Game Theory: Algorithms and Applications

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What is Game Theory?

Multi-agent Decision Theory:

- Classical Usage: Economics (auction theory), political science, biology to study
  Competition and Cooperation (Collusion)
  Role of Taxes/regulations etc.
- Zero-Sum games model Pursuit-evasion games
- Recent interest: Network Systems/Inefficiency of queuing systems
  Online advertising: Sponsored Search Auctions/ Spectrum Auctions
  Spread of influence and beliefs in Social Networks

Elements of game theory. The simplest form is that of a strategic game.

A **strategic game** consists of

1. a set of n **players**, $\mathcal{N}$
2. for each player, a set of **actions**, $\mathcal{A}$
3. for each player, **preferences** over the *outcomes* of joint actions, i.e. over $A = A_1 \times A_2 \dots A_n$

A preference relation $\succeq$ is used to represent preferences.

It is more useful to express preference relation by cardinal information.

For every player, p , outcomes are mapped to numbers that represent the value of the outcome.

$u_i(x)$ is the utility of player i depending on x, a vector representing the action of all the players

Example: Prisoners Dilemna

- Two suspects in a major crime are held in separate cells.
- Enough evidence to convict each of them of a minor offense.
- Not enough evidence to convict either of them of the major crime unless one of them acts as an informer against the other(*Confess*).
- If they both stay *quiet* then each will be convicted of the minor offense and spend one year in prison.
  If one and only one of them confess, she will be freed and used as a witness against the other, who will spend four years in prison.
  If they both confess then each will spend three years in prison.

Example: Prisoners Dilemma-Game Model

We can represent the suspects' preference orderings by using a payoff functions in a table as follows.

Table : Prisoner's Dilemma

Payoff	Quiet	Confess
Quiet	-1,-1	-4,0
Confess	0,-4	-3,-3

What would they do?

Example: Matching Pennies

Two players choose sides of a coin each. The resulting game is

Table : Matching Pennies

Payoff	Heads	Tail
Heads	1,-1	-1,1
Tail	-1,1	1,-1

Is there a stable point?

Stable Strategies

- Each Player attempts to choose a strategy which is "best" given that the other players strategies are fixed.
- Is there a strategy profile (strategy of all the players) that is stable, i.e. no player wishes to deviate from its chosen strategy?
- The simultaneous choice of best strategies by every player constitutes **Nash Equilibrium**

Nash Equilibrium and Best Response

Mathematically, $(N, (A_i), u_i)$ where

- $N, (A_i), u_i$ represents a set of players,
- action(strategy) set of player i,
- utility function $u_i : A_1 \times A_2 \times ... \times A_N \to R$.

Definition

A Strategic game has a Nash equilibrium if there is a strategy $(a_1^*, a_2^*, ..., a_N^*)$ s.t. $\forall i, a_i(\in A_i) : u_i(a_1^*, a_2^*, ..., a_i^*, ..., a_N^*) \geq u_i(a_1^*, a_2^*, ..., a_i, ..., a_N^*) \Rightarrow \forall i, a_i(\in A_i) : u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*)$.
Denote by a_{-i}^* the strategies of the all players except player i.

Best Response Strategy

Let us define $B_i(a^*_{-i})$ as the best response to player i when other players has strategy a^*_{-i}. Then,

$$B_i(a^*_{-i}) = a^*_i$$

if and only if

$$\forall a_i \in A_i : u_i(a^*_i, a^*_{-i}) \geq u_i(a_i, a^*_{-i})$$

Definition

A Strategic game has a Nash equilibrium if there is a strategy $(a^*_1, a^*_2, ..., a^*_N)$ s.t. $\forall i : a^*_i = B_i(a^*_{-i})$

Dominant Strategy at Equilibrium

Definition

A strategy $a_i \in A_i$ is a **dominant strategy** for player i if

$$u_i(a_i, a_{-i}) \geq u_i(a', a_{-i}) \; \forall a' \in A_i, \; \forall a_{-i}$$

Definition

A strategy profile a^* is a **dominant strategy equilibrium** if for each player i, $a*_i$ is a dominant stratgey.

Dominant Strategy at Equilibrium

Definition

A strategy $a_i \in A_i$ is a **Weakly Dominated Strategy** for player i if $\exists a' \in A_i$ such that

$$u_i(a_i, a_{-i}) \leq u_i(a', a_{-i}) \quad \forall a_{-i}$$

and

$$u_i(a_i, a_{-i}) < u_i(a', a_{-i}) \quad \exists a_{-i}$$

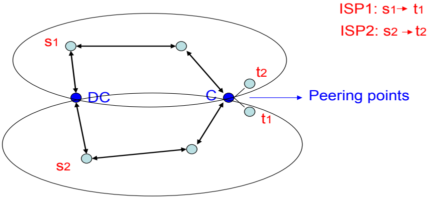
Definition

A strategy profile a_i is a **strictly dominated strategy** for player i, if $\exists a' \in A_i$ such that

$$u_i(a_i, a_{-i}) < u_i(a', a_{-i}) \quad \forall a_{-i}$$

Example: Routing Game

In this game there are two Broadband providers with their own network. ISP 1 has to route from $s1$ to $t1$ using either DC or C. Her cost is the route to either peering point. Each link costs 1 unit. Her selfish option is to route to DC-called Hot potato routing. A similar situation holds for ISP 2.



- If ISP 1 chooses DC and ISP 2 chooses DC then total cost incurred by both is 4 units.
- If ISP 1 and ISP 2 both choose C then the total cost incurred by both is 2 units.

Example: Routing Choice-Game Model

We can represent the routing choice by using a payoff functions in a table as follows.

Table : Routing Game

Payoff	HotPotato	Planned
HotPotato	-4,-4	-1,-5
Planned	-5,-1	-2,-2

What would they do?

Example: Load Balancing Game

4 players and two machines. Each player has a job . Player 1 and 2 have a job that requires 2 units of time, while player 3 and 4 have a job that requires 1 unit of time.
Possible Scenarios

- Player 1 and 3 choose machine 1. Player 2 and 4 choose machine 2.
- Player 1 and 2 choose machine 1. Player 3 and 4 choose machine 2.

What are the other choices. Set-up a game model and analyze it.

Existence and Efficiency of Nash Equilibrium: Illustrated by the load balancing game.

- n Players, P, and m Machines, M
  Each player has a job with weight w_i.
- The load on a machine j, $l_j(A) = \sum_{i \in A_j} w_i$ where A_j is the set of of jobs assigned to machine i in an assignment A and players to machines.
- The cost seen by a player i is $c_i(A) = l_j(A)$ where i is assigned to machine j, i.e. $A(i) = j$.
- An assignment $A : [n] \to [m]$ is at pure Nash equilibrium iff $\forall i \in [n], \forall k \in [m], c_i(A) \leq c_i(A')$ where A' differs in the assignment of player i.
  In effect this implies $c_i^{A(i)} \leq c_i^k$ where c_i^j is the cost of assigning i to j keeping all other player's strategies fixed.

Existence and Efficiency of Nash Equilibrium

Theorem

Every instance of the load balancing game has a pure Nash equilibrium

Proof.

Consider the sorted load vector of the machines $(\alpha_1, \alpha_2 \ldots \alpha_m)$. If an assignment is not at Nash Equilibrium, then this vector can only decrease lexicographically. Since the load vector is positive , this implies convergence to a load vector which cannot be improved. $\square$

Auction Games-Nash Equlibrium

Second Price Auction:

The Problem:

- One object O , n players in $\mathcal{P}$. Player i's valuation function V_i. Assume $V_1 > V_2 > \ldots > 0$. Assume that everybody knows all the valuations.
- Simultaneous bids, $b_1, \ldots b_n$. Winner is the player with the highest bid. Payment from player: Second highest Bid.
- Utility $U_i = V_i - b_i$, if i is the winner otherwise 0.

Second Price Auctions

Theorem

Truthful bidding, $b_i = V_i$ is a Nash equilibrium.

Proof.

- Player 1 has payoff $V_1 - V_2$. No need to deviate
- Other players will not deviate because they need to bid higher than V_1 to win and then they get negative utility.

$\square$

Other Nash Equilibria?
The strategy $(V_1, 0, 0, ..., 0)$ is also a Nash Equilibrium.
The strategy $(V_2, V_1, 0, 0, ..., 0)$ is also a Nash Equilibrium.

Second Price Auctions

The truthful equilibrium, $B_i = V_i, \forall i$, is a Weakly Dominant Nash Equilibrium
Consider any player i. If it is Player 1 then he wins the auction and and deviating is not going to increase his value. Similarly other players will not outbid him since they will not gain.

Tragedy of Commons-Sharing a common Bandwidth

Let us define a utility function U_i of $i-$th ISP as follows:

$$U_i = X_i - 5(X_1 + X_2 + ... + X_N)/N$$

where X_i is the unit of traffic sent by ISP i. Keep in mind that all ISP's desire to maximize their own utility. That means all ISP's send traffic until marginal utility ≥ 0. The marginal utility of $U_i = 1 - 5/N$. Thus, if $N \geq 5$ then all ISP's always send traffic. However, this situation has that any capacity will be destroyed. To prevent from being over capacitated , we can add taxes. After adding taxes, $U_i =$

$$X_i - TX_i - 5(X_1 + X_2 + ... + X_N)/N$$

The marginal utility of $U_i = 1 - T - 5/N$. For all ISP to keep increasing traffic, it is required that $1 - T - 5/N \geq 0 \Rightarrow T \leq 1 - 5/N$. Only if we set $T \leq 1 - 5/N$ then the capacity will be overcome.

A more realistic model:

$$U_i = X_i(1 - \sum X_j)$$

Cournot Games

- Games with infinite strategy sets
- Two ISP's (players) offering bandwidth
- Utility of player i

$$u_i(x_1, x_2) = x_i p(x_1 + x_2) - cx_i$$

where $p(r) = \max\{0, 2 - r\}$ is the price of good as a function of supply r and c is the unit cost of offering bandwidth.

- Best response of player i is given by

$$B_i(x_{-i}) \in \arg\max_{x \in S_i} u_i(x_i, x_{-i})$$

$$= \arg\max_{x_i \geq 0} (x_i(2 - x_1 - x_2) - cx_i)$$

$$= \begin{cases} \frac{(2-c)-x_{-i}}{2}, & \text{if } x_{-i} \leq 2 - c \\ 0, & \text{otherwise} \end{cases}$$

Cournot Games-Nash Equlibrium

- To find the Nash Equilibrium solve the equation

$$x_1^* = \frac{2 - c - x_2}{2}$$

Using the symmetry $x_1^* = x_2^*$ we get

$$x_1^* = \frac{2 - c}{3}$$

- The price is $2 - r$ or $2 - 2(2 - c)/3$ or equivalently

$$P^* = \frac{2 + 2c}{3}$$

- What happens if the firms collude?

When Does (Pure) Nash Equilibrium Exist

Theorem

Kakutani's Fixed point theorem *Let X be a compact convex set in $\mathbb{R}^n$ and let $f : X \to X$ be a correspondence such that*

- *$f(x)$ is non-empty and convex*
- *$f(x)$ is upper-hemicontinuous, i.e. image of f is compact and has a closed graph (i.e. for all sequences $\{x_n \in X\}$ such that x_n converges to x and $\{y_n \in f(x_n)\}$, such that y_n converges to y , then $y \in f(x)$*

Then there exists a x^* such that $x^* \in f(x^*)$

compactness: for euclidean spaces, the subset is closed (contains all limit points or whose complement is an open set) and bounded (contained in a ball of finite radius)

Existence of Nash Equillibrium

Theorem

The strategic game $(N, (A_i), (u_i))$ has a Nash equilibrium if for all players i,

- *The actions A_i are each non-empty compact convex sets in n-dimensional Euclidean space.*
- *The utility function u_i is continuous and quasi-convave on A_i.*

Proof.

Consider the best response function: For each player i, the correspondence $B_i(a_i, a_{-i})$, to obtain $B(a) = \times_i B_i(a_{-i})$, i.e a correspondence $B : A \to A$.This correspondence satisfies the assumptions of Kakutani's theorem. The set $B_i(a_{-i})$ is nonempty (since $u_i()$ is continuous and A_i is compact) and is a convex set since u_i is a concave function. B has a closed graph since $u_i()$ is continuous. Thus a fixed point exists

$$a^* = B(a^*)$$

Example of Fixed point/Convex Optimization

- $$f(y) = y - (y - \sqrt(x))$$
  At the fixed point $f(\sqrt{x}) = \sqrt{x}$.
- $$f(x^*) = \max\{f(x) : x \in \mathcal{X}\}$$
  for some feasible set $\mathcal{X} \subset \mathbb{R}^n$ where $f(x) : \mathbb{R}^n \to \mathbb{R}$. The optimization problem is called a convex optimization problem if $\mathcal{X}$ is a convex set and $f(x)$ is a concave function defined on $\mathbb{R}^n$

- Concave function $f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y)$
- Quasi-concave
  A function $f()$ is quasi-concave if
  $f(\alpha y + (1-\alpha)x) \geq \min\{f(y), f(x)\}$

Rainbow Triangulation Problem

A triangulation of a triangle is a subdivision of the triangle into small triangles. A Sperner Labeling is a labeling of a triangulation of a triangle with the numbers 1,2 and 3 such that

(1) The three corners are labeled 0, 1 and 2.

(2) Every vertex on the line connected corner vertex i and corner vertex j is labeled i or j.

A rainbow triangle is an inside triangle which is labeled (0,1,2).

Lemma (Sperner's Lemma)

Every Sperner Labeling contains a rainbow triangle.

Sperner's Lemma

$$\begin{aligned} x &= \text{number of edges on the boundary colored } (0,1); \\ y &= \text{number of edges inside colored } (0,1); \\ Q &= \text{number of triangles colored } (0,0,1) \text{ or } (0,1,1); \\ R &= \text{number of rainbow triangles.} \end{aligned}$$

Lemma

The number of edges on the boundary line (0,1) of the triangle is odd.

Now we prove the number of rainbow triangles is odd. Each triangle of type Q gives two $(0,1)$ edges. Each triangle of type R gives one $(0,1)$ edge. Each edge inside colored $(0,1)$ is shared by two triangles. Then we can get

$$2Q + R = x + 2y$$

Since x is odd by claim, R is odd.