HW1 Solution, CS330 Discrete Structures, Spring 2015

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1 Page.15 #32

1.1 (b)

$$p \leftrightarrow \neg p$$

p	$\neg p$	$p \leftrightarrow \neg p$
T	F	F
F	T	F

1.2 (d)

$$(p \wedge q) \rightarrow (p \vee q)$$

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

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Prove or disprove: $\lceil \lceil x/2 \rceil /2 \rceil = \lceil x/4 \rceil$ for all real numbers x.

Proposition 2.1. *The above statement is true.*

Proof. Since x is a real number, let $x = 4n + m$ where $n \in \mathbb{Z}$ and $0 < m \leq 4$. Then, we have the following for the LHS of the equation:

$$\begin{aligned}\lceil \lceil x/2 \rceil /2 \rceil &= \left\lceil \frac{\left\lceil \frac{4n+m}{2} \right\rceil}{2} \right\rceil \\ &= \left\lceil \frac{\left\lceil 2n + \frac{m}{2} \right\rceil}{2} \right\rceil\end{aligned}$$

Since $0 < m \leq 4$, $0 < m/2 \leq 2$. Then, we have the following two cases.

2.1 $0 < m \leq 2$

In this case, $0 < m/2 \leq 1$, and we have the following equations:

$$\begin{aligned}\left\lceil \frac{\left\lceil 2n + \frac{m}{2} \right\rceil}{2} \right\rceil &= \left\lceil \frac{2n+1}{2} \right\rceil \\ &= \left\lceil n + \frac{1}{2} \right\rceil \\ &= n+1\end{aligned}$$

2.2 $2 < m \leq 4$

In this case, $1 < m/2 \leq 2$, and we have the following equations:

$$\begin{aligned}\left\lceil \frac{\left\lceil 2n + \frac{m}{2} \right\rceil}{2} \right\rceil &= \left\lceil \frac{2n+2}{2} \right\rceil \\ &= \lceil n + 1 \rceil \\ &= n+1\end{aligned}$$

In both cases, LHS is equal to $n+1$. Next we look at the RHS of the equation.

$$\begin{aligned}\left\lceil \frac{x}{4} \right\rceil &= \left\lceil \frac{4n+m}{4} \right\rceil \\ &= \left\lceil n + \frac{m}{4} \right\rceil \\ &= n+1\end{aligned}$$

In conclusion, LHS=RHS. $\square$

3 Given an input sequence $v_1, v_2, \cdots, v_n$ of yes/no votes, describe an algorithm to determine the majority vote. Determine how much time your algorithm takes in the best and worst cases, expressing your answer with the big-theta notation.

The time complexity of this algorithm is $\Theta(n)$ for both best and worst cases.

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No. We can find a counterexample. If $f(x) = 3n, g(x) = n$, we have $f(x) = O(g(x))$. However, $2^{f(x)} = 2^{3n} = 8^n$ while $2^{g(x)} = 2^n$. In this case, $2^{f(x)} \neq O(2^{g(x)})$.

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We prove by induction. (The range of n is not mentioned in the book, so we assume $n \in \mathbb{Z}^+$.)

5.1 Base case

When $n = 1$, LHS $= H_1 = 1$ and RHS $= (1+1)H_1 - 1 = 1$. LHS=RHS is true in the base case.

```
y := 0;
n := 0;
for i = 1 → n do
    if v_i = yes then
        y ← y + 1
    end
    else
        n ← n + 1
    end
end
if y > n then
    Print 'Majority vote is yes'
end
else if y < n then
    Print 'Majority vote is no'
end
else
    Print 'yes and no are tied'
end
```

5.2 Inductive Hypothesis

Assume that when $n = k$, LHS=RHS for some integer k.

5.3 Inductive Step and its Proof

When $n = k + 1$, we have the followings:

$$\begin{aligned} LHS &= H_1 + H_2 + \cdots + H_k + H_{k+1} \\ &= (k+1)H_k - k + H_{k+1} \end{aligned}$$

Also, we have the following from the definition of the Harmonic Numbers:

$$H_n = H_{n-1} + \frac{1}{n} \Rightarrow H_{n-1} = H_n - \frac{1}{n}$$

Then, we further have the following equations:

$$\begin{aligned} LHS &= (k+1)H_k - k + H_{k+1} \\ &= (k+1)(H_{k+1} - \frac{1}{k+1}) - k + H_{k+1} \\ &= (k+1)H_{k+1} - (k+1) + H_{k+1} \\ &= (k+2)H_{k+1} - (k+1) \end{aligned}$$

When $n = k + 1$, we find the RHS is:

$$RHS = (k+2)H_{k+1} - (k+1)$$

That is, RHS=LHS.

5.4 Conclusion

If RHS=LHS is true when $n = k$ for some integer k, RHS=LHS also holds when $n = k + 1$. Combined with the base case where RHS=LHS when $n = 1$, we can conclude that RHS=LHS for every integer $n \geq 1$.

6 Bob and Carol are each secretly assigned consecutive positive integers; they each know their own number and that the numbers are consecutive, but they do not know each other's number. They are told to sit in a room with a clock that chimes every hour. They cannot communicate in any way, but are told to wait in the room until they can deduce the other's number and then leave the room at the next chime of the clock. Prove by induction that the person with the smaller number, n, will leave the room after the n-th strike of the clock. (Hint : Reason as in the "hat problem" of Lecture 3, January 21.)

6.1 Base case

When $n = 1$, since the person with 1 knows the other one must have 2 (the only consecutive positive number is 2), she/he will leave the room right after the first strike of the clock.

6.2 Inductive Hypothesis

When $n = k$ for some integer k (i.e., the smaller number is k), assume the person with the smaller number n will leave the room after the n-th strike of the clock.

6.3 Inductive Step and its Proof

When $n = k+1$ (i.e., the smaller number is $k+1$), at the k-th strike, the person with the smaller number $n = k + 1$ sees that no one left the room before, which means no one has the number less than $k + 1$. Then, he knows right away that he has the smaller number and the other person's number is $k+2$. Then, he leaves the room after the $k + 1$-th strike.

6.4 Conclusion

Combining the base case and the inductive step, we can conclude that the person with the smaller number n will leave the room after the n-th strike of the clock.