Game Theory: Algorithms and Applications
CS 539
Fall 2019
HomeWork 1 Solutions

1. In general, consider there are n players in total. First we notice that no cycle will exist in the equilibrium subgraph, for it costs a lab but brings no benefit. Therefore any Nash equilibrium is a tree or a forest. If a player pays for an edge, the edge must connect her to d players, where $d > c_1/C$ in order to keep the marginal utility positive. When $c_1 \geq C$, one possible Nash equilibrium is that no player connects to other players, for the marginal utility of buying any edge is not positive.

   On the other hand, consider a "stable" tree T that does not contain all the vertices, "stable" in the sense that in T all the players want to keep the currently paid edges. We must have $|T| > d$. Then a player out of T would like to pay for an additional edge to connect to T. Therefore this process continues until every outside player connects to T to form 1 spanning tree. Any spanning tree in which an edge connects the buyer to $d \geq c_1/C$ players is a Nash equilibrium. This also implies that $c_1 \leq (n-1)C$, otherwise no tree could satisfy $d \geq c_1/C$.

   We observe that when $\frac{n-1}{2}C \leq c_1 \leq (n-1)C$, the equilibrium is a star, where every leaf pays for the edge to connect to the center of the star.

   In the case of 5 labs, simply substitute n in the result above by 5.

2. We assume that a conflict is resolved in favor of the player with the lesser index.

   - *First-Price:* A Nash equilibrium is obtained when p_1 (player 1) bids a value $b_1 \in [v_1, v_2)$ and other players $p_i, i = \{2, 3, \cdots, n\}$ bid any value v_i where $v_i \in [0, v_i]$.
     Proof: By contradiction: Assume that b_1 is less than v_2, then p_2 can bid her valuation and win the auction, and this would, therefore, not be a Nash equlibrium.
     Special place for p_1: She may bid any value over v_2 and win the auction.
   - *Second-Price:*
     $b_1 \in [b_2, \infty)$ if $v_2 \leq b_2 \leq v_1$ and $b_3 \leq v_2$.
     $b_2 \in [v_1, \infty)$ if $b_1 \leq v_2$ and $b_3 \leq v_2$.
     $b_3 \in [v_1, \infty)$ if $b_1 \leq v_3$ and $b_2 \leq v_3$
     Proof: omitted

3. i At a Nash equilibrium, all consumers get distributed evenly accross all service providers. The difference between the number of consumers on any two providers is at most 1.

   ii Assume $N \geq P$. First we observe that every provider must serve at least 1 consumer. Start with an arbitrary assignment of consumers. Consider the vector of transmission rates sorted in increasing order

   $$R = [R_1, R_2, \cdots R_P], R_1 \leq R_2 \leq \cdots \leq R_P$$

   If a consumer currently with provider i wants to switch to provider j for some $i < j$, denote by R_i, R_j the current rates and by $R'_i = \frac{B_i}{n_i - 1}, R'_j = \frac{B_j}{n_j + 1}$ the rates after the switch, we must have

   $$R_i < R'_j \;\&\; R_i < R'_i$$

   Note that whether $R'_i < R'_j$ or not, the new sorted vector R' must increase in lexicographic order. Since each R_i is upper-bounded by $\frac{B_i}{1}$, the vector can only increase to a bounded value, therefore the system will converge to a Nash equilibrium.

4. Player i will keep increasing traffic until her marginal utility becomes zero (or less). Marginal utility is given by

$$\frac{\partial U_i}{\partial X_i} = K - 2\alpha X_i$$

The game is symmetric, $X_i = X_{i'}, \forall i, i'$ at Nash equilibrium.

$$\therefore X_i = \frac{K}{2\alpha}, \forall i \text{ @ NE}$$

5. Assume that $K \geq \sqrt{x_1 + x_2}$. Differentiating U_i,

$$\begin{aligned}\frac{\partial U_i}{\partial x_i} &= K - \sqrt{x_i + x_{i'}} + x_i \cdot \frac{\partial(K - \sqrt{x_i + x_{i'}})}{\partial x_i} - 2x_i \\ &= K - \sqrt{x_i + x_{i'}} - x_i \cdot \frac{1}{2\sqrt{x_i + x_{-i}}} - 2x_i \\ &\qquad \text{where } i' \text{ is the } \textit{other} \text{ player}\end{aligned}$$

Since the game is symmetric between the players, $x_i = x_{i'}$ at NE. Using KKT conditions,

$$\begin{aligned}K - \frac{\sqrt{50}}{4}\sqrt{x_i} - 2x_i = 0 \\ \implies \sqrt{x_i} = \frac{\pm\sqrt{50 + 128K} - \sqrt{50}}{16}\end{aligned}$$

Since $\sqrt{x_i} \geq 0$, $x_i = \left(\frac{\sqrt{50+128K} - \sqrt{50}}{16}\right)^2 = \frac{25+32K-5\sqrt{25+64k}}{64}, i = 1, 2$ @NE.