

Notes on Kiefer 2012

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1 Why quantum gravity

2 Covariant approaches to quantum gravity

3 Parametrised and relational systems

3.1 Particle systems

Gitman and Tyutin 1990; Prokhorov and Shabanov 2009; H. J. Rothe and K. D. Rothe 2010 can also be helpful.

3.1.1 Parametrised non-relativistic particle

The advantage of the complication below is to clarify the steps at which the constraints are not to be imposed.

Un-parametrised system The action reads

$$S[q] := \int_{t_1}^{t_2} dt L(q, q'), \quad (3.1)$$

where

$$q' := \frac{dq}{dt}. \quad (3.2)$$

It can be rephrased (see below) in the velocity formalism Gitman and Tyutin 1990, ch. 2

$$S[q, p, v] := \int_{t_1}^{t_2} dt \{p(q' - v) + L^v(q, v)\}, \quad (3.3)$$

in which

$$L^v(q, v) := L(q, q'). \quad (3.4)$$

Variation of eq. (3.3) with respect to (q, p, v) gives

$$\frac{\partial L^v}{\partial q}(q, v) = p', \quad \frac{\partial L^v}{\partial v}(q, v) = p, \quad q' = v, \quad (3.5)$$

respectively. Eliminating (p, v) in eq. (3.5) recovers the Euler–Lagrange equation.

To move to the Hamiltonian formalism, transform the action in eq. (3.3)

$$\begin{aligned} S[q, p, v] &\equiv \int_{t_1}^{t_2} dt \{pq' - (pv - L^v(q, v))\} \\ &=: \int_{t_1}^{t_2} dt \{pq' - H^v(q, p, v)\}. \end{aligned} \quad (3.6)$$

One may solve v by (q, p) from the second equation in eq. (3.5) by partially inverse the function $\frac{\partial L^v}{\partial v}(q, v)$, and denote the solution by

$$v = \bar{v}(q, p). \quad (3.7)$$

In the current case, all velocity can be solved, and one passes to the *canonical Hamiltonian* in one leap

$$H^c(q, p) := H^v(q, p, \bar{v}(q, p)) \equiv p\bar{v}(q, p) - L^v(q, \bar{v}(q, p)). \quad (3.8)$$

Parametrised system Parametrising t in eq. (3.1) gives

$$S[q, t] = \int_{t_1}^{t_2} d\tau \tilde{L}(q, t, \dot{q}, \dot{t}), \quad (3.9)$$

where

$$\tilde{L}(q, t, \dot{q}, \dot{t}) := tL\left(q, \frac{\dot{q}}{\dot{t}}\right), \quad \dot{f}(q, t) := \frac{df}{d\tau}(q, t). \quad (3.10)$$

The corresponding velocity formalism reads

$$\begin{aligned} S[q, t, p_q, p_t, u, N] &:= \int_{t_1}^{t_2} d\tau \{p_q(\dot{q} - u) + p_t(\dot{t} - N) + \tilde{L}^v(q, t, u, N)\} \\ &= \int_{t_1}^{t_2} d\tau \left\{p_q(\dot{q} - u) + p_t(\dot{t} - N) + NL^v\left(q, \frac{u}{N}\right)\right\}. \end{aligned} \quad (3.11)$$

Let us try to move to the Hamiltonian formalism. Variation of eq. (3.11) with respect to u gives

$$p_q = \frac{\partial \tilde{L}^v}{\partial u} \equiv \frac{\partial L^v}{\partial v} \left(q, \frac{u}{N} \right), \quad (3.12)$$

whereas the variation with respect to N leads to

$$p_t = \frac{\partial \tilde{L}^v}{\partial N} \equiv L^v \left(q, \frac{u}{N} \right) - \frac{u}{N} \cdot \frac{\partial L^v}{\partial v} \left(q, \frac{u}{N} \right). \quad (3.13)$$

Can one in this case solve both (u, N) by (q, t, p_q, p_t) from eqs. (3.12) and (3.13)? The answer is negative. With the help of eq. (3.7), one finds

$$\frac{u}{N} = \bar{v}(q, p_q) \quad \Leftrightarrow \quad u = \bar{u}(q, p_q, N) := N \bar{v}(q, p_q). \quad (3.14)$$

from eq. (3.12) (compare it with the second equation in eq. (3.5)!). Inserting eqs. (3.7) and (3.14) into eq. (3.13) yields

$$p_t = L^v(q, \bar{v}(q, p_q)) - \bar{v}(q, p_q) \cdot p_q \equiv -H^c(q, p_q), \quad (3.15)$$

where used is made of eq. (3.8) in the last step. One sees that eq. (3.15) contains no more velocity, and N cannot be solved by the positions and momenta.

Inserting the *partially solved set of velocities* (and not eq. (3.15), which is incompetent to eliminate the velocity), the action eq. (3.11) reads

$$\begin{aligned} S[q, t, p_q, p_t, N] &:= S[q, t, p_q, p_t, \bar{u}, N] \\ &= \int_{t_1}^{t_2} \mathbb{d}\tau \left\{ p_q \dot{q} + p_t \dot{t} - [p_q \bar{u} + p_t N - \tilde{L}^v(q, t, \bar{u}, N)] \right\} \\ &=: \int_{t_1}^{t_2} \mathbb{d}\tau \left\{ p_q \dot{q} + p_t \dot{t} - \tilde{H}^p(q, t, p_q, p_t, N) \right\}, \end{aligned} \quad (3.16)$$

in which

$$\tilde{H}^p = N \tilde{H}_\perp(q, t, p_q, p_t), \quad \tilde{H}_\perp := p_t + H^c(q, p_q) \quad (3.17)$$

are the *Hamiltonian with primary constraints* and *Hamiltonian constraint*, respectively. The canonical Hamiltonian in this case vanishes, since there are only constraints in \tilde{H}^p and nothing else left. Equation (3.15) is a (*primary constraint*) and is to be solved along with the Hamiltonian equations of motion, which can be found in Gitman and Tyutin 1990, ch. 2.

Homogeneous Lagrangian in the Hamiltonian formalism

$$\begin{aligned} &S[q^i, \pi_a, P_\alpha, N^a, V^\alpha] \\ &:= \int_{t_1}^{t_2} \mathbb{d}t \left\{ \pi_a (\dot{q}^a - N^a) + P_\alpha (\dot{q}^\alpha - V^\alpha) + L^v(q^i, N^a, V^\alpha) \right\}. \end{aligned} \quad (3.18)$$

Variation of (π_a, P_α) gives

$$\pi_a = \frac{\partial L^v}{\partial N^a}(q^i, N^a, V^\alpha), \quad (3.19)$$

$$P_\alpha = \frac{\partial L^v}{\partial V^\alpha}(q^i, N^a, V^\alpha). \quad (3.20)$$

$\{v^i\}$ is divided into $\{N^a\}$ and $\{V^\alpha\}$ such that the latter is the maximal set of primary solvable velocities. Inverting eq. (3.20) yields the solution

$$V^\alpha = \bar{V}^\alpha(q^i, P_\alpha, N^\alpha). \quad (3.21)$$

Inserting eq. (3.21) into eq. (3.19) does not yield any further solvable velocity, and the results are

$$\pi_a - f_a(q^i, P_\alpha) := \pi_a - \frac{\partial L^v}{\partial N^a}(q^i, N^a, \bar{V}^\alpha(q^i, P_\alpha, N^\alpha)) = 0. \quad (3.22)$$

Note that in eq. (3.22) all N^a cancels out; otherwise they may be further solved, which contradicts the maximality of $\{V^\alpha\}$. Equation (3.22) will be imposed at the dynamical level.

If L^v is linear in (N^a, V^α) , it is equivalent by Euler's Homogeneous Function Theorem Border 2000 that

$$L^v(q^i, N^a, V^\alpha) = \frac{\partial L^v}{\partial N^a} N^a + \frac{\partial L^v}{\partial V^\alpha} V^\alpha, \quad (3.23)$$

or

$$L^v(q^i, N^a, \bar{V}^\alpha) = \frac{\partial L^v}{\partial N^a}(q^i, N^b, \bar{V}^\alpha) N^a + \frac{\partial L^v}{\partial V^\alpha}(q^i, N^a, \bar{V}^\beta) \bar{V}^\alpha; \quad (3.24)$$

inserting eqs. (3.20) and (3.21) (these are equivalent!) yields

$$L^v(q^i, N^a, \bar{V}^\alpha) = f_a(q^i, P_\alpha) N^a + P_\alpha \bar{V}^\alpha. \quad (3.25)$$

Now the Hamiltonian with primary constraint can be calculated

$$H^p := \pi_a N^a + P_\alpha \bar{V}^\alpha - L^v(q^i, N^a, \bar{V}^\alpha) \quad (3.26)$$

$$\equiv (\pi_a - f_a(q^i, P_\alpha)) N^a, \quad (3.27)$$

where eq. (3.26) is the definition of H^p in general theory Gitman and Tyutin 1990, ch. 2. One sees that the Hamiltonian with primary constraints consists of constraints only, hence the canonical Hamiltonian H^c vanishes.

On the other hand, if H^c vanishes, one finds that

$$H^p = H^c + (\pi_a - f_a(q^i, P_\alpha)) N^a \quad (3.28)$$

$$\begin{aligned} &= (\pi_a - f_a(q^i, P_\alpha)) N^a \equiv \left(\pi_a - \frac{\partial L^v}{\partial N^a}(q^i, N^b, \bar{V}^\alpha) \right) N^a \\ &= \pi_a N^a + P_\alpha \bar{V}^\alpha + \frac{\partial L^v}{\partial N^a}(q^i, N^b, \bar{V}^\beta) N^a - P_\alpha \bar{V}^\alpha \\ &\equiv \pi_a N^a + P_\alpha \bar{V}^\alpha + \frac{\partial L^v}{\partial N^a}(q^i, N^b, \bar{V}^\beta) N^a - \frac{\partial L^v}{\partial V^\alpha}(q^i, P_\alpha, \bar{V}^\beta) \bar{V}^\alpha. \end{aligned} \quad (3.29)$$

where ?? is a result of the general theory *ibid.*, ch. 2. Comparing eqs. (3.26) and (3.29) gives eqs. (3.23) and (3.24), which implies the homogeneity of L^v in (N^a, V^α) .

3.1.2 The relativistic particle

Derivation with linear constraint In Gitman and Tyutin 1990, the primary constraints are *defined* by

$$\Phi_a \equiv \pi_a - \frac{\partial L^v}{\partial N^a}(q^i, N^b, \bar{V}^\alpha), \quad (3.30)$$

and are always linear with respect to the momenta $\{p_a\}$.

- 3.2 The free bosonic string
- 3.3 Parametrised field theories
- 3.4 Relational dynamical systems
- 3.5 General remarks on constrained systems
- 4 Hamiltonian formulation of general relativity
- 5 Quantum geometrodynamics
- 6 Quantum gravity with connections and loops
- 7 Quantisation of black holes
- 8 Quantum cosmology
- 9 String theory
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