Illinois Institute of Technology
Department of Computer Science

Second Examination

CS 330 Discrete Mathematics
Spring, 2015

11:25am–12:40pm, Wednesday, March 25, 2015
113 Stuart Building

Print your name and student ID, *neatly* in the space provided below; print your name at the upper right corner of *every* page. Please print legibly.

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This is an *open book* exam. You are permitted to use the textbook (hardcopy only), any class handouts, anything posted on the web page, any of your own assignments, and anything in your own handwriting. Foreign students may use a dictionary. *Nothing else is permitted*: No calculators, laptops, cell phones, Ipads, Ipods, communicators, GPSes, etc.!

Do all five problems in this booklet. *All problems are equally weighted, so do not spend too much time on any one question.*

Show your work! You will not get partial credit if the grader cannot figure out how you arrived at your answer.

Question	Points	Score	Grader
1	20	20	
2	20	8	
3	20	20	
4	20	10	
5	20	20	
Total	100	78	TA

 Name: Rachael Affenit

1. **Probabilty.**

A parking lot has a row of n parking spaces; k cars arrive and park in random spaces. An SUV, which needs two adjacent empty parking spaces, arrives. What is the probability that the SUV can park? (*Hint*: The hardest element of this problem was problem 3 of the first exam.)

$\binom{n}{n-k} = \binom{n}{k}$ arrangements of empty spaces

choose n-k empty from n spaces

K+1 places to insert n-k empty spaces between K cars

$\therefore \binom{K+1}{n-k}$ arrangements where SUV cannot park

insert n-k empty into K+1 potential slots

$|u| = \binom{n}{k}$

$|u - SUV| = \binom{K+1}{n-k}$

$P(SUV \text{ can't park}) = \dfrac{\binom{K+1}{n-k}}{\binom{n}{k}}$

$$\boxed{P(SUV \text{ CAN park}) = 1 - \dfrac{\binom{K+1}{n-k}}{\binom{n}{k}}}$$

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2. **Probabilistic Analysis of Duels**

Two people, P and Q, stand opposite each other, each with a loaded pistol. Their duel proceeds in rounds: at each round P fires a single shot at Q, whereupon Q dies if hit. If P misses, Q fires a single shot at P, whereupon P dies if hit. Suppose they have, respectively, probabilities p and q of hitting (and thus killing) their opponent and these probabilities remain constant as the duel progresses. Assume that P fires first and that they never run out of bullets. $\rightarrow \infty$

(a) Prove that the probability that P kills Q at the kth shot is $p[(1-p)(1-q)]^{k-1}$.

(b) Compute the probability that P survives, killing Q. $\rightarrow$ on any round k

(c) Compute the expected number of shots that P fires.

a) $\boxed{p[(1-p)(1-q)]^{k-1}}$

COMBINATORIAL PROOF

p kills q on k^{th} shot AND p misses all previous k-1 shots AND q misses all previous k-1 shots

RULE OF PRODUCT

b) $P(P \text{ kills } Q) = \boxed{\prod_{k=1}^{\infty} p[(1-p)(1-q)]^{k-1}}$ $\leftarrow$ Product of probabilities

c) Expected shots by P = $\boxed{\sum_{k=1}^{\infty} k\,p[(1-p)(1-q)]^{k-1}}$ $\leftarrow$ Sum of probabilities

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3. Conditional Probabilty.

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From the discussion in the lecture on February 23 we have:

$$P(BC) = \mathbf{Pr}\{\text{breast cancer in one's forties}\} = \frac{40}{10000}$$

$$P(+|BC) = \mathbf{Pr}\{\text{positive mammogram among breast cancer patients}\} = \frac{32}{40}$$

$$P(+) = \mathbf{Pr}\{\text{positive mammogram among all women}\} = \frac{1028}{10000}.$$

Use Bayes' theorem to find the conditional probability that a woman in her forties *does* have breast cancer, given that she had a *negative* mammogram?

BAYES' THM

$$\boxed{P(E|F) = \frac{P(E)P(F|E)}{P(F)}}$$

$$P(BC|+) = \frac{P(BC)P(+|BC)}{P(+)}$$

$$P(-) = 1 - P(+)$$

$$P(-|BC) = 1 - P(+|BC)$$

$$P(BC|-) = \frac{P(BC)P(-|BC)}{P(-)} = \frac{P(BC)(1 - P(+|BC))}{1 - P(+)}$$

$$= \frac{\frac{40}{10,000}\left(1 - \frac{32}{40}\right)}{1 - \frac{1028}{10,000}} = \frac{\frac{40}{10,000}\left(\frac{8}{40}\right)}{\frac{8,972}{10,000}}$$

$$= \frac{\cancel{40}}{\cancel{10,000}}\left(\frac{8}{\cancel{40}}\right)\left(\frac{\cancel{10,000}}{8,972}\right)$$

$$\boxed{P(BC|-) = \frac{8}{8,972}}$$

$\phi = \frac{1+\sqrt{5}}{2}$

$\hat{\phi} = \frac{1-\sqrt{5}}{2}$

4. **Linear Recurrences**

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Fill in the ten missing entries in the following table:

Annihilator	Growth Rate	Sample Recurrence
$(E+2)(E-4)$	$\Theta(4^n)$	$S_n = 2S_{n-1} + 8S_{N-2}$
$(E-5)$	$\Theta(5^n)$	$S_n = 5S_{n-1}$
$(E^2-1)(E-1)^3$	$\Theta(n^2)$	$S_n = S_{n-2} + n^2$
$(E-2)^2$	$\Theta(n2^n)$	$S_n = 4S_{n-1} \cancel{-} 2S_{n-2}$
does not work	$\Theta(3^{-n}) = \Theta(1/3^n)$	$S_n = \frac{1}{3S_{n-1}}$
$(E-\phi)(E-\hat{\phi})(E-1)^2$	$\Theta(F_n)$	$S_n = S_{n-1} + S_{n-2} + n$

$(E^2-1) = (E-1)(E+1)$

$E^2 - 4E + 2$

$E^2 - 2E + 1$

F_n is the Fibonnaci sequence

5. **Divide-and-Conquer Multiplication of Integers**

(a) On March 23 Professor Reingold presented an algorithm to multiply two n-digit numbers in time $T(n)$ defined by

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 3T(n/2) + Kn \end{aligned}$$

for some value of K and he showed that $T(n) = cn + \hat{c}n^{\lg 3} = \Theta(n^{\lg 3}) \approx \Theta(n^{1.57})$. The TA saw a way to modify the algorithm so that the time required would be described by

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 10T(n/4) + kn \end{aligned}$$

where k is much, much smaller than K. Reingold said that the TA's algorithm is inferior. Explain why.

(b) The TA then improved his algorithm so that it would only take time

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 25T(n/8) + 2^K n \end{aligned}$$

What was Reingold's reaction (and why)?

~~a) let $n = 4^i$ and $t_i = T(4^i)$~~

~~$t_i = 10t_{i-1} + k4^i$~~

~~$(E-4)(E-10)$~~

~~$T(n) = cn + \hat{c}n^{\log 10} \approx \Theta(n^{\log 10}) > \Theta(n^{\log 3})$~~

~~$n = 2^i, \quad t_i = T(2^i)$~~

~~$t_i = 10t_{i-2} + k2^i$~~

~~$(E-2)(E^2-10)$~~

~~The value of k has far less impact than the coefficient of the recurrence on the time complexity.~~

~~b) let $n = 2^i$ and $t_i = T(2^i)$~~

~~$t_i = 25t_{i-3} + 2^k 2^i = 25t_{i-3}$~~ $+ 2^{k+i}$

~~$(E-2)(E^3-25)$~~

~~$T(n) = cn +$~~

See back

a) $T(n) = 10T(n/4) + kn$

$\uparrow a \quad \uparrow b \quad \uparrow f(n)$

$$\frac{a f(n/b)}{f(n)} = \frac{\frac{10\cancel{kn}}{4}}{\cancel{kn}} = \frac{10}{4} = \frac{5}{2} > 1$$

MASTER THM: $T(n) = \Theta(n^{\log_4 10}) > \Theta(n^{\log 3})$

f(n) cancels out, so k does not impact the overall time complexity.

(However, the ~~a+~~ b vals of 10 + 4 increase complexity)

b) $T(n) = 25T(n/8) + 2^k n$

$\uparrow a \quad \uparrow b \quad \uparrow f(n)$

$$\frac{a f(n/b)}{f(n)} = \frac{\frac{25(\cancel{2^k n})}{8}}{\cancel{2^k n}} = \frac{25}{8} > 1$$

MASTER THM: $\boxed{T(n) = \Theta(n^{\log_8 25}) < \Theta(n^{\log 3})}$

The professor would approve of this algorithm, as it decreases the time complexity