CS 330 Homework 4

1) a) a_n = # ways to climb n stairs if you can take 1, 2, or 3 at a time.

a_{n-1} = # " " " n-1 " ⟶

To climb (n-2) stairs, # ways = a_{n-2}, and the remaining 2 can be completed in a single step, for a total of a_{n-2} ways.

To climb (n-3) stairs, # ways = a_{n-3}, + remaining 3 can be completed in one step ⟶ a_{n-3} ways

Recurrence relation:

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} \quad \text{where } n \geq 3$$

Initial Conditions

$a_0 = 1, \quad a_1 = 1, \quad a_2 = 2$

ways to climb 8 stairs

$a_3 = 4$	$a_6 = 24$
$a_4 = 7$	$a_7 = 44$
$a_5 = 13$	$a_8 = 81$ ways

$a_n = Cn^2 + dn + e$

1) b) Find all solutions of the recurrence relation $\underbrace{a_n = 2a_{n-1}}$ $+ 2n^2$

$a_n^{(G)} = \alpha_1 2^n$

$r - 2 = 0 \longrightarrow r = 2$

$a_{n-1} = C(n-1)^2 + d(n-1) + e$

$a_n = Cn^2 + dn + e = 2(C(n-1)^2 + d(n-1) + e) + 2n^2$

$C = 2C + 2 \longrightarrow C = -2$

$2d - 4C = d \rightarrow d = 8$

$2C - 2d + 2e = e \rightarrow e = 20$

$a_n^P = -2n^2 + 8n + 20$

$\boxed{a_n = \alpha_1 2^n - 2n^2 + 8n + 20}$

Find the solution when $a_1 = 4$

$a_1 = 4 = \alpha_1 2^1 - 2 + 8 + 20$

$2\alpha_1 = -22$

$\alpha_1 = -11$

$\boxed{a_n = -11(2^n) - 2n^2 + 8n + 20}$

c) Solve $\underbrace{a_n = 3a_{n-1} + 2b_{n-1}}$, $\underbrace{b_n = a_{n-1} + 2b_{n-1}}$

$r - 3 = 0 \rightarrow r = 3$

$a_n^G = \alpha 3^n$

$s - 1 = 0 \longrightarrow s = 1$

$b_n^G = \beta 1^n$

$a_n^P = \beta$

$b_n^P = \alpha 3^n$

$a_n = a_n^G + a_n^P = \alpha 3^n + \beta$

$\alpha 3^n = (\alpha 3^{n-1} + \beta) + 2\alpha 3^{n-1}$

$\alpha 3^n = 3\alpha 3^{n-1} + \beta$

$\alpha 3^n = \alpha 3^n + \beta \longrightarrow \beta = 0$

$a_n = \alpha 3^n$

$b_n = \alpha 3^n$

Given $b_0 = 2$ and $a_0 = 1$,

$a_0 = \alpha 3^0 \longrightarrow \alpha = 1$

$b_0 = \alpha 3^0 = 2 \longrightarrow \alpha = 2$

$\boxed{a_n = 1(3^n)}$

$\boxed{b_n = 2(3^n)}$

2) a) Find $f(n)$ when $n=2k$, where f satisfies the RR
$f(n) = 8f(n/2) + n2$, and $f(1) = 1$

<u>Master Theorem</u>

$$f(n) = af(n/b) + cn^d$$

$$f(n) = \begin{cases} \Theta(n^d) & a < b^d \\ \Theta(n^d \log n) & a = b^d \\ \Theta(n^{\log_b a}) & a > b^d \end{cases}$$

<u>If $a \neq b^d$</u>

$$f(n) = C_1 n^d + C_2 n^{\log_b a}$$

$$C_1 = \frac{b^d c}{(b^d - a)} \qquad C_2 = f(1) + \frac{b^d c}{(a - b^d)}$$

$a = 8$ $\quad c = 1$

$b = 2$ $\quad d = 2$

$C_1 = -1$

$C_2 = 2$

$$f(n) = -n^2 + 2n^{\log_2 8} = -n^2 + 2n^{\log_2 2^3}$$

$$\boxed{f(n) = -n^2 + 2n^3}$$

b) Derive the Master Theorem

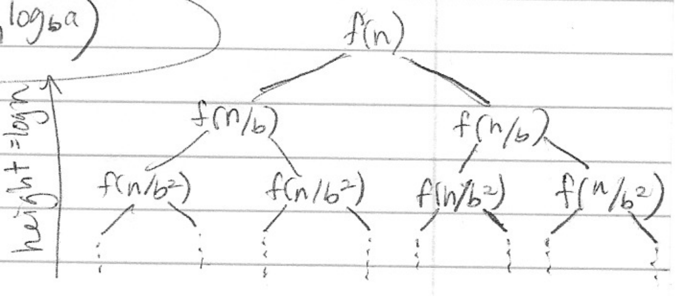
$T(n) = aT(n/b) + f(n)$ where a+b are constants

A) If $f(n) = \Theta(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, $T(n) = \Theta(n^{\log_b a})$

B) If $f(n) = \Theta(n^{\log_b a})$, $\quad T(n) = \Theta(n^{\log_b a} \log n)$

C) If $f(n) = \Theta(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$, $T(n) = \Theta(f(n))$

$$T(n) = \sum_{i=0}^{\log_b n} a^i f(n/b^i) + O(n^{\log_b a})$$



$$T(n) = \sum_{i=0}^{\log_b n} a^i f(n/b^i) + O(n^{\log_b a}) \leq$$

$$\sum_{i=0}^{\log_b n} a^i (n/b^i)^{\log_b(a-\varepsilon)} + O(n^{\log_b a})$$

$$\sum_{i=0}^{\log_b n} a^i (n/b^i)^{\log_b(a-\varepsilon)} = n^{\log_b(a-\varepsilon)} \sum_{i=0}^{\log_b n} a^i a^{-i} b^{i\varepsilon} = n^{\log_b(a-\varepsilon)} \left(\frac{n^\varepsilon b^\varepsilon - 1}{b^\varepsilon - 1}\right) \leq n^{\log_b(a)} \left(\frac{b^\varepsilon}{b^\varepsilon - 1}\right)$$

$$= n^{\log_b a} \left(\frac{b^\varepsilon}{b^\varepsilon - 1}\right) = \boxed{O(n^{\log_b a}) = T(n)}$$ CASE B

2) b) $\sum_{i=0}^{\log_b n} a^i (n/b^i)^{\log_b a} = n^{\log_b a} \sum_{i=0}^{\log_b n} a^i a^{-i} = n^{\log_b a}(\log_b n + 1) = \Theta(n^{\log_b a} \log n)$

$\boxed{T(n) = \Theta(n^{\log_b a} \log n)}$ CASE C

$f(n) = n^{\log_b(a+\varepsilon)}$ for any $\varepsilon > 0$ w/ $c = b^{-\varepsilon} < 1$

$a f(n/b) = a(n/b)^{\log_b(a+\varepsilon)} = a n^{\log_b(a+\varepsilon)} b^{-\log_b a} b^{-\varepsilon} = f(n) b^{-\varepsilon}$

$a^i f(n/b^i) \le c^i f(n)$

$\rightarrow T(n) = \sum_{i=0}^{\log_b n} a^i f(n/b^i) + O(n^{\log_b a}) \le \sum_{i=0}^{\log_b n} c^i f(n) + O(n^{\log_b a})$

$\le f(n) \sum_{i=0}^{\infty} c^i + O(n^{\log_b a}) = f(n) \frac{1}{1-c} + O(n^{\log_b a})$

$\boxed{T(n) = O(f(n))}$ CASE A

Cornell.cs → proof of Master Method

c) Solve $T(n) = \sqrt{n}\, T(\sqrt{n}) + n$ by secondary recurrance + annihilator.

MASTER THEOREM: $T(n) = \Theta(n \log(\log(n)))$

$T(n_i) = \sqrt{n_i}\, T(n_{i-1}) + n_i$ $\quad n_{i-1} = \sqrt{n_i} \rightarrow n_i = n_{i-1}^2$

$t_i = 2t_{i-1} + \alpha 2^i$ $\quad (E-2)$

$\downarrow$

$(E-2)^2$ $\quad n_i = \alpha 2^i$

$n/\alpha = 2^i$

$t_i = \alpha 2^i + \beta 2^i$ $\quad \log_2(n/\alpha) = i$

$T(n) = \alpha 2^{\log_2(n/\alpha)} + \beta 2^{\log_2(n/\alpha)} = \alpha(n/\alpha) + \beta(n/\alpha)$

$\boxed{T(n) = n + \frac{\beta n}{\alpha}}$

3) a) Prove that $1/89 = 1/10^2 + 1/10^3 + 2/10^4 + 3/10^5 + 5/10^6 + \ldots$
(Hint: $89 = 10^2 - 10 - 1$)

$$\frac{1}{89} = \underbrace{\sum_{i=1}^{\infty}}_{\text{Sum of all Fibonacci}} \underbrace{F(i)/10^{i+1}}_{10^2 \to 10^{i+1}}$$

gives a decimal representation of Fibonacci sequence $= 1/89$

$$\frac{1}{89} = \sum_{i=1}^{\infty} \frac{F(i)}{10^{i+1}} \approx 0.011235\ldots$$

b) Prove that $1/55 = 1/8^2 + 1/8^3 + 2/8^4 + 3/8^5 + 5/8^6 + \ldots$

$$\frac{1}{55} = \sum_{i=1}^{\infty} \frac{F_i}{8^{i+1}} \approx 0.01818 18\ldots$$

Sum of Fibonacci over base 8 to the number in the sequence

How are we supposed to prove this??