Game Theory: Algorithms and Applications
CS 539

Spring 2018
HomeWork 4 Solutions

1. (a) Compute the Pigou bound for linear functions of the form $c_e(x) = a_e x + b_e, a, b \geq 0$ which form a class $\mathcal{C}$. Let the Pigou bound be

$$\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{x,r \geq 0} \frac{rc(r)}{xc(x) + (r-x)c(r)}$$

(b) Show that the Pigou bound for nondecreasing, nonnegative, concave functions is at most 4/3.

Solution: Pigou bound defined as

$$\alpha(c) = \sup_{c} \sup_{x,r \geq 0} \gamma_c(r,x) \text{ where } \gamma_c(r,x) = \frac{rc(r)}{xc(x) + (c-x)c(r)}$$

Non-negativity of $c \implies$ if $r \geq x$, $c(r) \geq c(x)$

Concavity $\implies \forall r, x \in [0, \infty)$ we have

$$(\lambda r + (1-\lambda)x) \geq \lambda c(r) + (1-\lambda)c(x) \tag{1}$$

Let $f \in \mathcal{C}$ s.t. $\gamma_f(r,x) > 1$ where $\exists r, x \geq 0$.

Claim 1. *If $r, x \geq 0$ maximizes $\gamma_f(r,x)$ then $r > x > 0$*

Proof. At $x = 0$, $\gamma_f(r,x) = \frac{rf(r)}{rf(r)} = 1$. Suppose that $\gamma_f(r,x) > 1$ when $x > 1$. Then, $rf(r) > xf(x) + rf(r) - xf(r) \implies 0 > x(f(x) - f(r))$ which is not possible since f is non-decreasing. Therefore, $r > x > 0$ □

Let $r > x > 0$ and $c(r) \neq 0$. Now, set $a = r$ and $b = 0$. Since $\frac{x}{r} \leq 1$, we define $\lambda = \frac{x}{r}$ and then plug these values into the definition of *concavity* $(f(\lambda a + (1-\lambda)b) \geq \lambda f(a) + (1-\lambda)f(b))$

$$c(x) \geq \frac{x}{r}c(r) + \frac{r-x}{r}c(0) \geq \frac{x}{r}c(r) \implies 1 - \frac{c(x)}{c(r)} \leq 1 - \frac{x}{r}$$

$$\implies \frac{x}{r}(1 - \frac{c(x)}{c(r)}) \leq \frac{x}{r}(1 - \frac{x}{r}) \implies \frac{1}{1 - \frac{x}{r}(1 - \frac{c(x)}{c(r)})} \leq \frac{1}{1 - \frac{x}{r}(1 - \frac{x}{r})}$$

$$\implies \frac{rc(r)}{xc(x) + (r-x)c(r)} \leq \sup_{r > x > 0} \frac{1}{1 - \frac{x}{r}(1 - \frac{x}{r})}$$

$$\implies \gamma_f(r,x) \leq \sup_{1 > y > 0} \frac{1}{1 - y(1-y)} \text{ where } y = \frac{x}{r}$$

Let g(y) = y(1-y). Then,

$$\sup_{1>y>0} \frac{1}{1-g(y)} = \frac{1}{1-\sup_{1>y>0} g(y)}$$

Deriving and setting it equal to 0,

$$0 = \frac{\partial g(y)}{\partial y} = 1-2y \implies y = \frac{1}{2} \implies g(y) = \frac{1}{4}$$

$$\implies \alpha(c) = \sup_{1>y>0} \frac{1}{1-g(y)} \leq \frac{1}{1-\frac{1}{4}} \leq \frac{4}{3}$$

This answers the question for 1(b).

Since linear functions fit the description of concave functions, $\alpha(c) \leq \frac{4}{3}$ for linear functions. Lastly, let $c(x) = x, r = 2$ and $x = 1$. Then, $\gamma_c(r,x) = \frac{4}{3}$

$$\frac{4}{3} \leq \sup_{x,r\geq 0} \frac{rc(r)}{xc(x)+(c-x)c(r)}$$

Thus, $\alpha(c) \geq \frac{4}{3}$ and $\alpha(c) \leq \frac{4}{3} \implies \alpha(c) = \frac{4}{3}$ (tight)

2. Use the following figure to show that Nash equilibria need not exist in Atomic Congestion Games. Use two players between s and t, with requirement 1 and 2 units.

Solution: The following matrix shows that a stable point doesn't exist $\implies$ a pure NE doesn't exist

	$\{s,t\}$	$\{s,v,t\}$	$\{s,w,t\}$	$\{s,v,w,t\}$
$\{s,t\}$	$141, 282$	$47, 120$	$47, 122$	$47, 124$
$\{s,v,t\}$	$48, 188$	$80, 160$	$48, 122$	$72, 154$
$\{s,w,t\}$	$47, 188$	$47, 120$	$75, 150$	$73, 150$
$\{s,v,w,t\}$	$22, 188$	$46, 150$	$48, 148$	$120, 240$

3. Let (G, K, c) be an atomic instance with affine cost functions where G defines the network, K the set of source-sink pairs and c the cost function. Show that (G, K, c) admits at least one equilibrium flow. Use the following potential function

$$\Phi(f) = \sum_{e\in E}(c_e(f_e)f_e + \sum_{i\in P} c_e(r_i)r_i)$$

where P is the set of players that choose a path that includes e. Note the difference w.r.t. the theorem proved in class. That was specific to all flow requirements being the same value R.

Solution: Potential function: $\phi(f) = \sum_{e \in E}[c_e(f_e) + \sum_{i \in Paths} c_e(r_i) r_i]$. Let flow f be a global optimum for $\phi(f)$.

Claim 2. *f is an equilibrium flow.*

Proof. Assume that a player i (say) could strictly decrease her cost by deviating from path p to p', yielding a new flow f'. Then

$$0 > c_{p'}(f') - c_p(f) = [\sum_{e \in p'_i \setminus p} c_e(f_e + r_i) - \sum_{e \in p_i \setminus p'} c_e(f_e)]$$

Consider the impact of player i's deviation on the potential function $\phi()$,

$$\begin{aligned}
\phi(f') - \phi(f) = & \sum_{e \in p' \setminus p} [c_e(f_e + r_i)(f_e + r_i) + \sum_{j: e \in p_j} c_e(r_j) r_j + c_e(r_i) r_i] \\
& + \sum_{e \in p \setminus p'} [c_e(f_e - r_i)(f_e - r_i) + \sum_{j: e \in p_j} c_e(r_j) r_j - c_e(r_j) r_j] \\
& - \sum_{e \in p' \setminus p} [c_e(f_e) f_e + \sum_{j: e \in p_j} c_e(r_j) r_j] \\
& - \sum_{e \in p \setminus p'} [c_e(f_e) f_e + \sum_{j: e \in p_j} c_e(r_j) r_j] \\
= & \sum_{e \in p' \setminus p} [c_e(f_e + r_i)(f_e + r_i) - c_e(f_e) f_e + c_e(r_i) r_i] \\
& + \sum_{e \in p \setminus p'} [c_e(f_e - r_i)(f_e - r_i) - c_e(f_e) f_e - c_e(r_i) r_i]
\end{aligned}$$

Noting the affine characteristic of the cost function $c_e = a_e x + b_e$, the above equation becomes

$$\begin{aligned}
&= 2r_i \sum_{e \in p' \setminus p} (a_e r_i + a_e f_e + b_e) - 2r_i \sum_{e \in p \setminus p'} (a_e f_e + b_e) \\
&= 2r_i (\sum_{e \in p' \setminus p} c_e(f_e + r_i) - \sum_{e \in p \setminus p'} c_e(f_e)) \\
&= 2r_i (c_{p'} - c_p(f) < 0
\end{aligned}$$

$\implies$ $\phi(f')$ is strictly less than $\phi(f)$, which contradicts the fact that f yields the global optimum for the potential function. $\square$

4. (a) In the local connection game where a node incurs cost $\alpha d_u + \sum_v dist(u, v)$, where d_u is the degree of node u and $dist(u, v)$ the shortest distance from u to v, show that if $\alpha > n^2$ then all Nash Equilibria are trees. Show that the *Price of Anarchy* is bounded by a constant.
(b) Show that the Price of stability is bounded by 4/3.

Solution: Suppose there exists an NE graph which is not a tree. Then, it must contain at least one cycle. Then, breaking the cycle must increase the cost of node u (a node who loses an incident edge).

$$\implies \alpha d_u^c + \sum_v dist^c(u,v) < \alpha d_u^b + \sum_v dist^b(u,v)$$

where

- d_u^c : degree of node u when it is a part of a cycle
- d_u^b : degree of node u when the cycle is broken
- $dist^c$: distance function when a cycle exists
- $dist^b$: distance function when the cycle is broken

$$\implies \alpha(d_u^c - d_u^b) < \sum_v dist^b(u,v) - \sum_v dist^c(u,v)$$

Since $d_u^c - d_u^b = 1$

$$\alpha < \sum_v dist^b(u,v) - \sum_v dist^c(u,v)$$

But

$$\max_{\text{all subgraph constructions with } n \text{ nodes}} [\ \sum_v dist^b(u,v) - \sum_v dist^c(u,v)\ ]$$

(MAX = a path $P_n -$ a cycle C_n)

$$= [1 + 2 + \cdots + (n-1)] - [2(1 + 2 + \cdots + (\frac{n-1}{2}))] = \frac{(n-1)^2}{2}$$

$\implies$ the above inequality is not possible, since $\alpha > n^2 \implies$ our assumption is wrong.

Since all NE are trees, the upper bound of the cost of a tree provides us with a worst NE cost. Note that the local connection game defines that the edge is paid for by only one of its two endpoint nodes.

The maximum cost incurred is given by a "path" P_n

$$\implies worstNEcost \le \alpha(n-1) + \sum_{u,v \in V} dist(u,v) = n(n^2-1) + \alpha(n-1)$$

The opt. social cost is obtained by minimizing the social cost function

$$\implies Optcost \ge \alpha(n-1) + 2 \sum_{\{u,v\}:dist(u,v)=1} 1 + 2 \sum_{\{u,v\}:dist(u,v)\ge 2} 2$$

$$= \alpha(n-1) + 2m + 4(\frac{n(n-1)}{2} - m)$$

Let us use $m = n-1$ to get a lower bound. Then,

$$Optcost \ge (\alpha + 2 + 4(\frac{n}{2} - 1))(n-1) \ge (n^2 + 2n - 2)(n-1)$$

Thus,

$$PoA \leq \frac{worstNEcost}{Optcost} \leq \frac{(\alpha + n(n+1))(n-1)}{(\alpha + 2n - 2)(n-1)} = \frac{\alpha + n^2 + n}{\alpha - 2 + 2n}$$

If $\alpha = n^2$ then $PoA \leq \frac{n^2+n^2+n}{n^2+2n-2} \approx 2$ as $n \to \infty$ and ≈ 1 as $\alpha \to \infty$. Therefore, $PoA \leq 2$, a constant.

5. Global Connection game:
(a) Show that the price of anarchy in the global connection game can never exceed k where k is the number of players.
(b) In the connection game, where players (having same source and sink) are to choose a flow path each from a set of flow paths, which are disjoint, and split the benefits if they happen to share a path. Suppose players have weights $w_i \geq 0$ and the benefit from sharing a path is proportional to the weights of the players, i.e. the gain from a path for player i is w_i/W where $W = \sum_j w_j$. Does Nash equilibrium exist? Can you design a potential function for this problem?

Solution:

(a) Let $c_i(NE)$ be the cost to player i at Nash Equilibrium on path p_i.
Let $c_i(SO)$ be the cost of path at social optimum along path p_i^*.
Let $c_e(S)$ be the cost of edge e to any player i at solution S.
Let $C(NE)$ and $C(SO)$ be the cost of the Nash and Social optimum solutions, respectively.
Then,

$$c_i(NE) \leq \sum_{e \in p_i^*} c_e(NE') \tag{2}$$

where NE' is the solution where the ith commodity changes path from p_i to p_i^*. Note that

$$\sum_{e \in p_i^*} c_e(NE') \leq \sum_{e \in p_i^*} c_e$$

Furthermore,

$$\sum_{e \in p_i^*} c_e \leq n_e(SO) \sum_{e \in p_i^*} c_e(SO) \leq k \sum_{e \in p_i^*} c_e(SO)$$

where $n_e(SO)$ is the number of players at Social Optimum on edge e.
Summing over all players we get

$$C(NE) \sum_i c_i(NE) \leq \sum_i \sum_{e \in p_i^*} c_e(NE') \leq k \sum_i \sum_{e \in p_i^*} c_e^i(SO)) \leq kC(SO)$$

The price of anarchy bound follows.

(b) Let $\phi = \sum_{p \in paths} (W_p)^2$ where $W_p = \sum_{j \in W_p} w_j$. If a player with weight i switches links to improve benefits, the change is

$$\phi_{new} - \phi_{old} = W_{old}^2 + (W_{new} + w_i)^2 - (W_{old} + w_i)^2 + W_{new}^2 = 2w_i(W_{new} - W_{old}) < 0$$

This is because $W_{new} > W_{old}$ (i wouldn't switch otherwise). Lastly, we know that ϕ is lower bounded at 0 and hence, the potential function cannot keep decreasing infinitely $\implies$ NE exists.