Notes on Kiefer 2012

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Contents

1 Parametrised and relational systems

1.1 Particle systems

Gitman and Tyutin 1990; Prokhorov and Shabanov 2009; H. J. Rothe and K. D. Rothe 2010 can also be helpful.

1.1.1 Parametrised non-relativistic particle

The advantage of the complication below is to clarify the steps at which the constraints are not to be imposed.

Un-parametrised system The action reads

$$S[q] := \int_{t_1}^{t_2} \mathrm{d}t\, L(q, q'), \tag{1.1}$$

where

$$q' := \frac{\mathrm{d}q}{\mathrm{d}t}. \tag{1.2}$$

It can be rephrased (see below) in the velocity formalism Gitman and Tyutin 1990, ch. 2

$$S[q, p, v] := \int_{t_1}^{t_2} \mathrm{d}t\, \{p(q' - v) + L^{\mathrm{v}}(q, v)\}, \tag{1.3}$$

in which

$$L^{\mathrm{v}}(q, v) := L(q, q'). \tag{1.4}$$

Variation of eq. (1.3) with respect to (q, p, v) gives

$$\frac{\partial L^{\mathrm{v}}}{\partial q}(q, v) = p', \quad \frac{\partial L^{\mathrm{v}}}{\partial v}(q, v) = p, \quad q' = v, \tag{1.5}$$

respectively. Eliminating (p, v) in eq. (1.5) recovers the Euler–Lagrange equation.

To move to the Hamiltonian formalism, transforms the action in eq. (1.3)

$$
\begin{aligned}
S[q,p,v] &\equiv \int_{t_1}^{t_2} \mathrm{d}t \, \{pq' - (pv - L^{\mathrm{v}}(q,v))\} \\
&=: \int_{t_1}^{t_2} \mathrm{d}t \, \{pq' - H^{\mathrm{v}}(q,p,v)\}.
\end{aligned} \tag{1.6}
$$

One may solve v by (q, p) from the second equation in eq. (1.5) by partially inverse the function $\frac{\partial L^{\mathrm{v}}}{\partial v}(q, v)$, and denote the solution by

$$
v = \bar{v}(q,p). \tag{1.7}
$$

In the current case, all velocity can be solved, and one passes to the *canonical Hamiltonian* in one leap

$$
H^{\mathrm{c}}(q,p) := H^{\mathrm{v}}(q,p,\bar{v}(q,p)) \equiv p\bar{v}(q,p) - L^{\mathrm{v}}(q,\bar{v}(q,p)). \tag{1.8}
$$

Parametrised system Parametrising t in eq. (1.1) gives

$$
S[q,t] = \int_{t_1}^{t_2} \mathrm{d}\tau \, \tilde{L}(q,t,\dot{q},\dot{t}), \tag{1.9}
$$

where

$$
\tilde{L}(q,t,\dot{q},\dot{t}) := \dot{t} L\Big(q, \frac{\dot{q}}{\dot{t}}\Big), \quad \dot{f}(q,t) := \frac{\mathrm{d}f}{\mathrm{d}\tau}(q,t). \tag{1.10}
$$

The corresponding velocity formalism reads

$$
\begin{aligned}
S\big[q,t,p_q,p_t,u,N\big] &:= \int_{t_1}^{t_2} \mathrm{d}\tau \, \Big\{p_q(\dot{q}-u) + p_t(\dot{t}-N) + \tilde{L}^{\mathrm{v}}(q,t,u,N)\Big\} \\
&= \int_{t_1}^{t_2} \mathrm{d}\tau \, \Big\{p_q(\dot{q}-u) + p_t(\dot{t}-N) + NL^{\mathrm{v}}\Big(q, \frac{u}{N}\Big)\Big\}.
\end{aligned} \tag{1.11}
$$

Variation of eq. (1.11) with respect to u gives

$$
p_q = \frac{\partial \tilde{L}^{\mathrm{v}}}{\partial u} \equiv \frac{L^{\mathrm{v}}}{v}\Big(q, \frac{u}{N}\Big), \tag{1.12}
$$

whereas the variation with respect to N leads to

$$
p_t = \frac{\partial \tilde{L}^{\mathrm{v}}}{\partial N} \equiv L^{\mathrm{v}}\Big(q, \frac{u}{N}\Big) - \frac{u}{N} \cdot \frac{\partial L^{\mathrm{v}}}{\partial v}\Big(q, \frac{u}{N}\Big). \tag{1.13}
$$

Can one in this case solve both (u, N) by $\big(q, t, p_q, p_t\big)$ from eqs. (1.12) and (1.13)? The answer is negative. With the help of eq. (1.7), one finds

$$
\frac{u}{N} = \bar{v}\big(q, p_q\big) \quad \Leftrightarrow \quad u = \bar{u}\big(q, p_q, N\big) := N\bar{v}\big(q, p_q\big). \tag{1.14}
$$

from eq. (1.12) (compare it with the second equation in eq. (1.5)!). Inserting eqs. (1.7), (1.12) and (1.14) into eq. (1.13) yields

$$
p_t = L^{\mathrm{v}}\big(q, \bar{v}\big(q, p_q\big)\big) - \bar{v}\big(q, p_q\big) \cdot p_q \equiv H^{\mathrm{c}}\big(q, p_q\big). \tag{1.15}
$$

One sees that eq. (1.15) contains no more velocity, and N cannot be solved by the positions and momenta.

Inserting the partially solved set of velocities, the action eq. (1.11) reads

$$\begin{aligned}
S\big[q,t,p_q,p_t,N\big] &:= S\big[q,t,p_q,p_t,\bar{u},N\big] \\
&= \int_{t_1}^{t_2} \mathbb{d}\tau \Big\{ p_q\dot{q} + p_t\dot{t} - \Big[p_q\bar{u} + p_tN - \tilde{L}^{\mathrm{v}}(q,t,\bar{u},N)\Big]\Big\} \\
&=: \int_{t_1}^{t_2} \mathbb{d}\tau \Big\{ p_q\dot{q} + p_t\dot{t} - \tilde{H}^{\mathrm{p}}\big(q,t,p_q,p_t,N\big)\Big\},
\end{aligned} \tag{1.16}$$

in which

$$\tilde{H}^{\mathrm{p}} = N\tilde{H}_{\perp}\big(q,p_q\big), \qquad \tilde{H}_{\perp} := p_t + H^{\mathrm{c}}\big(q,p_q\big) \tag{1.17}$$

are the *Hamiltonian with primary constraints* and *Hamiltonian constraint*, respectively. The canonical Hamiltonian in this case vanishes, since there are only constraints in H^{p} and nothing else left.

References

Gitman, Dmitriy M. and Igor V. Tyutin (1990). *Quantization of Fields with Constraints*. Springer Berlin Heidelberg. DOI: `10.1007/978-3-642-83938-2`. URL: `https://doi.org/10.1007%2F978-3-642-83938-2`.

Kiefer, Claus (2012). *Quantum Gravity*. 3rd. Oxford University Press. DOI: `10.1093/acprof:oso/9780199585205.001.0001`. URL: `https://doi.org/10.1093%2Facprof%3Aoso%2F9780199585205.001.0001`.

Prokhorov, Lev V. and Sergei V. Shabanov (2009). *Hamiltonian Mechanics of Gauge Systems*. Cambridge University Press. DOI: `10.1017/cbo9780511976209`. URL: `https://doi.org/10.1017%2Fcbo9780511976209`.

Rothe, Heinz J and Klaus D Rothe (2010). *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*. WORLD SCIENTIFIC. DOI: `10.1142/7689`. URL: `https://doi.org/10.1142%2F7689`.