Game Theory: Algorithms and Applications

Sanjiv Kapoor

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Mixed Equilibrium

Instead of playing one strategy, what if the players were allowed to play multiple strategies, i.e. either play one or the other with a probability assigned to them.
Recall that we have $(N, (A_i), u_i)$ representing Pure Nash Equilibrium where

- $N, (A_i), u_i$ represents a set of players,
- action(strategy) set of player i, referred to as *Pure Strategies*
- utility function $u_i : A_1 \times A_2 \times ... \times A_N \to R$.

In addition we specify a probability distribution over each players strategy set, termed *Lottery*.
Let $(N, ((\Delta(A_i)), u_i)$ represent the mixed extension of the strategic game above
where $\Delta(A_i)$ represents the probability distributions over A_i.

Each player maximizes his expected utility:
For each player, the probability of playing mixed strategy he could use is represented by the function π^i where $\pi^i(a)$ is the probability that the players plays strategy $a, a \in A_i$. The utility of each player i is

$$u_i(\pi) = \sum_{\tilde{A}} u_i(\tilde{A})\pi(\tilde{A}),$$

where $\tilde{A} \in \times_n A$ and π is a probability distribution, (π^i).
Here $\pi(\tilde{A})$ is the probability that $\tilde{A}$ is used, and

$$\pi(\tilde{A}) = \prod_i \pi^i(\tilde{A}_i),$$

where $\pi^i(\tilde{A}_i)$ is the probability that the ith player plays strategy $\tilde{A}_i$.

Co-ordination Game

Payoff	Bach	Stravinsky
Bach	(2,1)	(0,0)
Stravinsky	(0,0)	(1,2)

Let $\pi^i(B)$ and $\pi^i(S)$ be the probability that the ith player uses strategy B and S respectively. Then the expected utility of player1 is

$$
\begin{aligned}
u_1(\pi) &= \sum_{\tilde{A}} u_1(\tilde{A})\pi(\tilde{A}) \\
&= \sum_{\tilde{A}} u_1(\tilde{A})\pi^1(\tilde{A}_1)\pi^2(\tilde{A}_2) \\
&= \pi^1(B)\pi^2(B)u_1(B,B) + \pi^1(B)\pi^2(S)u_1(B,S) \\
&+ \pi^1(S)\pi^2(B)u_1(S,B) + \pi^1(S)\pi^2(S)u_1(S,S) \\
&= 2\pi^1(B)\pi^2(B) + \pi^1(S)\pi^2(S)
\end{aligned}
$$

Nash Equilibrium in the mixed strategy game:
For all players find π_i and $\pi = \times_i \pi_i$, s.t. no player is induced to change his distribution, given fixed strategies of other players, or equivalently

$$u_i(\pi) \equiv u_i(\pi_i, \pi_{-i}) \geq u_i(\pi'_i, \pi_{-i}), \quad \forall \pi'_i,$$

where π_i is the probability distribution of the ith player, and π_{-i} is the one for the other players.

Definition

For a mixed strategy π at Nash Equilibrium, define **support set** of player i as $S_i(\pi) = \{a | a \in A, s.t. \ \pi_i(a) > 0\}$.

Support Set theorem

Let $u_i(\pi)\,|_a$ be the utility when player i plays the pure strategy a, i.e.

$$u_i(\pi)\,|_a = \sum_{\tilde{A_{-i}}} u_i(a, \tilde{A_{-i}})\pi(\tilde{A_{-i}})$$

Theorem

At Nash Equilibrium, player i is "indifferent" to all strategies in his support set, that is,

$$u_i(\pi)\,|_a = u_i(\pi)\,|_b \ , \quad \forall a, b \in S_i(\pi)$$

where $u_i(\pi)\,|_a$ is the utility of player i playing pure strategy a, given other players' strategies fixed.

Proof of Support set Theorem

Proof.

Assume π is a mixed Nash Equilibrium strategy, then

$$\begin{aligned} u_i(\pi) &= \sum_{\tilde{A}} u_i(\tilde{A})\pi(\tilde{A}) \\ &= \sum_{a \in \tilde{A}_i} \sum_{\tilde{A}_{-i}} u_i(a, \tilde{A}_{-i})\pi(a, \tilde{A}_{-i}) \\ &= \sum_{a \in \tilde{A}_i} \pi_i(a) \sum_{\tilde{A}_{-i}} u_i(a, \tilde{A}_{-i})\pi(\tilde{A}_{-i}) \end{aligned}$$

Let $u_i^a(\pi_i) = \sum_{\tilde{A}_{-i}} u_i(a, \tilde{A}_{-i})\pi(\tilde{A}_{-i})$ denote the utility of player i playing strategy a while others playing $\tilde{A}_{-i}$. Then by the local stability of Nash Equilibrium, for any strategy in the support set, $u_i^a(\pi_i)$ must be the same. (Otherwise say $u_i^a(\pi_i)$ is higher than the others, than player i could raise $\pi_i(a)$ to increase $u_i(\pi)$, which contradicts to our assumption that π is Nash Equilibrium.) $\square$ $\square$

Use the above result we can find the Nash Equilibrium of the battle of sexes game.

$$\begin{aligned} u_1(\pi) &= \pi_1(B)\,(2\cdot\pi_2(B)+0\cdot\pi_2(S)) \\ &+ \pi_1(S)\,(0\cdot\pi_2(B)+1\cdot\pi_2(S)) \end{aligned}$$

By Theorem,

$$2\pi_2(B) = u_1^B(\pi_{-i}) = u_1^S(\pi_{-i}) = \pi_2(S),$$

and $\pi_2() + \pi_2(S) = 1$, so $\pi_2(B) = 1/3, \pi_2(S) = 2/3$. Similarly we have $\pi_1(B) = 2\pi_1(S)$ and then $\pi_1(B) = 2/3, \pi_1(S) = 1/3$.

Inspection Game

Payoff	Inspect	Not Inspect
Shirk	(0,-h)	(w,-w)
Work	(w-g,v-w-h)	(w-g,v-w)

where h is the cost of inspection, v is the value of the good produced, g is the cost to the worker and w is the wage and assume $w > g, w > h$, if not, $(w, -w)$ would be a pure Nash Equilibrium.

Let $x = \pi_1(S), y = \pi_2(I)$, then we have
$0 \cdot y + w(1-y) = (w-g)y + (w-g)(1-y)$. Thus $y = g/w$.
Similarly $-hx + (v-w-h)(1-x) = -wx + (v-w)(1-x)$
Thus $x = h/w$.
So a mixed equilibrium exists.

Matching Pennies

Payoff	Head	Tail
Head	(1,-1)	(-1,1)
Tail	(-1,1)	(1,-1)

Let $\pi^i(H)$ and $\pi^i(T)$ be the probability that the ith player uses strategy H and T respectively. Then the expected utility of player1 is

$$
\begin{aligned}
u_1(\pi) &= \sum_{\tilde{A}} u_1(\tilde{A})\pi(\tilde{A}) \\
&= \sum_{\tilde{A}} u_1(\tilde{A})\pi^1(\tilde{A}_1)\pi^2(\tilde{A}_2) \\
&= \pi^1(H)\pi^2(H)u_1(H,H) + \pi^1(H)\pi^2(T)u_1(H,T) \\
&+ \pi^1(T)\pi^2(H)u_1(T,H) + \pi^1(T)\pi^2(T)u_1(T,T)
\end{aligned}
$$

Thus $\pi^2(H) - (1 - \pi^2(H)) = \pi^2(H)(-1) + (1 - \pi^2(H))$
$4pi^2(H) = 2$
$\pi^i(H) = 1/2, i = 1, 2$

Load Balancing:
We consider a **randomized strategy**.
Let's p_i^j be the probability that task i is assigned to machine j.
Consider the following **assumptions**:

- Number of **tasks** $= n$
- Number of **machines** $= m$
- **Tasks** require $W_1 W_2 ... W_n$ units of "effort"
- **Machine** have speed $S_1 S_2 ... S_m$

So time W_i / S_j is the time for executing task i on machine j.

The **expected load** on a machine is given by :

$$E(l_j) = E(\sum_i X_i^j (W_i/S_j))$$

With $X_i^j = 1$ if task i is assigned on machine j, 0 otherwise.
To simplify let $W_i = S_j = 1$.

$$E(l_j) = \sum_i p_i^j [X_i^j W_i/S_j]$$

$$E(l_j) = \sum_i p_i^j$$

Definition

Strategy Profile

$$P = ((P_1^1 P_1^2 ... P_1^m)(P_2^1 P_2^2 ... P_2^m)...(P_n^1 P_n^2 ... P_n^m))$$

$$P = (P_i^j)_{i \in [n], j \in [m]}$$

Property 1: *A strategy profile P is a Nash Equilibrium iff*:

$$\forall i, \forall j \; P_i^j > 0 \Longrightarrow C_i^j \leq C_i^k$$

Where C_i^j is the expected cost of scheduling given that task i is assigned to machine j.

$$C_i^j = E[l_j | i \text{ is assigned to machinej}]$$

$$C_i^j = 1 + \sum_{k \neq i} p_k^j$$

Claim: $\forall i, \forall j \; P_i^j = 1/m$ (uniformly distributed) is a Mixed Nash Equilibrium.

$$C_i^j = 1 + \sum_{k \neq i} p_k^i$$

$$C_i^j = 1 + \sum_{k \neq i} 1/m$$

$$C_i^j = 1 + \frac{n-1}{m}$$

$\forall j, \forall k \; C_i^j = C_i^k$. So it satisfies $C_i^j \leq C_i^k$. By **property 1**, a mixed nash equilibrium always exists.

Existence of Mixed Equilibria.

Yes! via Kakutani's fixed point theorem again.

Rainbow Triangulation Problem

A triangulation of a triangle is a subdivision of the triangle into small triangles. A Sperner Labeling is a labeling of a triangulation of a triangle with the numbers 1,2 and 3 such that

(1) The three corners are labeled 0, 1 and 2.

(2) Every vertex on the line connected corner vertex i and corner vertex j is labeled i or j.

A rainbow triangle is an inside triangle which is labeled (0,1,2).

Lemma (Sperner's Lemma)

Every Sperner Labeling contains a rainbow triangle.

Sperner's Lemma

$$\begin{aligned} x &= \text{number of edges on the boundary colored } (0,1); \\ y &= \text{number of edges inside colored } (0,1); \\ Q &= \text{number of triangles colored } (0,0,1) \text{ or } (0,1,1); \\ R &= \text{number of rainbow triangles.} \end{aligned}$$

Lemma

The number of edges on the boundary line (0,1) of the triangle is odd.

(Proof Omitted)

Now we prove the number of rainbow triangles is odd. Each triangle of type Q gives two $(0,1)$ edges. Each triangle of type R gives one $(0,1)$ edge. Each edge inside colored $(0,1)$ is shared by two triangles. Then we can get

$$2Q + R = x + 2y$$

Since x is odd by claim, R is odd.

Theorem

(Nash) Every finite game has a mixed strategy Nash equilibrium.

A mixed strategy profile π^* is a NE if
$U_i(\pi_i^*, \pi_{-i}^*) > U_i(\pi_i', \pi_{-i}^*), \forall \pi_i' \in A_i$.
Or, $\pi_i^* \in B_i(\pi^*),\ \forall i$,
where $B_i(\pi^*)$ is the best response of player i, given that the other players' strategies are fixed at π_{-i} .
We define the correspondence $B : \Sigma \to \Sigma$ such that for all $\pi \in \Sigma$, we have

$$B(\pi) = (B_i(\pi_{-i}))$$

Determining Nash equilibrium is equivalent to determining a mixed strategy π such that $\pi \in B(\pi)$.

A set in a Euclidean space is compact if and only if it is bounded and closed. A set S is convex if for any $x, y \in S$ and any $\alpha \in [0, 1]$, $\alpha x + (1 - \alpha)y \in S$.

Theorem

(Kakutani) Let A be a non-empty subset of a finite dimensional Euclidean space. Let $f : A \to A$ be a correspondence, with $f(x) \subseteq A$, satisfying the following conditions:

A is a compact and convex set.

$f(x)$ is non-empty for all $x \in A$.

$f(x)$ is a convex-valued correspondence: $\forall x in A, f(x)$ is a convex set.

$f(x)$ has a closed graph: i.e. if $\{x_n, y_n\} \to \{x, y\}$with $y_n \in f(x_n)$, then $y \in f(x)$.

Then, f has a fixed point, that is, there exists some $x \in A$, such that $x \in f(x)$.

Proof of Nash's Equilibrium Existence

We use the correspondence $B : \Sigma \to \Sigma$.
Σ is compact, convex, and non-empty.
$B_i(\pi_i, \pi_{-i})$ is a linear function hence non-empty and convex valued.
And since $B_i()$ is a linear optimization it is a continuous map (Can you prove this formally), hence B has a closed graph.
Thus $B()$ has a fixed point.

Dominant Strategy at Equilibrium

Definition

A strategy $a_i \in A_i$ is a **dominant strategy** for player i if

$$u_i(a_i, a_{-i}) \geq u_i(a', a_{-i}) \; \forall a' \in A_i, \; \forall a_{-i}$$

Definition

A strategy profile a^* is a **dominant strategy equilibrium** if for each player i, $a{*}_i$ is a dominant stratgey.

Dominant Strategy at Equilibrium

Definition

A strategy $a_i \in A_i$ is a **Weakly Dominated Strategy** for player i if $\exists a' \in A_i$ such that

$$u_i(a_i, a_{-i}) \leq u_i(a', a_{-i}) \quad \forall a_{-i}$$

and

$$u_i(a_i, a_{-i}) < u_i(a', a_{-i}) \quad \exists a_{-i}$$

Definition

A strategy profile a_i is a **strictly dominated strategy** for player i, if $\exists a' \in A_i$ such that

$$u_i(a_i, a_{-i}) < u_i(a', a_{-i}) \quad \forall a_{-i}$$

Strictly dominated strategies

Weakly Dominated Strategy Can survive

Table : Example Game

Payoff	S1	S2
S1	1,1	2,0
S2	0,2	2,2

For Mixed NE:

Theorem

Every strategic game with finite number of actions has a MNE in which no player's strategy is weakly dominated.

Algorithm for Mixed Nash equilibrium

Brute Force: Try all possible Support Sets

M be the set of the pure strategies of player1 and let N be the set of the pure strategies of player 2. $M = \{1, \ldots, m\}$, $N = \{m+1, ..., m+n\}$.

Lemma

(Best response condition) Let x and y be mixed strategies of player 1 and 2, respectively. Then x is a best response to y iff $\forall i \in M$,

$$x_i > 0 \implies (Ay)_i = u = \max\{(Ay)_k | k \in M\}.$$

Proof.

$(Ay)_i$ is the ith component of Ay, which is the expected payoff to player 1 when playing row i. Then

$$x^T Ay = \sum_i x_i (Ay)_i = \sum_i x_i (u - (u - (Ay)_i) = u - \sum_i x_i (u - (Ay)_i)$$

So $x^T Ay \leq u$ since $x_i \geq 0$ and $u - (Ay)_i \geq 0$

Mathematical Preliminaries

Affine combination of points $z_1, \ldots z_k$ in some Euclidean space is of the form $\sum_i z_i \lambda_i$ where $\lambda_1 \ldots, \lambda_k$ are reals with $\sum_i \lambda_i = 1$. It is called a convex combination if $\lambda_i \geq 0$ for all i.

A set of points is convex if it is closed under forming convex combinations.

A set of points is affinely independent if none of these points are an affine combination of the others.

A convex set has dimension d if and only if it has $d + 1$, but no more, affinely independent points.

A polyhedron $\mathcal{P} \in R^d$ is a set

$$\{z \in R^d | Cz \leq q\}$$

for some matrix C and vector q.
It is called full-dimensional if it has dimension d.
It is called a polytope if it is bounded.
A face of $\mathcal{P}$ is $\{z \in \mathcal{P} | c^T z = q_0\}$ for some $c \in R^d$ where $\forall z \in \mathcal{P} c^T z \leq q_0$
A vertex of $\mathcal{P}$ is the unique element of a zero-dimensional face of $\mathcal{P}$.
An edge of $\mathcal{P}$ is a one-dimensional face of $\mathcal{P}$.
A facet of a d-dimensional polyhedron $\mathcal{P}$ is a face of dimension $d-1$.
A facet is characterized by a single binding inequality which is irredundant; i.e., the inequality cannot be omitted without changing the polyhedron.
A d-dimensional polyhedron $\mathcal{P}$ is called simple if no point belongs to more than d facets of P, which is true if there are no special dependencies between the facet-defining inequalities.

A 2-player game is not degenerate if no mixed strategy of support size k has more than k best responses.
In any bimatrix games A, B, Nash equilibrium (x, y) has equal support set sizes. (x is the mixed equilibria of player 1 and y of player 2)
Best Response Polyhedron

$$\bar{P} = \{(x, v) \in R^M R | x \geq 0, \mathbf{1}^T x = 1, B^T x \leq \mathbf{1} v\}$$
$$\bar{Q} = \{(y, u) \in R^N R | Ay \leq \mathbf{1} u, y \geq 0, \mathbf{1}^T y = 1\}$$

.

A point (y, u) of $\bar{Q}$ has label $k \in M \cup N$ if the kth inequality in the definition of $\bar{Q}$ is binding, which for $k = i \in M$ is the ith binding inequality.
And for $k \in N$ the binding inequality is $y_j = 0$.
A point (x, v) of $\bar{P}$ has label $i \in M$ if $x_i = 0$, and label $j \in N$ if $b_{ij} x_i = v$.
With these labels, an equilibrium is a pair (x, y) of mixed strategies so that with the corresponding expected payoffs v and u, the pair $((x, v), (y, u))$ in P and Q is completely labeled, which means that every label $k \in M \cup N$ appears as a label either of (x, v) or of (y, u).

Get new polyhedron as follows: Divide $\sum b_{ij} x_i \leq v$ by v and $Ay \leq 1u$ by u. Then

$$P = \{x \in R^M | x \geq 0, B^T x \leq 1\}$$

and

$$Q = \{y \in R^N | Ay \leq \mathbf{1}, y \geq 0\}$$

.

Algorithm for a non degenerate bimatrix game.

For each vertex x of $P - \{0\}$, and each vertex y of $Q - \{0\}$ do

1. Determine labels on the pair (x, y).
2. if (x, y) is completely labeled from the set $M \cup N$, output the Nash equilibrium $(x \cdot 1/\mathbf{1}^T x, y \cdot 1/\mathbf{1}^T y)$

Lemke-Howson Method

Define graphs G_P and G_Q as follows:
Vertices of the graphs are the vertices of P and Q.
Edge between v_1 and v_2 in G_1 if and only if v_1 and v_2 are adjacent corner points of P.
Similarly for G_2.
Label each vertex v of G_1 with the indices of the tight constraints in P, i.e.
$L_1(x) = \{i \in M \text{ if } x_i = 0, \text{ and label } j \in N \text{ if } b_{ij}x_i = v\}$.
Similarly for G_2
$L_2(y) = \{j \in N \text{ if } y_j = 0, \text{ and label } i \in M \text{ if } a_{ij}y_j = v\}$.
We have shown that:
Theorem: A pair (x, y) is a Nash equilibrium if and only $L(x) \cup L(y) = M \cup N = \{1, 2, \dots m + n\}$.

Let $G = G1xG2$ (Cartesian Product– i.e.,vertices of G are defined as $v = (v1, v2)$ where $v_1 \in V(G1)$ and $v2 \in V(G2)$.
There is an edge between $v = (v_1, v_2)$ and $v' = (v_1', v_2')$ in G if and only if $(v_1, v_1') \in E(G1)$ and $v2 = v_2'$ or $v_1 = v_1'$ and $(v_2, v_2') \in E(G2)$.
Define for each vertex $v = (v1, v2) \in V(G), v_1 \in G1, v_2 \in G_2$, a label $L(v) = L(v_1) \cup L(v_2)$.
Let $S_k = \{(v \in V(G))$ such that $L(v) \cup \{k\} = M \cup N$, i.e. $L(v)$ may lack only k.

Lemma

For a given label k, the set S_k along with induced edges form a set of cycles and paths

The vertex $(0,0)$ is in S_k (it has all labels)
Every Nash equilibrium vertex is in S_k and has exactly one neighbor where k is replaced by a duplicate label (in the direction way from the hyperplane defining k.
Every vertex in S_k, not including k, has a duplicate label and two edges : one that adds the duplicate label and the other direction that removes it.

Lemke-Howson Algorithm

Choose k: missing label.
Let $(x, y) = (0, 0) \in P\mathtt{x}Q$.
Drop label k (from $x \in P$ if $k \in M$, from $y \in Q$ if $k \in N$).
Repeat:
Call the new vertex pair (x, y).
Let l be the label that is picked up.
If $l = k$, terminate with Nash equilibrium (x, y) (rescaled as mixed strategy pair).
Otherwise, drop l in the other polytope and repeat.

Congestion games

- n players
- Elements $E = \{e_1 \ldots e_t\}$
- $S_i \subseteq 2^E$ is strategy set of player i
- Cost to player i of playing S_i is $\sum_{e_j \in S_i} c_j(x)$, where $c_j(x)$ is the cost of element e_j when number of player's selecting e_j is x. Cost function is monotonically non-decreasing.
  Does Nash Equilibrium exist.

Examples

Network of roads. n people travel from different source to destination. Time of travel is an increasing function of the number of people traveling. Each person travels in the minimum amount of time.
(1) roads are elements
(2) strategies are paths

Theorem

All congestion games have at least one pure strategy Nash Equilibria

Proof.

Consider the program:

$$\min \sum_{k=1}^{k=t} \sum_{y=0}^{x_k} c_k(y)$$

subject to:

$$\forall i, \ \sum_{s \in S_i} x_s^i = 1$$

$$x_k = \sum_i \sum_{k \in s} x_s^i$$

$$x_s^i \in \{0, 1\}, \quad s \in S_i, \forall i$$

x_s^i represents the use of strategy s by ith player. $x_s^i = 1$ if player i uses strategy s, 0 otherwise $\square$

Proof.

(Continued) Let $(\hat{x}_r^i)$ be a strategy that is a solution to the above. Then it is at equilibrium. If not then consider a shift in strategy. Then there exists a player j with strategy s_j such that

$$\sum_{k \in s_j} c_k(\hat{x}_k + 1) < \sum_{k \in \hat{s}_j} c_k(\hat{x}_k)$$

But then

$$\sum_{k=1}^{k=t} \sum_{y=0}^{x_k} c_k(y) < \sum_{k=1}^{k=t} \sum_{y=0}^{\hat{x}_k} c_k(y)$$

which is a contradiction.

□

Theorem

Congestion games are potential games.

As an example consider the road network problem: In this case a strategy profile is of the form $P = (P_1, P_2 \ldots P_k)$ where P_i is a path from source s_i to sink t_i. The potential function is

$$\Phi = \Phi(P_1, \ldots P_k) = \sum_{e \in E} \sum_{k=1}^{k=n_e} f(k)$$

where n_e is the number of paths that use the edge e in the strategy profile P. $f(k)$ is the delay function: it is assumed to be positive and nondecreasing . Let $\Phi' = \Phi'(P_1, \ldots P_i' \ldots P_k)$ be the potential function when the strategy of the ith player changes. Then the delay that the ith player faces changes as follows:

$$D_i' - D_i = \sum_{e \in P_i' - P_i} f(n_e + 1) - \sum_{e \in P_i - P_i'} f(n_e)$$

The change in potential is

$$\Phi' - \Phi = \sum_{e \in P_i' - P_i} \left(\sum_{k=1}^{n_e+1} f(k) - \sum_{k=1}^{n_e} f(k) \right) + \sum_{e \in P_i - P_i'} \left(\sum_{k=1}^{n_e-1} f(k) - \sum_{k=1}^{n_e} f(k) \right)$$

$$= \sum_{e \in P_i' - P_i} f(n_e + 1) - \sum_{e \in P_i - P_i'} f(n_e)$$

$$= D_i - D_i$$

Thus the potential function changes match the change in delay when a player improves his delay. Since the potential function is lower bounded, this change in player delays is bounded and the process of improving delays is bounded.