CS430 – Introduction to Algorithms

Spring Semester, 2017

1. Use the widget on page 4 of Professor Reingold's graph coloring slides to prove that 3-coloring a planar graph is NP-complete. Remember, to prove NP-completeness you must prove NP-hardness and that the problem is in the class NP.

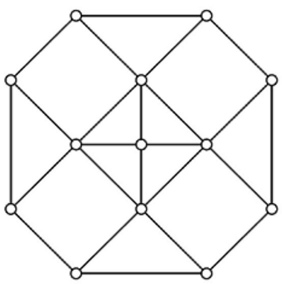
   Sol:

   In order to 3-color a planar graph, we need to consider the case of reducing 3-colorability of an arbitrary graph to the planar case.

   Let's consider an undirected graph G = (V,E), possibly nonplanar, embed the graph in the plane arbitrarily, letting edges cross if necessary.

   We will replace each edge crossing with the planar widget W provided below.

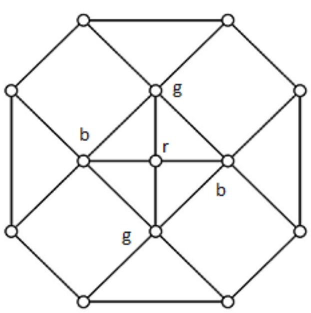
W =



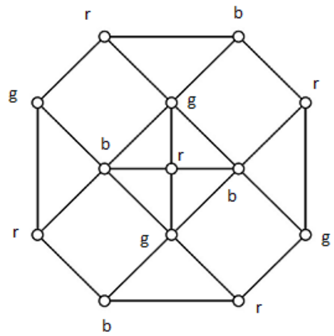
The widget W is a planar graph with the following interesting properties:

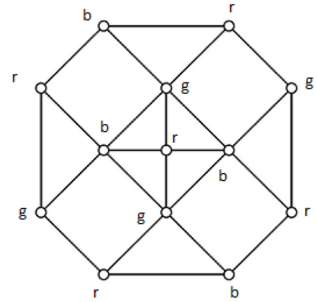
I. In an legal 3-coloring of W, the opposite corners are forced to have the same color
II. Any assignment of colors to the corners such that opposite corners have the same color extends to a 3-coloring of all of W

To see this, color the center of W red; then the vertices adjacent to the center must be colored blue or green alternately around the center



Now the northeast vertex can be colored either red or green. In either case, the colors of all the remaining vertices are forced (proceed counterclockwise to obtain the left hand coloring and clockwise to obtain the right hand coloring) as follows





All other colorings are obtained from these by permuting the colors.
For each edge(u,v) in E, replace each point at which another edge crosses (u,v) in the embedding with a copy of W. Identify the adjacent corners of these copies of W and identify the outer corner of the external copies with u and v, all except for one pait, which are connected by an edge.
The resulting graph G' – (V',E') is planar. If c: V' -> {red,blue,green} is a 3-coloring of G', then property (I) of W implies that c restricted to V is a 3-coloring of G. Conversely, if c: V->{red,blue,green} is a 3-coloring of G, then property(II) of W allows c to be extended to a 3-coloring of G'. Thus we redued 3-Colorable $\leq_p$ Planar 3-Colorable. Through this reduction we can say that Planar 3-Colorable graph is NP-hard. Since it is NP as well, we can conclude that it is NP-complete.

Citation: http://www.cs.umd.edu/~gasarch/BLOGPAPERS/npcproblems.pdf

2. Exercise 34.5-7
The longest-simple-cycle problem is the problem of determining a simple cycle (no repeated vertices) of maximum length in a graph. Formulate a related decision problem, and show that the decision problem is NP-complete.
Sol: A related decision problem is – Given a graph G and a integer k decide if there is a simple cycle of length at least k in the graph G. To see that this problem is in NP, just let the certificate be the cycle itself. It is really easy just to walk along this cycle, keeping track of what vertices you have already seen, and making sure that they don't get repeated.
To see that it is NP-hard, we will be doing a polynomial reduction to Hamilton cycle. Suppose we have a graph G and want to know if it is Hamilton. We then create an instance of the decision problem asking if the graph has a simple cycle of length at least |V| vertices. If it does then there is a Hamiltonian cycle. If there is not, then there cannot be any Hamiltonian cycle.
But, we already know that the Hamilton cycle problem is NP-complete. So, it is easy to see that the Hamilton-cycle problem $\leq_p$ the longest-simple-cycle problem because graph G = (V,E) admits a Hamilton cycle if and only if the length of its longest simple cycle equals |V|. This shows that the decision problem is NP-complete.

Citation: http://www.math.rutgers.edu/~ajl213/CLRS/Ch34.pdf