Continuous-time stochastic models

Modern Techniques in Modelling

LONDON SCHOOL of HYGIENE & TROPICAL MEDICINE

Introduction

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Overview

– Introduce continuous-time stochastic models

– Implement Gillespie algorithm and analyse stochastic model output
– Implement a stochastic model with the `adaptivetau` package

– Discussion and concluding remarks

Overview

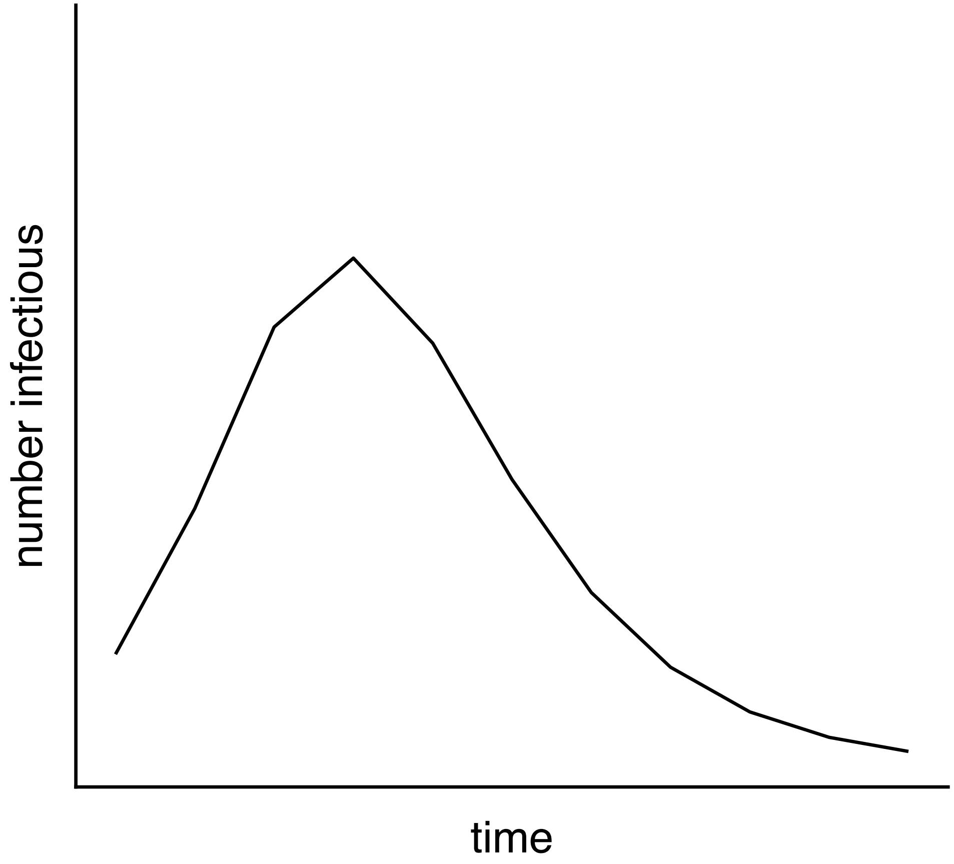
– Introduce continuous-time stochastic models
  (~20 minutes)

– Implement Gillespie algorithm and analyse stochastic model output
  (~60 minutes)

– Implement a stochastic model with the `adaptivetau` package
  (~20 minutes)

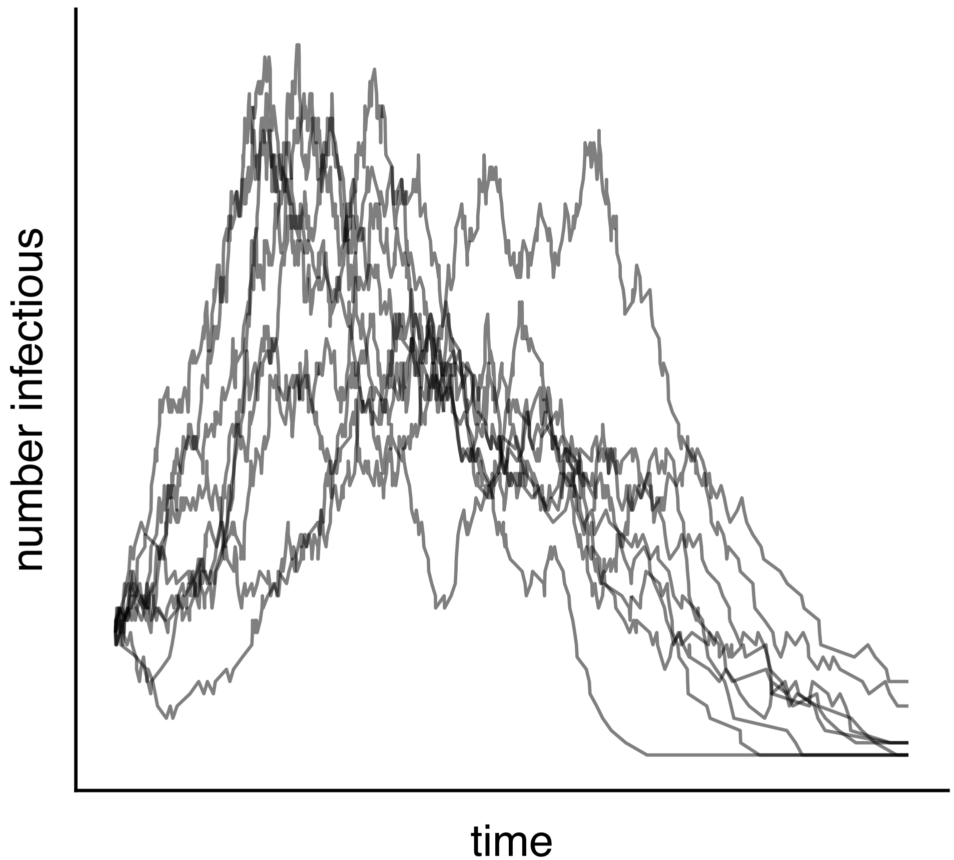
– Discussion and concluding remarks
  (~20 minutes)

Deterministic models



One set of parameters -> one trajectory

Stochastic models



One set of parameters -> many trajectories

Types of model

– discrete vs continuous *time*

– discrete vs continuous *compartments*

– deterministic vs stochastic *dynamics*

Example 1: difference equations

– discrete vs continuous *time*

– discrete vs continuous *compartments*

– deterministic vs stochastic *dynamics*

$$
\begin{aligned}
S(t+1) &= S(t) - \beta S(t) I(t) \\
I(t+1) &= I(t) + \beta S(t) I(t) - \gamma I(t) \\
R(t+1) &= R(t) + \gamma I(t)
\end{aligned}
$$

Discrete-time deterministic models

- **discrete** vs continuous *time*
- discrete vs **continuous** *compartments*
- **deterministic** vs stochastic *dynamics*

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\end{aligned}
$$

Example 2: ODEs

- discrete vs continuous *time*
- discrete vs continuous *compartments*
- deterministic vs stochastic *dynamics*

$$
\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= -\beta SI/N \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \beta SI/N - \gamma I \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \gamma I
\end{aligned}
$$

Continuous-time deterministic models

– discrete vs **continuous** *time*

– discrete vs **continuous** *compartments*

– **deterministic** vs stochastic *dynamics*

$$
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\frac{\mathrm{d}I}{\mathrm{d}t} &= \beta S I / N - \gamma I \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \gamma I
\end{aligned}
$$

Example 3: individual-based model

– discrete vs continuous *time*

– discrete vs continuous *compartments*

– deterministic vs stochastic *dynamics*

```
For each ts from 1 to T {
  lambda <- beta * I/N

  For each i from 1 to N {
    If individual i is susceptible:
      with prob 1-exp(-lambda·Δt) make infected.
    Else-if individual i is infected:
      with prob 1-exp(-gamma·Δt) make susceptible.
  }
  Record population state
}
```

Example 3: individual-based model

- **discrete** vs continuous *time*
- **discrete** vs continuous *compartments*
- deterministic vs **stochastic** *dynamics*

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   Record population state
}
```

Event-based view

– **discrete** vs continuous *time*

– **discrete** vs continuous *compartments*

– deterministic vs **stochastic** *dynamics*

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  }
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}
```

Continuous-time stochastic models

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Continuous-time stochastic models

– discrete vs **continuous** *time*
– discrete vs continuous *compartments*
– deterministic vs **stochastic** *dynamics*

Stochastic differential equations (SDEs)

– discrete vs **continuous** *time*

– discrete vs **continuous** *compartments*

– deterministic vs **stochastic** *dynamics*

$$
\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= -\beta SI/N - \sqrt{\beta SI/N}\,\mathrm{d}W_1 \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \beta SI/N - \gamma I + \sqrt{\beta SI/N}\,\mathrm{d}W_1 - \sqrt{\gamma I}\,\mathrm{d}W_2 \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \gamma I + \sqrt{\gamma I}\,\mathrm{d}W_2
\end{aligned}
$$

Can be solved with Euler method (see Session 7).

Continuous-time discrete stochastic models

– discrete vs **continuous** *time*

– **discrete** vs continuous *compartments*

– deterministic vs **stochastic** *dynamics*

Continuous-time discrete stochastic models

– discrete vs **continuous** *time*

– **discrete** vs continuous *compartments*

– deterministic vs **stochastic** *dynamics*

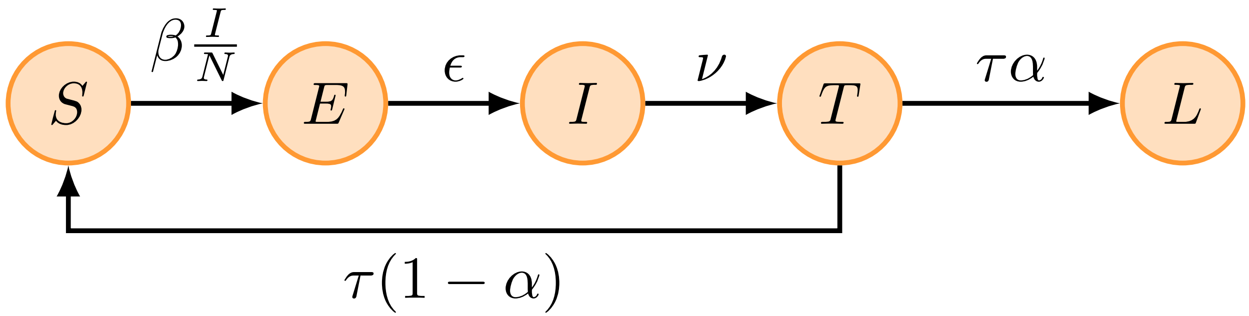
Event-based view:

– infection: $(S, I, R) \rightarrow (S-1, I+1, R)$ with rate $\beta SI/N$

– recovery: $(S, I, R) \rightarrow (S, I-1, R+1)$ with rate γI

We model these as a so-called **continuous-time Markov chain**.

Example: Influenza with short and long immunity



$$
\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= -\beta S \frac{I}{N} + (1-\alpha)\tau T \\
\frac{\mathrm{d}E}{\mathrm{d}t} &= \beta S \frac{I}{N} - \epsilon E \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \epsilon E - \nu I \\
\frac{\mathrm{d}T}{\mathrm{d}t} &= \nu I - \tau T \\
\frac{\mathrm{d}L}{\mathrm{d}t} &= \alpha \tau T
\end{aligned}
$$

QUESTION: Which events can happen, and at what rates?

Time

DISCRETE (session on individual-based models)

```
for (ts in 1:steps){ … EVENTS … }
```

CONTINUOUS (here)

```
while (time < finaltime) { …
   time <- time + rexp(n = 1, rate = sum(rates))
         if (time <= finaltime) { … EVENTS … }
}
```

Waiting times between events in a Poisson process are exponentially distributed

Gillespie algorithm

[Gillespie, 1976]

Repeat until end time:

1. Calculate event rates

```
rates <- c()
rates["infection"] <- beta * S * I / N
rates["recovery"] <- gamma * I
```

2. Choose how long nothing happens

```
rexp(n = 1, rate = sum(rates))
```

3. Choose which event happens

```
sample(x = length(rates), size = 1, prob = rates)
```

and update system state according to event.

Now, put it in R!

Practical part 1

Stochastic simulation using the Gillespie algorithm

– Objective: use the Gillespie algorithm to simulate the SIR model; process outputs from stochastic models

adaptive tau-leaping

Algorithm:

- Identifies periods during which all rates are not expected to change, and all variables are far from 0
- “Leaps” over these periods of time
- Adds the net effect of the Poisson-distributed number of transitions that should have occurred in that period

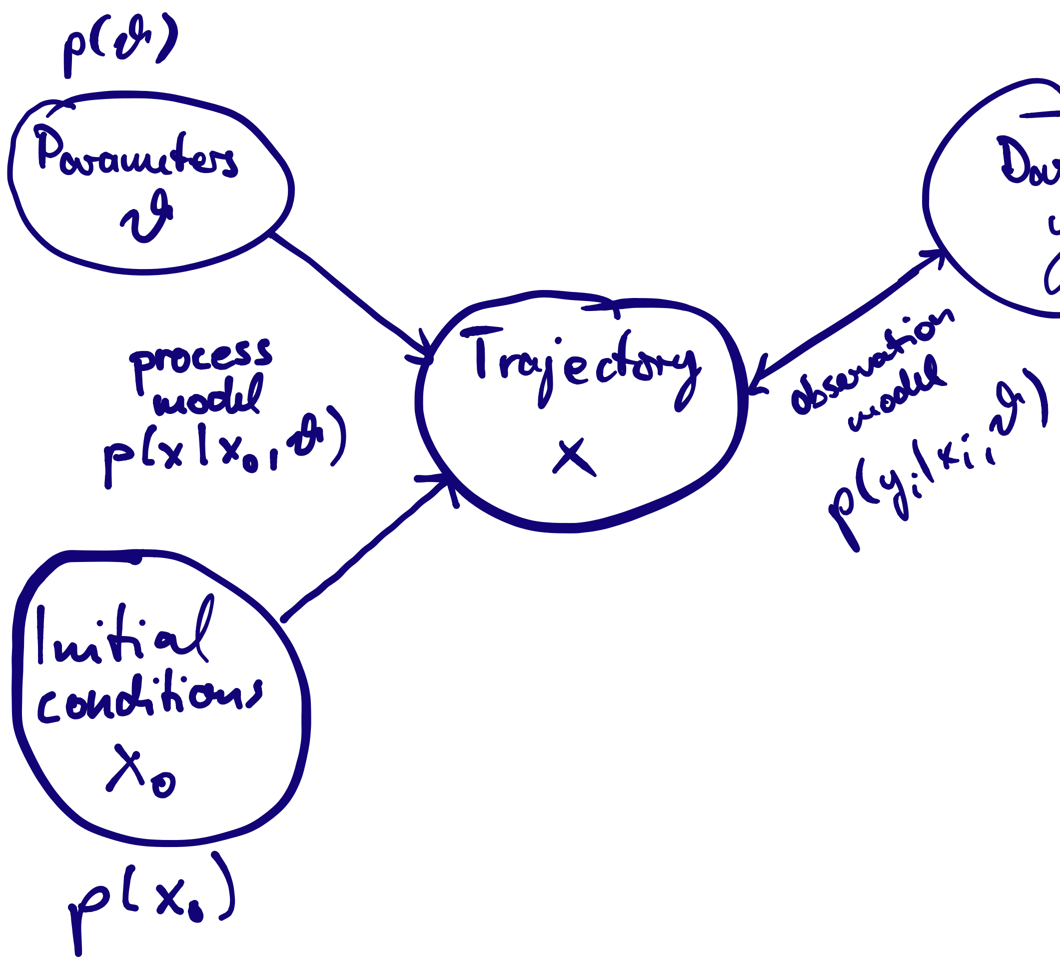
In R, the `ssa.adaptivetau` package implements this (and generates fast C code).

Practical part 2

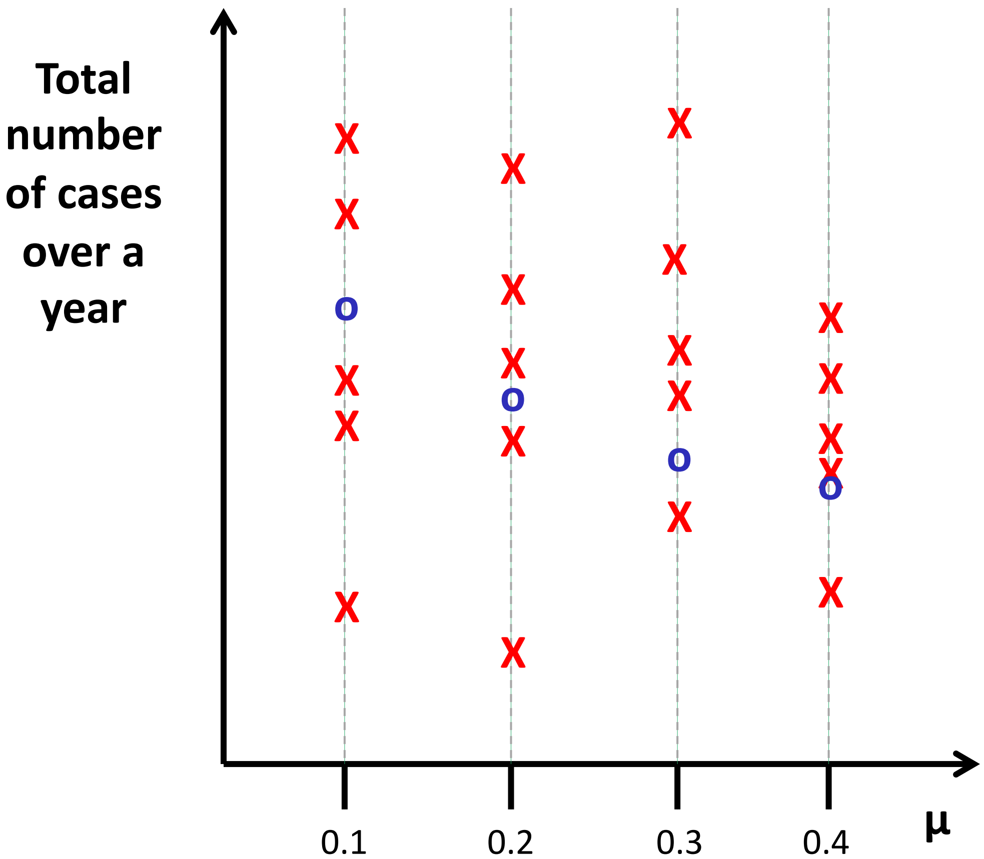
A faster alternative: the `adaptivetau` package

– Objective: use the `adaptivetau` packages to simulate the SIR and SEITL models

Representing uncertainty



Parameter variation

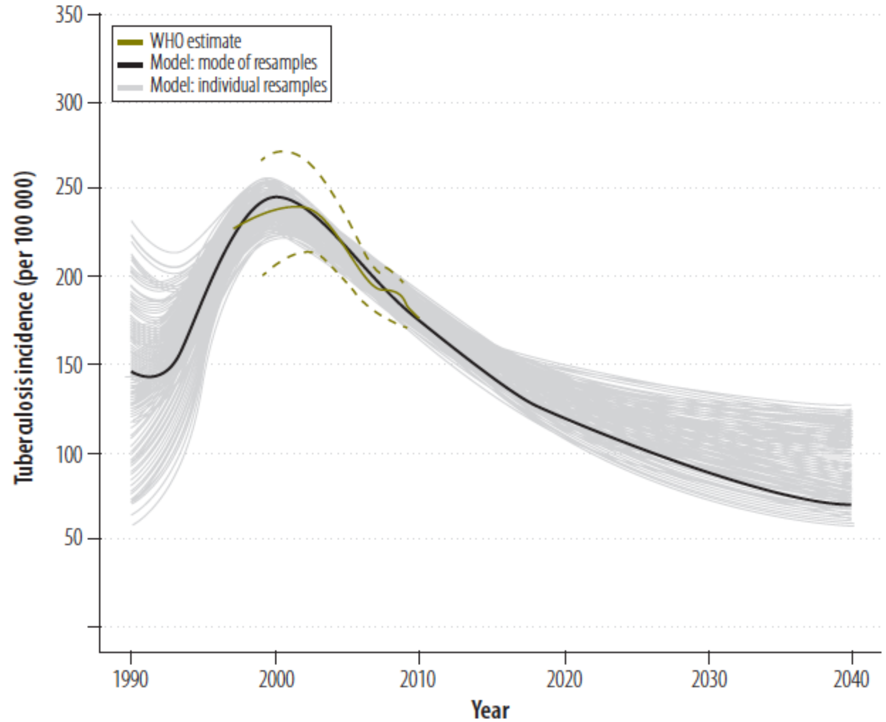


At each of 4 μ values,
5 runs of a model that is:

Stochastic = o / X ?
Deterministic = o / X ?

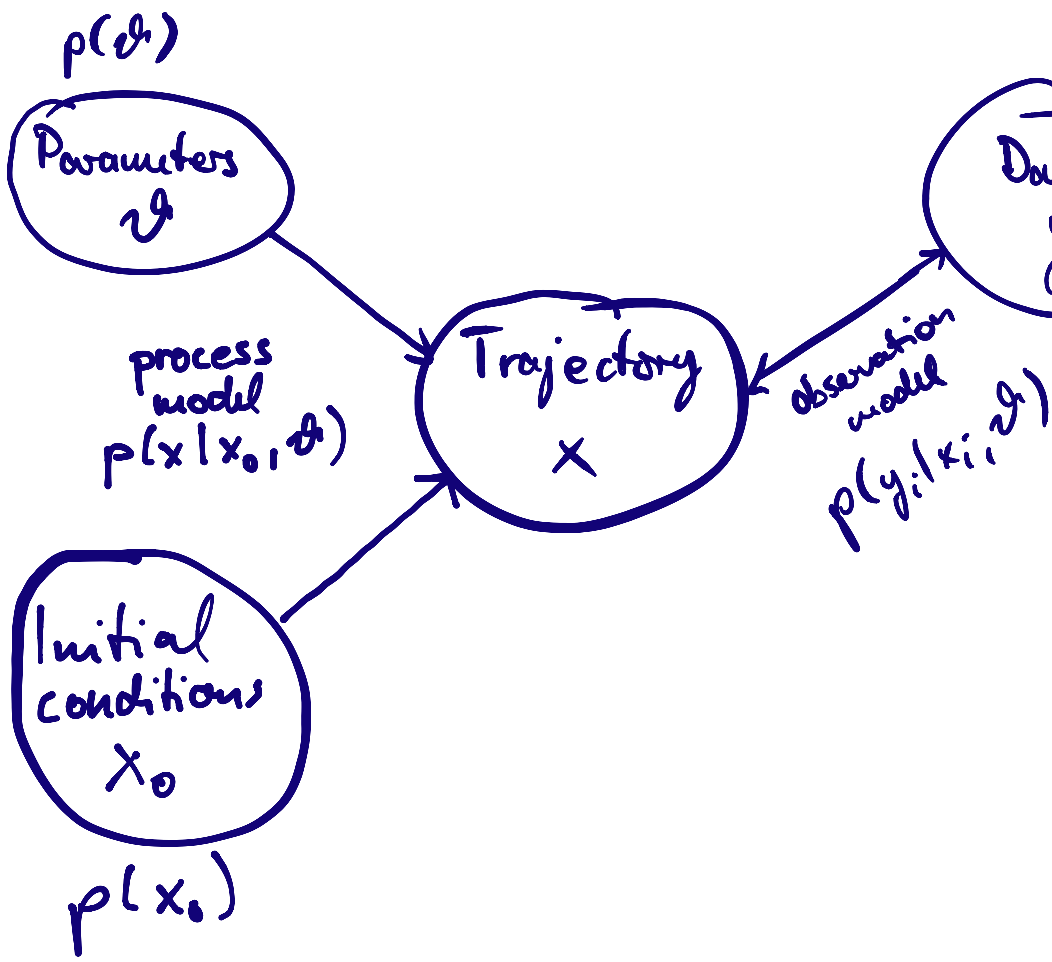
Parameter variation

Fig. 2. **Incidence of tuberculosis (all forms) in the United Republic of Tanzania based on WHO estimates and projected incidence based on the calibrated epidemic model**



[Lin, 2012]

Representing uncertainty



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