Problem: The velocity of a falling skydiver is given by the equation below.

$$v(t) = \sqrt{\frac{gm}{c_d}} \tanh\left(\left(\sqrt{\frac{gc_d}{m}}\right)t\right)$$

Assuming that m = 68.1 kg and c_d = 0.25 kg/m, how far will the skydiver fall between t = 4 seconds and t = 7 seconds?

Ideal solution:

$$s = \int_4^7 v(t)dt = \int_4^7 \sqrt{\frac{gm}{c_d}} \tanh\left(\left(\sqrt{\frac{gc_d}{m}}\right)t\right)dt$$

$$= \left|\frac{m}{c_d}\ln\left[\cosh\left(\left(\sqrt{\frac{gc_d}{m}}\right)t\right)\right]\right|_4^7 = \frac{m}{c_d}\ln\left[\cosh\left(\left(\sqrt{\frac{gc_d}{m}}\right)7\right)\right] - \frac{m}{c_d}\ln\left[\cosh\left(\left(\sqrt{\frac{gc_d}{m}}\right)4\right)\right]$$

$$= 119.5867 \text{ m}$$

Analytic solutions are best when they are possible:

Having a closed form solution can be useful
Solution is not subject to inaccuracies inherent in numerical methods

Matlab Solution:

Code Required:

```
cd = 0.25; m = 68.1; g = 9.81;

v = @(t) sqrt(g * m / cd) * tanh(sqrt(g * cd/m)*t);

s = integral (v, 4, 7);  % integrate v(t) for t from 4 to 7

fprintf ('The skydiver will fall %f m\n', s);
```

In this case the result of 119.5867 m is exact to four decimal places

Function integral performs numerical integration

Numerical integration only approximates the true value of an integral (but the approximation can be very good and exact in some cases).

Function *integral:*

Usage: answer = integral(FUN,A,B,PARAM1,VAL1,PARAM2,VAL2,...) ;

FUN = function to be integrated

- should accept vectors (i.e. dot operators should be used as required)

a, *b* = limits of integration

Parameters can be:

'AbsTol', absolute error tolerance ((default = 10^{-6})
'RelTol', relative error tolerance

It is also possible to integrate vector valued functions. See the documentation.

Integration given data points:

In some cases the function is unknown and only data points are available

Time (s)	4.0	4.5	5.0	5.5	6.0	6.5	7.0
Vel (m/s)	33.1118	35.8309	38.2154	40.2881	42.0762	43.6086	44.9145

Matlab code:

```
t = [4.0   4.5   5.0   5.5   6.0   6.5   7.0];
vel = [33.1118   35.8309   38.2154   40.2881   42.0762   43.6086   44.9145];

s = trapz (t, vel)

fprintf ('The  distance travelled is  %f m\n', s);
```

These data points correspond to the skydiver problem.

The answer obtained (119.5162 m) is imperfect but pretty close.

More closely spaced data points would give an even more accurate answer.

Function *trapz*:

Basic usage: answer = trapz (x, y)

- *x*, *y* = vectors containing *x* and *y* values for data points
- Input vectors must be of the same length
- Result is integral of *y* with respect to *x*

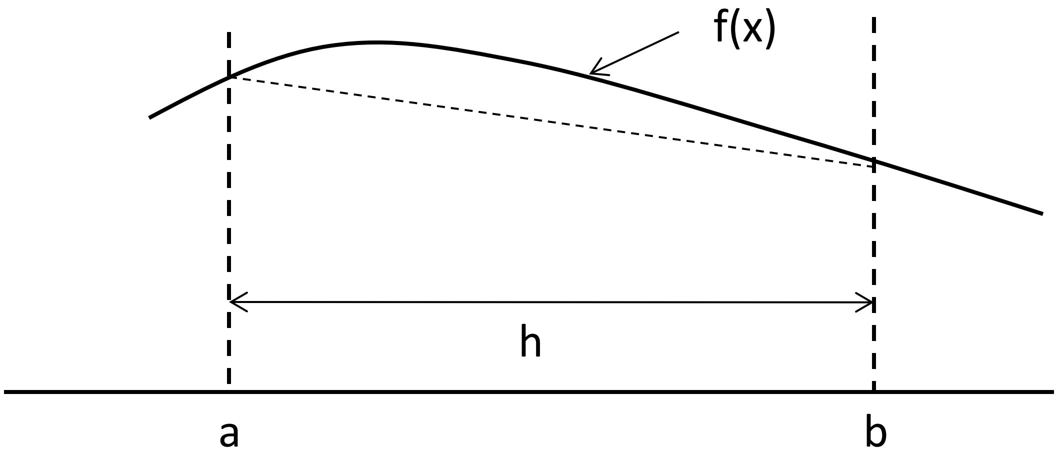
Alternate usage: answer = trapz (y)

- Assumes that *x* values are uniformly spaced and one apart from each other
- For other uniformly spaced *x* values multiply result by actual spacing
- Example: s = 0.5 * trapz (vel); % time values are 0.5 seconds apart

trapz uses *trapezoidal integration*

- Equivalent to fitting a linear spline and integrating the area under it

Trapezoidal integration (single application):



$$I = (b-a)\frac{f(a)+f(b)}{2} = h\frac{f(a)+f(b)}{2}$$

$$\varepsilon \cong -\frac{(b-a)^3}{12} f''(\zeta) = -\frac{h^3}{12} f''(\zeta)$$

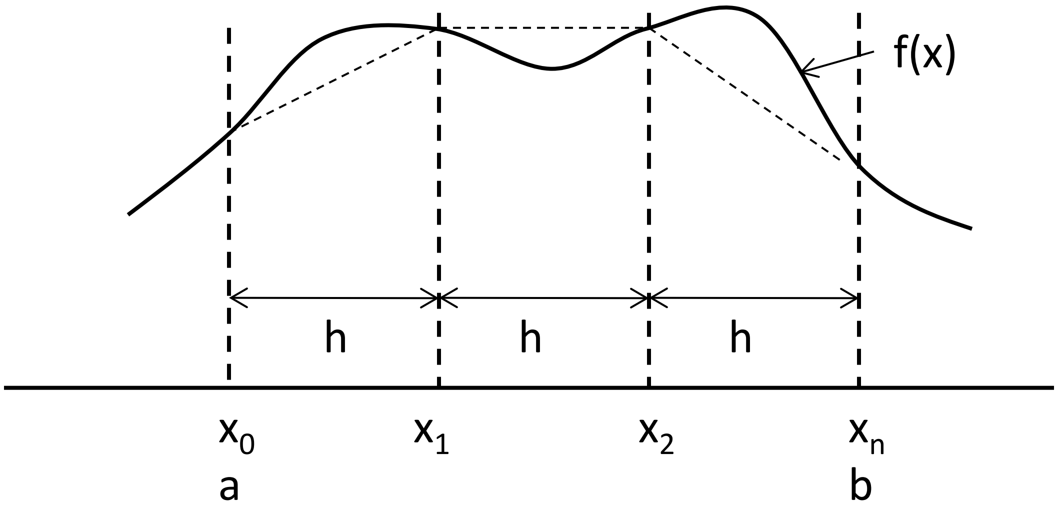
Integral approximated by fitting a straight line and integrating under this line

- more intuitively: area = average height times base

The error is related to the second derivative at some point in the interval

- error is zero if the $f''(x)$ is zero throughout the interval (linear function)

Trapezoidal integration (composite application):



n intervals

$n+1$ data points

$h = (b-a)/n$

$$I = h\frac{f(x_0)+f(x_1)}{2} + h\frac{f(x_1)+f(x_2)}{2} + \ldots + h\frac{f(x_{n-1})+f(x_n)}{2}$$

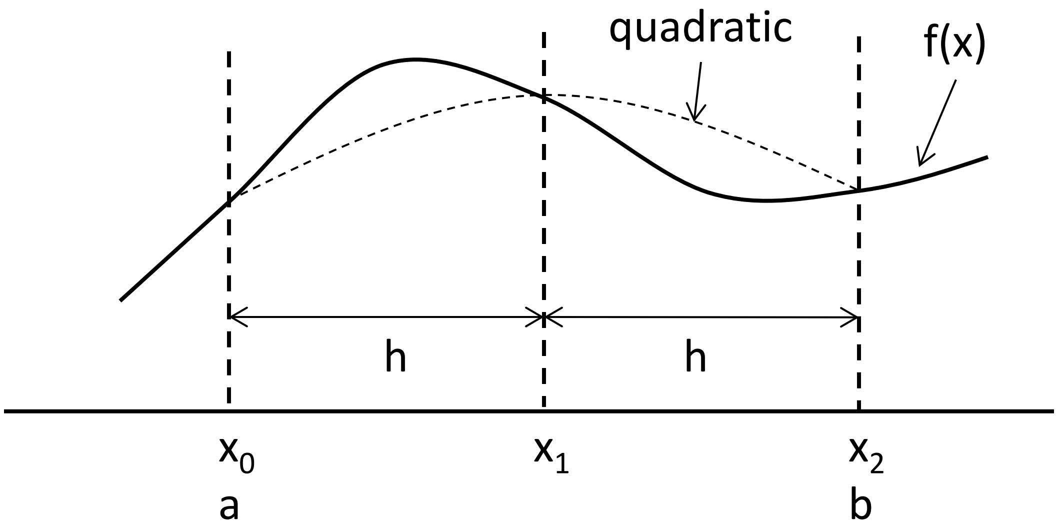
$$= \frac{h}{2}\left(f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)\right)$$

$$\varepsilon \cong -\frac{(b-a)^3}{12n^2}\,\overline{f}''(\zeta) = -\frac{nh^3}{12}\,\overline{f}''(\zeta) \qquad \overline{f}''(\zeta) = \text{"average" second derivative}$$

The error is order (1/n^2): doubling n given one quarter of the error

In general more intervals are better but there is a practical limit beyond which round-off errors become dominant (see text page 104)

Simpson's 1/3 Rule (single application):



$$I = (b-a)\frac{f(x_0)+4f(x_1)+f(x_2)}{6} = \frac{h}{3}\left(f(x_0)+4f(x_1)+f(x_2)\right)$$

$$\varepsilon \cong -\frac{(b-a)^5}{2880} f^{(4)}(\zeta) = -\frac{h^5}{90} f^{(4)}(\zeta)$$

Integral approximated by fitting a quadratic and integrating under this curve

- the math produces the final result given above

The error is related to the fourth derivative at some point in the interval

- error is zero if the $f^{(4)}(x)$ is zero throughout the interval (e.g. a cubic)

The fact that Simpson's 1/3 rule gives perfect results for cubics even though it uses a quadratic interpolating polynomial is surprising enough to warrant a demonstration.

$$I = \int_2^4 x^3 - 2x^2 + 4x - 3dx$$

$$= \left| \frac{x^4}{4} - \frac{2x^3}{3} + \frac{4x^2}{2} - 3x \right|_2^4 = 40.66\overline{6}$$

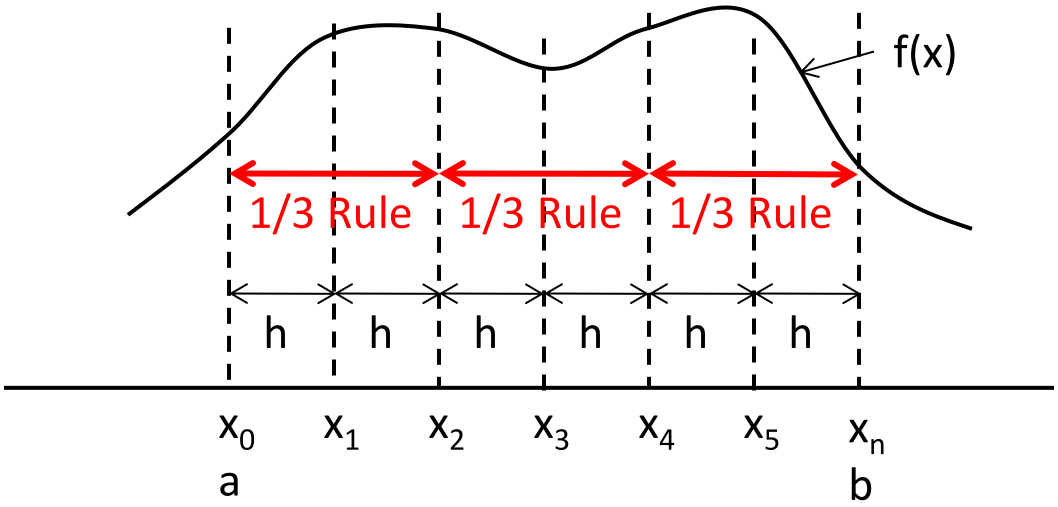
Using Simpson's 1/3 rule:

x	2	3	4
$f(x)$	5	18	45

$$I = \frac{h}{3}\left(5 + 4(18) + 45\right) = \frac{1}{3}\left(5 + 4(18) + 45\right) = 40.66\overline{6}$$

As predicted the result is exact.

Simpson's 1/3 rule (composite application):



n intervals

$n/2$ applications of 1/3 rule

$n+1$ data points

$h = (b-a)/n$

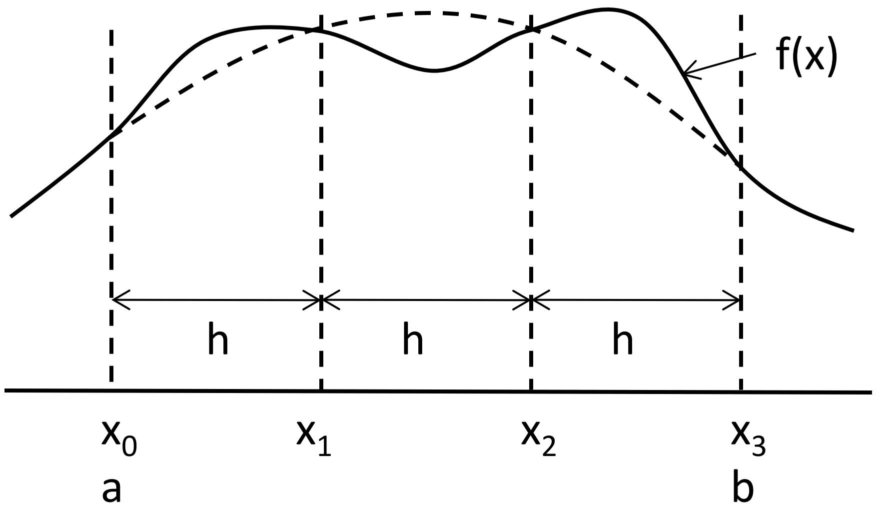
$$I = \frac{h}{3}\left(f(x_0)+4f(x_1)+f(x_2)\right)+\frac{h}{3}\left(f(x_2)+4f(x_3)+f(x_4)\right)+\ldots+\frac{h}{3}\left(f(x_{n-2})+4f(x_{n-1})+f(x_n)\right)$$

$$= \frac{h}{3}\left(f(x_0)+4\sum_{i=1,3,5}^{n-1} f(x_i)+2\sum_{i=2,4,6}^{n-2} f(x_i)+f(x_n)\right)$$

$$\varepsilon \cong -\frac{(b-a)^5}{180n^4}\bar{f}^{(4)}(\zeta) = -\frac{nh^5}{180}\bar{f}^{(4)}(\zeta) \qquad \bar{f}^{(4)}(\zeta) = \text{"average" fourth derivative}$$

The error is order ($1/n^4$): doubling n given one sixteenth of the error

Simpson's 3/8 rule (single application):



$$I = (b-a)\frac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8}$$

$$= \frac{3h}{8}\left(f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right)$$

$$\varepsilon \cong -\frac{(b-a)^5}{6480} f^{(4)}(\zeta) = -\frac{3h^5}{80} f^{(4)}(\zeta)$$

Integral approximated by fitting a cubic and integrating under this curve

- the math produces the final result given above

The error is related to the fourth derivative at some point in the interval

- no better than 3/8 rule in this respect (although denominator larger)

Useful when dealing with an even number of data points (odd number of intervals)

- use 3/8 rule for first or last three intervals, use composite 1/3 rule for the rest

Higher-order Newton-Cotes formulas:

All techniques employ same basic idea (find interpolating polynomial for points and the integrate under this polynomial

- trapezoidal = first order polynomial
- Simpson's 1/3 Rule = second order polynomial
- Simpson's 3/8 Rule = third order polynomial

All are *Newton-Cotes closed integration formulas*

Idea extends to higher order polynomials (e.g. Boole's rule = fourth order)

See text p 411

Integrating with unevenly spaced points:

Apply basic trapezoidal rule to each interval and add up results.