1. Show $cost(A) = \phi \cdot opt(G)$ exists where $\phi$ is the price of anarchy.

$$l = \sum \frac{weight}{speed}$$

$cost(A) = \max(l_j)$ over all machines $j$

if machine $j \neq j^*$ has a load less $l_{j*} - \frac{w_i}{s_j}$ for any task $i$,

$$l_j \leq l_{j*} - \frac{w_i}{s_j} \leq cost(A) - \frac{w_i}{s_j}$$

by $cost(A) = \max(l_j)$

$$\frac{w_1}{s_1} = \frac{w_2}{s_2} \text{ @ eq}$$

$$w_1 s_2 = w_2 s_1$$

$$\frac{w_1}{w_2} = \frac{s_1}{s_2}$$

We know $opt(G) \geq \frac{\sum l_j}{2}$.

In order to ensure that the loads are as balanced as possible, the higher weighted of the two tasks should be given to the faster machine.

so now, $opt(G) \geq \frac{2cost(A) - \frac{w_1}{s_1} - \frac{w_2}{s_2}}{2}$, where $w_1 > w_2$ if $s_1 > s_2$

$$cost(A) \leq opt(G) + \frac{w_1}{2s_1} + \frac{w_2}{2s_2}$$

$$4 + 4.75 = \frac{8.75}{4} = 2.18$$

Say $w_1 = 6$, $w_2 = 4$

$s_1 = 3$, $s_2 = 2$

$$\frac{6}{3} + \frac{4}{2} = 4$$

$$\frac{4}{3} + \frac{6}{2} = 4.75$$

$$cost(A) \leq 2.18 \, opt(G)$$

$\Downarrow$

this condition can be satisfied with $cost(A) = 1.618 \, opt(G)$, the golden ratio.

2. n traders

m goods

utility $(i) = v_i x_{ij}$, where $v_i \in \mathbb{Z}^+$ and $x_{ij} \in \mathbb{R}^+$ (good j)

If each trader has the same utility for all goods, then no trader would experience a net gain in utility from a trade. Therefore, any distribution of goods when traders have the same utility for all goods is a market equilibrium.

3) Reverse auction

Buy k identical items, each bidder supplies one.

A mechanism for this auction could be saying that the win is given to the k lowest bids, but at a price equal to the average bid. This promises that a bidder would be paid more than their bid, but not by how much. The higher the number of bidders who are competing, lower the final price will get. As long as $n > k$ and the avg price is $>$ the valuation for the bidder, the best bid is lower. Eq occurs when all players make a bid equal to their valuation, since they do not know each other's valuation. This is the lowest bid for each bidder.

$$\frac{\text{bid total}}{\#\text{ bidders } n} = \text{price avg}$$

total cost = k * price

$$\text{utility}_i = \begin{cases} \text{WIN: avg price} - \text{valuation} \\ \text{LOSE: } \emptyset \end{cases}$$

PROOF

Utility$_i$ = ∅ if their bid wins, and ~~∅~~ if it does not.

- If they increase their bid the likelihood that they will lose increases and utility decreases overall.
- If they decrease their bid, winning will yield negative utility.

4) 3-player majority game: is there a core?

$|S| \geq 1$ $\left[ v(1) = \varnothing \right.$

$\forall |S| \geq 2$ $\left[ \begin{array}{l} v(2) = 1 \\ v(3) = 1 \end{array} \right.$

$x_1 + x_2 \geq 2$ → these would not produce core, because one of v(2) and v(3) is worse off.

$x_2 + x_3 \geq 2$

$x_1 + x_3 \geq 2$ →

$x_1 + x_2 + x_3 = 2$

Both of these can be considered a core, as the payoff for each player is equal to its individual payoff.

No player is better off alone, therefore no one will break coalition.

CONTRADICTION: assume $v(S) >$ coalition payoff, then one must be true:

$v(1) > x_3 + x_2 - v(3) - v(2)$, OR

$v(2) > x_3 + x_2 - v(3) - v(1)$, OR

$v(3) > x_3 + x_2 - v(2) - v(1)$, OR

$v(1) > x_1 + x_2 + x_3 - v(2) - v(3)$, OR

$v(2) > x_1 + x_2 + x_3 - v(1) - v(3)$, OR

$v(3) > x_1 + x_2 + x_3 - v(1) - v(2)$

NONE of these is true, therefore the core holds