# CS539 HW1

Christopher Marcom

① Since $C > c_i$ where $1 \le c_i \le 4C$ for all nodes, we can infer that since $C$ is "a large constant," Each node $u$ will gain $k(C - c_i)$ where any spanning tree has a load vector of $v = (v_1, v_2, \ldots, v_n)$ where $\forall v_i \in V = k_i(C - c_i)$, to maximize revenue. That is: any action where node $u_i$ disconnects an edge decreases profits and any action adding an edge increases profit for $u_i$ by $(C - c_i)$.
hence any graph that is a mesh topology would be a Nash equilibrium.
So, for 5 nodes, a mesh topology would be profit of $5(C - c_i)$ for all nodes, which is ideal since we assume $C$ is at least $k$ times larger than $c_i$.

② For first-price auctions: (assume $v_1 > v_2 > v_3 > \ldots > v_n$)
Since all players $p_i$ can bid any amount $b_i \in [0, v_i]$, nash equilibrium occurs when $p_1$ bids any value $b_1 \in [v_2, v_1)$. and all other players $p_i \mid i \ge 2$ bid $b_i \in (0, v_i) \; \forall i \ge 2$.
$p_1$ will always be at the top since $p_1$ can bid any value over $v_2$ & will win.
↑ this condition constitutes all possible Nash equilibriums. ↑ hence $p_1$'s "special place"

For second price auctions:
A pure Nash Equilibrium $(E)$ for this game occurs when:

$$E = \begin{cases} \text{if } P_1 \text{ bids } b_1 \in [b_2, \infty) \text{ and } b_2 \le P_1\text{'s value } v_1 \text{ and } P_3 \text{ bids } b_3 \in [0, v_2]. \\ \text{if } P_2 \text{ bids } b_2 \in [b_1, \infty) \text{ and } b_1 \le P_2\text{'s value } v_2 \text{ and } P_3 \text{ bids } b_3 \in [0, v_2]. \\ \text{if } P_3 \text{ bids } b_3 \in [b_1, \infty) \cap [b_2, \infty) \text{ \& } b_1 \text{ \& } b_2 \le P_3\text{'s value } v_3. \end{cases}$$

This is because we can characterize the total gain for each case as the difference in truthful bids $\le v_i \in P$ such that each $P_i$ has a value $v_i$ which is the max one can bid

(3) Given: $N$ consumers assigned to $P$ providers || Find $(B_i = B)\ \forall$ base stations

$T_{rate} \rightarrow R_i = \frac{B_i}{n_i}$

→ Assume that the load for each customer $= l_i$. Rate of transmission:
$R_i = B_i / \sum l_i$ for provider $i$. Hence, we need to find a balanced load vector $\langle B_i \rangle$ to find pure equilibrium.

– Similar to question 1, this game is load balanced, meaning for $M_i$ base stations in provider $i$, $B_i = \sum B_{M_i}$. For $P$ providers, pure Equilibrium is reached when $N$ consumers are evenly distributed across providers such that

$$R_1 \leq R_2 \leq \cdots \leq R_p \quad \text{where } B_i = B \text{ for all providers}$$

where the largest difference between $R_{p-1}$ & $R_p = \frac{B_p}{n_p} - \frac{B_{p-1}}{n_{p-1}}$.

– Should a customer decide to change providers, we can prove that in the general case, the system will converge to a ~~pure~~ Nash.
A customer $C$ will move from provider $j$ to $k$ (if $j<k$) if provider $k$ has a better $R_i$ value. The lexicon of load vectors expressed as:

$$V = \{\, R_1, R_2, \ldots, R_p \} \quad \text{will continue to decrease}$$

as
$$R_j = \frac{B_j}{n_j} = \frac{B_j}{\sum_{j \in P_j} l_j} < R_k = \frac{B_k}{n_{k+1}} = \frac{B_k}{1 + \sum_{k \in P_k} l_k}$$

this implies that the vector $V$ can only decrease as $j < k$ so

$$V_{initial} \leq V_{final}$$

where
$$R_j \in V_{initial} < R_j \in V_{final} \quad \text{and}$$
$$R_k \in V_{initial} \geq R_k \in V_{final}$$

since $j<k \in V$.

Meaning $V$ will converge to a Nash Equilibrium.

(4) Tragedy of Commons Game: ($x_i$ - units of traffic for Player $i \in N$)

Given:

$$u_i = x_i K - \alpha \sum_{j \in N} x_j^2 = x_i K - \alpha(x_1^2 + x_2^2 + \dots x_j^2)$$

Players increase traffic until Marginal utility $(U_i) \leq 0$ : $(i \leq j)$

$$\frac{\partial u_i}{\partial x_i} = U_i = K - \alpha(2x_i) + 0 \text{ since } \frac{\partial U}{\partial x_i}\left(\sum_{j \neq i} x_j^2\right) = 0$$

so player $i$ increases traffic as long as:

$$K - 2\alpha x_i \geq 0 \quad \text{so} \quad x_i \leq \frac{K}{2\alpha}$$

Since the game is symmetrics each player $x_i$ $\forall i \in N$ has a Nash Equilebrium such that:

$$x_i \leq \frac{K}{2\alpha} \quad \forall i \text{ at Nash Equilibrium}$$

(5) Cournot Game w/ $P_1$ & $P_2$: (Note: game is symmetric for $P_1$ & $P_2$)

let: $u_i = x_i \cdot p(x_1 + x_2) - c_i$ : $c_i = x_i^2$, $P(y) = \max\{0, K - \sqrt{y}\}$.

↳ $u_i$ becomes: $u_i = x_i \cdot (K - \sqrt{x_1 + x_2}) - x_i^2$ (assume $K \geq \sqrt{y}$)

Marginal Utility $(U_i)$ for each Player $i$ $\Rightarrow \frac{\partial u_i}{\partial x_i} = U_i = K - \sqrt{x_1 + x_2} + \frac{x_i}{2\sqrt{x_1 + x_2}} - 2x_i$

if $x_1 = x_2$ (since the game is symmetric) : $U_i = K - \sqrt{2x_i} + \frac{x_i}{2\sqrt{2x_i}} - 2x_i$

since $K \geq \sqrt{x_1 + x_2}$, let $K - \sqrt{2x_i} = A$. set $U_i = 0$

$$0 = A + \frac{\sqrt{2x_i}}{4} - 2x_i, \quad \text{using quadratic rules:}$$

$$0 = 8x_i - \sqrt{2}x_i - 4A \longrightarrow x_i = \frac{\sqrt{2} \pm \sqrt{2 - 128A}}{16}$$

(sub: $A = K - \sqrt{2x_i}$)

$$\text{So: } x_i = \frac{\sqrt{2} \pm \sqrt{2 - 128(K - \sqrt{2x_i})}}{16} \quad \forall i \text{ at Nash Equilibrium}$$