S}$  works as follows: given  $(z_0, z_1) \in \{0, 1\}^{2n}$ , it answers each query  $x$  by returning  $G_{x_n}(\cdots G_{x_2}(z_{x_1}) \cdots)$ .

It is easy to check that the simulator emulates the desired hybrids. First, if  $(z_0, z_1) = (G_0(s), G_1(s))$  for  $s \leftarrow U_n$ , then  $\mathcal{S}(z_0, z_1)$  answers each query  $x \in \{0, 1\}^n$  as

$$G_{x_n}(\cdots G_{x_2}(G_{x_1}(s)) \cdots) = f_s(x),$$

exactly as in  $H_0$ . Similarly, if  $(z_0, z_1) \leftarrow (U_n, U_n)$ , then  $\mathcal{S}$  answers each query exactly as in  $H_1$ . Now because  $G(U_n) \stackrel{c}{\approx} U_{2n}$  and  $\mathcal{S}$  is efficient, by the hybrid lemma we conclude that  $H_0 \stackrel{c}{\approx} H_1$ .

Unfortunately, this approach does not seem to scale too well when we go down to the deeper layers of the tree, because the simulator  $\mathcal{S}$  would need to take as input an exponential number of input strings. However, we can make two observations:

- In  $H_i$ , all the subtrees growing from the  $i$ th level are symmetric, i.e., they are identically distributed and independent.
- The polynomial-time distinguisher  $\mathcal{D}$  attacking the PRF can make only a polynomial number of queries to its oracle.

The key point is that the simulator then only needs to simulate  $q(n) = \text{poly}(n)$  number of subtrees in order to answer all the queries of the distinguisher correctly.

For the hybrids  $H_{i-1}$  and  $H_i$ , Algorithm 2.2 defines a simulator that takes  $q(n) = \text{poly}(n)$  pairs of  $n$ -bit strings.

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Algorithm 1 Simulator  $\mathcal{S}_i$  for emulating either  $H_{i-1}$  or  $H_i$ .

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Input:  $(z_0^1, z_1^1), \dots, (z_0^q, z_1^q) \in \{0, 1\}^{2n}$  for some large enough  $q(n) = \text{poly}(n)$

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1:  $j \leftarrow 1$ 
2: while there is a query  $x \in \{0, 1\}^n$  to answer do
3:   if prefix  $x_1 \cdots x_i$  is not yet associated with any  $k$  then
4:     associate  $j$  to  $x_1 \cdots x_i$ 
5:      $j \leftarrow j + 1$ 
6:   end if
7:   look up the  $k$  associated with prefix  $x_1 \cdots x_i$ 
8:   answer  $G_{x_n}(\cdots G_{x_{i+1}}(z_{x_i}^k) \cdots)$ 
9: end while
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We analyze the behavior of  $\mathcal{S}_i$ . Suppose that the distinguisher  $\mathcal{D}$  (making queries to  $\mathcal{S}_i$ ) makes at most  $q$  queries, so the counter  $j$  never “overflows.” Now, if each of the pairs  $(z_0^j, z_1^j) \leftarrow G(U_n^j)$  are independent pseudorandom strings, then  $\mathcal{S}_i$  answers each query  $x$  by  $G_{x_n}(\cdots (G_{x_{i+1}}(G_{x_i}(U_n^k))) \cdots)$ , for a distinct  $k$  associated uniquely with the  $i$ -bit prefix of  $x$ . By construction,  $\mathcal{S}_i$  therefore emulates  $H_{i-1}$  exactly. Similarly, if the  $(z_0^j, z_1^j) \leftarrow U_{2n}^j$  are uniformly random and independent, then  $\mathcal{S}_i$  simulates  $H_i$ .

At this point, we would like to conclude that  $H_{i-1} \stackrel{c}{\approx} H_i$ , but can we? To do so using the hybrid lemma, we would need to show that the two types of inputs to  $\mathcal{S}_i$  (namely, a sequence of  $q = \text{poly}(n)$  independent pairs  $(z_0, z_1)$  each drawn from either  $G(U_n)$  or  $U_{2n}$ ) are indistinguishable. This can be shown via a straightforward hybrid argument, using the hypothesis that  $G$  is a PRG, and is left as an exercise.  $\square$

### 2.3 Consequences for (Un)Learnability

A family of functions is said to be learnable if any member of the family can be reconstructed efficiently (i.e., as code), given oracle access to the function. In this sense, a PRF family is completely unlearnable, in that no efficient adversary can determine anything about the values of the function (given oracle access) on any of the unqueried points. As a consequence, if a class of functions is expressive enough to “contain” a PRF family, then this class is unlearnable. E.g., under standard assumptions, the class  $\text{NC}^1$  can implement PRFs, hence it is unlearnable.