Illinois Institute of Technology
Department of Computer Science

Solutions to Homework Assignment 4

CS 330 Discrete Structures
Spring Semester, 2015

1. Solve the following problems using the operator method discussed in class and given in the notes. If the roots of the recurrence are not integers, just get the form of the solution in each case for symbolic roots, together with the simultaneous equations that determine the coefficients.

   (a) Page 511, exercise 12.

   Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time.

   Let a_n be the number of ways to climb n stairs. Using the rule of sum it can be seen that to climb n stairs one can initially take either one step with a_{n-1} remaining ways, or take two steps with a_{n-2} remaining ways, or take three steps with a_{n-3} remaining ways to climb the rest of the stairs.

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

   with initial conditions

$$a_0 = 1, a_1 = 1, a_2 = 2$$

   In order to annihilate the sequence:

$$\begin{aligned} E^3 \langle a_n \rangle &= E^2 \langle a_n \rangle + E \langle a_n \rangle + \langle a_n \rangle \\ (E^3 - E^2 - E - 1) \langle a_n \rangle &= \langle 0 \rangle \end{aligned}$$

   This would be a challenge to factor, use your favorite computer aided algebra software and you would find that the roots are not integers, and that they are around

$$\begin{aligned} r_1 &\approx 1.8393 \\ r_2 &\approx -0.4196 + 0.6063i \\ r_3 &\approx -0.4196 - 0.6063i \end{aligned}$$

   The important thing to note is that they are all distinct. So we know the solution is of the form:

$$a_n = \alpha r_1^n + \beta r_2^n + \gamma r_3^n$$

   In order to find a_8 we can just solve the recurrence iteratively

$$\begin{aligned}
a_0 =&1\\
a_1 =&1\\
a_2 =&2\\
a_3 =&1+1+2=4\\
a_4 =&1+2+4=7\\
a_5 =&2+4+7=13\\
a_6 =&4+7+13=24\\
a_7 =&7+13+24=44\\
a_8 =&13+24+44=81
\end{aligned}$$

(b) Page 525, exercise 28
Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$.
The homogeneous part is annihilated by $E-2$. The residue is eliminated by $(E-1)^3$. So the annihilator is $(E-2)(E-1)^3$. The general form is then:

$$a_n = \alpha 2^n + \beta n^2 + \gamma n + \delta$$

(c) Page 526, exercise 40
Solve the simultaneous recurrence relations

$$\begin{aligned}
a_n =&3a_{n-1} + 2b_{n-1}\\
b_n =&a_{n-1} + 2b_{n-1}
\end{aligned}$$

with the initial conditions $a_0 = 1$ and $b_0 = 2$
First re-write the second equation as

$$\begin{aligned}
a_{n-1} =&b_n - 2b_{n-1}\\
a_n =&b_{n+1} - 2b_n
\end{aligned}$$

Substitute these back into the first equation

$$\begin{aligned}
b_{n+1} - 2b_n =&3(b_n - 2b_{n-1}) + 2b_{n-1}\\
b_{n+1} =&5b_n - 4b_{n-1}\\
b_n =&5b_{n-1} - 4b_{n-2}
\end{aligned}$$

This is annihilated by $E^2 - 5E + 4 = (E-1)(E-4)$ which means $b_n = \alpha + \beta 4^n$ and using our previous result

$$\begin{aligned}
a_n =&b_{n+1} - 2b_n\\
=&\alpha + \beta 4^{n+1} - 2(\alpha + \beta 4^n)\\
=&4\beta 4^n - 2\beta 4^n - \alpha\\
=&2\beta 4^n - \alpha
\end{aligned}$$

plugging in our initial conditions we get

$$\begin{aligned} a_n =& 2(4^n) - 1 \\ b_n =& 4^n + 1 \end{aligned}$$

2. Solve the following problems using secondary recurrences together with the operator method discussed in class and given in the notes:

(a) Page 536, exercise 36

$$\begin{aligned} f(1) =& 1 \\ f(n) =& 8f(n/2) + n^2 \end{aligned}$$

Let $n = n_i$ be the i^{th} argument to f from the base case.

$$\begin{aligned} n_0 =& 1 \\ n_{i-1} =& n_i/2 \\ n_i =& 2n_{i-1} && \text{annihilated by (E-2)} \\ n_i =& 2^i \end{aligned}$$

Let $t_i = f(n_i) = 8t_i + 4^i$. t_i is then annihilated by $(E-8)(E-4)$ So $t_i = \alpha 2^{3i} + \beta 2^{2i}$. Also note $i = lg(n)$

$$\begin{aligned} f(n) =& \alpha 2^{3lg(n)} + \beta 2^{2lg(n)} \\ =& \alpha n^3 + \beta n^2 \end{aligned}$$

We then need an additional initial condition which we can calculate from the original recurrence $f(2) = 12$. Which gives $f(n) = 2n^3 - n^2$

(b) Derive all three cases of the Master Theorem on page 532 (we did parts of this in class)

Recall the Master Theorem states

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

- if $af\left(\frac{n}{b}\right) = Kf(n)$ for some $K < 1$ then $T(n) \in \Theta(f(n))$
- if $af\left(\frac{n}{b}\right) = Kf(n)$ for some $K > 1$ then $T(n) \in \Theta(n^{\log_b a})$
- if $af\left(\frac{n}{b}\right) = f(n)$ then $T(n) \in \Theta(f(n)\log_b n)$
- if none of these three cases apply, then you're on your own.

We begin by setting up the recurrence: $T(n) = aT\left(\frac{n}{b}\right) + f(n)$

We will replace n with a linear recurrence $T(u_m) = aT\left(\frac{u_m}{b}\right) + f(n) = aT(u_{m-1}) + f(n)$, so $u_m = bu_{m-1}$. This recurrence is easy to solve: it's $n = u_m = b^m$. We'll also note the inverse: $m = \log_b n$

From the three cases in the Master Theorem, we also know that $\frac{a}{K}f\left(\frac{n}{b}\right) = f(n)$. If we rewrite this in terms of u_m, $\frac{a}{K}f(u_{m-1}) = f(u_m)$, and rewrite it further still as $\frac{a}{K}f'_{m-1} = f'_m$, we can see that $f'_m = \left(\frac{a}{K}\right)^m$.

We are now ready to rewrite our man recurrence $T(n)$ to t_n. By substituting our above derivations, $t_m = at_{m-1} + f'_m$. Since f'_m is already solved, this is equal to $t_m = at_{m-1} + \left(\frac{a}{K}\right)^m$

Now we can solve the recurrence. There are two cases here.

The first case is when $K \neq 1$, and it corresponds to the first *two* cases in the Master Theorem. The annihilator for t_m is $(E-a)\left(E-\frac{a}{K}\right)$, so $t_m = \alpha a^m + \beta\left(\frac{a}{K}\right)^m = \alpha a^m + \beta f'_m$. Undoing all of our substitutions, $T(n) = \alpha a^{\log_b n} + \beta f(n)$. And the exponent can be rearranged making $T(n) = \alpha n^{\log_b a} + \beta f(n)$.

If $K > 1$, then $f(n)$ grows more slowly than $n^{\log_b a}$. So $T(n) \in \Theta(n^{\log_b a})$ If $K < 1$, then $f(n)$ grows more quickly than $n^{\log_b a}$. So $T(n) \in \Theta(f(n))$

In the second case, when $K = 1$, the annihilator $(E-a)\left(E-\frac{a}{K}\right) = (E-a)^2$, so $t_m = (\alpha + \beta m)a^m = (\alpha+\beta m)f'_m$. Undoing all of our substitutions, this tells us that $T(n) = (\alpha+\beta\log_b n)f(n) \in \Theta(f(n)\log_b n)$

(c) In the notes, the recurrence $T(n) = \sqrt{n}\cdot T(\sqrt{n}) + n$ is solved by a recursion tree, and again by the "guess-and-confirm" method in section 1.6.3. Solve it by means of a secondary recurrence (section 1.5.3) together with the operator method.

$$T(n) = \sqrt{n}T(\sqrt{n}) + n$$

Divide the whole equation by n.

$$\frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1$$

Let's substitute $n = 2^m$ and $\frac{T(n)}{n} = \frac{T(2^m)}{2^m} = g(m)$. We note that $m = \lg n$.

$$g(m) = g\left(\frac{m}{2}\right) + 1$$

Now, we can use a secondary recurrence, by letting $m = u_k$

$$g(u_k) = g\left(\frac{u_k}{2}\right) + 1 = g\left(u_{k-1}\right) + 1$$

This means $u_k = 2u_{k-1}$ so we solve u_k and find $m = u_k = 2^k$ and therefore $k = \lg m$.

Now, let $g'_k = g(u_k)$.

$$g'_k = g'_{k-1} + 1$$

We can see that the annihilator for this equation is $(E-1)^2$, so $g_k = \alpha + \beta k$.

Now, all that's left is to undo all of the substitutions and the division that we started with:

$$\begin{aligned} g_k &= \alpha + \beta k \\ g(u_k) = g(m) &= \alpha + \beta \lg m \\ \frac{T(n)}{n} &= \alpha + \beta \lg\lg n \\ T(n) &= \alpha n + \beta n \lg\lg n \end{aligned}$$

3. (a) Prove that $1/89 = 1/10^2 + 1/10^3 + 2/10^4 + 3/10^5 + 5/10^6 + 8/10^7 + \ldots$. (*Hint:* $89 = 10^2 - 10 - 1$)

First observe that the numerators form the Fibonacci numbers. Writing the above in summation form gives:

$$\sum_{i=1}^{\infty} F_i \left(\frac{1}{10}\right)^{i+1}$$

looking ahead to part (b) notice that the only change when interpreting the above sum as base 8 is the $10(10_8 = 8_{10})$. The Fibonacci number will have the same value. So we can write the above equation as a function of the base, and save some work.

$$\begin{aligned}
f(x) =& \sum_{i=1}^{\infty} F_i \left(\frac{1}{x}\right)^{i+1} \\
=& \frac{1}{x}\sum_{i=1}^{\infty} F_i \left(\frac{1}{x}\right)^{i} \\
=& \frac{1}{x}\left[F_1\left(\frac{1}{x}\right)^1 + \sum_{i=2}^{\infty} F_i \left(\frac{1}{x}\right)^{i}\right] \\
=& \frac{1}{x^2} + \frac{1}{x}\sum_{i=2}^{\infty} F_i \left(\frac{1}{x}\right)^{i} \\
=& \frac{1}{x^2} + \frac{1}{x}\sum_{i=2}^{\infty} (F_{i-1} + F_{i-2}) \left(\frac{1}{x}\right)^{i} \\
=& \frac{1}{x^2} + \frac{1}{x}\sum_{i=2}^{\infty} F_{i-1} \left(\frac{1}{x}\right)^{i} + \frac{1}{x}\sum_{i=2}^{\infty} F_{i-2} \left(\frac{1}{x}\right)^{i}
\end{aligned}$$

Now we let $j = i - 1$ in the first summation and $k = i - 2$ in the second and remember that $F_0 = 0$

$$\begin{aligned}
f(x) =& \frac{1}{x^2} + \frac{1}{x}\sum_{j=1}^{\infty} F_j \left(\frac{1}{x}\right)^{j+1} + \frac{1}{x}\sum_{k=0}^{\infty} F_k \left(\frac{1}{x}\right)^{k+2} \\
=& \frac{1}{x^2} + \frac{1}{x^2}\sum_{j=1}^{\infty} F_j \left(\frac{1}{x}\right)^{j} + \frac{1}{x^3}\sum_{k=1}^{\infty} F_k \left(\frac{1}{x}\right)^{k} \\
=& \frac{1}{x^2} + \frac{f(x)}{x} + \frac{f(x)}{x^2} \\
=& \frac{1}{x^2 - x - 1}
\end{aligned}$$

This is what we expect, given our hint.

(b) What reciprocal plays the same role base 8? Prove it.

With our initial observations above we can just write $f(8) = 1/55 = 1/(67_8)$

The puzzle, by Jonah Kagan and edited by Will Shortz, is from the *New York Times* of May 11, 2011. Here is the solution:

NY Times, Wed, May 11, 2011 Jonah Kagan / Will Shortz

S	T	E	M		D	E	A	R		R	E	A	M	S
P	I	S	A		A	W	R	Y		E	A	T	I	N
A	R	T	I	C	H	O	K	E		P	R	O	L	E
T	E	D	T	A	L	K	S			A	N	N	I	E
			A	L	I	S		L	D	R		A	E	R
O	P	T	I	M	A		N	A	U	T	I	L	U	S
A	H	A				E	I	T	H	E	R			
F	I	B	O	N	A	C	C	I	S	E	R	I	E	S
			P	O	L	L	E	N				R	E	C
I	N	N	E	R	E	A	R		I	O	L	A	N	I
S	O	A		E	X	T		A	T	R	A			
L	O	G	I	A			D	I	S	A	R	M	E	D
E	G	G	O	S		S	U	N	F	L	O	W	E	R
T	I	E	T	O		R	A	G	U		S	A	R	A
S	E	D	A	N		A	D	E	N		A	H	O	Y

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ACROSS

1. Pipe part
5. Babycakes
9. Paper orders
14. Leonardo of ___ (mathematician who wrote about the 33-Across)
15. Out of kilter
16. Have dinner at home
17. Food with a heart
19. Blue-collar worker
20. Popular online lectures about "ideas worth spreading"
21. Alvy's love interest in a Woody Allen film
22. Father-and-daughter boxers
23. Pres., e.g.
25. ___ Lingus
26. Midsize Kia
29. Cephalopod known for its shell
31. "Now I see!"
32. "Both work for me"
33. Mathematical sequence related to a pattern found in a 17-, 29-, 42- or 58-Across, informally
40. Sneeze producer
41. Cam button
42. Cochlea locale
45. Honolulu's ___ Palace
48. "___ man walks into a bar ..."
49. No. after a no.
50. Razor brand
51. Sayings of Jesus
53. No longer able to shoot
57. Some waffles
58. Van Gogh subject
59. Link with
60. Sauce that's made "Old World Style"
61. Singer Bareilles with the 2007 top 10 hit "Love Song"
62. 26-Across, e.g.
63. Yemeni port
64. Ship-to-ship communication

DOWN

1. Short row
2. Ring around a rim
3. Cornerstone abbr.
4. Tiki bar staple
5. "The Black ___" (2006 film)
6. They thought C-3PO was a god in "Return of the Jedi"
7. Scroll storers
8. Pumpernickel grain
9. Badinage
10. Merit
11. Like much Schoenberg music
12. Surroundings
13. Mocking looks
18. Pacify
23. Homo sapiens, e.g.
24. Stupidity syllables
26. Speaker of stupid syllables
27. Greek letter associated with the 33-Across
28. Web browser subwindow
29. More pleasant
30. Like oddly conjugated verbs: Abbr.
32. Razzle-dazzle
34. Unbarred, to a bard
35. "Because I felt like it"
36. "A Clockwork Orange" narrator
37. Glass behind a radio microphone
38. Night of poetry
39. Chem. or biol.
42. They may start as sandbars
43. Knuckle-head move?
44. Pestered
45. "You'll enjoy this"
46. Said aloud
47. Old-time crooner Julius
50. N.B.A. player/manager Danny
52. Itsy-bitsy bit
53. Pair
54. Sound of an air kiss
55. Architect Saarinen
56. Heavy cart
58. Miguel's Mrs.

All the theme answers relate to the FIBONACCI SERIES (33 across) of Leonardo of PISA (14 across). Patterns found in nature such as the flowering of an ARTICHOKE (17 across), shell of a NAUTILUS (29 across), cochlea in the INNER EAR (42 across), and florets in a SUNFLOWER (58 across) can all be mathematically described the Fibonacci series. The circled squares in the puzzle, which spell GOLDEN RATIO, show a Fibonacci spiral, a pattern created using the Fibonacci series. And, of course, the golden ratio, intimately tied to the Fibonacci series, is usually represented by the Greek letter PHI (33 across), $\phi = (1 + \sqrt{5})/2 \approx 1.61801$.