

Continuous-time stochastic models

Modern Techniques in Modelling

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Introduction

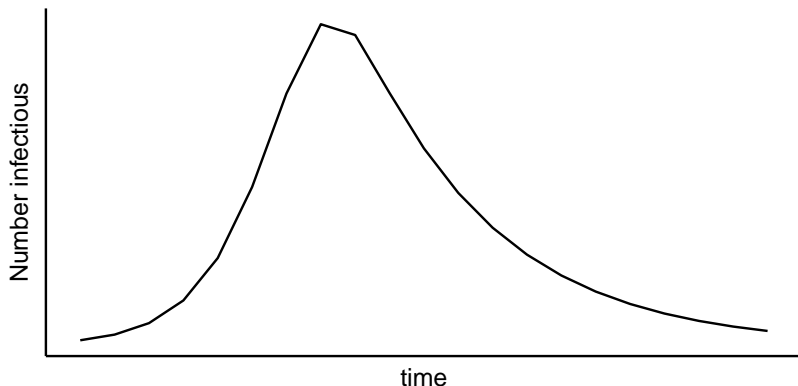
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- Introduce continuous-time stochastic models (~20 minutes)
- Implement the Gillespie algorithm and analyse stochastic model output (~60 minutes)
- Implement a stochastic model with the `adaptivetau` package (~20 minutes)
- Discussion and concluding remarks (~20 minutes)

Deterministic models

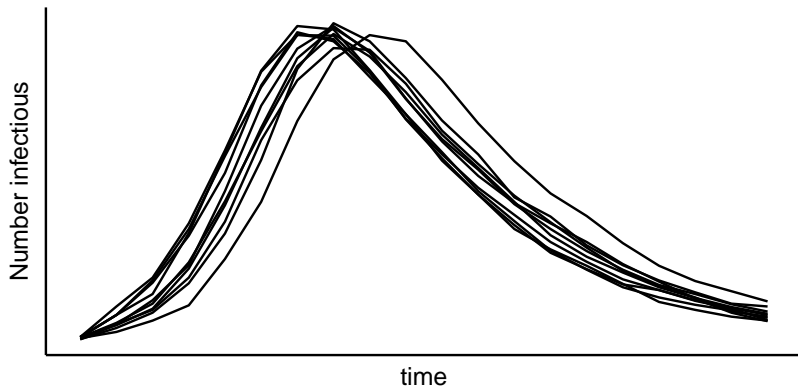
SIR model with $I_0 = 10, \beta = 1.3, \gamma = 0.3$



One set of parameters \rightarrow one trajectory

Stochastic models

SIR model with $I_0 = 10, \beta = 1.3, \gamma = 0.3$



One set of parameters \rightarrow many trajectories

Types of model

- *discrete* vs *continuous* time
- *compartment-* vs *individual*-based
- *deterministic* vs *stochastic* dynamics

Session 3: Discrete-time deterministic models

- *discrete* vs *continuous* time
- *compartment-* vs *individual*-based
- *deterministic* vs *stochastic* dynamics

$$S(t+1) = S(t) - \beta S(t)I(t)$$

$$I(t+1) = I(t) + \beta S(t)I(t) - \gamma I(t)$$

$$R(t+1) = R(t) + \gamma I(t)$$

Session 3: Discrete-time deterministic models

- **discrete** vs ~~continuous~~ time
- **compartment-** vs ~~individual~~-based
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Session 4: Ordinary differential equations

- *discrete* vs *continuous* time
- *compartment-* vs *individual*-based
- *deterministic* vs *stochastic* dynamics

$$dS/dt = -\beta SI$$

$$dI/dt = \beta SI - \gamma I$$

$$dR/dt = \gamma I$$

Session 4: Ordinary differential equations

- ~~discrete~~ vs **continuous** time
- **compartment-** vs ~~individual~~-based
- **deterministic** vs ~~stochastic~~ dynamics

$$dS/dt = -\beta SI$$

$$dI/dt = \beta SI - \gamma I$$

$$dR/dt = \gamma I$$

Session 8: Stochastic individual-based models

- *discrete vs continuous* time
- *compartment- vs individual-based*
- *deterministic vs stochastic* dynamics

```
For each ts from 1 to T {  
  lambda <- beta * I/N  
  For each i from 1 to N {  
    If individual i is susceptible:  
      with prob 1-exp(-lambda·t) make infected.  
    Else-if individual i is infected:  
      with prob 1-exp(-gamma·t) make susceptible.  
  }  
  Record population state  
}
```

Session 8: Stochastic individual-based models

- **discrete** vs ~~continuous~~ time
- ~~compartment-~~ vs **individual**-based
- ~~deterministic~~ vs **stochastic** dynamics

```
For each ts from 1 to T {  
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```

Continuous-time stochastic models

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Continuous-time stochastic models

- ~~discrete~~ vs **continuous** time
- *compartment-* vs *individual*-based
- ~~deterministic~~ vs **stochastic** dynamics

Stochastic differential equations (SDEs)

- ~~discrete~~ vs **continuous** time
- **compartment-** vs ~~individual~~-based
- ~~deterministic~~ vs **stochastic** dynamics

$$dS/dt = -\beta SI$$

$$dI/dt = \beta SI - \gamma I$$

$$dR/dt = \gamma I$$

! Important

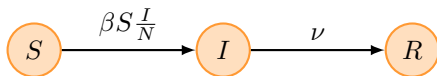
Not covered in this course.

Continuous-time discrete stochastic models

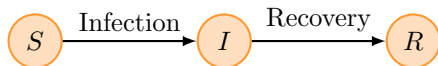
- ~~discrete~~ vs **continuous** time
- ~~compartment-~~ vs **individual-**based
- ~~deterministic~~ vs **stochastic** dynamics

We model these as a so-called **continuous-time Markov chains**.

Event-based view



Event-based view



- **infection:** $(S, I, R) \rightarrow (S - 1, I + 1, R)$ with rate $\beta SI/N$
- **recovery:** $(S, I, R) \rightarrow (S, I - 1, R + 1)$ with rate γI

Moving to continuous time

Discrete time

```
for (ts in 1:steps) {  
  update all compartments  
}
```

(see Session 8: Stochastic individual-based models)

Continuous time

```
while (time < finaltime) {  
  advance time and record next event  
}
```

(this session)

Gillespie algorithm

Repeat until end time:

1. Calculate rates of all possible events

```
rates <- c(  
  infection = beta * S * I / N,  
  recovery = gamma * I  
)
```

2. Determine time of next event

```
rexp(n = 1, rate = sum(rates))
```

3. Determine which event happens

```
sample(x = length(rates), size = 1, prob = rates)
```

and update system state according to event.

Now, put it in R

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Practical structure

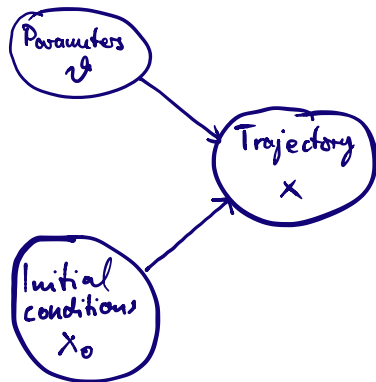
- Part 1: Stochastic simulations using the Gillespie algorithm
- Part 2: A faster alternative: the **adaptivetau** package

Representing uncertainty

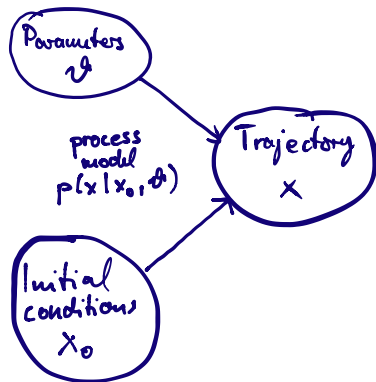
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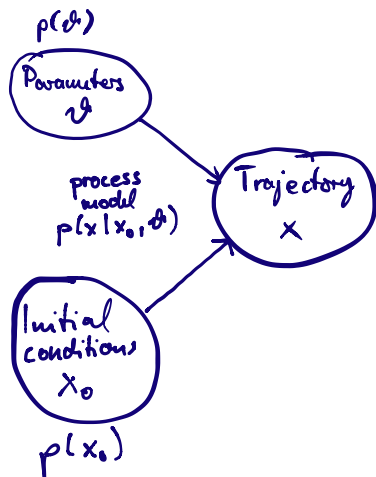
The deterministic view



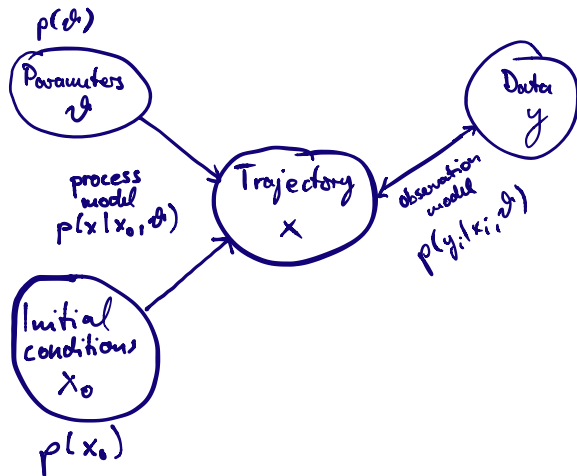
The stochastic view



Other types of uncertainty



Linking models to data



See LSHTM short course on Model Fitting and Inference for Infectious Disease Dynamics.

Further reading

- L.J.S. Allen (2017). A primer on stochastic epidemic models: Formulation, numerical simulation, and analysis. *Infectious Disease Modelling*, 2(2):128–142. <https://doi.org/10.1016/j.idm.2017.03.001>
- M.J. Keeling, P. Rohani (2017). *Modeling Infectious Diseases in Humans and Animals*. Princeton University Press.
- D.T. Gillespie (1976). A general method for numerically simulating the stochastic time evolution of coupled chemical reactions. *J Comput Phys*, 22(4):403–434, 1976. ISSN 0021-9991. [https://doi.org/10.1016/0021-9991\(76\)90041-3](https://doi.org/10.1016/0021-9991(76)90041-3)
- Y. Cao, D.T. Gillespie, and L.R. Petzold (2007). Adaptive explicit-implicit tau-leaping method with automatic tau selection. *J Chem Phys*, 126(22):224101 URL <https://doi.org/10.1063/1.2745299>
- A.A. King, M. Domenech de Cellès, F.M.G. Magpantay and Pejman Rohani (2015). Avoidable errors in the modelling of outbreaks of emerging pathogens, with special reference to Ebola. *Proc Roy Soc B* 282(1806). <https://doi.org/10.1098/rspb.2015.0347>