NUMERICAL METHODS FOR UNIVERSITY OF SOUTH ALABAMA

A Brief Textbook Presented to the
Student Body of the University of South Alabama

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Current Edition

This manuscript was last updated on December 5, 2025. The latest edition can be found on Github

Manuscript Changes

1. December 10th, 2020 - Moved to public Github Repo separate from MATLAB
2. March 20th, 2024 - Added color to hyperlinks and added section above called current edition
3. April 17th, 2024 - Added Graphics to repo and recompiled
4. December 5th, 2025 - Edited changes needed below to point to Github

Changes Needed

Needed changes are now tracked on Github

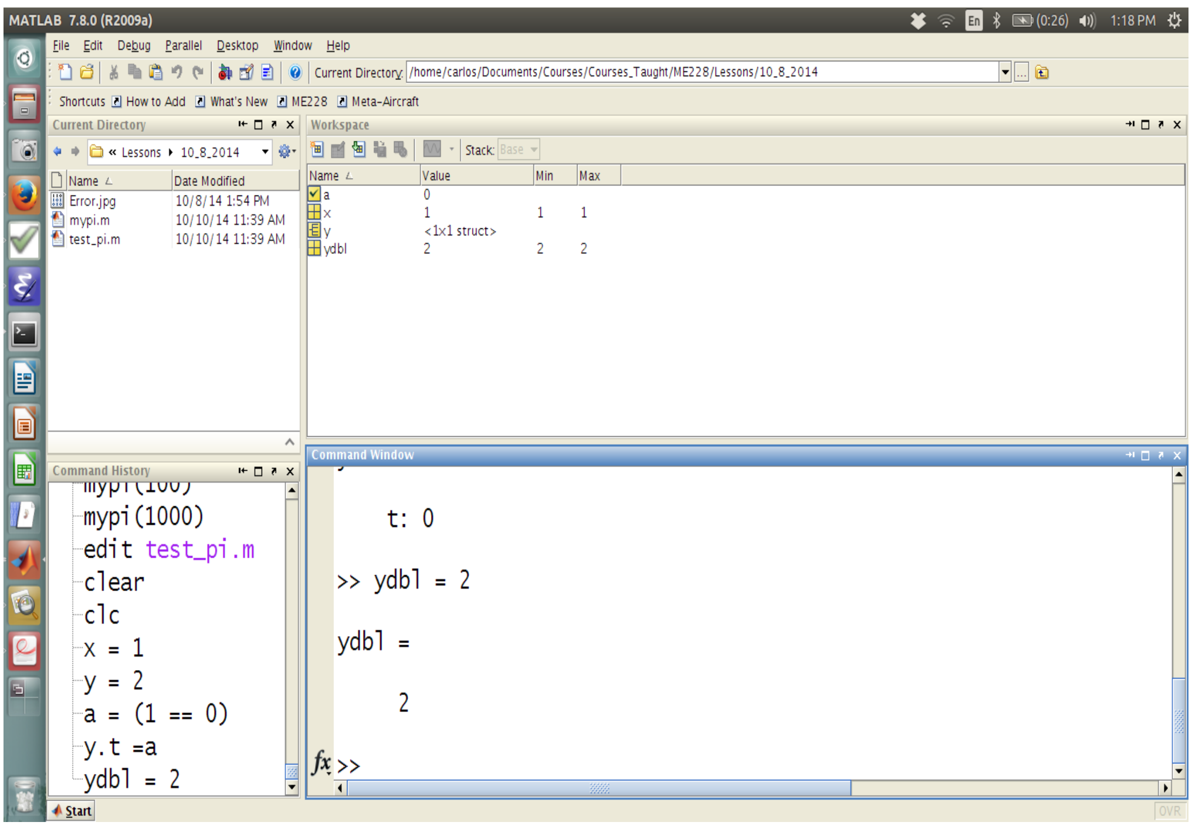
Contents

1. INTRODUCTION TO MATLAB

1.A. Basics of MATLAB - Palm - Chapter 1

1. **MATLAB(Current Directory, Workspace, Command Window, File) interface**

   When you double click MATLAB you will have a number of windows present. The configuration of all windows in your MATLAB version may be different but the names are all the same.



   The very top of the MATLAB window is our toolbar. It has similar buttons such as File, Edit, etc. Below this is your current directory. This lists the current folder that you are operating in. The window in the upper right is the Current Directory as well which lists all the files currently in the folder. In this example I have an error.jpg file, a mypi.m and test_pi.m file. Note that a .jpg is a picture file and a .m file is a MATLAB script file. The window directly to the right is the Workspace. This lists all the current variables in the workspace. The lower left window is my Command History. It contains the history of all of my inputs into the Command Window. The command window is where I can use MATLAB like a calculator. If I type the following command

   >> x = 1

   into the command window, MATLAB will create a variable 1 and save it into x. Everytime I type in x it will now use 1 as the value. Thus, if I type in

   >> y = 2*x

   the variable y will be evaluated as 2*x where x = 1 thus 2*1 = 2 and y will be 2.

2. **Order of operations**

MATLAB works just like your calculator does. Thus, you need to make sure parentheses are in the correct spot. For example

>> x = 1/(2+1)

will set x = to one third. However,

>> x = 1/2+1

will set x = 0.5 + 1 = 1.5. Try this in MATLAB to see the difference. It is important to remember PEMDAS (Parentheses, Exponents,Multiplication/Division, Addition/Substraction) That is, MATLAB evaluates those operators in that order.

3. **Learn format of output**

   There may be times when you need to see more than 4 significant digits. For example,

   >> x = sqrt(2)

   will output x = 1.4142. However it may be beneifical to see more significant digits. To do that simply type

   >> format long g

   into MATLAB. Then when you type 'x' into MATLAB you should see 13 digits. To return to normal output simply type

   >> format short g

4. **What variables do I have?**

   Sometime through your coding you will have alot of variables in your workspace. For example it you type

   >> x = 2

   >> y = 2*x

   >> z = x^2

   In order to see all of your variables you can simply type

   >> whos

   into MATLAB. This will output all variable names in your workspace, their size, the number of bytes they are taking up and the class which is also known as type. Another way to see the current variables is to simply look at the workspace window.

5. **Variable Types**

   There are numerous types of variables that MATLAB can handle. Think of them as different currencies. For example I could give you a number which is known as a double

   >> x = 42

   or I could give you a set of characters known as a char.

   >> s = 'Hello World'

   If I then type in

   >> whos

   into MATLAB I see that I have a variable x with a size of 1x1, using 8 bytes, and a class of double. I also will have a variable s with a size of 1x11, and a class of char. The size of s is 1x11 because I have 11 characters in the variable. The space is included. The number of bytes is equal to 2*11 or 22. That is because MATLAB needs 2 bytes for every letter and thus needs 22 bytes to represent that number. A double always needs 8 bytes. That is, the number 42 needs 8 bytes, the number 6789 needs 8 bytes. The name double comes from the fact that numbers are converted to binary which is of base 2. If I wish to convert variables from one type to another I use the functions num2str and str2num.

   >> x = 1

   >> y = num2str(x)

   The code above will convert the variable x to a string and save it in y. Note that if you type whos into MATLAB x will still be a double but y will be a string. Try using str2num to see what happens.

   Note that the reason why MATLAB uses 2 bytes per character is because it uses the unicode UTF-16 standard. Remember that 2 bytes is 16 bits hence UTF-16. How many characters can UTF-16 represent? Check out the entire list here.

   http://www.fileformat.info/info/charset/UTF-16/list.htm

   Obviously there are other formats such as UTF-7 and UTF-8 which stand for 7 and 8 bit respectively. They all have their strengths and weaknesses.

6. **Scripting**

   Alot of times you will have to compute something such as the number of seconds in a day. This can be easily accomplished by typing.

   >> seconds_min = 60

   >> seconds_hour = 60*seconds_min

   >> seconds_day = 24*seconds_hour

   However, if you make a mistake you will have to clear everything

   >> clear

   and start over. This can be very tedious. Thus scripting was invented.

   (a) **Open a File...**

   To open a file you can simply click File → New → Blank M-File. Once the file is open you must save the file by clicking File → Save. You can also type

   >> edit filename.m

   into MATLAB and it will create a file called filename.m. If you then return to the MATLAB window notice that you will have the file filename.m in your current directory. If you type

   >> ls

   into MATLAB you should also see your filename displayed there as well. With the script open you can then create a script.

   (b) **Example Script**

   Below is a simple script calculating the number of seconds in a day. The first three lines clears the workspace, the command window and closes all open figures. Note that the % is a comment that can be used to make notes in your script.

```
clear
clc
close all
%%%This is a comment
seconds_min = 60
seconds_hour = 60*seconds_min
seconds_day = 24*seconds_hour
```

   (c) **Execute Script**

   There are three ways to execute a script.

   i. Hit the play button from the top of the editor
   ii. Hit F5 on the keyboard
   iii. Type
   >> filename
   in the command window. When you type filename in the command window, MATLAB will search for a file called filename.m and execute the contents of the script. If the file does not exist, MATLAB will throw an error.

7. **Directories**

   Further along you may download a file from the internet and open it in MATLAB. When you do this the file will be opened in a editor in MATLAB. However, when you attempt to run this script, MATLAB will ask you whether or not you would like to change the current directory. If you do this the current directory at the top toolbar will change to the location of the file you just downloaded. Note that if you then run another file you will have to change the directory again. I urge you to try and organize your files such that everything makes intuitive sense.

8. **Help!**

   If at any point you ever lose your way simply type

>> help

into MATLAB. If you know of a function but you forget what it does or how to use it simply type

>> help <variable name>

for example

>> help num2str

will list what num2str does and how to use it.

9. **Functions Learned**

   whos

   clear

   clc

   format long g

   ls

   num2str, str2num

   disp

1.B. Arrays - Palm - Chapter 2

1. **Inputting Arrays**

   Whenever you type a variable into MATLAB

   >> x = 1

   MATLAB will create an array. The variable x currently is a 1x1 array. That is it has 1 row and 1 column. It is possible however to create a vector of numbers.

   >> x = [1 2 3 4]

   This will create a vector with 1 row and 4 columns. It is possible to make the same vector with only 1 column by using a semicolon.

   >> x = [1 ; 2 ; 3 ; 4]

   This will create a variable with 4 rows and 1 column. It is also possible to create a column vector by using the transpose value.

   >>x = [1 2 3 4]'

   The ' will transpose the vector. In order to create a matrix you simply combine the semicolons and spaces.

   >> A = [1 2 ; 3 4]

   This will create a 2x2 matrix. It is also possible to create rules for vectors. For example, say you want all even number from 2 to 100. This can be accomplished by using the form

2. **Linspace vs. Increment**

   start:increment:end

   >> x = 2:2:100

   The line above will start at 2, increment by 2 and stop when it hits 100. You can also use the linspace command which uses the form

   linspace(start,end,number of elements)

   >> x = linspace(1,10,5)

   The line above will create a vector that starts at 1 and end at 10 but contains 5 elements.

3. **Length Command**

   >> L = length(x)

   Length will compute the number of elements in the vector. In this case L will be equal to 5. If you type whos in MATLAB you will see that x is a vector 1x5 and L is a 1x1 but its value is 5.

4. **Creating Empty Matrices**

   To create empty matrices you may use the zeros() and ones() command.

   >> A = zeros(5,3)

   The line above will create a matrix of zeros with 5 rows and 3 columns.

   Note that adding matrices is very simple.

   >> B = zeros(5,3) + 5

   will create a matrix of zeros with 5 rows and 3 columns and then add the number 5 to every element in the matrix.

5. **Add/Substract and Multiply/Divide**

   >> C = A + B

   this will then add the contents of A to B. Similar results are seen for substraction. Multiplication is tricky. The following line will throw an error.

   >> C = A*B ← ERROR

   The reason is that A is a 5x3 and B is a 5x3. As such the inner matrix dimensions do not match.

   >> x = [1 ; 3]

   >> A = [1 2 ; 4 5]

   >> b = A*x

   the lines above are totally valid because x is a 2x1 and A is a 2x2 thus b is a 2x1. Type in whos to verify.

6. **Dot Operator**

   It may be beneficial to compute the square of the numbers from say 1 to 10.

   >> x = 1:1:10

   will create the numbers from 1 to 10 incrementing by 1.

   >> x = xˆ2 ← ERROR

   This again will throw an error because x is a 1x10. A 1x10 times a 1x10 does not have the inner matrix dimensions that match. In order to multiply these matrices you must use the dot operator.

   >> x = x.ˆ2

   You can also use the dot operator for muliplication and division.

   >> y = 2*ones(1,10)

   >> z = x./y

   This will create a vector of ones (1x10) and then multiply each element by 2. Thus y will be a vector of 2's. z will then be every element in x divided by y. Try this out to see if it works.

7. **Reference Elements of an Array**

   Finally you may find yourself trying to use the second row and third column of a matrix.

   >> A = magic(5)

   will create a 5x5 matrix using the magic algorithm. If you then wish to select the second element and third column you simply type

   >> a32 = A(3,2)

   You can also grab the entire column by typing

   >> a2 = A(:,2)

   which will grab the entire second column.

8. **Semicolons!**

   Notice that when you type

   >> x = 0:0.01:100

MATLAB will spit out a lot of numbers. When creating a script it is common practice to stop the output of computation by putting a plug on the computation. This is accomplished by using a semi-colon.x

>> x = 0:0.01:100;

will not output anything to the Command Window.

9. **Solving Multiple Algebraic Equations**

   The task of solving the equation below is trivial to do by hand however it is easy to make simple mistakes.

$$
\begin{gathered}
5x - 3y + 4z = 41 \\
12x + 6y - 7z = -26 \\
-4x + 2y + 6z = 14
\end{gathered}
$$

   In order to do this by hand we simply type

   >> A = [5 -3 4 ; 12 6 -7 ; -4 2 6]

   which will create an A matrix of the coefficients of our system. Then

   >> b = [41;-26;14]

   which will create a vector of the solutions. Finally assuming the form Ax=b we can solve this by multiplying the left hand side by the inverse of A.

   >> x = inv(A)*b

10. **Evaluating Functions**

    Assume you have the function $y = x^2 cos(x/2)$ and you wish to evaluate the function at x = $\pi/4$. That would simply be

    >> x = pi/4

    >> y = x^2*cos(x/2)

    However if you wish to evaluate the function over an interval say -pi to pi then you need to use vector math.

    >> x = linspace(-pi,pi,100)

    which creates a vector of 100 elements from -π to π. Then you can evaluate y using the dot operator.

    >> y = x.^2.*cos(x/2)

    Note that you do not need a dot operator when dividing by 2 because 2 is a scalar.

11. **Other Arrays**

    There are two other types of arrays. One is a cell array and the other is a structure. I will go over them briefly here.

    A structure is a bucket that holds multiple attributes of information. To make a structure you simply type in

    >> computers.name = 'Toshiba'

    >> computers.CPU = 3.9

    This will create a structure called computers. The structure has two attributes, name and CPU. If you would like to add another element to the structure simply type

    >> computers(2).name = 'Lenovo'

    >> computers(2).CPU = 2.0

    You can then call the structure in two ways. You can list all attributes of an element by typing

    >> computers(2)

    or you can list all the names of the computers by typing

    >> computers.name

    Finally you can access the 1 computers CPU by typing in

    >> computers(1).CPU

    A cell array can be created which contains any all variable type embedded. For example

    >> x = [43 56 77]

>> s = 'Hello'

>> u = 0

>> computers.name = 'Toshiba'

>> computers(2).name = 'Lenovo'

creates 4 variables. The first is a double with 3 elements. The second is a string with 5 elements. The third is a double with 1 element and the last is a structure with 2 elements. These can be combined into a cell array by using curly braces {}.

>> c = {x,s,u,computers}

Note however that the length of c is 4. Furthermore if you wish to access part of the cell array you need to use curly braces.

>> c{1}

will output the contents of the first cell which in this case is x.

Functions Learned

magic, ones, zeros, linspace, length, inv

1.C. Example Problems

1.) Using rules of mathematical precedence, solve for the following with MATLAB (writing it exactly as it is printed). Compare answers by hand. Did you get the same answer?

a) 81^(3/4)+5*2^2/2-5 b) 12*4/3-8^2/2-15/(5-2)

2.) Suppose that x=4, and y=3. Use MATLAB to compute the following. Check your answers by hand.

$$a)\frac{yx^3}{x-y} \quad b)\frac{x^5}{y^5-1} \quad c)2\pi x^2 y \quad d)\frac{4\sqrt{y-2}}{3x-5}$$

3.) Evaluate the following expressions in MATLAB for the given value of x. Check your work by hand.

$$a)y = 5x^3 \quad x = 3 \quad b)y = 2\frac{sinx}{7} \quad x = 15^0 \quad c)y = 6x^{1/3} + \frac{2x}{3} \quad x = 27$$

4.) Suppose that x=-6-5i and y=4+2i. Use MATLAB to compute the following. Check your work by hand.

$$a)x + y \quad b)x/y \quad c)xy$$

5.) Use MATLAB to calculate the following, and check your answers with a calculator.

$$a)e^{-2.3^2} + 2.86log_{10}(14) + \sqrt[6]{516} \quad b)\frac{5\ ln7}{2} + \sqrt{5^2 + 4^3} \quad c)cos(\frac{3\pi}{8})(sinh(\frac{3\pi}{4}))^2$$

6.) The Richter scale is a measure of the intensity of an earthquake. The energy E (in Joules) released by the quake is related to the magnitude M on the Richter scale as follows:

$$E = 10^{4.4}10^{1.5M}$$

Use MATLAB to determine how much more energy is released by a magnitude 7.5 quake than a 6.1 quake.

7.) Convert the following strings to numbers using the str2num() function. What happens?

a) '42'	b) 'hello'	c) 'h'
d) '4.2'	e) '4/2'	f) 'cos(pi)'

8.) Convert the following numbers to strings using num2str() function. What happens? Using the length function compute the length of each string.

a) 42	b) -15	c) exp(1)
d) 4.2	e) 4/2	f) cos(pi)

9.) Using the function exist() test whether or not these variables exist. Note depending on how you save your workspace you may get different answers however verify the output using whos.

a) x b) y c) z

If they do exist using the disp() function print the following to the command window:

Yes the variable _____ exists.

10.) Type this matrix in MATLAB.

$$A = \begin{bmatrix} 2 & 7 & -3 & 9 \\ -3 & 5 & 15 & 3 \\ 4 & 11 & 8 & 13 \\ 16 & 4 & -5 & -11 \\ 5 & -2 & 18 & 3 \end{bmatrix}$$

Use MATLAB to answer the following questions.

a) Create a vector consisting of the elements in the third column of **A**.
b) Create a vector consisting of the elements in the second row of **A**
c) Create a submatrix encompassing the lower right 3x3 matrix only.
d) Find the value of a_{32}.

11.) Given the matrices

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 5 & -3 \\ 4 & -10 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 11 & 0 & -3 \\ 5 & -12 & 4 \\ 2 & 3 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 7 & 15 & 1 \\ 10 & 3 & -2 \\ 9 & -5 & 8 \end{bmatrix}$$

Compute the following. Check your work by hand.

a) Compute **A+B+C**
b) Compute **A-B-C**
c) Does $5(\mathbf{A} + \mathbf{C}) = 5\mathbf{A} + 5\mathbf{C}$?
d) Does **A*C** = **C*A**?
e) Does $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$?
f) Does **A/B** = **A./B**?

12.) Use matrices to solve for x,y and z.

$$\begin{aligned} 5x - 3y + 4z &= 41 \\ 12x + 6y - 7z &= -26 \\ -4x + 2y + 6z &= 14 \end{aligned}$$

Check your answer by hand.

13.) Plot the equation below. Use the bounds [-2 2]. Label your axes, make a title, turn on the grid and make the background of your plot white. Plot a circle where the graph crosses the x axis.

$$y = 5x^3 + 2x^2 - 5x - 20$$

14.) Plot the equation from above. Use the bounds [-2 2]. Label your axes, make a title, turn on the grid and make the background of your plot white. Plot a circle where the graph crosses the x axis. Is this the same as your answer in problem 13?

15.) Assume that an aircraft is flying at a certain speed (mi/hr) for 4 different legs. Assume that this speed is constant for a certain amount of time as given by the table below.

Leg	1	2	3	4
Speed (mi/hr)	200	250	400	300
Time (hr)	2	5	2.5	1.2

a.) Input all the data into a cell array including the table header. For example, the first row should read:

aircraft_data = {'Leg',1,2,3,4};

b.) Compute the distance traveled by the aircraft in each leg using the cell array you defined in part a.
d.) Input all data into a structure assuming the following fields: 'Leg', 'Speed_mph' and 'Time_hr'.
e.) Compute the distance traveled and add a field to the structure from part d called 'Distance_mi'

1.D. Functions - Palm - Chapter 3

1. **Create a Function**

   You have already used a number of functions. When you use num2str, linspace or inv. These are all built-in MATLAB functions that take in a number of arguments and then outputs the result to the workspace. If is possible to create functions that do whatever you wish. For example, the following function computes the area of a circle.

```
function out = areac(in)
out = in^2*pi;
```

   The first line of the code is the function header. The variable out is the output of the function and areac is the name of the function and finally in is the name of the input variable. If we try and run this function from the editor MATLAB will throw an error. This is because the input variable in is undefined. You must run the function from the command window or a script

   >> areac(5)

   will compute the area of a circle with a radius of 5.

2. **Nested Functions**

   It is possible to call a function within a function. The function below will use the function areac to compute the volume of a cylinder.

```
function out = areacyl(radius,height)
out = areac(radius)*height;
```

   To call this function we then simply type

   >> areacyl(5,4)

   which will compute the volume of a cylinder with a radius of 5 and a height of 4. Note that the following is also acceptable as well.

   >> r = 5;

   >> h = 4;

   >> V = areacyl(r,h);

   This will save the volume of the same cylinder into a variable V.

3. **Multiple Outputs**

   Finally it is possible to output multiple values. The function below computes the area of a circle and the volume of a cylinder.

```
function [Ac,Vcyl] = areacyl(radius,height)
Ac = areac(radius);
Vcyl = Ac*height;
```

To call this you would need to use the brackets to accept two variables.

```
>> [A,V] = areacyl(r,h);
```

1.E. Example Problems

1. Create a function that has the following function header.

   ```
   function [mass,volume,weight] = myellipse(name, major_axis, minor_axis, transverse_axis,density)
   ```

   This function will compute the volume, mass, and weight(on earth) of an ellipse. Furthermore, place an if statement in the function that will display(using the disp() function) “The ellipse: (blank) is too heavy”, if the ellipse if heavier than 5 kg. (Use SI units for everything)

   Experiment with at least 3 different types of ellipses to ensure your function is working properly. Include your inputs and outputs in your document.

2. Create a function that adds up the entries of a vector. Here is an example function header

   ```
   function s = mysum(x)
   ```

   Remember you may not use the function sum() in MATLAB however you may use it to check your answers. Test this on 3 example vectors to ensure your functions are working properly.

3. Create a function that finds the minimum value in a vector. Here is an example function header

   ```
   function m = mymin(x)
   ```

   Remember you may not use the functions min() or find() in MATLAB however you may use them to check your answers. Test this on 3 example vectors to ensure your functions are working properly.

4. Use the method of loops to compute the fibonacci sequence until the number of digits in the sequence is more than 2 (so 100 or bigger). Save the sequence into a vector as you move through the sequence and include it in your document. The fibonacci sequence can be written using the equation below.

   $$\lambda_{i+1} = \lambda_i + \lambda_{i-1} \tag{1}$$

   To start the sequence assume that $\lambda_{-1} = 0$ and $\lambda_0 = 1$. The first three steps of the sequence are shown below.

   $$\begin{aligned} \lambda_1 &= \lambda_0 + \lambda_{-1} = 1 + 0 = 1 \\ \lambda_2 &= \lambda_1 + \lambda_0 = 1 + 1 = 2 \\ \lambda_3 &= \lambda_2 + \lambda_1 = 2 + 1 = 3 \end{aligned} \tag{2}$$

5. Assume I create a structure that chracterizes the properties of an ellipse. Assume that the current fields of the structure are: name, major_axis, minor_axis, transverse_axis, and density. Create a function that will compute the volume, mass, and weight(on earth) of all ellipses in the structure and add it to a field called by a similar name and output it to the workspace. Furthermore, place an if statement in the function that will display “The ellipse: (blank) is too heavy”, if the ellipse if heavier than 5 kg.

   Use the following lines of code to test your function (Notice the difference in units!)

   ```
   ellipses(1).name = 'Kids ball';
   ellipses(1).major_axis = 21.59; %cm
   ellipses(1).minor_axis = 21.59; %cm
   ellipses(1).transverse_axis = 21.59; %cm
   ellipses(1).density = 3.0; %g/cm^3
   ellipses(2).name = 'Hindenburg';
   ellipses(2).major_axis = 803.8/3.28; %meters
   ellipses(2).minor_axis = 135.1/3.28; %meters
   ```

ellipses(2).transverse_axis = 135.1/3.28; %meters

ellipses(2).density = 1.2; $\%kg/m^3$

6. The following website has some useful information about the orbit of the earth:

   "http://en.wikipedia.org/wiki/Earth's_orbit"

   Assuming the Sun is located at 0,0 and the orbit of the earth is flat (2D). Plot the orbit of the earth around the sun. Where is the earth right now? Plot a large blue circle where the earth currently is. Assume today is September 3rd, 2014. Furthermore, plot a large yellow circle where the sun is.

7. Create a function that will plot the trajectory of a sphere. Assume the inputs are the angle from the horizontal in radians and the speed in m/s. Neglect aerodynamic drag and assume the sphere is thrown on earth. Furthermore, assume the sphere is launched at time t = 0 and compute the time the ball hits the ground. Experiment with this function and determine the optimal angle from the horizontal that will throw the sphere the farthest assuming a constant speed. Then compute the optimal angle that will give the ball the most hang time for a given speed. Is it different than the farthest distance? Does this make sense?

1.F. Project

By now you must have solved some pretty difficult problems in your other classes. Every engineering problem can be cast into the form

dependent variable = function(independent variable,parameters)

The example of computing the terminal velocity of a cat is a perfect example of this. Your task it to team up with 2 more individuals and write a MATLAB code to solve three problems from your other classes.

The first step is to identify three problems that can be solved by hand but have parameters and independent variables that change. In the cat falling example, our parameters were the area and weight of the cat. We could perhaps keep the size of the cat fixed but vary the weight. This MATLAB code can then be used to compute the terminal velocity of the cat just by changing the weight of the cat.

Your deliverable for this assignment will be to write a report detailing your 3 functions. The sections included in your report will be the following

1. **Introduction** Explain what the problems are. Why do we care? Why is this important? Give some background on this type of problem.

2. **Mathematical Model** Explain the theory on how these problems are solved. Include equations in your report. Do not screenshot equations or just type them in. You are engineers. It's time to learn how to use Equation Editor. Finally, include all pertinent data required to run your code. Are there fixed parameters that do not vary? Include them in this section.

3. **Results** Explain your inputs to your code and your outputs. Do not copy and paste MATLAB output. Write your results in normal english. For example, "When the weight of the cat is 5 lbs the terminal velocity is 50 ft/s. If the weight of the cat is increased to 10 lbs the terminal velocity of the cat is 80 ft/s".

4. **Appendix MATLAB Code** Copy and paste your MATLAB code. This is the only place the word MATLAB should be. No supporting text required, simply copy and paste your code into this section.

1.G. Loops and Logicals - Palm - Chapter 4

1. **Logical Operators**

   For now MATLAB has just been a calculator. Howevever MATLAB is much more than that. One main thing that is done in MATLAB is to test whether or not something is true. If it is the code does one thing and then if not it does another. For example

   \>\> x = (1 == 0)

   tests whether or not 1 is equal to zero. In this case it is not thus the value of x is false or 0. However it is possible to do other tests such as

   \>\> x = linspace(0,10,5);

   \>\> t = length(x) > 2

   In this case the length(x) is 5 which is greater than 2 thus the value of t is true or 1. Notice that if you type in whos you will see that t is a logical variable rather than a double. A logical can only be a zero or a 1 thus it only needs 1 byte rather than 8 bytes. The other logical tests are >=, <=, ∼=. The last is 'not equal to'.

2. **If/Else/End Statements**

   If/else/end statements are called logical statements. They are typically used in a script. The basic structure of an if statement is as follows

   if *statement*

       execute this block of code if true

   else

       execute this block of code if false

   end

   For example the block of code below is a script that uses the if/else/end structure to do one thing or the other. Note that this function takes one input but has no outputs.

```
function logicals(N)
x = linspace(0,10,N);
if length(x) > 5
    disp('The vector is longer than 5')
else
    disp('The vector is shorter than or equal to 5')
end
```

   Try coding this example and see what it does for different values of N.

3. **For Loops**

   A for loop has the following structure

   for *index = start:increment:end*

   end

   The block of code above will create an index that starts at the variable start, increments by increment and stops at end. For example the code below will add up the numbers from 1 to 10

```
I = 0;
for idx = 1:1:10
    I = I + idx;
end
```

4. **While Loops**

   A while is used when you don't know how far you are looping. For example, assume you want to loop until the square of a number exceeds 100. Then you would need a while loop. The code below will stop when the square of the index exceeds 100. Note that you will have to keep track of the index rather than the for loop which does it intrinsically.

```
I = 0; idx = 1;
while idx*idx <= 100
    I = idx;
    idx = idx + 1;
end
```

   Running the script above will stop when idx = 11 since 11 squared is greater than 100. The value of I will be 10 however since the code will break out of the loop before I is set to 11. Try it out and see for yourself.

1.H. Example Problems

1. Use your script from last week to plot the fibonacci sequence as a function of iteration number. Include the figure in your word document however make sure your graph looks pretty (i.e. label your axes, add a grid, etc). The fibonacci sequence can be written using the equation below.

$$\lambda_{n+2} = \lambda_{n+1} + \lambda_n \tag{3}$$

   To start the sequence assume that $\lambda_1 = 0$ and $\lambda_2 = 1$.

2. Simulate the system below for three different values of Δx. Use 1,0.5 and 0.1. However simulate the system until $x_{n+1} == 5$. Let $x(1) = 0$ and $y(1) = 1$.

$$\begin{aligned} y_{n+1} &= (1 - 2\Delta x)y_n \\ x_{n+1} &= x_n + \Delta x \end{aligned} \tag{4}$$

   Create a figure and plot y on the y-axis and x on the x-axis. Include all three lines on your figure. Remember to plot each line in a different color and for this example you will need to add a legend. Again make your plot look nice and include it in your document.

3. Make a 3-Dimensional object of your choosing using mesh just like I did in my youtube video. You can make an ellipse, or a bowl, or a cup, or a ball, a pyramid, etc. Any 3-dimensional object. Again, make the graph look nice and include your figure in your document.

4. Using your function from problem 1, Homework 3 edit the function to plot the ellipsoid that is read in using mesh(). That is, in your for loop create a figure and mesh an ellipsoid. Label your axes, make the background white, create a title with the name of the ellipse and set the viewport to [-27,30]. Use the view() command. The run script will be the same as problem 1 from the previous homework thus your function should create two ellipses in this example.

5. I have uploaded an excel spreadsheet with information about all planets in the solar system including Pluto (As far as I'm concerned it's still a planet). Edit your function in problem 2 Homework 3 to read in the excel spreadsheet and loop through all planets and plot the orbits of all planets. You should end up generating a plot with 9 orbits. There is no need to plot the location of the planets in this example. This function will have no outputs or inputs. It merely needs to generate a plot.

6. Using your function from problem 3 Homework 3 edit the function to make a movie of the ball traveling through the air. Name the movie file M for my convenience and make it the output of the function. There should only be one output (the movie file M).

1.I. Advanced Plotting - Palm - Chapter 5

1. **2-Dimensional Plots**

   Here we will focus on plotting. We already know how to evaluate functions using the dot operator. For example, the code below will evaluate the trajectory of a projectile launched at 45 degrees.

```
theta = pi/4;
V = 10;
vy = V*sin(theta); vx = V*cos(theta);
timestep = 0.1;
t = 0:timestep:3;
x0 = 0; y0 = 0;
x = x0 + vx*t;
g = 9.81;
y = y0 + vy*t - (1/2)*g*t.^2;
```

   To plot this function we merely use the plot command.

   \>> plot(x,y)

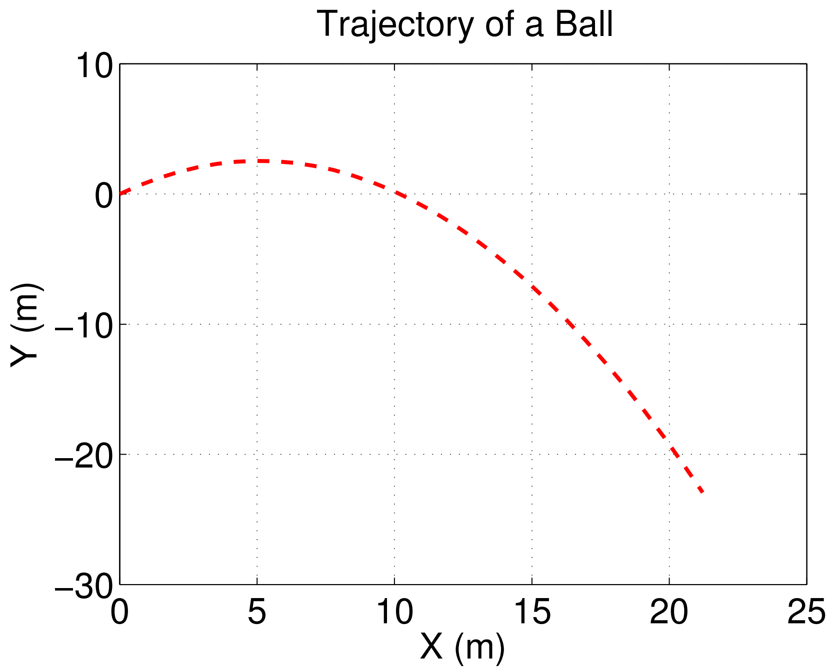
   Running this command from the command window will open a default plot with a blue line. However, the plot will be pretty mundane. The code below will make the figure look alot better.

```
% Close all other figures then, create a figure and name it Trajectory
close all
fig = figure('Name','Trajectory');
% Set the background of the figure to white.
set(fig,'color','white')
% Change the fontsize of the figure to 18
set(axes,'FontSize',18)
% Plot x and y with a red dashed line with a line width of 2
plot(x,y,'r--','LineWidth',2)
% Turn on the grid
grid on
% Label the axes and make the font size 18
xlabel('X (m)')
ylabel('Y (m)')
title('Trajectory of a Ball');
```

   Running the code above produces the figure below.

   Notice, that the projectile falls through the ground. It is beneficial to restrict the axis so as to only see the plot when the ball hits the ground. Extra code must be added to find when the ball hits the ground. The code below computes when the ball hits the ground and then restricts the axis to this window. In addition, the maximum height is computed.

```
tground = 2*vy/g;
tmax_height = tground/2;
```




```
xground = x0 + vx*tground;
ymax = y0 + vy*tmax_height - (1/2)*g*tmax_height^2;
axis([0 xground 0 ymax])
```

The result of the code above creates the figure below.

2. **3-D Plotting**

   MATLAB can also plot in three dimension. Let's assume for example we wish to plot a helical pattern. The equation of a helix can be done by creating a linear equation for the z-coordinate and a circular pattern for the x and y coordinates.

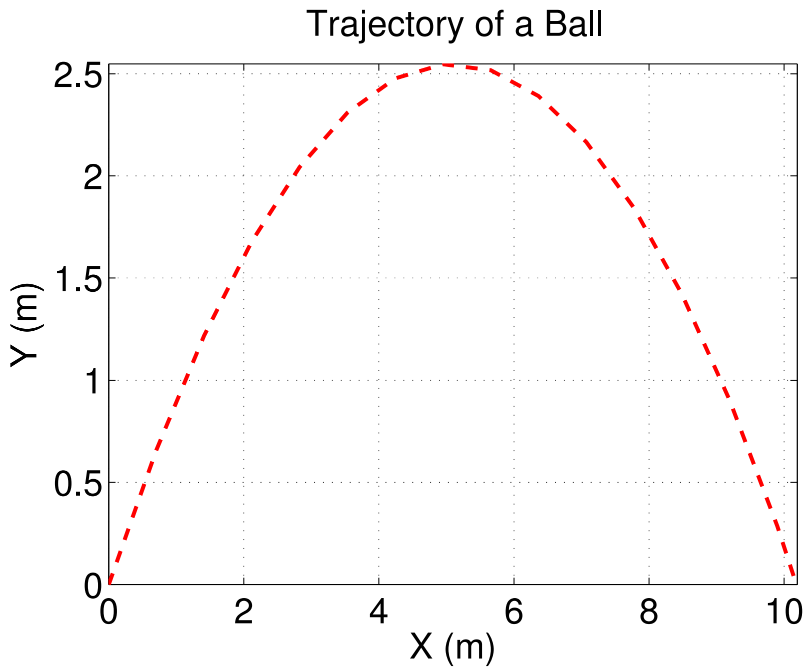
```
theta = 0:0.01:(6*pi);
x = cos(theta);
y = sin(theta);
z = linspace(-1,1,length(theta));
plot3(x,y,z)
```

After including grids, and labels the figure below is created.

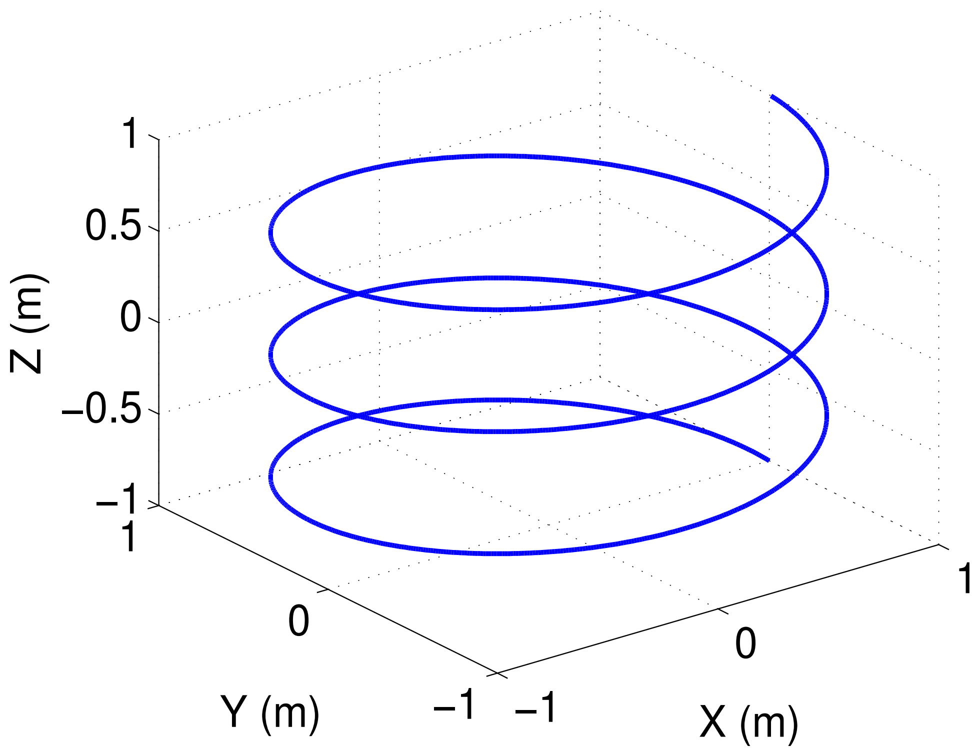
3. **Meshing**

   Meshing can be used to plot 3D surfaces as well as 3D solids. The code below creates a 3D surface and then meshes the figure shown below. The line meshgrid is used to create an array from the 1-D x and y vectors.

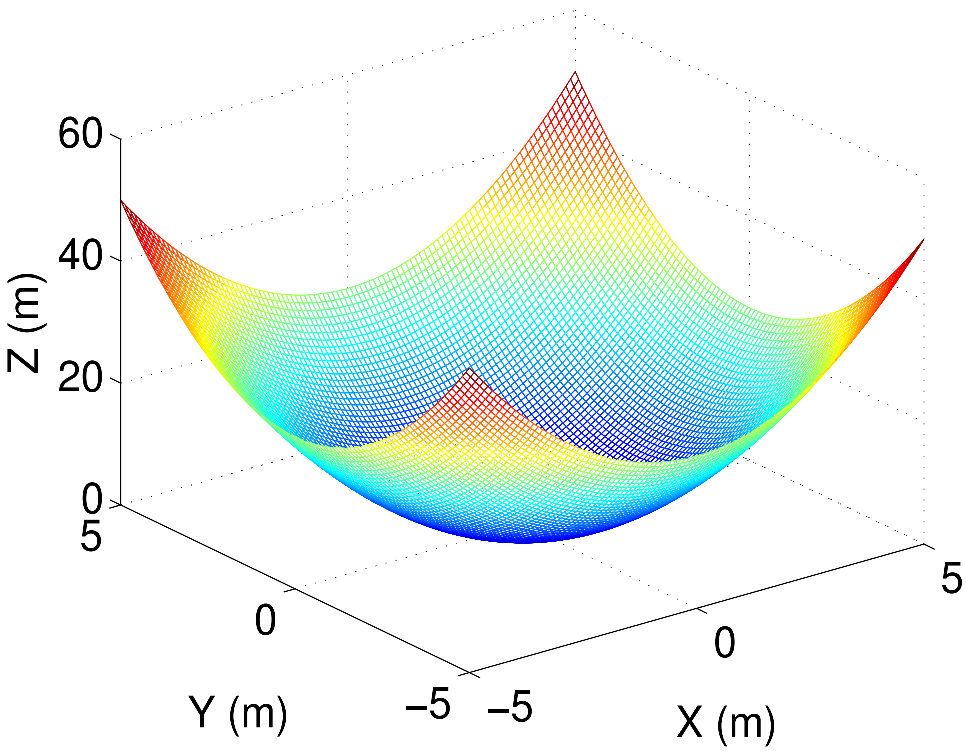
```
x = linspace(-5,5,100);
y = linspace(-5,5,100);
[xx,yy] = meshgrid(x,y);
zz = xx.^2 + yy.^2;
mesh(xx,yy,zz);
```



Note again, that extra code was added to create a grid and labels, etc.



1
0.5
0
−0.5
−1
Z (m)
1
0
−1
Y (m)
−1
0
1
X (m)



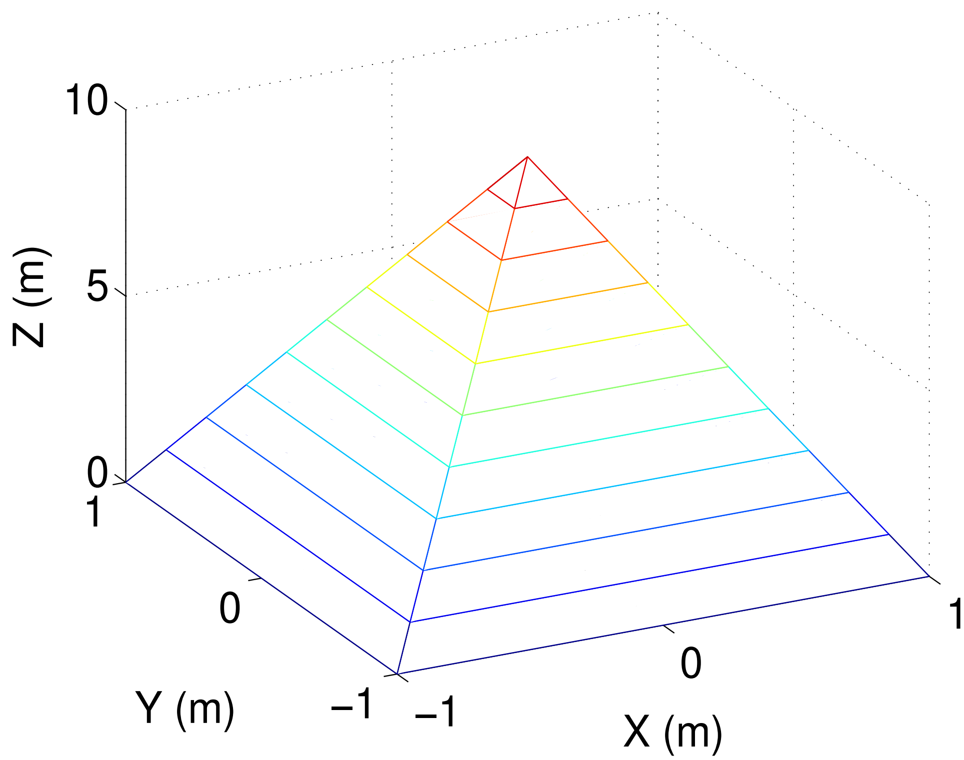
It is also possible to create 3-D solids using a stacking algorithm. That is, it is possible to stack 2-D objects on top of each other. The code below plots a 2-D square in 3-dimensions.

```
x = [-1 1 1 -1 -1];
y = [-1 -1 1 1 -1];
z = [0 0 0 0 0];
plot3(x,y,z);
```

Stacking more squares on top of each other you can create a rectangular prism. However, if you stack smaller and smaller squares it is possible to make a pyramid. The code below creates a pyramid using a stacking algorithm. Note that the [ ] brackets are used to create empty vectors. The last line of code is used to orient the camera to view the pyramid at an elevation of 30 degrees and an azimuth of -27.

```
radius = 1;
xx = [ ];yy = [ ];zz = [ ];
while radius >= 0
    xx = [xx;radius*x];
    yy = [yy;radius*y];
    zz = [zz;z];
    z = z + 1;
    radius = radius - 0.1;
end
mesh(xx,yy,zz)
```

```
view(-27,30)
```



4. **Animating**

   Animating is actually very simple. The code below animates the ball flying through the air assuming you have the x and y values defined from above.

```
for idx = 1:length(x)
    cla;
    plot(x(1:idx),y(1:idx),'b-','LineWidth',2)
    hold on
    plot(x(idx),y(idx),'bs','MarkerSize',10)
    axis([0 xground 0 ymax])
    drawnow
end
```

The way this code works is by intelligently using cla and drawnow. cla clears the current figure. The lines inbetween cla and drawnow serve to plot the ball at the current position in the loop. Drawnow is then used to create a frame such that you and I can see it in real time. That is, every loop in the for loop creates a frame. A movie is really just a sequence of frames plotted one after the other.

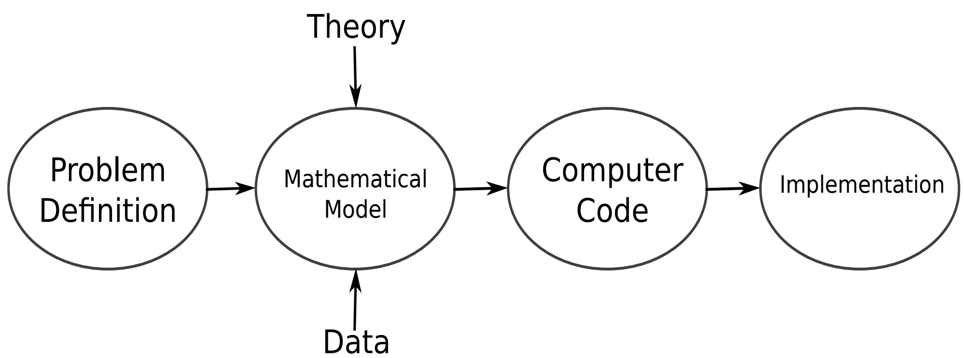
2. INTRODUCTION TO ENGINEERING AND COMPUTERS

2.A. Computers

Students are expected have a basic understanding of their computer. For example, they should be able to logon to Sakai(https://ecampus. navigate to assignments and download their assignment. Most operating systems will download this file to either the Downloads folder or the Desktop. I encourage all students to create a folder title ME228 with all pertinent documents inside. Each assignment must be completed in MATLAB however all text and figures must be placed into a WORD document. The document must the be converted to a PDF by clicking File>Save&Send>Create PDF Document>Create PDF. Once your document is converted to a PDF you must upload your document to a PDF. Obviously having a WORD processor and MATLAB on your computer is required for this course.

2.B. Mathematical Modeling and Engineering Problem Solving - Chapter 1

1. **General Formulation of Engineering Problems** The general formulation of an engineering problem has 4 parts. Problem Definition, Mathematical Model, Computer Code and Implementation. The flow of the steps is shown in the figure below. Notice that determining the Mathematical Model involves Theory and Data. Every problem starts with the



definition of a problem. Using this problem a mathematical model is created to solve the problem. This mathematical model will require data and theory. Using this mathematical model a computer tool will be created. The code is then implemented to analyze the problem defined and make decisions based on the output of the model.

2. **Example Problem**

Problem Definition: How do I get on top of my roof?

Mathematical Model is the angle of the Ladder.

Problem Definition: A round ball is launched straight into the air. The initial velocity, mass and initial height are given. If air drag is neglected what is the maximum height of the ball and what is the final time before it hits the ground.

The mathematical model is this case takes the form

$$Dependent\ variable = f(independent\ variable, parameters, forcing\ function)$$

In the example above our independent variable is time. The parameters are v_0,m, and y_0. These are the initial velocity, mass and height of the ball. The forcing function is gravity or $-mg$. The theory associated with this system is Newton's equation of motion and the data are the givens. Here Newton's equation of motion can be solved to obtain the height of the ball at time t.

$$y(t) = y_0 + v_0 t - (1/2)gt^2 \tag{5}$$

A computer tool can be used to plot the location of the ball as a function of time. The code can then be implemented in order to obtain the maximum height of the ball and the time the ball hits the ground.

3. **False Assumptions**

   John takes $T_J = 6\ min$ to wash a car and Mark takes $T_m = 8\ min$ to wash a car. How long does it take to wash the car together. It is not 6 or 7. To solve this assume it's a velocity problem.

$$V_J = \frac{1\ car}{T_J\ min} V_m = \frac{1\ car}{T_m\ min} \tag{6}$$

   We can then add the velocities together to equal 1 car where T_{JM} is the time it takes to wash 1 car.

$$V_J T_{JM} + V_M T_{JM} = 1car \tag{7}$$

   Solving for T_{JM} and substituting in T_J and T_M yields

$$T_{JM} = (\frac{1}{T_J} + \frac{1}{T_M})^{-1} \tag{8}$$

   So the solution is the same as 2 resistors in parallel. People assume it's like a series problem and you just divide by 2 but that's not true. Both people operate independently and thus the time decreases.

2.C. Approximations and Round Off Errors - Chapter 3

1. **Significant Figures**

   Significant Figures or Digits is the number or amount of information we have to convey a number. For example, take a look at your phone or your watch. What time is it? My watch says 13:11 PM. That is it is the 13th hour of the day, the 11th minute of the hour. How many seconds on the minute is it? I don't know. My watch doesn't tell me. Next time you get into your car look at how many miles your car has. Mine says 67,611.1, that is pretty accurate but what if I converted that number to feet? 5280 feet = 1 mile so 67,611.1 = 356,986,608 feet. Where did the decimal place go? Certainly I haven't traveled exactly the many feet. The reason is because I don't have enough significant digits. What about pi? How many decimal places can you list? How many does your calculator list? The fact is we all make approximation and computers are no different. They hold a certain number of digits in its memory to represent a number. The goal is that the number has enough numbers to be accurate as well as precise.

2. **Accuracy and Precision**

   Accuracy: how closely a computed or measured value agrees with the true value *Precision*: how closely multiple measurements agree with each other

   Let's go back to the watch again. How accurate is your watch? My watch says 1:17PM, according to Google it is 1:18:22 PM. So my watch was off by 1 minute and 22 seconds. My phone says 1:18PM so my phone is only off by 22 seconds. Thus, my phone is more accurate then my watch but this is because my phone is automatically synchronized with the world clock. However, both my watch and my phone are arguably just as precise.

3. **Error Definitions**

   Let's assume for the moment that I am 6'0" tall. I go to the doctor and the doctor tells me I am 72.25". Using the equation below we can determine the absolute error in my doctor's measurement.

$$absolute\_error = |actual - computed| \tag{9}$$

   6 feet is 72 inches thus the absolute error is 0.25". Sometimes however it is not enough to compute the absolute error. For example, assume you are measuring a 1000 foot building. Does it really matter that you are off by a quarter of an inch? It might matter given the circumstances but it matters a lot more when you are measuring a 2 foot long beam to put in your house. Thus, the percent error is defined as such to account for the difference in scale.

$$percent\_error = 100 * absolute\_error/actual \tag{10}$$

   Thus the percent error in my height is 0.347%. Notice, that I only saved four significant digits since my measurement was only accurate to 4 significant digits.

4. **Computer Representation of Numbers**

Computers although powerful have the same fundamental problem as above. Let's assume for the sake of this argument that you have a book in your hand and a pencil. You are only allowed to write two numbers on each page. Some one asks you what 5 times 5 is. You open up page 1 and write the number 5 on page 1. Then you turn to page two and write the number 5 on page 2. You then compute 5 times 5 in your head and arrive at the number 25 and write 25 on page three. The same person then says what is 25 times 5. You again compute in your head that 5 times 25 is 125. You turn to page 4, you pause and scratch your head, you can only write down two numbers, what should you do? Your most logical result is to throw out the 1 and write down 25. You then say that 25 times 5 is 25. This is how a computer works. The person asking you questions is a mouse and/or keyboard. The signals from the mouse and keyboard are sent through the motherboard to the Central Processing Unit (CPU) much like your brain. The motherboard is like your entire body connecting everything together. Think of it as your central nervous system. When you computed 5 times 5 you did that in your head but note that your brain has long term and short term memory. The short term memory is like the computers RAM or Random Access Memory. The long term memory is like the Hard disk drive or Harddrive (HDD). The pages in your book are different locations or hexadecimal addresses the computers harddrive. So then how does the computer actually compute? Well let's start with how we represent numbers in base 10. If I write 5 in base 10 I would have

$$0 \times (10^1) + 5 \times (10^0) \tag{11}$$

the number 25 would then be

$$2 \times (10^1) + 5 \times (10^0) \tag{12}$$

You can now see why the number 125 is impossible to represent if I can only hold two digits. If I could hold 3 digits I could represent 125 using this equation

$$1 \times (10^2) + 2 \times (10^1) + 5 \times (10^0) \tag{13}$$

Fractions are simple too. Say we return to my height of 72.25". In base 10 that would be

$$7 \times (10^1) + 2 \times (10^0) + 2 \times (10^{-1}) + 5 \times (10^{-2}) = 72.25 \tag{14}$$

We however have adopted base 10 so easily that we just leave off the 10^x and just report the numbers. Wouldn't it be great if we could just get computers to do this? It would only have to represent the number 0-9 and then it would be able to represent any number in the number line provided it could store that many significant digits. The problem with this is in the circuitry. Computers are either on (1) or off (0). This means they can't represent the number 0-9 they can only represent the numbers 0 and 1. Most computers operate on +5V for on and -5V for off. Thus if the computer receives +5V it is a 1 and -5V for an off. So if 0-9 (10 numbers) is base 10, what is 0-1 (2 numbers)? That's base 2 which is called binary. The conversion from binary to base 10 is difficult so let's just start with a few examples. Say I want to convert 101_2 to base 10. The subscript 2 indicates base 2. Well that would be

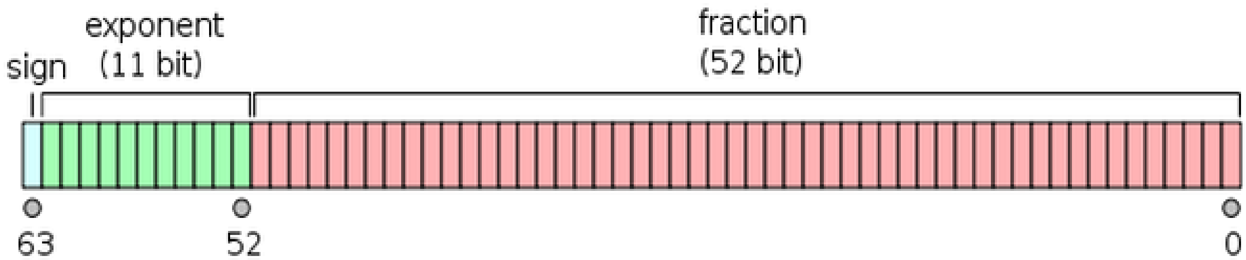
$$1 \times (2^2) + 0 \times (2^1) + 1 \times (2^0) = 5 \tag{15}$$

This number above is 3 bits of data. 8 bits equals 1 byte. 8 bytes is equal to 64 bits. So if you have a 64 bit machine it means that the computer is capable of reading 64 bits of data. My computer represents all number uses 8 bytes or 64 bits. MATLAB on the other hand using floating point precision. So, the 0-51st bit is used for fractions, 2^{-1} all the way to 2^{-51}, the 52st bit to the 62nd bit is used for the exponent except it is biased by 1023 thus the maximum number that MATLAB can predict is 2^{1024}. Finally, the 63rd bit is the sign of the number.

Thus a number would be represented using the formula below.

$$(-1)^{(bit_{63})}(1 + \sum_{i=1}^{52} bit_{52-i} 2^{-i}) \times 2^{e-1023} \tag{16}$$

where e is determined by bits 52-63. The maximum number can be computed by noting that the exponent has 11 bits. The maximum exponent number represented by 11 bits is $2^{11} - 1 = 2047$. The biased exponent is then $2047 - 1023 = 1024$. Note that if you type 2^{1024} in MATLAB you will get Inf. However if you type $2^{1023} * 1.99999$ you will get the value of



realmax. The largest number is still 2^{1024}. The smallest number is actually 2^{-1022}. This is represented with a 1 in the exponent field such that e=1 and e-1023 = -1022. It is actually possible to get smaller numbers on a computer by using denormalized numbers. That is when using denormalized number the exponent is assumed to be zero. Thus e=0 and the number can be as small as 2^{-1023}. However, with this representation it is possible to truncate the equation above to the following denormalized version

$$(-1)^{(bit_{63})}(\sum_{i=1}^{52} bit_{52-i}2^{-i}) \times 2^{-1023} \tag{17}$$

Thus the smallest number is actually $2^{-1023} * 2^{-52}$. Again, MATLAB will return a zero if you type this number in but $2^{-1023} * 2^{-51} * (1.999)^{-1}$ is pretty close.

5. **Computer Representation of Characters**

Numbers are all well and good but I also want to be able to write text. So if I write 'hello' how does my computer represent it? Well back in the 1960s when binary representation was first coming out they came up with the American Standard or Character Information Interchange (ASCII). If you take the upper/lower case alphabet (26*2), plus punctuation like ? or # and some non-printing characters like *backspace* or *del* you find that you need something like 100 characters. In binary that means you need 7 bits which leads to $2^7 - 1 = 127$. In order to organize the characters properly the mathematicians decided to make the letter 'A' = 65. The reason why can be explained by computing the binary equivalent for A.

$$\begin{gathered} A = 65_{10} = 10\ 00001_2 \\ B = 66_{10} = 10\ 00010_2 \\ C = 67_{10} = 10\ 00011_2 \\ \ldots \\ etc, etc, etc \end{gathered} \tag{18}$$

Similary, lower case 'A' is 'a' or 97. Again converting to binary leads to

$$\begin{gathered} a = 97_{10} = 11\ 00001_2 \\ b = 98_{10} = 11\ 00010_2 \\ c = 99_{10} = 11\ 00011_2 \\ \ldots \\ etc, etc, etc \end{gathered} \tag{19}$$

Thus, it is really easy to compute the letters based on the binary number. If you'd like to put a string together you can just combine binary numbers. For example, let's convert 'cat' to binary.

$$\begin{gathered} c = 99_{10} = 11\ 00011_2 \\ a = 97_{10} = 11\ 00001_2 \\ t = 116_{10} = 11\ 10100_2 \end{gathered} \tag{20}$$

What's really cool is that if you want to capitalize the word 'cat' in binary you simply need to decrement the second bit to 0 thus 'CAT' in binary is simply $10\ 00011_2, 10\ 00001_2, 10\ 110100_2$. In MATLAB/Octave it is really easy to check your work when performing these actions. To get the ASCII code simply type

$$double('inert\_letter\_here') \tag{21}$$

Try it out and see what you get for different letters. You can also do strings like your name for example.

$$double('carlos') = 99_{10}, 97_{10}, 114_{10}, 108_{10}, 111_{10}, 115_{10} \tag{22}$$

Even cooler you can convert from decimal to binary using the *dec2bin* function thus you can type

$$dec2bin(double('carlos')) = \begin{matrix} 11\ 00011_2 \\ 11\ 00001_2 \\ 11\ 10010_2 \\ 11\ 01100_2 \\ 11\ 01111_2 \\ 11\ 10011_2 \end{matrix} \tag{23}$$

Note, you can also convert to characters by typing in a number from 0 to 127 using the *char* function. For example if I type *char(33)* I get the exclamation point. Try a few and see what you get.

$$char(33) = \ ! \tag{24}$$

So for a long time this was all well and good but then people wanted to be able to represent other characters like latin, greek, roman for computation. Well 127 characters wasn't enough but 8 bit computers came out and then we had 255 characters. At the time this was enough for every language. The problem is that the Norwegians adopted a different standard rather than ASCII, the Japanese came up with their own multibyte data structure. This was fine back in the 70s because typically back then you would just print your document and fax it or mail it. Imagine if you write something in ASCII and send the binary file to someone using a different format. It's like speaking a different language they wouldn't understand. Well this didn't matter until the late 70s early 80s when the World Wide Web opened up and then suddenly people are reading languages in different binary format.

6. **Unicode Format - UTF**

So what did people do? They adopted the Unicode format which is a variable length format. Currently UTF-8 is the most standard (circa 2015) which uses 1-4 8 bits strings = 8-32 bits = 1-4 bytes per number. Using this variable length format you can represent all characters you can imagine. All 1,112,064 characters in all the human languages. However, MATLAB/Octave said they do not need 32 bits of information so Octave decided to use 8 bits (255 characters) and MATLAB decided to use 16 bits (65,535 characters). In MATLAB you can test this by typing *uint16* for example

$$x = uint16('g') = 103_{10} = 2\ bytes \tag{25}$$

yields the number 103 using 16 bits or 2 bytes. You can also similarly use the 8 bit standard using the function *uint8*

$$x = uint8('g') = 103_{10} = 1\ byte \tag{26}$$

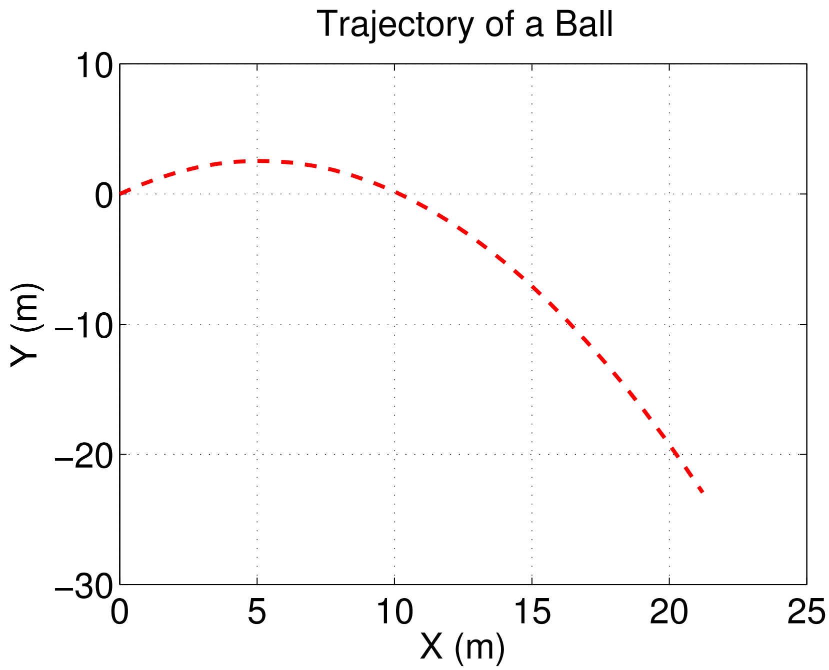
The standard character format in MATLAB is the character which uses 2 bytes. Octave's standard for characters is 1 byte. This is just what happens when you have multiple people writing different code. MATLAB as it is however allows you to use multiple different formats should you so desire. Some examples include 'UTF-7','UTF-8','UTF-16' or 'ISO-8859-1' to name a few.

3. ROOT FINDING METHODS

3.A. Bracketing Methods - Chapter 5

1. **Graphical Approach**

It is possible to obtain the root of a function by simply plotting it and zooming in on the graph. If we return to our example where we simulate the ball flying through the air and zoom in on the graph we see that the ball collides with the ground around 10 meters. This results in a time of 1.4142.



Numerical computation of the point where the ball hits the ground is equal to 10.194 meters which corresponds to a time of 1.4416. Thus graphically we were not off by very much. However, this does not help us very much when it comes to finding a better solution.

2. **Bisection Method**

The best way to describe the bi-section method is the shower heat problem. When you jump in your always turn the knob all the way up to heat the water up. But you jump in and it's way too hot. So you turn it down. More than likely it's too cold now. So you turn it back but not as much as before. You repeat this turning back and forth until you dial in the right temperature. This is much like the bi-section method.

When you looked at the graph and searched for the root you inherently looked for where the line changed sign. That is when the value of y went from positive to negative you picked off the point. It is possible to do this while using a stepping algorithm. That is we can start at an initial guess of t = 1. $y(t = 1) = v_y t - (1/2)gt^2 = 2.166 > 0$. Since y is greater than 0 we can try a new guess such that $t_1 = t_0 + \Delta t$. If we choose $\Delta t = 0.5$ our new guess is then $t_1 = 1.5$. Our height is then $y(t = 1.5) = -0.42965 < 0$. Thus our time that we hit the ground must be inbetween 1 and 1.5 seconds. At this point we can divide our timestep by 2 and compute a new timestep of 0.25. Our new time is then $t = 1.25$ and $y(t = 1.25) = 1.1748$ which is greater than zero thus our estimate lies inbetween 1.25 and 1.5. We again half our step such that we compute our time at $t = 1.375$ and $y = 0.4492$. Our value of y is still greater than zero thus we step forward one more time and get $t = 1.5$. We know that this value has a height that is less than zero which means our function has switched signs and we then halve our timestep again to 1/16 and compute $t = 1.5 - 1/16 = 1.4375$. This process repeats itself over and over again until a convergence criterion is met. A computer code can be used to run the bisection method. The algorithm is given below.

```
function ti = bisection()
clc
close all
%Initial guess
ti = 1;
%Initial timestep
dt = 0.5;
%Initial Function Value
```

```
f1 = myfunc(ti);
f2 = f1;
%Set an error threshold
threshold = 1e-2;
%Loop while error is greater > threshold
while abs(f1-0) > threshold
    %Step forward until function changes sign
    while sign(f2) == sign(f1)
        ti = ti+dt;
        f2 = myfunc(ti);
    end
    %If the loop breaks out we change the sign of dt and halve it
    dt = -0.5*dt;
    %Furthermore we change f1 to f2
    f1 = f2;
end
function y = myfunc(t)
V = 10;
theta = pi/4;
vy = V*sin(theta);
a = -9.81;
y = vy*t + (1/2)*a*t^2;
```

Running the code below produces the following table of data. Notice that to get our error to 1e-2 only requires 10 iterations.

Iteration	1	2	3	4	5	6	7	8	9	10
t (sec)	1.5	1.25	1.375	1.5	1.4375	1.469	1.4531	1.4375	1.445	1.441
dt (sec)	0.5	-0.25	0.125	0.125	-0.0625	0.031	-0.016	-0.016	0.008	-0.004
y (m)	-0.429	1.175	0.449	-0.429	0.029	-0.196	-0.082	0.029	-0.026	0.0014

3. **Bisection Method Alternate Approach**

   It is possible to simple use an iterative method to solve for the bisection method which students find to be much simpler. The algorithm is shown below:

   (a) Define upper(xU) and lower(xL) bounds

   (b) Set initial conditions

   $$\Delta x_1 = (xU - xL)/2 \qquad x_1 = xL \qquad y_1 = f(x_1)$$

   (c) Perform iterations using the following two iterative equations

   $$x_{n+1} = x_n + \Delta x_n \qquad \Delta x_{n+1} = \Delta x_n/2$$

   (d) Change the sign of Δx_{n+1}, if $sign(f(x_n)) \sim= sign(f(x_{n+1}))$

If you're looking for a fun game to test out your bi-section skills check out the clock game from the Price is Right. Here are two really fun links.

Terrible Bi-Section Contestant: https://www.youtube.com/watch?v=oc9H8bo8yg0
Million Dollar Winner: https://www.youtube.com/watch?v=RJw1rlmJ81U

4. **The Parachutist**

An interesting example that uses the Bi-Section method is the falling parachutist with unknown drag coefficient. If we use the simplest form of Newton's Second Law we have

$$\sum F = ma \tag{27}$$

where F is the forces on the parachutist. In this example the only forces are gravity and aerodynamic forces. To simplify this problem we assume Newtonian drag such that $F_d = -cv$ where c is a drag coefficient and v is the velocity of the parachutist. We can also use the relationship that $\dot{v} = a$ so that our equation of motion is given as

$$\dot{v} + (c/m)v = g \tag{28}$$

This equation can be solved analytically using methods from Differential equations. I recommend you brush up on your differential equation skills before you move on. The analytical solution is given below to check your work where v_0 is the initial velocity of the parachutist.

$$v(t) = v_0 e^{-ct/m} + (1 - e^{-ct/m})mg/c \tag{29}$$

So where does the bi-section method come in? Let's say that a velocity sensor is put on the parachutist so that v_0 is known, and $v(t = 4)$ is also known.

$$v_0 e^{-4c/m} + (1 - e^{-4c/m})mg/c - v(t = 4) = 0 \tag{30}$$

In the equation above the only unknown is then the drag coefficient since gravity, and the mass of the parachutist are known. Since the equation above is in the form $v(c) = 0$ the equation can be solved using the bi-section method. The solution is left as an example to the reader.

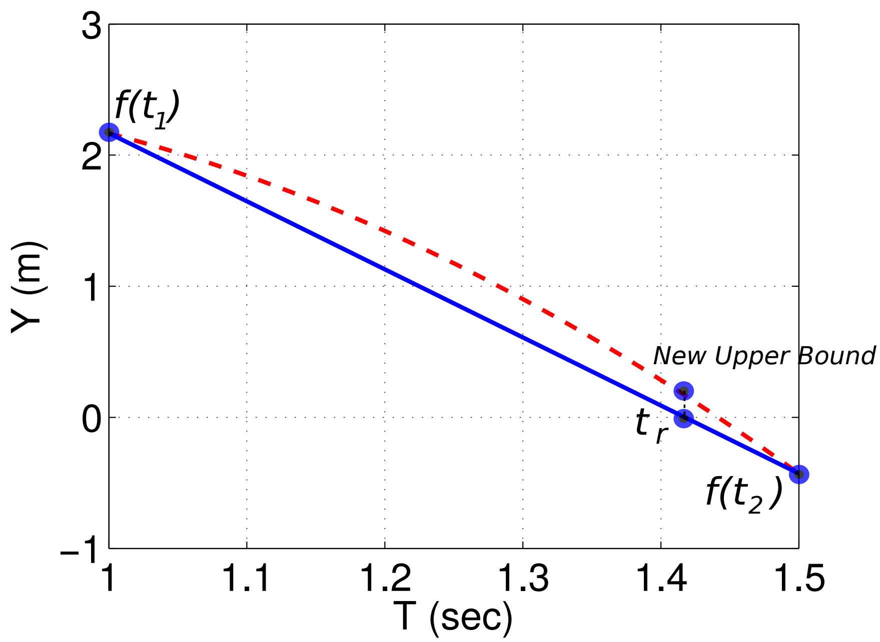
5. **False Position Method**

The false position method starts with an initial guess that yields a result that is less than zero and another value that is greater than zero. A line is then connected between the two and where this line intersects the zero line is the new value of the lower or upper bound depending on the sign of the result. For example, the graph below shows the first iteration of the example problem above.

Here $f(t_1)$ is positive and $f(t_2)$ is negative. A straight line is then connected and t_r is solved using the equation below which is derived using the equation of a line.

$$t_r = t_2 - \frac{f(t_2)(t_1 - t_2)}{f(t_1) - f(t_2)} \tag{31}$$

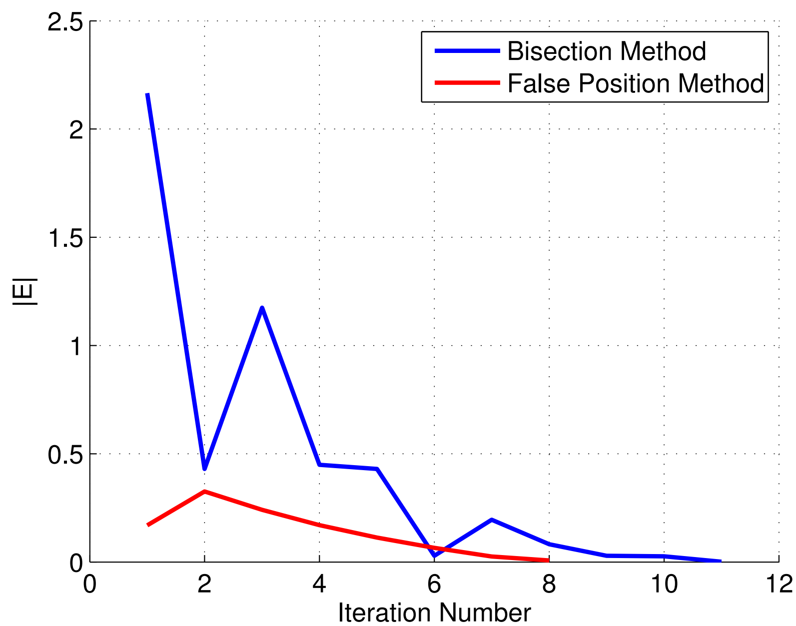
Using the new value of t the sign of $f(t_r)$ is computed. Since it is the same as $f(t_1)$, t_r becomes the new upper bound and $t_r = t_1$. A simple code can also be written to run the false position method and is given below. Running this code only requires 8 iterations. It is apparent then that this method is faster for the given problem. Doing a simply tic,toc test yields this result. Note that a tic,toc test must be done numerous times such as a for loop to test for convergence.




```
function falseposition()
clc
close all
%Initial guess
t1 = 1;
%Initial timestep
dt = 0.5;
t2 = t1 + dt;
%Initial Function Value
f1 = myfunc(t1);
f2 = myfunc(t2);
fr = f1;
%Set an error threshold
threshold = 1e-2;
%Loop while error is greater than threshold
while abs(fr-0) > threshold
    %Compute tr based on f1 and f2
    tr = t2 - f2*(t1-t2)/(f1-f2);
    fr = myfunc(tr);
    %Test to see which bound we will replace
    if sign(fr) == sign(f1)
        t1 = tr;
    else
        t2 = tr;
```

```
        end
    end
```

Another interesting test is to plot the error as a function of iteration number. The figure below shows this. Notice how the bisection method is all over the place whereas the false position method is fairly linear.



3.B. Open Methods - Chapter 6

Open methods are deemed open because they are not guaranteed to converge. The benefit is that if they do converge they converge much faster than all other methods. That is except for fixed-point iteration.

1. **Simple Fixed-Point Iteration**

   Assume for the moment we are trying to solve

$$x^2 - 2x - 3 = 0 \tag{32}$$

   This equation can be re-written intelligently such that

$$x = \frac{x^2 - 3}{2} \tag{33}$$

   Notice that the form of this equation is $x = g(x)$ which can actually be used in an iterative sense. Let us start with an initial guess of x=0. $g(0) = -3/2$ Thus we set x=-3/2. We then compute $g(-3/2) = -0.375$ and again we set x=-0.375. When then compute $g(-0.375)$. This process is repeated over and over again until our change in x is below a certain threshold. The code for this iteration method can be seen below.

```
function xnew = fixed_point()
x0 = 0;
xnew = g(x0);
```

```
xold = x0;
thresh = 1e-2;
while abs(xnew - xold) > thresh
    xold = xnew;
    xnew = (xold^2-3)/2;
end
```

The pitfalls of this code are that it takes a large number of iterations to converge. The example above requires 79,982 iterations.

2. **Simple Fixed Point Iteration Iterative Algorithm**

   The algorithm goes like this:

   (a) Convert the Problem into the SFPI form.

   $$f(x) = 0 \rightarrow x = g(x)$$

   (b) Use the following iterative method

   $$x_{n+1} = g(x_n)$$

3. **The Newton-Raphson Method**

   The Newton-Raphson Method is the standard in numerical root solving. Many additions can be made to the Newton-Raphson Method however the basic algorithm starts with an initial guess x_i, the function $f(x_i)$, and the derivative of the function $f'(x_i)$. Then, just like the false position method you create a line between $f(x_i)$ and zero using a slope equal to $f'(x_i)$ and find where this line crosses the y-axis. This point is your new point. This can be seen graphically below and using the equation

   $$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{34}$$

   There is an issue associated with the Newton-Raphson technique and that is convergence. A way to mitigate this issue is by employing a step size instead of using the entire magnitude of $f'(x_i)$. Thus, the iterative method becomes

   $$x_{i+1} = x_i - \alpha\frac{f(x_i)}{f'(x_i)} \tag{35}$$

   where α is a step size that is less than 1. It is possible to have this step size be a variable just like in the bisection method.

4. **The Secant Method (Numerical Version of Newton-Raphson)**

   Often times the first derivative of a function is not known. Thus the Newton-Raphson method cannot be used or rather the first derivative of the function is replaced with $\tilde{f}$ which is the numerical derivative of the first derivative.

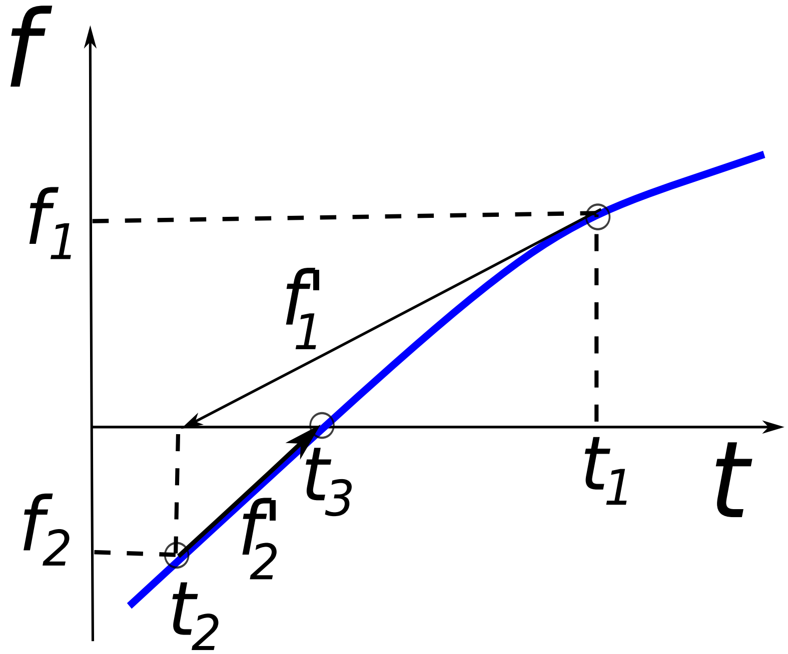
5. **Error in Newton-Raphson**

   Just as with the Trapezoidal rule it is possible to obtain an error estimate in the Newton-Raphson method. If we write the taylor series expansion we have

   $$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + f''(\zeta)\frac{(x_{i+1} - x_i)^2}{2!} \tag{36}$$

   The Newton-Raphson method is derived by setting $f(x_{i+1}) = 0$ and solving for x_{i+1} assuming that $f'' = 0$.

   $$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i) \tag{37}$$



Re-arranging the equation above yields the Newton-Raphson method. If instead we assume that $x_{i+1} = x_r$ or rather in one step we get the correct answer and let $f'' \neq 0$ we would have

$$0 = f(x_i) + f'(x_i)(x_r - x_i) + f''(\zeta)\frac{(x_r - x_i)^2}{2!} \tag{38}$$

since $f(x_r) = 0$. The error in our estimate is then the difference between the two estimates where $f'' = 0$ and $f'' \neq 0$ we would have

$$0 = f'(x_i)(x_r - x_{i+1}) + f''(\zeta)\frac{(x_r - x_i)^2}{2!} \tag{39}$$

Substituting $E_{i+1} = x_r - x_{i+1}$ and assume that $\zeta = x_r$.

$$E_{i+1} = \frac{-f''(x_r)}{2f'(x_r)}E_i^2 \tag{40}$$

What this means is that if the second derivative of a function is zero the Newton-Raphson technique will compute the answer in one iteration.

3.C. Example Problems

1. You have hopefully learned 4 different search techniques. The Bisection Method, the False Position Method, Fixed Point Iteration and Newton-Raphson. The function you defined above should have a zero somewhere between $-\pi$ to π. If it doesn't, change your frequency ω so that it does. Then program **ONE** of the search techniques you've learned and have the code solve for the zero. Your function header should look like one of the following.

```
function xest = mybisection(xL,xR,maxIter)
function xest = myfalseposition(xL,xR,maxIter)
function xest = myfixedpoint(x0,maxIter)
function xest = myNewton(x0,alfa,maxIter)
```

where xL and xR are your initial guesses for bracketing methods, and x0 is your initial guess for the open methods. alfa is a parameter you can use to make sure your Newton-Raphson method converges properly. maxIter is the maximum number of iterations your code will go through. Plot your function from problem 1 and the output of your answer on the same graph. The code below will plot a square at your solution.

```
plot(xest,0,'ks','MarkerSize',10)
```

2. You are designing a spherical tank to hold water for a small village in a developing country. The volume of liquid it can hold can be computed as

$$V = \pi h^2 \frac{3R - h}{3} \tag{41}$$

where V = volume (m^3), h = depth of water (m), and R = the tank radius (m). If R = 3 m, to what depth must the tank be filled so that it holds 30 m^3? Use three iterations of the false-position to determine your answer. Employ initial guesses of 0 and R.

3. According to *Archimedes Principle*, the buoyancy force is equal to the weight of fluid displaced by the submerged portion of an object. In order for the sphere to be in a state of equilibrium the buoyancy force must be equal and opposite to the force of gravity.

$$F_b = \rho_{water} V_{submerged} g = \rho_{sphere} V_{sphere} g \tag{42}$$

Assume you have a sphere that is submerged into the water. Determine the height h of the portion of the sphere that is above water using the bisection method. Use the following values for your computation: $R = 1\ m$,$\rho_{sphere} = 200\ kg/m^3$,$\rho_{water} = 1000 kg/m^3$. Choose an initial height of 2*R(fully outside the water) and a step size of -2*R. Note that the volume of a sphere is simply

$$V_{sphere} = \frac{4}{3}\pi R^3 \tag{43}$$

The volume of the submerged portion is then

$$V_{submerged} = \frac{4}{3}\pi R^3 - \frac{\pi h^2}{3}(3R - h) \tag{44}$$

Perform as many iterations required to reach an error of 1e-2.

4. I would like to know the speed for minimum drag for the aircraft that I am currently simulating. Use the Newton-Raphson technique to solve for this speed. Assume an initial guess of 30 m/s. The drag of an aircraft can be calculated using the following equations.

$$\begin{gathered} D = \tfrac{1}{2}\rho V^2 S (C_{D0} + K C_L^2) \\ C_L = \tfrac{2W}{\rho V^2 S} \end{gathered} \tag{45}$$

Use the following values in your solution: $\rho = 1.225\ kg/m^3$, $W = 55\ N$, $S = 0.6558\ m^2$, $C_{D0} = 0.028$ and $K = 0.0502$. Iterate until the change in your velocity values is 1e-2.

5. The mathematical model of a pendulum is set up with a single degree of freedom. The pendulum can be modeled as a ball of mass "m", connected to a rigid massless link of length "L", as shown in the Figure below. The angle from the vertical is denoted θ and can attain values from 0 to 2π. The equations of motion are second order and are written in terms of all parameters in the system and are given by the equation below.

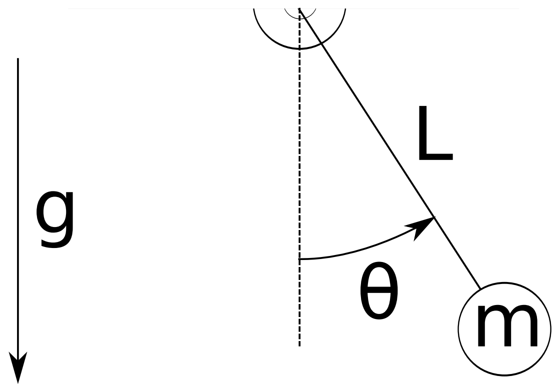
$$mL^2\ddot{\theta} + mgLsin(\theta) = 0 \tag{46}$$

It is possible to solve for the analytic solution for small angles. That is, if θ is small the approximation that $sin(\theta) = \theta$ is valid. Using this result, the equation above reduces to the following. Note the equation was also divided by mL^2.

$$\ddot{\theta} + (g/L)\theta = 0 \tag{47}$$

The equation above is now simply a spring mass system. The analytical solution is then

$$\theta(t) = \dot{\theta}_0 \sqrt{\frac{L}{g}} sin(\sqrt{\frac{g}{L}} t) + \theta_0 cos(\sqrt{\frac{g}{L}} t) \tag{48}$$



6. Plot the analytical solution for $\theta_0 = \pi/4$ and $\dot{\theta}_0 = 0$. Assume $m = 1\ kg$, $g = 9.81\ m/s^2$, and $L = 4\ m$.

7. Use Euler's integration technique and compute the numerical solution to the simplified expression in equation 47. Plot the analytical solution on the same graph as the numerical solution. Did you get the same thing? Is your timestep small enough?

8. Use Euler's integration technique and compute the numerical solution to the full expression in equation 208. Plot the analytical solution, the numerical solution from problem 2 and 3 on the same graph. Do they all look the same? Try it again for $\theta_0 = pi/10$. Do they look the same now? What is happening?

9. Bonus: For extra credit create an animation of the pendulum and upload it along with your word document. Look up movie2avi for help or request a screen cast. (+50pts)

10. Recall the parachutist falling with an unknown drag coefficient. Estimate the drag coefficient using the Newton-Raphson technique. Choose an appropriate initial guess. Iterate until your percent error is less than 1%

11. Perform the same problem as above except this time use the bi-section method. Create a plot of your g(K) function to prove that your answer is correct. Have the bi-section method iterate until you are within 1%. Make a plot showing the error in your newton-raphson estimate, your right bound estimate and your left bound estimate.

4. OPTIMIZATION

4.A. One-Dimensional Unconstrained Optimization - Chapter 13

Often times it is important to find the most efficient or cost effective solution to a particular design. The Newton-Raphson method depicted below is a widely used method for optimization. Note however that any search method can be adapted for use in optimization.

1. **Classical Optimization**

   Classically optimization is written in the form like the equation below

$$find\ x\ such\ that\ min(f(x)) \tag{49}$$

   A very simple example is where $f(x) = x^2 - 1$. The solution to this equation can be solved by computing where the derivative $\partial f/\partial y = 2x = 0$. In this case the function is a minimum when $x = 0$ and the minimum value is negative -1. Care must be taken however when using this method because finding where the partial derivative is zero is actually finding a local extrema. For example, if $f(x) = -x^2 + 8x - 12$, the first derivative is zero when $x = 4$ which yields $f(4) = 4$ however simple inspection shows that when $x = 0$, $f(0) = -12$ which is smaller than 4. This is because the method above has found a maximum instead of a minimum. Suppose however that you cannot solve the first derivative? For example, try and solve for the minimum below

$$f(x) = 5(2 + \frac{3}{x^4})x^2 \tag{50}$$

For problems such as this it is sometimes easier to employ a numerical method. The most widely popular method is the Newton-Raphson method.

2. **Newton-Raphson**

   The goal of an optimization problem is similar to a root finding problem. The difference is that instead of finding where $f(x) = 0$ optimization problems seek to find where $f'(x) = 0$. Thus the Newton-Raphson technique can be easily modified to write

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)} \tag{51}$$

   If the relationship $g(x) = f'(x)$ is defined then the equation becomes

$$x_{i+1} = x_i - \frac{g(x_i)}{g'(x_i)} \tag{52}$$

   which is exactly the same equation as the original Newton-Raphson technique using $g(x)$ instead of $f(x)$. The issue is that if the first derivative does not exist, it means the second derivative also does not exist. Thus, the two derivatives must both be determined numerically in order to use Newton-Raphson for optimization problems. Typically the second spatial derivative is replaced such that

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{\Delta x^2} \tag{53}$$

   and

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{\Delta x} \tag{54}$$

   The method above is called the secant method.

3. **Convergence Criteria**

   Optimization methods are iterative methods just like everything else in numerical methods. The stop criterion is setup such that convergence is reached when $\epsilon_a \leq \epsilon_s$ where ϵ_s is a user defined parameter whereas ϵ_a is equal to the equation below.

$$\epsilon_a = \frac{current\_estimate - last\_estimate}{current\_estimate} \tag{55}$$

4. **Gradient Search**

   Often times computing the second derivative for the Newton-Raphson technique is difficult. Another method which is very similar is the gradient or steepest descent method.

$$x_{n+1} = x_n - \alpha \nabla f(x_n) \tag{56}$$

   where α is a number between 0 and 1 which is used to slow down the descent search to ensure the iteration steps to not diverge.

5. **Parabolic Interpolation**

   Although, least squares regression is not covered until section 6.A., it is possible to obtain a close form expression for a parabola that approximates the equation trying to be optimized. To do this, three coordinates (x_0, x_1, x_2) are taken that straddle the assumed optimum. The equation below is then used iteratively until the system converges.

$$x_3 = \frac{f(x_0)(x_1^2 - x_2^2) + f(x_1)(x_2^2 - x_0^2) + f(x_2)(x_0^2 - x_1^2)}{2f(x_0)(x_1 - x_2) + 2f(x_1)(x_2 - x_0) + 2f(x_2)(x_0 - x_1)} \tag{57}$$

6. **Adaptation of Bi-Section - Grid Search**

   Notice however that when using the Newton-Raphson method, the second derivative of the function must be known. It is possible to use the secant method and numerically obtain the second derivative however if the system is well behaved, a simple grid search can be used to solve the problem.

   The bi-section method can be adapted to perform optimization by changing the switch criteria from this

   Change the sign of Δx_{n+1}*, if* $sign(f(x_n)) \sim= sign(f(x_{n+1}))$

   to this

   Change the sign of Δx_{n+1}*, if* $f(x_{n+1}) \leq f(x_n)$

   The algorithm basically amounts to keep stepping forward until your function becomes smaller. Once it does switch the sign and halve the distance until the step size is smaller than an exit criteria.

7. **Ladder Problem**

   Assume a ladder is to be used to paint a wall. Unfortunately there is a fence a distance d from the wall with a height of h. Solve for the minimum distance L of the ladder. **Draw a picture to help**.

   To start solving this problem we define a coordinate system and let the ladder be defined by $y = mx + b$ where b is the position of the ladder on the wall and $-b/m$ is the x distance from the wall. Obviously m will be negative. Since this is a minimization problem, the smallest ladder will be touching the fence such that $h = md + b$ or rather $b = h - md$

   Using the Pythagorean Theorem, we know that $L^2 = b^2 + (b^2/m^2)$. Substituting $b = h - md$ into the equation for length yields

$$L^2 = (h - md)^2 + (h - md)^2/m^2 \tag{58}$$

   Since the height of the fence is known $h = 4\ m$ and the distance from the wall is known $d = 4\ m$, the equation above is in the form $find\ m\ to\ min(L(m))$. At this point we can use Grid Search, Newton-Raphson or Gradient Search to solve this problem.

4.B. Multi-Dimensional Unconstrained Optimization

1. **Analytic Method**

   Suppose that we want to minimize

$$p = -8\rho + \rho^2 + 12T + 4T^2 - 2\rho T \tag{59}$$

   Analytically we would simply set the gradient equal to zero.

$$\nabla p = \begin{Bmatrix} \frac{\partial p}{\partial \rho} \\ \frac{\partial p}{\partial T} \end{Bmatrix} = \begin{Bmatrix} -8 + 2\rho - 2T \\ 12 + 8T - 2\rho \end{Bmatrix} = \vec{0} \tag{60}$$

   This yields a system of equations which can be solved using Gaussian substitution or matrix inversion. The solution is $\rho = 3.33$ and $T = -0.667$.

2. **Grid Search**

   In order to perform grid search in 1 dimensions you simply evaluate the function one step to the left and to the right. Formally this can be written as $f(x_0-\Delta x)$ and $f(x_0+\Delta x)$. However in two dimensions, the number of function evaluations becomes 8. That is, $f(x_0+\Delta x, y_0)$, $f(x_0-\Delta x, y_0)$, $f(x_0, y_0+\Delta y)$, $f(x_0, y_0-\Delta y)$, $f(x_0-\Delta x, y_0-\Delta y)$, $f(x_0+\Delta x, y_0-\Delta y)$, $f(x_0-\Delta x, y_0+\Delta y)$, and $f(x_0+\Delta x, y_0+\Delta y)$. The rules are the same, continue to move towards the smallest of the 8 and if you cannot reduce the function size anymore, halve the step size until an exit criteria. This method becomes large rather quickly. The function evaluations is equal to $\sum_{i=0}^{D-1} 2^{i+1}(D-i)$ where D is the order. When $D=1$ the function evaluations is 2D = 2, when $D=2$ the function evaluations is 2D+4(D-1) = 4+4 = 8, when $D=3$ the function evaluations is 2D+4(D-1)+8(D-2) = 6+12+8=26, $D=4$ yields 2D+4(D-1)+8(D-2)+16(D-3) = 8+12+16+16 = 52. That's alot of function evaluations. A simple computer program can be programmed to do this but the gradient search is alot better.

3. **Gradient Search**

   The gradient search in multiple dimension is extremely easy to employ and often the easiest to code as well.

$$\vec{x}_{n+1} = \vec{x}_n - \alpha \nabla f(\vec{x}_n) \tag{61}$$

   The simplicity lies in the equations ability to handle multiple dimensions much easier than other methods such as Grid Search or Newton-Raphson.

4.C. Linear Constrained Optimization - Chapter 15

All engineering problems have constraints. Time constraints, storage constraints, load constraints, you name it. It is impossible to make something too big. At some point there will be diminishing returns and it will be better to make less of something than more of something. Most problems are non-linear and require the use of Lagrange Multipliers but that is beyond the scope of this course. Thus, only linear constrained optimization problems will be considered in this course.

1. **Gas-Processing Plant**

   Suppose a gas-processing plant receives a fixed amount of raw gas each week. Gas is processed into two grades regular and premium. Only 1 type of gas can be processed at one time. The facility is open 80 hrs/week using two shifts and receives 77 m^3 of raw gas per week. The regular gas requires 7 m^3 of raw gas per tonne and the premium gas requires 11 m^3 of raw gas per tonne. The regular gas requires 10 hrs to produce 1 tonne while the premium gas requires 8 hrs to produce 1 tone. The plant has storage constraints and can only store 9 tonnes of regular gas and 6 tonnes of premium gas. Assume the company sells all the regular gas produced at the end of the week at $150/tonne and the premium gas at $175 per tonne. How much regular and premium gas should be made to maximize profit.

   If x_1 = amount of regular gas and x_2 is the amount of premium gas the problem can be set up using the equations to the left. In order to solve this problem using a Mathematics toolbox however requires problem to be cast into the form shown on the right where the profit function has been inverted, becomes a minimization problem and the inequalities are cast in the form of $g(x) \leq 0$

$$\begin{array}{ll} \text{Maximize} & f(x_1,x_2) = 150\,x_1 + 175\,x_2 \\ \text{subject to} & 7\,x_1 + 11\,x_2 \leq 77 \\ & 10\,x_1 + 8\,x_2 \leq 80 \\ & x_1 \leq 9 \\ & x_2 \leq 6 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{array} \quad \Rightarrow \quad \begin{array}{ll} \text{Minimize} & f(x_1,x_2) = -(150\,x_1 + 175\,x_2) \\ \text{subject to} & 7\,x_1 + 11\,x_2 - 77 \leq 0 \\ & 10\,x_1 + 8\,x_2 - 80 \leq 0 \\ & x_1 - 9 \leq 0 \\ & x_2 - 6 \leq 0 \\ & -x_1 \leq 0 \\ & -x_2 \leq 0 \end{array}$$

2. **MATLAB Solution**

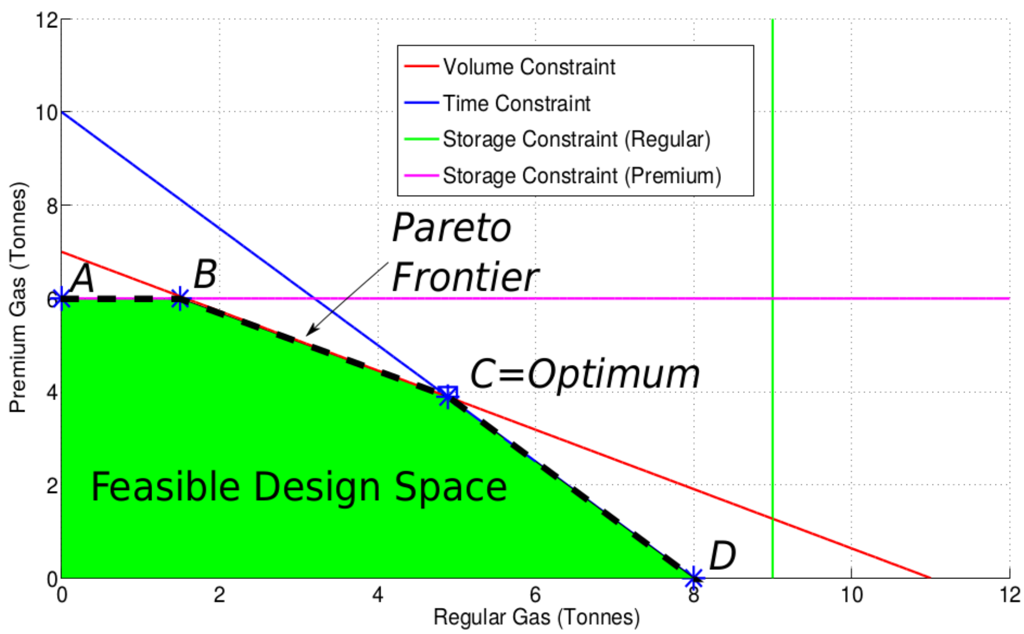
   Using the form on the right, the equation can be solved using the function $fmincon$. Unfortunately, this function is not included in Octave (at least at the time this text was written in 2015). In order to solve this problem by hand, skip to the next section.

```
function Ex15_1

x0 = [1,1];
options = optimset('LargeScale','off');
[x,fmin] = fmincon(@objfun,x0,[],[],[],[],[],[],@confun,options);
x_o = x
fmax =-fmin

function f = objfun(x)
f = -(150*x(1) + 175*x(2));

function [c,ceq] = confun(x)
c = [7*x(1) + 11*x(2) - 77;
  10*x(1) + 8*x(2) - 80;
  x(1) - 9;
  x(2) - 6;
  -x(1);
  -x(2)];
ceq = [];
```

In the function above the first function Ex15_1 sets up the problem. objfun is the objective function to be solved and confon is the constraints where c contains the inequality constraints and ceq contains the equality constraints.

3. **Graphical Solution**

In order to solve linear programming problems, the constraints must first be graphed.



The graph has a lot of information on it so let's walk through this graph slowly. The x-axis is the amount of regular gas in tonnes and the y-axis is the amount of premium gas in tonnes. The green and magenta lines indicate storage constraints since the plant can only store 9 tonnes of regular gas and 6 tonnes of premium gas. The blue line shows the time constraint. Since the plant is only open 80 hrs/week and only 1 gas can be made at a time, the blue line indicates the limits on this shortcoming of the factory. The red line is similar only it indicates the volume constraint since the plant only receives 77 m^3 of raw gas per week. The green line is a redundant constraint in that if it is removed, the problem stays the same; however, the other 3 lines create the feasible design space as indicated by the large green space. The intersection of each of the constraints lines with each other and the x and y-axes are critical design points which designate the Pareto Frontier. The optimum design choice is one of these 4 points. Each point can be solved for explicitly using Gaussian subsitution and plugging each coordinate into the profit function. Inspection shows that point C is the optimum design choice. $x_1 = 4.89$, $x_2 = 3.89$ and $Profit = \$1413.89$.

4.D. Example Problems

1. Solve the ladder problem using Grid Search, Newton-Raphson and Gradient search. Plot the length of the ladder as a function of the slope.

2. In MATLAB type *help fminbnd* and *help fminsearch*. Use the help function to learn how to use these functions and use them to solve the ladder problem. Compare your results to your own numerical method.

3. Solve the Density and Temperature problem using Grid Search and Gradient Search.

4. You are asked to design a covered conical pit to store 50 m^3 of waste liquid. Assume excavation costs at $\$100/m^3$, side lining costs at $\$50/m^2$, and cover costs at $25/m^2$. Determine the dimensions of the pit that minimizes cost. If the slope is unconstrained and if the side slope must be less than 45^o. Here are some equations that might help you

$$\begin{gathered} Excavation\ Volume = \pi r^2 h/3 \\ Side\ Lining\ Area = \pi r\sqrt{h^2 + r^2} \\ Cover\ Area = \pi r^2 \end{gathered} \tag{62}$$

   Create plots of Cost vs. Radius as well as cone angle vs. radius. Solve the minimization problem using the built in MATLAB function fminbnd. Is the constraint a limiting factor? If so, what is the radius with the constraint?

5. A company makes two types of products, A and B. These products are produced during a 40-hr work week and then shipped out at the end of the week. They require 20 and 5 kg of raw material per kg of product, respectively, and the company has access to 9500 kg of raw material per week. Only one product can be created at a time with production times for each of 0.04 and 0.12 hr, respectively. The plant can only store 550 kg of total product per week. Finally, the company makes profits of \$45 and \$20 on each unit of A and B, respectively. Each unit of product is equivalent to a kg.

   Create a mesh plot of the objective function and plot all constraint equations on it. Evaluate all limiting cases by taking a look at your Pareto Frontier. Once you have evaluated which limiting conditions is the minimum use the fmincon function like we did in class to solve for the solution. Note that fmincon does not exist in Octave so you will have to team up with someone who owns MATLAB and solve the problem that way. Plot a blue square on the solution of this problem just as we did in class.

6. You are an engineer planning to airlift $M_t = 2000kg$ worth of supplies to people in need. The supplies are being dropped at an altitude of $500m$. The price of a parachute is given using the equation below

$$c_{chute} = 200 + 56l + 0.1A^2 \tag{63}$$

   where $l = \sqrt{r}$,$A = 2\pi r^2$ and r is the radius of the parachute. It is possible to put all 2000 kg worth of material into 1 parachute however the parachute must be really big because the velocity that the product strikes the ground must be less than $20m/s$ to ensure no one is injured and the product is not damaged. We know from the previous section on Numerical Integration that a parachute in free fall is governed by the equation below

$$\ddot{z} + (c/m)\dot{z} = g \tag{64}$$

   where z is the altitude, $c = 3$ is the drag coefficient and $g = 9.81$ is the gravitational constant on Earth. This equation can be solved analytically to obtain $z(t)$ to compute when the parachute collides with the ground. The time can then be substituted into the velocity equation to ensure that the velocity is under $20m/s$. However, it is also possible to break up the single parachute into multiple chutes thereby decreasing the radius of each parachute. The mass of each parachute than becomes $m = M_t/N$ and the total cost of the mission is $Cost = Nc_{chute}$. Compute the number of parachutes and radius of each parachute that minimizes the cost of the entire mission. Create plots to support your answer.

Hint: If you solve for the velocity analytically you should obtain this equation below

$$v(t) = \frac{-gm}{c}(1 - e^{-ct/m}) \tag{65}$$

integrating one more time yields the position

$$z(t) = z_0 - \frac{gm}{c}t + \frac{gm^2}{c^2}e^{-ct/m} \tag{66}$$

Thus, rather than integrating the equations numerically until the parachute hits the ground you can turn this problem into a root finding problem to determine when the parachute collides with the ground.

4.E. Project

As you know every engineering problem can be cast into the form

dependent variable = function(independent variable,parameters)

The problem is that alot of times the independent variable can change. So the question is what is the best solution? What is the criteria for optimal solutions? Cost? Weight? Size? It's up to you, you are the engineer. Your task is to find an optimization problem out in the world and figure out what the best solution is. Restrict yourself to a 2-D problem. 1D Problems will not be accepted. I encourage you to use fmincon in MATLAB if not you can try using a grid search and/or try N-R however N-R in 2D is pretty complex.

Your deliverable for this assignment will be to write a report detailing your optimization problem. The sections included in your report will be the following:

1. **Introduction** Explain what the problems are. Why do we care? Why is this important? Give some background on this type of problem.
2. **Mathematical Model** Explain the theory on how these problems are solved. Include equations in your report. Do not screenshot equations or just type them in. You are engineers. It's time to learn how to use Equation Editor. Finally, include all pertinent data required to run your code. Are there fixed parameters that do not vary? Include them in this section.
3. **Results** Explain your inputs to your code and your outputs. Do not copy and paste MATLAB output. Write your results in normal english. For example, "When the weight of the cat is 5 lbs the terminal velocity is 50 ft/s. If the weight of the cat is increased to 10 lbs the terminal velocity of the cat is 80 ft/s".
4. **Appendix MATLAB Code** Copy and paste your MATLAB code. This is the only place the word MATLAB should be. No supporting text required, simply copy and paste your code into this section.

5. MATRIX SUPPLEMENTALS

5.A. Gaussian Elimination - Chapter 9 and 10

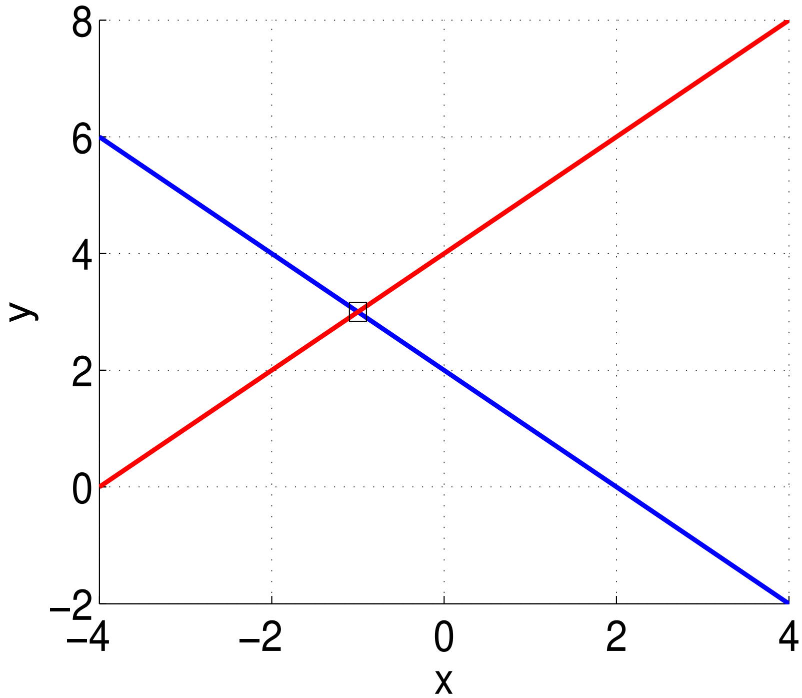
1. **Solving Multiple Algebraic Equations**

   Solving multiple algebraic equations is very simple. For example if we were to solve

$$\begin{aligned} x + y &= 2 \\ y - x &= 4 \end{aligned} \tag{67}$$

   We would simply set $x = 2 - y$ and plug this into the equation below yielding $y - (2 - y) = 4$ and thus $y = 3$ and $x = -1$. It is also possible to solve these equations graphically. If the first equation is written in the form $y = 2 - x$ and the second equation is written in the form $y = 4 + x$ the solution to this system of equations is where the graphs intersect.

   However, it is also possible to take the first equation and add it to the second equation. This would yield $2y = 6$ and thus $y = 3$. The second method is known as Gaussian Elimination. This can be written in matrix form using the equation below



$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 2 \\ 4 \end{Bmatrix} \tag{68}$$

This is the general form of $A\vec{x} = \vec{b}$. To use Gaussian Elimination in Matrix form an augmented matrix is formed such that $\tilde{A} = [A|b]$. This would yield the following equation.

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 4 \end{bmatrix} \tag{69}$$

Adding the first row to the second would yield

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix} \tag{70}$$

Dividing the last equation by 2 would yield

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \tag{71}$$

Finally, we take the bottom equation and subtract it from the first. That would yield.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix} \tag{72}$$

The form above is known as reduced row echelon form. The first matrix is the identity matrix and we can use it to directly solve for x and y. Here it is clear that $x = -1$ and $y = 3$.

2. **Equations with No Solutions - The Matrix Determinant**

   Note however that often times the equations have no solution. Take for example the system.

$$\begin{aligned} 2x + 2y &= 1 \\ x + y &= 1 \end{aligned} \tag{73}$$

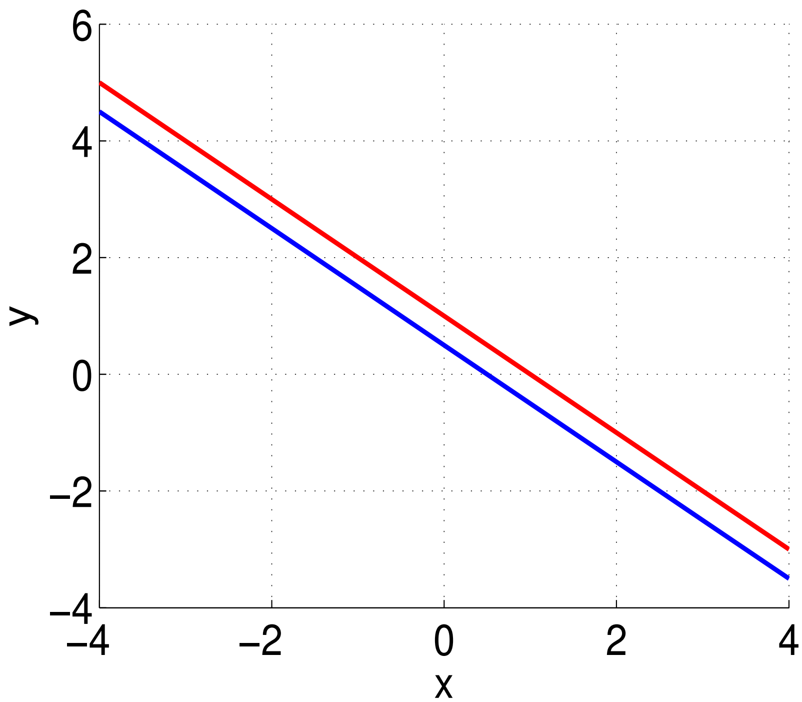
Using the "Naive" Gaussian Elimination technique would yield the following augmented matrix.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{74}$$

The rref of the matrix above is then

$$\begin{bmatrix} 1 & 1 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix} \tag{75}$$

Notice here that the last row is all zeros. This doesn't make sense. 0 does not equal 0.5. In order to visualize what's happening we need to look at this graphically. If we plot the first equation in the form $y = (1 - 2x)/2$ and the second equation in the form $1 - x$ we can see that the lines below do not intersect. This is why the rref of the augmented matrix produces a row of zeros. Rather than performing Gaussian Elimination it is sometimes useful to compute the matrix



determinant in order to see if a solution exists. For a 2x2 matrix the determinant of a matrix is given by

$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \tag{76}$$

The determinant is a useful property because if the det = 0 the solution does not exist and there will be a row of zeros in the rref of the matrix. For example, let's take the first example where

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{77}$$

The det(A) is then simply 1*1-1*(-1) = 2 thus the solution exists. However, if we take the second example

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \tag{78}$$

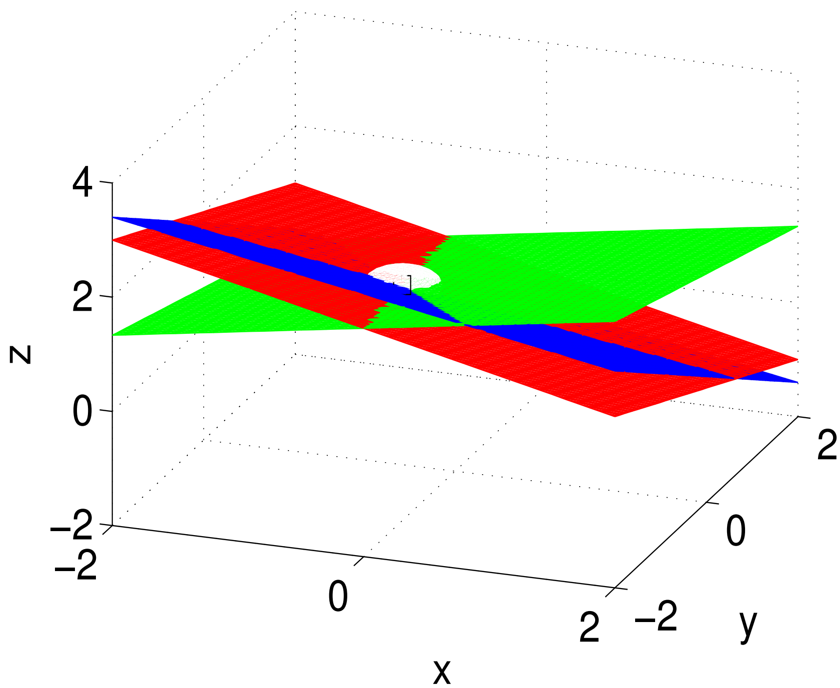
the det(A) is 2*1-2*1 = 0 and thus the solution does not exist. This can be helpful when doing systems of multiple variables like the example below

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 2 \\ 4 \\ 5 \end{Bmatrix} \tag{79}$$

Using MATLAB we can write det(A) in the command window which returns -8 which means the solution exists. Performing Gaussian elimination on this problem yields

$$\begin{bmatrix} 1 & 0 & 0 & -0.3750 \\ 0 & 1 & 0 & -0.1250 \\ 0 & 0 & 1 & 1.25 \end{bmatrix} \tag{80}$$

We can graph this result rewriting each equation in the form. $z = (2 - x - y)/2$, $z = (4 + x - y)/3$ and $z = (5 - 2x - 4y)/5$. These are equations of planes and are plotted in MATLAB below. The white ball is the intersection of all three planes



and the solution to the system of equations. If instead we take a look at the system below.

$$\begin{bmatrix} 1 & 1 & 2 \\ -2 & -2 & -4 \\ 2 & 4 & 5 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 2 \\ 4 \\ 5 \end{Bmatrix} \tag{81}$$

and compute det(A) in MATLAB the system returns 0 which means there is no solution. This system can be plotted as well in MATLAB. Here it is clear that the red and green planes intersect however the green plane is parallel to the red plane and thus there is no solution.

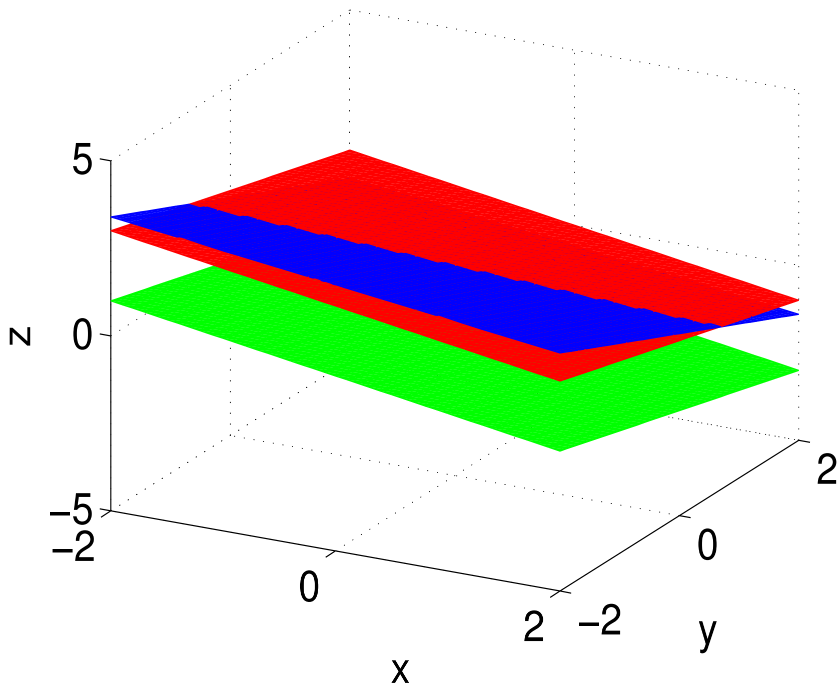
3. **Eigenvalues(λ) and Eigenvectors($\vec{\nu}$)**

Eigenvalues and Eigenvectors are used in a variety of different applications including differential equations. Traditionally they are used to solve problems of the form $A\vec{\nu} = \lambda\vec{\nu}$. In order to solve these problems we subtract the right hand side to achieve the form $(A - \lambda I)\vec{\nu} = 0$. The solution is then either $(A - \lambda I) = 0$ or $\vec{\nu} = 0$. However if the eigenvector is equal to zero we obtain a trivial solution thus we would like $(A - \lambda I)$ to be zero but it is impossible for this to be true since the identity matrix is diagonal. Thus, the only way for this solution to hold is for $det(A - \lambda I) = 0$. The determinant is a scalar equation and can be solved just like any other quadratic equation. Using the eigenvalue it is possible to solve for the eigenvector which is not unique but is typically normalized to unity. As an example we will compute the eigenvalues of the matrix

$$\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \tag{82}$$

If we compute the determinant of this matrix we find that it is -3 which means the matrix has a solution if paired with a vector. Computing $(A - \lambda I)$ yields

$$\begin{bmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix} \tag{83}$$



The determinant is then simply $(1-\lambda)^2 - 4$. Solving this equation is simply and yields two values, -1 and 3. Notice that the determinant is the product of the eigenvalues. To find the eigenvectors we then simply plug the eigenvalues into the matrix $(A - \lambda I)$. This yields two matrix equations.

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} v_x \\ v_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{84}$$

If you notice, the determinant of the matrix is now zero thus the rref is simply

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} v_x \\ v_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{85}$$

Thus we can pick any eigenvector $\vec{\nu} = [1, 1]^T$. Doing the same calculation for the eigenvalue of 3 yields an eigenvector of $\vec{\nu} = [-1, 1]^T$. If we normalize these vectors and place them in matrix form we can write the eigenvector matrix as

$$V = \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \tag{86}$$

The interesting property of this matrix is that it can be used to decompose the matrix A into the form $A = V\Lambda V^{-1}$ where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \tag{87}$$

This form is useful for the Matrix Inverse. The last thing to note is that if the determinant of the matrix is zero it means on of the eigenvalues is also zero.

4. **The Matrix Inverse**

The last way to solve equation is by using the matrix inverse. If the equation is written in matrix form $A\vec{x} = \vec{b}$ the equation can be multiplied by a special form of A called the inverse such that $A^{-1}A = I$ where I is the identity matrix. The equation is then $\vec{x} = A^{-1}\vec{b}$. For a 2x2 matrix the solution for the matrix inverse is

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} b & -c \\ -d & a \end{bmatrix} \tag{88}$$

Notice, that if det(A) is equal to zero the inverse does not exist. This applies to all size matrices. However, for matrices with bigger dimensions analytical solutions do not exist. Using the augmented matrix it is possible to solve for the inverse of a matrix. For example. Let's assume we are trying to solve for the inverse of the matrix.

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \tag{89}$$

The augmented matrix to find an inverse involves putting the identity matrix on the right-hand side such that $\tilde{A} = [A|I]$

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{bmatrix} \tag{90}$$

With this augmented matrix Gaussian Elimination can be performed. First the first row is multiplied by 4 and then substracted from the second equation yielding

$$\begin{bmatrix} 4 & 12 & 4 & 0 \\ 0 & 10 & 4 & -1 \end{bmatrix} \tag{91}$$

You can then divide the bottom equation by 10 and the top equation by 12.

$$\begin{bmatrix} 1/3 & 1 & 1/3 & 0 \\ 0 & 1 & 2/5 & -1/10 \end{bmatrix} \tag{92}$$

Then subtract the second equation from the first and then multiply the top equation by 3.

$$\begin{bmatrix} 1 & 0 & -1/5 & 3/10 \\ 0 & 1 & 2/5 & -1/10 \end{bmatrix} \tag{93}$$

Thus our inverse is

$$\begin{bmatrix} -1/5 & 3/10 \\ 2/5 & -1/10 \end{bmatrix} \tag{94}$$

Using this matrix you can solve any problem of the form $A\vec{x} = \vec{b}$. For example,

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{95}$$

The solution is then simply,

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} -1/5 & 3/10 \\ 2/5 & -1/10 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1/10 \\ 3/10 \end{Bmatrix} \tag{96}$$

We can also compute the rref of the augmented matrix to check the solution.

$$rref(\begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix}) = \begin{bmatrix} 1 & 0 & 1/10 \\ 0 & 1 & 3/10 \end{bmatrix} \tag{97}$$

Finally, it is also possible to compute the inverse of the matrix using the eigenvalue, eigenvector form. Remember the eigenvector, eigenvalue form is $A = V\Lambda V^{-1}$. The interesting property about the matrix V is that $V^{-1} = V^{-1}$. Thus $A^{-1} = V\Lambda^{-1}V^{-1}$. The computation of Λ^{-1} is trivial and can be done using the equation below.

$$\Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix} \tag{98}$$

5.B. Example Problems

1. Write a MATLAB code that computes the inverse of a matrix using gaussian elimination. Your function header should look like this

   `function invA = myinv(A)`

   where A is your input matrix of any size. Test it with three random matrices. Provide the answer in your assignment. You may not use inv(),rref() or ^(-1). You may use the eye() function to create the identity matrix. In your function check that the determinant of the matrix is not equal to zero using the det() function. If the determinant is equal to zero return a matrix of zeros and display 'The inverse of the matrix is undefined'.

2. Using your inverse routine from above write a routine that will solve a system of equations. For example, use it to solve the system of equations below.

$$\begin{aligned} 2u + v &= -3 \\ u - 5v &= 2 \end{aligned} \tag{99}$$

Come up with two more random systems of equations including one with three variables (u,v and w) and use your routine to solve it. All system of equations can be written in the form

$$A\vec{x} = \vec{b} \tag{100}$$

Using the form above your function header should look like the following header.

```
function x = my_sys_solve(A,b)
```

6. REGRESSION AND INTERPOLATION

6.A. Least Squares Regression - Chapter 17

1. **Linear Regression**

Linear Regression attempts to fit a line to a function f such that

$$y = f(x) \approx a_0 + a_1 x \tag{101}$$

The solution of the coefficients a_0 and a_1 are found by minimizing the square error between the actual value of y and the estimated value. That is,

$$min[\sum_{i=0}^{N}(y_i - a_0 - a_1 x_i)^2] \tag{102}$$

The easiest way to solve this is by using gaussian elimination. First let Y be a vector of all actual values y_i, X be a vector of all actual values x_i and let $A = [a_0\ a_1]^T$.

$$Y = [1\ X]A \tag{103}$$

We can then say that $H = [1\ X]$. The problem above has been solved by Gauss. The idea is to obtain the least squares estimate of A using the form $Y = HA$. The solution is

$$A^* = (H^T H)^{-1} H^T Y \tag{104}$$

Notice how A has been replaced with A^* this is because $H^T H$ is a 2x2 matrix. The problem has been reduced from a system of N points to a 2x2 system and thus information has been lost because the order of the system has been truncated.

2. **Polynomial Regression**

The benefit of using the formula above is that it can be extended to include polynomials. That is assume we are trying to fit

$$y \approx a_0 + a_1 x + a_2 x^2 + ... a_n x^N \tag{105}$$

Using Gauss' formula we can write

$$Y = [1\ X\ X^2\ ...\ X^N]A \tag{106}$$

where $A = [a_0\ a_1\ a_2\ ...\ a_N]^T$. Notice that our problem is still in the form $Y = HA$ and thus we can still use the formula above for linear regression.

3. **Basis Function Regression**

   It is even possible to use the method above to fit a function using basis functions. For example assume we are trying to fit

$$y \approx a_0 + a_1 f_1(x) + a_2 f_2(x) + ...a_n f_N(x) \tag{107}$$

   Again this problem can simply be written as

$$Y = [1\ F_1\ F_2\ ...\ F_N]A \tag{108}$$

   and solved just as before.

4. **Goodness of Fit**

   Often times it is beneficial to compute the goodness of fit which is a number that represents how well the regression curve fits the data. To do this we must first compute the values of y computed by the regression line. This is very simple once you have computed H, and A^*.

$$\tilde{Y} = HA^* \tag{109}$$

   This is equation gives you the y-coordinates as computed by the regression line. Using this it is possible to compute the total residuals or the error in the fit and the measured/given data.

$$S_r = \sum_{i=1}^{N} (Y_i - \tilde{Y}_i)^2 \tag{110}$$

   This value will get bigger when the error between the fit and the data are large. However, notice that this value changes with the number of data points. If there are more data points the error can get quite large which doesn't necessarily help. Instead what we do is compute the error between the mean of Y which is denoted as $\bar{Y}$.

$$S_t = \sum_{i=1}^{N} (Y_i - \bar{Y})^2 \tag{111}$$

   and then use S_r and S_t to normalize everything and compute r which is the correlation coefficient.

$$r = \sqrt{\frac{S_t - S_r}{S_t}} \tag{112}$$

   This number can only assume values between 0 and 1. If the value of r is 1 then the fit is said to be perfect. If the value of r is 0 the fit is not good.

6.B. Example Problems

1. Write a routine that will use Polynomial Least Squares regression. The routine will have a function header like this

```
function coeff = myregression(Y,X,N)
```

   where Y is the vector of sampled point y_i and X is a vector of independent points x_i. N is the order of the fit. When N = 0, the fit is zero order, when N = 1 the fit is linear and when N = 2 the fit is quadratic. For the data below plot the data given on a figure and compute the least squares fit. Note, you will have to plot the data to figure out which order to pick. Pick an order that reduces to the least square error to a minimum. Once you have the fit, plot the fit on the graph and put the least square error in the title.

   In modeling an oil reservoir, it may be necessary to find a relationship between the equilibrium constant of a reaction and the pressure at constant temperature.

$K-value$	$Pressure$
7.5	0.635
5.58	1.035
4.35	1.435
3.55	1.835
2.97	2.235
2.53	2.635
2.2	3.035
1.93	3.435
1.7	3.835
1.46	4.235
1.28	4.635
1.11	5.035
1.0	4.435

2. Write a routine that will compute a 2nd-order planar fit. That is, assume the form

$$z = a_0 + a_1 sin(2x) + a_2 sin(2y)$$

Use Gauss' equation to solve for the coefficients and create a mesh of the solution. The data will be provided in a text file with columns x,y and z. First plot the data using blue stars. These can be used to solve for the coefficients a_0, a_1 and a_2. To create a mesh use the following code.

```
xest = linspace(-pi,pi,100);
yest = linspace(-pi,pi,100);
[xx,yy] = meshgrid(xest,yest);
zz = a0 + a1*sin(2*xx) + a2*sin(2*yy);
mesh(xx,yy,zz)
```

Your results should have 1 graph with the provided data and the mesh. You need to also include your coefficients in your answer as well.

3. The data below describes the growth of a population following a logistical model. This system is non-linear however the equation can be converted to a linear system.

$$y = \frac{1}{1+e^{ax+b}}$$
$$z = 1/y - 1$$
$$zz = ln(z)$$

Use the equations above to solve for the coefficients a and b and plot the solution on the same graph.

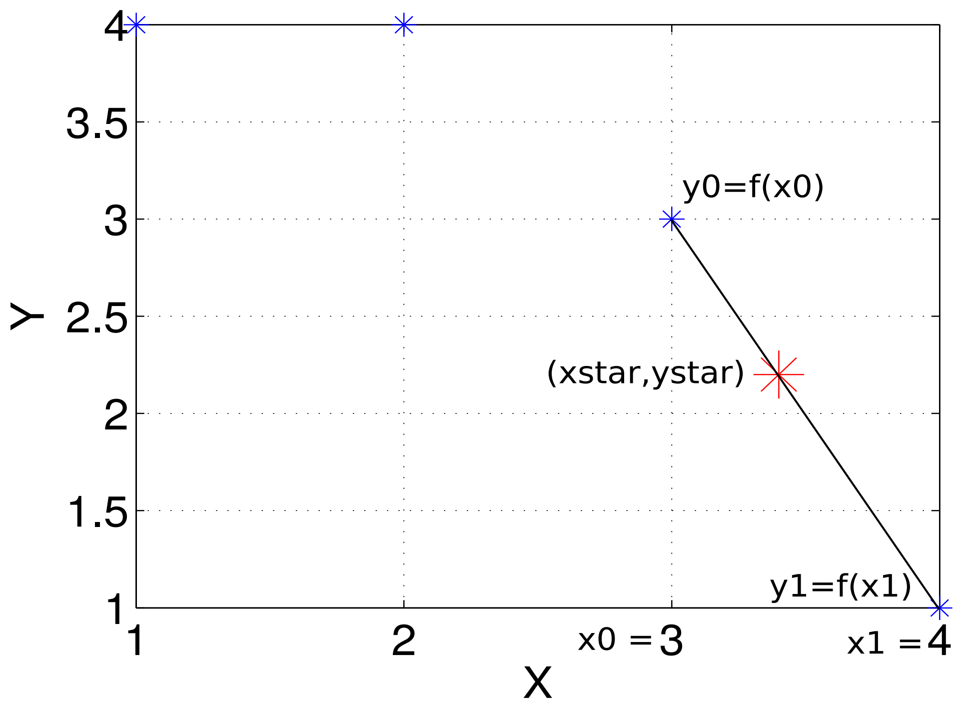
X	Y
-1.0	0.05
-0.8	0.08
-0.6	0.14
-0.4	0.23
-0.2	0.35
0.0	0.50
0.2	0.65
0.4	0.77
0.6	0.86
0.8	0.92
1.0	0.95

6.C. Interpolation - Chapter 18

1. **Linear Interpolation**

The simplest form of interpolation is by creating a line in between points x_1 and x_0 such that

$$y^* = f(x^*) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x^* - x_0) \tag{113}$$

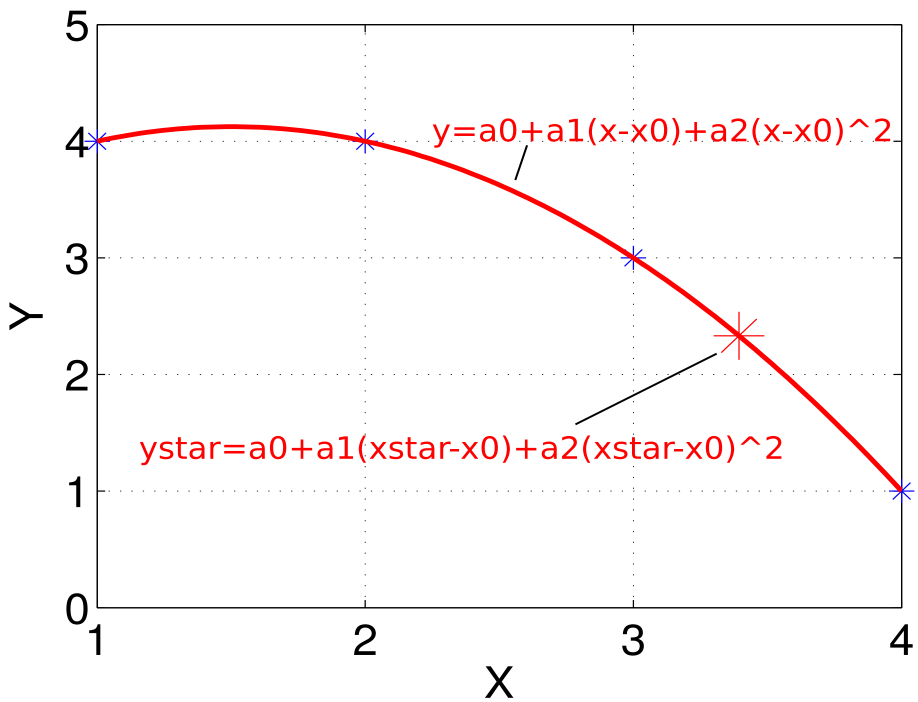


2. **Polynomial Interpolation**

Note that the problem can be explicitly derived using Linear Regression. That is we are trying to solve the problem $f(x) = a_0 + a_1(x - x_0)$. If this problem is set up the solution would yield $a_0 = f(x_0)$ and $a_1 = slope$. Thus it is possible to extend the interpolation method to higher order polynomials. Our polynomial is then

$$\tilde{f}(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ...a_N(x - x_0)^N \tag{114}$$

Note however that in order to interpolate using the equation above your need N+1 data points. For example, in order to fit a line the method requires two points to solve for a_0, a_1. In order to fit a quadratic the method requires three points to solve for a_0, a_1 and a_2.



3. **Linear Splines**

Splines are a way to approximate the curve with more than one model. For example, if you have a curve that looks fourth order you could fit the entire data set as fourth order polynomial or you could simply fit the line to 4 linear polynomials. In certain situations this would actually create a better fit. The equation for linear splines is simply

$$\tilde{f}(x) = f(x_{i-1}) + m_{i-1}(x - x_{i-1}) \quad x_{i-1} \leq x \leq x_i \tag{115}$$

where

$$m_{i-1} = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \tag{116}$$

4. **Quadratic Splines**

The equation for quadratic splines gets a little more messy so the derivation is left for the student. Instead rules are listed to help you with the derivation. The basic formula of a quadratic spline is

$$\tilde{f}(x) = a_i x^2 + b_i x + c_i \quad x_{i-1} \leq x \leq x_i \tag{117}$$

The rules for quadratic splines are then

(a) The function values of adjacent splines must be equal to each other

$$a_i x_i^2 + b_i x_i + c_i = a_{i+1} x_i^2 + b_{i+1} x_i + c_{i+1} \tag{118}$$

(b) The first and last function must pass through the end points

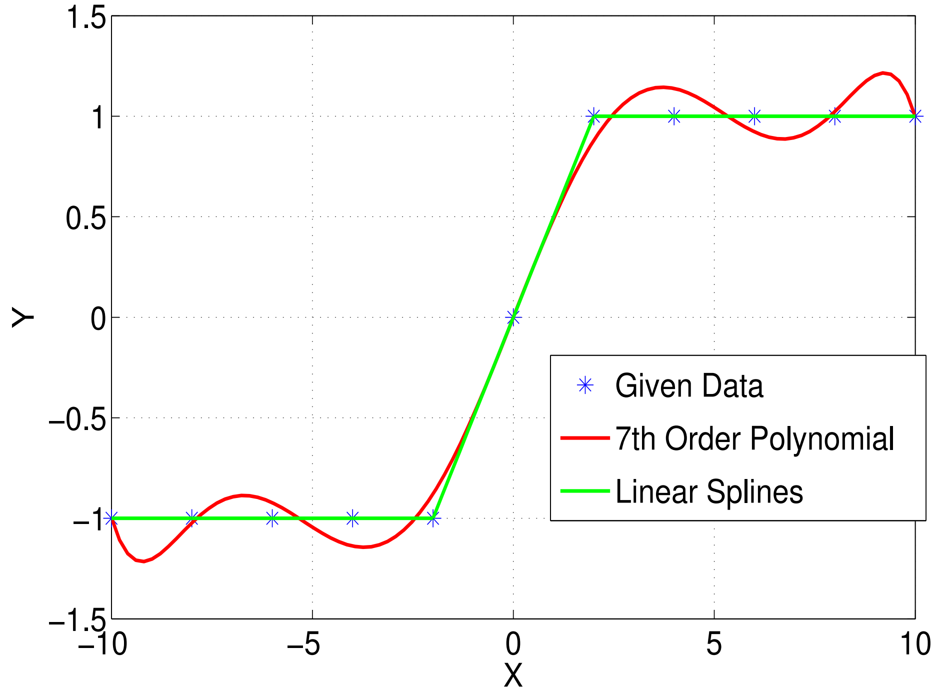
$$\begin{aligned} a_1 x_0^2 + b_1 x_0 + c_1 &= f(x_0) \\ a_n x_n^2 + b_n x_n + c_n &= f(x_n) \end{aligned} \tag{119}$$

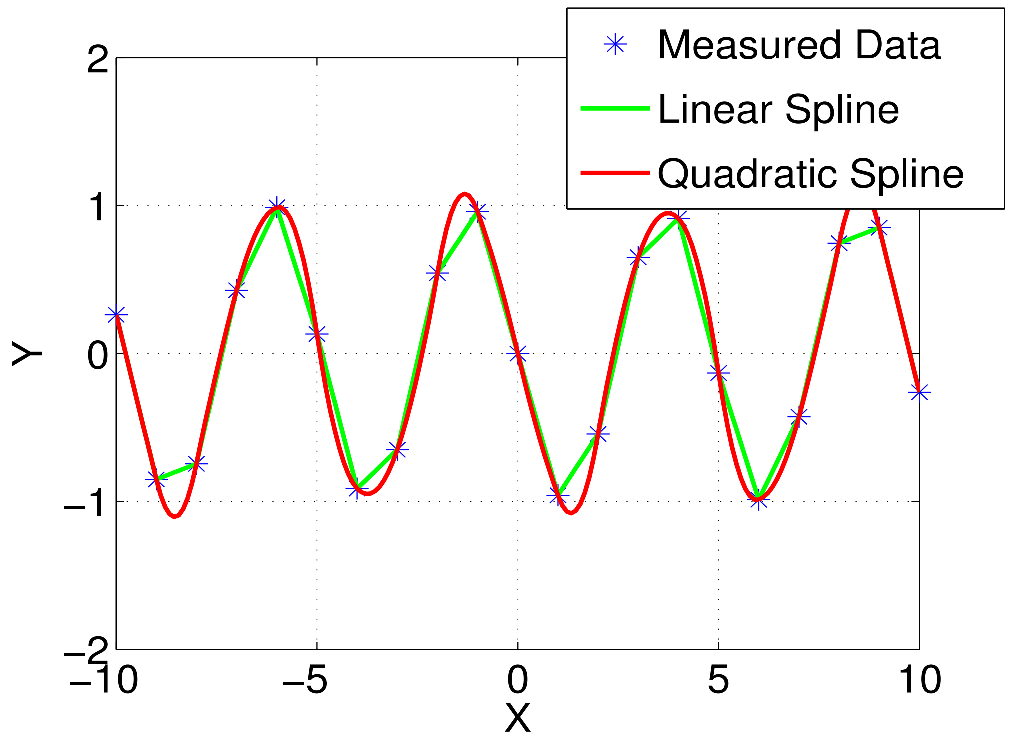
(c) The first derivative of adjacent splines must be equal to each other

$$2a_i x_i + b_i = 2a_{i+1} x_i + b_{i+1} \tag{120}$$

(d) The second derivative of the first spline at $x_0 = 0$

$$a_1 = 0 \tag{121}$$





5. **2-D Interpolation**

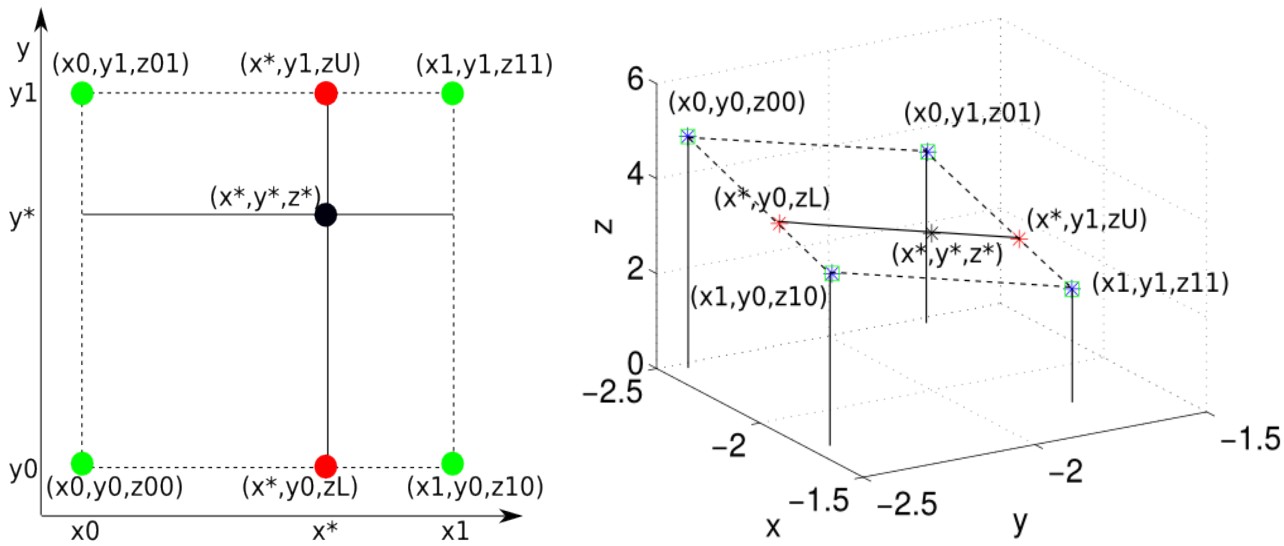
It is often the case that rather than your system being of the form $y = f(x)$ you have an equation of the form $z = f(x, y)$. In this case you need to use 2-D interpolation and here we will discuss linear 2-D interpolation called bi-linear interpolation for short. This interpolation scheme is three separate interpolations. Assume you are trying to interpolate $z^* = \tilde{f}(x^*, y^*)$ where $x_{i-1} \leq x^* \leq x_i$ and $y_{i-1} \leq y^* \leq y_i$. The equations to solve for z^* is then

$$z_U = f(x_{i-1}, y_i) + \frac{f(x_i, y_i) - f(x_{i-1}, y_i)}{x_i - x_{i-1}}(x^* - x_{i-1}) \tag{122}$$

$$z_L = f(x_{i-1}, y_{i-1}) + \frac{f(x_i, y_{i-1}) - f(x_{i-1}, y_{i-1})}{x_i - x_{i-1}}(x^* - x_{i-1}) \tag{123}$$

$$z^* = z_L + \frac{z_U - z_L}{y_i - y_{i-1}}(y^* - y_{i-1}) \tag{124}$$

The basic solution is simply three linear interpolations to get your solution.



6.D. Example Problems

1. Write an interpolation program that will create Linear Splines for your data set. You merely need to compute the point slope form equations for every data point. Use your splines to compute the value of ystar.

```
function ystar = myinterp(X,Y,xstar)
```

Note, X is a vector of data points and Y is a vector of sampled points occuring at the values of X. Test your code using the table of data below. Let xstar = 1.5, 3.5 and 5.5. Again make your code general enough to handle any number of data points.

$K - value$	$Pressure$
7.5	0.635
5.58	1.035
4.35	1.435
3.55	1.835
2.97	2.235
2.53	2.635
2.2	3.035
1.93	3.435
1.7	3.835
1.46	4.235
1.28	4.635
1.11	5.035
1.0	4.435

2. Edit your code from problem 1 and use polynomial interpolation instead of linear splines. Use the entire data set to compute the polynomial fit. Feel free to use an expansion point. It is entirely up to you. Your function header will look like this.

```
function ystar = interpoly(X,Y,xstar,N)
```

All inputs are the same except N is now the order of the approximation. Thus if N=0 you use a zero order method and if N=1 you use a linear approximation. Test your code using the data below and set N=1, did you get something different than problem 1? Explain why if so. Then set N=2. What do you get? Test it using the same values of xstar in problem 1.

3. Problem 2 from last weeks homework could technically be used as a 2-D polynomial interpolation method.

$$z = a_0 + a_1 sin(2x) + a_2 sin(2y)$$

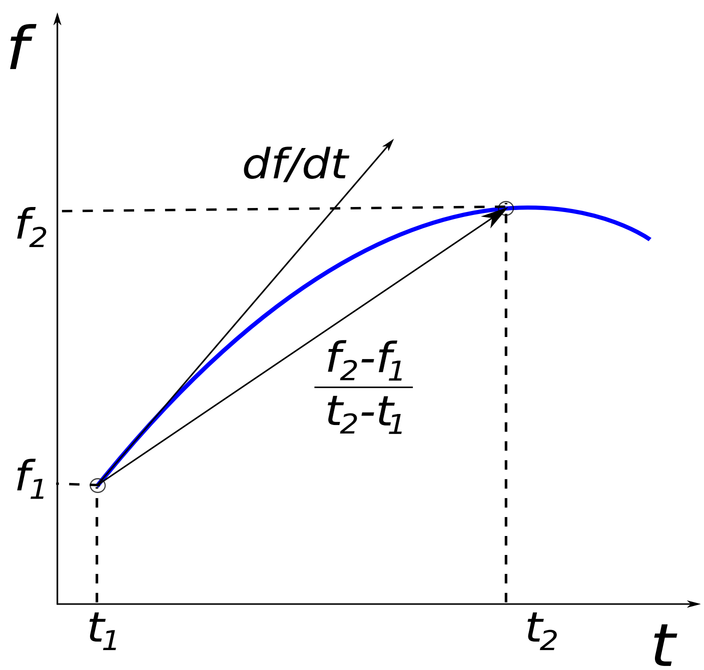
In this problem, you were given X,Y, and Z and you found A^*. Use this fit to compute z^* for $x^* = 1, y^* = 1$ and compare it to a simple 2-D linear interpolation method. Explain the difference in your answers.

7. NUMERICAL CALCULUS

7.A. Taylor Series - Chapter 4

1. **First Order Approximation - Euler's Method**

Euler's Method of Integration is a method used to integrate equations of motion. Because the method is first order, Euler's method tends to be largely inaccurate. The basic idea of Euler's Method is to approximately determine a function f while only knowing $\dot{f} = \frac{df}{dt}$. That is, it is beneficial to integrate $\dot{f}$ to obtain f. If we plot a general plot of f(t) we obtain the graph above.



Here, the blue line is the analytical equation for f. Using this graph it is possible to write that

$$\dot{f}_1 \approx \frac{f_2 - f_1}{t_2 - t_1} \tag{125}$$

if we then use the relationship that $t_2 = \Delta t + t_1$ we can write that

$$\dot{f}_1 \approx \frac{f_2 - f_1}{\Delta t} \tag{126}$$

Note that in the limit as $\Delta t \to 0$ we have

$$\lim_{\Delta t \to 0} \frac{f_2 - f_1}{\Delta t} = \dot{f}_1 \tag{127}$$

thus as we reduce our timestep in our measurement we create a more and more accurate representation of f. Returning to our approximate equation for $\dot{f}$ we can rearrange this equation to write.

$$f_2 \approx f_1 + \dot{f}_1 \Delta t \tag{128}$$

Thus the value of f_2 can be estimated if the derivative and the initial value are known. It is simple to extrapolate to another timestep such that

$$f_3 \approx f_2 + \dot{f}_2 \Delta t \tag{129}$$

Thus as long as the derivative of f and the initial condition is known, f can be numerically integrated using the recursive algorithm.

$$f_{i+1} \approx f_i + \dot{f}_i \Delta t \tag{130}$$

2. **The Series**

Recall that Euler's method uses a simple approximation for the derivative using simple forward differencing. That is,

$$\frac{df}{dt} \approx \frac{f_{i+1} - f_i}{t_{i+1} - t_i} \tag{131}$$

We then said that $f(t_{i+1}) = f(t_i) + \dot{f}(t_i)\Delta t$. This is a first order approximation to the function f. It is possible to create higher order approximations using more terms such that

$$f(t_{i+1}) \approx f(t_i) + f'(t_i)\Delta t + f''(t_i)\frac{\Delta t^2}{2!} + f'''(t_i)\frac{\Delta t^3}{3!} + ... + f^{(N)}(t_i)\frac{\Delta t^N}{N!} \tag{132}$$

3. **Error**

The Taylor series always contains error since it is impossible to compute an infinite series. The easiest way to derive the error is by way of logical extension. If we write the zero order approximation to f we have

$$f(t_{i+1}) \approx f(t_i) \tag{133}$$

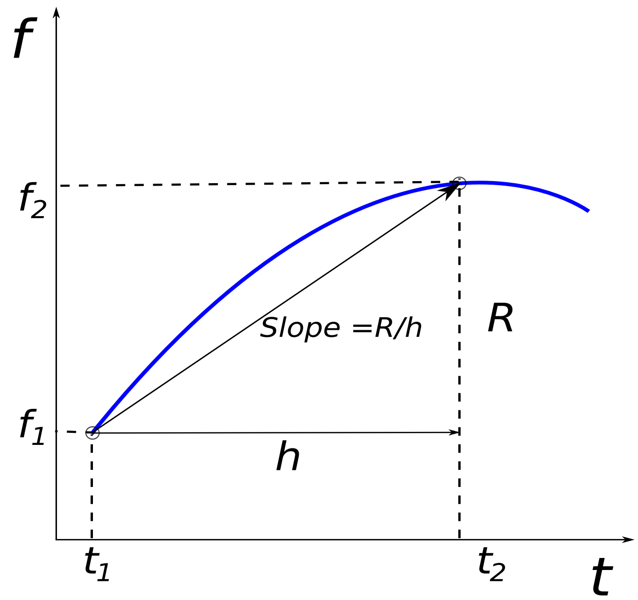
our error would then be

$$R_0 \approx f'(t_i)\Delta t + f''(t_i)\frac{\Delta t^2}{2!} + f'''(t_i)\frac{\Delta t^3}{3!} + ... + f^{(N)}(t_i)\frac{\Delta t^N}{N!} \tag{134}$$

however graphically we can show that in fact our error can simply be written as

$$R_0 = f'(\zeta)\Delta t \tag{135}$$

using the graph below.



where ζ is a number between B and A. It is easy to extend this to $R_1 = f''(\zeta)\Delta t^2/2!$ or generally

$$R_N = \frac{f^{(N+1)}(\zeta)\Delta t^{N+1}}{(N+1)!} \tag{136}$$

4. **The Non-linear Parachutist**

Let's return to the parachutist. In that problem we had

$$\dot{v} + (c/m)v = g \tag{137}$$

Since the drag was Newtonian the analytical solution could be obtained using simple differential equation techniques.

$$v(t) = v_0 e^{-ct/m} + (1 - e^{-ct/m})mg/c \tag{138}$$

However, if Bernoulli drag is used instead the equations of motion become

$$\dot{v} + (c/m)v^2 = g \tag{139}$$

This unfortunately has no analytical solution. The solution then is to solve the problem numerically using Euler's method. If the initial condition is known v_0 the problem reduces to the following equations

$$\begin{gathered}
\dot{v}_0 = g - (c/m)v_0^2 \\
v_1 = v_0 + \dot{v}_0 \Delta t \\
\dot{v}_1 = g - (c/m)v_1^2 \\
v_2 = v_1 + \dot{v}_1 \Delta t \\
\vdots \\
\dot{v}_N = g - (c/m)v_N^2 \\
v_{N+1} = v_N + \dot{v}_N \Delta t
\end{gathered} \tag{140}$$

Provided the timestep is small enough the solution will converge to the actual solution.

7.B. Newton-Cotes Integration - Chapter 21

1. **The Standard Reimmann Sum**

   The standard Reimmann sum can be used to integrate equations. For example, assume for the moment that the velocity of a ball in free-flight is equal to $v(t) = v_0 + at$. In calculus it is shown that the area under the curve $v(t)$ is equal to the position. That is,

$$x(t) = x_0 + \int_{t=0}^{t} v(t)dt \approx x_0 + \sum_{i=1}^{N} v(t_i)\Delta t \tag{141}$$

   The position is merely the integral of the velocity curve. In addition, the position can be approximated by the area under the curve using a series of rectangles to approximate the area under the curve. Note that the equation above is a similar representation or for Euler's method. That is, Euler's method is a first order method for integrating equations of motion and a Reimmann sum is a zero-order method for obtaining the area under the curve.

2. **Car Example**

   So let's say we're driving our care and we take a few measurements as we head to work. We also note the time when we take the measurement.

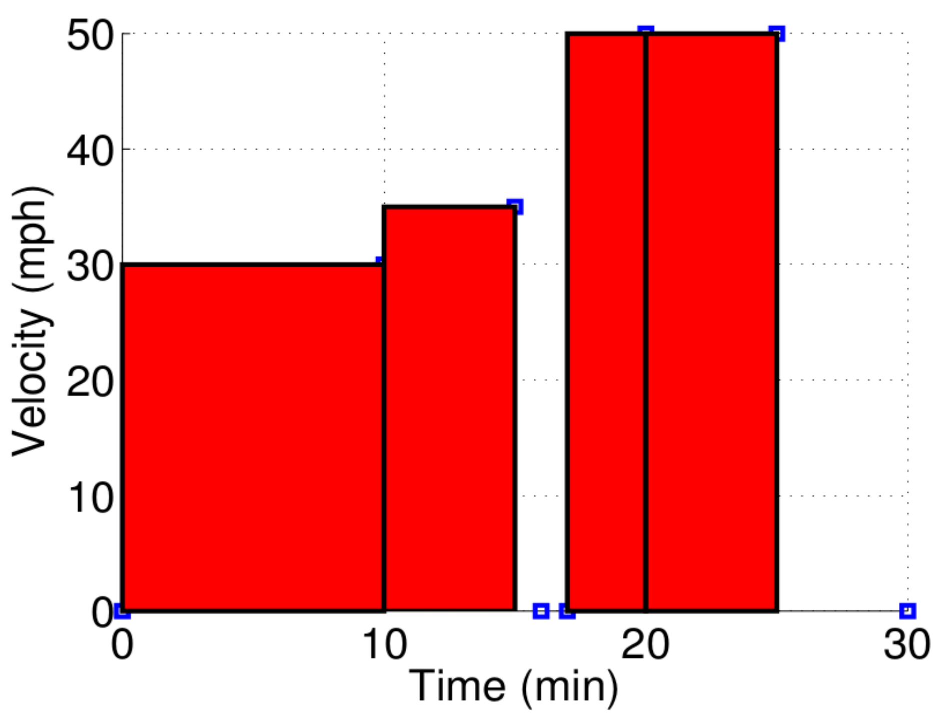
$$\begin{gathered} V = [30, 35, 0, 0, 50, 50, 0]\ mph \\ T = [10, 15, 16, 17, 20, 25, 30]\ min \end{gathered} \tag{142}$$

   The solution to this problem is then given by using the Reimann Sum. Note that we are using the left Reimann sum since we assume that the initial velocity and time are both zero.

$$X = V_0(T_0 - 0) + V_1(T_1 - T_0) + \ldots + V_N(T_N - T_{N-1}) \tag{143}$$

$$X = (30*(10-0)+35*(15-10)+0*(16-15)+0*(17-16)+50*(20-17)+50*(25-20)+0*(30-25))/60 \tag{144}$$

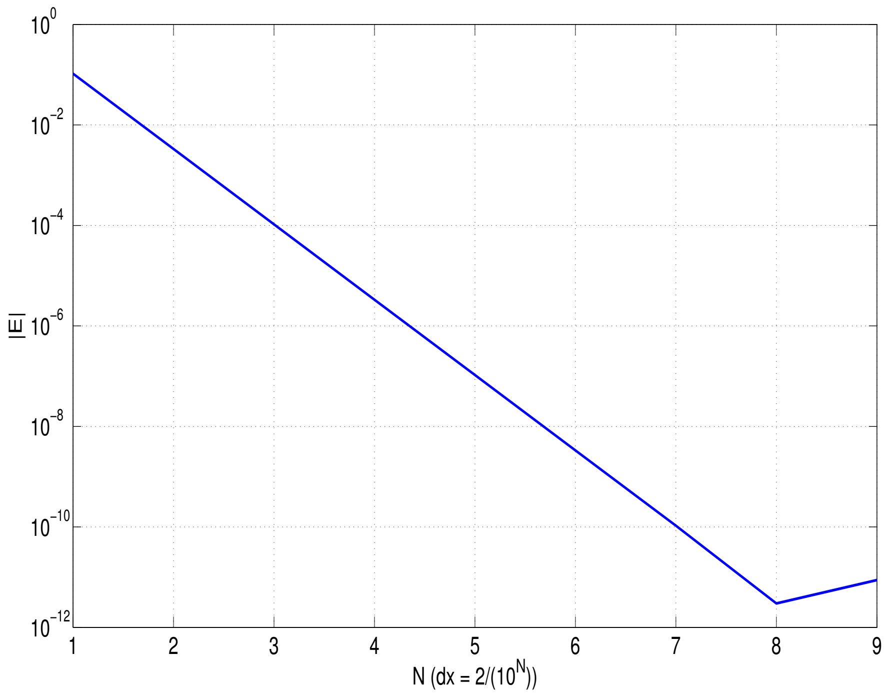
   Which results in about 14.58 miles. We divide by 60 because our time is in minutes and our velocity is in miles/hour. This solution can be visualized in the graphic below.



Notice though that there is quite a bit of error. It is possible to reduce the error by taking more measurements thus reducing Δt.

3. **Round-Off Error**

   The equation above has a very fundamental term in the equation above and that is the timestep (Δt). The solution becomes more approximate as the timestep decreases. This reduction in the timestep and the increase in accuracy is known as round-off error. As the timestep becomes smaller and smaller you reduce the amount of round off error in your system because more decimal places are used in the approximation. Note, however that there is a limit to how small you can make the timestep. If the timestep is reduced to levels below the threshold of your computer you will find that the error in your estimate actually begins to increase. This limit is known as the limit of precision. That is, the CPU cannot be more precise given the timestep you have chosen and the value of your error begins to rise.



   However, in our example above with the car driving, we cannot decrease Δt because we only have a discrete set of measurements. Instead we can increase the order of our method.

4. **Trapezoidal Rule**

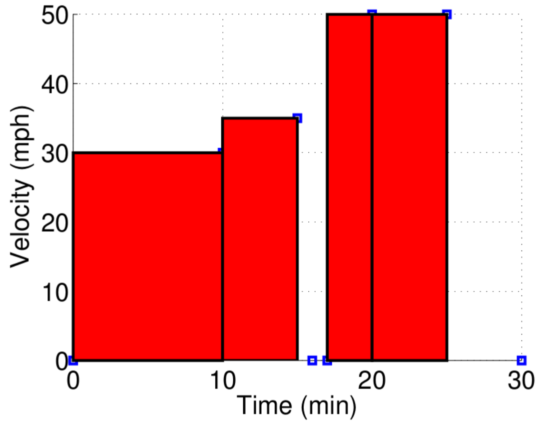
   Similar to the Reimmann Sum the trapezoidal rule sum individual Trapezoids instead of rectangles. The area of a trapezoid is given by $(1/2)(b_1 + b_2)h$. Where b is the base and h is the width. For our integration of functions we have

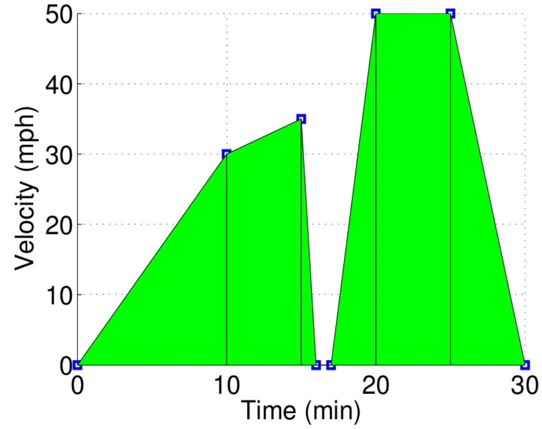
$$I \approx \sum_{i=1}^{N} \frac{1}{2}(f(t_i) + f(t_i + \Delta t))\Delta t \tag{145}$$

   Note that the Standard Reimmann sum is a zero order approximation whereas the trapezoidal rule is a linear or first order approximation. Because the trapezoidal rule is a first order approximation, errors are still produced but they are not as bad.

5. **Car Example Returned**

   Let's return to the car example. Instead of adding rectangles let's use trapezoids and add them together using the trapezoidal rule. The graphic below depicts the solution using trapezoids. Notice how much error is reduced when using trapezoids instead of rectangles. I've placed the Reimman Sum and Trapezoidal rule problem side by side to show the benefit.





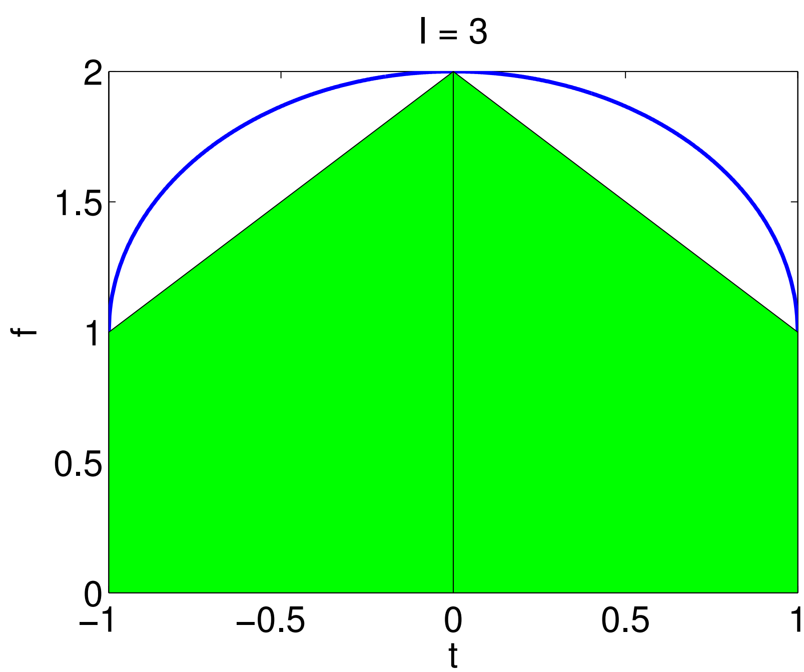
The solution of adding the triangles is realized in the equation below.

$$
\begin{gathered}
X = \tfrac{1}{2}(V_0 + V_1)(T_1 - T_0) + \tfrac{1}{2}(V_1 + V_2)(T_2 - T_1) + \ldots + \tfrac{1}{2}(V_N + V_{N-1})(T_N - T_{N-1}) \\
X = \tfrac{1}{2}((0 + 30)(10 - 0) + (30 + 35)(15 - 10) + (30 + 35)(15 - 10) + \\
(35 + 0)(16 - 15) + (0 + 50)(20 - 17) + (50 + 50)(25 - 20) + (50 + 0)(30 - 25))/60
\end{gathered} \tag{146}
$$

The solution to using the trapezoidal rule is 15.71 miles which is alot different than the Reimman sum solution.

6. **Compute Truncation Error of Trapezoidal Rule**

The trapezoidal rule is not perfect though. The figure below shows the inherent problem of using the trapezoidal rule for functions with a second derivative. That is, the order of the system is larger than 2.



Here the green shaded region can be defined as the area of the trapezoid. The blue line is the analytical solution thus the white region is the error created by the trapezoidal rule. In a general sense the truncation error can be written using the following equation below.

$$E_{Ti} = \int_a^b f(t)\ dt - \frac{1}{2}(b-a)(f(a)-f(b)) \tag{147}$$

As noted previously it is possible to write f(t) as a taylor series expansion and evaluate the first integral. Doing this yields the equation below.

$$E_{Ti} = \frac{-1}{12} f''(\zeta)(b-a)^3 \tag{148}$$

Remember from Chapter 4 that ζ is a number in between the interval a,b. The total error than for the trapezoidal rule is then just the sum of all errors. Note that $\Delta x = (b-a)$.

$$E_T = \sum_{i=1}^{N} E_{Ti} = \sum_{i=1}^{N} \frac{-1}{12} f''(\zeta)\Delta x^3 \tag{149}$$

Making the substitution that $\Delta x = (B-A)/N$ where A and B are x_1 and x_N respectively the total error is written as

$$E_T = \frac{-1}{12N^2} \bar{f}''(B-A)^3 \tag{150}$$

Here $\bar{f}''$ is the average second derivative over the interval A,B.

7. **Simpson's 1/3 and 3/8 Rule**

Clearly it is possible to fit a polynomial much like the RK2 techniques and the taylor series expansion. Fitting second order and third order polynomials yields Simpson's 1/3 and 3/8 rule.

$$I \approx= \sum_{i=1}^{N} \frac{1}{6}(f(t_i) + 4f(t_i + \Delta t/2) + f(t_i + \Delta t))\Delta t \tag{151}$$

$$I \approx= \sum_{i=1}^{N} \frac{1}{8}(f(t_i) + 3f(t_i + \Delta t/3) + 3f(t_i + 2\Delta t/3) + f(t_i + \Delta t))\Delta t \tag{152}$$

8. **Improper Integrals**

Improper integrals pose a problem for numerical integration becase there is a ∞ in the integral. For example,

$$I = \int_0^\infty f(t)dt \tag{153}$$

Numerically, ∞ does not exist. A computer can represent a very large number but it cannot represent ∞. Thus, the solution is to convert the integral to two integrals.

$$I = \int_0^B f(t)dt + \int_B^\infty f(t)dt \tag{154}$$

In order to remove the ∞ in the second integral a change of variables is done such that $x = 1/t$. Going through the change of variables yields the following equation.

$$I = \int_0^B f(t)dt + \int_0^{1/B} \frac{1}{x^2} f(1/x)dx \tag{155}$$

Now the two integrals above can be evaluated with two separate numerical integration schemes.

7.C. Integration of Equations - Chapter 22

1. **Richardson's Extrapolation**

Recall that the error from the trapezoidal rule is given by

$$E_T = \frac{-1}{12N^2}\bar{f}''(B-A)^3 \tag{156}$$

We know that moving to a higher order method can increase the accuracy. However because of the law of diminishing returns it is not always practical to move to a higher order method. Thus it is useful to write the actual computed value as a ratio from 1 increment to another. That is it is possible to write

$$I = I(N_1) + E(N_1) \tag{157}$$

where $I(N_1)$ is the solution of the trapezoidal rule for N_1 intervals. Because of the method certain errors are accrued as we've seen and is given as $E(N_1)$. Now it is equally valid to assume that $N_2 = 2N_1$ then

$$I = I(N_2) + E(N_2) \tag{158}$$

Thus, $I(N_1) + E(N_1) = I(N_2) + E(N_2)$. The ratio of errors can then be written as

$$\frac{E(N_1)}{E(N_2)} = \left(\frac{N_2}{N_1}\right)^2 \tag{159}$$

substituting this into our two equations of I and solving for $E(N_2)$ we have

$$E(N_2) = \frac{I(N_1) - I(N_2)}{1 - \left(\frac{N_2}{N_1}\right)^2} \tag{160}$$

we can then plug this into the equation involving N_2 and collect terms to arrive at

$$I = \frac{4}{3}I(N_2) - \frac{1}{3}I(N_1) \tag{161}$$

Remember that $N_2 = 2N_1$. The result of this derivation is that is possible to compute $I(N_2)$ and $I(N_1)$ and obtain a more accurate result of I. It can be shown that if $I(N_1)$ is of $O(\Delta x^2)$, I is $O(\Delta x^4)$. Essentially you have increased the order of the method without actually using another method.

2. **Romberg Integration**

Romberg Interpolation is simply an extension of Richardson's Extrapolation.

$$I_{j,k+1} = \frac{4^k I_{j+1,k} - I_{j,k}}{4^k - 1} \tag{162}$$

j is the interval number and k is the order of the method. The trapezoidal method is of order 1 thus k = 1. When k = 1 and j = 1, Romberg integration reduces to

$$I_{1,2} = \frac{4I_{2,1} - I_{1,1}}{4-1} \tag{163}$$

Here $I_{2,1}$ is $I(N_2)$ using the trapezoidal rule. $I_{1,1}$ is $I(N_1)$ using the trapezoidal rule. $I_{1,2}$ is the equivalent of $I(N_1)$ using a second order method. The power of Romberg Integration is that it does not stop there. It is possible to increase k and j until you get the estimate you like. For example, the table below shows the flow of Romberg Integration to compute $I_{1,4}$.

Interval	First Order		Second Order		Third Order		Fourth Order
N_1	$I_{1,1}$	$\longrightarrow$	$I_{1,2}$	$\longrightarrow$	$I_{1,3}$	$\longrightarrow$	$I_{1,4}$
$N_2 = 2N_1$	$I_{2,1}$	$\nearrow$	$I_{2,2}$	$\nearrow$	$I_{2,3}$	$\nearrow$	
$N_3 = 2N_2$	$I_{3,1}$	$\longrightarrow$	$I_{3,2}$	$\nearrow$			
$N_4 = 2N_3$	$I_{4,1}$	$\nearrow$					

This shows that it is possible to obtain accuracy on the order of Δx^8 by only using methods that have accuracy on the order of Δx^2.

3. **Gauss Quadrature**

7.D. Example Problems

1. Create an empty function that will compute a sine wave of the form

$$y = Asin(\omega x) \tag{164}$$

Your function header will be as follows

function y = myfunction(x,w,A)

where x can be a scalar or a vector, A is the amplitude of your wave and w is the frequency of your sine wave. Test your function by plotting it from a value of $-\pi$ to π. Make sure to use enough data points such that the figure is smoothly plotted. As always label your axes and make it as pretty as possible.

2. Take your function from problem 1 and compute the Taylor Series expansion. Find a way to have MATLAB compute the Taylor series at any arbitrary point x. Your function header will look like this.

function yest = mytaylor(x,w,A,N)

where x is the input variable which can be a vector or a scalar, w is the frequency of the sine wave, A is the amplitude and N is the order of your Taylor Series expansion. Choose an expansion point of zero to make the math simple. Starting with your plot from problem 1, add 3 more lines showing a Taylor Series expansion of order 1,3 and 5. You should have 4 lines on your graph. The first should be the output of myfunction, the other three should be the output of your taylor series expansion.

3. Assume I have a NACA 0012 airfoil across a wing. The total lift can be given using the equation below

$$L = 2\int_0^{b/2} L' dy \tag{165}$$

where L' is the lift per unit span. Let $L' = \frac{1}{2}\rho V^2 c C_l$ where ρ is the atmospheric density, 1.225 kg/m^3, c is the chord 0.3125 m, and V is the velocity, 20 m/s and b is the span of the wing, 2.04 m. Let c_l the lift coefficient be defined using the equation below where $C_{l0} = 1.0$

$$C_l = C_{l0}\sqrt{1-(2y/b)^2} \tag{166}$$

Use the trapezoidal rule to compute the total lift across the wing. Plot the lift as a function of span from -b/2 to b/2.

4. Write a loop that will compute the taylor series expansion for exp(x) from -2 to 2. How many orders does it take to get within 10% of the true value? How many orders does it take to get within 1% of the true value. Make plots of exp(x) along with your two fits. Then create plots of percent error between your two fits to prove that the taylor series expansion converges.

5. The solution to the integrand below is shown. Write a computer code that will use Simpson's 1/3 Rule to compute the integral (dt = 0.01). When transforming the integrand use (dx = 0.001). Furthermore, split the integral at t = 100. Compute the absolute error between the analytical solution and the numerical solution.

$$I = \int_0^{\infty} \frac{1}{t^2+1} dt = \pi/2 \tag{167}$$

6. Using Romberg integration, compute $I_{1,4}$ for the equation below starting with the Trapezoidal Rule. Let $n_1 = 10$. Compute the error for all estimates ($I_{j,k}$). Note in order to compute the error you will need to compute the solution to this equation analytically.

$$I = \int_{0}^{\pi} (5 + 3\ sin\ x)dx \tag{168}$$

7. Determine the distance traveled for the following data:

t (minutes)	0	1	2	3.25	4.5	6	7	8	9	9.5	10
v (m/s)	0	5	6	5.5	7	8.5	8	6	7	7	5

You may use a computer code to solve this or simply solve it by hand. It is up to you, however you must use the trapezoidal rule.

8. Integrate the following equation by hand. In addition, write a computer code to compute the integral with the Standard Reimmann Sum (dx = 0.01,dy = 0.01). Compute the absolute error between your analytical solution and your numerical solution.

$$\int_{-1}^{1}\int_{0}^{2} (x^2 - 2y^2 + xy^3)dx\ dy \tag{169}$$

Finally, create of Mesh of the equation above. That is, use the mesh() function in MATLAB and generate a plot that shows the surface from x = [-1,1] and y = [0,2]. Label all your axes and include the figure in your homework.

9. Consider the differential equation below:

$$\ddot{y} = -2\dot{y} - y \tag{170}$$

Euler's method is an iterative method that can be used to solve differential equations. The iterative equations are shown below.

$$\begin{aligned} \dot{y}_{n+1} &= \dot{y}_n + (-2\dot{y}_n - y_n)\Delta t \\ y_{n+1} &= y_n + \dot{y}_n \Delta t \\ t_{n+1} &= t_n + \Delta t \end{aligned} \tag{171}$$

Write a MATLAB code that will use Euler's method to compute y until t is equal to 10 seconds. The function header is shown below.

```
function myEuler(deltat)
```

Assume deltat is the timestep Δt. Let t(1) = 0, y(1) = 2 and $\dot{y}$(1) = -2; Run the function for smaller and smaller timesteps until your graph does not change. Put in your report what timestep you chose and why. Obviously include your final graph in your homework assignment.

7.E. Numerical Integration Methods - Chapter 25

1. **Truncation Error in Euler's Method**

The inherent problem associated with Euler's method is the assumption that the derivative is linear in between the interval Δt. That is, Euler's method assumes the slope $\dot{f}_1$ is constant over time 1 to 2.

$$f_2 \approx f_1 + \dot{f}_1 \Delta t \tag{172}$$

However, the taylor series expansion clearly states that f can be approximated as an expansion of multiple terms. The equation below is the taylor series expansion starting at $f(t_0)$.

$$f(t_1) \approx f(t_0) + \dot{f}(t_0)\Delta t + \frac{\ddot{f}(t_0)}{2!}\Delta t^2 + ... + \frac{f^{(N)}(t_0)}{N!}\Delta t^N \tag{173}$$

Notice that Euler's method is merely the first two terms. Thus we can write that the error accrued by Euler's method is

$$E_T = \frac{\ddot{f}(t_0)}{2!}\Delta t^2 + ... + \frac{f^{(N)}(t_0)}{N!}\Delta t^N \tag{174}$$

This type of error is called Truncation Error and is a direct result of truncating the higher order terms (HOTs) in the taylor series expansion.

2. **Taylor Series Approximation to Euler's Method**

   Note however, that it is extremely easy to include higher order terms in Euler's method. That is we can simply include three terms instead of just two and suddenly Euler's method becomes second order.

$$f(t_{i+1}) \approx f(t_i) + \dot{f}(t_i)\Delta t + \frac{\ddot{f}(t_i)}{2!}\Delta t^2 \tag{175}$$

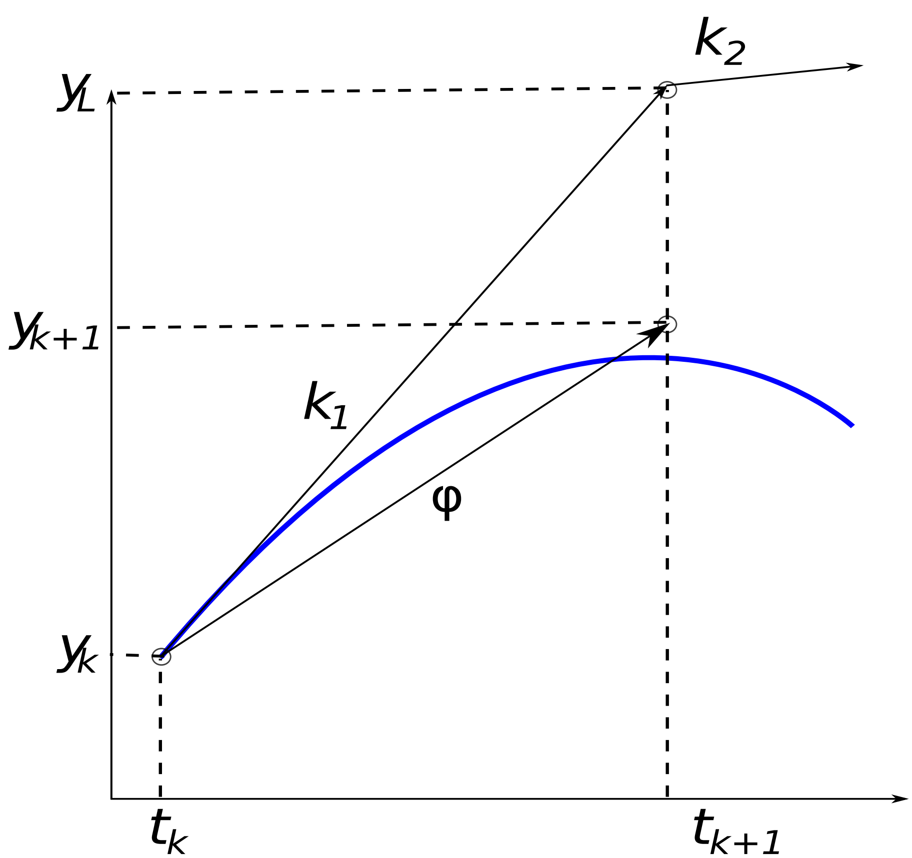
   The issue with this method is that the second derivative must be know. Often this derivative is not known and the method cannot be used. To mitigate this a class of higher order methods is typically used.

3. **Heun's Method**

   Heun's method is derived such that $p_1 = 1$, $q_{11} = 1$, and $a_1 = a_2 = 1/2$. Heun's method is then given using the equation below.

$$\begin{gathered} y_{k+1} = y_k + \phi\Delta t \\ \phi = \tfrac{1}{2}(k_1 + k_2) \\ k_1 = f(t_k, y_k) \\ k_2 = f(t_k + \Delta t, y_k + k_1\Delta t) \\ \dot{y} = f(t, y) \end{gathered} \tag{176}$$

   This can also be represented graphically as shown below.



4. **Runge-Kutta**

Runge-Kutta techniques are a class of higher order integration methods. They all have the same form using the equation below.

$$
\begin{gathered}
y_{k+1} = y_k + \phi \Delta t \\
\phi = a_1 k_1 + a_2 k_2 + ... + a_n k_n \\
k_1 = f(t_k, y_k) \\
k_2 = f(t_k + p_1 \Delta t, y_k + q_{11} k_1 \Delta t) \\
k_3 = f(t_k + p_2 \Delta t, y_k + q_{21} k_1 \Delta t + q_{22} k_2 \Delta t) \\
\vdots \\
k_n = f(t_k + p_n \Delta t, y_k + q_{n1} k_1 + ... q_{nn} k_{n-1}) \\
\dot{y} = f(t, y)
\end{gathered} \tag{177}
$$

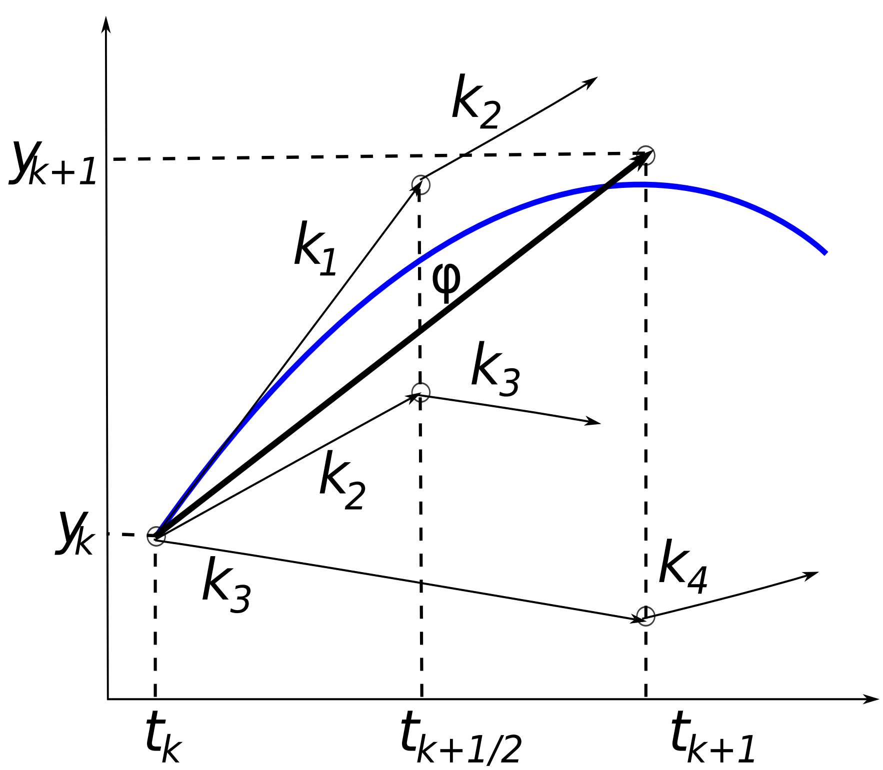
The coefficients a_n and q_n must be solved using the taylor series expansion. The number n is the order of the method. Thus if $n = 2$ the order of the method is quadratic and is called an RK2 technique.

5. **Runge-Kutta-4**

The Runge-Kutta-4 (RK4) algorithm is the standard in numerical simulations. This method uses 4 function calls and is computed using the equation below.

$$
\begin{gathered}
y_{k+1} = y_k + \phi \Delta t \\
\phi = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
k_1 = f(t_k, y_k) \\
k_2 = f(t_k + \Delta t/2, y_k + k_1 \Delta t/2) \\
k_3 = f(t_k + \Delta t/2, y_k + k_2 \Delta t/2) \\
k_4 = f(t_k + \Delta t, y_k + k_3 \Delta t) \\
\dot{y} = f(t, y)
\end{gathered} \tag{178}
$$

Again, the RK4 method can be represented graphically as shown below.



6. **Shower Problem**

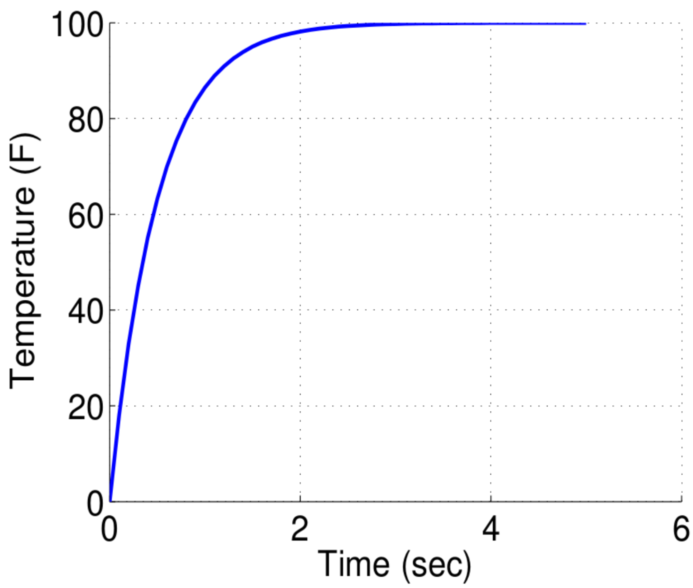
Let's return to our shower problem from the bi-section method except this time we want to model the rise in temperature as a function of time. Let's assume the problem is first order as shown below where

$$\dot{T} + \tau T = \tau T_c \tag{179}$$

where τ is a time constant and T_c is a commanded temperature. Again the solution to this equation is simple and can be determined using differential equation techniques to achieve the solution below

$$T(t) = T_0 e^{-\tau t} + T_c(1 - e^{-\tau t}) \tag{180}$$

The solution to this equation is shown below. Notice the rise time in the graph here which is dictated by τ.



Now what if we make the problem a bit more complex and let $T_c = k\theta$ where θ is the angle of the shower knob. This would mean our equations of motion become

$$\dot{T} + \tau T = \tau k\theta \tag{181}$$

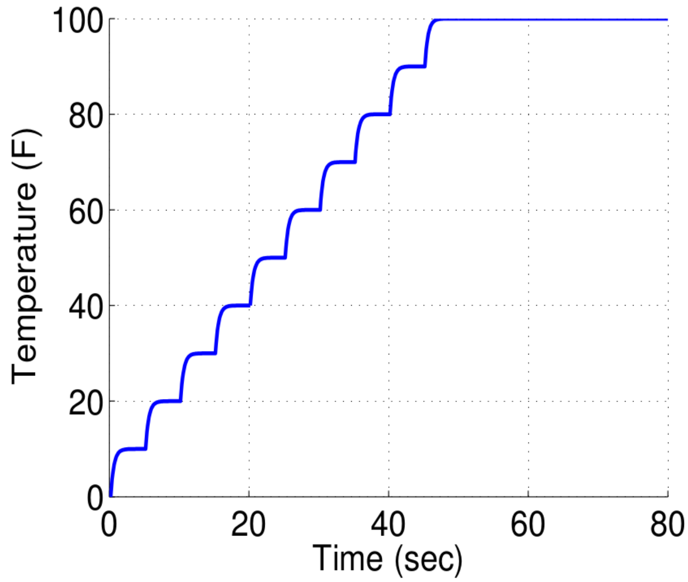
So what's θ? Well we would like to turn the knob bigger when the water is too cold and turn it down when the water is too hot. Something like this

$$\theta_{n+1} = \begin{Bmatrix} \theta_n + 2, T < T_c \\ \theta_n - 2, T > T_c \end{Bmatrix} \tag{182}$$

So if the water is too cold we turn the knob 2 degrees and if it's too hot we turn it down 2 degree. To make it more complicated we only check every 5 seconds and if the water is within 5 degrees we stop changing the temperature. So what does that analytic solution look like? Well this would require Euler's method or RK4. Let's do Euler's method first, let $\theta_0 = 0\ deg$, $k = 5\ F/deg$, $\tau = 2\ /sec$ and $T_0 = 0$ and $\Delta t = 0.1\ s$. Let, $T_c = 100\ F$.

$$\begin{gathered} \dot{T}_0 = \tau k\theta_0 - \tau T_0 \\ T_1 = T_0 + \dot{T}_0\Delta t \\ \dot{T}_1 = \tau k\theta_1 - \tau T_1 \\ T_2 = T_1 + \dot{T}_1\Delta t \\ \vdots \\ \dot{T}_N = \tau k\theta_N - \tau T_N \\ T_{N+1} = T_N + \dot{T}_N\Delta t \end{gathered} \tag{183}$$

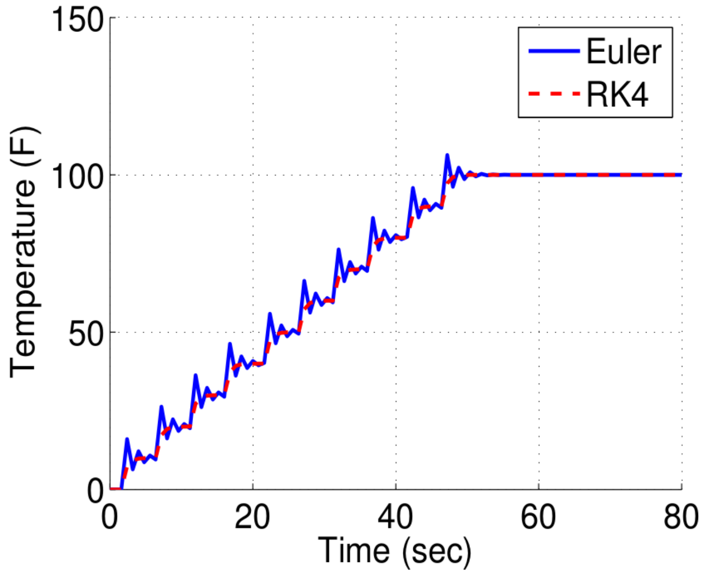
The code for this is a bit more complex but it's left out as an exercise. If the timestep is sufficiently small (0.1 seconds) Euler's method will converge to the solution and produce the result below.



Notice that every 5 seconds we turn up the temperature until we arrive at 100 degrees. You can see that this doesn't really have an analytical solution. We can then do the same thing for the Runge Kutta Method. Here our iterative method becomes

$$
\begin{gathered}
k_1 = \tau k\theta - \tau T_0 \\
k_2 = \tau k\theta - \tau(T_0 + k_1\Delta t/2) \\
k_3 = \tau k\theta - \tau(T_0 + k_2\Delta t/2) \\
k_4 = \tau k\theta - \tau(T_0 + k_3\Delta t) \\
\phi = (1/6)(k_1 + 2k_2 + 2k_3 + k_4) \\
T_1 = T_0 + \phi\Delta t \\
\vdots
\end{gathered}
\tag{184}
$$

The solution using the iterative method above is the same the difference is that we can increase the timestep and still get good results. The figure below shows the solution using Euler's and RK4 using a much larger timestep (0.8 seconds). Notice that RK4 still converges and Euler's method begins to have abnormal oscillations.



7. **Camera Tracker**

Assume you'd like to track a moving target. We will restrict ourselves to a ball being launched vertically such that $z(t) = z_0 + v_0 t - 0.5gt^2$. The camera is always pointed towards the object but requires an elevation angle command.

$$\theta_c = tan^{-1}(z/x_s) \tag{185}$$

where x_s is the horizontal distance from the camera and the ball. Let's let the dynamics of the elevation angle be determined using the equation below

$$\ddot{\theta} = \frac{T - c\dot{\theta}}{J} \tag{186}$$

where J is the moment of inertia of the camera and c is a friction coefficient which models the bearing the camera sits on. In order to track the object properly we need to apply proportional feedback control such that

$$T = -k_p(\theta - \theta_c) \tag{187}$$

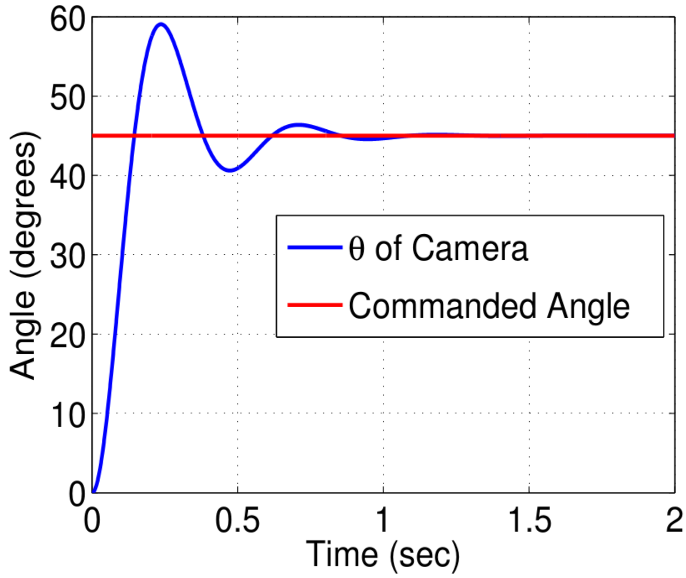
Thus our equation of motion becomes

$$\ddot{\theta} = \frac{-k_p(\theta - \theta_c) - c\dot{\theta}}{J} \tag{188}$$

Again, if θ_c is a constant the solution becomes

$$\theta(t) = (\alpha_1 cos(\omega t) + \alpha_2 sin(\omega t))e^{\sigma t} + \theta_C \tag{189}$$

where α_1 and α_2 are determined using initial conditions and $\sigma \pm \omega i$ is the solution to the characteristic equation when solving the differential equation. If $\theta_0 = \dot{\theta}_0 = z_0 = 0$, $v_0 = 20\ m/s$, $J = 0.1kg - m^2$, $k_p = 20$, $c = 1$, $x_s = 40\ m$, and $\theta_c = 45degrees$ the analytic solution is shown below.



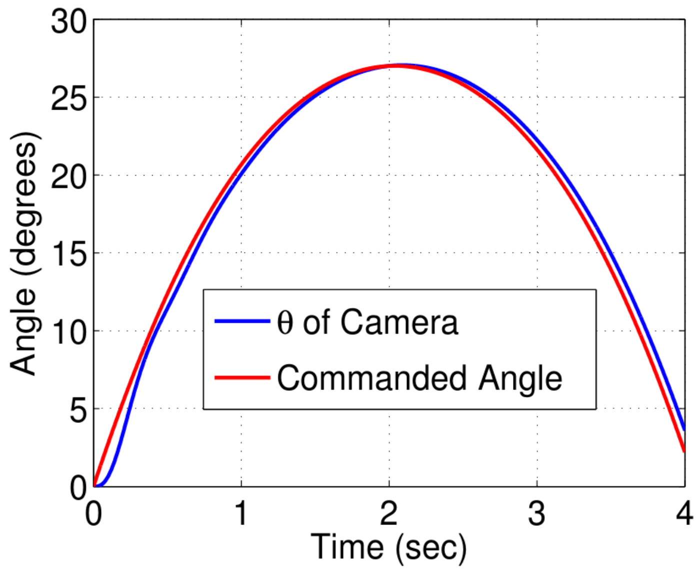
It looks like our camera over shoots by a bit but that dynamics can be tuned easily in our simulation to obtain the desired response. Now, if θ_C is not constant the solution is not analytic and the solution must be obtained numerically. The easiest way to solve this system of equations numerically is to put the equation in first order form. To do this we set $x_1 = \theta$ and $x_2 = \dot{\theta}$. Thus $\dot{x}_1 = \dot{\theta} = x_2$ and $\dot{x}_2 = \ddot{\theta} = -c\dot{\theta}/J - k_p(\theta - \theta_c)/J = -cx_2/J - k_p(x_1 - \theta_c)/J$. This can be put in matrix form to yield the equation below

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_p/J & -c/J \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ k_p/J \end{Bmatrix} \theta_c \tag{190}$$

This can be written in more compact form as $\vec{\dot{x}} = A\vec{x} + Bu$ where A and B are matrices. The great thing about this equation is that the solution to this equation if $u = 0$ is simply $\vec{x}(t) = e^{At}\vec{x}_0$ where $\vec{x}_0$ is the initial condition vector. If this matrix form is used for Euler's method the iterative sequence becomes

$$\begin{aligned} \dot{x}_0 &= Ax_0 + Bu_0 \\ x_1 &= x_0 + \dot{x_0}\Delta t \\ \dot{x}_1 &= Ax_1 + Bu_1 \\ x_2 &= x_1 + \dot{x_1}\Delta t \\ &\vdots \\ \dot{x}_N &= Ax_N + Bu_N \\ x_{N+1} &= x_N + \dot{x_N}\Delta t \end{aligned} \tag{191}$$

hence why the matrix form equation is so powerful. The A and B matrices are constant and thus the iterative method is extremely simple. If the iterative method is used to compute the solution where θ_c is not constant the solution can be seen in the figure below.



Notice the slight lag in the camera which is just a product of active feedback.

7.F. Example Problems

1. A hockey puck is sliding across the ice. The equations of motion are defined using the equation below.

$$\dot{v} + (c/m)v = 0 \tag{192}$$

   Solve the equation analytically and using Euler's method. Then plot the two equations side by side. Let c = 2, m = 1 and $v_0 = 5\ m/s$.

2. Let a parachute be falling from 500 meters. The equations of motion of a parachute can be simplified using the equations below

$$\dot{v} + (c/m)v = g \tag{193}$$

   Solve the equation analytically and using Euler's method. Then plot the two equations side by side. Let c = 2, m = 1 and $v_0 = 0\ m/s$ and of course $g = 9.81\ m/s^2$.

3. Code the Camera Tracking problem using the RK4 method.

4. The mathematical model of a spring mass damper is set up with a single degree of freedom. The mass can be modeled as a cube of mass "m", connected to a spring and damper. The distance from the wall is defined simply as "x". The equations of motion are second order and are written in terms of all parameters in the system and are given by the equation below.

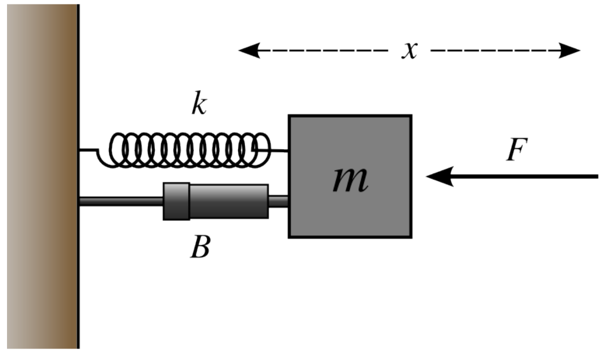
$$m\ddot{x} + B\dot{x} + kx = 0 \tag{194}$$

   It is possible to solve for the analytic solution for spring mass dampers. The general equation is given below.

$$x(t) = e^{(-\zeta w_n t)} \left(C sin(w_d t) + D cos(w_d t) \right) \tag{195}$$

   where

$$\begin{gathered} w_d = w_n\sqrt{1-\zeta^2} \qquad w_n = \sqrt{\tfrac{k}{m}} \qquad \zeta = \tfrac{B}{2mw_n} \\ C = \tfrac{\dot{x}_0 + \zeta w_n x_0}{w_d} \qquad D = x_0 \end{gathered} \tag{196}$$



(a) Plot the analytical solution for $x_0 = 5\ m$ and $\dot{x}_0 = 50 m/s$. Assume $m = 1\ kg$, $k = 100\ N/m$, and $B = 2\ N - s/m$. Simulate the system for 10 seconds.

(b) Use Euler's First Order Method, Heun's Method and Runge-Kutta-4 method to compute the numerical solution to the full expression in equation 194. Plot all equations on top of each other for a timestep of 0.1 (Remove Euler's Method from the plot if it does not converge). Do they all look the same? Why not? What if you make your timestep 0.01? What happens now?

(c) Compute the absolute error for all methods in problem 2 and plot the error on the same graph for a timestep of 0.1 (Removing Euler's Method if it does not converge) and a timestep of 0.01. Is the error the same for all methods? Which method is the best?

7.G. Parameter Estimation Problem

This assignment will dive down the realm of "parameter estimation" Your task is to measure the length of a string without using a ruler. To do this you will need to create a penduluum. I will provide supplies for you to build a pendulum. Once your pendulum is built you need to hold a protractor behind the pendulum and video tape the pendulum oscillating. You can also just hold a piece of paper behind the pendulum provided it contains angle lines on it.

Open the video in windows movie maker on your computer (you may have to download this) On a Mac you have numerous free video editing programs iMovie, Cheese, Handbrake, Blender. I'm not sure which one will be the best so we may just have to figure this out the day of the lab.

With the video open write down the time and angle as you parse through each frame in the video.

1. Create a plot of the angle vs time using the data you obtained from your experiment.

2. The equations of motion for a pendulum have been derived in class. Use RK4 to simulate your pendulum. The initial angle can be measured from your video, assume the initial angular velocity is zero and assume gravity is 9.81 m/s^2.

3. Your task then is to change L until your data and your graphs match up. Once you have converged on a value for L go measure your pendulum and comment on how close your estimated length was to your actual length.

YOU MAY AGAIN WORK WITH YOUR GROUP (MAX THREE MEMBERS) TO COMPLETE THE ASSIGNMENT. I recommend building the experiment together and sampling the data together. YOU MUST THEN WRITE YOUR OWN LAB REPORT ABOUT THE ASSIGNMENT.

7.H. Eigenvalue Problems - Chapter 27

1. **Error Propagation in Euler's Method**

   Recall that Euler's method is first order thus Euler's method assumes that the first derivative is linear in between timesteps. For high order systems, this is not necessarily the case. Let us return to the equation for f_2. In many systems the derivative is simply $\dot{f} = af$. Using this result the equation for f_2 simplifies to

$$f_2 \approx f_1(1 + a\Delta t) \tag{197}$$

Similarly

$$f_3 \approx f_2(1 + a\Delta t) \tag{198}$$

Substituting f_2 into this equation reduces f_3 to

$$f_3 \approx f_1(1 + a\Delta t)^2 \tag{199}$$

There is a pattern here.

$$f_N \approx f_1(1 + a\Delta t)^{N-1} \tag{200}$$

This equation is the stability criterion for Euler's method. It says that if $|1 + a\Delta t| > 1$ the system will blow up and tend to infinity. Thus, not only will the system not converge but it won't even be bounded. Thus care must be taken to choose a timestep small enough to ensure that this value is less than 1. Try simulating $\dot{v} = -2v$ to see the affect of the timestep.

2. **Application to Differential Equations**

   Euler's method has a unique property in that it convertes a continuous differential equation such as the one below

$$\ddot{y} + 2\dot{y} + 4y = 0 \tag{201}$$

into a discrete differential equation like the form below.

$$\begin{aligned} y_{n+1} &= y_n + \dot{y}_n \Delta t \\ \dot{y}_{n+1} &= \dot{y}_n + (-2\dot{y}_n - 4y)\Delta t \end{aligned} \tag{202}$$

We've done this problem a million times but what we haven't done is placed Euler's method into the following form.

$$\begin{Bmatrix} y_{n+1} \\ \dot{y}_{n+1} \end{Bmatrix} = \begin{bmatrix} 1 & \Delta t \\ -4\Delta t & (1 - 2\Delta t) \end{bmatrix} \begin{Bmatrix} y_n \\ \dot{y}_n \end{Bmatrix} \tag{203}$$

In this form it is possible to use vector algebra to compute the solution to the differential equation.

$$\vec{y}_{n+1} = A\vec{y}_n \tag{204}$$

the solution to the differential equation is simply

$$\vec{y}_k = A^k \vec{y}_0 \tag{205}$$

An interesting result is to compute the stability of Euler's method. In order to do that we decompose the matrix A into the eigenvalue, eigenvector form $A = V\Lambda V^{-1}$. It is easy to show that $A^k = V\Lambda^k V^{-1}$. Furthermore, the computation of Λ^k is

$$\Lambda^k = \begin{bmatrix} {\lambda_1}^k & 0 \\ 0 & {\lambda_2}^k \end{bmatrix} \tag{206}$$

What you should immediately notice is that if the eigenvalues of the matrix A are bigger than one, Euler's method will not converge.

3. **Solution to ODEs using Eigenvalues**

   It is possible to use matrices and eigenvalues to solve ODEs. Take for instance a penduluum with 1 degree of freedom. This system is second order as given by the equation below:

$$mL^2\ddot{\theta} + mgLsin(\theta) = 0 \tag{207}$$

In order to solve this equation you need two initial conditions $\theta(t = 0) = \pi/4$ and $\dot{\theta}(t = 0) = 0$. Once you have that information you can solve it. This equation unfortunately is non-linear but it can be linearized by assuming $sin(\theta) \approx \theta$ which yields

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \tag{208}$$

To solve this you assume that $\theta = ce^{st}$. Substituting this equation leads to the chracteristic equation where $s = \pm i\sqrt{g/L}$. This yields the following equation for $\theta(t)$.

$$\theta(t) = c_1 e^{i\sqrt{g/L}t} + c_2 e^{-i\sqrt{g/L}t} \tag{209}$$

If you open any physics textbook you'll see a slightly different solution for a pendulum. This is because introductory physics texts use Euler's formula

$$e^{ix} = cos(x) + isin(x) \tag{210}$$

Thus the equation for $\theta(t)$ can be simplified to

$$\begin{gathered} \theta(t) = c_1(cos(\sqrt{g/L}t) + isin(\sqrt{g/L}t)) + c_2(cos(\sqrt{g/L}t) - isin(\sqrt{g/L}t)) \\ \theta(t) = (c_1 + c_2)cos(\sqrt{g/L}t) + (c_1 - c_2)isin(\sqrt{g/L}t) \\ \theta(t) = Acos(\sqrt{g/L}t) + Bsin(\sqrt{g/L}t) \end{gathered} \tag{211}$$

Where $A = c_1 + c_2$ and $B = i(c_1 - c_2)$. The values of c_1 and c_2 are irrelevant and thus most textbooks will simply report the final equation for $\theta(t)$. In order to solve for A and B we use the initial conditions $\theta(t = 0) = \pi/4 = A$ thus $A = \pi/4$ since the $sin(0) = 0$. It is easy to check that $B = 0$ since $\dot{\theta}(t = 0) = 0$ and thus $\theta(t) = \frac{\pi}{4}cos(\sqrt{g/L}t)$. Using $g = 9.81 m/s^2$ and $L = 4.905m$ yields $\theta(t) = \frac{\pi}{4}cos(\sqrt{2}t)$.

It is possible to convert the system to matrix form by letting $x_1 = \theta$ and $x_2 = \dot{\theta}$ which leads to $\dot{x_1} = x_2$ and $\dot{x_2} = \ddot{\theta} = -g/L$. This can be put in matrix form as shown in the equation below:

$$\begin{Bmatrix} \dot{x_1} \\ \dot{x_2} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -g/L & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{212}$$

or in more compact form $\dot{\vec{x}} = A\vec{x}$. The solution to this equation can be obtained the same way as we did before. If we assume that the solution $\vec{x}(t) = e^{St}\vec{c}$ where S is a matrix of unknowns and $\vec{c}$ is a vector of constants we can plug this into the equation above noting that $\dot{\vec{x}}(t) = Se^{St}\vec{c}$.

$$\begin{gathered} \dot{\vec{x}} - A\vec{x} = 0 \\ Se^{St}\vec{c} - Ae^{St}\vec{c} = 0 \\ e^{St}(S - A)\vec{c} = 0 \end{gathered} \tag{213}$$

The last equation leads to a few results. First we could have $e^{St} = 0$ or $\vec{c} = 0$ but this would leave to a solution of the form $\vec{x}(t) = 0$ thus the only solution is that $S = A$ which leads to the general form of differential equations $\vec{x}(t) = e^{At}\vec{c}$. The coefficients in $\vec{c}$ are then found by using the initial conditions $\vec{x}(t = 0) = \vec{x}_0 = e^0\vec{c}$ and thus $\vec{x}_0 = \vec{c}$.

$$\vec{x}(t) = e^{At}\vec{x}_0 \tag{214}$$

In the pendulum example problem we have $\vec{x}_0 = [\theta_0, \dot{\theta}_0]^T$. The question then becomes, how do you compute an exponential of a matrix? The easiest way to compute this is to decompose A into it's eigenvalue form where $A = V\Lambda V^{-1}$. Plugging this into our equation for $\vec{x}(t)$ yields

$$\begin{gathered} \vec{x}(t) = e^{V\Lambda V^{-1}t}\vec{x}_0 \\ \vec{x}(t) = Ve^{\Lambda t}V^{-1}\vec{x}_0 \end{gathered} \tag{215}$$

This relationship can be derived by noting that V is invertible and Λ is diagonal. Remember that Λ has the following form

$$\Lambda = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_N \end{bmatrix} \tag{216}$$

thus $e^{\Lambda t}$ can be written like

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_N t} \end{bmatrix} \tag{217}$$

If we then write $V = [\vec{v}_1 \ \dots \ \vec{v}_N]$ and $V^{-1}\vec{x}_0 = [a_1 \ \dots \ a_N]^T$ we can write $\vec{x}(t)$ in the following form

$$\vec{x}(t) = [\vec{v}_1 \ \dots \ \vec{v}_N] \begin{bmatrix} e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_N t} \end{bmatrix} \begin{Bmatrix} a_1 \\ \vdots \\ a_N \end{Bmatrix} = [\vec{v}_1 \ \dots \ \vec{v}_N] \begin{Bmatrix} a_1 e^{\lambda_1 t} \\ \vdots \\ a_N e^{\lambda_N t} \end{Bmatrix} \tag{218}$$

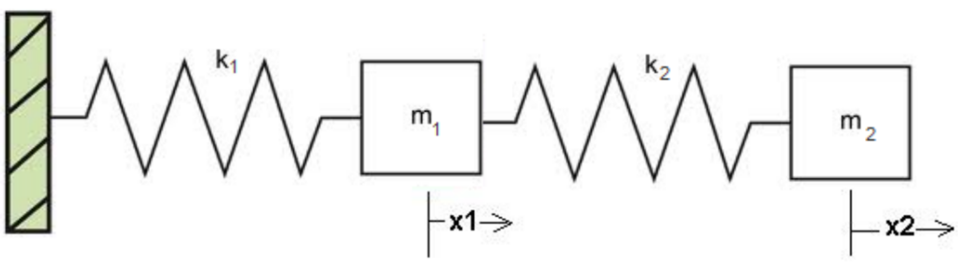
Carrying out the last matrix multiplication leads to a very powerful result as given by the equation below.

$$\vec{x}(t) = a_1 \vec{v}_1 e^{\lambda_1 t} + \dots + a_N \vec{v}_N e^{\lambda_N t} = \sum_{n=1}^{N} a_n \vec{v}_n e^{\lambda_n t} \tag{219}$$

This result says that a general solution to a differential equation is a summation of a dynamic systems individual mode shapes given by $e^{\lambda_n t}$.

4. **Example Eigenvalue Solution**

An example of this can be done by analyzing the double spring mass damper system as shown in the figure below.



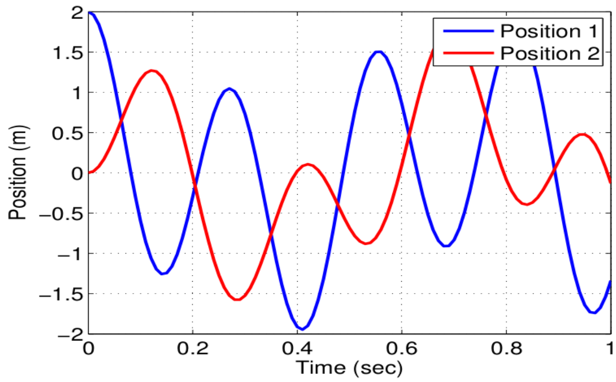
Here we have two degrees of freedom that is x_1 and x_2. The velocities of the masses are then $\dot{x_1}$ and $\dot{x_2}$. The accelerations are found by doing a free body diagram which results in the equations below. The derivation of the equations below are left as an exercise to the reader.

$$\begin{Bmatrix} \dot{x_1} \\ \ddot{x_1} \\ \dot{x_2} \\ \ddot{x_2} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (-k_1 - k_2)/m_1 & 0 & k_2/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k_2/m_2 & 0 & -k_2/m_2 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ \dot{x_1} \\ x_2 \\ \dot{x_2} \end{Bmatrix} \tag{220}$$

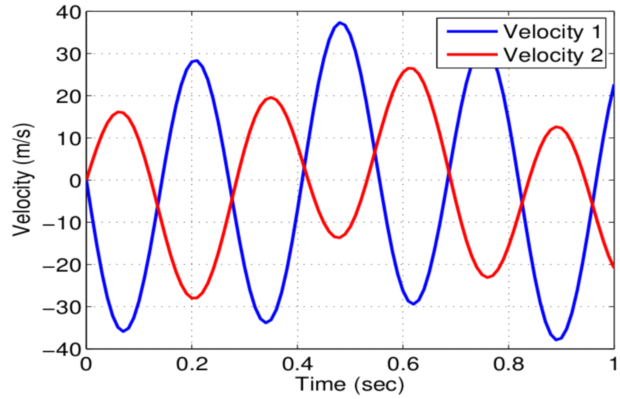
This can easily be put into the form $\dot{\vec{x}} = A\vec{x}$. The solution to this equation is then simply $\vec{x}(t) = e^{At}\vec{x}_0$. Using values $k_1 = k_2 = 200N/m$ and $m_1 = m_2 = 1kg$ along with initial conditions $\vec{x}_0 = [2, 0, 0, 0]$, the solution can be plotted using the MATLAB programming language.

The basic code required to plot the solution is shown below.

```
x0 = [2;0;0;0];
k1 = 200;k2 = 200;
m1 = 1;m2 = 1;
A = [0 1 0 0;(-k1-k2)/m1 0 k2/m1 0;0 0 0 1;k2/m2 0 -k2/m2 0];
```



Position (m) vs Time (sec)



Velocity (m/s) vs Time (sec)

Position (m) vs Time (sec)

Velocity (m/s) vs Time (sec)

```
t = 0:0.01:1;
x = zeros(4,length(t));
for idx = 1:length(t)
    x(:,idx) = expm(A.*t(idx))*x0;
end
plot(t,x)
```

This problem can also be solved using the eigenvalue solution. Since there are four mode shapes the solution is given by

$$\vec{x}(t) = a_1\vec{v}_1 e^{\lambda_1 t} + a_2\vec{v}_2 e^{\lambda_2 t} + a_3\vec{v}_3 e^{\lambda_3 t} + a_4\vec{v}_4 e^{\lambda_4 t} \tag{221}$$

The solution to obtaining the eigenvalues and eigenvectors for a 4x4 matrix by hand is beyond the scope of this course. Thus a numerical solver will be used to obtain the eigenvalues and eigenvectors. Using the parameters previously defined our A matrix is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -400 & 0 & 200 & 0 \\ 0 & 0 & 0 & 1 \\ 200 & 0 & -200 & 0 \end{bmatrix} \tag{222}$$

Using the function $[V, L] = eig(A)$ in MATLAB produces the following result

$$L = \begin{bmatrix} 22.9i & 0 & 0 & 0 \\ 0 & -22.9i & 0 & 0 \\ 0 & 0 & 8.74i & 0 \\ 0 & 0 & 0 & -8.74i \end{bmatrix} \tag{223}$$

where the diagonal components are the four eigenvalues. Notice that the eigenvalues are actually complex eigenvalues which means there are actually only two mode shapes and four eigenvalues. The two sets of mode shapes are complex conjugates of each other. The $eig()$ function in MATLAB also produces the eigenvectors

$$V = \begin{bmatrix} -0.04i & 0.04 & -0.06i & 0.06i \\ 0.85 & 0.85 & 0.52 & 0.52 \\ 0.02i & -0.02i & -0.10i & 0.10 \\ -0.53 & -0.53 & 0.85 & 0.85 \end{bmatrix} \tag{224}$$

where each column of the V matrix is an eigenvector of the system. Note again that the real components of columns 1 and 2 are the same while the imaginary components are different. The same is true with columns 3 and 4. The a coefficients are solved by using the formula $\vec{a} = V^{-1}\vec{x}_0$. This can also be solved by simply using a numerical solver.

$$\vec{a} = \begin{Bmatrix} 19.5i \\ -19.5i \\ 4.63i \\ -4.63i \end{Bmatrix} \tag{225}$$

Using these coefficients, eigenvalues and eigenvectors it is possible to write the solution of the system.

$$\vec{x}(t) = 19.5i \begin{Bmatrix} -0.04i \\ 0.85 \\ 0.02i \\ -0.53 \end{Bmatrix} e^{22.9it} - 19.5i \begin{Bmatrix} 0.04i \\ 0.85 \\ -0.02i \\ -0.53 \end{Bmatrix} e^{-22.9it} + 4.63i \begin{Bmatrix} -0.06i \\ 0.52 \\ -0.10i \\ 0.85 \end{Bmatrix} e^{8.74it} - 4.63i \begin{Bmatrix} 0.06i \\ 0.52 \\ 0.10i \\ 0.85 \end{Bmatrix} e^{-8.74it} \tag{226}$$

Remember that if Euler's formula is used the two exponent's will be combined to produce *sines* and *cosines*. Still, this result is very powerful because it tells you the two fundamental frequencies associated with this system. This solution can also be implemented easily in a numerical program such as MATLAB.

```
[V,L] = eig(A);
a = inv(V)*x0;
xEigen = zeros(4,length(t));
for idx = 1:length(t)
    for n = 1:4
        xEigen(:,idx) = xEigen(:,idx) + a(n)*V(:,n)*exp(L(n,n)*t(idx));
    end
end
plot(t,xEigen)
```

7.I. Example Problems

1. Euler's method has a unique property in that it convertes a continuous differential equation such as the one below

$$\ddot{y} + 2\dot{y} + 4y = 0 \tag{227}$$

into a discrete differential equation like the form below.

$$\begin{aligned} y_{n+1} &= y_n + \dot{y}_n \Delta t \\ \dot{y}_{n+1} &= \dot{y}_n + (-2\dot{y}_n - 4y)\Delta t \end{aligned} \tag{228}$$

We've done this problem a million times but what we haven't done is placed Euler's method into the following form.

$$\begin{Bmatrix} y_{n+1} \\ \dot{y}_{n+1} \end{Bmatrix} = \begin{bmatrix} 1 & \Delta t \\ -4\Delta t & (1 - 2\Delta t) \end{bmatrix} \begin{Bmatrix} y_n \\ \dot{y}_n \end{Bmatrix} \tag{229}$$

In this form it is possible to use vector algebra to compute the solution to the differential equation.

$$\vec{y}_{n+1} = A\vec{y}_n \tag{230}$$

First, take your function from before spring break

```
function myEuler(deltat)
```

and edit it to use the matrix form of Euler's method. Do the same thing you did last time where you vary the timestep until the graph does not change. However, this time compute the eigenvalues of the matrix A. What do you notice about eigenvalues of the matrix as the timestep gets smaller? What happens to the eigenvalues when the timestep is too big and the graph goes unstable?

2. Simulate the multi body system derived in this chapter. Use the following data below.

   m1 = 2;
   m2 = 3;
   k1 = 50;
   k2 = 100;
   x1(t=0) = 5;
   xdot1(t=0) = 0;
   x2(t=0) = 10;
   xdot2(t=0) = 0;

   You will create the solution to the differential equation using three different methods.

   (a) RK4
   (b) z(t) = expm(A*t)*z0 - The analytical solution
   (c) z(t) = V*expm(L*t)*inv(V)*z0 - Eigenvalue solution

   where V is the eigenvalues and L is the eigenvectors. You can use the eig function

   For plotting, plot all velocities (xdot1 and xdot2) for all three solutions on the same graph. All lines should be on top of each other.

   In addition plot all positions (x1 and x2) for all three solutions on the same graph. Again, all lines should match.

   LIST THE EIGENVALUES OF THE A MATRIX IN YOUR REPORT. Explain what the eigenvalues mean in your own words. How many eigenvalues are there? Why are there so many? How many degrees of freedom does your system have?

7.J. Numerical Differentiation - Chapter 23

1. **Numerical Differentiation**

   Recall that the taylor series expansion for $f(x)$ is

$$f(x_1) = f(x_0) + f'(x_0)\Delta x + f''(x_0)\Delta x^2/2! + ... + O(\Delta^3) \tag{231}$$

   If the equation above is truncated to first order it is possible to solve for $f'(x_0)$

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{\Delta x} \tag{232}$$

   The equation above is known as the first order approximation to the first derivative. There are other ways to compute the first derivative. The equations below are known as backward differencing and midpoint differencing.

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{\Delta x} \tag{233}$$

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} \tag{234}$$

   Note that the equations above only work for the first derivative. Assume however I have a function f(x) and 3 data points x_0, x_1, x_2. How would one obtain $f''(x_1)$ using finite differencing. First assume that $\Delta x = x_2 - x_1 = x_1 - x_0$ to make the math easier so $x_2 - x_0 = 2\Delta x$. First we start by writing $f(x_2)$ using a 1st order expansion about x_1.

$$f(x_2) = f(x_1) + f'(x_1)\Delta x \tag{235}$$

Then using that equation the first order derivative is derived as shown below

$$f'(x_1) \approx \frac{f(x_2) - f(x_1)}{\Delta x} \tag{236}$$

The next step involves estimating $f(x_0)$ using x_1 as the expansion point.

$$f(x_0) = f(x_1) - f'(x_1)\Delta x \tag{237}$$

Note the minus sign. This equation can be used to get the backward differencing equations.

$$f'(x_1) \approx \frac{f(x_1) - f(x_0)}{\Delta x} \tag{238}$$

Using the forward and backward finite differencing equations the midpoint formula can be derived as shown

$$f'(x_1) \approx \frac{f(x_2) - f(x_0)}{2\Delta x} \tag{239}$$

Then, we estimate $f(x_2)$ using a second order expansion about x_1.

$$f(x_2) = f(x_1) + f'(x_1)\Delta x + \frac{f''(x_1)}{2!}\Delta x^2 \tag{240}$$

Using the equation above, substituting in the midpoint differencing formula and solving for $f''(x_1)$ yields the equation below.

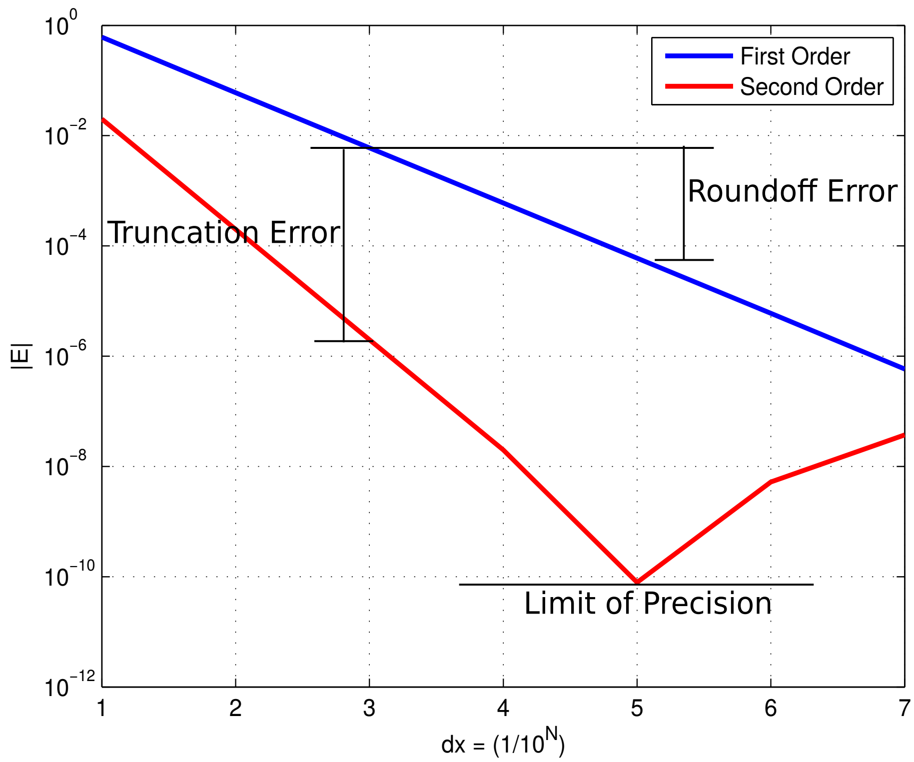
$$f''(x_1) \approx \frac{f(x_2) - 2f(x_1) + f(x_0)}{\Delta x^2} \tag{241}$$

2. **Higher Order Differentiation**

The equations above were simply first order approximations to the first and second derivatives. It is possible to truncate the taylor series to second order and solve for the first derivative. Thus the second order approximation to the first derivative is

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2\Delta x} \tag{242}$$

Note that these methods show a perfect example of truncation error and roundoff error. Assume for the moment that we would like to differentiate (x^3) at $x = 2$. It is easy to see that the analytical solution is just 12. However if we use the first and second order approximations and plot the error as a function of our increment we get the following graph.



Notice here that we can directly see that increasing our timestep reduces our round-off error. We can reduce our truncation error by using a higher and higher order method however we will always hit our limit of precision.

3. **Bicycle Sensor**

Let's say I'd like to know my speed and distance while riding a bike. The easiest thing to do would be to install a sensor on the fork and the tire. When the sensor on the fork is in sync the main controller saves the time this happens in the form

$$T_{sync} = [T_1 \dots T_N] \tag{243}$$

The task is then to extract velocity and distance from this equation. The first order derivative of velocity can be written in the form

$$V_i = \frac{x_{i+1} - x_i}{T_{i+1} - T_i} = \frac{\Delta x}{T_{i+1} - T_i} \tag{244}$$

The question is what is Δx? Δx is the distance the wheel travels during one click which is simply the circumference of the tire so we arrive at the equation below.

$$V_i = \frac{2\pi r}{T_{i+1} - T_i} \tag{245}$$

To obtain the distance travelled we can apply the standard Reimmann Sum to the velocity equation above we obtain

$$x_N = x_0 + \sum_{i=0}^{N} V_i \Delta T_i = \sum_{i=0}^{N} \frac{2\pi r}{T_{i+1} - T_i} \Delta T_i = \sum_{i=0}^{N} 2\pi r \tag{246}$$

where N is the number of times the sensor passes the main fork. Notice now that the sum does not depend on i thus the distance is simply

$$x_N = x_0 + N2\pi r \tag{247}$$

Which means we now have a method to determine velocity and distance traveled. As an example, try and verify the velocity and distance travelled using the timestamps below. Assume you're riding 14 inch tires.

$$\begin{aligned} T &= [0, 1.1320, 4.1921, 6.4561, 8.3252, 10.0598]\ sec \\ V &= [6.4758, 2.3955, 3.2377, 3.9219, 4.2260]\ ft/s \\ X &= [0, 7.3304, 14.6608, 21.9911, 29.3215, 36.6519]\ ft \end{aligned} \tag{248}$$

4. **Problems with Derivatives**

Let's say I have a GPS sensor on an airplane traveling 800 ft/s. This GPS sensor gives me two measurements. First the position in feet (Not really but we can just assume this for classroom example purposes).

$$X = [x_1 \dots x_N] \tag{249}$$

The GPS is also timestamped and thus returns the time coordinate

$$T = [t_1 \dots t_N] \tag{250}$$

This seems like a simple problem. Just differentiate it and get

$$V_i = \frac{x_{i+1} - x_i}{t_{i+1} - t_i} \tag{251}$$

The problem is real sensors have noise. So I can't actually know what X is really and instead I obtain $\hat{X}$ which is my sampled measurement or output from the GPS sensor. Let's assume that my GPS is accurate to 25 ft. I can model this as:

$$\hat{X} = X + N(0, 25) \tag{252}$$

where $N(0, 25)$ is a random uniform number with a mean of zero and a standard deviation of 25 ft. If we return to our numerical derivative we obtain

$$\hat{V}_i = \frac{\hat{x}_{i+1} - \hat{x}_i}{t_{i+1} - t_i} \tag{253}$$

Ok so how bad is this? Let's compute the error between V_i and $\hat{V}_i$.

$$V_i - \hat{V}_i = \frac{\hat{x}_{i+1} - \hat{x}_i}{t_{i+1} - t_i} - \frac{x_{i+1} - x_i}{t_{i+1} - t_i} = \frac{\hat{x}_{i+1} - \hat{x}_i - x_{i+1} + x_i}{t_{i+1} - t_i} \tag{254}$$

If we substitute in equation 252 we obtain

$$V_i - \hat{V}_i = \frac{x_{i+1} + N_{i+1} - x_i - N_i - x_{i+1} + x_i}{t_{i+1} - t_i} = \frac{N_{i+1} - N_i}{t_{i+1} - t_i} \tag{255}$$

Ok so let's substitute real number in here and obtain the maximum error remembering that GPS updates at 4 Hz = 0.25 sec

$$|V_i - \hat{V}_i|_{max} = \frac{|N_{i+1} - N_i|_{max}}{0.25sec} \tag{256}$$

So what is $|N_{i+1} - N_i|_{max}$. Well, the maximum of N_i is 25 ft which means that max of $|N_{i+1} - N_i|_{max} = 25\ ft$ which leads to

$$|V_i - \hat{V}_i|_{max} = \frac{25\ ft}{0.25\ sec} = 100ft/s \tag{257}$$

Remember, we were flying at 800 ft/s and we could be off by as much as 100 ft/s! That's over 12.5%. Notice however that, the velocity error is independent of flight speed so what would happen if we were riding a bike and our speed was say 10 ft/s? This would mean our velocity measurement was off by an order of magnitude.

5. **Filters**

We've shown in the previous derivation that derivatives introduce alot of noise in the measurement. So what do we do? The easiest thing to implement is what's called a complimentary filter. It is a simplification of the Kalman Filter created by Rudolf Kalman in 1960. The basic derivation goes like this, assume the input signal is $\hat{y}$ and the output filtered signal is $\tilde{y}$. This is the setup for all types of filters. In this setup we use the equation below

$$\tilde{y}_{i+1} = \sigma\hat{y}_{i+1} + (1-\sigma)\tilde{y}_i \tag{258}$$

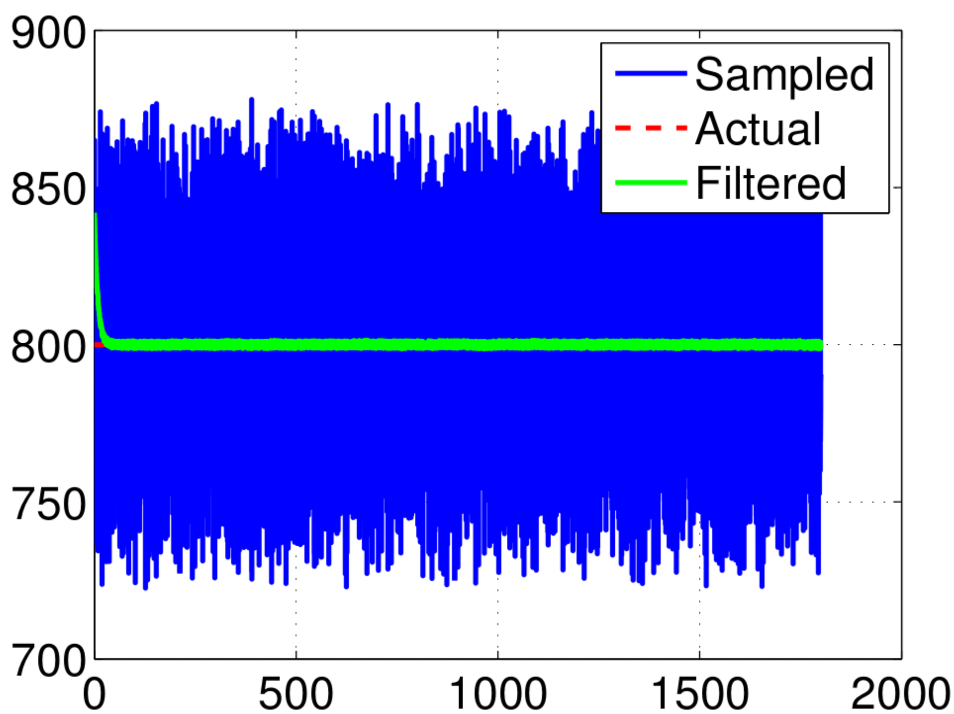
This may seem simple and it is but it's pretty powerful. Remember that the system is iterative so

$$\begin{aligned}\tilde{y}_1 &= \sigma\hat{y}_1 + (1-\sigma)\tilde{y}_0 \\ \tilde{y}_2 &= \sigma\hat{y}_2 + (1-\sigma)\tilde{y}_1 \\ \tilde{y}_3 &= \sigma\hat{y}_3 + (1-\sigma)\tilde{y}_2 \\ &\vdots \\ \tilde{y}_N &= \sigma\hat{y}_N + (1-\sigma)\tilde{y}_{N-1}\end{aligned} \tag{259}$$

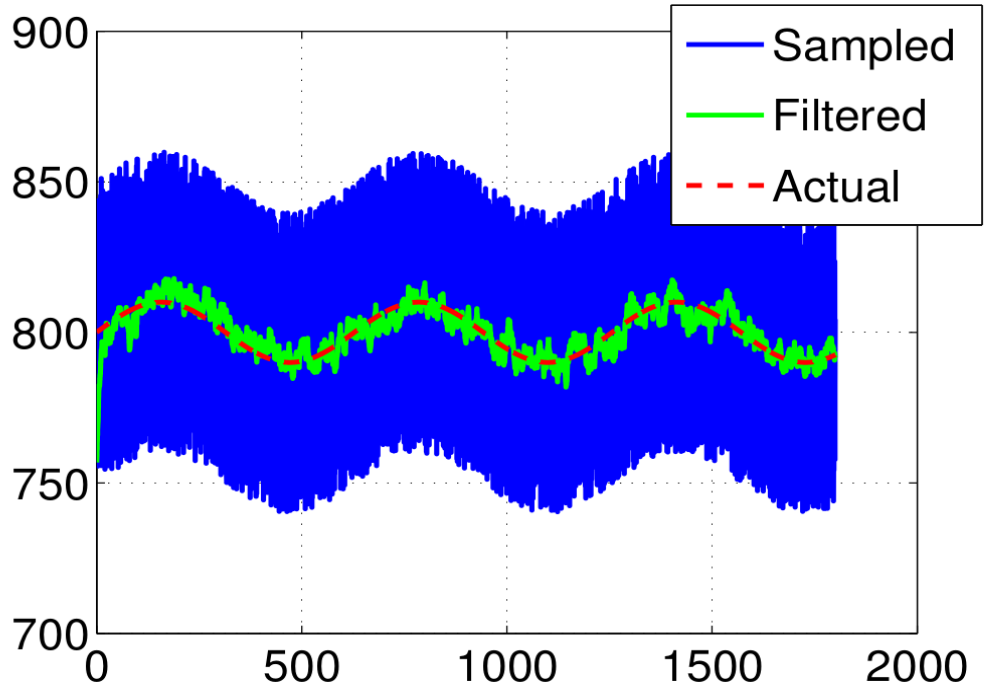
Notice that in this equation $\tilde{y}_0$ cannot be defined thus a separate equation is used for the initial condition.

$$\tilde{y}_0 = \hat{y}_0 \tag{260}$$

With this setup we can now show some examples. Let's assume that $\sigma = 1$. In this scenario $\tilde{y}_i = \hat{y}_i$ which means the signal is not filtered. If $\sigma = 0$, $\tilde{y}_{i+1} = \tilde{y}_i$. Since $\tilde{y}_0 = \hat{y}_0$ it means that that $\tilde{y}_i = \hat{y}_0$. However, if we return to the example where we have our GPS sensor on board and set $\sigma = 0.03$ we obtain the following result. The other thing we can do of course is simply average the result.



Now averaging the result would be easy when the speed is constant but airplanes rarely fly at constant speeds. What if the speed was more sinusoidal reflecting the changes in flight speed from atmospheric winds.



In the graph you can see that averaging the result does not simply capture the changes in the velocity of the airplane hence why the complimentary filter is so useful.

7.K. Driving Lab Experiment

Your task is to team with two other lab partners and go for a ride around the town or get some coffee or just run an errand. Here are the rules.

1. The driver is responsible for one thing and that is driving. Do not take your eyes off the road or text while driving. You must be safe at all times and get all of your passengers to and from your destination safely. For this assignment the driver must own a vehicle with a functioning odometer and speedometer. I would recommend resetting your trip meter. If your car does not have a trip meter you must write down the initial mileage of your vehicle before you depart.

2. Passenger number 1. Your task is to read off the drivers speed every 10 seconds on the departure leg and every 30 seconds on the return leg. I would bring a pencil and paper or your computer with excel open and write down the time in intervals of 10 seconds on the departure leg and 30 seconds for the return leg. While the car is driving you can then write down the speed by looking at the speedometer and writing the speed in mph. I would recommend converting mph to miles/second so that your units will match up.

3. Passenger number 2 is responsible for measuring the distance traveled in units of 0.1 miles using the odometer. I would recommend synchronizing watches with passenger number 1 and passenger number 2 so that your time vectors line up. Your table should have time in one column and position in increments of 0.1 miles in another column.

Once all of this data is compiled you need to generate the following plots.

1. Velocity versus time

2. Position versus time

3. Use first order differentiation and compute velocity versus time using part b. Plot this alongside part a and compare your results. For the down leg and the return leg. Which is more accurate?

4. Use the trapezoidal rule to compute position versus time using part a and plot it alongside part b. Compare your results with the down leg and the return leg. Which is more accurate?

   NOTE ALL THREE OF YOU MUST COMPLETE THE ASSIGNMENT. YOU MAY TAKE DATA AND HAVE FUN TOGETHER BUT EVERYONE MUST STILL COMPLETE THE LAB.

7.L. Output Error Method

1. **Output Error Method**

Newton-Raphson works great when you have one output and one variable trying to be determined using a direct correlation. For example, let's assume I place an aircraft in a wind tunnel and I vary the wind tunnel speed while changing the angle of attack such that Lift is equal to Weight. I have a load sensor on board that measures Drag such that I can plot Drag as a function of velocity. Explicitly I would have $D = f(V)$. Using this function, I could compute the drag coefficient using the simple formula.

$$D_i = \frac{1}{2}\rho V_i^2 S C_{Di} \tag{261}$$

The subscript i is in the formula above because I have multiple velocity and drag readings. The method defined above is typical for operators in a wind tunnel. Using the drag as a function of velocity you can obtain the drag coefficient as a function of velocity. Using the drag coefficient I could also find the speed for minimum drag using the standard Newton-Raphson technique.

$$V_{i+1} = V_i - \frac{f'(V_i)}{f''(V_i)} \tag{262}$$

The issue is that you can only obtain the drag coefficient and nothing else using a wind tunnel. The reason for this is that the aircraft in the wind tunnel is static and you can only measure certain coefficients at a time. For example, you could measure lift as well and estimate the lift coefficient but let's assume you want damping coefficients. To do this you need to use the output error method. Let's assume instead that I actually fly the airplane and instead of measuring velocity and drag, I measure time and distance. Thus I have $x = f(t)$. Here, I want to find the drag coefficient but notice that I do not have have the drag anymore. Thus, I can't use Newton-Raphson technique explicitly but with some massaging I can cast the problem into a better form. First if we assume the aircraft flies in one dimension I can write the equations of motion of the aircraft to be

$$\ddot{\tilde{x}} = -\frac{1}{2m}\rho\dot{\tilde{x}}^2 S\tilde{C}_D \tag{263}$$

Notice, here that suddenly our drag coefficient pops up. What we can do then is create an initial guess for our drag coefficient, then use a numerical integration technique to get an estimate for x. Using this we can actually create an error estimate as a function of time.

$$\tilde{E}(t) = x(t) - \tilde{x}(t) \tag{264}$$

from here we can create an average error square estimate such that

$$J = \frac{1}{N}\sum_{i=1}^{N} |\tilde{E}(t_i)|^2 \tag{265}$$

Let's recap all we've done. We now have an error estimate J which is a function of our drag estimate $\tilde{C}_D$. Formally we have $J = f(\tilde{C}_D)$. Typically J is called a cost function. If we choose $\tilde{C}_D$ that is the same as the actual value of C_D then we would have a cost of 0. This is our ideal situation. It might be tempting to simply use a root finding method but in fact we actually want to minimize J. Often times we cannot reach a zero cost given modeling errors, truncation errors, sensor errors, and round off errors. However we can still estimate C_D using the recursive algorithm as defined in the Newton-Raphson method.

$$\tilde{C}_{D,i+1} = \tilde{C}_{D,i} - \frac{J'(\tilde{C}_{D,i})}{J''(\tilde{C}_{D,i})} \tag{266}$$

Note that the first and second derivatives of the cost function must be computed numerically. This is a very involved process with multiple parts from this course. This entire method is called the Output Error Method.

2. **Example**

This assignment will involve the students splitting into groups to write codes that perform the Output Error Method. The Output Error Method is a way to solve for unknowns where there is no analytical solution available. For example, let us examine the parachute in free fall. The equations of motion are

$$\ddot{z} = -g + \frac{1}{2m}\rho \dot{z}^2 S C_D$$

This equation can be simulated on a modest computer assuming all parameters are known. Assume for the moment that the data $z(t)$ has been given in a table format. Then assume that the initial conditions and all parameters except C_D are given:

$$z_0 = 100 \text{ m} \quad \dot{z}_0 = 0 \text{ m/s} \quad \text{g} = 9.81\ m/s^2 \quad \rho = 1.225\ kg/m^3 \quad \text{S} = 1\ m^2 \quad \text{m} = 10 \text{ kg}$$

The question then is how to solve for the drag coefficient C_D. To do this the output error method is used.

(a) The method starts with an initial guess $\tilde{C_D}$.

(b) This initial guess can be used to obtain $\tilde{z}$.

(c) The total error between the computed value $\tilde{z}$ and the measured data from the input file z can be computed using the equation below.

$$E = \frac{1}{N}\sum_{i=1}^{N}(z(t_i) - \tilde{z}(t_i))^2 \tag{267}$$

The goal then naturally is to perform an optimization technique and compute $\tilde{C_D}$ such that the error drops to zero or at least a minimum.

(d) To minimize this problem the Newton-Raphson technique is used such that

$$\tilde{C}_{D,i+1} = \tilde{C}_{D,i} - \frac{E'(\tilde{C}_{D,i})}{E''(\tilde{C}_{D,i})} \tag{268}$$

Obviously the first and second derivatives will need to be computed numerically.

To solve this problem the students will break into groups of 6 or 7. Each member of the group will have to complete one of the codes below. Built in MATLAB functions are allowed unless noted specifically in the text. Note that the group may opt to work as a team on certain sections of the code.

(a) function [t,z] = readdata(filename), this function will use the dlmread() command to read the text file of data provided and return the t vector, z vector and the timestep used to generate the data.

(b) function ploteverything(t,z,ztilde), this function will take a vector t, z and ztilde and plot everything in a nice pretty graph

(c) function zdot = Derivs(t,z,CDtilde), this function will compute the derivative of z for a given value of CDtilde. Remember that z will be a vector containing z and $\dot{z}$ thus zdot will be a vector containing $\dot{z}$ and $\ddot{z}$

(d) function [ttilde,ztilde] = RK4(CDtilde), this function will take a value of the drag coefficient and use the RK4 method to compute the height of the parachute as a function of time. Note that you need to make sure you use the same timestep as the one used in the data set provided. You may not use the ode45() function.

(e) function E = compute_error(CDtilde,z), this function will compute the total error between the estimated height ztilde and the actual height z.

(f) function fprime = Eprime(CDtilde,z), this function will compute the first derivative of the error function at the value of CDtilde. Remember that you will have to choose the value of Δ.

(g) function fdblprime = Edoubleprime(CDtilde,z), this function will compute the second derivative of the error function at the value of CDtilde. Again you must choose Δ

(h) function main(), this function will perform the optimization. I have written this part of the code for you.

```
function main()
filename = 'drc_data.txt';
[t,z] = readdata(filename);
CDtilde = 0.1;
Ei = compute_error(CDtilde,z);
while abs(Ei) > 1e-2
    CDtilde = CDtilde - Eprime(CDtilde,z)/Edoubleprime(CDtilde,z);
    Ei = compute_error(CDtilde,z);
end
[ttilde,ztilde] = RK4(CDtilde);
ploteverything(t,z,ztilde);
```

The winning team must solve for C_D and email the instructor a graph showing z and ztilde on top of each other. The first team to correctly email the instructor with the value of C_D and the correct graph will get a prize. Ready go!

8. FINITE DIFFERENCE/ELEMENTS METHOD

1. **Boundary-Value Problems**

The heat equation on the other hand is a boundary value problem. Here we represent the pipe without time such that $T = f(x)$ instead of time.

$$\frac{d^2T}{dx^2} - h'(T - T_a) = 0 \tag{269}$$

Thus the difference between boundary value problems and initial condition problems is really the independent variable. To solve this we set $T = Ce^{sx} + T_a$. We substitute this equation into the equation above and obtain the characteristic equation and solve for s = $\pm\sqrt{h'}$. This yields the analytical solution $T(x) = Ae^{\sqrt{h'}x} + Be^{-\sqrt{h'}x} + T_a$. Again the boundary conditions can be used to solve for A and B. For example, $T(x = 0) = 40$ and $T(x = L) = 200$. Note that solving for A and B yields a system of equations. The solution to A and B has been left for the reader.

Now, in order to solve this equation numerically you must replace all derivatives with finite difference approximations. Below is a center first order approximation of the second derivative.

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} - h'(T_i - T_a) = 0 \tag{270}$$

or

$$-T_{i-1} + (2 + h'\Delta x^2)T_i - T_{i+1} = h'\Delta x^2 T_a \tag{271}$$

If the rod is discretized into 6 beads ($\Delta x = 2$ and L = 10 m) such that $T(0) = T_0 = 40$ and $T(L) = T_5 = 200$, letting h'=0.01 and $T_a = 20$ yields a system of equations

$$\begin{bmatrix} 2.04 & -1 & 0 & 0 \\ -1 & 2.04 & -1 & 0 \\ 0 & -1 & 2.04 & -1 \\ 0 & 0 & -1 & 2.04 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 40.8 \\ 0.8 \\ 0.8 \\ 200.8 \end{Bmatrix} \tag{272}$$

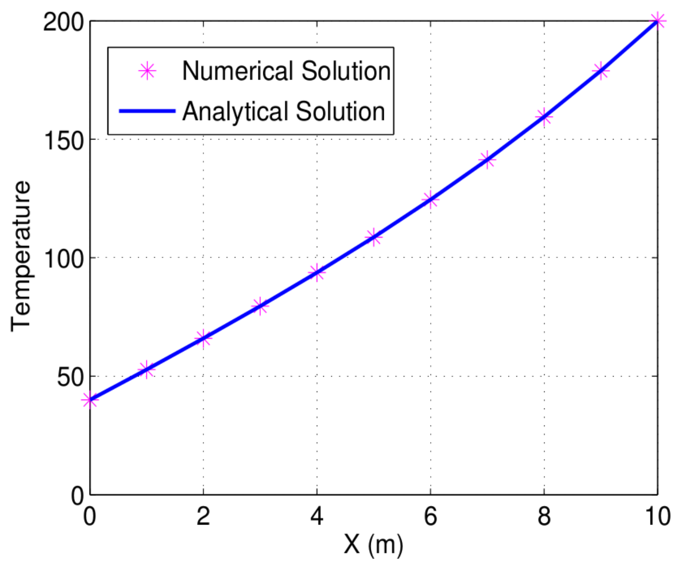
which can be solved with any numerical solver. In this case $T = [65.970, 93.778, 124.538, 159.480]$. Note, this can also be done using an iterative method. If the candidate equation is written such that

$$T_i = \frac{T_{i-1} + h'\Delta x^2 T_a + T_{i+1}}{2 + h'\Delta x^2} \tag{273}$$

The problem above evolves into a Simple Fixed Point Iteration Problem where the output is computed for every location i along the rod and the solution is used to compute the next solution. The code required to implement the Simple Fixed Point Iteration technique is shown below.

```
x = 0:2:10;
Tguess = 0*x;
Tguess(1) = 20;
Tguess(end) = 200;
Ta = 20;
hprime = 0.01;
for iter = 1:100
    for idx = 2:length(Tguess)-1
        Tguess(idx) = (Tguess(idx-1) + hprime*delx^2*Ta + Tguess(idx+1))/(2+hprime*delx^2);
    end
end
```

The benefit of this code over solving the system using $A\vec{x} = B$ is that the number of beads can be arbitrarily increased without any change to the code. The solution to the code above can be plotted alongside the analytical solution and is shown in the equation below. Note that the solution below has 11 beads rather than 6 beads.



8.A. Partial Differential Equations - Chapter 29

1. **Overview of PDEs**

   All equations we've dealt with thus far involves functions of one dimension $f(t)$ (initial value problems) or $f(x)$ (boundary value problems). In engineering, problems are rarely one-dimensional and usually involve multiple independent variables $f(x, y, z, t)$ (spatially and temporally varying wind field for example). Most systems encountered in engineering are second order and can be expressed using the equation below.

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0 \tag{274}$$

If $B^2 - 4AC < 0$ the equation is elliptic. If $B^2 - 4AC = 0$ the equation is parabolic and if $B^2 - 4AC > 0$ the equation is hyperbolic.

2. **Elliptic Equations**

The 2-D heat transfer governing equation is

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + Q = \frac{2h}{t}(T - T_\infty) \tag{275}$$

where k is the thermal conductivity, T is the temperature, Q is the heat source, h is the convection coefficient and t is the plate thickness. If the plate is insulated on its lateral surfaces(top and bottom), $k_x = k_y = k$ and $Q = 0$, the governing equation becomes

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{276}$$

In the general form this means that A = 1, B = 0, C = 1 and C = 0 thus $B^2 - 4AC = -4 < 0$ which means the equation is elliptic. In order to solve this equation the system is discretized into a set of grid points. Then the partial derivatives can be approximated using central difference formulas.

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} = 0 \tag{277}$$

If $\Delta x = \Delta y$ the equation reduces to

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} - 4T_{i,j} + T_{i,j-1} = 0 \tag{278}$$

Assume you start with grid point $i = 1$ and $j = 1$. The equation is then

$$T_{2,1} + T_{0,1} + T_{1,2} - 4T_{1,1} + T_{1,0} = 0 \tag{279}$$

If boundary conditions are used such that $T_{0,j} = 75^o$, $T_{i,0} = 0^o$, $T_{4,j} = 50^o$ and $T_{i,4} = 100^o$ (assume a 3x3 grid) the equation above reduces to

$$\begin{gathered} T_{2,1} + 75 + T_{1,2} - 4T_{1,1} + 0 = 0 \\ T_{2,1} + T_{1,2} - 4T_{1,1} = -75 \end{gathered} \tag{280}$$

Note that when boundary conditions are given this is known as Dirichlet boundary conditions, that is the temperature is held constant. If instead the sides are insulated it means the heat flux $(\partial T/\partial x)$ is zero. Thus the derivative is given instead of the value. This is known as a Neumann condition.

Repeating equation 280 for all 9 grid points yields a system of the form $A\vec{x} = b$ where A is a 9x9 matrix. This can then be solved by any numerical solver just like the single dimension boundary value problem.

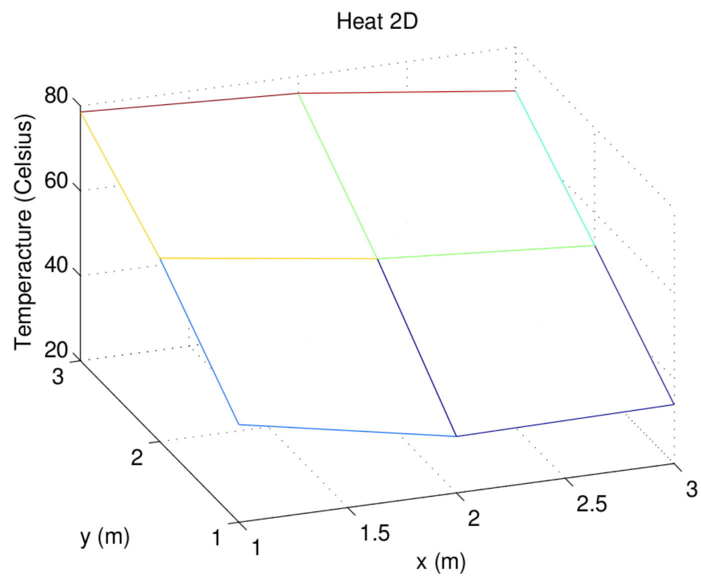
The solution is shown in the matrix below

$$T(x, y) = \begin{bmatrix} 43.0000 & 33.3000 & 33.8900 \\ 63.2100 & 56.1100 & 52.3400 \\ 78.5900 & 76.0600 & 69.7100 \end{bmatrix} \tag{281}$$

This can be plotted in MATLAB using the function below

```
x = 1:3;
y = 1:3;
[xx,yy] = meshgrid(x,y);
Tsolution = [43 33.3 33.89;63.21 56.11 52.34;78.59 76.06 69.71];
mesh(xx,yy,Tsolution)
```

Notice that MATLAB matrices are top to bottom so make sure you enter the solution in correctly otherwise you will get incorrect results.

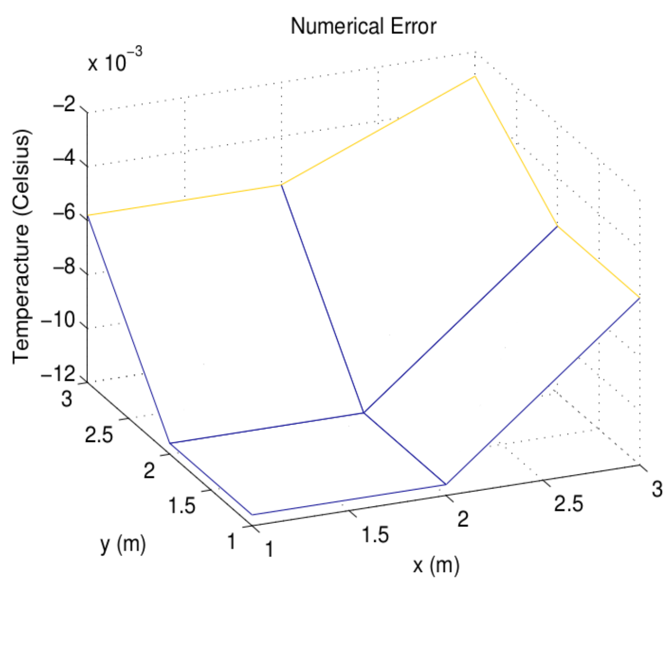


3. **Gauss-Seidel Method**

Just as in the single dimensional heat equation the process of creating the matrices above is very tedious when performing this by hand. Thus it is beneficial to create a routine that can iterate until convergence. Taking equation 278 and rearranging for $T_{i,j}$ yields

$$T_{i,j} = (T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1})/4 \tag{282}$$

This again can be easily implemented into a numerical solver just as before. The code has been left out so that the reader may attempt to create this code as an exercise however the error between the iterative solution and the solution from above even after 10 iterations is on the order of $10^{-}3$ as shown by the mesh plot of the error below. Of course this is specific to the initial guess but the iterative method is still robust to changes in grid size. This is known as the Gauss-Seidel Method.



4. **Derivative Boundary Conditions**

   If derivative boundary conditions are introduced we no longer have the value of T at the boundary. Instead we have the value of the derivative. In order to incorporate the derivative boundary the central difference equation is written for the boundary point such that

$$T_{1,j} + T_{-1,j} + T_{0,j+1} + T_{0,j-1} - 4T_{0,j} = 0 \tag{283}$$

   Notice that the variable $T_{-1,j}$ has been introduced which is outside the boundary. However this variable can be used to substitute the derivative into the finite difference equations.

$$\frac{\partial T_{0,j}}{\partial x} = \frac{T_{1,j} - T_{-1,j}}{2\Delta x} \tag{284}$$

   The equation above can be solved for $T_{-1,j}$ and substituted into the $0, j$ equation to yield

$$2T_{1,j} - 2\Delta x \frac{\partial T_{0,j}}{\partial x} + T_{0,j+1} + T_{0,j-1} - 4T_{0,j} = 0 \tag{285}$$

   The equation above can be used for all nodes with a derivative boundary condition to solve for the heat flux in a plate.

5. **Vibrating String**

   Another elliptic equation is the vibrating string which can be solved in a similar fashion.

$$T\frac{\partial^2 v}{\partial x^2} - m\frac{\partial^2 v}{\partial t^2} = 0 \tag{286}$$

   In the equation above, T is the tension in the string and m is the mass per unit length of the rod. $v(x,t)$ is the amount of deflection in the string as a function of space and time. Notice that in this equation the time derivative is second order. Thus Euler's method is not accurate enough to compute the time derivative and a higher order method such as a Runge-Kutta-4 scheme must be used in order to converge to the solution.

6. **Parabolic Equations**

   An example parabolic equation is again a heat equation however here the equation is a function of x and time. The equation is given below

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{287}$$

   Just as before the spatial derivative is approximated using central finite differencing however a superscript has been added to denote the time variable. That is T_1^3 is the value of temperature at the first node at time t = 3.

$$\frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{\Delta x^2} = \frac{\partial^2 T}{\partial x^2} \tag{288}$$

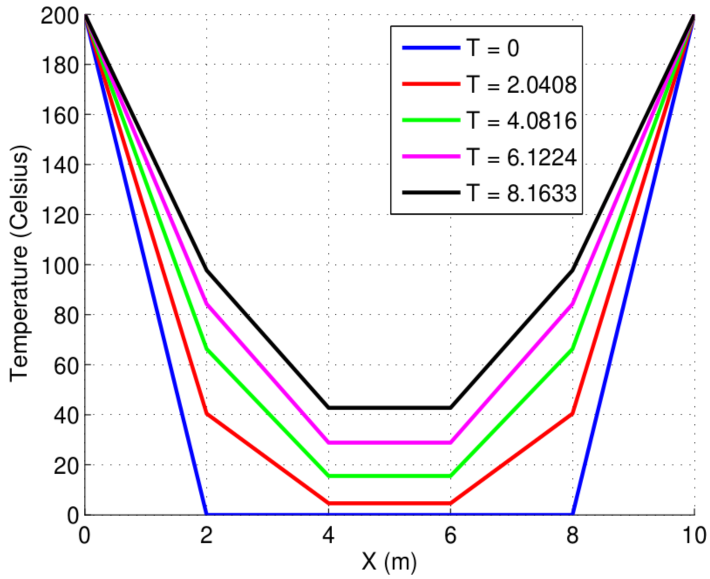
   Similarly Euler's Method can be used to approximate the time derivative of T.

$$\frac{\partial T}{\partial t} = \frac{T^{l+1} - T^l}{\Delta t} \tag{289}$$

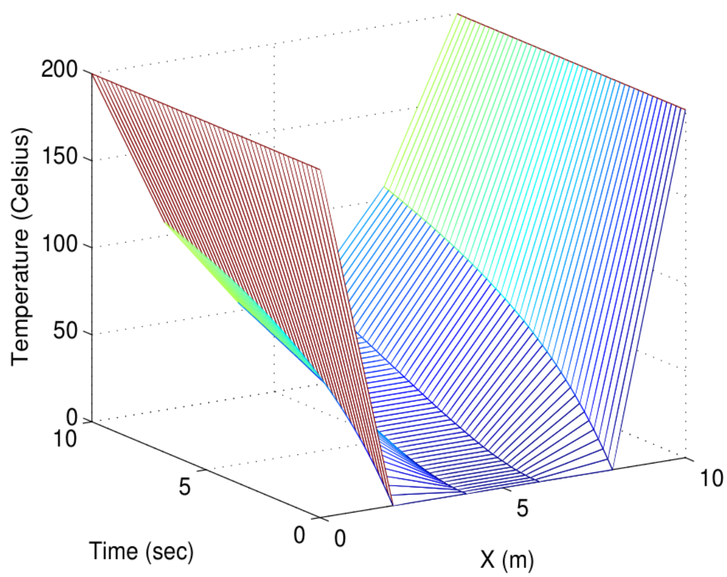
   If equations 289 and 288 are substituted in equation 287 and rearranged for T_i^{l+1} the Finite Difference Method equation becomes

$$T_i^{l+1} = T_i^l + \lambda(T_{i+1}^l - 2T_i^l + T_{i-1}^l) \tag{290}$$

   where $\lambda = \frac{k\Delta t}{\Delta x^2}$. Note, just like Euler's method, there are limits on stability for this method. In Carnahan et al. 1969 it was proved that the system is stable if $\Delta t \leq \frac{\Delta x^2}{2k}$. The iterative method above thus combines the time dependent and spatially dependent portion of the rod. Thus, let us return to the simple static case we saw in section 1 where the rod is discretized into 6 beads ($\Delta x = 2m$ and L = 10 m) such that $T(0) = T_0 = 40$ and $T(L) = T_5 = 200$. However, in this case let's assume that the rod has no heat at all except at the end points such that T = [200 0 0 0 0 200]. The iterative procedure above is used to propogate the system forward to see how the heat in the rod evolves over time where $k' = 0.49 cal/(s - cm -^o C)$. Again the code has been removed and left for an exercise to the reader. The temperature distribution as a function of time is shown in the figure below.



This can also be visualized in three dimensions where a mesh plot is created. Here the x axis is the length of the rod, the y axis is the time variable and the z axis is the temperature along the rod. It is clear that the temperature in the rod is slowly heating up to $200^o C$.



7. **Crank-Nicolson Method**

The Crank-Nicolson method is an implicit method rather than an explicit method. That is, rather than computing the next time step forward one at a time for each spatial coordinate, each timestep is computed simultaneously for the entire rod. The equations are obviously more complex however, it does not have the stability problem that the explicit method has because the spatial and time derivatives are second order accurate. First, the time derivative is approximated at the midpoint of time using the equation below.

$$\frac{\partial T}{\partial t} = \frac{T_i^{l+1} - T_i^l}{\Delta t} \tag{291}$$

The spatial derivative is then approximated as an average of the forward point $l+1$ and the current time point l.

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{2}\left[\frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{\Delta x^2} + \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{\Delta x^2}\right] \tag{292}$$

Substituting these two equations into equation 287 results in

$$-\lambda T_{i-1}^{l+1} + 2(1+\lambda)T_i^{l+1} - \lambda T_{i+1}^{l+1} = \lambda T_{i-1}^l + 2(1-\lambda)T_i^l + \lambda T_{i+1}^l \tag{293}$$

where $\lambda = \frac{k\Delta t}{\Delta x^2}$. For the first point above $i = 1$ the equation becomes

$$2(1+\lambda)T_1^{l+1} - \lambda T_2^{l+1} = \lambda T_0^l + \lambda T_0^{l+1} + 2(1-\lambda)T_1^l + \lambda T_2^l \tag{294}$$

for the last point $i = m$, the equation becomes

$$-\lambda T_{m-1}^{l+1} + 2(1+\lambda)T_m^{l+1} = \lambda T_{m+1}^{l+1} + \lambda T_{m-1}^l + 2(1-\lambda)T_m^l + \lambda T_{m+1}^l \tag{295}$$

where T_0 and T_{m+1} are boundary conditions. These equations can be stacked in matrix form to yield the following equations.

$$\begin{bmatrix} 2(1+\lambda) & -\lambda & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & -\lambda & 2(1+\lambda) & -\lambda & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & -\lambda & 2(1+\lambda) \end{bmatrix} \begin{Bmatrix} T_1^{l+1} \\ T_2^{l+1} \\ \vdots \\ T_{i-1}^{l+1} \\ T_i^{l+1} \\ T_{i+1}^{l+1} \\ \vdots \\ T_{m-1}^{l+1} \\ T_m^{l+1} \end{Bmatrix} = \vec{b} \tag{296}$$

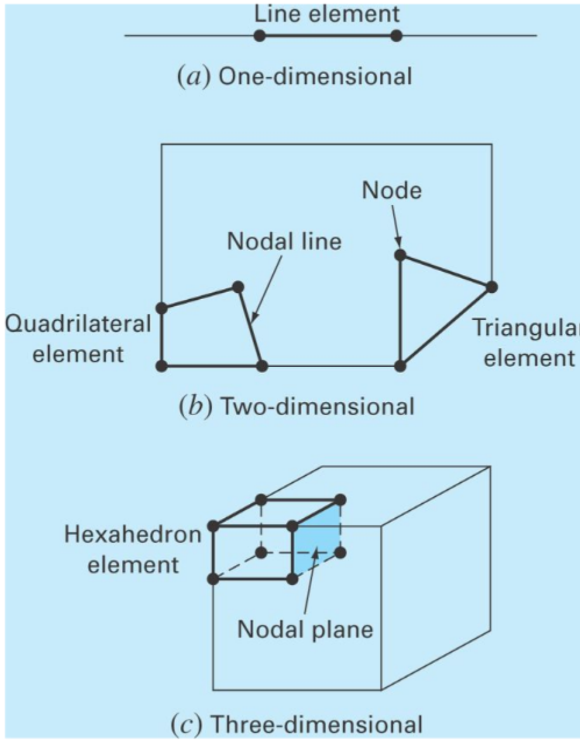
where $\vec{b}$ is

$$\vec{b} = \begin{Bmatrix} \lambda T_0^l + \lambda T_0^{l+1} + 2(1-\lambda)T_1^l + \lambda T_2^l \\ \vdots \\ \lambda T_{i-1}^l + 2(1-\lambda)T_i^l + \lambda T_{i+1}^l \\ \vdots \\ \lambda T_{m-1}^l + \lambda T_{m+1}^{l+1} + 2(1-\lambda)T_m^l + \lambda T_{m+1}^l \end{Bmatrix} \tag{297}$$

The assumption here is that every coordinate at T^l is known and the equation above is used to compute the entire bar at the next timestep. In addition, the equation is of the form $A\vec{x} = \vec{b}$. The solution to this equation is $A^{-1}\vec{b}$ however since the A matrix is constant this equation can be solved iteratively extremely quickly.

8.B. Finite Element Analysis - Bars and Trusses - Chapter 31 (Kind of)

In finite element methods as opposed to finite difference approaches the bodies are broken up into nodes rather than discretized into equal parts. These bodies can be one, two or even three dimensional objects as shown in the figure below.



These nodes are not restricted to be linear and thus offer more capabilities over finite difference methods. The chapters below will begin with the derivation of a 1-D beam with examples to follow

1. **Finite Element Analysis of a 2-Node Bar**

   For the moment let's consider the one dimensional object. Typically approximation functions are created to approximate the nodes such that

$$u(x) = a_0 + a_1 x \tag{298}$$

where $u(x)$ is whatever the independent variable can be and the coefficients a are constants to be solved for. When a one-dimensional bar is split into different nodes we must enforce the constraint that $u(x_1) = u_1$ is equal to $u(x_2) = u_2$ where x_1 and x_2 are the coordinates of the nodes along the beam. This constraint for a two node beam would yield the following two equations.

$$\begin{aligned} u_1 &= a_0 + a_1 x_1 \\ u_2 &= a_0 + a_1 x_2 \end{aligned} \tag{299}$$

These equations can be easily solved for using substitution or Gaussian Elimination.

$$a_0 = \tfrac{u_1 x_2 - u_2 x_1}{x_2 - x_1} \quad a_1 = \tfrac{u_2 - u_1}{x_2 - x_1} \tag{300}$$

These equations can then be further reduced by setting

$$N_1 = \frac{x_2 - x}{x_2 - x_1} \tag{301}$$

and

$$N_2 = \frac{x - x_1}{x_2 - x_1} \tag{302}$$

such that equation 298 becomes

$$u = N_1 u_1 + N_2 u_2 \tag{303}$$

The equation above is called a *shape function* and N_1 and N_2 are called *interpolation functions*. The solution does not seem very fancy however it leads to very interesting results in differentiation and integration. For example, if equation 303 is differentiated to obtain

$$\frac{du}{dx} = \frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} u_2 \tag{304}$$

where

$$\tfrac{dN_1}{dx} = -\tfrac{1}{x_2 - x_1} \quad \tfrac{dN_2}{dx} = \tfrac{1}{x_2 - x_1} \tag{305}$$

Noting that u_1 and u_2 are not functions of x. The variable $u(x)$ is a function of x but u_1 and u_2 are constants w.r.t x. Substituting this into the equation for the derivative yields

$$\frac{du}{dx} = \frac{u_2 - u_1}{x_2 - x_1} \tag{306}$$

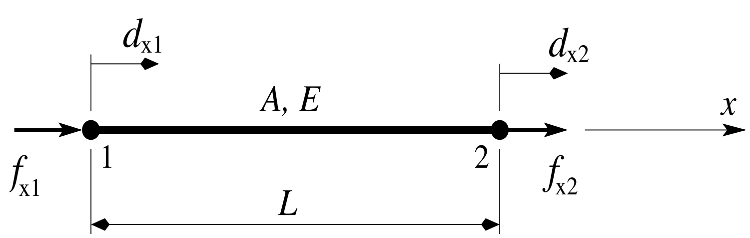
which is simply the slope of the line. Similarly, the integral of $u(x)$ can be expressed as

$$\int_{x_1}^{x_2} u \; dx = \int_{x_1}^{x_2} N_1 u_1 + N_2 u_2 \; dx = \frac{u_1 + u_2}{2}(x_2 - x_1) \tag{307}$$

Close inspection reveals that the equation above is simply the trapezoidal rule. Thus, using shape functions create a simple equation for both the derivative and the integral. Something that can be used when creating the differential equations for governing bodies. The task then becomes to solve for the value of $u(x)$ at all node locations.

2. **Stiffness Matrix for a Bar Element**

   As a first example, let's consider the uniform bar being loaded axially below. If this bar is discretized into N or more beads the problem becomes considerably more complex than if the bar was simply discretized into two nodes (the left and right dots labeled 1 and 2). In this fashion the loads are computed at 1 and 2 and are assumed to be uniform throughout the entire member.



Consider again the uniform prismatic bar element of length L, cross-sectional area A and Young's modulus E. Assume for the moment this bar is just a uniform bar and can resist only axial load, thus nodes are allowed to displace only in the axial direction. The displacement-force relation and the equation of static equilibrium in the x-direction are respectively given by Hooke's law where Force = stiffness * displacement

$$k(d_{x2} - d_{x1}) = f_{x2} \tag{308}$$

where $k = EA/L$ is the axial stiffness constant, and

$$f_{x1} = -f_{x2} \tag{309}$$

If put into matrix form with the forces on the right hand side the equations become

$$k\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}d_{x1}\\ d_{x2}\end{Bmatrix}=\begin{Bmatrix}f_{x1}\\ f_{x2}\end{Bmatrix} \tag{310}$$

As a result, the stiffness matrix for a bar element can be found as

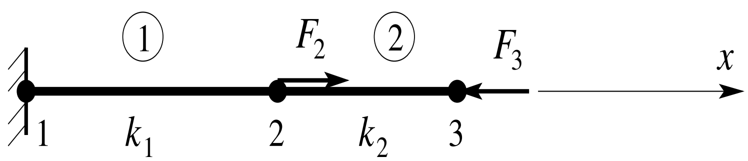
$$\left[K_{(e)}\right]=k\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \tag{311}$$

3. **Stiffness Matrix for a Bar Assemblage**

With the stiffness matrix of one bar known it is now possible to create a stiffness matrix of an entire assemblage. The structure stiffness matrix $[K]$ may be obtained by assembling $\left[K_{(e)}\right]$ of n individual bar elements.

$$[K]=\sum_{e=1}^{n}\left[K_{(e)}\right] \tag{312}$$

For example, consider the bar below with two elements connected to a wall.



The stiffness matrices can be written for both bars using equation 311.

$$\left[K_{(1)}\right]=k_1\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}d_{x1}\\ d_{x2}\end{Bmatrix} \qquad \left[K_{(2)}\right]=k_2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}d_{x2}\\ d_{x3}\end{Bmatrix} \tag{313}$$

In order to obtain the total stiffness matrix $[K]$ the 2x2 systems must be expanded to 3x3 systems.

$$\left[K_{(1)}\right]=\begin{bmatrix}k_1 & -k_1 & 0\\ -k_1 & k_1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}d_{x1}\\ d_{x2}\\ d_{x3}\end{Bmatrix} \qquad \left[K_{(2)}\right]=\begin{bmatrix}0 & 0 & 0\\ 0 & k_2 & -k_2\\ 0 & -k_2 & k_2\end{bmatrix}\begin{Bmatrix}d_{x1}\\ d_{x2}\\ d_{x3}\end{Bmatrix} \tag{314}$$

$$\rightarrow \quad [K]=\left[K_{(1)}\right]+\left[K_{(2)}\right]=\begin{bmatrix}k_1 & -k_1 & 0\\ -k_1 & (k_1+k_2) & -k_2\\ 0 & -k_2 & k_2\end{bmatrix}\begin{Bmatrix}d_{x1}\\ d_{x2}\\ d_{x3}\end{Bmatrix} \tag{315}$$

The full system of equations is then simply $[K]\vec{d}=\vec{F}$

$$\begin{bmatrix}k_1 & -k_1 & 0\\ -k_1 & (k_1+k_2) & -k_2\\ 0 & -k_2 & k_2\end{bmatrix}\begin{Bmatrix}d_{x1}\\ d_{x2}\\ d_{x3}\end{Bmatrix}=\begin{Bmatrix}R_{x1}\\ F_2\\ -F_3\end{Bmatrix} \tag{316}$$

The unknowns in the equation above are the displacements and the reaction force R_{x1}. In this example, F_1 and F_2 are known forcing functions. Even with these two values there are still 4 unknowns and only 3 equations. The last equation is obtained by using the boundary conditions of the system. This is the fact that the beam cannot deflect at the attachment point 1. Thus $d_{x1}=0$. Using this result the systems of equations reduces to

$$\begin{bmatrix}(k_1+k_2) & -k_2\\ -k_2 & k_2\end{bmatrix}\begin{Bmatrix}d_{x2}\\ d_{x3}\end{Bmatrix}=\begin{Bmatrix}F_2\\ -F_3\end{Bmatrix} \tag{317}$$

This system is in the form $A\vec{x}=\vec{b}$ and can now be easily solved by any computer program. Since the system is 2x2 it can also be easily solved by hand. Knowing the values of $\vec{d}$, it is possible to obtain the tension/compression force of each bar

$$\{f_{(e)}\} = [K_{(e)}]\ \{d_{(e)}\} \tag{318}$$

For example bar 1 is given as

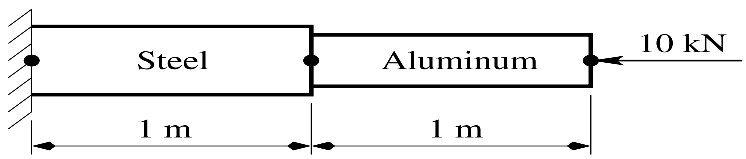
$$\{f_{(1)}\} = [K_{(1)}]\ \{d_{(1)}\} \tag{319}$$

$$\left\{ \begin{array}{c} f_{x1} \\ f_{x2} \end{array} \right\} = k_1 \left[\begin{array}{rr} 1 & -1 \\ -1 & 1 \end{array} \right] \left\{ \begin{array}{c} d_{x1} \\ d_{x2} \end{array} \right\} \tag{320}$$

Remember though that $f_{x1} = -f_{x2}$.

4. **FEA of a Bar Example Problem**

Determine the nodal displacements, the forces in each element, and the reactions. Let ($E_{st} = 200$ GPa, $A_{st} = 4 \times 10^{-4}$ m^2, $E_{al} = 70$ GPa, $A_{al} = 2 \times 10^{-4}$ m^2).



(a) Displacements of nodes 2 and 3

$$[K_{(1)}] = k_1 \left[\begin{array}{rr} 1 & -1 \\ -1 & 1 \end{array} \right] \qquad [K_{(2)}] = k_2 \left[\begin{array}{rr} 1 & -1 \\ -1 & 1 \end{array} \right] \tag{321}$$

where
$$\begin{aligned} k_1 &= \frac{E_{st}A_{st}}{L} = \frac{(200 \times 10^6)(4 \times 10^{-4})}{1} = 8 \times 10^4 \text{ kN/m} \\ k_2 &= \frac{E_{al}A_{al}}{L} = \frac{(70 \times 10^6)(2 \times 10^{-4})}{1} = 14 \times 10^3 \text{ kN/m} \end{aligned}$$

Use of the BCs and loading conditions yields

$$\left[\begin{array}{ccc} k_1 & -k_1 & 0 \\ -k_1 & (k_1+k_2) & -k_2 \\ 0 & -k_2 & k_2 \end{array} \right] \left\{ \begin{array}{c} 0 \\ d_{x2} \\ d_{x3} \end{array} \right\} = \left\{ \begin{array}{c} R_{x1} \\ 0 \\ -P \end{array} \right\} \quad \text{or} \quad 10^3 \left[\begin{array}{rrr} 80 & -80 & 0 \\ -80 & 94 & -14 \\ 0 & -14 & 14 \end{array} \right] \left\{ \begin{array}{c} 0 \\ d_{x2} \\ d_{x3} \end{array} \right\} = \left\{ \begin{array}{c} R_{x1} \\ 0 \\ -10 \end{array} \right\} \tag{322}$$

Results:

$$10^3 \left[\begin{array}{rr} 94 & -14 \\ -14 & 14 \end{array} \right] \left\{ \begin{array}{c} d_{x2} \\ d_{x3} \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ -10 \end{array} \right\} \quad \rightarrow \quad \left\{ \begin{array}{c} d_{x2} \\ d_{x3} \end{array} \right\} = 10^{-3} \left[\begin{array}{rr} 94 & -14 \\ -14 & 14 \end{array} \right]^{-1} \left\{ \begin{array}{c} 0 \\ -10 \end{array} \right\} \tag{323}$$

Thus,

$$\left\{ \begin{array}{c} d_{x2} \\ d_{x3} \end{array} \right\} = 10^{-4} \left\{ \begin{array}{c} -1.25 \\ -8.39 \end{array} \right\} \text{ m} = \left\{ \begin{array}{c} -0.125 \\ -0.839 \end{array} \right\} \text{ mm} \tag{324}$$

(b) Forces in each element: $\{f_{(e)}\} = [K_{(e)}]\ \{d_{(e)}\}$

$$\text{Element 1:} \quad \left\{ \begin{array}{c} f_1 \\ f_2 \end{array} \right\} = 8 \times 10^4 \left[\begin{array}{rr} 1 & -1 \\ -1 & 1 \end{array} \right] \left\{ \begin{array}{l} d_{x1} = 0 \\ d_{x2} = -1.25 \times 10^{-4} \end{array} \right\} = \left\{ \begin{array}{r} 10 \\ -10 \end{array} \right\} \text{kN} \tag{325}$$

$$\text{Element 2:} \quad \left\{ \begin{array}{c} f_2 \\ f_3 \end{array} \right\} = 14 \times 10^3 \left[\begin{array}{rr} 1 & -1 \\ -1 & 1 \end{array} \right] \left\{ \begin{array}{l} d_{x2} = -1.25 \times 10^{-4} \\ d_{x3} = -8.39 \times 10^{-4} \end{array} \right\} = \left\{ \begin{array}{r} 10 \\ -10 \end{array} \right\} \text{kN} \tag{326}$$

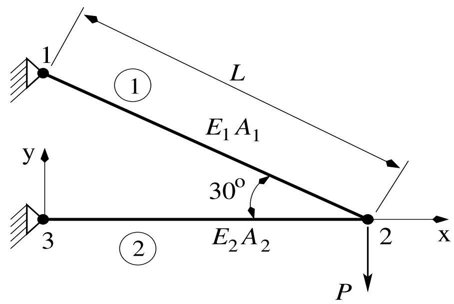
Notice that in Element 1, $f_2 = -10kN$ but in Element 2, $f_2 = 10kN$. This is because the force f_2 is seen as compression for Element 1 and tension for Element 2. This is a direct consequence of Newton's 3rd Law.

(c) Reaction R_{x1} (using the first equation in (322))

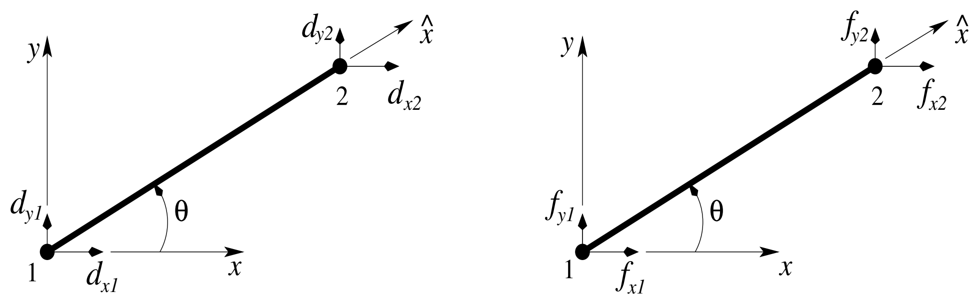
$$R_{x1} = -k_1\, d_{x2} = (-8 \times 10^4)(-1.25 \times 10^{-4}) = 10 \text{ kN} \tag{327}$$

5. **Introduction to Trusses**

Trusses as opposed to bars are not restricted to one dimension. That is, a truss is a system of interconnected bars who can still only support axial loading but in their local reference frame. A drawing of a 2 beam truss in 2-dimensions is shown below.



In the problem above Let $A_1 = A_2 = 5$ in.^{2}, $E_1 = E_2 = 10^6$ psi, $L = 100$ in., and $P = 10$ kip (klbf). In order to solve for the displacement at node 2 the stiffness matrices of each member must be translated to a global inertial coordinate system. In the problems with 1-D beams the local body frame of each beam was identical to the global inertial frame. However in the 2D truss problem, each beam is rotated through an angle θ as shown by the problem below.



The nomenclature for the two frames is a superscript $[I]$ for inertial and $[B]$ for body or beam frame. The first step is to write the stiffness in the local body frame as

$$\left[K_{(e)}^{[B]}\right] \vec{d_{(e)}}^{[B]} = \vec{f_{(e)}}^{[B]} \tag{328}$$

Note that this is a vector equation thus as long as the superscript [ ] is the same on all variables the equation is satisfied. This equation can also be written in component form as

$$k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \delta_{x1} \\ \delta_{x2} \end{Bmatrix} = \begin{Bmatrix} g_{x1} \\ g_{x2} \end{Bmatrix} \tag{329}$$

Notice that δ and g are used to denote the components of displacement and force in the local body frame. Clearly there is a relationship between the body frame components and the inertial frame components. To do this a rotation matrix T_{IB} is used to rotate from one coordinate system to the other. This can be written as

$$\vec{d_{(e)}}^{[I]} = [T_{IB}]\, \vec{d_{(e)}}^{[B]} \tag{330}$$

The rotation matrix has the property of being an orthonormal basis in R^N thus $T_{IB}^{-1} = T_{IB}^T = T_{BI}$. Thus the equation above can be written simply as

$$\vec{d_{(e)}}^{[B]} = \left[T_{IB}^{T}\right] \vec{d_{(e)}}^{[I]} \tag{331}$$

The derivation of T_{IB} is left out for space but the reader is encouraged to consult his/her dynamics textbook on 2D rigid body rotations to obtain the T_{IB} matrix below. It is advised not to just memorize this matrix and learn the steps to derive the matrix.

$$[T_{IB}] = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \tag{332}$$

If equation 331 is substituted into equation 328 the equation becomes

$$\left[K_{(e)}^{[B]}\right] \left[T_{IB}^{T}\right] \vec{d_{(e)}}^{[I]} = \left[T_{IB}^{T}\right] \vec{f_{(e)}}^{[I]} \tag{333}$$

Multiplying both sides of the equation by T_{IB} yields

$$\left[T_{IB}^{T}\right] \left[K_{(e)}^{[B]}\right] \left[T_{IB}^{T}\right] \vec{d_{(e)}}^{[I]} = \vec{f_{(e)}}^{[I]} \tag{334}$$

Notice then that all terms contain the superscript $[I]$ except for the stiffness matrix. This can be solved by substituting in the equation below.

$$\left[K_{(e)}^{[I]}\right] = \left[T_{IB}^{T}\right] \left[K_{(e)}^{[B]}\right] \left[T_{IB}^{T}\right] \tag{335}$$

This equation relates the local body frame coordinate system for the global inertial coordinate system. Using this equation yields the final vector equation.

$$\left[K_{(e)}^{[I]}\right] \vec{d_{(e)}}^{[I]} = \vec{f_{(e)}}^{[I]} \tag{336}$$

At this point the solution to the problem is identical to a 1D bar problem. Thus the example presented above is left as an exercise to the reader. Remember that in order to solve this the constraint that beams can only hold axial loads must be held.

8.C. Finite Element Analysis - Heated Rods - Chapter 31

Let's re-investigate the heat conduction equation we encountered in the finite difference methods Section 1. The rod is discretized in the normal fashion into 5 nodes where the endpoint temperature values are known as shown in the figure below.

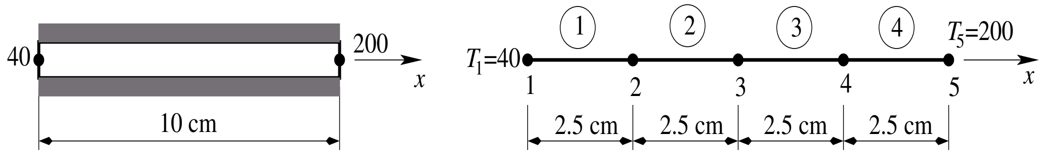


Figure 1: **1-D Heat Rod for Finite Element Analysis**

If the body obey's Fourier's law

$$q = -k\frac{dT}{dx} \tag{337}$$

If k is a constant

$$k\frac{d^2T}{dx^2} + Q(x) = 0 \tag{338}$$

where $Q(x) = dq(x)/dx$ is an internal uniform heat source.

1. **Analytic Solution with Q(x) = 0**

   The easiest finite element solution is the direct method as has been done with bars and trusses. This method only applies to $Q(x) = 0$. such that

$$\frac{d^2T}{dx^2} = 0 \tag{339}$$

   To solve the equation the equation above is integrated twice to yield

$$T(x) = c_1 x + c_2 \tag{340}$$

   The boundary conditions are then used to solve for the coefficients c_1 and c_2.

$$\begin{gathered} T(x=0) = T_1 = c_2 = 40 \\ T(x=10) = T_2 = c_1(10) + 40 = 200 \rightarrow c_1 = 16 \end{gathered} \tag{341}$$

   Thus the analytical solution is

$$T(x) = 16x + 40 \tag{342}$$

2. **Direct Method Solution**

   Although this equation can be solved for explicitly it is a good problem to do for its simplicity. Again the question is to determine the temperature at the nodes in the rod. First let's examine one element where

$$T(x) = N_1 T_1 + N_2 T_2 \tag{343}$$

   In this fashion

$$\frac{dT}{dx} = T^{'} = \frac{T_2 - T_1}{x_2 - x_1} \tag{344}$$

   which was derived in the previous section. This leads to the heat flux at node 1 equal to the following.

$$q_1 = -k\frac{T_2 - T_1}{x_2 - x_1} = -kT^{'} \tag{345}$$

   In the finite element bar example subject to an axial load, the forces applied to the beam are equal and opposite. Here similar constraints are imposed such that $q_1 = -q_2$. This implies that the heat flux flowing out of 1 bar is equal to the heat flux flowing into 2. Thus,

$$q_2 = kT^{'} = k\frac{T_2 - T_1}{x_2 - x_1} \tag{346}$$

   Writing this in matrix form yields the element matrix equation

$$\frac{-k}{x_2 - x_1}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = k\begin{Bmatrix} -T^{'} \\ T^{'} \end{Bmatrix} \tag{347}$$

   Dividing out the thermal coefficient k, distributing the minus sign and noting that $T^{'}$ at node 1 is $T^{'} = T_1^{'}$ and $T^{'} = T_2^{'}$ at node 2 yields the following element matrix.

$$\frac{1}{x_2 - x_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} -T_1^{'} \\ T_2^{'} \end{Bmatrix} \tag{348}$$

   At this point the solution is the same as a bar subject to axial loads.

3. **Heat Conduction Bar Example**

Let's now solve the heat conduction problem for the rod in Figure 1. First let's write the element stiffness matrix for elements 1-4. Note that there are 5 nodes and 4 elements where $x_2 - x_1 = \Delta x = 2.5$.

$$\begin{bmatrix} 0.4 & -0.4 \\ -0.4 & 0.4 \end{bmatrix} \begin{Bmatrix} 40 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} -T_1^{'} \\ T_2^{'} \end{Bmatrix} \tag{349}$$

$$\begin{bmatrix} 0.4 & -0.4 \\ -0.4 & 0.4 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} -T_2^{'} \\ T_3^{'} \end{Bmatrix} \tag{350}$$

$$\begin{bmatrix} 0.4 & -0.4 \\ -0.4 & 0.4 \end{bmatrix} \begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} -T_3^{'} \\ T_4^{'} \end{Bmatrix} \tag{351}$$

$$\begin{bmatrix} 0.4 & -0.4 \\ -0.4 & 0.4 \end{bmatrix} \begin{Bmatrix} T_4 \\ 200 \end{Bmatrix} = \begin{Bmatrix} -T_4^{'} \\ T_5^{'} \end{Bmatrix} \tag{352}$$

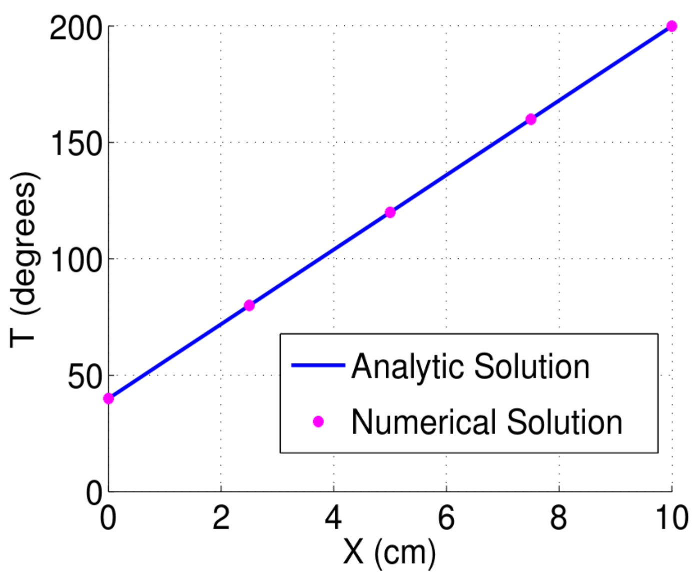
Just as before this is combined to a full bar element matrix such that

$$\begin{bmatrix} 0.4 & -0.4 & 0 & 0 & 0 \\ -0.4 & 0.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 0.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 0.8 & -0.4 \\ 0 & 0 & 0 & -0.4 & 0.4 \end{bmatrix} \begin{Bmatrix} 40 \\ T_2 \\ T_3 \\ T_4 \\ 200 \end{Bmatrix} = \begin{Bmatrix} -T_1^{'} \\ 0 \\ 0 \\ 0 \\ T_5^{'} \end{Bmatrix} \tag{353}$$

Notice that the internal heat flux was canceled due to equal and opposite reactions. The formulation of these equations led to two unknowns being introduced. That is, $T_1^{'}$ and $T_5^{'}$ are now unknowns. The equations must be rearranged to yield the following equations.

$$\begin{bmatrix} 1 & -0.4 & 0 & 0 & 0 \\ 0 & 0.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 0.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 0.8 & 0 \\ 0 & 0 & 0 & -0.4 & -1 \end{bmatrix} \begin{Bmatrix} T_1^{'} \\ T_2 \\ T_3 \\ T_4 \\ T_5^{'} \end{Bmatrix} = \begin{Bmatrix} -16 \\ 16 \\ 0 \\ 80 \\ -80 \end{Bmatrix} \tag{354}$$

This equation is of the form $A\vec{x} = \vec{b}$ which can be solved explicitly for the temperature. The figure below shows the result of the analytic solution from equation 342 and the numerical solution from above.



4. **Method of Weighted Residuals**

The direct method above works well if $Q(x) = 0$; however, it is it's major shortcoming. Let's re-investigate the same problem above however setting $Q(x)/k = 10$. This equation can again be solved analytically since

$$\frac{d^2T}{dx^2} = -10 \tag{355}$$

which leads to the temperature solution shown below where the boundary conditions $T(0) = 40$ and $T(10) = 200$ are used to solve for the undetermined coefficients.

$$T(x) = -5x^2 + 66x + 40 \tag{356}$$

In order to solve this using finite element analysis the heat equation is written as

$$\frac{d^2T}{dx^2} + f(x) = 0 \tag{357}$$

Then using equation 343 which is an approximate solution leads to

$$\frac{d^2\tilde{T}}{dx^2} + f(x) = R \tag{358}$$

where R is a residual since the equation is only an approximation. The method of weighted residuals then requires

$$\int_D RW_i dD = 0 \tag{359}$$

where W_i are weights force the integrand to zero. D is the entire control volume. For this case the control volume is the length of the rod and the weighted functions are the interpolation functions N_i. This method is called Galerkin's method.

$$\int_D RN_i dD = 0 = \int_{x_1}^{x_2} \left[\frac{d^2\tilde{T}}{dx^2} + f(x)\right] N_i \; dx = 0 \tag{360}$$

which can also be written as

$$\int_{x_1}^{x_2} \frac{d^2\tilde{T}}{dx^2} N_i \; dx = -\int_{x_1}^{x_2} f(x) N_i dx \tag{361}$$

The integrand on the left can be evaluated using integration by parts

$$\int_{x_1}^{x_2} \frac{d^2\tilde{T}}{dx^2} N_i \; dx = N_i \frac{d\tilde{T}}{dx}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d\tilde{T}}{dx}\frac{dN_i}{dx} dx \tag{362}$$

The first term on the right hand side can be evaluated to yield

$$N_1 \frac{d\tilde{T}}{dx}\bigg|_{x_1}^{x_2} = N_1(x_2)T_2^{'} - N_1(x_1)T_1^{'} \tag{363}$$

where $i = 1$. Remember that $N_1(x_2) = 0$ and $N_1(x_1) = 1$ so

$$N_1 \frac{d\tilde{T}}{dx}\bigg|_{x_1}^{x_2} = -T_1^{'} \tag{364}$$

Similarly with $i = 2$

$$N_2 \frac{d\tilde{T}}{dx}\Big|_{x_1}^{x_2} = T_2^{'} \tag{365}$$

using equations 361 through 363 leads to the following two equations for $i = 1$ and $i = 2$

$$\int_{x_1}^{x_2} \frac{d\tilde{T}}{dx}\frac{dN_1}{dx}dx = -T_1^{'} + \int_{x_1}^{x_2} f(x)N_1 dx \tag{366}$$

$$\int_{x_1}^{x_2} \frac{d\tilde{T}}{dx}\frac{dN_2}{dx}dx = T_2^{'} + \int_{x_1}^{x_2} f(x)N_2 dx \tag{367}$$

The first terms in the left-hand side are simple to evaluate using the shape functions.

$$\int_{x_1}^{x_2} \frac{d\tilde{T}}{dx}\frac{dN_1}{dx}dx = -\frac{T_2 - T_1}{x_2 - x_1} \tag{368}$$

$$\int_{x_1}^{x_2} \frac{d\tilde{T}}{dx}\frac{dN_2}{dx}dx = \frac{T_2 - T_1}{x_2 - x_1} \tag{369}$$

If the equation above is written in matrix form the equation is identical to the element matrix written in 348 except there is an external forcing function added that must be evaluated.

$$\frac{1}{x_2 - x_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} -T_1^{'} \\ T_2^{'} \end{Bmatrix} + \begin{Bmatrix} \int_{x_1}^{x_2} f(x)N_1 dx \\ \int_{x_1}^{x_2} f(x)N_2 dx \end{Bmatrix} \tag{370}$$

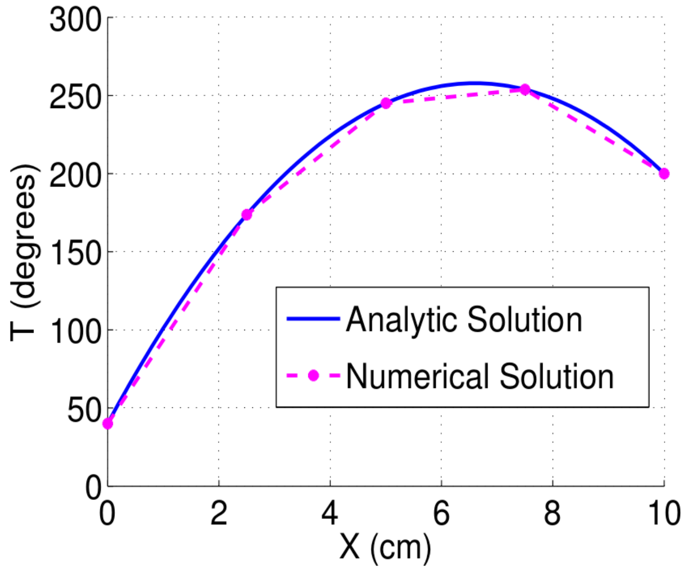
The solution to the example with $Q(x)/L = 10$ is solved in the same fashion as before only the forcing functions must be evaluated for each element. Just as before this is done for all 4 elements to yield 8 equations. The equations are then stacked together to yield only 5 equations as shown in the equations below. The equations are again identical to 353 only the forcing function adds a bit extra to the equation.

$$\begin{bmatrix} 0.4 & -0.4 & 0 & 0 & 0 \\ -0.4 & 0.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 0.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 0.8 & -0.4 \\ 0 & 0 & 0 & -0.4 & 0.4 \end{bmatrix}\begin{Bmatrix} 40 \\ T_2 \\ T_3 \\ T_4 \\ 200 \end{Bmatrix} = \begin{Bmatrix} -T_1^{'} + 12.5 \\ 25 \\ 25 \\ 25 \\ T_5^{'} + 12.5 \end{Bmatrix} \tag{371}$$

Just as before the equations are altered to solve for the introduced unknowns $T_1^{'}$ and $T_2^{'}$ which leads to the equation

$$\begin{bmatrix} 1 & -0.4 & 0 & 0 & 0 \\ 0 & 0.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 0.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 0.8 & 0 \\ 0 & 0 & 0 & -0.4 & 1 \end{bmatrix}\begin{Bmatrix} T_1^{'} \\ T_2 \\ T_3 \\ T_4 \\ T_5^{'} \end{Bmatrix} = \begin{Bmatrix} -3.5 \\ 41 \\ 25 \\ 105 \\ -67.5 \end{Bmatrix} \tag{372}$$

Again, this can be solved on any numerical computer and the solution is shown in the figure below.



Notice however that the solution is not quite exact since the shape functions are linear. Obviously there are solutions where the shape functions are quadratic but these are beyond the scope of this text.

9. References

1. Anh-Vu Phan, Professor, Mechanical Engineering, University of South Alabama
2. *"Numerical Methods for Engineers, 7th Ed."*, S. Chapra and R. Canale, McGraw-Hill, ISBN 9781259318726.