Notes on Kiefer 2012

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3.1 Particle systems

Gitman and Tyutin 1990; Prokhorov and Shabanov 2009; H. J. Rothe and K. D. Rothe 2010 can also be helpful.

3.1.1 Parametrised non-relativistic particle

The advantage of the complication below is to clarify the steps at which the constraints are not to be imposed.

Un-parametrised system The action reads

$$S[q] := \int_{t_1}^{t_2} \mathbb{d}t \, L(q, q'), \tag{3.1}$$

where

$$q' := \frac{\mathbb{d}q}{\mathbb{d}t}. \tag{3.2}$$

It can be rephrased (see below) in the velocity formalism Gitman and Tyutin 1990, ch. 2

$$S[q, p, v] := \int_{t_1}^{t_2} \mathbb{d}t \, \{p(q' - v) + L^{\mathrm{v}}(q, v)\}, \tag{3.3}$$

in which

$$L^{\mathrm{v}}(q, v) := L(q, q'). \tag{3.4}$$

Variation of eq. (3.3) with respect to (q, p, v) gives

$$\frac{\partial L^{\mathrm{v}}}{\partial q}(q, v) = p', \quad \frac{\partial L^{\mathrm{v}}}{\partial v}(q, v) = p, \quad q' = v, \tag{3.5}$$

respectively. Eliminating (p, v) in eq. (3.5) recovers the Euler–Lagrange equation.

To move to the Hamiltonian formalism, transform the action in eq. (3.3)

$$\begin{aligned} S[q, p, v] &\equiv \int_{t_1}^{t_2} \mathbb{d}t \, \{pq' - (pv - L^{\mathrm{v}}(q, v))\} \\ &=: \int_{t_1}^{t_2} \mathbb{d}t \, \{pq' - H^{\mathrm{v}}(q, p, v)\}. \end{aligned} \tag{3.6}$$

One may solve v by (q, p) from the second equation in eq. (3.5) by partially inverse the function $\frac{\partial L^{\mathrm{v}}}{\partial v}(q, v)$, and denote the solution by

$$v = \bar{v}(q, p). \tag{3.7}$$

In the current case, all velocity can be solved, and one passes to the *canonical Hamiltonian* in one leap

$$H^{\mathrm{c}}(q, p) := H^{\mathrm{v}}(q, p, \bar{v}(q, p)) \equiv p\bar{v}(q, p) - L^{\mathrm{v}}(q, \bar{v}(q, p)). \tag{3.8}$$

Parametrised system Parametrising t in eq. (3.1) gives

$$S[q, t] = \int_{t_1}^{t_2} \mathbb{d}\tau \, \tilde{L}(q, t, \dot{q}, \dot{t}), \tag{3.9}$$

where

$$\tilde{L}(q, t, \dot{q}, \dot{t}) := \dot{t} L\left(q, \frac{\dot{q}}{\dot{t}}\right), \quad \dot{f}(q, t) := \frac{\mathbb{d}f}{\mathbb{d}\tau}(q, t). \tag{3.10}$$

The corresponding velocity formalism reads

$$\begin{aligned} S\big[q, t, p_q, p_t, u, N\big] &:= \int_{t_1}^{t_2} \mathbb{d}\tau \left\{p_q(\dot{q} - u) + p_t(\dot{t} - N) + \tilde{L}^{\mathrm{v}}(q, t, u, N)\right\} \\ &= \int_{t_1}^{t_2} \mathbb{d}\tau \left\{p_q(\dot{q} - u) + p_t(\dot{t} - N) + N L^{\mathrm{v}}\left(q, \frac{u}{N}\right)\right\}. \end{aligned} \tag{3.11}$$

Let us try to move to the Hamiltonian formalism. Variation of eq. (3.11) with respect to u gives

$$p_q = \frac{\partial \tilde{L}^{\mathrm{v}}}{\partial u} \equiv \frac{\partial L^{\mathrm{v}}}{\partial v}\Big(q, \frac{u}{N}\Big), \tag{3.12}$$

whereas the variation with respect to N leads to

$$p_t = \frac{\partial \tilde{L}^{\mathrm{v}}}{\partial N} \equiv L^{\mathrm{v}}\Big(q, \frac{u}{N}\Big) - \frac{u}{N} \cdot \frac{\partial L^{\mathrm{v}}}{\partial v}\Big(q, \frac{u}{N}\Big). \tag{3.13}$$

Can one in this case solve both (u, N) by $\big(q, t, p_q, p_t\big)$ from eqs. (3.12) and (3.13)? The answer is negative. With the help of eq. (3.7), one finds

$$\frac{u}{N} = \bar{v}\big(q, p_q\big) \quad \Leftrightarrow \quad u = \bar{u}\big(q, p_q, N\big) := N\bar{v}\big(q, p_q\big). \tag{3.14}$$

from eq. (3.12) (compare it with the second equation in eq. (3.5)!). Inserting eqs. (3.7) and (3.14) into eq. (3.13) yields

$$p_t = L^{\mathrm{v}}\big(q, \bar{v}(q, p_q)\big) - \bar{v}\big(q, p_q\big) \cdot p_q \equiv -H^{\mathrm{c}}\big(q, p_q\big), \tag{3.15}$$

where used is made of eq. (3.8) in the last step. One sees that eq. (3.15) contains no more velocity, and N cannot be solved by the positions and momenta.

Inserting the *partially solved set of velocities* (and not eq. (3.15), which is incompetent to eliminate the velocity), the action eq. (3.11) reads

$$\begin{aligned} S\big[q, t, p_q, p_t, N\big] &:= S\big[q, t, p_q, p_t, \bar{u}, N\big] \\ &= \int_{t_1}^{t_2} \mathrm{d}\tau \left\{ p_q \dot{q} + p_t \dot{t} - \big[p_q \bar{u} + p_t N - \tilde{L}^{\mathrm{v}}(q, t, \bar{u}, N)\big] \right\} \\ &=: \int_{t_1}^{t_2} \mathrm{d}\tau \left\{ p_q \dot{q} + p_t \dot{t} - \tilde{H}^{\mathrm{p}}\big(q, t, p_q, p_t, N\big) \right\}, \end{aligned} \tag{3.16}$$

in which

$$\tilde{H}^{\mathrm{p}} = N\tilde{H}_{\perp}\big(q, t, p_q, p_t\big), \qquad \tilde{H}_{\perp} := p_t + H^{\mathrm{c}}\big(q, p_q\big) \tag{3.17}$$

are the *Hamiltonian with primary constraints* and *Hamiltonian constraint*, respectively. The canonical Hamiltonian in this case vanishes, since there are only constraints in $\tilde{H}^{\mathrm{p}}$ and nothing else left. Equation (3.15) is a *(primary) constraint* and is to be solved along with the Hamiltonian equations of motion, which can be found in Gitman and Tyutin 1990, ch. 2.

Homogeneous Lagrangian in the Hamiltonian formalism

$$\begin{aligned} &S[q^i, \pi_a, P_\alpha, N^a, V^\alpha] \\ &:= \int_{t_1}^{t_2} \mathrm{d}t \left\{ \pi_a(\dot{q}^a - N^a) + P_\alpha(\dot{q}^\alpha - V^\alpha) + L^{\mathrm{v}}(q^i, N^a, V^\alpha) \right\}. \end{aligned} \tag{3.18}$$

Variation of (π_a, P_α) gives

$$\pi_a = \frac{\partial L^{\mathrm{v}}}{\partial N^a}(q^i, N^a, V^\alpha), \tag{3.19}$$

$$P_\alpha = \frac{\partial L^{\mathrm{v}}}{\partial V^\alpha}(q^i, N^a, V^\alpha). \tag{3.20}$$

$\{v^i\}$ is divided into $\{N^a\}$ and $\{V^\alpha\}$ such that the latter is the maximal set of primary solvable velocities. Inverting eq. (3.20) yields the solution

$$V^\alpha = \bar{V}^\alpha(q^i, P_\alpha, N^\alpha). \tag{3.21}$$

Inserting eq. (3.21) into eq. (3.19) does not yield any further solvable velocity, and the results are

$$\pi_a - f_a(q^i, P_\alpha) := \pi_a - \frac{\partial L^{\mathrm{v}}}{\partial N^a}\big(q^i, N^a, \bar{V}^a(q^i, P_\alpha, N^\alpha)\big) = 0. \tag{3.22}$$

Note that in eq. (3.22) all N^a cancels out; otherwise they may be further solved, which contradicts the maximality of $\{V^a\}$. Equation (3.22) will be imposed at the dynamical level.

If L^{v} is linear in (N^a, V^α), one has

$$L^{\mathrm{v}}(q^i, N^a, V^\alpha) = \frac{\partial L^{\mathrm{v}}}{\partial N^a}N^a + \frac{\partial L^{\mathrm{v}}}{\partial V^\alpha}V^\alpha; \tag{3.23}$$

inserting eqs. (3.20) and (3.21) (these are equivalent!) yields

$$L^{\mathrm{v}}\big(q^i, N^a, \bar{V}^\alpha\big) = f_a(q^i, P_\alpha)N^a + P_\alpha \bar{V}^\alpha. \tag{3.24}$$

Now the Hamiltonian with primary constraint can be calculated

$$\begin{aligned} H^{\mathrm{p}} &= \pi_a N^a + P_\alpha \bar{V}^\alpha - L^{\mathrm{v}}\big(q^i, N^a, \bar{V}^\alpha\big) \\ &\equiv (\pi_a - f_a(q^i, P_\alpha))N^a. \end{aligned} \tag{3.25}$$

Border 2000

4 Hamiltonian formulation of general relativity

5 Quantum geometrodynamics

6 Quantum gravity with connections and loops

7 Quantisation of black holes

8 Quantum cosmology

9 String theory

10 Phenomenology, decoherence, and the arrow of time

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