Notes on Kiefer 2012

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3.1 Particle systems

Gitman and Tyutin 1990; Prokhorov and Shabanov 2009; H. J. Rothe and K. D. Rothe 2010 can also be helpful.

3.1.1 Parametrised non-relativistic particle

The advantage of the complication below is to clarify the steps at which the constraints are not to be imposed.

Un-parametrised system The action reads

$$S[q] := \int_{t_1}^{t_2} \mathrm{d}t\, L(q, q'), \tag{3.1}$$

where

$$q' := \frac{\mathrm{d}q}{\mathrm{d}t}. \tag{3.2}$$

It can be rephrased (see below) in the velocity formalism Gitman and Tyutin 1990, ch. 2

$$S[q, p, v] := \int_{t_1}^{t_2} \mathrm{d}t\, \{p(q' - v) + L^{\mathrm{v}}(q, v)\}, \tag{3.3}$$

in which

$$L^{\mathrm{v}}(q, v) := L(q, q'). \tag{3.4}$$

Variation of eq. (3.3) with respect to (q, p, v) gives

$$\frac{\partial L^{\mathrm{v}}}{\partial q}(q, v) = p', \quad \frac{\partial L^{\mathrm{v}}}{\partial v}(q, v) = p, \quad q' = v, \tag{3.5}$$

respectively. Eliminating (p, v) in eq. (3.5) recovers the Euler–Lagrange equation.

To move to the Hamiltonian formalism, transform the action in eq. (3.3)

$$\begin{aligned} S[q, p, v] &\equiv \int_{t_1}^{t_2} \mathrm{d}t\, \{pq' - (pv - L^{\mathrm{v}}(q, v))\} \\ &=: \int_{t_1}^{t_2} \mathrm{d}t\, \{pq' - H^{\mathrm{v}}(q, p, v)\}. \end{aligned} \tag{3.6}$$

One may solve v by (q, p) from the second equation in eq. (3.5) by partially inverse the function $\frac{\partial L^{\mathrm{v}}}{\partial v}(q, v)$, and denote the solution by

$$v = \bar{v}(q, p). \tag{3.7}$$

In the current case, all velocity can be solved, and one passes to the *canonical Hamiltonian* in one leap

$$H^{\mathrm{c}}(q, p) := H^{\mathrm{v}}(q, p, \bar{v}(q, p)) \equiv p\bar{v}(q, p) - L^{\mathrm{v}}(q, \bar{v}(q, p)). \tag{3.8}$$

Parametrised system Parametrising t in eq. (3.1) gives

$$S[q, t] = \int_{t_1}^{t_2} \mathrm{d}\tau\, \tilde{L}(q, t, \dot{q}, \dot{t}), \tag{3.9}$$

where

$$\tilde{L}(q, t, \dot{q}, \dot{t}) := \dot{t} L\Big(q, \frac{\dot{q}}{\dot{t}}\Big), \quad \dot{f}(q, t) := \frac{\mathrm{d}f}{\mathrm{d}\tau}(q, t). \tag{3.10}$$

The corresponding velocity formalism reads

$$S\big[q,t,p_q,p_t,u,N\big] := \int_{t_1}^{t_2} \mathbb{d}\tau \,\Big\{p_q(\dot q - u) + p_t(\dot t - N) + \tilde L^{\mathrm v}(q,t,u,N)\Big\}$$
$$= \int_{t_1}^{t_2} \mathbb{d}\tau \,\Big\{p_q(\dot q - u) + p_t(\dot t - N) + N L^{\mathrm v}\Big(q,\frac{u}{N}\Big)\Big\}. \tag{3.11}$$

Let us try to move to the Hamiltonian formalism. Variation of eq. (3.11) with respect to u gives

$$p_q = \frac{\partial \tilde L^{\mathrm v}}{\partial u} \equiv \frac{\partial L^{\mathrm v}}{\partial v}\Big(q,\frac{u}{N}\Big), \tag{3.12}$$

whereas the variation with respect to N leads to

$$p_t = \frac{\partial \tilde L^{\mathrm v}}{\partial N} \equiv L^{\mathrm v}\Big(q,\frac{u}{N}\Big) - \frac{u}{N}\cdot\frac{\partial L^{\mathrm v}}{\partial v}\Big(q,\frac{u}{N}\Big). \tag{3.13}$$

Can one in this case solve both (u,N) by $\big(q,t,p_q,p_t\big)$ from eqs. (3.12) and (3.13)? The answer is negative. With the help of eq. (3.7), one finds

$$\frac{u}{N} = \bar v\big(q,p_q\big) \quad \Leftrightarrow \quad u = \bar u\big(q,p_q,N\big) := N\bar v\big(q,p_q\big). \tag{3.14}$$

from eq. (3.12) (compare it with the second equation in eq. (3.5)!). Inserting eqs. (3.7) and (3.14) into eq. (3.13) yields

$$p_t = L^{\mathrm v}\big(q,\bar v\big(q,p_q\big)\big) - \bar v\big(q,p_q\big)\cdot p_q \equiv -H^{\mathrm c}\big(q,p_q\big), \tag{3.15}$$

where used is made of eq. (3.8) in the last step. One sees that eq. (3.15) contains no more velocity, and N cannot be solved by the positions and momenta.

Inserting the *partially solved set of velocities* (and not eq. (3.15), which is incompetent to eliminate the velocity), the action eq. (3.11) reads

$$S\big[q,t,p_q,p_t,N\big] := S\big[q,t,p_q,p_t,\bar u,N\big]$$
$$= \int_{t_1}^{t_2} \mathbb{d}\tau \,\Big\{p_q\dot q + p_t\dot t - \big[p_q\bar u + p_t N - \tilde L^{\mathrm v}(q,t,\bar u,N)\big]\Big\}$$
$$=: \int_{t_1}^{t_2} \mathbb{d}\tau \,\Big\{p_q\dot q + p_t\dot t - \tilde H^{\mathrm p}\big(q,t,p_q,p_t,N\big)\Big\}, \tag{3.16}$$

in which

$$\tilde H^{\mathrm p} = N\tilde H_\perp\big(q,t,p_q,p_t\big), \qquad \tilde H_\perp := p_t + H^{\mathrm c}\big(q,p_q\big) \tag{3.17}$$

are the *Hamiltonian with primary constraints* and *Hamiltonian constraint*, respectively. The canonical Hamiltonian in this case vanishes, since there are only constraints in $\tilde H^{\mathrm p}$ and nothing else left. Equation (3.15) is a *(primary) constraint* and is to be solved along with the Hamiltonian equations of motion, which can be found in Gitman and Tyutin 1990, ch. 2.

Homogeneous Lagrangian in the Hamiltonian formalism

$$\begin{aligned}&S[q^i,\pi_a,P_\alpha,N^a,V^\alpha]\\ &:=\int_{t_1}^{t_2}\mathrm{d}t\,\{\pi_a(\dot q^a-N^a)+P_\alpha(\dot q^\alpha-V^\alpha)+L^{\mathrm{v}}(q^i,N^a,V^\alpha)\}.\end{aligned}\tag{3.18}$$

Variation of (π_a, P_α) gives

$$\pi_a=\frac{\partial L^{\mathrm{v}}}{\partial N^a}(q^i,N^a,V^\alpha),\tag{3.19}$$

$$P_\alpha=\frac{\partial L^{\mathrm{v}}}{\partial V^\alpha}(q^i,N^a,V^\alpha).\tag{3.20}$$

$\{v^i\}$ is divided into $\{N^a\}$ and $\{V^\alpha\}$ such that the latter is the maximal set of primary solvable velocities. Inverting eq. (3.20) yields the solution

$$V^\alpha=\bar V^\alpha(q^i,P_\alpha,N^\alpha).\tag{3.21}$$

Inserting eq. (3.21) into eq. (3.19) does not yield any further solvable velocity, and the results are

$$\pi_a-f_a(q^i,P_\alpha):=\pi_a-\frac{\partial L^{\mathrm{v}}}{\partial N^a}\big(q^i,N^a,\bar V^a(q^i,P_\alpha,N^\alpha)\big)=0.\tag{3.22}$$

Note that in eq. (3.22) all N^a cancels out; otherwise they may be further solved, which contradicts the maximality of $\{V^a\}$. Equation (3.22) will be imposed at the dynamical level.

If L^{v} is linear in (N^a, V^α), it is equivalent by Euler's Homogeneous Function Theorem Border 2000 that

$$L^{\mathrm{v}}(q^i,N^a,V^\alpha)=\frac{\partial L^{\mathrm{v}}}{\partial N^a}N^a+\frac{\partial L^{\mathrm{v}}}{\partial V^\alpha}V^\alpha,\tag{3.23}$$

or

$$L^{\mathrm{v}}\big(q^i,N^a,\bar V^\alpha\big)=\frac{\partial L^{\mathrm{v}}}{\partial N^a}\big(q^i,N^b,\bar V^\alpha\big)N^a+\frac{\partial L^{\mathrm{v}}}{\partial V^\alpha}\big(q^i,N^a,\bar V^\beta\big)\bar V^\alpha;\tag{3.24}$$

inserting eqs. (3.20) and (3.21) (these are equivalent!) yields

$$L^{\mathrm{v}}\big(q^i,N^a,\bar V^\alpha\big)=f_a(q^i,P_\alpha)N^a+P_\alpha\bar V^\alpha.\tag{3.25}$$

Now the Hamiltonian with primary constraint can be calculated

$$H^{\mathrm{p}}:=\pi_aN^a+P_\alpha\bar V^\alpha-L^{\mathrm{v}}\big(q^i,N^a,\bar V^\alpha\big)\tag{3.26}$$

$$\equiv(\pi_a-f_a(q^i,P_\alpha))N^a,\tag{3.27}$$

where eq. (3.26) is the definition of H^{p} in general theory Gitman and Tyutin 1990, ch. 2. One sees that the Hamiltonian with primary constraints consists of constraints only, hence the canonical Hamiltonian H^{c} vanishes.

On the other hand, if H^{c} vanishes, one finds that

$$\begin{aligned}H^{\mathrm{p}}&=H^{\mathrm{c}}+(\pi_a-f_a(q^i,P_\alpha))N^a &(3.28)\\ &=(\pi_a-f_a(q^i,P_\alpha))N^a\equiv\Big(\pi_a-\frac{\partial L^{\mathrm{v}}}{\partial N^a}\big(q^i,N^b,\bar V^a\big)\Big)N^a\\ &=\pi_aN^a+P_\alpha\bar V^\alpha+\frac{\partial L^{\mathrm{v}}}{\partial N^a}\big(q^i,N^b,\bar V^\beta\big)N^a-P_\alpha\bar V^\alpha\\ &\equiv\pi_aN^a+P_\alpha\bar V^\alpha+\frac{\partial L^{\mathrm{v}}}{\partial N^a}\big(q^i,N^b,\bar V^\beta\big)N^a-\frac{\partial L^{\mathrm{v}}}{\partial V^\alpha}\big(q^i,P_\alpha,\bar V^\beta\big)\bar V^\alpha. &(3.29)\end{aligned}$$

where eq. (3.28) is a result of the general theory Gitman and Tyutin 1990, ch. 2. Comparing eqs. (3.26) and (3.29) gives eqs. (3.23) and (3.24), which implies the homogeneity of L^{v} in (N^a, V^α).

3.1.2 The relativistic particle

Derivation with linear constraint In ibid., the primary constraints are *defined* by

$$\Phi_a \equiv \pi_a - \frac{\partial L^{\mathrm{v}}}{\partial N^a}(q^i, N^b, \bar{V}^a), \tag{3.30}$$

and are always linear with respect to the momenta $\{p_a\}$.

Gauge structure and constraint

3.2 The free bosonic string

3.3 Parametrised field theories

3.4 Relational dynamical systems

3.5 General remarks on constrained systems

4 Hamiltonian formulation of general relativity

5 Quantum geometrodynamics

6 Quantum gravity with connections and loops

7 Quantisation of black holes

8 Quantum cosmology

9 String theory

10 Phenomenology, decoherence, and the arrow of time

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