Game Theory: Algorithms and Applications
CS 539

Spring 2018
HomeWork 5 Solutions

1. Given that the price of anarchy for pure equilibria on instances of the load balancing game with two tasks and two machines with possibly different speeds corresponds to the golden ratio $\phi = (1+\sqrt{5})/2$. Show that there is a game instance G admitting an equilibrium assignment A with $cost(A) = \phi \cdot opt(G)$.

   Solution: Instance: $n = 2$,$m = 2$. we have: $\phi = \frac{1+\sqrt{5}}{2}$ Let $w_1 = 1, w_2 = \phi, s_1 = 1, s_2 = \phi$

   NE: $A \equiv A(1) = 2, A(2) = 1$ because

$$\frac{w_2}{s_1} = \frac{\phi}{1} \leq \frac{1+\phi}{\phi} = \frac{w_1 + w_2}{s_2},$$

$$\frac{w_1}{s_2} = \frac{1}{\phi} \leq \frac{1+\phi}{1} = \frac{w_1 + w_2}{s_1}$$

$$\implies cost(A) = \max\{\frac{w_2}{s_1}, \frac{w_1}{s_2}\} = \frac{w_2}{s_1} = \phi$$

   The opt. is given by

$$Opt(G) = \min\{\frac{w_1 + w_2}{s_2}, \max\{\frac{w_1}{s_2}, \frac{w_2}{s_1}\}, \max\{\frac{w_1}{s_1}, \frac{w_2}{s_2}\}, \frac{w_1 + w_2}{s_1}\}$$

$$\implies Opt(G) = \max\{\frac{w_2}{s_2}, \frac{w_1}{s_1}\} = 1$$

   Therefore,

$$\frac{cost(A)}{Opt(G)} = \phi \implies \text{result}$$

2. In the market equilibrium model where traders are endowed with money, let M be a market with n traders and m goods, with each trader i having utility $v_i x_{ij}$ for good j where $v_i \in \mathbb{Z}^+$, and $x_{ij} \in \mathbb{R}^+$,i.e., each trader has the same utility for all the goods. Can you provide a solution for the market equilibrium in this case?

   Solution: The price of goods will be the same since the players individually have equal utility for the goods (equivalently one can consider the goods to be one good). Thus

$$p_j = \frac{\sum_{i=1}^{n} e_i}{\sum_{j=1}^{m} a_j}$$

   where e_i is the endowment and a_j is the availability.

Let $A = \sum_j a_j$ and $E = \sum_i e_i$. Player i will thus get a fraction of each good defined as:

$$x_{ij} = a_j \frac{e_i}{E}$$

and trader i spends all money since

$$\sum_j x_{ij} p_j = \sum_j \frac{a_j}{A} e_i = e_i$$

This is also optimum for each player since his bang-per-buck is the same across all goods. Every trader spends all its money and all the goods are sold, since

$$\sum_i x_{ij} = a_j \sum_i \frac{e_i}{E} = a_j$$

providing us with a market equilibrium.

3. A reverse auction is when an item is to be bought at the lowest cost from a set of bidders. Suppose you want to buy k identical items, when each bidder can only supply one item. Design an incentive compatible mechanism to run this auction. Proofs required.

   Solution: Let N denote the set of bidders, c_i denote the "cost bid" of bidder i and $\hat{c}_i$ denote the reported or asking bid of bidder i.

   The objective is to pick a subset of bidders and define a payment mechanism.

   According to the VCG mechanism, the winners are determined as

$$S^* = \arg \min_{S,\ |S|=k} c(S)$$

   where

$$c(S) = \sum_{i \in S} \hat{c}_i$$

   This is the set of k best (lowest) bidders.

   The payment for each bidder is:

$$p_i = \alpha_i - \sum_{j \neq i, j \in S^*} \hat{c}_j$$

   where $\alpha_i = \min_{\{S', i \notin S'\}} c(S') = \min_{\{S', i \notin S'\}} \sum_{j \in S'} \hat{c}_j$. This indicates that p_i is the bid of the $k + 1^{st}$ bidder , when i is in the winning set and 0 otherwise.

4. 3-player majority game:
   $N = \{1, 2, 3\}$, i.e. 3 players.
   Valuation for subset of players $v(S) = 1, \forall |S| \geq 2$. $v(S) = 0, |S| \leq 1$. Is there a core of this coalition game? Prove.

   Solution: Suppose there exists a core, then

$$\begin{aligned}
v(S) = 0, |S| = 1 &\implies x_1, x_2, x_3 \geq 0 &(1)\\
v(S) = 1, |S| = 2 &\implies x_1 + x_2 \geq 1 &(2)\\
&\quad\ \ x_2 + x_3 \geq 1 &(3)\\
&\quad\ \ x_1 + x_3 \geq 1 &(4)\\
v(S) = 0, |S| = 3 &\implies x_1 + x_2 + x_3 = 1 &(5)
\end{aligned}$$

Then,

$$\begin{aligned} (2)+(3)+(4) &\implies \quad 2(x_1+x_2+x_3) \geq 3 \\ 2\cdot(5) &\implies \quad 2(x_1+x_2+x_3) = 2 \end{aligned}$$

which is not possible $\implies$ there is no core