Illinois Institute of Technology
Department of Computer Science

First Examination

CS 330 Discrete Mathematics
Spring, 2015

11:25am–12:40pm, Wednesday, February 11, 2015
113 Stuart Building

Print your name and student ID, *neatly* in the space provided below; print your name at the upper right corner of *every* page. Please print legibly.

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This is an *open book* exam. You are permitted to use the textbook (hardcopy only), any class handouts, anything posted on the web page, any of your own assignments, and anything in your own handwriting. Foreign students may use a dictionary. *Nothing else is permitted*: No calculators, laptops, cell phones, Ipads, Ipods, communicators, GPSes, etc.!

Do all five problems in this booklet. *All problems are equally weighted, so do not spend too much time on any one question.*

Show your work! You will not get partial credit if the grader cannot figure out how you arrived at your answer.

Question	Points	Score	Grader
1	20	20	
2	20	14	
3	20	10	
4	20	16	
5	20	19	
Total	100	79	

1. **Growth rates.**

Rank the following five functions in decreasing order of growth rate as n gets large:

$$\sqrt{H_n} \quad \ln n! \quad n^n \quad 1/n \quad (3/2)^n$$

that is, find an arrangement of the functions $f_1, f_2, \ldots, f_5$ such that $f_1(n) = \Omega(f_2(n))$, $f_2(n) = \Omega(f_3(n))$, $f_3(n) = \Omega(f_4(n))$, and $f_4(n) = \Omega(f_5(n))$. Justify your ordering.

$\sqrt{H_n} \rightarrow$ H_n is known to have a growth rate $= \Theta(\log n)$
$\therefore \sqrt{H_n}$ has growth rate $= \boxed{\Theta(\sqrt{\log n})}$

$\ln(n!) \rightarrow$ $n!$ is known to have a GR of $\Theta(\sqrt{n}\,(n/e)^n)$ by Stirling
$\therefore \ln(n!)$ has a GR $= \boxed{\Theta(\log(\sqrt{n}\,(n/e)^n))}$

$n^n \longrightarrow$ Growth rate of $n^n = \boxed{\Theta(n^n)}$

$1/n \longrightarrow$ Growth rate of $1/n = \boxed{0}$ since $1/n$ is a shrinking term, and goes to 0 as n gets large.

$(3/2)^n \longrightarrow$ Growth rate of $\boxed{\Theta(1.5^n)}$ since no constants, constant multiples, or shrinking terms are present.

GR Logic: $x^x > a^x > x^a > \log_a x$,
$\therefore \Theta(n^n) > \Theta(1.5^n) > \Theta(\log(\sqrt{n}\,(n/e)^n)) > \Theta(\sqrt{\log n}) > 0$,
so the final arrangement according to growth rate from largest to smallest is:

$$\boxed{n^n, (3/2)^n, \ln(n!), \sqrt{H_n}, 1/n}$$

2. **Rules of Sum and Product.**

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In class on February 2 (and repeated on February 4), Professor Reingold showed the number of different landscapes that could formed with n myriorama cards to be

choose k of n cards
AND
place cards one at a time (Rule of Sum)

$$\sum_{k=1}^{n} \frac{n!}{(n-k)!} = \sum_{k=1}^{n} \binom{n}{k} k!$$

Now, suppose that the cards are two-sided and can be used either side up. How many different landscapes can be formed with n such myriorama cards? (*Hint*: This was suggested as a good study problem in the February 4 lecture.)

choose k of n cards to arrange
AND
choose whether each card is face up or face down
AND
place cards one at a time (Rule of Sum)

-6

$$\sum_{k=1}^{n} \binom{n}{k}\cancel{\binom{k}{u}} k! = \sum_{K=1}^{n} \binom{n}{K} k!\, 2^k = \sum_{K=1}^{n} 2^n \cancel{2^K}\, k = \sum_{K=1}^{n} \cancel{2^n}\, k! \binom{K}{u}$$

choose whether each card is in or out of group

choose whether each card in group is face up or face down

n cards
group of k cards
u face-up cards

place cards one at a time

choose u out of k cards in the group to be face up

(10)

3. **Combinations.**

(a) A parking lot has a row of n parking spaces; k cars arrive to park. How many different arrangements are there for *empty parking spaces*?

one SUV

(b) An SUV needs two adjacent empty parking spaces. If k cars arrive to park, how many different arrangements are there in which an SUV *cannot* park? (That is, how many arrangements of empty parking spaces do not have two adjacent empty parking spaces?) (*Hint*: Imagine k cars parked adjacently; how many ways can $n-k$ empty spaces be inserted? A version of this was suggested as a good exam problem in the lecture on February 9.)

a) $\boxed{\sum_{k=0}^{n} \binom{n}{n-k}}$ arrangements for empty spaces −2

↑ choose n−k empty from n spaces

↑ for all possible numbers of cars k

b) $\boxed{\underbrace{\sum_{k=0}^{n} \binom{n}{n-k}}_{\text{All possible arrangements of empty spaces}} - \underbrace{\sum_{k=0}^{n-2} \binom{n-2}{k}}_{}}$ arrangements in which an SUV cannot find two adjacent parking spots.

↓ All possible arrangements of empty spaces

↓ If 2 of n spots (adjacent) are reserved for the SUV, choose k of the n−2 remaining spots to park cars

↓ Gives all arrangements where an SUV can park

−8

4. **Algorithms; mathematical induction**

(a) Suppose you compute $\binom{n}{i}$ with the recurrence

$$\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$$

Assume $i \le n$.

-4

using the following recursive function:

```
FUNCTION Combination(n,i)
  BEGIN
    IF i=0 OR i=n THEN RETURN 1;
    ELSE RETURN Combination(n-1, i-1) + Combination(n-1, i);
  END
```

Analyze the number of additions needed to compute $\binom{n}{i}$.

(b) Show how to compute $\binom{n}{i}$ in $O(i)$ arithmetic operations (not necessarily additions).

(c) Using (b), analyze the number of arithmetic operations used to compute the sum $\sum_{i=0}^{k} \binom{n}{i}$.

a) This function adds up 1 until it reaches the value $\binom{n}{i}$, so there must be $\boxed{\binom{n}{i} - 1 \text{ additions}}$ to add $\binom{n}{i}$ elements.

b) FUNCTION Combination (n,i)
BEGIN
IF i=0 OR i=n THEN RETURN 1;
ELSE temp1 := Combination (n-1,i-1);
temp2 := Combination (n-1, i);
RETURN temp1 + temp2;
END

X −4

This does one arithmetic operation each recursive calls, and there will be i−1 recursive calls.

c) Each addition takes O(i) from (b), so when summed together, this can be expressed as $\sum_{i=0}^{k} O(i) = O\left(\sum_{i=0}^{k} i\right) = \boxed{O(k^2)}$

∴ it takes k^2 additions (arithmetic operations) to compute the sum.

5. **Binomial Coefficients.**

Find the coefficient of x^7 in each of the following polynomials. Do not simplify powers, factorials, or binomial coefficients.

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(a) $(1 + 2x)^{2015}$

(b) $(1 - x)^{-2015}$

(c) $(1 - x^3 + x^9 - x^{27} + \cdots)^{2015}$

(d) $(1 - 2015x)^{2n+1}$

a) binomial thm: coeff of x^k in $(1 + rx)^n = r^k\binom{n}{k}$

$\therefore$ coeff of x^7 in $(1+2x)^{2015} = \boxed{2^7\binom{2015}{7}}$

b) binomial thm: coeff of x^k in $(1-x)^{-n} = \binom{n+k-1}{k}$

$\therefore$ coeff of x^7 in $(1-x)^{-2015} = \binom{2015+7-1}{7} = \boxed{\binom{2021}{7}}$

c) ~~[illegible]~~ coeff of x^7 in $(1 - x^3 + x^9 - x^{27} + \ldots)^{2015} = \boxed{0}$

because it does not contain an x^7 term.

d) binomial thm: coeff of x^k in $(1+rx)^n = r^k\binom{n}{k}$

$\therefore$ coeff of x^7 in $(1-2015x)^{2n+1} = \boxed{2015^7\binom{2n+1}{7}}$

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