Game Theory: Algorithms and Applications
CS 539
Spring 2018
HomeWork 1 Solutions

1. n supercomputers nodes wish to communicate data with each other. They want to add edges so as to create a connected graph amongst themselves. All possible edges can be used. Each node gains revenue C, a large constant, when the graph is connected. However, a node u pays the cost of adding an edge (u, v) with node v if it wishes to add that edge. Suppose the cost of each edge is the same. What Nash equilibrium subgraphs can be produced?
What is a Nash equilibrium if edges have non-uniform cost? What is a solution that has minimum total cost, where total cost is the sum of the costs of all edges.
Hint: It helps to define the strategy sets and pay-offs for each node.

Solution: We assume C is larger than twice the sum of all the costs.

Any spanning tree where the cost (degree $\times$ cost of an edge) of a node is less than C is a Nash equilibrium subgraph. Consider any spanning tree; a player stands to gain ($C-$ cost of connections she paid for) as payoff in her current state, and any action -

i Letting go of one of the edges it is connected to (this disconnects the graph)

ii Buying another edge (this creates a cycle)

will decrease her total payoff.

When the costs are non-uniform, every player will choose the least cost outgoing edge to connect to the spanning-tree. This leads to the subgraph becoming a MST.

2. In a sealed-bid e-bay auction, bidders submit bids based on their valuations. Let there be n bidders with valuations $v_1 > v_2 > \ldots > v_n > 0$. The object is won by the bidder with the highest price. In first price auctions the winner is assigned the item at the price that he bids. Find all pure Nash Equilibrium of this game. Does player 1 have a special place?
Also consider the second-price auction where the highest bidder gets the item at the price of the second-highest bid. Characterize all possible Nash equilibrium for a 3 player game with valuations v_1, v_2 and v_3 respectively.

Solution: We assume that a conflict is resolved in favor of the player with the lesser index.

- *First-Price:* A Nash equilibrium is obtained when p_1 (player 1) bids a value $b_1 \in [v_1, v_2)$ and other players $p_i, i = \{2, 3, \cdots, n\}$ bid any value v_i where $v_i \in [0, v_i]$.
  Proof: By contradiction: Assume that b_1 is less than v_2, then p_2 can bid her valuation and win the auction, and this would, therefore, not be a Nash equlibrium.
  Special place for p_1: She may bid any value over v_2 and win the auction.
- *Second-Price:*
  $b_1 \in [b_2, \infty)$ if $b_2 \leq v_1$ and $b_3 \in [0, v_2]$.
  $b_2 \in [b_1, \infty)$ if $b_1 \leq v_2$ and $b_3 \leq v_2$.
  $b_3 \in [\max\{b_1, b_2\}, \infty)$ if $b_1, b_2 \leq v_3$
  Proof: omitted

3. N consumers are to be assigned to P service providers to access the internet. Define the rate of transmission given to a consumer by service provider i to be $R_i = B/\sum_j L_j$ where L_j is the load of the jth consumers accessing service provider i, B is the bandwidth. Consumers can choose their providers with payoff provided by the rate offered, i.e. R_i. (i) Determine a Nash equilibrium when everyone has the same load. (ii) Show that a Nash equilibrium exists when loads are different.

Solution:

i At a Nash equilibrium, all consumers get distributed evenly accross all service providers. The difference between the number of consumers on any two providers is at most 1.

ii Consider a vector $V = \{(\sum_{j\in 1} L_j), (\sum_{j\in 2} L_j), \cdots, (\sum_{j\in P} L_j), \}$, where

$$\{(\sum_{j\in 1} L_j) > (\sum_{j\in 2} L_j) > \cdots > (\sum_{j\in P} L_j), \}$$

whose components are the sum of the loads of the consumers using the service providers in non-increasing order.

Every time a consumer k decides to make a switch from provider i to i' to increase her rate, we have the following condition -

$$R_i^{pre}(\text{pre-move}) = \frac{B}{\sum_{j\in i} L_j(\text{including } k)} < \frac{B}{\sum_{j\in i'} L_j + L_k} = R_{i'}^{post}(\text{post-move})$$

$$\Rightarrow \sum_{j\in i'} L_j + L_k < \sum_{j\in i} L_j(\text{including } k)$$

With any switch, the value of the lexicographic ordering of the vector decreases because either of the two following cases occur:

I Post-move position of i' in V shifts to the left of the post-move position of i. However, the load on i' after the switch is $\leq$ the load on i before the switch.

II Post-move position of i' in V remains on the right of the post-move postion of i.
Even in this case, we have (from the inequality above) $\sum_{j\in i'} L_j + L_k < \sum_{j\in i} L_j$ (including k) , ie., the load on i' after the switch is $\leq$ the load on i before the switch.

Next, we know that the load on any provider is lower bounded by

$$L_j : j = \arg \min_{j=\{1,\cdots,N\}} L_j$$

which implies that the vector can only decrease to a bounded value $\Rightarrow$ the system will converge to a Nash equilibrium.

4. Determine Nash Equilibrium for the "Tragedy of Commons/Bandwidth" sharing game involving n players when the utility function for player i is given by

$$U_i = X_i(K - \alpha \sum_j X_j)$$

where X_i is the units of traffic sent by player i.

Solution:

Player i will keep increasing traffic until her marginal utility becomes zero (or less). Marginal utility is given by

$$\frac{\delta U_i}{\delta X_i} = K - 2\alpha X_i - \alpha \sum_{j \neq i} X_j$$

So, player i will keep increasing her traffic so long as

$$K - \alpha(X_i + \sum_j X_j) \geq 0 \Rightarrow \quad K - \alpha \sum_{j \neq i} X_j \geq 2\alpha X_i \Rightarrow \quad X_i \leq \frac{K - \alpha \sum_{j \neq i} X_j}{2\alpha}$$

The game is symmetric, $X_i = X_{i'}, \forall i, i'$ at Nash equilibrium.

$$\therefore X_i \leq \frac{K}{\alpha(n+1)}, \forall i \text{ @ NE}$$

5. Determine Nash Equilibrium in a Cournot Game involving 2 players where the cost function for producers is given by

$$c_i = \ln x_i$$

where the utility for the i^{th} player is defined to be $U_i = x_i \cdot p(x_1 + x_2) - c_i$, and $p(y) = \max\{0, K - y\}$, K being a constant (almost as defined in class).

Solution: Assume that $K \geq x_1 + x_2$. Differentiating U_i,

$$\frac{\delta U_i}{\delta x_i} = K - 2x_i - x_{i'} - \frac{1}{x_i} \text{ where } i' \text{ is the } \textit{other} \text{ player}$$

Since the game is symmetric between the players, $x_1 = x_2$ at NE. Using KKT conditions,

$$K - 2x_i - x_{i'} - \frac{1}{x_i} = 0 \Rightarrow Kx_i - 2{x_i}^2 - x_i x_{i'} = 1 \Rightarrow 3{x_i}^2 - Kx_i + 1 = 0 \Rightarrow x_i = \frac{K \pm \sqrt{K^2 - 12}}{6}, \forall i \text{ @ NE}$$