CS539-Spring 2018, Mid-Exam
1.50-3.05pm, 20th March, 2018.
Closed Book.

Return this exam paper before you leave the class

Prove All Results. Do as many problems/sub-problems as possible. Take the rest home at reduced credit (60%). Total Points: 100 + 5 bonus.

1. (i) Find the mixed equilibrium of the game in terms of the values of c. (15 pts)

Table 1: Strategies of Countries

Payoff	Defense spending	Education
Defense Spending	$(-c, -c)$	$(c-1, -2c)$
Education	$(-2c, c-1)$	(c, c)

(ii) Characterize and find a correlated equilibrium in Problem 1 (i) which is not a mixed Nash Equilibrium. (15 pts)

Solution:

i Mixed NE: $-cp_D^2 + (c-1)p_2^2 = -2cp_D^2 + cp_2^2 \implies cp_D^2 = p_2^2$

$$p_D^2 = \frac{1}{1+c}, p_2^2 = \frac{c}{1+c}$$

$$p_D^1 = \frac{1}{1+c}, p_2^1 = \frac{c}{1+c}$$

ii Correlated NE:
P_1

$$\sum_{S_{-i}} \pi(S_i, S_{-i})[u(S_i, S_{-i}) - u(S_i, S_{-i})] \geq 0$$

$$\pi(D_1, D_2)[u_1(D_1, D_2) - u_1(E_1, D_2)] + \pi(D_1, E_2)[u_1(D_1, E_2) - u_1(E_1, E_2)] \geq 0$$

$$\pi(E_1, D_2)[u_1(E_1, D_2) - u_1(D_1, D_2)] + \pi(E_1, E_2)[u_1(E_1, E_2) - u_1(D_1, E_2)] \geq 0$$

$\implies$

$$\pi(D_1, D_2)c + \pi(D_1, E_2) \geq 0$$

$$-\pi(E_1, D_2)c + \pi(E_1, E_2) \geq 0$$

P_2

$$\pi(D_2, D_1)[u_2(D_2, D_1) - u_2(E_2, D_1)] + \pi(D_2, E_1)[u_2(D_2, E_1) - u_2(E_2, E_1)] \geq 0$$

$$\pi(E_2, D_1)[u_2(E_2, D_1) - u_2(D_2, D_1)] + \pi(E_2, E_1)[u_2(E_2, E_1) - u_2(D_2, E_1)] \geq 0$$

$\implies$

$$\pi(D_2, D_1)c + \pi(D_2, E_1) \geq 0$$

$$-\pi(E_2, D_1)c + \pi(E_2, E_1) \geq 0$$

and

$$\pi(D_1, D_2) + \pi(D_1, E_2) + \pi(D_2, E_1) + \pi(E_2, E_1) = 1$$

A probability distribution satisfying the above conditions:

$$\pi(D_1, D_2) = 0, \pi(D_1, E_2) = 0, \pi(D_2, E_1) = \frac{1}{2}, \pi(E_2, E_1) = \frac{1}{2}$$

2. Two companies bid on a project. The project work could be apportioned to the two companies. Company A has the following profit function $P_A = p(x_A, x_B)x_A - c_A x_A$ and $P_B = p(x_A, x_B)x_B - c_B x_B$ where x_A, x_B is the units of work put in by company A and B, respectively.

   (i) Determine the strategic units of work that the companies contribute at Nash equilibrium, when $p(x_A, x_B) = 2x_A + x_B$. (10 pts)
   (ii) What are the Stackelberg equilibrium strategies if company A determines its strategy first? (10 pts)
   (iii) What if Company A is unsure of the profit function of company B? $P_B = f_b(x_A, x_B)x_B - c_B x_B$ where f_b is $2x_A + x_B$ with probability $3/4$ and is $x_A + 3x_B$ with probability $1/4$. (15 pts)

   Solution:

   i This utility function is convex and increasing $\implies$ unbounded.
   The following method therefore, ends up finding the minimum value.
   $P_A = (2x_A + x_B)x_A - c_A x_A$, $P_B = (2x_A + x_B)x_B - c_B x_B$.
   Differentiation and equating to 0
   $\implies 4x_A + x_B - c_A = 0$ and $2x_A + 2x_B - c_B = 0$
   $\implies x_A = \frac{2c_A - c_B}{6}, x_B = \frac{4c_B - 2c_A}{6}$

   ii x_A^* is already decided $\implies p_B = 2x_A^* x_B + x_B^2 - c_B x_B \implies \frac{\partial p_B}{\partial x_B} = 2x_A^* + 2x_B - c_B = 0 \implies x_B = \frac{c_B - 2x_A^*}{2}$.
   $\implies p_A = x_A^2 + \frac{c_B x_A}{2} - c_A x_A \implies \frac{\partial p_A}{\partial x_A} = 2x_A + \frac{c_B}{2} - c_A = 0 \implies x_A = \dfrac{2c_A - c_B}{4}$
   Hence, $x_B = \frac{c_B}{2} - \frac{2c_A - c_B}{4} = \dfrac{3c_B - 2c_A}{4}$.

   iii If A is unsure about B,
   $\dfrac{3}{4} \longrightarrow x_B = \frac{c_B - 2x_A^*}{2}$
   $\dfrac{1}{4} \longrightarrow x_B = \frac{c_B - x_A^*}{6}$
   Expected value of $X_B \longrightarrow E(x_B) = \frac{3}{4}(\frac{c_B - 2_A^*}{2}) + \frac{1}{4}(\frac{c_B - x_A^*}{6}) = \frac{10c_B - 19x_A^*}{24}$

   $p_A = 2x_A^2 + x_A x_B - c_A x_A$

   Differentiating and equating to 0 $\longrightarrow x_A = \dfrac{12c_A - 5c_B}{29} \implies x_B = \max \begin{cases} \dfrac{39c_B - 24c_A}{58} \\ \dfrac{17c_B - 6c_A}{87} \end{cases}$

3. Load balancing game: There are 2 communication links, each providing rate $1/(\sum_{i \in link_j} d_i)$ where d_i is the data rate of user i on the links (assumed to be the same on the two links). There are 4 users with data rates $d_D = d_2$ and $d_3 = 2d_D$, $d_4 = 2d_D$. Find all Nash equilibrium in this setting. What is the worst case loss of efficiency at equilibrium (PoA) as compared to social optimum when the objective function is the sum of rates of all the users. (20 pts)

   Solution: We check all 8 possible assignments to find an NE, and then obtain:
   P_1 and P_3 on a link and P_2 and P_4 on the other link $\implies$ worst case NE $\frac{4}{3d_1}$.
   Social opt. is obtained when P_1 is alone on one link and the rest are on the other link $\implies$ opt value

$\frac{8}{5d_1}$.

$\implies PoA = \frac{5}{6} \approx 0.83$

4. **SoftMic v/s Logog:** Software Giant SoftMic faces the possibility of entry by a challenger Logog. First the challenger chooses whether to enter. If it does not enter, neither firm has any further action; the incumbent's payoff is TM (it obtains the profit M in each of the following $T \geq 1$ periods) and the challenger's payoff is 0. If the challenger enters, it incurs an entry cost $f > 0$, and in each of T periods SoftMic, the incumbent, plays first and either commits to fight or cooperate with the challenger in that period. Next Logog chooses whether to stay in the industry or to exit. If, in any period, the challenger stays in, each firm obtains in that period the profit $-F < 0$ if the incumbent fights and $C > \max\{F, f\}$ if it cooperates. If, in any period, the challenger exits, the incumbent obtains the profit $M > 2C$ and the challenger the profit 0 in the current and every subsequent period. Once the challenger exits, it cannot subsequently reenter. Each firm wishes to optimize the sum of its profits. Find the subgame perfect equilibria of the extensive game that models this situation when $T = 2$. (20 pts)

Solution: