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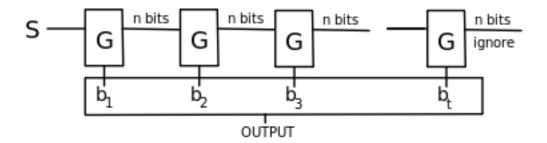
1 Expanding a PRG

In the last lecture we saw the definition of a Pseudorandom Generator (PRG) as a deterministic function that, given a seed of size n, outputs a pseudorandom string of length $\ell(n)$. We can ask the following question: how big can $\ell(n)$ be? Is there any limit on how much pseudorandom data we can generate starting from a seed of a certain size? We will find out that if we can get a PRG with even one bit of expansion (i.e., $\ell(n) = n + 1$), then we can get a PRG with any polynomial output length.

Theorem 1.1. Suppose that there exists a PRG G with output of length $\ell(n) = n + 1$. Then for any $\ell(n) = \text{poly}(n)$ (where $\ell(n) > n$), there exists a PRG G_t with output length $\ell(n)$.

Remark 1.2. The size of the set $\{G_t(s): s \in \{0,1\}^n\}$ is at most 2^n (because G_t is deterministic), while the number of possible t(n)-bit string is $|\{0,1\}^{t(n)}| = 2^{t(n)}$. The ratio of possible output strings that could actually be output by the PRG is at most $2^n/2^{t(n)} = 2^{n-t(n)}$, which is absurdly small when $t(n) \geq 2n$ (and even smaller when $t(n) = n^{10}$, say).

Proof. We will construct $G_t(s)$ from G. Our construction will apply the function $G(\cdot)$ t(n) times, outputting one new bit at each step and reusing the other n bits of the previous step's output as a seed. See the following picture for intuition about the construction.



Formally, G_t is defined as follows (note that it always applies G on string of the same length, n bits):

```
Algorithm 1 G_t(s)

if t = 0 then

return \varepsilon (the empty string)

else

let (x|b) = G(s), where x \in \{0,1\}^n, b \in \{0,1\}

return b|G_{t-1}(x)

end if
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By construction, $|G_t(s)| = t$. We want to show that G_t is a PRG. The function clearly runs in polynomial time (each call to G can be resolved in polynomial time and we only use a polynomial number of steps), so what's left is to prove that $\{G_t(U_n)\} \stackrel{c}{\approx} \{U_{t(n)}\}$. We need to be careful: we

already know that G is a PRG, but in our construction we are giving a pseudorandom seed to G, instead of a truly random seed. We will see that this fact will not affect the pseudorandomness of G_t . Intuitively, because no efficient algorithm can tell a pseudorandom seed apart from a random string, then in particular neither can G.

To prove that G_t is a PRG we will define a set of "hybrid experiments." We will build a sequence of distributions, where the first is equal to our "real" construction $G_t(U_n)$, the last is equal to the "ideal" truly uniform distribution $U_{t(n)}$, and each consecutive pair of distributions are computationally indistinguishable. By the hybrid lemma, we conclude that $\{G_t(U_n)\}$ and $\{U_{t(n)}\}$ are computationally indistinguishable and that G_t is a PRG, thus proving the theorem.

To give some intuition about how we design the hybrid experiments, we imagine that instead of invoking the first G on n uniform bits, what if we replaced its output with n+1 truly uniform bits? Intuitively, these two cases should not be distinguishable, because G is a PRG. And then what if we replaced the first two invocations of G, and so on? Eventually, we would end up with t = t(n) truly uniform output bits, as desired.

Formally, the hybrid experiments are defined as follows:

- $H_0 = G_t(U_n)$
- $H_1 = U_1 | G_{t-1}(U_n)$
- In general, $H_i = U_i | G_{t-i}(U_n)$ for $i \in \{0\} \cup [t]$
- $H_t = U_t$

We now show that for all $i \in [t-1]$, $H_i \stackrel{c}{\approx} H_{i+1}$. We will do this by using the simulation/composition lemma, and the fact that G is a PRG. For each i, we design a PPT "simulator" algorithm S_i such that $S_i(G(U_n)) = H_{i-1}$, and $S_i(U_{n+1}) = H_i$. Since we know that $G(U_n) \stackrel{c}{\approx} U_{n+1}$ from the fact that G is a PRG, the composition lemma implies that $H_{i-1} \stackrel{c}{\approx} H_i$.

We define S_i as follows:

```
Algorithm 2 S_i(y \in \{0,1\}^{n+1})
parse y as (x|b) for x \in \{0,1\}^n, b \in \{0,1\}
return U_{i-1}|b|G_{t-i}(x)
```

This algorithm clearly runs in polynomial time. We need to check that it maps $G(U_n)$ to H_{i-1} , and U_{n+1} to H_i . First suppose that the input of S_i comes from U_{n+1} :

$$S_i(U_{n+1}) = U_{i-1}|b|G_{t-i}(x) = U_{i-1}|U_1|G_{t-i}(U_n) = U_i|G_{t-i}(U_n) = H_i.$$

Now suppose that the input comes from $G(U_n)$. By the definition of G_t , we can see the following:

$$S_i(G(U_n)) = U_{i-1}|b|G_{t-i}(x) = U_{i-1}|G_{t-i+1}(U_n) = H_{i-1}.$$

This completes the proof.

To recap, our proof of Theorem 1.1 took the following path:

- We defined a construction: the actual PRG G_t having a pseudorandom output of polynomial length.
- We defined the sequence of hybrid ("imaginary") experiments. In each step, we replaced one "real" invocation of a crypto primitive $(G(U_n))$ with its "ideal" counterpart (U_{n+1}) .
- We proved that consecutive pairs of hybrids are computationally indistinguishable, using the composition lemma and the security properties of the underlying primitives (i.e., that G is a PRG).
 - To apply the composition lemma, we defined a "simulator" (reduction) for each pair of adjacent hybrids and analyzed its behavior.

2 Obtaining a PRG

Thanks to Theorem 1.1, we know that all we need is to obtain a PRG with one extra bit of output. We describe a number-theoretic construction, due to Blum and Micali, of such an object.

2.1 Number Theory Background

Theorem 2.1 (Euler's theorem). Let G be a finite abelian (i.e., commutative) multiplicative group. For every $a \in G$, we have $a^{|G|} = 1$.

Proof of Theorem 2.1. Consider the set $A = a \cdot G = \{ax : x \in G\}$. Because G is a group, a is invertible, and we have A = G. Taking products over all elements in A = G, we have

$$\prod_{x \in G} (ax) = \prod_{x \in G} x.$$

Because G is commutative, the LHS is $a^{|G|} \cdot \prod_{x \in G} x$, and we can multiply by the inverse of the RHS to obtain $a^{|G|} = 1$.

When $G = \mathbb{Z}_p^*$ for a prime p, we have $|\mathbb{Z}_p^*| = \varphi(p) = p - 1$, so we obtain the following corollary:

Corollary 2.2 (Fermat's "little" theorem). Let p be a prime. For any $a \in \mathbb{Z}_p^*$, we have $a^{p-1} = 1 \mod p$.

The following structural theorem will be very useful. (Its proof is elementary bur rather tedious, so we won't go through it today.)

Theorem 2.3. Let p be a prime. The multiplicative group \mathbb{Z}_p^* is cyclic, i.e., there exists some generator $g \in \mathbb{Z}_p^*$ such that $\mathbb{Z}_p^* = \langle g \rangle := \{g^1, g^2, \dots, g^{p-1} = 1\}.$

2.2 Discrete Logarithm Problem and One-Way Function

Theorem 2.3 leads naturally to the so-called discrete logarithm problem, which is: given $y \in \mathbb{Z}_p^*$ (and prime p and generator g of \mathbb{Z}_p^*), find $\log_g y$, i.e., the $x \in \{1, \ldots, p-1\}$ for which $y = g^x \mod p$. This problem is believed to be infeasible for large values of p.

Conjecture 2.4 (Discrete logarithm assumption). Let $S(1^n)$ be a PPT algorithm that outputs some prime p and generator g of \mathbb{Z}_p^* . For every non-uniform PPT algorithm \mathcal{A} ,

$$\Pr_{(p,g) \leftarrow \mathsf{S}(1^n), y \leftarrow \mathbb{Z}_p^*} [\mathcal{A}(p,g,y) = \log_g y] = \mathrm{negl}(n).$$

We would like to design a collection of OWFs based on the discrete logarithm assumption. The collection is made up of the functions $f_{p,g}:\{1,\ldots,p-1\}\to\mathbb{Z}_p^*$ (for prime p and generator g of \mathbb{Z}_p^*), defined as

$$f_{p,q}(x) = g^x \bmod p$$
.

Moreover, these functions are even permutations if we identify $\{1, \ldots, p-1\}$ with \mathbb{Z}_p^* in the natural way.

It is a tautology that the collection is one-way under the discrete logarithm assumption. It is also clear that we can efficiently sample from the domain of $f_{p,g}$. But we still need to check that $f_{q,p}$ can be evaluated efficiently, and that (p,g) can be generated efficiently.

For the first, we use the standard "repeated squaring" technique for exponentiation, which requires O(|x|) multiplications modulo p. The solution to the second issue is not entirely straightforward. Given only the prime p, it is unknown (in general) how to find a generator g of \mathbb{Z}_p^* efficiently. However, given the factorization of p-1, which can be generated along with p, it is possible: every element in \mathbb{Z}_p^* has order dividing p-1, so g is a generator if and only if $g^{(p-1)/q} \neq 1 \mod p$ for every prime divisor q of p-1. The number of non-generators is at most the sum of (p-1)/q over all prime divisors q of p-1, so the density of generators is typically large enough. An often-used special case is p=2q+1 for prime q, for which there are q=(p-1)/2 generators. However, it is not even known whether there exist infinitely many such "Sophie Germain" primes of this form! (Empirically, though, they are abundant.)

2.3 Blum-Micali PRG

We now present a PRG that uses the ideas presented in the previous section. From Section 1 we know that if we have a PRG that is able to generate one extra bit of randomness, we can generate a polynomial number of pseudorandom bits. Our goal will be the following: we want to construct a PRG $G_{p,g}: \mathbb{Z}_p^* \to \mathbb{Z}_p^* \times \{0,1\}.$

Our solution is a function with the following form:

$$G_{p,q}(x) = (f_{p,q}(x) = g^x \mod p$$
 , $h(x)$.

Note that $f_{p,g}(x)$ performs the modular exponentiation function (which is a one-way under the discrete log assumption), while $h: \mathbb{Z}_p^* \to \{0,1\}$ is some function (yet to be defined) that provides the additional bit.

Looking at the function, we can make the following observation: if $x \in \mathbb{Z}_p^*$ is chosen uniformly at random, then also $f_{p,g}(x)$ is uniform (because $f_{p,g}$ is a permutation). We still need to choose the function h, keeping in mind what we want from from the function: h(x) should "look like a random bit," even given $f_{p,g}(x)$. That is, h(x) should compute "something about x" that f hides completely.

¹To be pedantic, $G_{p,g}$ is a "collection" of PRGs, where the input seed comes from a set that depends on the function index (p,g). It it easy to check that our construction from Section 1 is compatible with this collection $G_{p,g}$.

We can think of many possible candidates for h: apply the xor function to all the bit of x; take the least significant bit of x (though you will show that this does not meet our requirements!); take the "most significant bit" of x (more precisely, test if $x > \frac{p-1}{2}$). The last function will be the one we will use.

Let's formalize the security properties we want from h(x): given f(x), no algorithm should be able to guess h(x) with much better than $\frac{1}{2}$ probability.

Definition 2.5 (Hardcore predicate). A predicate $h: \{0,1\}^* \to \{0,1\}$ is hard-core for f if for all non-uniform PPT algorithms \mathcal{A} ,

$$\Pr_{x}[\mathcal{A}(f(x)) = h(x)] \le \frac{1}{2} + \operatorname{negl}(n).$$

Exercise: Prove that if h is hard-core for a one-way permutation $f: D \to D$, then

$$(f(x), h(x)) \stackrel{c}{\approx} (U(D), U_1),$$

where $x \leftarrow D$. This means that $G(x) = (f(x), h(x)) \in D \times \{0, 1\}$ is a PRG that expands by one bit.

Next time, we will show that under the discrete log assumption, the "most significant bit" predicate $h(x) = [x > \frac{p-1}{2}]$ is hard-core for $f_{p,g}$.