Illinois Institute of Technology
Department of Computer Science

Solutions to First Examination

CS 330 Discrete Structures
Fall Semester, 2013

Fifty three students took the exam; the statistics were:

Minimum	13
Maximum	88
Median	43
Average	45.85
Std Dev	17.12

1. Mathematical Induction.

   Suppose we have a statement S_n and we know the following facts:

   (a) S_1 is true.

   (b) If S_n is true, then so is S_{n-1}.

   (c) If S_n is true, then so is S_{2n}

   Prove by induction that S_n is true for all integers $n \geq 1$.

   First, (a) and (c) tell us (by induction) that S_{2^k} is true for all $k \geq 0$. Now we prove by induction on i that S_{2^k-i} is true for all $k \geq 0$ and $0 \leq i < 2^k$. For $i = 0$, this follows because S_{2^k} is true for all $k \geq 0$, as we just showed. If $i < 2^k - 1$ and S_{2^k-i} is true, then by (b) $S_{2^k-(i+1)}$ is true, completing the induction.

2. Growth rates.

   (a) Does $\binom{2n}{n}$ grow slower than, the same as, or faster than n^n? Prove your answer.

   We calculate $\binom{2n}{n}$ using Stirling's formula; ignoring lower order terms (which we never discussed):

   $$\binom{2n}{n} = \frac{2n!}{(n!)^2} = \Theta\left(\frac{c\sqrt{2n}(2n/e)^{2n}}{(c\sqrt{n}(n/e)^n)^2}\right) = \Theta(4^n/\sqrt{n})$$

   which grows much slower than n^n.

   (b) Does $n(H_n)^2$ grow slower than, the same as, or faster than $n \log_2 n$? Prove your answer.

   We showed in class that $H_n = \Theta(\log n)$ so that $n(H_n)^2 = \Theta(n \log^2 n)$ which grows faster than $n \log_2 n$.

3. Algorithms.

   (a) Suppose you compute 2^n using the following recursive function:

```
FUNCTION PowerOfTwo(n)
  BEGIN
    IF i=0 THEN RETURN 1;
    ELSE RETURN PowerOfTwo(n-1) + PowerOfTwo(n-1);
  END
```

Analyze the number of additions needed to compute 2^n.

$2^n - 1$ additions are used (by indcution). The function does nothing more than add $1+1+1+\cdots+1$.

(b) Show how to compute 2^n in $O(n)$ additions.

Just avoid the duplicate call by using a temporary variable:

```
FUNCTION PowerOfTwo(n)
  BEGIN
    IF i=0 THEN RETURN 1;
    ELSE temp := PowerOfTwo(n-1);
    RETURN temp + temp;
  END
```

This does one addition per recursive call and there are $n-1$ such calls (by induction).

(c) Using (b), analyze the number of additions to compute the sum $\sum_{i=0}^{k} 2^k$.

Each of the summands takes $O(i)$ from part (b), so together they take $\sum_{i=0}^{k} O(i) = O(\sum_{i=0}^{k} i) = O(k^2)$.

4. Rules of Sum Product.

Use the rules of sum and product to give combinatorial interpretations of the identity

$$\left[\sum_{k=0}^{n} \binom{n}{k}\right]^2 = \sum_{k=0}^{2n} \binom{2n}{k} = 2^{2n} = 4^n.$$

With n couples there are $2n$ people. By the Rule of Product, 2^{2n} is the number of subsets (of any size, including zero) of a set of the $2n$ people (each person can be in or out of the subset).

For each couple, we can have just the man, just the woman, both of them, or neither of them in the subset, four possibilities. There are n couples, so the Rule of Product tells us there are 4^n ways to select the subset of the couples.

Such a subset can be formed by choosing m, the number of men in the subset, then that number of men, followed by choosing w, the number of women in the subset, then that number of women. Using the Rules of Sum and Product,

$$\left[\sum_{m=0}^{n} \binom{n}{m}\right] \left[\sum_{w=0}^{n} \binom{n}{w}\right] = \left[\sum_{k=0}^{n} \binom{n}{k}\right]^2.$$

Such a subset can also be formed by choosing k, the number of people in the subset, followed by choosing that number of people from the $2n$ men and women. Using the Rule of Sum this is

$$\sum_{k=0}^{2n} \binom{2n}{k}.$$

5. Binomial Coefficients.

Find the coefficient of x^2 in each of the following polynomials. You need not simplify powers, factorials, or binomial coefficients.

(a) $(1+2x)^{2013}$

By the binomial theorem, $2^2\binom{2013}{2}$.

(b) $(1-x)^{-2013}$

By the binomial theorem extended to negative exponents, $\binom{2013+2-1}{2} = \binom{2014}{2}$.

(c) $(1-x^3+x^9-x^{27}+\cdots)^{2013}$

Zero, because the polynomial only has mulotiples of three powers of x.

(d) $(1-2013x)^{2n+1}$

By the binomial theorem, $2013^2\binom{2n+1}{2}$.