Lecture 13

Markets, Mechanisms and Machines

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Motivation

- Explains price formation
  - Walrasian Auction
- Widely used selling game

Explore strategic behavior of:

- Bidders (usually buyers)
  - What bid to submit?
- Sellers
  - Which auction format to use?
  - Which selling game
  - Whether to restrict participation
  - Whether to charge entry fees

Examples

- Auctions used for many transactions in the Ancient world (marriage auctions in Mesopotamia, auctions for debt claims in ancient Greece)
- Art auctions for the last 500 years, (Christie's, Sotheby's)
- Real estate, treasury bills, electricity, livestock
- Large corporations are sold at auction
- Government procurement (highway construction), spectrum licenses
- Online advertising auctions

Auction formats

- A variety of formats are used to sell items
- Single item auction formats
  - English auction
    - Bidders call out successively higher prices until one bidder remains (Sotheby's and Christy's: Hammer auctions)
    - Japanese auction: seller continuously increases price, bidders drop out gradually and irrevocably by pressing a button
  - Vickrey or 2nd price auction:
    - Bidders submit sealed bids; high bidder wins and pays second highest bid
  - Dutch or descending price auction
    - opposite of English auction, Price falls until one bidder presses button, bidder gets object at the current price (Dutch flower auction)

Auction formats

- First Price sealed bid auction
  - Bidders submit sealed bids; high bidder wins and pays his bid
  - Construction contracts, governmental procurement
- Multi items auction formats
- Discriminatory Auction
  - A seller has an supply of items (possibly increasing in p)
  - Buyers submit downward sloping demand schedules (p; q combinations)
  - Equilibrium supply where aggregate demand equals supply
  - Buyers pay their bid for sold items

Auction formats

- Uniform Price
  - A seller has an supply of items (possibly increasing in p)
  - Buyers submit downward sloping demand schedules (p; q combinations)
  - Equilibrium supply where aggregate demand equals supply
  - Buyers pay the equilibrium price (where aggregate demand equals supply)
- Vickrey Auction
  - Win k units, then pay k highest opponents' losing bids (first highest losing bid for top unit, second highest losing bid for second unit, ...)

Auction formats

- Simultaneous ascending price auction (Milgrom (2000))
  - Each bidder demands one unit,
  - Bids are raised in multiple rounds,
  - In each round bidders specify which object that they are bidding for, and may switch from bidding for one object to bidding for another object
  - Auction closes when no further bids are raised
- Combinatorial Auction
  - Submit bids for stand-alone items and also for combination of items
  - Most expensive bidder/item allocation wins

Strategic equivalence

- When do auctions yield the same outcome? When are the bidding strategies identical?
- Example:
  - First price and Dutch auctions
  - Rational bidders, think about bidder giving instructions to an agent
  - In Dutch auction: a price at which to jump in.
  - Would do the same in a first price auction
  - Intuition: no information is revealed in a Dutch auction.
- Under some conditions there is also a strategic equivalence between the 2nd price and the English auction

Informational environment

- Private values:
  - Each bidder *i* values the item at a (privately) known value v_i
  - Other bidders do not know vi but know that v_i is drawn from some probability distribution
  - Example: construction contract in which firms know their own cost but not other firms' costs
- Common values
  - Same value for all bidders
  - Each bidder has a signal of the true value
  - Example: oil field as the value of oil is the same to everyone

Informational environment

- Affiliated values
  - a mixture between private and common values
- Interdependent values
- Reserve price: R
  - seller announces a minimum price prior to the auction, $b \geq R$
  - Reserve price may be kept secret

Vickrey (2nd price) auction

- Rule: High bidder wins and pays the second highest bid
- N bidders
- Common model: private values, each bidder's value $v_i \in [0,V]$ known to bidder i but not known to other bidders
- Bidder i wins if her bid is the highest
  - Gets payoff $v_i - b_{(2)}$ ($b_{(1)}, b_{(2)}, \ldots, b_{(N)}$ are order statistics of the set of submitted bids)
- Otherwise she gets 0

Vickrey (2nd price) auction

Theorem: *Every bidder bids their true value is a dominant strategy equilibrium.*

Proof:

- If $v_i < R$ (R is the reserve price): optimal to bid your value as otherwise pay at least R and $v_i - R < 0$
- *I*f $v_i > R$: strategy $b_i = v_i$
- Consider a deviation:
  - $b_i > v_i$, for $b_{(2)} \leq v_i$ pay b(2) and get the same payoff or for $b_i > b_{(2)} > v_i$ make a loss.
  - $b_i < v_i$, for $b_{(2)} < b_i$ pay $b_{(2)}$ and get the same payoff or for $b_i < b_{(2)} < v_i$ loose the auction

Vickrey (2nd price) auction

- Vickrey auction is efficient (item is allocated to bidder with the highest value)
- Expected Revenues (for the seller) of the Vickrey auction equal the expected second highest valuation
- Other equilibria?
  - yes
- Suppose $b_1 = V$ and all other bidders bid $b_i = 0$
- This is an equilibrium as nobody benets from deviating, but it is not a dominant strategy equilibrium

English auction

Setup (with private values):

- Button auction with continuously increasing prices
- Observe other bidders drop out prices
- No bidding costs

- Strategy: Press button until the price reaches your value v_i
- Is this an equilibrium?
  - Follow the proof for the Vickrey auction
- Other variants of the English auction may feature:
  - Discrete price increases; Open access: bidders may re-enter later-on; Bidding costs.

First price sealed bid auction

Setup:

- One object auctioned off and $R = 0$
- N bidders with private values $v_i \in [0,1]$ (normalization of support of values)
- Beliefs of other bidders about v_i drawn from distribution with density f: $[0, 1] \to \mathbf{R}_+$ (cdf $F(x) = \int_0^x f(z)dz$)
- Note: $v_i \perp v_j$
- Pay-your-bid payment rule:
  - If b_i is the highest bid, payoff is $v_i - b_i$
  - Otherwise, payoff is 0

First price sealed bid auction

- Expected payoff (interim utility)

  $U_i(b_i\,; v_i) = (v_i - b_i)\Pr(b_i > b_j\ \forall j \neq i)$
- Strategy $\beta_i : [0, 1] \rightarrow \mathbf{R}_+$ (maps values to bids)
- In BNE: each bidder chooses b_i that maximizes expected payoff given v_i and beliefs regarding values of other bidders
- Symmetric equilibrium: bidder i with value b_i picks $b_i = \beta(v_i)$
- Claim: if density $f > 0$ on its support then $\beta(v)$ is strictly monotone

First price sealed bid auction

- Determining the probability of winning

$$\Pr(b_i > b_j\ \forall j \neq i) = \Pr(b_i > \beta(v_j)\ b_j\ \forall j \neq i)$$

$$= \Pr(\beta^{-1}(b_i) > \beta^{-1}(\beta(v_j))\ \forall j \neq i)$$

$$= \Pr(\beta^{-1}(b_i) > v_j\ \forall j \neq i) \qquad \text{(monotonicity)}$$

$$= \Pr(\beta^{-1}(b_i) > v_1, \beta^{-1}(b_i) > v_2, \ldots, \beta^{-1}(b_i) > v_N)$$

$$= \Pr(\beta^{-1}(b_i) > v_1) \ldots \Pr(\beta^{-1}(b_i) > v_N) \qquad \text{(independence)}$$

$$= F(\beta^{-1}(b_i)) \ldots F(\beta^{-1}(b_i)) = F^{N-1}(\beta^{-1}(b_i)) \quad \text{(symmetry+beliefs)}$$

- Expected payoff (interim utility)

$$U_i(b_i\,; v_i) = (v_i - b_i)\Pr(b_i > b_j\ \forall j \neq i) = (v_i - b_i)\ F^{N-1}(\beta^{-1}(b_i))$$

First price sealed bid auction

- Utility is determined by interim allocation rule $x_i(b_i)= F^{N\text{-}1}(\beta^{\text{-}1}(b_i))$ (allocation probability as a function of bid)
- In a symmetric BNE: Bids maximize utilities; same values should lead to same bids
- Thus
  - $\frac{d}{db_i} U_i(b_i\,; v_i)=0$, which produces $b_i= \beta(v_i)$
  - $\beta^{\text{-}1}(b_i)= v_i$ and $F^{N\text{-}1}(\beta^{\text{-}1}(b_i))= F^{N\text{-}1}(v_i)$

First price sealed bid auction

- Now

$$\frac{d}{dv_i} U_i(\beta(v_i);\, v_i) = \frac{\partial}{\partial v_i} U_i(\beta(v_i);\, v_i) + \frac{\partial}{\partial b_i} U_i(\beta(v_i);\, v_i)\, \frac{\partial \beta(v_i)}{\partial v_i}$$

$$= x_i(\beta(v_i)) = F^{N\text{-}1}(v_i) \quad \text{(allocation for values)}$$

  - Because, FOC holds and $\frac{\partial}{\partial b_i} U_i(b_i;\, v_i)=0$
  - Because, $U_i(b_i;\, v_i)=x_i(b_i)(v_i - b_i)$ and $\frac{\partial}{\partial v_i}U_i(b_i;\, v_i)=x_i(b_i)$

- This means that $U_i(\beta(v_i);\, v_i) = \int_0^{v_i} \frac{\partial U_i(\beta(z);z)}{\partial z}\, dz$

$$= \int_0^{v_i} F^{N\text{-}1}(z)\, dz$$

First price sealed bid auction

- Collecting information
  - $U_i(\beta(v_i); v_i) = (v_i - b_i)\, F^{N\text{-}1}(v_i)$
  - $U_i(\beta(v_i); v_i) = \int_0^{v_i} F^{N-1}(z)\, dz$
- This produces explicit expression for bid function

$$\beta(v_i) = v_i - \frac{\int_0^{v_i} F^{N-1}(z) dz}{F^{N-1}(v_i)}$$

- This demonstrates that bid function is indeed differentiable and monotone
- This function shows that first-price auction leads to "bid shading"

First price sealed bid auction

- **Example**: 2 bidders with values uniformly distributed on [0,1]
- **Theorem:** $\beta(v)$=E[$v_{(2)}$ | $v_{(2)} < v$]
- Note that $\Pr(v_{(2)} < v) = G(v) = F^{N\text{-}1}(v_i)$
- The density $f_{v(2)}(v) = G'(v)$ and

$$f_{v(2)}(v|\ v_{(2)} < v) = \frac{f_{v(2)}(v)}{\Pr(v_{(2)} < v)}$$

- Thus $E[v_{(2)} \mid v_{(2)} < v] = \int_0^v \frac{zG'(z)}{G(v)} dz = v - \frac{\int_0^v G(z)dz}{G(v)} = \beta(v)$

Revenue equivalence

- How do auction formats compare?

Theorem: *The expected revenue to the seller is the same in a first-price and second-price auction.*

- First price auction bid equals the expected second highest valuation
- Second price auction payment equals the second highest valuation
- In expectation they are the same. Thus, seller's expected revenues are identical

Bidder surplus equivalence

- How do auction formats compare?
- Note that both first and second price auctions allocate to the highest value bidder (since the bid function in the first price auction is monotone in values)
- Thus, both auction formats lead to the same welfare
- Auction welfare is equal to the sum of revenue of the auctioneer and the surplus of bidders
- Since the revenue of the auctioneer is the same for both formats (by revenue equivalence theorem), the surplus of bidders has to be the same too

Revenue optimization

- First and second price auctions are efficient: they maximize social welfare by allocating the item to the bidder with the highest value
- Can auctioneer optimize her revenue by, possibly, sacrificing efficiency
- Yes, by setting a reserve price
    - Reserve price excludes some bidders (whose values are below the reserve)
    - BUT it also incentivizes the remaining bidders to raise their bids
    - This results in an increased revenue of the auctioneer

Revenue optimization

- Total social welfare with the reserve price R

$$N\int_R^1 zF^{N\text{-}1}(z)f(z)dz = \text{Revenue}(R)$$

$$+ N\int_R^1 \left(\int_0^z F^{N-1}(t)dt\right) f(z)dz$$

- Therefore

$$\text{Revenue}(R) = N\int_R^1 \left(v - \frac{1-F(v)}{f(v)}\right) F^{N\text{-}1}(v)f(v)dv$$

Revenue optimization

$$\text{Revenue}(R) = N\int_R^1 \left(v - \frac{1-F(v)}{f(v)}\right) F^{N\text{-}1}(v)f(v)dv$$

- Revenue is maximized when $R = (1 - F(R))/f(R)$
- Function $(1 - F(v))/f(v)$ is called Myerson's "virtual value"
- We maximize the revenue by discarding all bidders whose values are less than their virtual values
- Nobel prize, 2007