CS 539 HW 5

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① $\phi = \frac{1+\sqrt{5}}{2}$ $j = 2, m = 2$ where $cost(A) = \phi \cdot opt(G)$

Let 2 machines have weights where $\frac{w_i}{s_i} = 1$:.

$w_1 = \phi, s_1 = \phi$; $w_2 = 1$, $s_2 = 1$ where $PoA = \frac{cost(A)}{OPT(G)} = \phi$

Proof|

NE exists at $A(1) = 2$, $A(2) = 1$

where: $\frac{w_1}{s_2} = \frac{\phi}{1} \leq \frac{w_1 + w_2}{s_1} = \frac{1+\phi}{\phi} = \phi$

and $\frac{w_2}{s_1} = \frac{1}{\phi} \leq \frac{w_1 + w_2}{s_2} = \frac{1+\phi}{1}$

where $cost(A) = \max\left\{\frac{w_2}{s_1}, \frac{w_1}{s_2}\right\} = \phi$

this implies $OPT(G) = 1$

$$OPT(G) = \min\left\{\frac{\Sigma w}{s_1}, \frac{\Sigma w}{s_2}, \max\left\{\frac{w_1}{s_2}, \frac{w_2}{s_1}\right\}, \underline{\max\left\{\frac{w_1}{s_1}, \frac{w_2}{s_2}\right\}}\right\}$$

$$= \max\left\{\frac{w_1}{s_1}, \frac{w_2}{s_2}\right\} = \max\{1, 1\}$$

$$= 1$$

which holds $PoA = \frac{cost(A)}{opt(G)} = \phi/1 = \phi$

$M = M_{n \times m}$

(2) $u_i = a_j x_{ij}$ for $a_j \in \mathbb{Z}^+$ & $x_{ij} \in \mathbb{R}^+$

Claim:
$$p_{ij} = \frac{\sum_{i=1}^{n} e_i}{\sum_{j=1}^{m} a_j} = \frac{\sum_{\forall i} \text{endowment}}{\sum_{\forall i} \text{availability}}$$

Since u_i is the same for all traders, then:

$$x_{ij} = a_j \cdot \frac{e_i}{\Sigma e_i}$$ where trader$_i$ gets a fraction of the total endowment.

→ assuming traders spend all their money and sell all goods:

$$\sum_{\forall j} x_{ij} \cdot p_{ij} = \sum_{\forall j} \frac{a_j}{\sum_{\forall j} a_j} \cdot e_i = e_i \cdot \frac{1}{\sum_{\forall j} a_j} \cdot \sum_{\forall j} a_j = e_i$$

and
$$\sum_{\forall i} x_{ij} = \sum_{\forall i} a_j \cdot \frac{e_i}{\sum_{\forall i} e_i} = a_j \cdot \frac{\sum_{\forall i} e_i}{\sum_{\forall i} e_i} = a_j$$

which gives us a market equilibrium

(3) N-bidders (sellers) with cost bid: c_i. Their asking bid. is $\hat{c}_i$
let p_i be the payment for bidder i where each bidder sells only one item at a min bid. Assume a subset S of lowest bidders
Our modified VCG mechanism is:

$$S^* = \arg\min_{S} \left(\sum_{i=1}^{k} \hat{c}_i \right)$$

where S is the set of the lowest bidders and $|S| = k$.
where there are k identical items.

The payment for bidder i given $k-1$ bids in S^* is:

$$p_i = \min_{\{S', i \notin S\}} \left(\sum_{j \in S'} \hat{c}_j \right) - \left[\left(\sum_{j \in S^*} \hat{c}_j \right) - \hat{c}_i \right]$$

This shows that p_i is equal to the c_{k+1} where bidder $i \in S^*$ such that

$$p_i = \begin{cases} c_{k+1}, & \text{if } i \in S^* \\ 0, & \text{otherwise} \end{cases} \text{ where } |S| = k,\ i \in N$$