CS 539 HW 4

Christopher Marcom
A20385764

① Pigou Bound: $\alpha(C) = \sup_{c\in C} \sup_{x,r\geq 0} \underbrace{\frac{r\cdot c(r)}{x\,c(x) + (r-x)c(r)}}_{\text{let this be } \gamma_c(r,x)}$

constraints: non-negative: $r > x$, $c(r) \geq c(x)$

concavity definition: $\forall\, r, x \geq 0$, $\lambda r + (1-\lambda)x \geq \lambda c(r) + (1-\lambda)c(x)$

Let $f \in C$ where $\gamma_f(r,x) = \frac{r f(r)}{x f(x) + (r-x)f(r)} > 1 \;\; \forall\, r, x \geq 0$.

If $r \geq x \geq 0$ maximizes $\gamma_f(r,x)$, then $r > x > 0$.

Proof By contradiction:

if $x = 0$, then $\gamma_f = 1$.

claim $\boxed{\text{let } \gamma_f > 1 \text{ when } x > 1.}$ $\xrightarrow{\text{implies}}$ $r f(r) > x f(x) + (r-x)f(r)$

$\hookrightarrow = 0 > x f(x) - x f(r)$

$= 0 > x(f(x) - f(r))$

violates our claim since $f(\cdot)$ does not decrease, hence $\underline{r > x > 0}$ ①

if we assume ① for f s.t. f is concave, then:

$f(\lambda r + (1-\lambda)x) \geq \lambda f(r) + (1-\lambda)f(x)$

$\hookrightarrow f(x) \geq \frac{x}{r} f(r) + \frac{r-x}{r}\cdot f(0) \geq \frac{x}{r} f(r)$

$\hookrightarrow 1 - \frac{f(x)}{f(r)} \leq 1 - \frac{x}{r} \implies \underbrace{\frac{1}{1 - \frac{x}{r}\left(1 - \frac{f(x)}{f(r)}\right)}} \leq \underbrace{\frac{1}{1 - \frac{x}{r}\left(1 - \frac{x}{r}\right)}}$

$\underbrace{\frac{r f(r)}{x f(x) + (r-x) f(r)} \leq \sup_{r\geq x > 0} \left(\cdot \right)}$

let $A(x,r) = \frac{x}{r}\left(1 - \frac{x}{r}\right)$: $\gamma_f(r,x) \leq \sup_{r\geq x > 0} \frac{1}{1 - A(x,r)}$

let $\beta = \frac{x}{r}$ $\Rightarrow$ $\gamma_f(r,x) \leq \sup_{\beta\in(0,1)} \frac{1}{1 - A(\beta)}$

$\gamma_f(r,x) \leq \left(1 - \sup_{\beta\in(0,1)} A(\beta)\right)^{-1}$

$0 = \frac{\partial A}{\partial \beta} = 1 - 2\beta \rightarrow \beta = 1/2,\ A(\beta) = \frac{1}{4}$

let $\alpha(C) = \alpha(f)$

$\alpha(C) = \sup \frac{1}{1 - A(\beta)} \leq \frac{4}{3}$

$\alpha(C) = \frac{4}{3}$ at max for linear functions

if $c(x) = x$, $r = 2$, $x = 1$, $\rightarrow \gamma_c(r,x) = \frac{4}{3}$

implies a tight bound for $\alpha(C) = \frac{4}{3}$

(2) Let us list all paths from $s \to t$ and create a matrix comparing the cost to each player where P_1 & P_2 are on the Network

P_2

P_1	s,t	s,v,t	s,w,t	s,v,w,t
s,t	141, 282	47, 120	47, 122	47, 124
s,v,t	48, 188	80, 160	48, 122	72, 154
s,w,t	47, 188	47, 120	75, 150	73, 150
s,v,w,t	22, 188	46, 150	48, 148	120, 240

$\rightarrow$ example: P_1 on path $\{s,t\}$, P_2 on path $\{s,v,t\}$

$cost = n_i \cdot (3x_i^2 + (x_i^2+44))$

$\hookrightarrow n_2 = 2,\ x_2 = 2$

$cost = 2(3(2)^2 + ((2)^2+44)) = 120$

$\rightarrow P_1$ on $\{s,v,t\}$ with $P2 \Rightarrow \underset{n=1}{x = 2+1 = 3}$

$c_1 = 1(4x^2+44) = 80$

$c_2 = 2(4x^2+44) = 160$

$p_1\{s,v,w,t\}$ & $p_2\{s,v,t\}$

$c_1 = 6+13+1(3x^2) \overset{x=3}{\Rightarrow} 27+13+6 = 46$

$c_2 = (4+44)2 + 2(3(3)^2) = 96+54 = 150$

From the matrix, no Pure NE exists! (Only Mixed NE exists)

the decision submatrix circled in red proves this with no stable strategy where there is no Pure NE

(3) Given a potential function $\Phi(f) = \sum_{e \in E} \left[c_e(f_e) f_e + \sum_{i \in Paths} c_e(r_i) r_i \right]$ we want to show f is an equilibrium flow where f is a global optimum for $\Phi(f)$.

→ Proof by contradiction:

→ let p' be a path that decreases a player i's cost from path p and gives a flow f' such that:

$$(\star) \quad \boxed{0 > c_{p'}(f') - c_p(f)} \rightarrow 0 > \sum_{e \in P'-p} c_e(f_e + r_i) - \sum_{e \in P'-p} c_e(f_e)$$

see lemma on atomic flow

this implies $\phi(f') - \phi(f) = \Delta\phi(f, f')$

and $0 > \sum_{e \in P'/p} c_e(f_e + r_i) - c_e(f_e)$

Therefore taking $\Delta\phi(f,f')$, we can simplify to

$$\Delta\phi = \sum_{e \in P'/p} \left[c_e(f_e + r_i)\cdot(f_e + r_i) - \underbrace{c_e(f_e)(f_e) + c_e(r_i)(r_i)}_{} \right]$$

$$+ \sum_{e \in P'/p} \left[c_e(f_e - r_i)(f_e - r_i) - (\ \ '' \ \ ) \right]$$

factor out $2r_i$

Note that $c_e(x) = a_e(x) + b_e$

$$\Delta\phi = 2r_i \left[\sum_{e \in P'/p} (a_e r_i + a_e f_e + b_e) - \sum_{e \in P/p'} (a_e f_e + b_e) \right]$$

$$= 2r_i \left[\sum_{e \in P'/p} c_e(f_e + r_i) - \sum_{e \in P/p'} c_e(f_e) \right]$$

↳ $\phi(f') - \phi(f) = 2r_i \left(c_{p'}(f') - c_p(f) \right) < 0$

↳ this implies $\phi(f') < \phi(f)$

which contradicts the idea that f is global optimal for $\phi(\cdot)$ meaning f is <u>NOT</u> an equilibrium flow for $\phi(f)$.

④ a) i) Let $C_u = \alpha d_u + \sum_v dist(u,v)$ be the cost of node $u \to$ node v.

a) i) Proof by Contradiction:

Suppose NE exists in a graph without a tree. The graph must contain a cycle. Since NE exists only when there is a cycle, then breaking the cycle causes C_u to increase.

let d_u^t = no. edges of node u in a tree graph; d_u^c = no. edges for u in a cycle graph similarly $dist^t(u,v)$ & $dist^c(u,v)$ are distance functions respectively.

this implies:

$$C_u^t > C_u^c \Rightarrow \alpha d_u^t + \sum_v dist^t(u,v) > \alpha d_u^c + \sum_v dist^c(u,v)$$

$$= \alpha \underbrace{(d_u^c - d_u^t)}_{1} < \sum_v (dist^t - dist^c)$$

$$\therefore \quad \alpha < \underbrace{\sum_v (dist^t - dist^c)}$$

$$\alpha < \max_{u \in V} \{ \sum_v dist^t_{u,v} - \sum_v dist^c_{u,v} \}$$

implies $\alpha < \frac{(n-1)^2}{2}$ which violates $\alpha > n^2$ when a cycle exists

$\Rightarrow$ so all NE are trees for $\alpha > n^2$

Price of Anarchy (PoA): (for $\alpha = n^2$)

Note $\rightarrow$ $PoA \leq \frac{\text{worst NE cost}}{\text{opt. social cost}}$

worst NE cost $\leq \alpha(n-1) + \sum_{(u,v) \in G} dist(u,v) \Rightarrow$ worst NE cost $\leq \alpha(n-1) + n(n^2-1)$

optimum social cost $\geq \alpha(n-1) + 2(n-1) + 2(n\cdot(n-1) - 2(n-1)) = (\alpha + 2 + 4(\frac{n}{2}-1))(n-1)$

$\geq (n^2 + 2n - 2)(n-1)$

if $\alpha = n^2 \xrightarrow{\text{then}} PoA \leq \frac{\alpha(n-1) + n(n^2-1)}{(\alpha + 2 + 4(\frac{n}{2}-1))(n-1)} = \frac{2n^2+n}{n^2+2n-2}$

$\lim_{n\to\infty} \frac{2n^2+n}{n^2+n-2} = 2.$ $\lim_{\alpha \to \infty} PoA = 1$ therefore $PoA \leq 2$

(4) a) ::)

$$PoA = \frac{\alpha + n^2 + n}{\alpha - 2 + 2n}$$

Note: by lemma: if $\alpha \leq 1$ then the complete graph is a NE.

$$\lim_{\alpha \to 1^-} PoA = \frac{1+n^2+n}{-1+2n} = \frac{7}{4(2n-1)}$$

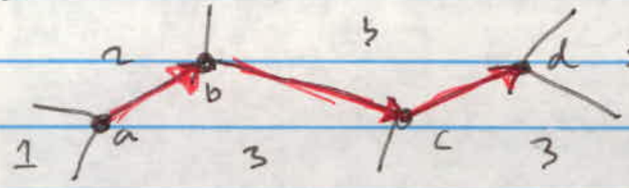
however Price of stability $(PoS) = 1$

hence, if player i stops paying for edges, his cost will increase by n. Hence the optimum is a complete graph

(4) b) assuming a graph with k vertices & k - almost complementing edges where k occurs at the start & end points. Any resultant path cannot repeat vertices without offering a second way to proceed when a vertex is first encountered. Hence a vertex can only be best visited once in a simple graph δ, meaning a graph's path terminates only when it is a NE.

(see LH method by Rahul Savani & Bernhard Stengel)
fig 1 from study

ex)



(5) a) for any player i:

$$\overline{C}(NE)\, c_i(NE) \le \sum_{e \in P_i^*} c_e(NE') \le k \sum_{e \in P_i^*} c_e^i(SO) \le k\,\overline{C}(SO)$$

where $c_i(NE)$, $c_i(SO)$ are cost for player i at NE or social optimum (SO)

$\overline{C}(NE)$, $\overline{C}(SO)$ is the cost of (NE) & (SO)

P_i^* is path of soc. opt. ; P_i is a path of NE

c_e^i is cost for player i on edge e

$c_e(NE')$ is

When we sum over all players, the PoA bound cannot exceed k since: PoA is bounded by $\overline{C}(NE)$ & $k\,\overline{C}(SO)$

$$\underline{\underline{\overline{C}(NE)\sum_i c_i(NE)}} \le \sum_i \sum_{e \in P_i^*} c_e(NE') \le k \sum_i \sum_{e \in P_i^*} c_e^i(SO) \le \underline{\underline{k\,\overline{C}(SO)}}$$

(5b) NE does exist since $\phi = \Sigma_{p \in paths}(W_p)^2 \ge 0$

where $W_p = \Sigma_{j \in w_p} w_j$ if players switch links

then $\phi_{new} - \phi_{old} < 0$ where ϕ is lower bounded at zero and ϕ cannot decrease infinitely