Continuous–time stochastic models

Modern Techniques in Modelling

Introduction

Overview

– Introduce continuous-time stochastic models

– Implement Gillespie algorithm and analyse stochastic model output

– Implement a stochastic model with the `adaptivetau` package

– Discussion and concluding remarks
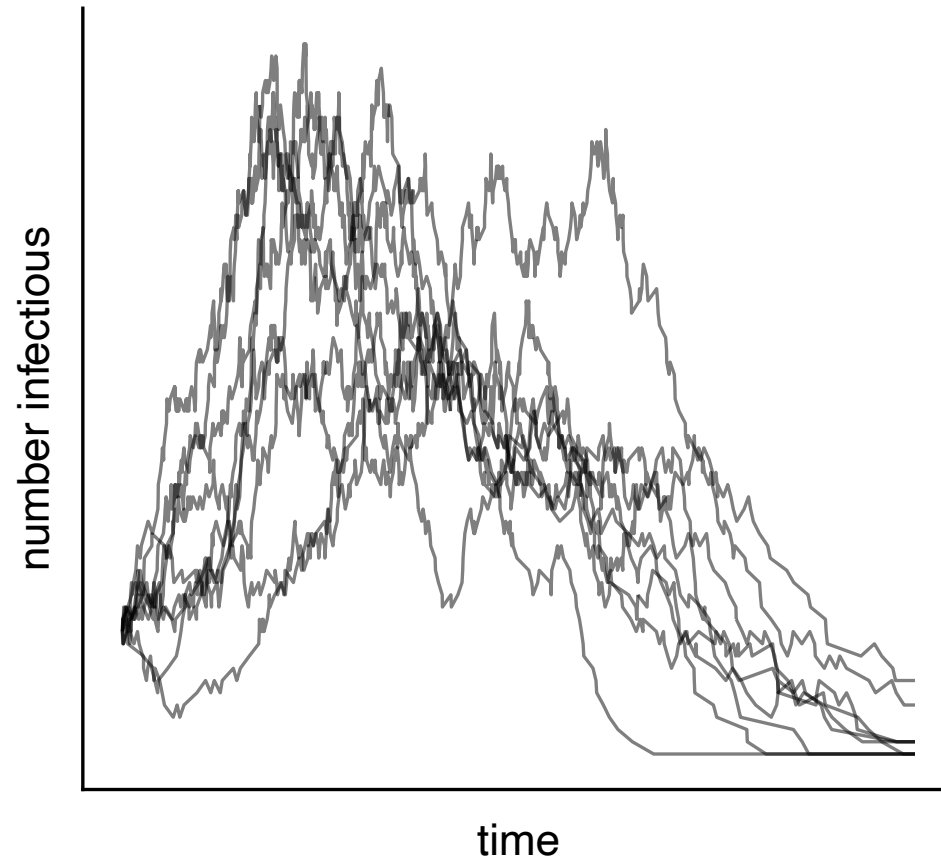
Overview

- Introduce continuous-time stochastic models
  (~20 minutes)

- Implement Gillespie algorithm and analyse stochastic model output
  (~60 minutes)

- Implement a stochastic model with the `adaptivetau` package
  (~20 minutes)

- Discussion and concluding remarks
  (~20 minutes)

Deterministic models
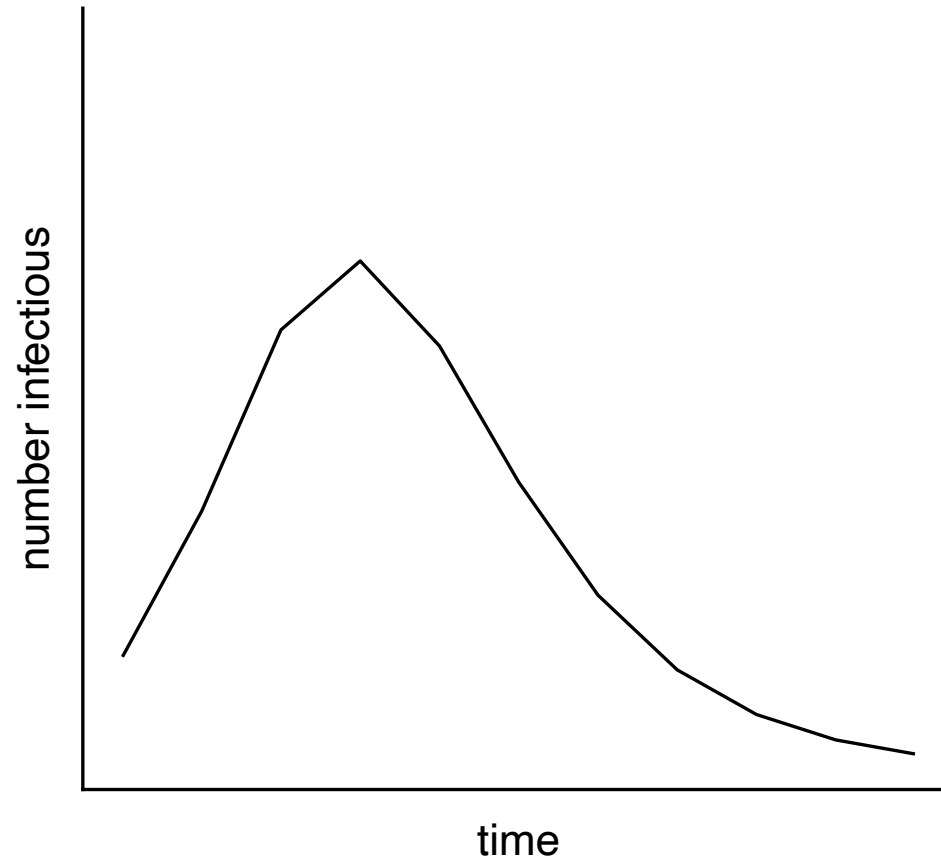
One set of parameters -> one trajectory

One set of parameters -> many trajectories

Types of model

- discrete vs continuous *time*

- discrete vs continuous *compartments*

- deterministic vs stochastic *dynamics*

– discrete vs continuous *time*

– discrete vs continuous *compartments*

– deterministic vs stochastic *dynamics*

$$S(t+1) = S(t) - \beta S(t)I(t)$$
$$I(t+1) = I(t) + \beta S(t)I(t) - \gamma I(t)$$
$$R(t+1) = R(t) + \gamma I(t)$$

- **discrete** vs continuous *time*

- discrete vs **continuous** *compartments*

- **deterministic** vs stochastic *dynamics*

$$
\begin{aligned}
S(t+1) &= & S(t) - \beta S(t)I(t) \\
I(t+1) &= & I(t) + \beta S(t)I(t) - \gamma I(t) \\
R(t+1) &= & R(t) + \gamma I(t)
\end{aligned}
$$

- discrete vs continuous *time*
- discrete vs continuous *compartments*
- deterministic vs stochastic *dynamics*

$$\frac{dS}{dt} = -\beta SI/N$$

$$\frac{dI}{dt} = \beta SI/N - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

– discrete vs **continuous** *time*

– discrete vs **continuous** *compartments*

– **deterministic** vs stochastic *dynamics*

$$\frac{dS}{dt} = -\beta SI/N$$

$$\frac{dI}{dt} = \beta SI/N - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

Example 3: individual-based model

- discrete vs continuous *time*
- discrete vs continuous *compartments*
- deterministic vs stochastic *dynamics*

```
For each ts from 1 to T {
   lambda <- beta * I/N

   For each i from 1 to N {
      If individual i is susceptible:
         with prob 1-exp(-lambda·Δt) make infected.
      Else-if individual i is infected:
         with prob 1-exp(-gamma·Δt) make susceptible.
   }
   Record population state
}
```

Example 3: individual-based model

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Event-based view

- **discrete** vs continuous *time*
- **discrete** vs continuous *compartments*
- deterministic vs **stochastic** *dynamics*

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}
```

Continuous–time stochastic models

– discrete vs **continuous** *time*

– discrete vs continuous *compartments*

– deterministic vs **stochastic** *dynamics*

Stochastic differential equations (SDEs)

— discrete vs **continuous** *time*

— discrete vs **continuous** *compartments*

— deterministic vs **stochastic** *dynamics*

$$\frac{dS}{dt} = -\beta SI/N - \sqrt{\beta SI/N}\, dW_1$$

$$\frac{dI}{dt} = \beta SI/N - \gamma I + \sqrt{\beta SI/N}\, dW_1 - \sqrt{\gamma I}\, dW_2$$

$$\frac{dR}{dt} = \gamma I + \sqrt{\gamma I}\, dW_2$$

Can be solved with Euler method (see Session 7).

Continuous-time discrete stochastic models

- discrete vs **continuous** *time*
- **discrete** vs continuous *compartments*
- deterministic vs **stochastic** *dynamics*

Continuous–time discrete stochastic models

- discrete vs **continuous** *time*
- **discrete** vs continuous *compartments*
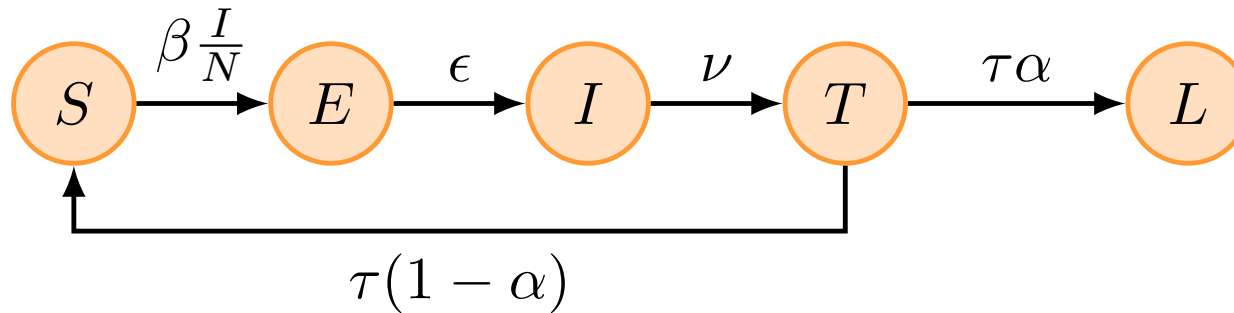- deterministic vs **stochastic** *dynamics*

Event-based view:

- infection: $(S, I, R) \rightarrow (S - 1, I + 1, R)$ with rate $\beta SI/N$
- recovery: $(S, I, R) \rightarrow (S, I - 1, R + 1)$ with rate γI

We model these as a so-called **continuous-time Markov chain**.

Example: Influenza with short and long immunity

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S \frac{I}{N} + (1-\alpha)\tau T$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \beta S \frac{I}{N} - \epsilon E$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \epsilon E - \nu I$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \nu I - \tau T$$

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \alpha \tau T$$

QUESTION: Which events can happen, and at what rates?

Time

DISCRETE (session on individual-based models)

```
for (ts in 1:steps){ … EVENTS … }
```

CONTINUOUS (here)

```
while (time < finaltime) { …
   time <- time + rexp(n = 1, rate = sum(rates))
      if (time <= finaltime) { … EVENTS … }
}
```

Waiting times between events in a Poisson process are exponentially distributed

Repeat until end time:

1. Calculate event rates

    ```
    rates <- c()
    rates["infection"] <- beta * S * I / N
    rates["recovery"] <- gamma * I
    ```

2. Choose how long nothing happens

    ```
    rexp(n = 1, rate = sum(rates))
    ```

3. Choose which event happens

    ```
    sample(x = length(rates), size = 1, prob = rates)
    ```

    and update system state according to event.

Now, put it in R!

- Part 1: Stochastic simulation using the Gillespie algorithm
- Part 2: A faster alternative: the `adaptivetau` package

- – Open the file
  `10_StochasticContinuousractical/01_Gillespie.R`
  and follow the instructions
- – Objective: use the Gillespie algorithm to simulate the SIR model; process outputs from stochastic models
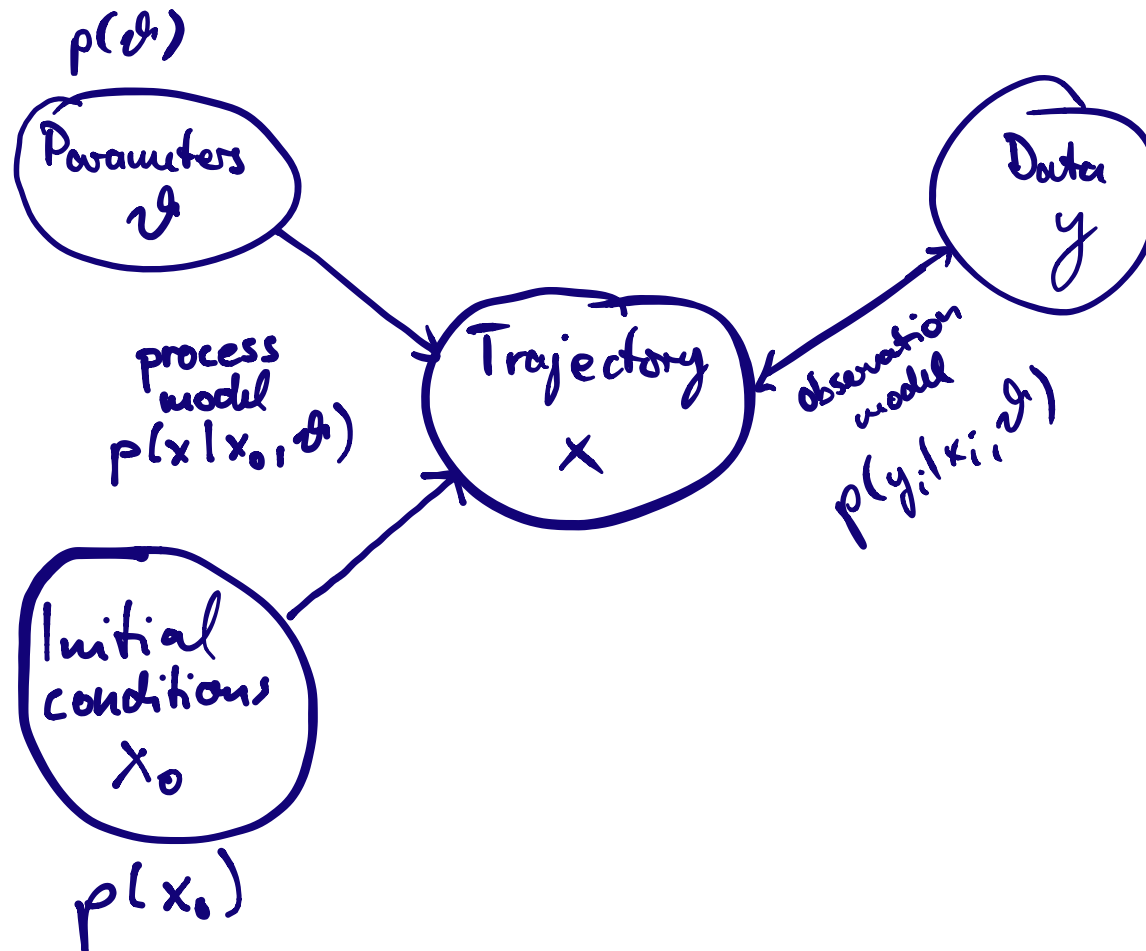
Algorithm:

- Identifies periods during which all rates are not expected to change, and all variables are far from 0

- "Leaps" over these periods of time

- Adds the net effect of the Poisson-distributed number of transitions that should have occurred in that period

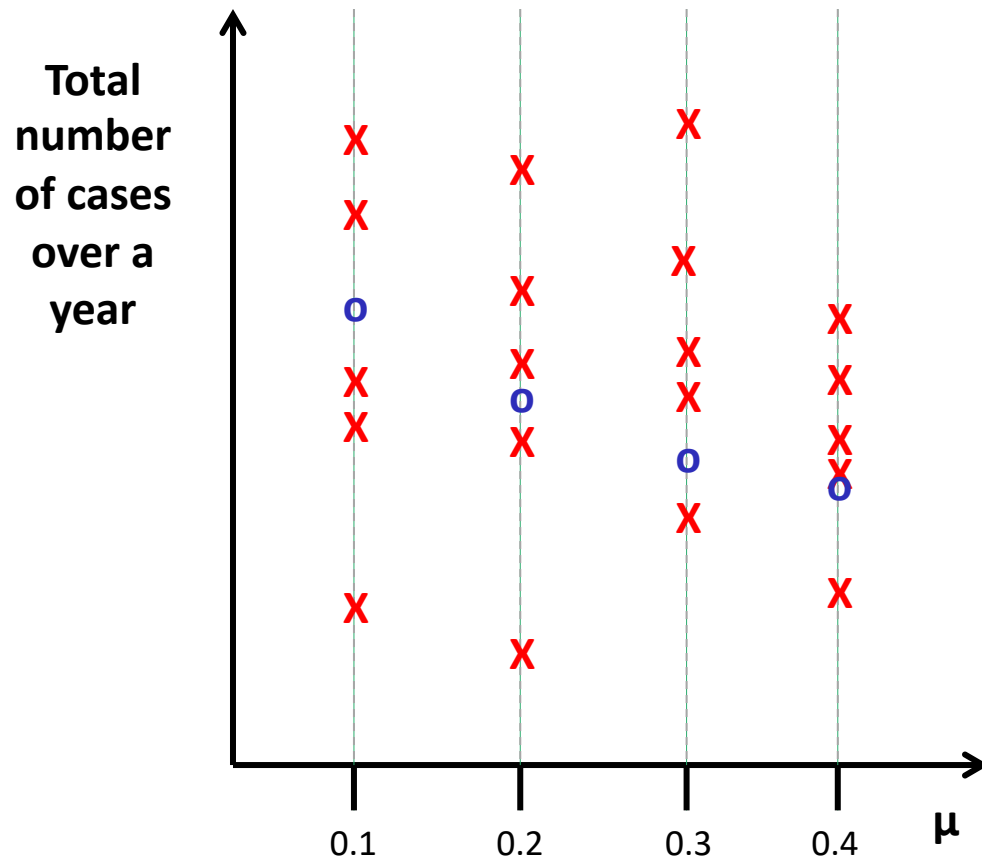In R, the `ssa.adaptivetau` package implements this (and generates fast C code).

- Open the file
  `10_StochasticContinuousractical/02_AdaptiveTau.R` and
  follow the instructions
- Objective: use the `adaptivetau` packages to simulate the SIR and
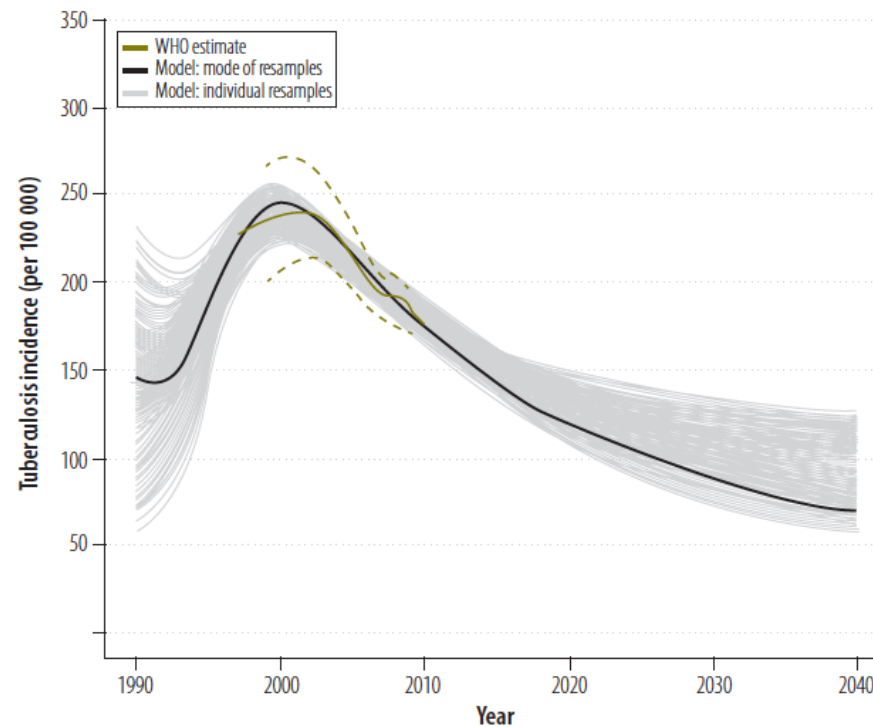 SEITL models

Total number of cases over a year

0.1   0.2   0.3   0.4   μ

At each of 4 μ values,
5 runs of a model that is:

Stochastic = o / X ?
Deterministic = o / X ?

Parameter variation

Fig. 2. **Incidence of tuberculosis (all forms) in the United Republic of Tanzania based on WHO estimates and projected incidence based on the calibrated epidemic model**

[Lin, 2012]

Representing uncertainty

$p(\vartheta)$

Parameters ϑ

process model $p(x \mid x_0, \vartheta)$

Trajectory x

observation model $p(y_i \mid x_i, \vartheta)$

Data y

Initial conditions x_0

$p(x_0)$

References
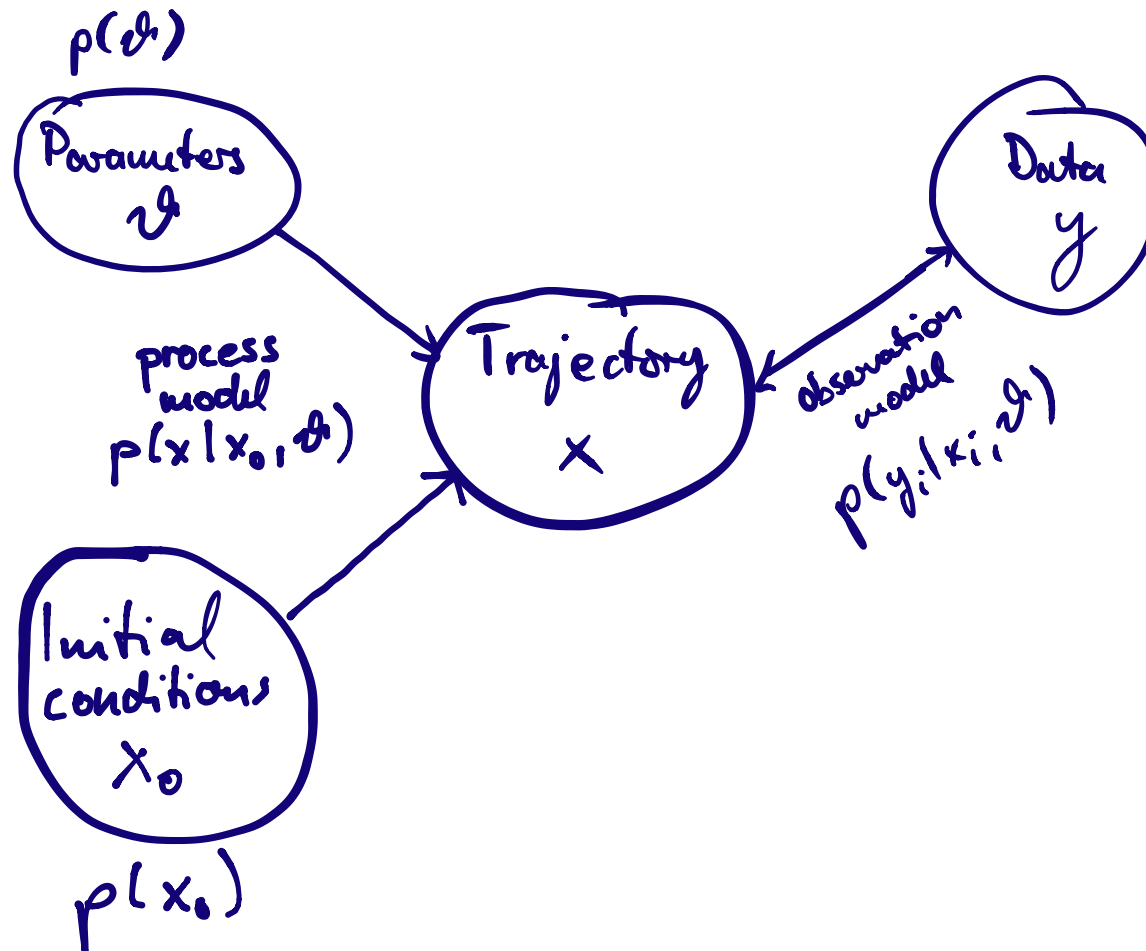
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