Illinois Institute of Technology
Department of Computer Science

Solutions to Second Examination

CS 330 Discrete Structures
Fall Semester, 2013

46 students took the exam; the statistics were:

Minimum	0
Maximum	90
Median	41
Average	42.26
Std Dev	22.37

1. Coin Tossing

   We have a peculiarly biased coin for which the probability of tossing heads changes after each flip. Initially (on the first flip), the probability of heads is 1/2. But for each successive toss the probability decreases by a multiplicative factor of 1/4, so that on the ith flip the probability of heads is $1/2\times(1/4)^{i-1}$. We toss the coin n times.

   (a) What is the probability that all n tosses were heads?

   It is the product of the probabilities of heads on each toss:

   $$\prod_{i=1}^{n}\frac{1}{2}\left(\frac{1}{4}\right)^{i-1}=\prod_{i=0}^{n-1}\frac{1}{2}\left(\frac{1}{4}\right)^{i}=\left(\frac{1}{2}\right)^{n}\left(\frac{1}{4}\right)^{0+1+2+\cdots+(n-1)}=\left(\frac{1}{2}\right)^{n}\left(\frac{1}{4}\right)^{n(n-1)/2}=\left(\frac{1}{2}\right)^{n^2}.$$

   (b) In general, what is the expected number of heads we will get?

   It is the sum of the probabilities of heads on each toss:

   $$\sum_{i=1}^{n}\frac{1}{2}\left(\frac{1}{4}\right)^{i-1}=\sum_{i=0}^{n-1}\frac{1}{2}\left(\frac{1}{4}\right)^{i}=\frac{1}{2}\left(\sum_{i=0}^{n-1}\left(\frac{1}{4}\right)^{i}\right)=\frac{1}{2}\left(\frac{1-(1/4)^n}{3/4}\right)=\frac{3}{2}\left(1-\frac{1}{4^n}\right).$$

2. Annihilators

   Prove by induction on k that the operator $(\mathbf{E}-a)^{k+1}$ annihilates any sequence $\langle P(i)a^i\rangle$, where $P(i)$ is any polynomial in i of degree k.

   This was suggested as a possible exam question in the lecture of October 16.

   For the basis, $k=0$ we have $(\mathbf{E}-a)^{0+1}=(\mathbf{E}-a)$ which we saw in class annihilates $\langle a^i\rangle$ and hence it also annihilates $c\langle a^i\rangle=\langle ca^i\rangle$ for any constant c; of course a polynomial of degree 0 is just a constant. If the statement is true for k, $k\geq 0$, then $(\mathbf{E}-a)^{k+1}$ annihilates any sequence $\langle P(i)a^i\rangle$ where $P(i)$ is any polynomial in i of degree k. Then $(\mathbf{E}-a)^{k+2}\langle P(i)a^i\rangle=(\mathbf{E}-a)^{k+1}[(\mathbf{E}-a)\langle P(i)a^i\rangle]$. So, what is $(\mathbf{E}-a)\langle P(i)a^i\rangle$ when $P(i)$ is a polynomial of degree of degree $k+1$, $P(i)=\sum_{j=0}^{k+1}c_j i^j$? $(\mathbf{E}-a)\langle P(i)a^i\rangle=(\mathbf{E}-a)\langle a^i\sum_{j=0}^{k+1}c_j i^j\rangle=\langle a^{i+1}\sum_{j=0}^{k+1}c_j(i+1)^j-aa^i\sum_{j=0}^{k+1}c_j i^j\rangle$ and the i^{k+1} terms cancel, leaving $\langle\hat{P}(i)a^i\rangle$ where $\hat{P}(i)$ is a polynomial of degree k. But by induction, then, $(\mathbf{E}-a)^{k+1}$ annihilates $\langle\hat{P}(i)a^i\rangle$ meaning that $(\mathbf{E}-a)^{k+2}\langle P(i)a^i\rangle=\langle 0\rangle$ which is what we wanted to prove.

3. Rolling a Die

   You roll a decimal die (that is, a ten-sided die) $n \geq 2$ times, recording the sequence of rolls.

   (a) Find a recurrence relation for the number of possible sequences in which there are no two consecutive sixes.

   Call such a sequence "valid" and let v_n be the number of valid sequences of length n; obviously, $v_0 = 1$ (the empty sequence) and $v_1 = 10$. If the first roll is not a 6 the sequence can be continued with any valid sequence of lenth $n-1$; there are v_{n-1} such ways to continue the sequence and 9 choices for the first roll, a total of $9v_{n-1}$ by the rule of product. On the other hand, if the first roll *is* a 6, the next roll must not be 6, and the remaining $n-2$ rolls can be any valid sequence; there are $9v_{n-2}$ was to continue after the first throw by the rule of product. By the rule of sum then, $v_n = 9v_{n-1} + 9v_{n-2}, n \geq 2$.

   (b) Solve the recurrence using annihilators; you need not solve the simultaneous equations from the initial conditions.

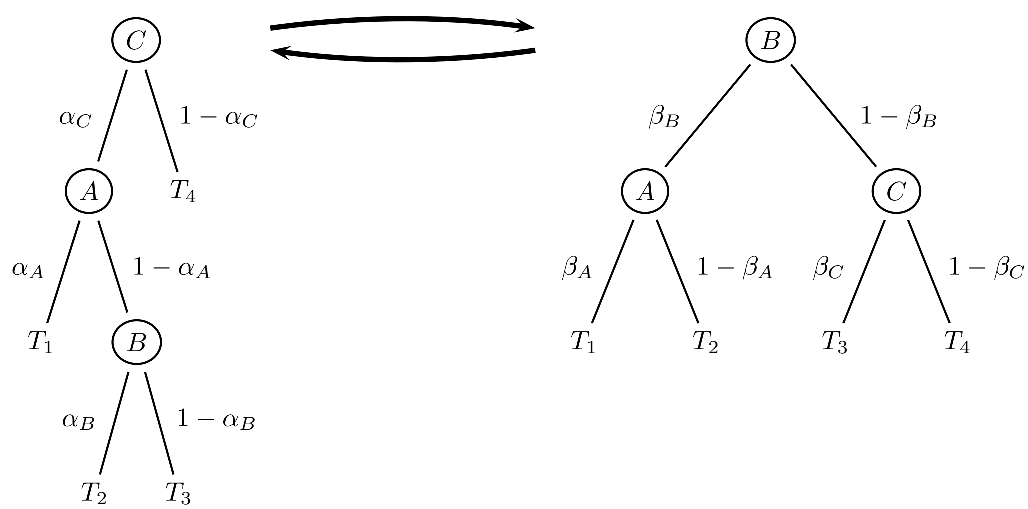
   The annihilator is $(E^2 - 9E - 9) = (E - r_1)(E - r_2)$ where $r_1 = (9 + \sqrt{117})/2$ and $r_2 = (9 - \sqrt{117})/2$, so the solution is $v_n = \alpha_1 r_1^n + \alpha_2 r_2^n$. The initial conditions give us $v_0 = 1 = \alpha_1 + \alpha_2$ and $v_1 = 10 = \alpha_1 r_1 + \alpha_2 r_2$.

   (c) How does the probability that a sequence of n rolls of the die with have no two consecutive sixes grow as $n \to \infty$?

   There are 10^n possible sequences of rolls of the die, of which $\alpha_1 \left(\frac{9+\sqrt{117}}{2}\right)^n + O(1)$ have no two consecutive sixes. The probability thus grows (shrinks, actually) like $\left(\frac{5+\sqrt{117}}{20}\right)^n$.

4. Double Rotations.

   Consider the double rotation of a binary search tree shown below.



   (a) Explain why $\beta_B = \alpha_C \alpha_A + \alpha_C(1 - \alpha_A)\alpha_B$.

   $\beta_B = \mathbf{Pr}\{x < B\}$ where x is what we are searching for. We need to go to either T_1 or T_2. To go to T_1 in the left tree we have to go through both C and A; by the rule of product that has probability $\alpha_C \alpha_A$. To go to T_2 in the left tree we have to go left through C and left through A, or left through C, right through A, and left through B; by the rule of product that has probability $\alpha_C(1 - \alpha_A)\alpha_B$. The rule of sum gives the answer.

(b) Express β_A and β_C in terms of α_A, α_B, and α_C.

Use Bayes' Theorem:

$$\begin{aligned}
\beta_C &= \mathbf{Pr}\{x < C | x > B\} \\
&= \frac{\mathbf{Pr}\{x < C\}\mathbf{Pr}\{x > B | x < C\}}{\mathbf{Pr}\{x > B\}} \\
&= \frac{\alpha_C(1-\alpha_A)(1-\alpha_B)}{\mathbf{Pr}\{x > B\}} \\
&= \frac{\alpha_C(1-\alpha_A)(1-\alpha_B)}{1-\mathbf{Pr}\{x < B\}} \\
&= \frac{\alpha_C(1-\alpha_A)(1-\alpha_B)}{1-\alpha_C\alpha_A-\alpha_C(1-\alpha_A)\alpha_B}
\end{aligned}$$

and

$$\begin{aligned}
\beta_A &= \mathbf{Pr}\{x < A | x < B\} \\
&= \frac{\mathbf{Pr}\{x < A\}\mathbf{Pr}\{x < B | x < A\}}{\mathbf{Pr}\{x < B\}} \\
&= \frac{\alpha_A\alpha_C 1}{\mathbf{Pr}\{x < B\}} \\
&= \frac{\alpha_A\alpha_C}{\alpha_C\alpha_A+\alpha_C(1-\alpha_A)\alpha_B} \\
&= \frac{\alpha_A}{\alpha_A+(1-\alpha_A)\alpha_B}
\end{aligned}$$

5. Divide-and-Conquer

Having seen the power of recursion and divide-and-conquer, the TA decided to write a program to compute x^n,

(a) His first attempt was

```
function Power(x, n)
  if n = 0 then
      return 1
  else if n is odd then
      return x * Power(x, ⌊n/2⌋) * Power(x, ⌊n/2⌋)
  else
      return Power(x, ⌊n/2⌋) * Power(x, ⌊n/2⌋)
  end if
```

Analyze the time required by this algorithm.

$T(0) = c$, $T(n) = 2T(n/2) + k$, where the constants c and k are, respectively, the small constant amounts of time used to return 1, or test the parity of n and to multiply the two recursively computed values and return the result.

Using the Master Theorem from page 11 of the notes for October 14–21, we have $a = b = 2$ and $f(n) = k$, so we have the second case because $af(n/b)/f(n) = 2k/k = 2 > 1$; hence $T(n) = \Theta(n^{\log_b a}) = \Theta(n)$. Or, with the Master Theorem as given in the text (page 532), $a = b = 2$

and $d = 0$, so that $a = 2 > 1 = b^0$, giving $T(n) = \Theta(n^{\log_b a}) = \Theta(n)$. Or, using the secondary recurrences and annihilators, $t_i = T(n_i)$, where $n_i = 2n_{i-1}$, $n_0 = 1$ so that (as in class), $n_i = 2^i$, the annihilator for $t_i = 2t_{i-1} + k$ is $(\mathbf{E} - 2)(\mathbf{E} - 1)$, and so $t_i = T(n_i) = 2^i = n_i$; that is, $T(n) = \Theta(n)$.

(b) His second attempt was

function Power(x, n)

```
if n = 0 then
   return 1
else
   integer t ← Power(x, ⌊n/2⌋)
   if n is odd then
      return x * t * t
   else
      return t * t
   end if
end if
```

Analyze the time required by this algorithm.

Here we have $T(0) = c$, $T(n) = T(n/2) + k$.

Using the Master Theorem from page 11 of the notes for October 14–21, we have $a = 1$, $b = 2$ and $f(n) = k$, so we have the third case because $af(n/b)/f(n) = k/k = 1$; hence $T(n) = \Theta(\log n)$. Or, with the Master Theorem as given in the text (page 532), $a = 1$, $b = 2$ and $d = 0$, so that $a = 1 = b^0$ and $n^d = 1$, giving $T(n) = \Theta(\log n)$. To use annihhilators, we use the same secondary recurrence as in part (a), which leads to $t_i = t_{i-1} + k$ which has annihilator $(\mathbf{E} - 1)^2$, so $t_i = T(n_i) = \Theta(i)$; that is $T(n) = \Theta(\log n)$.