

Aircraft Flight Mechanics

A Brief Textbook Presented to the
Student Body of the University of South Alabama

by

Carlos Montalvo

University of South Alabama

Copyright © Carlos José Montalvo

Last Update: Wednesday 30th December, 2020

Manuscript Changes

1. December 10th, 2020 - Moved manuscript to public Github repo
2. December 30th, 2020 - Added datetime to title page. Added section headers to RC aircraft design. Wrote text all the way through the airfoil selection. Added a references section.

Changes Needed

1. Need to finish section on how to design an RC aircraft
2. It would be nice to include some plots on RC aircraft design sections
3. I think an example RC aircraft design example would be good

Contents

I.	Particle Dynamics	6
1.	Linear Dynamics for Systems of Particles	6
2.	Rotational Dynamics for Systems of Particles	6
II.	Rigid Bodies	7
1.	Linear Dynamics	7
2.	Rotational Dynamics	8
III.	Aircraft Convention	9
IV.	Forces on Aircraft	10
1.	Gravity	11
2.	Aerodynamics	11
V.	Stability and Control	14
VI.	PID Control	15
VII.	Radio Controlled Aircraft Design	15
1.	Vehicle Type Selection and Requirements	15
2.	Initial Design - Hand Sketch and Aspect Ratio	16
3.	Weight Estimate - Tabular Approach	17
4.	Airfoil Selection and 2D and 3D Lift	17
5.	Wing Loading and Thrust to Weight Ratio	19
6.	Stability and Control, Center of Mass and Static Margin	19
7.	Re-Design - Iteration and Detailed Sketch	19
8.	Computer Aided Design	19
9.	Purchase Components	19
10.	Flying	19
VIII.	AIRCRAFT NOMENCLATURE	22
IX.	EQUATIONS	23

I. Particle Dynamics

For this formulation we start with Newton's Second Law with no approximations[1].

$$\sum_{i=0}^N \vec{F}_{ji} = \frac{d\vec{p}_j}{dt} \quad (1)$$

where \vec{p}_j is the momentum of a particle. \vec{F}_{ji} is a force on the particle. The statement above states that sum of all forces on a particle is equal to the time rate of change of momentum.

1. Linear Dynamics for Systems of Particles

If two particles are then considered the equation can be written for both particles.

$$\sum_{i=0}^N \vec{F}_{1i} + \vec{f}_{12} = \frac{d\vec{p}_1}{dt} \quad \sum_{i=0}^N \vec{F}_{2i} + \vec{f}_{21} = \frac{d\vec{p}_2}{dt} \quad (2)$$

Note that the forces \vec{f}_{12} and \vec{f}_{21} are internal forces experienced by each particle exerted on each other since they are rigidly connected. Newton's Third Law states that for every action there is an equal and opposite reaction. That is, $\vec{f}_{12} = -\vec{f}_{21}$. Thus, if both equations are added the following equation is created

$$\sum_{j=0}^P \sum_{i=0}^N \vec{F}_{ji} = \sum_{j=0}^P \frac{d\vec{p}_j}{dt} \quad (3)$$

where P is the number of particles. Typically the double summation in F is written just as \vec{F} .

2. Rotational Dynamics for Systems of Particles

Note that by construction, a system of particles rigidly connected can now rotate about a center point. The center of mass of a system of particles can be defined using the relationship below

$$\vec{r}_C = \frac{1}{m} \sum_{j=0}^P m_j \vec{r}_j \quad (4)$$

where

$$m = \sum_{j=0}^P m_j \quad (5)$$

This vector can then be used to create rotational dynamics starting with the linear dynamics.

$$\sum_{j=0}^P \sum_{i=0}^N \mathbf{S}(\vec{r}_{Cj}) \vec{F}_{ji} = \vec{M}_C = \sum_{j=0}^P \mathbf{S}(\vec{r}_{Cj}) \frac{d\vec{p}_j}{dt} \quad (6)$$

where $\mathbf{S}(\vec{r}_{Cj})$ is the skew symmetric matrix of the vector from the center of mass to the jth particle which results in a cross product.

II. Rigid Bodies

At this point, many assumptions are made about the system of particles.

1. The mass of each particle or rigid body is constant.
2. The Earth is assumed to be flat and not rotating. This allows the Earth to be used as an inertial frame.
3. The rigid body is not flexible and does not change shape. That is, the time rate of change of the magnitude of a vector \vec{r}_{PQ} is zero for any arbitrary points P and Q attached to the rigid body.

1. Linear Dynamics

Using all of these simplifications, the momentum term on the right can be simplified to

$$\sum_{j=0}^P \vec{p}_j = m\vec{v}_{C/I} \quad (7)$$

The derivation of the term above starts by deriving the position of the center of mass as the following equation.

$$\vec{r}_j = \vec{r}_C + \vec{r}_{Cj} \quad (8)$$

Taking one derivative results in the following equation

$$\vec{v}_{j/I} = \vec{v}_{C/I} + \frac{^B d\vec{r}_{Cj}}{dt} + \mathbf{S}(\vec{\omega}_{B/I})\vec{r}_{Cj} \quad (9)$$

where $\mathbf{S}(\vec{\omega}_{B/I})$ is the skew symmetric matrix of the angular velocity vector which results in a cross product. This equation comes from the derivative transport theorem. Since the body is a rigid body the term $\frac{^B d\vec{r}_{Cj}}{dt} = 0$ resulting in the equation below

$$\vec{v}_{j/I} = \vec{v}_{C/I} + \mathbf{S}(\vec{\omega}_{B/I})\vec{r}_{Cj} \quad (10)$$

which any dynamicist knows as the equation for two points fixed on a rigid body. This equation can then be substituted into the equation for momentum such that.

$$\sum_{j=0}^P \vec{p}_j = \sum_{j=0}^P m_j (\vec{v}_{C/I} + \mathbf{S}(\vec{\omega}_{B/I})\vec{r}_{Cj}) \quad (11)$$

The first term reduces to

$$\sum_{j=0}^P m_j \vec{v}_{C/I} = \vec{v}_{C/I} \sum_{j=0}^P m_j = m\vec{v}_{C/I} \quad (12)$$

the second term reduces to zero since the sum of all particles from the center of mass is by definition the center of mass and thus zero.

$$\sum_{j=0}^P \mathbf{S}(\vec{\omega}_{B/I}) m_j \vec{r}_{Cj} = \mathbf{S}(\vec{\omega}_{B/I}) \sum_{j=0}^P m_j \vec{r}_{Cj} = 0 \quad (13)$$

Plugging this result for momentum into Newton's equation of motion yields. This is typically called Newton-Euler equations of motion.

$$\vec{F}_C = m \left(\frac{^B d\vec{v}_{C/I}}{dt} + \mathbf{S}(\vec{\omega})_{B/I} \vec{v}_{C/I} \right) \quad (14)$$

2. Rotational Dynamics

Plugging in the expression for two points fixed on a rigid body results in a much different expression. First let's expand the rotational dynamic equations of particles using the assumptions made for a rigid body.

$$\vec{M}_C = \frac{d}{dt} \sum_{j=0}^P \mathbf{S}(\vec{r}_{Cj}) m_j \vec{v}_{j/I} \quad (15)$$

Then the equation of two points fixed on a rigid body can be introduced to obtain the following equation

$$\vec{M}_C = \frac{d}{dt} \sum_{j=0}^P \mathbf{S}(\vec{r}_{Cj}) m_j (\vec{v}_{C/I} + \mathbf{S}(\vec{\omega}_{B/I}) \vec{r}_{Cj}) \quad (16)$$

expanding this into two terms yields

$$\vec{M}_C = \frac{d}{dt} \left(\sum_{j=0}^P m_j \mathbf{S}(\vec{r}_{Cj}) \mathbf{S}(\vec{\omega}_{B/I}) \vec{r}_{Cj} + \sum_{j=0}^P \mathbf{S}(\vec{r}_{Cj}) m_j \vec{v}_{C/I} \right) \quad (17)$$

To simplify this further a useful equality is used for cross products. That is $\mathbf{S}(\vec{a})\vec{b} = -\mathbf{S}(\vec{b})\vec{a}$. The equation above then changes to

$$\vec{M}_C = \frac{d}{dt} \left(\left(- \sum_{j=0}^P m_j \mathbf{S}(\vec{r}_{Cj}) \mathbf{S}(\vec{r}_{Cj}) \right) \vec{\omega}_{B/I} - \mathbf{S}(\vec{v}_{C/I}) \sum_{j=0}^P \vec{r}_{Cj} m_j \right) \quad (18)$$

Notice, that parentheses were placed around the first term to isolate the angular velocity. This is because the angular velocity is constant across the system of particles. The term on the right has also been altered slightly to isolate the fact that the velocity of the center of mass is independent of the system of particles. With the equation in this form it is easy to see that the term on the right is zero because it is the definition of the center of mass. The equation then reduces to

$$\vec{M}_C = \frac{d}{dt} \left(\sum_{j=0}^P m_j \mathbf{S}(\vec{r}_{Cj}) \mathbf{S}(\vec{r}_{Cj})^T \right) \vec{\omega}_{B/I} \quad (19)$$

Notice again that minus sign has been removed. The skew symmetric matrix has an interesting property where the transpose is equal to the negative of the original matrix. The term in brackets is a well known value for rigid bodies and is known as the moment of inertia for rigid bodies.

$$\mathbf{I}_C = \sum_{j=0}^P m_j \mathbf{S}(\vec{r}_{Cj}) \mathbf{S}(\vec{r}_{Cj})^T \quad (20)$$

This results in the kinematic equations of motion for rigid bodies to the simple equation below.

$$\vec{M}_C = \frac{d}{dt} (\mathbf{I}_C \vec{\omega}_{B/I}) \quad (21)$$

With the equation in this form it is finally possible to carry out the derivative

$$\vec{M}_C = \frac{{}^B d(\mathbf{I}_C \vec{\omega}_{B/I})}{dt} + \mathbf{S}(\vec{\omega}_{B/I}) \mathbf{I}_C \vec{\omega}_{B/I} \quad (22)$$

The first term requires the chain rule to perform the derivative however the body frame derivative of the moment of inertia matrix is simply zero. Therefore the equation can simply be written as

$$\vec{M}_C = \mathbf{I}_C \frac{{}^B d(\vec{\omega}_{B/I})}{dt} + \mathbf{S}(\vec{\omega}_{B/I}) \mathbf{I}_C \vec{\omega}_{B/I} \quad (23)$$

III. Aircraft Convention

Aircraft convention involves using the Newton-Euler equations of motion to describe the aircraft[2]. Typically the position of the aircraft is written as

$$\mathbf{C}_I(\vec{r}_C) = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \quad (24)$$

The derivative of the position vector is the velocity vector is then written as

$$\mathbf{C}_I(\vec{v}_{C/I}) = \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} \quad (25)$$

However, body frame coordinates are typically used to describe the velocity vector such that

$$\mathbf{C}_B(\vec{v}_{C/I}) = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \quad (26)$$

In order to relate the body frame components of the velocity vector the inertial frame coordinates a transformation matrix is used. The transformation from the inertial frame to the body frame involves three unique rotations. The first is a rotation about the z-axis such that

$$\mathbf{C}_A(\vec{v}_{C/I}) = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{C}_I(\vec{v}_{C/I}) \quad (27)$$

this rotation is called the yaw or heading rotation. From here the intermediate frame is rotated about the y-axis such that

$$\mathbf{C}_{NR}(\vec{v}_{C/I}) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \mathbf{C}_A(\vec{v}_{C/I}) \quad (28)$$

this rotation is called the pitch angle rotation. Finally the no roll frame is rotated through the x-axis such that

$$\mathbf{C}_B(\vec{v}_{C/I}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{bmatrix} \mathbf{C}_{NR}(\vec{v}_{C/I}) \quad (29)$$

Putting all the equations together yields

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} = [\mathbf{T}_{IB}] \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \quad (30)$$

where

$$\mathbf{T}_{IB} = \begin{bmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{bmatrix} \quad (31)$$

Standard shorthand notation is used for trigonometric functions: $\cos(\alpha) \equiv c_\alpha$, $\sin(\alpha) \equiv s_\alpha$, and $\tan(\alpha) \equiv t_\alpha$. These three angles are known as the standard Euler angle rotation sequence of an aircraft in free flight. The angular velocity of a body is typically written as

$$\mathbf{C}_B(\vec{\omega}_{B/I}) = \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = p\hat{I}_B + q\hat{J}_B + r\hat{K}_B \quad (32)$$

There are no inertial components for the angular velocity vector. However, a relationship can be derived relating the derivatives of the Euler angles. The angular velocity can be written in vector form such that

$$\vec{\omega}_{B/I} = \dot{\psi}\hat{K}_A + \dot{\theta}\hat{J}_{NR} + \dot{\phi}\hat{I}_B \quad (33)$$

relating the unit vectors \hat{K}_A and \hat{J}_{NR} to the body frame using the planar rotation matrices results in the equation below. Note that NR is denoted as the “No-Roll” frame.

$$\begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} = [\mathbf{H}] \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} \quad (34)$$

where

$$\mathbf{H} = \begin{bmatrix} 1 & s_\phi t_\theta & c_\phi t_\theta \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi/c_\theta & c_\phi/c_\theta \end{bmatrix} \quad (35)$$

with all this information the Newton-Euler equations of motion can be used to form the equation below.

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Bmatrix} = \frac{1}{m} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} - \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \quad (36)$$

Using the rotational dynamic equations for rigid bodies, the equation for the derivative of angular velocity can be found as

$$\begin{Bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{Bmatrix} = \mathbf{I}_C^{-1} \left(\begin{Bmatrix} L \\ M \\ N \end{Bmatrix} - \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \mathbf{I}_C \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} \right) \quad (37)$$

Note that standard aircraft forces and moments are applied to the body. The forces are typically written as X, Y and Z while the moments are given as L, M and N. They can be written in component form using the equations below.

$$\mathbf{C}_B(\vec{F}_C) = \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = X\hat{I}_B + Y\hat{J}_B + Z\hat{K}_B \quad (38)$$

$$\mathbf{C}_B(\vec{M}_C) = \begin{Bmatrix} L \\ M \\ N \end{Bmatrix} = L\hat{I}_B + M\hat{J}_B + N\hat{K}_B \quad (39)$$

IV. Forces on Aircraft

To form the sum of the forces on an aircraft the assumptions are made that only gravity and aerodynamic forces act. For rockets, a propulsion model can be added. Still the gravitational force is shown below.

1. Gravity

The weight contribution is assumed to be a constant force applied to the aircraft. The equation below is the gravitational force applied in the body frame.

$$\begin{Bmatrix} X_W \\ Y_W \\ Z_W \end{Bmatrix} = mg \begin{Bmatrix} -s_\theta \\ s_\phi c_\theta \\ c_\phi c_\theta \end{Bmatrix} \quad (40)$$

2. Aerodynamics

Aircraft aerodynamics are written using a Taylor series expansion about a trim point [3, 4]. That is, the aerodynamic forces are given by

$$\vec{F} = \vec{F}_0 + \frac{\partial \vec{F}}{\partial \vec{x}} (\vec{x} - \vec{x}_0) \quad (41)$$

where $\vec{x} = [x, y, z, \phi, \theta, \psi, u, v, w, p, q, r]^T$. The partial derivative is thus expanded such that

$$\frac{\partial \vec{F}}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial \vec{F}}{\partial x} & \frac{\partial \vec{F}}{\partial y} & \dots & \frac{\partial \vec{F}}{\partial r} \end{bmatrix} \quad (42)$$

To find all of the partial derivative the forces are first written using a combination of dynamic pressure and coefficients that are functions of geometry and Reynolds number rather than speed, pressure and size. A general lift force can be written using the equation below

$$L = \frac{1}{2} \rho V_\infty^2 S C_L \quad (43)$$

where ρ is the atmospheric density, V_∞ is the free-stream velocity, S is the planform area of the wing and C_L is the lift coefficient.

$$V_\infty = \sqrt{u_a^2 + v_a^2 + w_a^2} \quad (44)$$

The subscript 'a' above denotes the velocity of the aircraft plus the atmospheric disturbance.

$$\begin{Bmatrix} u_a \\ v_a \\ w_a \end{Bmatrix} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} + \mathbf{T}^T_{IB} \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} \quad (45)$$

Note that the dynamic pressure is different for a rocket or projectile. A similar expression can be created for a generic moment such that

$$M = \frac{1}{2} \rho V_\infty^2 S \bar{c} C_M \quad (46)$$

where \bar{c} is the mean chord of the aircraft. The dynamic pressure $q_\infty = \frac{1}{2} \rho V_\infty^2$ can be used to non-dimensionalize the forces, thus $L/q_\infty = C_L$. This means that the equation involving partial derivatives can be written as

$$\frac{\partial \vec{C}_F}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial \vec{C}_F}{\partial x} & \frac{\partial \vec{C}_F}{\partial y} & \dots & \frac{\partial \vec{C}_F}{\partial r} \end{bmatrix} \quad (47)$$

If the vector is then expanded to include the components of the vector \vec{F} the partial derivatives expand to

$$\frac{\partial \vec{C}_F}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial C_X}{\partial x} & \frac{\partial C_X}{\partial y} & \dots & \frac{\partial C_X}{\partial r} \\ \frac{\partial C_Y}{\partial x} & \frac{\partial C_Y}{\partial y} & \dots & \frac{\partial C_Y}{\partial r} \\ \frac{\partial C_Z}{\partial x} & \frac{\partial C_Z}{\partial y} & \dots & \frac{\partial C_Z}{\partial r} \end{bmatrix} \quad (48)$$

shorthand can be adopted for the forces above such that $\frac{\partial C_Y}{\partial x} = C_{Yx}$. Using this shorthand the equation above can be written as.

$$\frac{\partial \vec{C}_F}{\partial \vec{x}} = \begin{bmatrix} C_{Xx} & C_{Xy} & \dots & C_{Xr} \\ C_{Yx} & C_{Yy} & \dots & C_{Yr} \\ C_{Zx} & C_{Zy} & \dots & C_{Zr} \end{bmatrix} \quad (49)$$

The coefficients listed above are standard coefficients that all aircraft have. A similar matrix can be formulated for the moments on an aircraft. When system identifying an aircraft all of these coefficients may be determined. However, many of these terms are zero. For example, all coefficients with respect to x y and z are zero. That is, $C_{Xx} = C_{Yx} = \dots C_{Nx} = C_{Xy} = \dots C_{Ny} = 0$. Other coefficients can be set to zero as well.

Aircraft Aerodynamics

For aircraft, some further simplifications are made. Some of the coefficients defined above are combined to be written as functions of the angle of attack(α) and sideslip(β).

$$\alpha = \tan^{-1} \left(\frac{w_a}{u_a} \right) \quad (50)$$

$$\beta = \sin^{-1} \left(\frac{v_a}{V_\infty} \right) \quad (51)$$

Transforming the equations into these formulations gives rise to coefficients such as $C_{L\alpha}$ which is the change in lift as a function of angle of attack and $C_{Y\beta}$ which is the change in Y-Force as a function of sideslip. Using all of the coefficients defined above taking into account the change to lift and drag, the body aerodynamic force is calculated using the equation below.

$$\begin{Bmatrix} X_A \\ Y_A \\ Z_A \end{Bmatrix} = \frac{1}{2} \rho V_\infty^2 S \begin{Bmatrix} C_L s_\alpha - C_D c_\alpha + C_{x\delta_t} \delta_t \\ C_{y\beta} \beta + C_{y\delta_r} \delta_r + C_{yp} \frac{pb}{2V_\infty} + C_{yr} \frac{rb}{2V_\infty} \\ -C_L c_\alpha - C_D s_\alpha \end{Bmatrix} \quad (52)$$

Where the lift and drag coefficients are:

$$\begin{Bmatrix} C_L \\ C_D \end{Bmatrix} = \begin{Bmatrix} C_{L0} + C_{L\alpha} \alpha + C_{Lq} \frac{q\bar{c}}{2V_\infty} + C_{L\delta_e} \delta_e \\ C_{D0} + C_{D\alpha} \alpha^2 \end{Bmatrix} \quad (53)$$

The body aerodynamic moment is also computed using an aerodynamic expansion.

$$\begin{Bmatrix} L_A \\ M_A \\ N_A \end{Bmatrix} = \frac{1}{2} \rho V_\infty^2 S \bar{c} \begin{Bmatrix} C_{l\beta} \beta + C_{lp} \frac{pb}{2V_\infty} + C_{lr} \frac{rb}{2V_\infty} + C_{l\delta_a} \delta_a + C_{l\delta_r} \delta_r \\ C_{m0} + C_{m\alpha} \alpha + C_{mq} \frac{q\bar{c}}{2V_\infty} + C_{m\delta_e} \delta_e \\ C_{np} \frac{pb}{2V_\infty} + C_{n\beta} \beta + C_{nr} \frac{rb}{2V_\infty} + C_{n\delta_a} \delta_a + C_{n\delta_r} \delta_r \end{Bmatrix} \quad (54)$$

The aerodynamic coefficients in equations (52), (53) and (54) can be obtained from flight data, aerodynamic modeling and windtunnel tests. Notice that the only coefficients remaining are coefficients from angle of attack, sideslip and angular velocities. Furthermore, the coefficients for angular velocities are also non-dimensionalized by terms such as $b/(2V_\infty)$ where b is the wingspan of the aircraft and \bar{c} is the mean chord of the aircraft. These terms are introduced to fully non-dimensionalize the coefficients. Notice, as well that four extra terms were also introduced. These will be discussed in more detail in the control section however the four terms are the aileron control surface δ_a , the elevator control surface δ_e , the rudder control surface δ_r and the thrust control value δ_t .

Projectile Aerodynamics

To fully define the projectile aerodynamics some more assumptions are made about the projectile.

1. The projectile is axially symmetric
2. The aerodynamic forces are not necessarily formulated at the center of mass
3. The projectile has the potential to be spinning rapidly thus interacting with the surrounding atmosphere

For a projectile the dynamic pressure is written as

$$Q = \frac{\pi}{8} \rho V_\infty^2 d^2 \quad (55)$$

The aerodynamic forces on the projectile are modeled using Taylor series ballistic expansions with known coefficients similar to the aircraft model only slightly different assumptions are made given the dynamics of the projectile. The subscripts in the equation below stand for steady and unsteady aerodynamics.

$$\begin{Bmatrix} X_A \\ Y_A \\ Z_A \end{Bmatrix} = \begin{Bmatrix} X_{SA} \\ Y_{SA} \\ Z_{SA} \end{Bmatrix} + \begin{Bmatrix} X_{UA} \\ Y_{UA} \\ Z_{UA} \end{Bmatrix} = Q \begin{Bmatrix} -C_{X_0} - C_{X_2} \frac{v^2 + w^2}{V^2} \\ -C_{Y_\beta} \frac{v}{V} \\ -C_{N_\alpha} \frac{w}{V} \end{Bmatrix} + Q \begin{Bmatrix} 0 \\ C_{Y_{p\alpha}} \frac{w}{V} \frac{pd}{2V} \\ C_{Z_{p\alpha}} \frac{v}{V} \frac{pd}{2V} \end{Bmatrix} \quad (56)$$

In this equation, Q is the dynamic pressure, d is the aerodynamic reference area, C_{X_0} is the zero-yaw axial force coefficient, C_{X_2} is the yaw-squared axial force coefficient, C_{N_α} is the normal force derivative coefficient, $C_{Y_{p\alpha}}$ is the Magnus force coefficient, and $V = \sqrt{u^2 + v^2 + w^2}$ is the total velocity of the projectile. The aerodynamic moments acting on the projectile are the pitching, pitch damping, Magnus, and roll damping moments. Pitching and Magnus moments are given by taking the cross product of the normal and Magnus forces given in (56) with the position vector from the center of mass to the center of pressure and location of Magnus force, respectively. The total aerodynamic moments are given in Eqn. (57).

$$\begin{Bmatrix} L_A \\ M_A \\ N_A \end{Bmatrix} = \mathbf{S}_B(\vec{r}_{CG,COP}) \begin{Bmatrix} X_{SA} \\ Y_{SA} \\ Z_{SA} \end{Bmatrix} + \mathbf{S}_B(\vec{r}_{CG,MCOP}) \begin{Bmatrix} X_{UA} \\ Y_{UA} \\ Z_{UA} \end{Bmatrix} + Qd \begin{Bmatrix} C_{l_p} \frac{pd}{2V} \\ C_{m_q} \frac{qd}{2V} \\ C_{n_r} \frac{rd}{2V} \end{Bmatrix} \quad (57)$$

Here, $\mathbf{S}_B(\vec{r}_{CG,COP})$ is the skew-symmetric operator acting on the position vector from the center of mass to the center of pressure expressed in the projectile body frame. Furthermore, $\mathbf{S}_B(\vec{r}_{CG,MCOP})$ is the skew-symmetric operator acting on the position vector from the center of mass to the Magnus center of pressure expressed in the projectile body frame. Typically the center of mass is defined from the rear of the projectile such that

$$\mathbf{C}_B(\vec{r}_{CG}) = \begin{Bmatrix} SL_{CG} \\ BL_{CG} \\ WL_{CG} \end{Bmatrix} \quad (58)$$

Similarly, the center of pressure is defined from the rear of the projectile such that

$$\mathbf{C}_B(\vec{r}_{COP}) = \begin{Bmatrix} SL_{COP} \\ BL_{COP} \\ WL_{COP} \end{Bmatrix} \quad (59)$$

The vector $\vec{r}_{CG,COP}$ is then simply the different between both vectors.

$$\vec{r}_{CG,COP} = \vec{r}_{COP} - \vec{r}_{CG} \quad (60)$$

The damping coefficient defined in equation (57) include C_{l_p} which is the roll damping coefficient while C_{m_q} is the pitch damping coefficient. These coefficients are added which essentially inhibit angular motion of the projectile. In addition, to these coefficients, sometimes magnus coefficients are given as pure moments rather than forces acting at a distance. This can be given in the equation below.

$$M_{UA} = Qd(-C_{M\alpha} \frac{v}{V} + C_{N_{p\alpha}} \frac{w}{V} \frac{pd}{2V}) \quad (61)$$

Where $C_{M\alpha}$ replaces the moment produced by $C_{N\alpha}$ and $C_{N_{p\alpha}}$ replaces the moment produced by $C_{Y_{p\alpha}}$. It is possible to derive an equation between the two different representations as given by the equations below.

$$\begin{aligned} C_{M\alpha} &= \frac{(SL_{COP} - SL_{CG})C_{N\alpha}}{d} \\ C_{N_{p\alpha}} &= \frac{(SL_{MAG} - SL_{CG})C_{Y_{p\alpha}}}{d} \end{aligned} \quad (62)$$

V. Stability and Control

Controllability is formally stated as a system where any initial state $x(0) = x_0$ and final state $x_1, t_1 > 0$, there exists a piecewise continuous input $u(t)$ such that $x(t_1) = x_1$. For a fixed wing aircraft the system has 12 states with 8 dynamic modes and 4 zero or rigid body modes. For a fixed wing aircraft the system has 12 states with 8 dynamic modes and 4 zero or rigid body modes. A conventional aircraft has 4 controls to control these 12 modes. The easiest way to test the controllability of a system is to compute the controllability matrix. However, the controllability matrix must be computed using a linearized model such that $\dot{\vec{x}} = A\vec{x} + B\vec{u}$. In order to do this the aircraft must be in equilibrium. For this example the aircraft is set with an initial velocity of 20 m/s at an altitude of 200 m. The altitude command is set to 200 m and the heading command is set to zero. Given the zero heading angle command and the symmetry of the configurations investigated the rudder and aileron commands are set to zero. Thus, only the thrust and elevator controls are activated for the trimming procedure. Each configuration is simulated for 200 seconds or until the derivatives of all states except \dot{x} are within a required tolerance. Using this equilibrium point a linear model can be computed by using forward finite differencing assuming that the aircraft model is put in the form $\dot{\vec{x}} = F(\vec{x}, \vec{u})$.

$$\dot{\delta x} = \frac{F(\vec{x}_0 + \Delta\vec{x}_0, \vec{u}_0) - F(\vec{x}_0, \vec{u}_0)}{\Delta\vec{x}} \delta\vec{x} + \frac{F(\vec{x}_0, \vec{u}_0 + \Delta\vec{u}) - F(\vec{x}_0, \vec{u}_0)}{\Delta\vec{u}} \delta\vec{u} \quad (63)$$

This linear model is the classic linear model where $\dot{\delta\vec{x}} = A\delta\vec{x} + B\delta\vec{u}$. Using this linear model, the controllability matrix can be computed as

$$W_C = [B \ AB \ A^2B \ A^3B \ \dots \ A^{N-1}B] \quad (64)$$

where N is the number of states in the system. With the controllability matrix formulated, the rank of the matrix is computed. If the $rank(W_C) = N$ the system is said to be controllable.

VI. PID Control

For a conventional PID controller of an aircraft, the rudder, elevator and aileron commands are set to

$$\begin{aligned}\delta_r &= -K_v v \\ \delta_e &= K_p(\theta - \theta_c) + K_d \dot{\theta} \\ \delta_a &= K_p(\phi - \phi_c) + K_d \dot{\phi}\end{aligned}\tag{65}$$

The Euler angle commands ϕ_c and θ_c are set using the following relationships:

$$\begin{aligned}\phi_c &= K_p(\psi - \psi_c) + K_d \dot{\psi} \\ \theta_c &= K_p(z - z_c) + K_d \dot{z} + K_I \int z - z_c dt\end{aligned}\tag{66}$$

The control scheme defined above is a conventional inner loop-outer loop control of a fixed wing aircraft using a PID tracking controller.

VII. Radio Controlled Aircraft Design

Many principles for large aircraft design can be applied to smaller radio controlled aircraft but understand that many of the aerodynamic principles are not quite well defined for slow and small aircraft. These aircraft have low Reynolds number which often exhibits odd phenomena. For example, airfoil selection is really not so much a design point other than ease of manufacturing rather than maximizing lift coefficient. Remember that Reynolds number can be defined from the equation below where ρ is the density at sea-level (1.225 kg/m^3 in SI units), V is the desired flight speed, \bar{c} is the mean aerodynamic chord and μ_∞ is the viscosity of air which in SI units is $1.81\text{e-}5 \text{ kg/(m-s)}$.

$$Re = \frac{\rho V \bar{c}}{\mu_\infty}\tag{67}$$

You can tell that flying a small aircraft (\bar{c}) and flying slow (V) results in a low Reynolds number. Either way the procedure below has produced some great aircraft and the tools you'll learn along the way will help you in your future aerospace engineering career. If you're not an engineer then this text will at least give you an appreciation for what goes into aircraft design. If you'd much rather watch youtube videos than read this document, feel free to watch my Youtube playlist on radio controlled aircraft design[5]

1. Vehicle Type Selection and Requirements

In the very beginning of your design you need to decide on the type of aircraft you want to build. Designing a glider versus an aerobatic airplane will result in vastly different engineering design decisions. For example, a glider is going to have very long slender wings while an aerobatic airplane is going to have somewhat shorter wings with large control surfaces. I suggest you select from the following types of aircraft and then move on: Gliders, Trainers, Sport Aerobatic, Racers. If you'd like to build a scale aircraft there isn't much to design since the shape of the aircraft is pretty much built. If you do go with a scale aircraft this textbook isn't really for you since you aren't really building a scratch build aircraft. You're more just copying someone else's design. In that case you may as well just buy a kit or watch some videos on balsa wood construction and an overview of all the electronics required for RC aircraft flight.

If you selected one of the other styles you're ready to move onto to the next stage which is requirements. There is so much literature on Systems Engineering, top level requirements, functional requirements and derived requirements. The bottom line is you need to determine

what you want your aircraft to do. Do you want it to fly straight up, upside down? Do you want the aircraft to be hand launched? Land on a runway? Determine what you want the aircraft to do and create a bulleted list of those requirements. Throughout the design you can refer to these requirements and make sure you are satisfying these requirements. If this is your first build then you may just have one requirement and that is to take off and land without crashing. But think a bit deeper. Do you want to turn the vehicle? Do you want full channel control for roll, pitch and yaw or just yaw control? Do you want landing gear? What sort of flying characteristics do you want? Be as specific as possible here.

2. Initial Design - Hand Sketch and Aspect Ratio

Once you have an idea of what the aircraft type is and what the requirements are it's time to hand sketch your aircraft. Try and use engineering paper, french curves, a ruler and a compass. Make this hand sketch look nice so you can use it in your future design. Draw your sketch to scale. You might be wondering, "how do I draw the aircraft without knowing what my wing loading or thrust to weight ratio is?". The answer comes from my late aircraft design professor Dr. Mikolowski (RIP). He would always say "If it looks good, it flies good". After designing so many aircraft and seeing so many scratch builds from my students I can honestly say that this is true 100%. If you're reading this now it means that there is already over 100 years of aircraft technology on the internet for you to research and see what other aircraft look like. Make your aircraft look like that but make sure it fits into your aircraft type and make sure it satisfies your requirements from above. If one of your requirements was to hand launch, then make sure your drawing reflects a vehicle without landing gear and a place to grab the aircraft. If you wanted full channel support make sure to include all the control surfaces. Think about where you want the propulsion system to go and how you're going to access the electronics before you fly. Think about what you want the wing to look like. Make it as big as you think it needs to fly. Use your intuition. This is an art. So much of engineering is an art.

Once your aircraft sketch is complete (make sure to do a front view, side view and top view of your aircraft), it's time to take down some wing characteristics. This is why you need to draw your sketch to scale. Measure the length of the wing (wingspan b) and the chord at the root (c_r) and the tip (c_t). Compute the area of the wing (S) using the area of a trapezoid or rectangle depending on the shape of your wing. Once you have the wingspan and area you can compute the aspect ratio of your aircraft.

$$AR = \frac{b^2}{S} \quad (68)$$

The general rule is that the larger the aspect ratio the more aerodynamically efficient your aircraft will be. This is why gliders have very long and slender wings. At the same time, high aspect ratio wings suffer from larger bending moments and can flex considerably in flight. Finally, it's important to compute the mean aerodynamic chord of the wing. This is basically the average chord of your wing. If you create a rectangular wing your mean aerodynamic chord is just the chord of the wing since it's constant. If not you'll need to integrate over the length of the wing using the formula below where $y = 0$ is the centerline of the vehicle and $y=b/2$ is the wingtip on the right side[6].

$$\bar{c} = \frac{2}{S} \int_0^{b/2} c^2 dy \quad (69)$$

3. Weight Estimate - Tabular Approach

Once you have an idea of the overall shape it's time to estimate how heavy the aircraft will be. First think about the fundamental components of an aircraft. If you're not familiar with any of the components below, just type the item into Google and you'll find numerous articles and Youtube Videos about each component.

1. ESC - Electronic Speed Controller
2. Battery - Assume for now that you'll be using a 1500mAh 3S Battery unless you're building a micro aircraft in which case you might end up using a 600 mAh 2S or even a 300 mAh 1S. Think about the size of your aircraft. You will do more sophisticated battery design in the future.
3. Motor and Propeller - Again select something that is in the ballpark of the aircraft you're building. You'll do a redesign later.
4. Servos
5. Receiver
6. Control Linkages and Servo Horns
7. Fuselage
8. Main Wing
9. Tail both Horizontal and Vertical
10. Payload

For each of the components above you need to estimate the weight of these components. The only way to do that is to either look up the weight of similar aircraft to the one you're designing or find components that you think will work for your aircraft and add up all of the weights. The most difficult part is going to be estimating the empty weight of the aircraft which is just the structure of the aircraft. For these estimates you need to decide what materials you plan on using. Is your aircraft made out of foam, balsa, carbon fiber or some type of combination. Perform an initial material selection and then use that to estimate your weight. Create a table in a spreadsheet type program or a numerical computer program so that you can go back and change weights as your design progresses. Use the spreadsheet or numerical program to compute the total weight of your aircraft. This is your maximum takeoff weight.

4. Airfoil Selection and 2D and 3D Lift

The section 2. details the aerodynamic forces on the aircraft. In this stage of design we will only be looking at the lift equation.

$$L = \frac{1}{2}\rho V^2 S C_L \quad (70)$$

The coefficient of lift C_L is the lift coefficient of the aircraft. This coefficient is a function of Reynolds number, angle of attack, airfoil shape and wing shape. Recall that the angle of attack is the angle between the zero lift line of the airfoil and the free stream air. Breaking the velocity vector into components yields the equation below where w is the airspeed along the z-axis of the vehicle and u is the velocity along the x-axis.

$$\alpha = \tan^{-1} \left(\frac{w}{u} \right) \quad (71)$$

Using the angle of attack as a parameter, the lift coefficient can be expanded to the following:

$$C_L = C_{L\alpha}(\alpha - \alpha_0) \quad (72)$$

The parameter α_0 is the angle of attack that results in zero lift. This is a function of the airfoil shape. The parameter $C_{L\alpha}$ is the lift curve slope and is a function of airfoil shape, wing shape and Reynolds number. In order to remove, wing shape from the design, the aspect ratio is used to convert the wing lift to a sectional airfoil coefficient.

$$C_{L\alpha} = \frac{C_{l\alpha}}{1 + \frac{C_{l\alpha}}{\pi e AR}} \quad (73)$$

The parameter e is an efficiency parameter which is often assumed to be 80-90%. The coefficient, $C_{l\alpha}$ is the airfoil lift curve slope which is different than the wing lift curve slope. The wing lift curve slope will always be smaller than the airfoil lift curve slope. This is because of an effect called wing tip vortices. Wing tip vortices are an aerodynamic effect where high pressure from the bottom of the wing moves around the wingtips to the area of low pressure. The only way to mitigate these vortices is by installing winglets or increasing the aspect ratio. Note that winglets increase drag and weight which is why you don't typically see them on RC aircraft. The coefficient $C_{l\alpha}$ is then simply a function of airfoil shape and Reynolds number. This is what is often referred to as 2D lift. It is the lift of airfoils which are two dimensional rather than 3D lift which is over an entire wing. It is at this point that airfoil selection and design can be used. The website airfoiltools.com is a great resource for plotting lift curve slopes of various airfoil shapes as a function of Reynolds number. I also have a great Youtube video on how to use XFLR5 (pronounced X-Flyer Five)[7]. A more comprehensive guide on XFLR5 can be found in [8] while the software itself can be downloaded in [9]. The basics of airfoil selection then break down into the following process.

First, using your max takeoff weight and cruise flight speed, compute the lift coefficient required in cruise. This assumes that lift equals weight.

$$C_L = \frac{2W}{\rho V^2 S} \quad (74)$$

Then using your flight speed and chord length, compute the Reynolds number of your aircraft. Use this Reynolds number and airfoiltools or XFLR5 to compute the sectional lift characteristics of various airfoils. Airfoil selection can be as complicated as you make it but you're looking for the highest lift to drag ratio airfoil. Another simple way is to just select the airfoil with the highest sectional lift coefficient. Another item to consider is manufacturing. Some airfoils may be feasible for full sized aircraft but not for RC flyers. My recommendation is to build an aircraft with a Clark-Y airfoil. They are easy to cut with balsa or shape with foam and have good lift to drag characteristics. Note that this airfoil is cambered. If you want to fly upside down I suggest you use a symmetric airfoil like a NACA 0012 or NACA 0014 if you need a bit more thickness to fit a wing spar through the airfoil. Once you have your airfoil selected, use the lift curve slope to estimate $C_{l\alpha}$ by fitting a linear trend line to the portion of the graph before stall. Also make sure to take note of what the zero lift angle of attack is. You can then compute the wing lift curve slope by using equation 73. Once you have $C_{L\alpha}$, you can compute the angle of attack needed for cruise. Make sure to convert α_0 to radians before you use the equation below.

$$\alpha = \frac{C_L}{C_{L\alpha}} + \alpha_0 \quad (75)$$

The resulting answer will be in radians and needs to be converted to degrees to make sure you are not close to stall. Typically in cruise, the angle of attack is only a few degrees with perhaps 8 degrees on the high end. If the answer you receive is higher than that it can mean a few things which may involve a redesign.

1. The simplest way to get more lift is to fly faster. The problem is you need a bigger motor which will increase your weight which will require more lift and more angle of attack. Flying faster will also change your Reynolds number.
2. You can increase the aspect ratio of your wing which will make your aircraft more efficient. The problem is that will increase the bending moment at the root creating the need for stronger materials at the root which also increases weight. This will also change your mean aerodynamic chord which will change your Reynolds number.
3. You can increase the area of the wing. This will also increase drag and weight but if the material you are using has a high lift to weight ratio then adding more wing area might be a good option. Depending on how you change the aircraft wing shape, the aspect ratio and/or the mean aerodynamic chord might change meaning you'll have to recompute the wing lift curve slope as well as the Reynolds number.
4. If you are using a symmetric airfoil it's possible you could forgo flying upside down and switch to a cambered airfoil which has more lift.

Regardless of what you do make sure you angle of attack in cruise is low which will reduce drag. It's optimal to fly at the aircraft's highest lift to drag ratio but radio controlled aircraft don't typically do that. Once you have redesigned your vehicle to fit within tolerable limits it's time to look at some aircraft performance characteristics which include the W/S (wing loading) and the T/W (thrust to weight ratio).

5. **Wing Loading and Thrust to Weight Ratio**
6. **Stability and Control, Center of Mass and Static Margin**
7. **Re-Design - Iteration and Detailed Sketch**
8. **Computer Aided Design**
9. **Purchase Components**
10. **Flying**

Bibliography

- [1] O. A. Bachau. *Flexible Multibody Dynamics*. Springer, 2008.
- [2] B. Etkins. *Dynamics of Atmospheric Flight*. Dover, Mineola, New York, 2000. pages 9-13,134-151,196-318.
- [3] Warren F. Phillips. *Mechanics of flight*. pages Preface,10–13,260–270,715–753, 2010.
- [4] Anderson D. *Fundamentals of Aerodynamics 4th Edition*. McGraw Hill Series, 2007.
- [5] Carlos Montalvo. Radio controlled aircraft design. accessed:12/30/2020 https://www.youtube.com/watch?v=RjC3Rou0wkA&list=PL_D7_GvGz-v3d8gRDXWjQaVYfvNOPVx97.
- [6] Caughey D.A. *Introduction to Aircraft Stability and Control Course Notes for M&AE 5070*. , edition, 2011.
- [7] Carlos Montalvo. Getting lift curve slopes and drag polars using xflr5. accessed 12/30/2020 https://www.youtube.com/watch?v=Yz7rmuhGYW0&list=PL_D7_GvGz-v3d8gRDXWjQaVYfvNOPVx97&index=4,.
- [8] Andr Deperrois. About stability analysis using xflr5 presentation document. June 2008. http://www.xflr5.tech/docs/XFLR5_and_Stability_analysis.pdf.
- [9] Xflr5. accessed 12/30/2020 <http://www.xflr5.tech/xflr5.htm>,.

VIII. AIRCRAFT NOMENCLATURE

x_i, y_i, z_i	components of the mass center position vector in the inertial frame of aircraft i (m)
ϕ_i, θ_i, ψ_i	Euler roll, pitch, and yaw of aircraft i (rad)
u_i, v_i, w_i	components of the mass center velocity vector in the body frame of aircraft i (m/s)
p_i, q_i, r_i	components of the mass center angular velocity vector in the body frame of aircraft i (rad/s)
$\vec{r}_{A \rightarrow B}$	position vector from a generic point A to a generic point B (m)
$\vec{V}_{A/B}$	velocity vector of a generic point A with respect to frame B (m/s)
\mathbf{T}_{AB}	generic transformation matrix rotating a vector from the frame B to frame A
\mathbf{H}_i	relationship matrix of Euler angle derivatives to body angular velocity components of aircraft i
m_i	mass of aircraft i (kg)
I_i	moment of inertia matrix of aircraft i taken about the mass center in the body frame ($kg - m^2$)
X_i, Y_i, Z_i	components of the total force applied to aircraft i in body frame (N)
L_i, M_i, N_i	components of the total moment applied to aircraft i in body frame (N-m)
X_{Wi}, Y_{Wi}, Z_{Wi}	total weight force applied to aircraft i (N)
L, D	Lift and Drag on Aircraft (N) - Not to be confused with Roll moment
g	gravitational constant on Earth (m/s^2)
ρ	atmospheric density (kg/m^3)
S_i	reference area of wing on aircraft i (m^2)
b_i	Wingspan of aircraft i (m)
\bar{c}_i	mean chord of wing on aircraft i (m)
α	Angle of attack (rad)
β	Slideslip angle (rad)
C_L, C_D, C_m	Lift, Drag and Pitch Moment coefficients
$\delta_t, \delta_a, \delta_r, \delta_e$	thrust, aileron, rudder, and elevator control inputs (rad)
$S_B(\vec{r})$	skew symmetric matrix operator on a vector expressed in the body frame.
K_p, K_d, K_I	proportional, derivative, and integral control gains
V	Total airspeed (m/s)
$\hat{p}, \hat{q}, \hat{r}$	Non-dimensional angular velocities
l	Distance from center of mass to aerodynamic center of the tail (m)
l_t	Distance from aerodynamic center of main wing to aerodynamic center of tail (m)
α_0	zero lift angle of attack (rad)
C_{L0}	Zero angle of attack lift coefficient
$C_{m\alpha}$	Pitch moment curve slope versus α
$C_{L\alpha}$	Lift curve slope
C_{mq}	Pitch damping coefficient
$C_{m\delta_e}$	Pitch moment curve slope versus elevator deflection angle
a_∞	Speed of sound (m/s)
μ_∞	Viscosity of Fluid $kg/(m - s)$

IX. EQUATIONS

Mach Number and Reynolds Number

$$\begin{aligned} M_\infty &= \frac{V}{a_\infty} \\ Re &= \frac{\rho V \bar{c}}{\mu_\infty} \end{aligned} \quad (76)$$

Total Velocity

$$V = \sqrt{u^2 + v^2 + w^2} \quad (77)$$

Angle of Attack and Sideslip

$$\begin{aligned} \alpha &= \tan^{-1} \left(\frac{w}{u} \right) \\ \beta &= \sin^{-1} \left(\frac{v}{V} \right) \end{aligned} \quad (78)$$

Lift Drag and Moment

$$\begin{aligned} Lift (L) &= \frac{1}{2} \rho V^2 S C_L \\ Drag (D) &= \frac{1}{2} \rho V^2 S C_D \\ Roll Moment (L) &= \frac{1}{2} \rho V^2 S b C_l \\ Pitch Moment (M) &= \frac{1}{2} \rho V^2 S \bar{c} C_m \\ Yaw Moment (N) &= \frac{1}{2} \rho V^2 S b C_n \end{aligned} \quad (79)$$

Lift and Drag Coefficients

$$\begin{aligned} C_L &= C_{L0} + C_{L\alpha} \alpha \\ C_L &= C_{L\alpha} (\alpha - \alpha_0) \\ C_D &= C_{D0} + C_{D\alpha} \alpha^2 \\ C_D &= C_{D0} + k C_L^2 \end{aligned} \quad (80)$$

Non-dimensional Angular velocities

$$\begin{aligned} \hat{p} &= pb/2V \\ \hat{q} &= q\bar{c}/2V \\ \hat{r} &= rb/2V \end{aligned} \quad (81)$$

Pitch Moment equation

$$C_m = C_{m0} + C_{m\alpha} \alpha + C_{m\delta_e} \delta_e + C_{mq} \hat{q} \quad (82)$$

$$\begin{aligned} C_{m0} &= C_{MAC} + C_{L0} \bar{x}_{sm} \\ \bar{x}_{sm} &= \frac{x_{cg}}{\bar{c}} - \frac{x_{ac} W}{\bar{c}} \\ C_{m\alpha} &= \left(C_{L\alpha, W} + \frac{S_t}{S} C_{L\alpha, t} \right) \bar{x}_{sm} - V_H C_{L\alpha, t} \\ V_H &= \frac{l_t S_t}{S \bar{c}} \\ C_{m\delta_e} &= \left(C_{L\delta_e} \frac{S_t}{S} \right) \bar{x}_{sm} - V_H C_{L\delta_e} \\ C_{mq} &= 2 C_{Lat} \frac{l^2}{\bar{c}^2} \end{aligned} \quad (83)$$

Max Lift to Drag Ratio (Only valid if $C_{L0} = 0$)

$$\alpha_{max, L/D} = \sqrt{\frac{C_{D0}}{C_{D\alpha}}} \quad (84)$$

Lift to Drag when $T = 0$ (Sum of Forces still zero)

$$\begin{aligned} \frac{D}{L} &= \tan(\alpha) \\ L \cos(\alpha) + D \sin(\alpha) &= W \end{aligned} \quad (85)$$

Airfoil and Wing Aerodynamics

$$\begin{aligned} x_{ac} &= c/4 \quad a = \frac{a_0}{1 + \frac{a_0}{\pi e AR}} \\ AR &= \frac{b^2}{S} \end{aligned} \quad (86)$$

Standard Atmosphere

$$\begin{aligned} \rho &= 1.225 \text{ kg/m}^3 = 0.00238 \text{ slugs/ft}^3 \\ \mu_\infty &= 1.81 \times 10^{-5} \text{ kg/(m} \cdot \text{s)} \\ a_\infty &= 331.3 \text{ m/s} \end{aligned} \quad (87)$$

General Notes

1. In trim or steady and level or cruise $q = 0, C_m = 0, L = W, T = D$
2. For symmetric airfoil $C_{MAC} = 0$ and $C_{L0} = 0$ thus $C_{m0} = 0$
3. For a flat plate all symmetric properties apply but $a_0 = 2\pi$
4. Tail surfaces are always assumed to be flat plates
5. For longitudinal problems, $\beta = 0$ so $v = 0$ (side velocity)