CS 330: Homework 2

1) Use the product rule to show that there are 2^{2^n} truth tables for propositions in n variables.

From product rule of counting: 2^n different types of 0's & 1's

$\downarrow$

In a Boolean function, a 0 or 1 is assigned to each of these 2^n functions

$\downarrow$

Each function has a truth table

$\downarrow$

$\boxed{\therefore\ 2^{2^n} \text{ truth tables for } n \text{ variables}}$

2) 40 T/F questions, 17 are true
How many possible answer keys?

$${}_{40}C_{17} = \frac{40!}{17!(40-17)!} = \boxed{887{,}323{,}7800 \text{ answer keys}}$$

3) ~~Proof~~ ${}_{n-1}C_{k-1}({}_{n}C_{k+1})({}_{n+1}C_{k}) = {}_{n-1}C_{k}({}_{n}C_{k-1})({}_{n+1}C_{k+1})$

LHS: $$\frac{(n-1)!}{((n-1)-(k-1))!\,(k-1)!}\left(\frac{n!}{(n-(k+1))!\,(k+1)!}\right)\left(\frac{(n+1)!}{((n+1)-k)!\,k!}\right)$$

$$= \frac{(n-1)!\,n!\,(n+1)!}{(n-1)!(k-1)!\,(n-1-k)!\,(k+1)!\,(n+1-k)!\,k!}$$

$$= \frac{(n-1)!}{((n-1)-k)!\,k!}\left(\frac{n!}{(n-(k+1))!\,(k-1)!}\right)\left(\frac{(n+1)!}{((n+1)-(k+1))!\,(k+1)!}\right)$$

$$= \boxed{{}_{n-1}C_{k}({}_{n}C_{k-1})({}_{n+1}C_{k+1}) = RHS}$$

The hexagon identity is a property of Pascal's triangle that states that any 6 numbers forming a hexagon will have a constant product of non-adjacent verticies, and the GCD of these verticies is also constant.

4) Prove that if n is a positive integer, then

$${}_{2n}C_2 = 2({}_nC_2) + n^2$$

a) Combinationally

2n people; n men, n women
choose 2 people from the 2n

$$\downarrow$$

$${}_{2n}C_2$$

choose k women and k men from n women + n men

$$= {}_nC_2\,{}_nC_0 + {}_nC_1\,{}_nC_1 + {}_nC_0\,{}_nC_2$$

$$= {}_nC_2 + {}_nC_1\,{}_nC_1 + {}_nC_2$$

$$= 2\,{}_nC_2 + ({}_nC_1)^2 \leftarrow {}_nC_1 = n$$

$$\boxed{{}_{2n}C_2 = 2({}_nC_2) + n^2}$$

b) Algebraically

LHS $= {}_{2n}C_2$

$$= \frac{2n!}{(2n-2)!\,2!}$$

$$= \frac{2n(2n-1)(2n-2)!}{(2n-2)!\,2!}$$

$$= n(2n-1)$$

$$\boxed{= 2n^2 - n}$$

RHS $= 2({}_nC_2) + n^2$

$$= 2\left(\frac{n!}{(n-2)!\,2!}\right) + n^2$$

$$= n(n-1) + n^2$$

$$\boxed{= 2n^2 - n}$$

c) Inductively

- base case: $n = 0$ ${}_0C_2 = 2({}_0C_2) + 0^2$

$$1 = 2 + 0$$

5) Deal 52 cards to 4 players. How many ways?

$= {}_{52}C_{13} \cdot {}_{39}C_{13} \cdot {}_{26}C_{13} \cdot {}_{13}C_{13}$

$= \frac{(52)!}{(13!)^4} = 53,644,737,765,488,792,839,237,440,000$

6) How many ways to sort 5 balls into ~~7~~ boxes if each box can only have one ball?

a) Both boxes + balls are labelled
Each ball could correspond to only one box, so there is only one way to sort correctly - put each of 5 balls in its correspondingly labelled box.

d) Balls + boxes are unlabelled
Each ball can correspond to ANY box, so there are ways to sort the balls.

${}_7C_5$

7) Prove ${}_{(kC2)}C_2 = 3\left({}_{k+1}C_4\right)$

a) Algebraically

$$\frac{k!}{(k-2)!\,2!}C_2 = 3\left(\frac{(k+1)!}{(k-3)!\,4!}\right)$$

$$\frac{\frac{k!}{(k-2)!\,2!}}{\left(\frac{k!}{(k-2)!\,2!}-2\right)!\,2!} = 3\left(\frac{(k+1)!}{(k-3)!\,4!}\right)$$

$$\frac{k!}{4(k-2)!\left(\frac{k!}{2(k-2)!}-2\right)!} = \frac{3(k+1)!}{(k-3)!\,4!}$$

$$\frac{\cancel{(k-2)!}\,(k-1)k}{4\cancel{(k-2)!}\left(\frac{k!}{2(k-2)!}-2\right)!} = \frac{\cancel{3}\cancel{(k-3)!}\,(k-2)(k-1)(k)(k+1)}{\cancel{(k-3)!}\,4\cdot\cancel{3}\cdot 2}$$

$$\frac{\cancel{(k-1)}\cancel{k}}{4\left(\frac{k!}{2(k-2)!}-2\right)!} = \frac{(k-2)\cancel{(k-1)}\cancel{(k)}(k+1)}{\cancel{8}}$$

$$\frac{2}{\left(\frac{k!}{2(k-2)!}-2\right)!} = (k-2)(k+1) = k^2-k-2$$

$$\frac{2}{\left(\frac{k(k-1)\cancel{(k-2)!}}{2\cancel{(k-2)!}}-2\right)!} = \frac{2}{\left(\frac{k^2-k}{2}-2\right)!} =$$

$$= (k-2)(k+1)$$

7) Prove combinatorially that ${}_{{}_kC_2}C_2 = 3({}_{k+1}C_4)$

1) Suppose you want to choose two colors of paint. Your options are ${}_kC_2$ possibilities, where k = number of paint choices.

But you must choose two options and allow your spouse to have a say. So to choose 2 of the ${}_kC_2$ options leaves you with ${}_{{}_kC_2}C_2$ ways to pick 2 combinations of colors.

2) There are k possible paints. So consider the k+1th paint color - if only one of the k are chosen and not the k+1th, there are 3 pairs of paint combos that could be chosen. ~~[illegible]~~ 4 total paints must be chosen of the k+1, so by the rule of product, $3\binom{k+1}{4}$ ways to choose a paint combo pair.