Illinois Institute of Technology
Department of Computer Science

Second Examination

CS 330 Discrete Mathematics
Fall, 2013

11:25am–12:40pm, Monday, October 28, 2013
IT 6C7-1 (sixth floor of IIT Tower, 35th & State)

Print your name and student ID, *neatly* in the space provided below; print your name at the upper right corner of *every* page. Please print legibly.

Name:	
Student ID:	

This is an *open book* exam. You are permitted to use the textbook (hardcopy only), hard copies of any class handouts, anything posted on the web page, any of your own assignments, and anything in your own handwriting. Foreign students may use a hardcopy dictionary. *Nothing else is permitted*: No calculators, laptops, cell phones, Ipads, Ipods, communicators, GPSes, etc.!

Do all five problems in this booklet. *All problems are equally weighted, so do not spend too much time on any one question.*

Show your work! You will not get partial credit if the grader cannot figure out how you arrived at your answer.

Question	Points	Score	Grader
1	20		
2	20		
3	20		
4	20		
5	20		
Total	100		

1. **Coin Tossing**

   We have a peculiarly biased coin for which the probability of tossing heads changes after each flip. Initially (on the first flip), the probability of heads is $1/2$. But for each successive toss the probability decreases by a multiplicative factor of $1/4$, so that on the ith flip the probability of heads is $1/2 \times (1/4)^{i-1}$. We toss the coin n times.

   (a) What is the probability that all n tosses were heads?

   (b) In general, what is the expected number of heads we will get?

2. **Annihilators**

   Prove by induction on k that the operator $(\mathbf{E}-a)^{k+1}$ annihilates any sequence $\langle P(i)a^i \rangle$, where $P(i)$ is any polynomial in i of degree k.

3. **Rolling a Die**

   You roll a decimal die (that is, a ten-sided die) $n \geq 2$ times, recording the sequence of rolls.

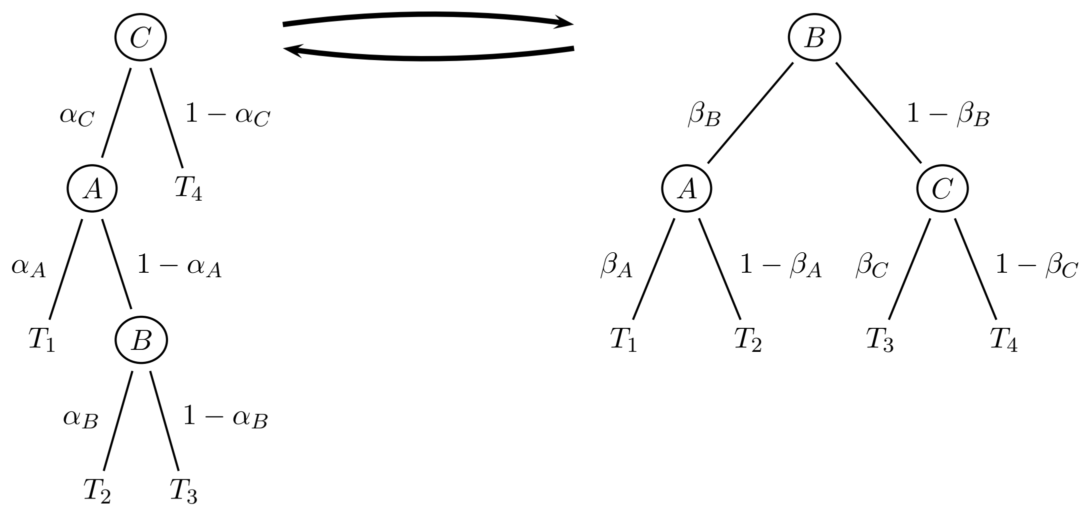
   (a) Find a recurrence relation for the number of possible sequences in which there are no two consecutive sixes.

   (b) Solve the recurrence using annihilators; you need not solve the simultaneous equations from the initial conditions.

   (c) How does the probability that a sequence of n rolls of the die with have no two consecutive sixes grow as $n \to \infty$?

4. **Double Rotations**

Consider the double rotation of a binary search tree shown below.



(a) Explain why $\beta_B = \alpha_C\alpha_A + \alpha_C(1-\alpha_A)\alpha_B$.

(b) Express β_A and β_C in terms of α_A, α_B, and α_C.

5. **Divide-and-Conquer**

Having seen the power of recursion and divide-and-conquer, the TA decided to write a program to compute x^n,

(a) His first attempt was

function Power(x, n)

```
if n = 0 then
    return 1
else if n is odd then
    return x * Power(x, ⌊n/2⌋) * Power(x, ⌊n/2⌋)
else
    return Power(x, ⌊n/2⌋) * Power(x, ⌊n/2⌋)
end if
```

Analyze the time required by this algorithm.

(b) His second attempt was

function Power(x, n)

```
if n = 0 then
    return 1
else
    integer t ← Power(x, ⌊n/2⌋)
    if n is odd then
        return x * t * t
    else
        return t * t
    end if
end if
```

Analyze the time required by this algorithm.

5. **Divide-and-Conquer, continued.**