**ECOR 2606 Final Exam Practice Material**

1/. The function $f(x) = x^2 - 12x + 32$ has roots at $x = 4$ and $x = 8$. Execute the first three steps of bisection, Regula Falsi, secant, and Newton's searches by hand. Choose appropriate starting points for each of your searches to find the lower root. sConfirm that your results are correct by using the supplied function m-files for these methods.

2/. The function $-\sin(x) * \exp(-0.5x)$ has a minimum between 0 and 2. Perform the first three steps of a golden section search by hand. Confirm your results by using the supplied function m-file.

3/. Use Gaussian eliminating (with pivoting) to solve the following series of equations by hand.

$$
\begin{aligned}
5x - y + z &= 6 \\
x + 3y - z &= 16 \\
3x + y - 5z &= 2
\end{aligned}
$$

After the coefficient matrix has been reduced to upper triangular form, complete the solution in two ways.

i) By back substitution

ii) By back elimination (Gauss Jordan)

The two approaches should, of course, give you the same answer.

Convert the equations into a form suitable for solution by iterative methods and do three iterations using both Jacobi and Gauss-Siedel starting with x = $\{1, 1, 1\}^T$. Calculate the value of the norm between the third and second iterations. Check your answer using the supplied MATLAB functions.

The A matrix above may be decomposed into L= [1 0 0; 0.2 1 0; 0.6 0.5 1] & U=[5 -1 1; 0 3.2 -1.2; 0 0 -5]. Solve the system Ax=b using L and U.

4/. Find the polynomial that passes through the following four points:

```
x:  1   3   6    10
y:  0  54  420  1944
```

Obtain an answer using i) Lagrange's form and ii) Newton's form. Check your answers by using "polyfit".

Manually find the "best fit" straight line for the four points using two different methods described in the course. Confirm your answer using "polyfit".

Manually calculate the correlation coefficient and compare it to the value derived from correlate.m

If we added another data point at (4, 100), what would be the degree of the new interpolating polynomial and what would be the value of the polynomial at x=6?

5/. Find the line that best fits the following four points:

| | | | | |
|---|---|---|---|---|
| x: | 1 | 3 | 6 | 10 |
| y: | 0 | 5 | 11 | 13 |

Obtain an answer using i) appropriate basis functions and the normal equations $Z^tZa = Z^ty$ and ii) the regression equations on the formula sheet. Check your answers by using "polyfit".

6/. The table below gives velocities of an object from $t$ = 0 to $t$ = 10 seconds.

| t(sec) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| v (m/s) | 452 | 410 | 377 | 350 | 322 | 301 | 283 | 269 | 257 | 251 | 250 |

Estimate the distance travelled by the object between $t$ = 0 and $t$ = 10. Use trapezoidal integration and the appropriate Simpson's rule(s).

Estimate the acceleration of the object at $t$ = 4 using forward, central, and backward differences.

7/. A 200kg cart that is initially moving at 10 m/s is subject to friction and a braking force.

$$friction = 0.5v \text{ N}$$

$$F_B = 500(1 - e^{-0.5t}) \text{ N}$$

The differential equation (which you should attempt to derive) is

$$\frac{dv}{dt} = -\frac{1}{200}\left(500(1 - e^{-0.5t}) + 0.5v\right)$$

Use Euler's method and Heun's method (without iteration) to predict the velocity of the object at $t$ = 0.5 seconds and $t$ = 1.0 seconds.

Using ode45 gives the results below for the first 5 seconds. How far will the cart travel during this time? Use numerical methods to estimate the deceleration at $t$ = 2.5 seconds? Compare your answer with the analytical value of dv/dt.

| t | v |
|---|---|
| 0 | 10.0000 |
| 0.5000 | 9.8436 |
| 1.0000 | 9.4428 |
| 1.5000 | 8.8522 |
| 2.0000 | 8.1140 |
| 2.5000 | 7.2613 |
| 3.0000 | 6.3197 |
| 3.5000 | 5.3092 |

```
4.0000    4.2454
4.5000    3.1404
5.0000    2.0038
```

8/. (From text 20.1)

$$\frac{dy}{dt} = yt^2 - 1.1y$$

Assuming that $y(0) = 1$, **manually** find $y(t)$ for $t = 0$ to $t = 2$ seconds

a) using Euler's method with $h = 0.5$ and $h = 0.25$
b) using Heuns' method (no iteration) with $h = 0.5$

Use function *ode45* to create a plot of $y(t)$ for $t = 0$ to $t = 2$ seconds and plot against the results of a, b and c.

Optional: Check your plot and your calculus by finding the analytical solution and plotting this as well.

9/. (From text 20.4) The following table gives the world population (in millions) for 1950 to 2000.

| | | | | | | |
|---|---|---|---|---|---|---|
| Year | 1950 | 1955 | 1960 | 1965 | 1970 | 1975 |
| Population | 2555 | 2780 | 3040 | 3346 | 3708 | 4087 |
| Year | 1980 | 1985 | 1990 | 1995 | 2000 | |
| Population | 4454 | 4850 | 5276 | 5686 | 6079 | |

One of the simplest growth models says that

$$\frac{dp}{dt} = k_g p$$

where $p$ is the population (in millions) and $t$ is time (in years).

Manually determine $dp/dt$ for each data point. Use first order central differences where this is possible and a first order forward or backward difference otherwise. Use function *gradient* to check your results.

Estimate $k_g$ by applying the formula below at each of the data points and averaging the results.

$$k_g = \frac{1}{p}\frac{dp}{dt}$$

Use the differential equation and the population is 1950 (your initial condition) to predict the world population for 1950 to 2000. Plot your results and the actual data (use markers) on the same graph.

10/. (From text 20.8) The *van der Pol equation* is a model of an electronic circuit that arose back in the days of vacuum tubes:

$$\frac{d^2y}{dt^2} - (1 - y^2)\frac{dy}{dt} + y = 0$$

Assuming that $y(0) = y'(0) = 1$, use function *ode45* to create a plot slowing the value of $y(t)$ and $y'(t)$ for $t = 0$ for $t = 10$ seconds.

Manually use Euler's method and a step size of 0.2 to calculate $y(t)$ over the same interval. Plot the results (using dashed line or different colours) along with the results obtained using *ode45*. Are the results for $t = 10$ of any use at all? Repeat the experiment using a step size of 0.01.

Write a MATLAB function that implements Euler's method. The first lines should be

```
function [t, y]=Euler(fp, y0, dt, t0, tf)
%EULER implements Euler's method for solving odes
% output time and output vectors
% input gradient function, initial y, time step, initial and final time
```

11/ Write a MATLAB function that implements three-point Gaussian quadrature. The first lines should be:

```
function [area] = Gauss3point (f, xl, xu)
%GAUSS3POINT: 3 point Gaussian quadrature integration
%Output area
%Input, function to be integrated, lower and upper limits of integration
```

12/. Write a MATLAB function that will take as input a function of a single variable ("f") and the lower ("a") and upper limits ("b") of integration and output the best estimate of the integral using three trapezoidal integrations using the full interval, half the interval and one quarter of the interval, and then combining these integrations in a Romberg Array calculation. The output should be the best estimate of the integral ("area") and its approximate error ("error"). The first and last statements of the function are shown below

```
function [area, error] = Romberg(f,a,b)
```

end

b) What would be the results of applying the function to

$$\int_{2}^{6} x^2 \sin(x)dx \qquad ?$$

13/. Find the values of a, b, c, d, e and f such that $s_1(x)$, $s_2(x)$ and $s_3(x)$ form a series of cubic splines that pass through (0,2), (1,1) and (3,-1). Show your work.

$$\boldsymbol{s_1(x) = ax^2 + b(x-1)^3 \quad x \in (-\infty, 1]}$$
$$\boldsymbol{s_2(x) = cx^2 + d \quad x \in (1, 2)}$$
$$\boldsymbol{s_3(x) = ex^2 + f(x-2)^3\,, \quad x \in [2, \infty)}$$