Notes on Kiefer 2012

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Contents

1 Parametrised and relational systems

1.1 Particle systems

Gitman and Tyutin 1990; Prokhorov and Shabanov 2009; H. J. Rothe and K. D. Rothe 2010 can also be helpful.

1.1.1 Parametrised non-relativistic particle

The advantage of the complication below is to clarify the steps at which the constraints are not to be imposed.

Un-parametrised system The action reads

$$S[q] := \int_{t_1}^{t_2} \mathbb{d}t \, L(q, q'), \tag{1.1}$$

where

$$q' := \frac{\mathbb{d}q}{\mathbb{d}t}. \tag{1.2}$$

It can be rephrased (see below) in the velocity formalism Gitman and Tyutin 1990, ch. 2

$$S[q, p, v] := \int_{t_1}^{t_2} \mathbb{d}t \, \{p(q' - v) + L^{\mathrm{v}}(q, v)\}, \tag{1.3}$$

in which

$$L^{\mathrm{v}}(q, v) := L(q, q'). \tag{1.4}$$

Variation of eq. (1.3) with respect to (q, p, v) gives

$$\frac{\partial L^{\mathrm{v}}}{\partial q}(q, v) = p', \quad \frac{\partial L^{\mathrm{v}}}{\partial v}(q, v) = p, \quad q' = v, \tag{1.5}$$

respectively. Eliminating (p, v) in eq. (1.5) recovers the Euler–Lagrange equation.

To move to the Hamiltonian formalism, transform the action in eq. (1.3)

$$\begin{aligned} S[q, p, v] &\equiv \int_{t_1}^{t_2} \mathbb{d}t \, \{pq' - (pv - L^{\mathrm{v}}(q, v))\} \\ &=: \int_{t_1}^{t_2} \mathbb{d}t \, \{pq' - H^{\mathrm{v}}(q, p, v)\}. \end{aligned} \tag{1.6}$$

One may solve v by (q, p) from the second equation in eq. (1.5) by partially inverse the function $\frac{\partial L^{\mathrm{v}}}{\partial v}(q, v)$, and denote the solution by

$$v = \bar{v}(q, p). \tag{1.7}$$

In the current case, all velocity can be solved, and one passes to the *canonical Hamiltonian* in one leap

$$H^{\mathrm{c}}(q, p) := H^{\mathrm{v}}(q, p, \bar{v}(q, p)) \equiv p\bar{v}(q, p) - L^{\mathrm{v}}(q, \bar{v}(q, p)). \tag{1.8}$$

Parametrised system Parametrising t in eq. (1.1) gives

$$S[q, t] = \int_{t_1}^{t_2} \mathbb{d}\tau \, \tilde{L}(q, t, \dot{q}, \dot{t}), \tag{1.9}$$

where

$$\tilde{L}(q, t, \dot{q}, \dot{t}) := \dot{t} L\Big(q, \frac{\dot{q}}{\dot{t}}\Big), \quad \dot{f}(q, t) := \frac{\mathbb{d}f}{\mathbb{d}\tau}(q, t). \tag{1.10}$$

The corresponding velocity formalism reads

$$\begin{aligned} S\big[q, t, p_q, p_t, u, N\big] &:= \int_{t_1}^{t_2} \mathbb{d}\tau \, \Big\{p_q(\dot{q} - u) + p_t(\dot{t} - N) + \tilde{L}^{\mathrm{v}}(q, t, u, N)\Big\} \\ &= \int_{t_1}^{t_2} \mathbb{d}\tau \, \Big\{p_q(\dot{q} - u) + p_t(\dot{t} - N) + NL^{\mathrm{v}}\Big(q, \frac{u}{N}\Big)\Big\}. \end{aligned} \tag{1.11}$$

Let us try to move to the Hamiltonian formalism. Variation of eq. (1.11) with respect to u gives

$$p_q = \frac{\partial \tilde{L}^{\mathrm{v}}}{\partial u} \equiv \frac{\partial L^{\mathrm{v}}}{\partial v}\Big(q, \frac{u}{N}\Big), \tag{1.12}$$

whereas the variation with respect to N leads to

$$p_t = \frac{\partial \tilde{L}^{\mathrm{v}}}{\partial N} \equiv L^{\mathrm{v}}\Big(q, \frac{u}{N}\Big) - \frac{u}{N} \cdot \frac{\partial L^{\mathrm{v}}}{\partial v}\Big(q, \frac{u}{N}\Big). \tag{1.13}$$

Can one in this case solve both (u, N) by $\big(q, t, p_q, p_t\big)$ from eqs. (1.12) and (1.13)? The answer is negative. With the help of eq. (1.7), one finds

$$\frac{u}{N} = \bar{v}\big(q, p_q\big) \quad \Leftrightarrow \quad u = \bar{u}\big(q, p_q, N\big) := N\bar{v}\big(q, p_q\big). \tag{1.14}$$

from eq. (1.12) (compare it with the second equation in eq. (1.5)!). Inserting eqs. (1.7) and (1.14) into eq. (1.13) yields

$$p_t = L^{\mathrm{v}}\big(q, \bar{v}(q, p_q)\big) - \bar{v}(q, p_q) \cdot p_q \equiv -H^{\mathrm{c}}(q, p_q), \tag{1.15}$$

where used is made of eq. (1.8) in the last step. One sees that eq. (1.15) contains no more velocity, and N cannot be solved by the positions and momenta.

Inserting the *partially solved set of velocities* (and not eq. (1.15), which is incompetent to eliminate the velocity), the action eq. (1.11) reads

$$\begin{aligned} S\big[q, t, p_q, p_t, N\big] &:= S\big[q, t, p_q, p_t, \bar{u}, N\big] \\ &= \int_{t_1}^{t_2} \mathbb{d}\tau \left\{ p_q \dot{q} + p_t \dot{t} - \Big[p_q \bar{u} + p_t N - \tilde{L}^{\mathrm{v}}(q, t, \bar{u}, N)\Big] \right\} \\ &=: \int_{t_1}^{t_2} \mathbb{d}\tau \left\{ p_q \dot{q} + p_t \dot{t} - \tilde{H}^{\mathrm{p}}\big(q, t, p_q, p_t, N\big) \right\}, \end{aligned} \tag{1.16}$$

in which

$$\tilde{H}^{\mathrm{p}} = N \tilde{H}_{\perp}\big(q, t, p_q, p_t\big), \qquad \tilde{H}_{\perp} := p_t + H^{\mathrm{c}}\big(q, p_q\big) \tag{1.17}$$

are the *Hamiltonian with primary constraints* and *Hamiltonian constraint*, respectively. The canonical Hamiltonian in this case vanishes, since there are only constraints in $\tilde{H}^{\mathrm{p}}$ and nothing else left. Equation (1.15) is a *(primary) constraint* and is to be solved along with the Hamiltonian equations of motion, which can be found in Gitman and Tyutin 1990, ch. 2.

References

Gitman, Dmitriy M. and Igor V. Tyutin (1990). *Quantization of Fields with Constraints*. Springer Berlin Heidelberg. DOI: `10.1007/978-3-642-83938-2`. URL: `https://doi.org/10.1007%2F978-3-642-83938-2`.

Kiefer, Claus (2012). *Quantum Gravity*. 3rd. Oxford University Press. DOI: `10.1093/acprof:oso/9780199585205.001.0001`. URL: `https://doi.org/10.1093%2Facprof%3Aoso%2F9780199585205.001.0001`.

Prokhorov, Lev V. and Sergei V. Shabanov (2009). *Hamiltonian Mechanics of Gauge Systems*. Cambridge University Press. DOI: `10.1017/cbo9780511976209`. URL: `https://doi.org/10.1017%2Fcbo9780511976209`.

Rothe, Heinz J and Klaus D Rothe (2010). *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*. WORLD SCIENTIFIC. DOI: `10.1142/7689`. URL: `https://doi.org/10.1142%2F7689`.