Aircraft Flight Mechanics

A Brief Textbook Presented to the
Student Body of the University of South Alabama

by

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Manuscript Changes

1. December 10th, 2020 - Moved manuscript to public Github repo

Changes Needed

1. Need a section on how to design an RC aircraft

Contents

I. Particle Dynamics

For this formulation we start with Newton's Second Law with no approximations.

$$\sum_{i=0}^{N} \vec{F}_{ji} = \frac{d\vec{p}_j}{dt} \tag{1}$$

where $\vec{p}_j$ is the momentum of a particle. $\vec{F}_{ji}$ is a force on the particle. The statement above states that sum of all forces on a particle is equal to the time rate of change of momentum.

1. Linear Dynamics for Systems of Particles

If two particles are then considered the equation can be written for both particles.

$$\sum_{i=0}^{N} \vec{F}_{1i} + \vec{f}_{12} = \tfrac{d\vec{p}_1}{dt} \quad \sum_{i=0}^{N} \vec{F}_{2i} + \vec{f}_{21} = \tfrac{d\vec{p}_2}{dt} \tag{2}$$

Note that the forces $\vec{f}_{12}$ and $\vec{f}_{21}$ are internal forces experienced by each particle exerted on each other since they are rigidly connected. Newton's Third Law states that for every action there is an equal and opposite reaction. That is, $\vec{f}_{12} = -\vec{f}_{21}$. Thus, if both equations are added the following equation is created

$$\sum_{j=0}^{P} \sum_{i=0}^{N} \vec{F}_{ji} = \sum_{j=0}^{P} \frac{d\vec{p}_j}{dt} \tag{3}$$

where P is the number of particles. Typically the double summation in F is written just as $\vec{F}$.

2. Rotational Dynamics for Systems of Particles

Note that by construction, a system of particles rigidly connected can now rotate about a center point. The center of mass of a system of particles can be defined using the relationship below

$$\vec{r}_C = \frac{1}{m} \sum_{j=0}^{P} m_j \vec{r}_j \tag{4}$$

where

$$m = \sum_{j=0}^{P} m_j \tag{5}$$

This vector can then be used to create rotational dynamics starting with the linear dynamics.

$$\sum_{j=0}^{P} \sum_{i=0}^{N} \mathbf{S}(\vec{r}_{Cj}) \vec{F}_{ji} = \vec{M}_C = \sum_{j=0}^{P} \mathbf{S}(\vec{r}_{Cj}) \frac{d\vec{p}_j}{dt} \tag{6}$$

where $\mathbf{S}(\vec{r}_{Cj})$ is the skew symmetric matrix of the vector from the center of mass to the jth particle which results in a cross product.

II. Rigid Bodies

At this point, many assumptions are made about the system of particles.

1. The mass of each particle or rigid body is constant.
2. The Earth is assumed to be flat and not rotating. This allows the Earth to be used as an inertial frame.
3. The rigid body is not flexible and does not change shape. That is, the time rate of change of the magnitude of a vector $\vec{r}_{PQ}$ is zero for any arbitrary points P and Q attached to the rigid body.

1. Linear Dynamics

Using all of these simplifications, the momentum term on the right can be simplified to

$$\sum_{j=0}^{P} \vec{p}_j = m\vec{v}_{C/I} \tag{7}$$

The derivation of the term above starts by deriving the position of the center of mass as the following equation.

$$\vec{r}_j = \vec{r}_C + \vec{r}_{Cj} \tag{8}$$

Taking one derivative results in the following equation

$$\vec{v}_{j/I} = \vec{v}_{C/I} + \frac{{}^B d\vec{r}_{Cj}}{dt} + \mathbf{S}(\vec{\omega}_{B/I})\vec{r}_{Cj} \tag{9}$$

where $\mathbf{S}(\vec{\omega}_{B/I})$ is the skew symmetric matrix of the angular velocity vector which results in a cross product. This equation comes from the derivative transport theorem. Since the body is a rigid body the term $\frac{{}^B d\vec{r}_{Cj}}{dt} = 0$ resulting in the equation below

$$\vec{v}_{j/I} = \vec{v}_{C/I} + \mathbf{S}(\vec{\omega}_{B/I})\vec{r}_{Cj} \tag{10}$$

which any dynamicist knows as the equation for two points fixed on a rigid body. This equation can then be substituted into the equation for momentum such that.

$$\sum_{j=0}^{P} \vec{p}_j = \sum_{j=0}^{P} m_j \left(\vec{v}_{C/I} + \mathbf{S}(\vec{\omega}_{B/I})\vec{r}_{Cj}\right) \tag{11}$$

The first term reduces to

$$\sum_{j=0}^{P} m_j\vec{v}_{C/I} = \vec{v}_{C/I}\sum_{j=0}^{P} m_j = m\vec{v}_{C/I} \tag{12}$$

the second term reduces to zero since the sum of all particles from the center of mass is by definition the center of mass and thus zero.

$$\sum_{j=0}^{P} \mathbf{S}(\vec{\omega}_{B/I})m_j\vec{r}_{Cj} = \mathbf{S}(\vec{\omega}_{B/I})\sum_{j=0}^{P} m_j\vec{r}_{Cj} = 0 \tag{13}$$

Plugging this result for momentum into Newton's equation of motion yields. This is typically called Newton-Euler equations of motion.

$$\vec{F}_C = m\left(\frac{{}^B d\vec{v}_{C/I}}{dt} + \mathbf{S}(\vec{\omega})_{B/I}\vec{v}_{C/I}\right) \tag{14}$$

2. Rotational Dynamics

Plugging in the expression for two points fixed on a rigid body results in a much different expression. First let's expand the rotational dynamic equations of particles using the assumptions made for a rigid body.

$$\vec{M}_C = \frac{d}{dt}\sum_{j=0}^{P}\mathbf{S}(\vec{r}_{Cj})m_j\vec{v}_{j/I} \tag{15}$$

Then the equation of two points fixed on a rigid body can be introduced to obtain the following equation

$$\vec{M}_C = \frac{d}{dt}\sum_{j=0}^{P}\mathbf{S}(\vec{r}_{Cj})m_j\left(\vec{v}_{C/I} + \mathbf{S}(\vec{\omega}_{B/I})\vec{r}_{Cj}\right) \tag{16}$$

expanding this into two terms yields

$$\vec{M}_C = \frac{d}{dt}\left(\sum_{j=0}^{P}m_j\mathbf{S}(\vec{r}_{Cj})\mathbf{S}(\vec{\omega}_{B/I})\vec{r}_{Cj} + \sum_{j=0}^{P}\mathbf{S}(\vec{r}_{Cj})m_j\vec{v}_{C/I}\right) \tag{17}$$

To simplify this further a useful equality is used for cross products. That is $\mathbf{S}(\vec{a})\vec{b} = -\mathbf{S}(\vec{b})\vec{a}$. The equation above then changes to

$$\vec{M}_C = \frac{d}{dt}\left(\left(-\sum_{j=0}^{P}m_j\mathbf{S}(\vec{r}_{Cj})\mathbf{S}(\vec{r}_{Cj})\right)\vec{\omega}_{B/I} - \mathbf{S}(\vec{v}_{C/I})\sum_{j=0}^{P}\vec{r}_{Cj}m_j\right) \tag{18}$$

Notice, that parentheses were placed around the first term to isolate the angular velocity. This is because the angular velocity is constant across the system of particles. The term on the right has also been altered slightly to isolate the fact that the velocity of the center of mass is independent of the system of particles. With the equation in this form it is easy to see that the term on the right is zero because it is the definition of the center of mass. The equation then reduces to

$$\vec{M}_C = \frac{d}{dt}\left(\sum_{j=0}^{P}m_j\mathbf{S}(\vec{r}_{Cj})\mathbf{S}(\vec{r}_{Cj})^T\right)\vec{\omega}_{B/I} \tag{19}$$

Notice again that minus sign has been removed. The skew symmetric matrix has an interesting property where the transpose is equal to the negative of the original matrix. The term in brackets is a well known value for rigid bodies and is known as the moment of inertia for rigid bodies.

$$\mathbf{I}_C = \sum_{j=0}^{P}m_j\mathbf{S}(\vec{r}_{Cj})\mathbf{S}(\vec{r}_{Cj})^T \tag{20}$$

This results in the kinematic equations of motion for rigid bodies to the simple equation below.

$$\vec{M}_C = \frac{d}{dt}\left(\mathbf{I}_C\vec{\omega}_{B/I}\right) \tag{21}$$

With the equation in this form it is finally possible to carry out the derivative

$$\vec{M}_C = \frac{{}^{B}d(\mathbf{I}_C\vec{\omega}_{B/I})}{dt} + \mathbf{S}(\vec{\omega}_{B/I})\mathbf{I}_C\vec{\omega}_{B/I} \tag{22}$$

The first term requires the chain rule to perform the derivative however the body frame derivative of the moment of inertia matrix is simply zero. Therefore the equation can simply be written as

$$\vec{M}_C = \mathbf{I}_C\frac{{}^{B}d(\vec{\omega}_{B/I})}{dt} + \mathbf{S}(\vec{\omega}_{B/I})\mathbf{I}_C\vec{\omega}_{B/I} \tag{23}$$

III. Aircraft Convention

Aircraft convention involves using the Newton-Euler equations of motion to describe the aircraft. Typically the position of the aircraft is written as

$$\mathbf{C}_I(\vec{r}_C) = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{24}$$

The derivative of the position vector is the velocity vector is then written as

$$\mathbf{C}_I(\vec{v}_{C/I}) = \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} \tag{25}$$

However, body frame coordinates are typically used to describe the velocity vector such that

$$\mathbf{C}_B(\vec{v}_{C/I}) = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \tag{26}$$

In order to relate the body frame components of the velocity vector the inertial frame coordinates a transformation matrix is used. The transformation from the inertial frame to the body frame involves three unique rotations. The first is a rotation about the z-axis such that

$$\mathbf{C}_A(\vec{v}_{C/I}) = \begin{bmatrix} cos(\psi) & sin(\psi) & 0 \\ -sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{C}_I(\vec{v}_{C/I}) \tag{27}$$

this rotation is called the yaw or heading rotation. From here the intermediate frame is rotated about the y-axis such that

$$\mathbf{C}_{NR}(\vec{v}_{C/I}) = \begin{bmatrix} cos(\theta) & 0 & -sin(\theta) \\ 0 & 1 & 0 \\ sin(\theta) & 0 & cos(\theta) \end{bmatrix} \mathbf{C}_A(\vec{v}_{C/I}) \tag{28}$$

this rotation is called the pitch angle rotation. Finally the no roll frame is rotated through the x-axis such that

$$\mathbf{C}_B(\vec{v}_{C/I}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\phi) & sin(\phi) \\ 0 & -sin(\phi) & cos(\phi) \end{bmatrix} \mathbf{C}_{NR}(\vec{v}_{C/I}) \tag{29}$$

Putting all the equations together yields

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} = [\mathbf{T}_{IB}] \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \tag{30}$$

where

$$\mathbf{T}_{IB} = \begin{bmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{bmatrix} \tag{31}$$

Standard shorthand notation is used for trigonometric functions: $cos(\alpha) \equiv c_\alpha$, $sin(\alpha) \equiv s_\alpha$, and $tan(\alpha) \equiv t_\alpha$. These three angles are known as the standard Euler angle rotation sequence of an aircraft in free flight. The angular velocity of a body is typically written as

$$\mathbf{C}_B(\vec{\omega}_{B/I}) = \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = p\hat{I}_B + q\hat{J}_B + r\hat{K}_B \tag{32}$$

There are no inertial components for the angular velocity vector. However, a relationship can be derived relating the derivatives of the Euler angles. The angular velocity can be written in vector form such that

$$\vec{\omega}_{B/I} = \dot{\psi}\hat{K}_A + \dot{\theta}\hat{J}_{NR} + \dot{\phi}\hat{I}_B \tag{33}$$

relating the unit vectors $\hat{K}_A$ and $\hat{J}_{NR}$ to the body frame using the planar rotation matrices results in the equation below. Note that NR is denoted as the "No-Roll" frame.

$$\begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} = [\mathbf{H}] \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} \tag{34}$$

where

$$\mathbf{H} = \begin{bmatrix} 1 & s_\phi t_\theta & c_\phi t_\theta \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi / c_\theta & c_\phi / c_\theta \end{bmatrix} \tag{35}$$

with all this information the Newton-Euler equations of motion can be used to form the equation below.

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Bmatrix} = \frac{1}{m} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} - \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \tag{36}$$

Using the rotational dynamic equations for rigid bodies, the equation for the derivative of angular velocity can be found as

$$\begin{Bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{Bmatrix} = \mathbf{I}_C^{-1} \left(\begin{Bmatrix} L \\ M \\ N \end{Bmatrix} - \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \mathbf{I}_C \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} \right) \tag{37}$$

Note that standard aircraft forces and moments are applied to the body. The forces are typically written as X,Y and Z while the moments are given as L,M and N. They can be written in component form using the equations below.

$$\mathbf{C}_B(\vec{F}_C) = \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = X\hat{I}_B + Y\hat{J}_B + Z\hat{K}_B \tag{38}$$

$$\mathbf{C}_B(\vec{M}_C) = \begin{Bmatrix} L \\ M \\ N \end{Bmatrix} = L\hat{I}_B + M\hat{J}_B + N\hat{K}_B \tag{39}$$

IV. Forces on Aircraft

To form the sum of the forces on an aircraft the assumptions are made that only gravity and aerodynamic forces act. For rockets, a propulsion model can be added. Still the gravitational force is shown below.

1. Gravity

The weight contribution is assumed to be a constant force applied to the aircraft. The equation below is the gravitational force applied in the body frame.

$$\begin{Bmatrix} X_W \\ Y_W \\ Z_W \end{Bmatrix} = mg \begin{Bmatrix} -s_\theta \\ s_\phi c_\theta \\ c_\phi c_\theta \end{Bmatrix} \tag{40}$$

2. Aerodynamics

Aircraft aerodynamics are written using a taylor series expansion about a trim point. That is, the aerodynamic forces are given by

$$\vec{F} = \vec{F}_0 + \frac{\partial \vec{F}}{\partial \vec{x}}(\vec{x} - \vec{x}_0) \tag{41}$$

where $\vec{x} = [x, y, z, \phi, \theta, \psi, u, v, w, p, q, r]^T$. The partial derivative is thus expanded such that

$$\frac{\partial \vec{F}}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial \vec{F}}{\partial x} & \frac{\partial \vec{F}}{\partial y} & \dots & \frac{\partial \vec{F}}{\partial r} \end{bmatrix} \tag{42}$$

To find all of the partial derivative the forces are first written using a combination of dynamic pressure and coefficients that are functions of geometry and Reynolds number rather than speed, pressure and size. A general lift force can be written using the equation below

$$L = \frac{1}{2}\rho {V_\infty}^2 S C_L \tag{43}$$

where ρ is the atmospheric density, V_∞ is the free-stream velocity, S is the planform area of the wing and C_L is the lift coefficient.

$$V_\infty = \sqrt{{u_a}^2 + {v_a}^2 + {w_a}^2} \tag{44}$$

The subscript 'a' above denotes the velocity of the aircraft plus the atmospheric disturbance.

$$\begin{Bmatrix} u_a \\ v_a \\ w_a \end{Bmatrix} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} + \mathbf{T}^T{}_{IB} \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} \tag{45}$$

Note that the dynamic pressure is different for a rocket or projectile. A similar expression can be created for a generic moment such that

$$M = \frac{1}{2}\rho {V_\infty}^2 S \bar{c} C_M \tag{46}$$

where $\bar{c}$ is the mean chord of the aircraft. The dynamic pressure $q_\infty = \frac{1}{2}\rho {V_\infty}^2 S$ can be used to non-dimensionalize the forces, thus $L/q_\infty = C_L$. This means that the equation involving partial derivatives can be written as

$$\frac{\partial \vec{C}_F}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial \vec{C}_F}{\partial x} & \frac{\partial \vec{C}_F}{\partial y} & \dots & \frac{\partial \vec{C}_F}{\partial r} \end{bmatrix} \tag{47}$$

If the vector is then expanded to include the components of the vector $\vec{F}$ the partial derivatives expand to

$$\frac{\partial \vec{C}_F}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial C_X}{\partial x} & \frac{\partial C_X}{\partial y} & \dots & \frac{\partial C_x}{\partial r} \\ \frac{\partial C_Y}{\partial x} & \frac{\partial C_Y}{\partial y} & \dots & \frac{\partial C_Y}{\partial r} \\ \frac{\partial C_Z}{\partial x} & \frac{\partial C_Z}{\partial y} & \dots & \frac{\partial C_Z}{\partial r} \end{bmatrix} \tag{48}$$

shorthand can be adopted for the forces above such that $\frac{\partial C_Y}{\partial x} = C_{Yx}$. Using this shorthand the equation above can be written as.

$$\frac{\partial \vec{C}_F}{\partial \vec{x}} = \begin{bmatrix} C_{Xx} & C_{Xy} & ... & C_{Xr} \\ C_{Yx} & C_{Yy} & ... & C_{Yr} \\ C_{Zx} & C_{Zy} & ... & C_{Zr} \end{bmatrix} \tag{49}$$

The coefficients listed above are standard coefficients that all aircraft have. A similar matrix can be formulated for the moments on an aircraft. When system identifying an aircraft all of these coefficients may be determined. However, many of these terms are zero. For example, all coefficients with respect to x y and z are zero. That is, $C_{Xx} = C_{Yx} = ...C_{Nx} = C_{Xy} = ...C_{Nz} = 0$. Other coefficients can be set to zero as well.

Aircraft Aerodynamics

For aircraft, some further simplifications are made. Some of the coefficients defined above are combined to be written as functions of the angle of attack(α) and sideslip(β).

$$\alpha = tan^{-1}\left(\frac{w_a}{u_a}\right) \tag{50}$$

$$\beta = sin^{-1}\left(\frac{v_a}{V_\infty}\right) \tag{51}$$

Transforming the equations into these formulations gives rise to coefficients such as $C_{L\alpha}$ which is the change in lift as a function of angle of attack and $C_{Y\beta}$ which is the change in Y-Force as a function of sideslip. Using all of the coefficients defined above taking into account the change to lift and drag, the body aerodynamic force is calculated using the equation below.

$$\begin{Bmatrix} X_A \\ Y_A \\ Z_A \end{Bmatrix} = \frac{1}{2}\rho {V_\infty}^2 S \begin{Bmatrix} C_L s_\alpha - C_D c_\alpha + C_{x\delta_t}\delta_t \\ C_{y\beta}\beta + C_{y\delta_r}\delta_r + C_{yp}\frac{pb}{2V_\infty} + C_{yr}\frac{rb}{2V_\infty} \\ -C_L c_\alpha - C_D s_\alpha \end{Bmatrix} \tag{52}$$

Where the lift and drag coefficients are:

$$\begin{Bmatrix} C_L \\ C_D \end{Bmatrix} = \begin{Bmatrix} C_{L0} + C_{L\alpha}\alpha + C_{Lq}\frac{q\bar{c}}{2V_\infty} + C_{L\delta_e}\delta_e \\ C_{D0} + C_{D\alpha}\alpha^2 \end{Bmatrix} \tag{53}$$

The body aerodynamic moment is also computed using an aerodynamic expansion.

$$\begin{Bmatrix} L_A \\ M_A \\ N_A \end{Bmatrix} = \frac{1}{2}\rho {V_\infty}^2 S\bar{c} \begin{Bmatrix} C_{l\beta}\beta + C_{lp}\frac{pb}{2V_\infty} + C_{lr}\frac{rb}{2V_\infty} + C_{l\delta_a}\delta_a + C_{l\delta_r}\delta_r \\ C_{m0} + C_{m\alpha}\alpha + C_{mq}\frac{q\bar{c}}{2V_\infty} + C_{m\delta_e}\delta_e \\ C_{np}\frac{pb}{2V_\infty} + C_{n\beta}\beta + C_{nr}\frac{rb}{2V_\infty} + C_{n\delta_a}\delta_a + C_{n\delta_r}\delta_r \end{Bmatrix} \tag{54}$$

The aerodynamic coefficients in equations (52), (53) and (54) can be obtained from flight data, aerodynamic modeling and windtunnel tests. Notice that the only coefficients remaining are coefficients from angle of attack, sideslip and angular velocities. Furthermore, the coefficients for angular velocities are also non-dimensionalized by terms such as $b/(2V_\infty)$ where b is the wingspan of the aircraft and $\bar{c}$ is the mean chord of the aircraft. These terms are introduced to fully non-dimensionalize the coefficients. Notice, as well that four extra terms were also introduced. These will be discussed in more detail in the control section however the four terms are the aileron control surface δ_a, the elevator control surface δ_e, the rudder control surface δ_r and the thrust control value δ_t.

Projectile Aerodynamics

To fully define the projectile aerodynamics some more assumptions are made about the projectile.

1. The projectile is axially symmetric
2. The aerodynamic forces are not necessarily formulated at the center of mass
3. The projectile has the potential to be spinning rapidly thus interacting with the surrounding atmosphere

For a projectile the dynamic pressure is written as

$$Q = \frac{\pi}{8}\rho V_\infty^2 d^2 \tag{55}$$

The aerodynamic forces on the projectile are modeled using taylor series ballistic expansions with known coefficients similar to the aircraft model only slightly different assumptions are made given the dynamics of the projectile. The subscripts in the equation below stand for steady and unsteady aerodynamics.

$$\begin{Bmatrix} X_A \\ Y_A \\ Z_A \end{Bmatrix} = \begin{Bmatrix} X_{SA} \\ Y_{SA} \\ Z_{SA} \end{Bmatrix} + \begin{Bmatrix} X_{UA} \\ Y_{UA} \\ Z_{UA} \end{Bmatrix} = Q \begin{Bmatrix} -C_{X_0} - C_{X_2}\frac{v^2+w^2}{V^2} \\ -C_{Y_\beta}\frac{v}{V} \\ -C_{N_\alpha}\frac{w}{V} \end{Bmatrix} + Q \begin{Bmatrix} 0 \\ C_{Y_{p\alpha}}\frac{w}{V}\frac{pd}{2V} \\ C_{Z_{p\alpha}}\frac{v}{V}\frac{pd}{2V} \end{Bmatrix} \tag{56}$$

In this equation, Q is the dynamic pressure, d is the aerodynamic reference area, C_{X_0} is the zero-yaw axial force coefficient,C_{X_2} is the yaw-squared axial force coefficient, C_{N_α} is the normal force derivative coefficient, $C_{Y_{p\alpha}}$ is the Magnus force coefficient, and $V = \sqrt{u^2+v^2+w^2}$ is the total velocity of the projectile. The aerodynamic moments acting on the projectile are the pitching, pitch damping, Magnus, and roll damping moments. Pitching and Magnus moments are given by taking the cross product of the normal and Magnus forces given in (56) with the position vector from the center of mass to the center of pressure and location of Magnus force, respectively. The total aerodynamic moments are given in Eqn. (57).

$$\begin{Bmatrix} L_A \\ M_A \\ N_A \end{Bmatrix} = \mathbf{S}_B(\vec{r}_{CG,COP}) \begin{Bmatrix} X_{SA} \\ Y_{SA} \\ Z_{SA} \end{Bmatrix} + \mathbf{S}_B(\vec{r}_{CG,MCOP}) \begin{Bmatrix} X_{UA} \\ Y_{UA} \\ Z_{UA} \end{Bmatrix} + Qd \begin{Bmatrix} C_{l_p}\frac{pd}{2V} \\ C_{m_q}\frac{qd}{2V} \\ C_{n_r}\frac{rd}{2V} \end{Bmatrix} \tag{57}$$

Here, $\mathbf{S}_B(\vec{r}_{CG,COP})$ is the skew-symmetric operator acting on the position vector from the center of mass to the center of pressure expressed in the projectile body frame. Furthermore, $\mathbf{S}_B(\vec{r}_{CG,MCOP})$ is the skew-symmetric operator acting on the position vector from the center of mass to the magnus center of pressure expressed in the projectile body frame. Typically the center of mass is defined from the rear of the projectile such that

$$\mathbf{C}_B(\vec{r}_{CG}) = \begin{Bmatrix} SL_{CG} \\ BL_{CG} \\ WL_{CG} \end{Bmatrix} \tag{58}$$

Similarly, the center of pressure is defined from the rear of the projectile such that

$$\mathbf{C}_B(\vec{r}_{COP}) = \begin{Bmatrix} SL_{COP} \\ BL_{COP} \\ WL_{COP} \end{Bmatrix} \tag{59}$$

The vector $\vec{r}_{CG,COP}$ is then simply the different between both vectors.

$$\vec{r}_{CG,COP} = \vec{r}_{COP} - \vec{r}_{CG} \tag{60}$$

The damping coefficient defined in equation (57) include C_{l_p} which is the roll damping coefficient while C_{m_q} is the pitch damping coefficient. These coefficients are added which essentially inhibit angular motion of the projectile. In addition, to these coefficients, sometimes magnus coefficients are given as pure moments rather than forces acting at a distance. This can be given in the equation below.

$$M_{UA} = Qd(-C_{M\alpha}\frac{v}{V} + C_{N_{p\alpha}}\frac{w}{V}\frac{pd}{2V}) \tag{61}$$

Where $C_{M\alpha}$ replaces the moment produced by $C_{N\alpha}$ and $C_{N_{p\alpha}}$ replaces the moment produced by $C_{Y_{p\alpha}}$. It is possible to derive an equation between the two different representations as given by the equations below.

$$\begin{aligned} C_{M\alpha} &= \tfrac{(SL_{COP} - SL_{CG})C_{N\alpha}}{d} \\ C_{N_{p\alpha}} &= \tfrac{(SL_{MAG} - SL_{CG})C_{Y_{p\alpha}}}{d} \end{aligned} \tag{62}$$

V. Stability and Control

Controllability is formally stated as a system where any initial state $x(0) = x_0$ and final state $x_1, t_1 > 0$, there exists a piecewise continuous input $u(t)$ such that $x(t_1) = x_1$. For a fixed wing aircraft the system has 12 states with 8 dynamic modes and 4 zero or rigid body modes. For a fixed wing aircraft the system has 12 states with 8 dynamic modes and 4 zero or rigid body modes. A conventional aircraft has 4 controls to control these 12 modes. The easiest way to test the controllability of a system is to compute the controllability matrix. However, the controllability matrix must be computed using a linearized model such that $\dot{\vec{x}} = A\vec{x} + B\vec{u}$. In order to do this the aircraft must be in equilibrium. For this example the aircraft is set with an initial velocity of 20 m/s at an altitude of 200 m. The altitude command is set to 200 m and the heading command is set to zero. Given the zero heading angle command and the symmetry of the configurations investigated the rudder and aileron commands are set to zero. Thus, only the thrust and elevator controls are activated for the trimming procedure. Each configuration is simulated for 200 seconds or until the derivatives of all states except $\dot{x}$ are within a required tolerance. Using this equilibrium point a linear model can be computed by using forward finite differencing assuming that the aircraft model is put in the form $\dot{\vec{x}} = F(\vec{x}, \vec{u})$.

$$\dot{\vec{\delta x}} = \frac{F(\vec{x_0} + \Delta\vec{x_0}, \vec{u_0}) - F(\vec{x_0}, \vec{u_0})}{\Delta\vec{x}}\vec{\delta x} + \frac{F(\vec{x_0}, \vec{u_0} + \Delta\vec{u}) - F(\vec{x_0}, \vec{u_0})}{\Delta\vec{u}}\vec{\delta u} \tag{63}$$

This linear model is the classic linear model where $\dot{\vec{\delta x}} = A\vec{\delta x} + B\vec{\delta u}$. Using this linear model, the controllability matrix can be computed as

$$W_C = [B \ AB \ A^2B \ A^3B \ ... \ A^{N-1}B] \tag{64}$$

where N is the number of states in the system. With the controllability matrix formulated, the rank of the matrix is computed. If the $rank(W_C) = N$ the system is said to be controllable.

VI. PID Control

For a conventional PID controller of an aircraft, the rudder, elevator and aileron commands are set to

$$\begin{aligned}\delta_r &= -K_v v \\ \delta_e &= K_p(\theta - \theta_c) + K_d\dot{\theta} \\ \delta_a &= K_p(\phi - \phi_c) + K_d\dot{\phi}\end{aligned} \tag{65}$$

The Euler angle commands ϕ_c and θ_c are set using the following relationships:

$$\begin{aligned}\phi_c &= K_p(\psi - \psi_c) + K_d\dot{\psi} \\ \theta_c &= K_p(z - z_c) + K_d\dot{z} + K_I \int z - z_c dt\end{aligned} \tag{66}$$

The control scheme defined above is a conventional inner loop-outer loop control of a fixed wing aircraft using a PID tracking controller.

VII. AIRCRAFT NOMENCLATURE

x_i, y_i, z_i	components of the mass center position vector in the inertial frame of aircraft i (m)
ϕ_i, θ_i, ψ_i	Euler roll,pitch, and yaw of aircraft i (rad)
u_i, v_i, w_i	components of the mass center velocity vector in the body frame of aircraft i (m/s)
p_i, q_i, r_i	components of the mass center angular velocity vector in the body frame of aircraft i (rad/
$\vec{r}_{A\to B}$	position vector from a generic point A to a generic point B(m)
$\vec{V}_{A/B}$	velocity vector of a generic point A with respect to frame B (m/s)
$\mathbf{T}_{AB}$	generic transformation matrix rotating a vector from the frame B to frame A
$\mathbf{H}_i$	relationship matrix of Euler angle derivatives to body angular velocity components of aircra
m_i	mass of aircraft i (kg)
I_i	moment of inertia matrix of aircraft i taken about the mass center in the body frame($kg -$
X_i, Y_i, Z_i	components of the total force applied to aircraft i in body frame(N)
L_i, M_i, N_i	components of the total moment applied to aircraft i in body frame(N-m)
X_{Wi}, Y_{Wi}, Z_{Wi}	total weight force applied to aircraft i (N)
L, D	Lift and Drag on Aircraft (N) - Not to be confused with Roll moment
g	gravitational constant on Earth (m/s^2)
ρ	atmospheric density(kg/m^3)
S_i	reference area of wing on aircraft i (m^2)
b_i	Wingspan of aircraft i (m)
$\bar{c}_i$	mean chord of wing on aircraft i (m)
α	Angle of attack (rad)
β	Slideslip angle (rad)
C_L, C_D, C_m	Lift, Drag and Pitch Moment coefficients
$\delta_t, \delta_a, \delta_r, \delta_e$	thrust, aileron, rudder, and elevator control inputs(rad)
$S_B(\vec{r})$	skew symmetric matrix operator on a vector expressed in the body frame.
K_p, K_d, K_I	proportional, derivative, and integral control gains
V	Total airspeed (m/s)
$\hat{p}, \hat{q}, \hat{r}$	Non-dimensional angular velocities
l	Distance from center of mass to aerodynamic center of the tail (m)
l_t	Distance from aerodynamic center of main wing to aerodynamic center of tail (m)
α_0	zero lift angle of attack (rad)
C_{L0}	Zero angle of attack lift coefficient
$C_{m\alpha}$	Pitch moment curve slope versus α
$C_{L\alpha}$	Lift curve slope
C_{mq}	Pitch damping coefficient
$C_{m\delta_e}$	Pitch moment curve slope versus elevator deflection angle
a_∞	Speed of sound (m/s)
μ_∞	Viscosity of Fluid $kg/(m-s)$

VIII. EQUATIONS

Mach Number and Reynolds Number

$$M_\infty = \frac{V}{a_\infty}$$

$$Re = \frac{\rho V \bar{c}}{\mu_\infty}$$

Total Velocity

$$V = \sqrt{u^2 + v^2 + w^2}$$

Angle of Attack and Sideslip

$$\alpha = tan^{-1}\left(\frac{w}{u}\right)$$
$$\beta = sin^{-1}\left(\frac{v}{V}\right)$$

Lift Drag and Moment

$$Lift\ (L) = \frac{1}{2}\rho V^2 S C_L$$
$$Drag\ (D) = \frac{1}{2}\rho V^2 S C_D$$
$$Roll\ Moment\ (L) = \frac{1}{2}\rho V^2 S b C_l$$
$$Pitch\ Moment\ (M) = \frac{1}{2}\rho V^2 S \bar{c} C_m$$
$$Yaw\ Moment\ (N) = \frac{1}{2}\rho V^2 S b C_n$$

Lift and Drag Coefficients

$$C_L = C_{L0} + C_{L\alpha}\alpha$$
$$C_L = C_{L\alpha}(\alpha - \alpha_0)$$
$$C_D = C_{D0} + C_{D\alpha}\alpha^2$$
$$C_D = C_{D0} + k{C_L}^2$$

Non-dimensional Angular velocities

$$\hat{p} = pb/2V$$
$$\hat{q} = q\bar{c}/2V$$
$$\hat{r} = rb/2V$$

Pitch Moment equation

$$C_m = C_{m0} + C_{m\alpha}\alpha + C_{m\delta_e}\delta_e + C_{mq}\hat{q}$$

$$C_{m0} = C_{MAC} + C_{L0}\bar{x}_{sm}$$
$$\bar{x}_{sm} = \frac{x_{cg}}{\bar{c}} - \frac{x_{acW}}{\bar{c}}$$
$$C_{m\alpha} = \left(C_{L\alpha,W} + \frac{S_t}{S}C_{L\alpha,t}\right)\bar{x}_{sm} - V_H C_{L\alpha,t}$$
$$V_H = \frac{l_t S_t}{S\bar{c}}$$
$$C_{m\delta_e} = \left(C_{Lt\delta_e}\frac{S_t}{S}\right)\bar{x}_{sm} - V_H C_{Lt\delta_e}$$
$$C_{mq} = 2C_{L\alpha t}\frac{l^2}{\bar{c}^2}$$

Max Lift to Drag Ratio (Only valid if $C_{L0} = 0$)

$$\alpha_{max,L/D} = \sqrt{\frac{C_{D0}}{C_{D\alpha}}}$$

Lift to Drag when $T = 0$ (Sum of Forces still zero)

$$\frac{D}{L} = tan(\alpha)$$
$$Lcos(\alpha) + Dsin(\alpha) = W$$

Airfoil and Wing Aerodynamics

$$x_{ac} = c/4 \quad a = \frac{a_0}{1 + \frac{a_0}{\pi e AR}}$$
$$AR = \frac{b^2}{S}$$

Standard Atmosphere

$$\rho = 1.225\ kg/m^3 = 0.00238\ slugs/ft^3$$
$$\mu_\infty = 1.81x10^{-5}\ kg/(m-s)$$
$$a_\infty = 331.3\ m/s$$

General Notes

1. In trim or steady and level or cruise $q = 0$, $C_m = 0$,$L = W$, $T = D$
2. For symmetric airfoil $C_{MAC} = 0$ and $C_{L0} = 0$ thus $C_{m0} = 0$
3. For a flat plate all symmetric properties apply but $a_0 = 2\pi$
4. Tail surfaces are always assumed to be flat plates
5. For longitudinal problems, $\beta = 0$ so $v = 0$ (side velocity)