CS539
HW 2.5

Christopher Morcom
A20385764

① Prove for zero-sum games that if:

$$V = \min_{X \in A_1} \max_{Y \in A_2} \left(\mathcal{U}_1(X, Y) \right) = \min_{Y \in A_2} \max_{X \in A_1} \left(\mathcal{U}_1(X, Y) \right)$$

s.t X^* & Y^* are maximizers for Players 1 & 2 respectively,
then: (X^*, Y^*) is a Nash Equilibrium.

ANS:

let x^* be the solution to $\max_{X \in A_1} \min_{Y \in A_2} (\mathcal{U}_1(x, y)) \Rightarrow \mathcal{U}_1(x^*, y) \geq V$

let y^* be the solution to $\max_{X \in A_1} \min_{Y \in A_2} (\mathcal{U}_1(x, y)) \Rightarrow \mathcal{U}_1(x, y^*) \geq V$

where there is a large positive constant (C), and total costs X_A & Y_A for players 1 & 2 respectively, then the profit of each player $\forall X_A \in A_1, Y_A \in A_2$ is:

$$\mathcal{U}_1(X, Y) = X_A(C + Y_A - X_A)$$
$$\mathcal{U}_2(X, Y) = Y_A(C + X_A - Y_A)$$

The best response for P_1 is given by solution of the derivative of the best response function where Y_A is fixed at:

$$\mathcal{U}_{1\,\max}(X, Y) = \max_{X \in A_1} X_A(C + \overset{(const)}{Y_A} - X_A)$$

$$\frac{\partial}{\partial X}(\mathcal{U}_1) = U_1(X, Y) = C + Y_A - 2X_A \therefore \text{Maximizer}_1 := X_A = \frac{C + Y_A}{2}$$

by symmetry: $\frac{\partial}{\partial Y}(\mathcal{U}_2) = U_1(X, Y) = C + X_A - 2Y_A \therefore \text{maximizer}_2 := Y_A = \frac{C + X_A}{2}$

there best response B_1 & B_2 by symmetry are the maximizers:

$$\left.\begin{aligned} B_X(Y_A) &= \frac{C + Y_A}{2} \\ B_Y(X_A) &= \frac{C + X_A}{2} \end{aligned}\right\} \text{where } B_1 \,\&\, B_2 = X^* \,\&\, Y^* \text{ such that: } (X^*, Y^*) = (B_X(Y_A), B_Y(X_A))$$

since the Nash Equilibrium is by definition, the Best response point (X^*, y^*)

then the property holds true for a zero sum game since $\forall$ functions $\mathcal{U}_1$, $\min(-\mathcal{U}_1) = -\max(\mathcal{U}_1)$ via symmetry across utilities for P_1 & P_2