Illinois Institute of Technology
Department of Computer Science

Solutions to First Examination

CS 330 Discrete Structures
Spring Semester, 2015

Exam Statistics

47 students took the exam. The range of scores was 19–86, with a mean of 54.87, a median of 56, and a standard deviation of 14.53. Very roughly speaking, if I had to assign final grades on the basis of this exam only, above 65 would be an A (11), 55–64 a B (15), 40–54 a C (13), 25–39 a D (6), below 25 an E (2).

Problem Solutions

1. Growth rates.

   The (decreasing) order by growth rates is

$$n^n \quad (3/2)^n \quad \ln n! \quad \sqrt{H_n} \quad 1/n$$

   - $n^n = \Omega((3/2)^n)$ because taking logarithms, we are comparing $n \log(3/2) = \Theta(n)$, linear growth, with $n \log n$; as n gets large $n \log n$ grows faster than linear.
   - $(3/2)^n = \Omega(\ln n!)$ because $\ln n! = \Theta(n \log n)$, as we saw by Stirling's formula, which grows more slowly than $\Theta(n^2)$, quadratic growth; as n gets large, $(3/2)^n$ grows exponentially, much faster than quadratic.
   - $\ln n! = \Theta(n \log n) = \Omega(\sqrt{H_n})$ because $H_n = \Theta(\log n)$ as we saw by induction January 14 and by calculus on January 28.
   - $\sqrt{H_n} = \Omega(1/n)$ because as n gets large $1/n$ shrinks to 0, while $\sqrt{H_n}$ grows unboundedly.

2. Rules of Sum and Product.

   Each card can now be placed face up or face down, two choices, so for k cards there are 2^k possibilities. The number of different landscapes that can formed with n myriorama cards is thus

$$\sum_{k=1}^{n} \binom{n}{k} k! 2^k.$$

   As an aside, we note that

$$\sum_{k=1}^{n} \binom{n}{k} k! 2^k = 2^n n! \sum_{k=1}^{n} \frac{2^{-(n-k)}}{(n-k)!} = 2^n n! \sum_{k=0}^{n-1} \frac{2^{-k}}{k!} \approx \frac{2^n n!}{e^2},$$

   using the Taylor series expansion $e^x = \sum_{k=0}^{\infty} x^k/k!$.

3. Combinations.

   (a) $\binom{n}{n-k} = \binom{n}{k}$.

   (b) I posed this problem to my 16-year old grandson; here is his solution: There are $k+1$ places to insert the $n-k$ empty spaces ($k-1$ between adjacent cars and one at each end). To prevent an SUV parking, at most one empty space can be inserted in these $k+1$ places. Hence the answer is $\binom{k+1}{n-k}$.

4. Algorithms; mathematical induction

   (a) If the recursion is expanded out, the computation is basically $1+1+...+1$, so there are $\binom{n}{k}-1$ additions. This also follows from the identity used, together with a double induction on n and i that states "Computing $\binom{n}{i}$ by the above code takes $\binom{n}{i}-1$ addition operations."

   (b) Simply write the binomial coefficient as the ratio of two products, $\binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i(i-1)\cdots 1}$. Evaluating the numerator takes i multiplications, as does the denominator, together with a single division.

   (c) Each of the summands takes $\Theta(i)$ from part (b), so together they take $\sum_{i=0}^{k}\Theta(i) = \Theta(\sum_{i=0}^{k} i) = \Theta(k^2)$.

5. Binomial Coefficients.

   (a) By the binomial theorem, $2^7\binom{2015}{7}$.

   (b) By the binomial theorem extended to negative exponents, $\binom{2015+7-1}{7} = \binom{2021}{7}$.

   (c) Zero, because the polynomial only has multiples of three powers of x.

   (d) By the binomial theorem, $-2015^7\binom{2n+1}{7}$.