Game Theory: Algorithms and Applications
CS 539

Spring 2018
HomeWork 3
Due 11:59pm, March 5th
Total: 160 points

1. Find a correlated equilibrium for the the following game

Payoff	A	B	C
X	(6,6)	(2,7)	(5,6)
Y	(7,2)	(0,0)	(2,2)

Is there a correlated equilibrium that has better expected utility (for at least one player) than a Mixed Nash Equilibrium?

Solution: Correlated:
For P_1,

$$\Pi(XA)(U_1(XA)-U_1(YA))+\Pi(XB)(U_1(XB)-U_1(YB))+\Pi(XC)(U_1(XC)-U_1(YC)) \geq 0$$

Similarly,

$$\Pi(YA)(U_1(YA)-U_1(XA))+\Pi(YB)(U_1(YB)-U_1(XB))+\Pi(YC)(U_1(YC)-U_1(XC)) \geq 0$$

We know, (sum of prob. of all possible events in a game)

$$\Pi(XA)+\Pi(XB)+\Pi(XC)+\Pi(YA)+\Pi(YB)+\Pi(YC)=1$$

For P_2,

$$\Pi(XA)(U_2(XA)-U_2(XB))+\Pi(YA)(U_2(YA)-U_2(YB)) \geq 0$$
$$\Pi(XA)(U_2(XA)-U_2(XC))+\Pi(YA)(U_2(YA)-U_2(YC)) \geq 0$$
$$\Pi(XB)(U_2(XB)-U_2(XA))+\Pi(YA)(U_2(YB)-U_2(YA)) \geq 0$$
$$\Pi(XB)(U_2(XB)-U_2(XC))+\Pi(YA)(U_2(YB)-U_2(YC)) \geq 0$$
$$\Pi(XC)(U_2(XC)-U_2(XA))+\Pi(YA)(U_2(YC)-U_2(YA)) \geq 0$$
$$\Pi(XC)(U_2(XC)-U_2(XB))+\Pi(YA)(U_2(YC)-U_2(YB)) \geq 0$$

There are multiple solutions to the above system of equations.
A feasible solution for Correlated-eqb. exists where

$$\Pi(XA)=0.4, \Pi(XB)=0.1, \Pi(XC)=0.1, \Pi(YA)=0.3, \Pi(YB)=0, \Pi(YC)=0.1;$$
$$U_1=5.4, U_2=4.5$$

Mixed:
Using the support set theorem, one mixed NE is given by

$$\Pi_1(X)=\frac{2}{3}, \Pi_1(Y)=\frac{1}{3}, \Pi_2(A)=\frac{2}{3}, \Pi_2(B)=\frac{1}{3}, \Pi_1(C)=0; U_1=\frac{14}{3}, U_2=\frac{14}{3}$$

2. Determine correlated equilibrium for the following Inspection Game:

Payoff	Inspect	Not Inspect
Shirk	(0,-h)	(w,-w)
Work	(w-g,v-w-h)	(w-g,v-w)

where h is the cost of inspection, v is the value of the good produced, g is the cost to the worker and w is the wage and assume $w > g, w > h$.

Solution: *Notation:* S : shirk, W : work, I : inspect, NI : not inspect.
For P_1,

$$\Pi(SI)(U_1(SI) - U_1(WI)) + \Pi(S, NI)(U_1(S, NI) - U_1(W, NI)) \geq 0$$

$$\implies \Pi(SI)(g - w) + \Pi(S, NI)g \geq 0$$

Similarly,

$$\Pi(WI)(w - g) + \Pi(W, NI)(-g) \geq 0$$

Next,

$$\Pi(SI) + \Pi(WI) + \Pi(S, NI) + \Pi(W, NI) = 1$$

We use the fact that a Mixed NE is also a correlated NE, and assume $w > g$ and $w > h$. Thus,

$$\Pi_2(NI) = \frac{w-g}{w} \text{ and } \Pi_2(I) = \frac{g}{w}$$

Using the same technique,

$$\Pi_1(S) = \frac{h}{w} \text{ and } \Pi_1(W) = \frac{w-h}{w}$$

Note: v, w, h must still be values that do not violate inequality condtions where the sum of probs. equals 1.

3. Prove that for a 2-player game with n strategies each, mixed Nash equilibrium is also a correlated equilibrium.

Solution: Let the two actions sets be $A_1 = \{s_1^1, s_2^1, \cdots, s_n^1\}$ and $A_2 = \{s_1^2, s_2^2, \cdots, s_n^2\}$. Let the probabilities for the strategies be denoted by $\{p_1^1, p_2^1, \cdots, p_n^1\}$ and $\{p_1^2, p_2^2, \cdots, p_n^2\}$ respectively. Probability of P_1 playing strategy i and P_2 playing strategy $j = p_i^1 \cdot p_j^2$.

Assume, w.l.o.g., that the players have all n strategies in their support sets.

MNE for P_1:

$$\sum_{j=1}^{n} p_j^2 u_{i,j}^1 = \sum_{j=1}^{n} p_j^2 u_{i',j}^1, \quad \forall i' \neq i \implies \sum_{j=1}^{n} p_j^2 [u_{i,j}^1 - u_{i',j}^1] = 0, \quad \forall i' \neq i$$

$$\implies \sum_{j=1}^{n} p_i^1 p_j^2 [u_{i,j}^1 - u_{i',j}^1] = 0, \quad \forall i' \neq i \implies \sum_{j=1}^{n} p_i^1 p_j^2 [u_{i,j}^1 - u_{i',j}^1] \geq 0, \quad \forall i' \neq i$$

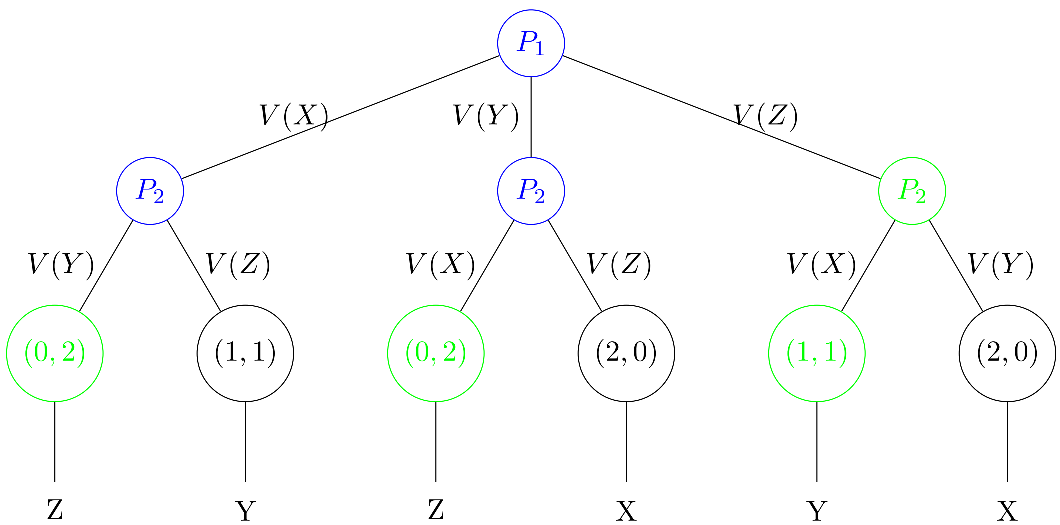
$\implies$ correlated equilibrium for P_1. Proof for P_2 is symmetric $\implies$ every MNE is a correlated equilibrium.

4. Two people select a policy that affects them both by alternatively vetoing policies until only one remains. First, person 1 vetoes a policy. If more than one policy remains, then person 2 vetoes a policy. If more than one policy still remains, person 1 then vetoes another policy. The process continues until a single policy remains unvetoed. Suppose there are three possible policies, X, Y and Z, person 1 prefers X to Y to Z, and person 2 prefers Z to Y to X. Model this situation as an extensive game and find its Nash Equilibria.

   Solution P_1 preference: $X > Y > Z$. W.l.o.g. let $U_1(X) = 2, U_1(Y) = 1, U_1(Z) = 0$.

   P_2 preference: $Z > Y > X$. W.l.o.g. let $U_2(X) = 0, U_2(Y) = 1, U_2(Z) = 2$.

   Tree construction:



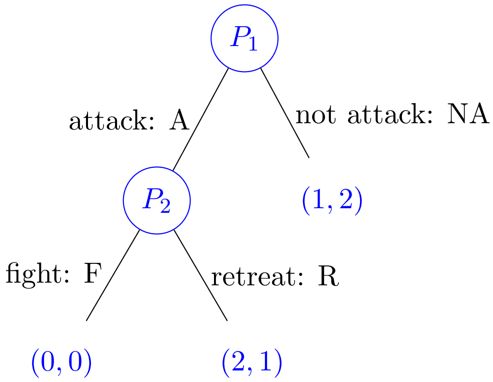
   We use SGPE theory: Each subtree of height 1 chooses his best option (strategy) $\implies$ $(V(Y), V(X), V(X))$. Then, for each subtree of height 2, P_1 makes his move $\implies$ $(V(Z), (V(Y), V(X), V(X)))$, which is the SGPE and NE with payoffs $= (1, 1)$.

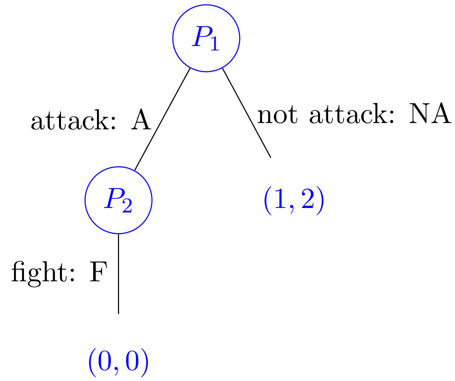
   We can exhaustively find the remaining NE:

   - $(V(Z), (V(Z), V(X), V(X)))$

5. Army 1, of country 1, must decide whether to attack army 2, of country 2, which is occupying an island between the two countries. In the event of an attack, army 2 may fight, or retreat over a bridge to its mainland. Each army prefers to occupy the island than not occupy it; a fight is the worst outcome for both armies. Model this situation as an extensive game with perfect information and show that army 2 can increase its subgame perfect equilibrium payoff(and reduce army 1s payoff) by burning the bridge to its mainland (assume this act entails no cost), eliminating its option to retreat if attacked.

   Solution:

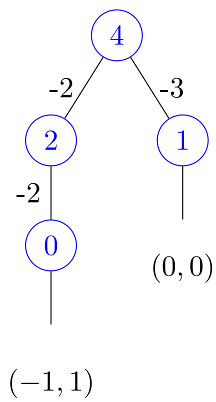




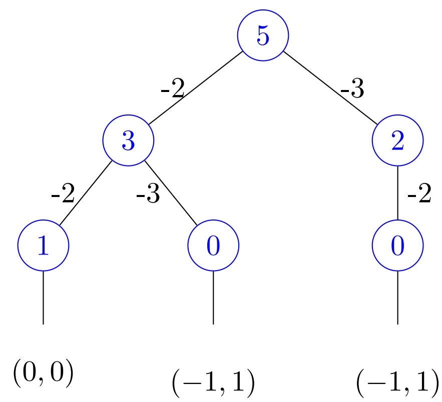
Left tree: Using SGPE theory, P_2 chooses his strategy in case of an attack $\implies P_2$ retreats $\implies (R)$. Then, P_1 chooses between $U_1 = 2$ for (A) and $I_1 = 1$ for (NA), and attacks $\implies$ SGPE is (A,R).

Right tree: When P_2 burns the bridge, the army can only choose (F) $\implies P_1$ will choose to not attack to get $U_1 = 1$ instead of 0 $\implies$ SGPE is (NA,F), ie., P_2 has a higher payoff.

6. 2 people take turns removing stones from a pile, initially containing n stones. Each can either remove 2 or 3 stones at a time. The person who removes the last stone is the winner , and the game ends in a draw if no player can make a move. Determine Nash equilibria and the sub-game perfect equilibria when $n = 4, 5$



$n = 4$



$n = 5$

$n = 4$**:** NE and SGPE $\equiv (-3, -2)$.

$n = 5$**:** NE and SGPE $\equiv (-2, (-3, -2))$ and $(-3, (-3, -2))$.

7. Show that a sub-game perfect equilibrium of an extensive game is also a subgame perfect equilibrium of a game obtained by deleting a subgame not reached in the equilibrium and assigning to the new terminal node an outcome which is an equilibrium of the deleted subgame.

Solution: Let G denote the extensive game and h be its history for an equilibrium. Let G' be the extensive game obtained by deleting subgame $G \setminus m$, rooted at node M; and let m denote the history which leads to node M.

From the question, we have the fact that the SGPE of G is not reached in subgame $G \setminus m$.

Let S_i^* be the strategy of i at equilibrium.

$$\implies O(S_i^*, S_{-i}^*) \geq O(S_i', S_{-i}^*)$$

where, $O()$ is the outcome function, S_i^* is i's strategy at equilibrium and S_i' is any other strategy of i.

Let $\bar{m}$ be the history in G starting at node M (call this G_m), which leads to an SGPE in G_m.

Proof by contradiction: In G', M is now a terminal with the outcome:

$$O(S_i^*(G_m), S_{-i}^*(G_m)) \tag{1}$$

Suppose history h *does not* lead to an equilibrium in G'. This change in SGPE $\bar{S}$ must be a consequence induced by the change in G to obtain G', ie.,

$$O(\bar{S}_i^*, \bar{S}_{-i}^*) = O(S_i^*(G_m), S_{-i}^*(G_m))$$

Then, the outcome of supposed equilibrium of G' is *greater than* the outcome of G' when profile with history h is followed, denoted as:

$$O(S_i^*(G_m), S_{-i}^*(G_m)) > O(\bar{S}_i^h), \bar{S}_{-i}^*) \tag{2}$$

But, we know

$$O(\bar{S}_i^h, \bar{S}_{-i}^*) = O(S_i^*, S_{-i}^*) \tag{3}$$

since h leads to the same outcome in both G and G'.

Thus, equations 2 and 3

$$\implies O(S_i^*(G_m), S_{-i}^*(G_m)) > O(S_i^*, S_{-i}^*)$$

and equation 1, and the fact that h led to a SGPE in G

$$\implies O(S_i^*(G_m), S_{-i}^*(G_m)) \leq O(S_i^*, S_{-i}^*)$$

which gives us a contradiction.

8. Show that more information may hurt a player by constructing a 2-player Bayesian game with the following features. Player 1 is fully informed while player 2 is not; the game has a unique Nash equilibrium, in which player 2's payoff is higher than his payoff in the unique equilibrium of any of the related games in which he knows player 1's type (i.e., player 2 has only 1 type).

Solution: P_1 has two types (Prob. $= \frac{1}{2}$ each): State 1 $\equiv$

	B	S
B	2,1	1,0
S	2,4	0,5

State 2 $\equiv$

	B	S
B	2,3	0,1
S	1,0	1,1

Expected utility of P_2:

	B	S
(B,B)	2	$\frac{1}{2}$
(B,S)	$\frac{1}{2}$	$\frac{1}{2}$
(S,B)	$\frac{7}{2}$	3
(S,S)	2	3

NE is obtained at (B,SB). P_2 has payoff $\frac{7}{2}$ as opposed to his payoff in states 1 and 2 (3 and 1 respectively).