**Game Theory: Algorithms and Applications**
**CS 539**

**Spring 2018**
**HomeWork 2 Solutions**
**Total: 100 points**

1. Generalized Matching Pennies: Two players choose a number between 1 and N. If the players choose the same number and it is even then player 2 pays \$1 to player 1, if odd then player 1 pays \$1 to player 2, otherwise no payment is made. Does this game have a pure NE? Find a mixed Nash Equilibrium.

   **Solution:** The payoff matrix of size $(n \times n)$ has $(-1, 1)$ and $(1, -1)$ alternating for all diagonal entries, and $(0, 0)$ for all non-diagonal entries.

   *Pure NE:* When $P_1$ chooses an even number and $P_2$ chooses an odd number, the utilities for both $= 0$ and neither can improve her utility by changing her strategy.

   *Mixed NE:* (Using the support set theorem) Strategies are dominated as *even* for $P_1$ and *odd* for $P_2$. $P_1$ will therefore have a uniform probability distribution over all the even choices, and similarly $P_2$ will have a uniform probability distribution over all the odd choices.

2. Consider a network where there are two players sending one unit of traffic using a single path. $P_i^j$ represents the path $j$ of player $i$ and each path costs $C$. If the path chosen by player 1 and the one by player 2 each share an edge, then both the players share the cost of the path, but introduce a congestion cost of $B$, also shared evenly. The objective of each player is to minimize the objective function defined by: *cost of the path + congestion.*

   i Find pure and mixed NE when $B = C$.

   ii What happens when $B < C$, and $B > C$.

   **Solution:**

   i $B = C$ A player's utility $= C$ for any path that she chooses. Let there be $n$ and $m$ paths for $P_1$ and $P_2$. Applying the support set theorem: (w.r.t $P_1$, $P_2$ is omitted)

   $$\Pi_1(1) = C\Pi_2(1) + \cdots + C\Pi_2(m)$$

   $$\Pi_1(2) = C\Pi_2(1) + \cdots + C\Pi_2(m)$$

   $$\vdots$$

   Also,

   $$\Pi_1(1) = \cdots = \Pi_1(n)$$

   and

   $$\Pi_1(1) + \cdots + \Pi_1(n) = 1$$

   One instance of a mixed NE is given by an equal distribution over all available paths, for either player. (Note that any probability distribution is a mixed NE.)

ii i. A player has less cost when her chosen path is congested, and therefore prefers it. Applying the support set theorem: (w.r.t $P_1$, $P_2$ is omitted)

$$\Pi_1(1) = \frac{B+C}{2}\Pi_2(1) + C\Pi_2(2) + \cdots + C\Pi_2(m)$$

$$\Pi_1(2) = C\Pi_2(1) + \frac{B+C}{2}\Pi_2(2) + \cdots + C\Pi_2(m)$$

$$\ddots$$

Also,

$$\Pi_1(1) = \cdots = \Pi_1(n)$$

and

$$\Pi_1(1) + \cdots + \Pi_1(n) = 1$$

$$\implies \text{there is a uniform distribution of } \Pi_1 \text{ and similarly, } \Pi_2$$

A pure NE is obtained when both players select paths which share an edge.

ii. We obtain uniform distributions of $\Pi_1$ and $P_2$. Since, both players prefer selecting paths which do not share any edges, a pure NE is one where non-sharing paths are selected by the players.

3. Consider the *hot-potato routing*/coordination routing game discussed in class. Find a mixed equilibrium of that game.

**Solution:** We obtain the following payoff matrix:

| ISP1 \ ISP2 | HPR | CR |
|---|---|---|
| HPR | -4, -4 | -1, -5 |
| CR | -5,-1 | -2,-2 |

Using the support set theorem,

$$u_1(HPR) = -4\Pi_2(HPR) - 1\Pi_2(CR)$$

$$u_1(CR) = -5\Pi_2(HPR) - 2\Pi_2(CR)$$

$$u_1(HPR) = u_1(CR)$$

$$\implies (-4-5)\Pi_2(HPR + (-1+2)\Pi_2(CR) = 0$$

$$\implies \Pi_2(HPR) = -1\Pi_2(CR)$$

$$\implies \Pi_2(HPR) + \Pi_2(CR) = 0$$

i.e., there is no interior solution for the problem of MNE $\implies$ each player has a probability of 0 on one of her strategies $\implies$ pure NE is the only MNE. This is because of domination ($P_1$ sees the 1st row dominate the 2nd).

4. *Games against Nature:* In a network there are $k$ paths, $P_1, P_2, P_3 \ldots P_k$ available to send data. One each path there is a compromised edge, on which the player might lose information, the probability of loss of information on each path $P_i$ being $l_i$ . Sending information along path $P_i$ gives a pay-off of $v_i$ . The strategy set of the player sending data is $\{P_1, P_2, \cdots, P_k\}$. *Mother Nature* is the other player with strategies in the set $\{\texttt{Lettinginformationflow}, \texttt{Probabilisticallylettinginformationflow}\}$. So path $P_i$ can either have payoff $v_i$ or payoff $v_i(1-l_i)$ to the first player. Find mixed Nash Equilibrium of this game.

**Solution:** Mother Nature's utility is the opposite of the player's utility. Given a strategy $i \in I$ (say) of player 1 ($I$ is the strategy set of $I$), mother nature stands to gain (or lose) either $-v_i$ or $-v_i(1-l_1)$ ( either $v_i$ or $v_i(1-l_1)$ ),

The player's utility is given by

$$u_1(\Pi) = \sum_{i=1}^{k} p_i^1 \cdot \left(v_i \cdot p_f^N + v_i(1-l_i) \cdot p_p^N\right)$$

where $p_f^N$ is Nature's probability of $\{\texttt{Lettinginformationflow}\}$ and $p_p^N = (1 - p^N f)$

Using the support set theorem:

$$u_1(i) = v_i p_f^N + v_i(1-l_i)p_p^N$$

Let $j$ be another strategy, then

$$u_1(j) = v_j p_f^N + v_j(1-l_j)p_p^N$$

$$\implies p_p^N = \frac{v_j - v_i}{v_j l_j - v_i l_i} \text{ and } p_f^N = 1 - \frac{v_j - v_i}{v_j l_j - v_i l_i}$$

Similarly, the player's distribution may be found by applying the support set theorem.

$$u_2(f) = -\sum_i v_i \cdot p_i^1$$

$$u_2(p) = -\sum_i v_i(1-l_i) \cdot p_i^1$$

Note, $u_2(p)$ dominates $u_2(f)$, ie., nature's strategy is dominated by $\{\texttt{Probabilisticallylettinginformationflow}\}$
$\implies p_f^N = 0$ and $p_p^N = 1$

$$\implies v_j = v_i$$

i.e., for the set of strategies where $v_j = v_i$ and $v_i > v_{i'}, \forall i' \neq i, j$, the player will have a uniform probability distribution (this is one mixed NE, any distibution over the set can be an MNE).

5. Consider the two-player Rock/Paper/Scissors (strategic) game. Remember that the game has the following rule: Rock beats Scissors, Paper beats Rock and Scissors beats paper. The winning player receives \$ 10 from the loser. A tie results in zero gain for both players. Find a mixed Nash equilibrium in this game.

**Solution:**

$$\pi_2(R) + \pi_2(P) + \pi_2(S) = 1 \tag{1}$$

$$u_1\pi(R) = u_1\pi(P) \implies -\pi_2(P) + \pi_2(S) = \pi_2(R) - \pi_2(S) \tag{2}$$

$$u_1\pi(P) = u_1\pi(S) \implies \pi_2(R) - \pi_2(S) = -\pi_2(R) - \pi_2(P) \tag{3}$$

$$\implies \pi_2(R) = \pi_2(P) = \pi_2(S) = \frac{1}{3}$$

and similarly, we have

$$\pi_1(R) = \pi_1(P) = \pi_1(S) = \frac{1}{3}$$