HW6-Extra Solution, CS330 Discrete Structures, Spring 2015

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1 A connected graph has an Euler circuit if and only if all of its vertices' degrees are even.

To prove this statement, we need to prove two:

$$\text{All degrees are even} \rightarrow \text{An euler circuit exists}$$

$$\text{An euler circuit exists} \rightarrow \text{All degrees are even}$$

1.1 An euler circuit exists→ All degrees are even

This is trivial. If an euler circuit exists, it means all vertices must have one or several pairs of (in,out) edges. Therefore, all vertices' degree must be even.

1.2 All degrees are even→ An euler circuit exists

This is not so straightforward. We prove it by induction on the number of vertices.

Let $P(n)$: "A connected graph with n vertices, whose degrees are all even, must have an euler circuit".

Base case $P(2)$

Degrees are even, there must be an even number of edges between the two vertices. Suppose the vertices are a and b, and there are $2k$ edges between them. We can go from a to b and then back from b to a to 'consume' 2 edges. Then, we can discover a euler circuit by making this trip k times.

Induction $P(n) \rightarrow P(n+1)$

1. Firstly, assume $P(n)$ is true. Then, let the graph with $n+1$ vertices of even degree be G_{n+1}. Then, there must exist a vertex removal of which does not disconnect G_{n+1}, say x.

2. Remove x and all incident edges from G_{n+1} to get G_n. Because the degree is even, x's incident edges can be paired up. For every removed pair of edges $e(a,x), e(x,b)$, we add one edge $e(a,b)$ to G_n, and call it as an augmented edge. After we have added all such edges, the degrees of G_n's remain same, and thus all the degrees are still even.

3. Because of our assumption, G_n must have an euler circuit, say C. Add x back to G_n and restore all edge changes to get back G_{n+1}. Then, replace all augmented edges $e(a, b)$ in C with $e(a, x)$ and $e(x, b)$. After all augmented edges are all replaced, the resulting new circuit C' is the euler circuit of G_{n+1}.

4. Therefore, if $P(n)$ is true, $P(n + 1)$ is true as well.