HW2 Solution, CS330 Discrete Structures, Spring 2015

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1 Page.398 #70

For each variable, we have 2 possible cases: T, F, therefore we have to have 2^n lines in one truth table. Then, for each line in the truth table, we could have T or F depending on the proposition. So, for various propositions in n variables, we could have 2^{2^n} different truth tables.

2 Page.414 #28

We only need to choose 17 positions to put 'true', so there are $\binom{40}{17}$ different answer keys.

3 Page.421 #20

Once we use the definition of the binomial coefficient, we can easily find that the both sides of the equality are equal to the following term:

$$\frac{(n-1)!n!(n+1)!}{(k-1)!k!(k+1)!(n-k-1)!(n-k)!(n-k+1)!}$$

It's called the hexagon identity because these six binomial coefficients make the vertices of a regular hexagon in Pascal's triangle.

4 Page.422 #28

4.1 (a)

LHS of the equality is the possible choices when we pick 2 from $2n$ choices. This is essentially equal to the addition of the following three cases: 1) choosing 2 from the first half ($\binom{n}{2}$); 2) choosing 2 from the other half ($\binom{n}{2}$); 3) choosing 1 from the first half and the other 1 from the other half (n^2). The addition is $2\binom{n}{2} + n^2$.

4.2 (b)

$$
\begin{aligned}
LHS &= \binom{2n}{2} \\
&= \frac{2n(2n-1)}{2} \\
&= n(2n-1) \\
&= 2n^2 - n \\
&= n^2 - n + n^2 \\
&= n(n-1) + n^2 \\
&= 2\frac{n(n-1)}{2} + n^2 \\
&= 2\binom{n}{2} + n^2
\end{aligned}
$$

4.3 Proof by Induction

1. **Base Case**
   When $n = 1$, LHS$= 1$ and RHS$= 1$. Therefore LHS=RHS.
2. **Inductive Hypothesis**
   When $n = k$, assume that the equality holds. That is, we assume that

$$\binom{2k}{2} = 2\binom{k}{2} + k^2$$

3. **Inductive Step & Proof**
   When $n = k + 1$, we have the following for the LHS:

$$
\begin{aligned}
LHS &= \binom{2k+2}{2} \\
&= \binom{2k+1}{1} + \binom{2k+1}{2} && \text{(Pascal's Identity)} \\
&= 2k + 1 + \binom{2k}{1} + \binom{2k}{2} && \text{(Pascal's Identity)} \\
&= 2k + 1 + 2k + \binom{2k}{2} \\
&= 2k + 2\binom{k}{2} + k^2 + 2k + 1 && \text{(Inductive Hypothesis)} \\
&= 2(k + \binom{k}{2}) + (k+1)^2 \\
&= 2(\binom{k}{1} + \binom{k}{2}) + (k+1)^2 \\
&= 2\binom{k+1}{2} + (k+1)^2 && \text{(Pascal's Identity)} \\
&= RHS
\end{aligned}
$$

Combining the Base Case, Inductive Hypothesis and the Inductive Step & Proof, we can deduce that for any integer $n \geq 1$, the equality holds.

5 Page.433 #42

In this game, one player gets 13 cards. Therefore, for the first player, we have $\binom{52}{13}$ choices. For the second player, we have $\binom{39}{13}$ choices. For the third player, we have $\binom{26}{13}$ choices. For the fourth player, we have $\binom{13}{13}$ choices. According to the rule of product, we have the following different ways to deal bridge hands to four players:

$$\begin{aligned}
&\binom{52}{13}\cdot\binom{39}{13}\cdot\binom{26}{13}\cdot\binom{13}{13}\\
=&\frac{52!}{39!\cdot 13!}\cdot\frac{39!}{26!\cdot 13!}\cdot\frac{26!}{13!\cdot 13!}\cdot\frac{13!}{13!\cdot 0!}\\
=&\frac{52!}{(13!)^4}
\end{aligned}$$

6 Page.434 #58

6.1 (a)

We can apply the rule of product. For the first ball, we have 7 choices. For the second one, we have 6 choices, and so forth. Then, we have $7\cdot 6\cdot 5\cdot 4\cdot 3 = 2520$ ways.

6.2 (d)

Since all the boxes are unlabeled, no matter how we put the balls in the boxes, all the ways are identical. Therefore, there is only one way.

7 Combinatorial identity problem

$$\binom{\binom{k}{2}}{2} = 3\binom{k+1}{4}$$

7.1 (a)

$$\begin{aligned}
LHS &= \binom{\binom{k}{2}}{2}\\
&= \binom{\frac{k(k-1)}{2}}{2}\\
&= \frac{\frac{k(k-1)}{2}\cdot(\frac{k(k-1)}{2}-1)}{2}\\
&= \frac{\frac{k(k-1)}{2}\cdot\frac{(k+1)(k-2)}{2}}{2}\\
&= 3\frac{(k+1)k(k-1)(k-2)}{4\cdot 3\cdot 2}\\
&= 3\binom{k+1}{4} = RHS
\end{aligned}$$

7.2 (b)

LHS of the equality is the number of different ways to choose 2 pairs among the pairs of k items where two items in each pair should be different.

Following the hint, we consider adding the $k+1$-st element "DUP" into the existing k items. If we pick 4 from these $k+1$ items, we have two different cases depending on whether "DUP" is picked or not.

If there is no "DUP" in these 4 items, firstly we choose 4 from k valid items and pick 0 from the "DUP", which leads to $\binom{k}{4}$ ways to do it. Then, we have $\binom{4}{2} \cdot \binom{2}{2}/2 = 3$ ways to have 2 pairs. The reason why we should divide 2 can be explained via an example. Suppose we have 4 numbers 1,2,3 and 4. When we pair them into 2 pairs, pairing (1,2) as a first pair and (3,4) as a second pair is identical to pairing (3,4) as a first pair and (1,2) as a second pair. However, in our term $\binom{4}{2} \cdot \binom{2}{2}$, this duplicity is not reflected. Therefore, there are $3\binom{k}{4}$ ways in this case

If one of them is the "DUP", which is choosing 3 from k valid items and 1 from "DUP" ($\binom{k}{3}$), any one of other 3 items can be duplicated to make the items into 4 items. In this case, we have 3 choice to make the duplication at the first step. After the duplication, since we have two duplicate items, the only way to pair them into two pairs is to put two duplicate item into each pair, which leads to only one way to pair these 4 items. So, there are $3\binom{k}{3}$ ways in this case.

The addition of the above two cases is equivalent to choosing 2 pairs among the pairs of k items. By the rule of sum, the number of final possibilities is $3\binom{k}{3} + 3\binom{k}{4} = 3\binom{k}{4}$ (Pascal's Identity).

Therefore, the equality $\binom{\binom{k}{2}}{2} = 3\binom{k+1}{4}$ holds.