HW3 Solution, CS330 Discrete Structures, Spring 2015

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February 25, 2015

1 Tzolkin Birthday Problem

For simplicity, denote the probability that no one has the same birthday when there are n people as $p(n)$. Then, what we want is

$$1 - p(n) \geq 0.5 \Rightarrow p(n) \leq 0.5$$

$$\begin{aligned}
\Rightarrow p(n) &= \frac{\text{\# of ways to have different birthday for everyone}}{\text{\# of ways of total assignments}} \\
&= \frac{\frac{260!}{(260-n)!}}{260^n} \\
&= 1 \cdot \frac{259}{260} \cdot \frac{258}{260} \cdot \dots \cdot \frac{260-n+2}{260} \cdot \frac{260-n+1}{260} \leq 0.5
\end{aligned}$$

Solving the above inequality leads to $n \geq 20$. Therefore we need 20 people.

2 Page.467 #30

Not containing a 0 essentially means that every bit is 1.

a) $\frac{1}{2^{10}}$

b) $(0.6)^{10}$

c) $\frac{1}{2} \cdot \frac{1}{2^2} \cdot \frac{1}{2^3} \cdots \frac{1}{2^{10}} = \prod_{i=1}^{10} \frac{1}{2^i} = \frac{1}{2^{\sum_{i=1}^{10} i}} = \frac{1}{2^{10 \cdot \frac{1+10}{2}}} = \frac{1}{2^{55}}$

3 Page.477 #22

a) The total arrived messages is $s + h$, therefore,

$$p(S) = \frac{s}{s+h}, p(\bar{S}) = \frac{h}{s+h}$$

b) If we denote the event that an incoming message contains w as W and the event that the incoming message is spam as S, the final probability is:

$$p(S|W) = \frac{p(W|S) \cdot p(S)}{p(W)} = \frac{p(W|S) \cdot p(S)}{p(W|S) \cdot p(S) + p(W|\bar{S}) \cdot p(\bar{S})} = \frac{p(w)\frac{s}{s+h}}{p(w)\frac{s}{s+h} + q(w) \cdot \frac{h}{s+h}} = \frac{p(w)s}{p(w)s + q(w)h}$$

4 Bulgarian Lottery

a) $\binom{49}{6}$, because the order does not matter.

b.i) $\left(\frac{1}{\binom{49}{6}}\right)^2$, because the chance we have one specific set of 6 numbers is $\frac{1}{\binom{49}{6}}$, and it happened twice.

b.ii) For this, we can first calculate the probability that every drawing is unique (*i.e.,* no identical combinations) out of n combinations, which is:

$$\frac{\binom{49}{6}}{\binom{49}{6}} \cdot \frac{\binom{49}{6}-1}{\binom{49}{6}} \cdot \frac{\binom{49}{6}-2}{\binom{49}{6}} \cdots \frac{\binom{49}{6}-n+1}{\binom{49}{6}} = \frac{\left(\frac{\binom{49}{6}!}{(\binom{49}{6}-n)!}\right)}{\binom{49}{6}^n}$$

Then, the probability that some pair have been the same for n drawings is

$$1 - \frac{\left(\frac{\binom{49}{6}!}{(\binom{49}{6}-n)!}\right)}{\binom{49}{6}^n}$$

which must be about 50%. With this equation, n can be solved easily by a program.

b.iii) Following the same theory as b.ii), we can first calculate the probability that every adjacent drawing is different for n drawings, which is:

$$\frac{\binom{49}{6}}{\binom{49}{6}} \cdot \frac{\binom{49}{6}-1}{\binom{49}{6}} \cdot \frac{\binom{49}{6}-1}{\binom{49}{6}} \cdot \frac{\binom{49}{6}-1}{\binom{49}{6}-1} \cdots = \frac{\binom{49}{6} \cdot (\binom{49}{6}-1)^{n-1}}{\binom{49}{6}^n} = \left(1 - \frac{1}{\binom{49}{6}}\right)^{n-1}$$

because every drawing only excludes one combination (the one appeared in the previous drawing). We assumed n is even without the loss of generality. Then, the probability that some adjacent pair are same is:

$$1 - \left(1 - \frac{1}{\binom{49}{6}}\right)^{n-1}$$

which must be about 50%. This equation can be solved easily by hand, and n is $9,692,844$.

5 Professor's joke

a) Since the coins are fair, the probability of showing head is $\frac{1}{2}$ and the one of showing tail is also $\frac{1}{2}$. Then, the probability of student i's coin showing head and $n-1$ showing tails is

$$\frac{1}{2}(\frac{1}{2})^{n-1} = \frac{1}{2^n}$$

The probability of student i's coin showing tail and $n-1$ showing heads is also $\frac{1}{2^n}$ due to the same reason. Then, according to the rule of sum, the probability of an odd toss happening at some student i's coin is

$$\frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}}$$

Since there are n students, the probability of an odd toss for any student is (according to the rule of sum)

$$n\frac{1}{2^{n-1}} = \frac{n}{2^{n-1}}$$

Therefore, $p = \frac{n}{2^{n-1}}$.

Also, ending on the k-th round means there were more than one odd toss in the previous $k-1$ rounds and one odd toss appeared at the k-th round, whose probability is:

$$(1-p)^{k-1}p$$

Therefore, the expected number of the rounds is:

$$\sum_{i=1}^{\infty} i \cdot (1-p)^{i-1}p = p\sum_{i=1}^{\infty} i(1-p)^{i-1} = p \cdot \frac{1}{(1-(1-p))^2} = \frac{1}{p} = \frac{2^{n-1}}{n}$$

b) Suppose the probability that the cheating student's coin shows head is $p(x)$. Then, the false coin shows tail by a probability $1-p(x)$. Then, in this case, the probability that a student i's (including cheating student) coin is an odd toss is

$$\frac{1}{2^{n-1}}p(x) + \frac{1}{2^{n-1}}(1-p(x)) = \frac{1}{2^{n-1}}$$

Therefore, the probability p remains same, and thus the answer does not change.

6 Couple and Child Plan

a) First of all, they must have at least two children to stop, which is obvious. The probability that the first child is a boy and then they stop having children after having the k-th child is

$$p^{k-1}(1-p)$$

The probability that the first child is a girl and then they stop having children after having the k-th child is

$$(1-p)^{k-1}p$$

Then, the probability that they will stop having children after having the k-th child is

$$p^{k-1}(1-p) + (1-p)^{k-1}p$$

which leads to the expected number of children

$$\begin{aligned}
&\sum_{i=2}^{\infty} i(p^{i-1}(1-p) + (1-p)^{i-1}p) \\
=&\sum_{i=2}^{\infty} ip^{i-1}(1-p) + \sum_{i=2}^{\infty} i(1-p)^{i-1}p \\
=&(1-p)\sum_{i=2}^{\infty} ip^{i-1} + p\sum_{i=2}^{\infty} i(1-p)^{i-1} \\
=&(1-p)(\frac{1}{(1-p)^2} - 1) + p(\frac{1}{p^2} - 1) \\
=&\frac{1}{1-p} - (1-p) + \frac{1}{p} - p = \frac{p+1-p}{p(1-p)} - 1 = \frac{1-p+p^2}{p(1-p)}
\end{aligned}$$

If $p = 0$ or $p = 1$, they will never stop since they will only have same sex children, therefore our answer does not make sense because we implicitly made an assumption that their k-th child will have a different sex.

b) Again, they must have at least two children to stop. The probability that the first child is a boy and then they stop having children after the k-th child is

$$p(1-p)^{k-2}p$$

The probability that the first child is a girl and then they stop having children after the k-th child is

$$(1-p)p^{k-2}(1-p)$$

Then, the total probability that they stop having children after the k-th child is

$$p^2(1-p)^{k-2} + (1-p)^2p^{k-2}$$

which leads to the expected number of children

$$\begin{aligned}
&\sum_{i=2}^{\infty} i(p^2(1-p)^{i-2} + (1-p)^2p^{i-2}) \\
=&\sum_{i=2}^{\infty} i \cdot p^2(1-p)^{i-2} + \sum_{i=2}^{\infty} i(1-p)^2p^{i-2} \\
=&\frac{p^2}{1-p}\sum_{i=2}^{\infty} i(1-p)^{i-1} + \frac{(1-p)^2}{p}\sum_{i=2}^{\infty} ip^{i-1} \\
=&\frac{p^2}{1-p}(\frac{1}{p^2} - 1) + \frac{(1-p)^2}{p}(\frac{1}{(1-p)^2} - 1) \\
=&\frac{1}{1-p} - \frac{p^2}{1-p} + \frac{1}{p} - \frac{(1-p)^2}{p} \\
=&\frac{1-p^3-(1-p)^3}{p(1-p)} \\
=&\frac{1-p^3-(1-3p+3p^2-p^3)}{p-p^3} \\
=&\frac{3p-3p^3}{p-p^3} = 3
\end{aligned}$$

This answer does not make sense either when $p=0$ or $p=1$ because when $k=2$, there is a term 0^0 in the $p^2(1-p)^{k-2} + (1-p)^2p^{k-2}$. 0^0 is not a valid number.

If $p=0$ or $p=1$, they will have only two children.