An Introduction to [WZU17]

An Attempt to Avoid the Old Cosmological Constant Problem

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Outline

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Overview

- If contributed by vacuum energy, the cosmological constant could differ 120 order-of-magnitude from observation.
- Wang, Zhu, and Unruh consider vacuum energy density instead, which is subject to fluctuation, and their results could moderate the problem.
- They assumed a localised RW metric and obtained FL-like equations for $a(t, \vec{x})$, whose coefficient contains fluctuation, which is evaluated from QFT in flat space-time.
- The equations leads to local Hubble parameters which fluctuate, but a global average is claimed be defined.
- The solution to the equations is evaluated with various approximations.
- The results are qualitatively supported by numerical calculation and further discussion.

Outline

Cosmological constant and vacuum energy

- GR + vacuum QFT

$$G_{\mu\nu} + \lambda_{\text{b}} g_{\mu\nu} = 8\pi G T^{\text{vac}}_{\mu\nu} \tag{1}$$

- In Special Relativity, Lorentz invariance requires

$$T^{\text{vac}}_{\mu\nu} = -\rho^{\text{vac}}_{\mu\nu} \eta_{\mu\nu}, \tag{2}$$

which generalises to curved space-time as

$$T^{\text{vac}}_{\mu\nu} = -\rho^{\text{vac}}_{\mu\nu} g_{\mu\nu}. \tag{3}$$

- Effectively, eq. (1) can be written as

$$G_{\mu\nu} + \lambda_{\text{eff}} g_{\mu\nu} = 0, \qquad \lambda_{\text{eff}} := \lambda_{\text{b}} + 8\pi G \rho^{\text{vac}}; \tag{4}$$

$$G_{\mu\nu} = -8\pi G \rho^{\text{vac}}_{\text{eff}} g_{\mu\nu}, \qquad \rho^{\text{vac}}_{\text{eff}} := \rho^{\text{vac}} + \frac{\lambda_{\text{b}}}{8\pi G}. \tag{5}$$

Hubble parameter and cosmological constant

- Homogeneity and isotropy: Robertson–Walker metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{6}$$

- Hubble parameter / expansion rate $H := \dot{a}/a$; eqs. (4) and (5) take the corresponding Friedmann–Lemaître form

$$3H^2 = \lambda_{\text{eff}} = 8\pi G \rho_{\text{eff}}^{\text{vac}}, \tag{7}$$

$$3\ddot{a} = \lambda_{\text{eff}} a = 8\pi G \rho_{\text{eff}}^{\text{vac}} a. \tag{8}$$

- Solution to eq. (8)

$$a(t) = a(t_0)\,\mathrm{e}^{H(t-t_0)} \tag{9}$$

The Old Cosmological Problem

- Contributions to $\rho_{\text{eff}}^{\text{vac}}$ or λ_{eff}: vacuum fluctuation of all quantum fields, Electroweak phase transition, etc.
- λ_{eff} by vacuum fluctuation evaluated in Minkowski space: taking a scalar field (also see below) and using sharp-momentum cut-off,

$$\langle \rho^{\text{vac}} \rangle = \frac{\Lambda^4}{16\pi^2}. \tag{10}$$

- Setting $\Lambda = E_{\text{P}}$ results in a surpass of the observed value of λ_{eff} by a factor of $\sim 10^{120}$.

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Matter: field and quantisation

- Matter model: free neutral massless scalar field in flat space-time

$$S_{\mathrm{m}}[\phi] := \int \mathrm{d}^4 x \left[-\frac{1}{2} \eta^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) \right] \tag{11}$$

- Canonical quantisation

$$\phi(t, \vec{x}) := \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} \left(a_{\vec{k}} \, \mathrm{e}^{-\mathrm{i}(\omega t - \vec{k} \cdot \vec{x})} + \text{h.c.} \right), \tag{12}$$

so that

$$T_{00} = \frac{1}{2} \left(\dot{\phi}^2 + (\nabla \phi)^2 \right) \tag{13}$$

$$= \int \mathrm{d}^3 k_1 \, \mathrm{d}^3 k_2 \, f \left(a_{\vec{k}_1} a^\dagger_{\vec{k}_2}, a^\dagger_{\vec{k}_1} a_{\vec{k}_2}, a_{\vec{k}_1} a_{\vec{k}_2}, a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} \right) \tag{14}$$

Matter: fluctuating vacuum energy density

- Hamiltonian

$$H := \int \mathrm{d}^3 x \, T_{00} = \frac{1}{2} \int \mathrm{d}^3 k \, \omega \Big(a_{\vec{k}} a^\dagger_{\vec{k}} + a^\dagger_{\vec{k}} a_{\vec{k}} \Big). \tag{15}$$

- $|0\rangle$ is an eigenstate of H, because it is defined by

$$a_{\vec{k}} \, |0\rangle := 0, \quad \forall \vec{k} \tag{16}$$

- This is not the case for T_{00}, and one has

$$\Big\langle (\Delta T_{00})^2 \Big\rangle = \frac{2}{3} \langle T_{00} \rangle^2, \qquad \langle T_{00} \rangle = \frac{\Lambda^4}{16\pi^2}. \tag{17}$$

Space-time: localised Robertson–Walker metric

- Inhomogeneity and isotropy

$$\boxed{\mathrm{d}s^2 := -\mathrm{d}t^2 + a(t,\vec{x})\,\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j.} \tag{18}$$

- Generalising the proper distance and Hubble parameter

$$L_{1\to 2}(t) := \int_{\vec{x}_1}^{\vec{x}_2} \sqrt{a^2(t,\vec{x})}\,\mathrm{d}l, \quad H_{1\to 2}(t) := \frac{\dot{L}_{1\to 2}}{L_{1\to 2}} \tag{19}$$

- Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{20}$$

where $T_{\mu\nu}$'s are viewed as (random c-)numbers (and subject to fluctuation)

Space-time: stochastic oscillator equation for a

- Taking the trace of eq. (20) leads to

$$\boxed{\ddot{a} + \Omega^2(t, \vec{x})\, a = 0,} \tag{21}$$

where

$$\Omega^2 := \frac{4\pi G}{3} g^{\mu\nu} T_{\mu\nu}, \tag{22}$$

which is viewed as a (random c-)number (and subject to fluctuation)

- $T_{\mu\nu}$ given by the quantised scalar field eqs. (11) and (12); evaluate Hubble parameter by solving eq. (21)

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Parametric oscillator

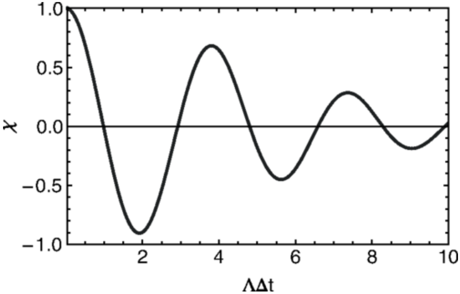
- If Ω were definite and periodic in t, eq. (21) would be a Hill's equation, describing parametric oscillation (see e.g. [LL76, §27] and Wikipedia)
- General solution

$$a(t,\vec{x}) = c_1 \mathrm{e}^{+H_{\vec{x}}\,t} P_1(t,\vec{x}) + c_2 \mathrm{e}^{-H_{\vec{x}}\,t} P_2(t,\vec{x}) \tag{23}$$

  where $H_{\vec{x}} > 0$ (thus second term exponentially suppressed at late time) and P_i's have the same period as Ω.
- It was claimed that Ω behaves quasi-periodically, because of $\mathrm{Cov}[\Omega^2(t_1,\vec{x}),\Omega^2(t_2,\vec{x})]$ (see below)

Quasi-periodicity

- Covariance of $\Omega^2(t)$, taken from [WZU17]



- The solution is modified (at late time) to be

$$\boxed{a(t,\vec{x}) \approx e^{\int_0^t dt'\, H_{\vec{x}}(t')} P(t,\vec{x}),} \tag{24}$$

where $P(t,\vec{x})$ behaves also only quasi-periodically (see below).

Proper distance and Hubble parameter revisited

- The expressions for proper distance and Hubble parameter in terms of eq. (24) are

$$L_{1\to 2}(t) = L_{1\to 2}(0)\, \mathbb{e}^{Ht}, \quad L_{1\to 2}(0) = \int_{\vec{x}_1}^{\vec{x}_2} \sqrt{P^2(t,\vec{x})}\, \mathbb{d}t, \tag{25}$$

$$\boxed{H = \frac{1}{t}\int_0^t H_{\vec{x}}(t')\, \mathbb{d}t'.} \tag{26}$$

- It looks like that the authors assumed averaging over a long distance / time smooths out the difference between different $\vec{x}$.
- It was claimed that leading-order WKB gives

$$a(t,\vec{x}) \approx \frac{A_0}{\sqrt{\Omega}} \cos\left(\int_0^t \mathbb{d}t'\, \Omega(t',\vec{x}) + \theta_{\vec{x}} \right). \tag{27}$$

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Review: canonical transformation

- Introducing the (type-2) generating function $G_2(q, P, t)$

$$S = \int p \, \mathrm{d}q - H \, \mathrm{d}t \equiv \int P \, \mathrm{d}Q - K \, \mathrm{d}t + \mathrm{d}(-QP + G_2) \quad (28)$$

where $H = H(q, p, t)$, $K = K(Q, P, t)$, so that

$$\int (p - \partial_q G_2) \, \mathrm{d}q + (Q - \partial_P G_2) \, \mathrm{d}P + (K - H + \partial_t G_2) \, \mathrm{d}t \equiv 0. \quad (29)$$

- Arbitrariness of the track in phase space requires

$$p = \partial_q G_2, \quad Q = \partial_P G_2, \quad K = H - \partial_t G_2. \quad (30)$$

- It can be shown that G_2 generates a Possion-bracket-keeping transformation from $(q, p; H)$ to $(Q, P; K)$.

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Action-angle variables

See e.g. [Hen93]

- Consider a $G_2(\varphi, I)$ which brings $(q, p; H(q,p))$ to $(\varphi, I; K(I))$; note the time-independence
- Dynamics in the new variables reads

$$\begin{aligned} \dot{\varphi} &= \partial_I K, & \dot{I} &= 0; & (31)\\ \Rightarrow \varphi &= (\partial_I K)\, t + \varphi_0, & I &\equiv I_0 & (32) \end{aligned}$$

- G_2 can be solved by the Hamilton–Jacobi-like equation

$$H\big(q, \partial_q G_2\big) = K(I). \tag{33}$$

- If the motion has a period T and one requires $\Delta\varphi := (\partial_I K)\, T \equiv 2\pi$, it can be shown that

$$I = \frac{1}{2\pi} \oint p \, \mathrm{d}q, \tag{34}$$

thus the name action-angle variable comes.

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Adiabatic approximation

See e.g. [Hen93]

- The variables above can be extended to $H = H(q, p; \lambda)$, where a perturbative expansion is made in terms of $\epsilon := \mathrm{d}\lambda/\mathrm{d}t$.
- It can be shown that the leading-order expansion gives

$$I(t) = I(0) + O(\epsilon), \quad 0 \leq t \leq \epsilon^{-1} \tag{35}$$

- Let ω be the angular frequency of the unperturbed system. If

$$\lambda^{-1}\epsilon = \frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t} \ll \omega, \tag{36}$$

then I varies only slightly during one quasi-period $2\pi/\omega$ and is thus a adiabatic invariant.

Adiabaticity of eq. (21)

- Using sharp-momentum cut-off Λ,

$$\langle \Omega^2 \rangle = \frac{8\pi G}{3} \langle \dot{\phi}^2 \rangle = \frac{G}{6\pi} \Lambda^4, \tag{37}$$

$$\langle \dot{\Omega}^2 \rangle = \frac{8\pi G}{3} \langle \ddot{\phi}^2 \rangle = \frac{G}{9\pi} \Lambda^6. \tag{38}$$

- Evaluating eq. (36):

$$\frac{\dot{\Omega}}{\Omega} \sim \sqrt{\frac{\langle \dot{\Omega} \rangle^2}{\langle \Omega^2 \rangle}} = \frac{2}{3}\Lambda, \quad \Omega \sim \sqrt{\langle \Omega^2 \rangle} = \frac{1}{\sqrt{6\pi}} \sqrt{G}\Lambda^2, \tag{39}$$

so the adiabatic condition is satisfied if $\sqrt{G}\Lambda \to +\infty$.

- It looks like that the cut-off needs to be much higher than the Planck scale.

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Action-angle variables of eq. (21)

- For fixed Ω,

$$I = \frac{E}{\Omega} = \frac{1}{2} A_0^2, \tag{40}$$

where A_0 is the amplitude in the WKB solution eq. (27). It was argued that one can replace $A_0 \to A_0 \mathrm{e}^{\int_0^t H_{\vec{x}}(t')\,\mathrm{d}t'}$.

- Transforming $(a, \dot{a})$ to (φ, I)

$$a = \sqrt{2I/\Omega}\,\sin\varphi, \quad \dot{a} = \sqrt{2I\Omega}\,\cos\varphi, \tag{41}$$

so the canonical equations are

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -I\frac{\dot{\Omega}}{\Omega}\cos 2\varphi, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \Omega + \frac{\dot{\Omega}}{2\Omega}\sin 2\varphi. \tag{42}$$

- Integration yields

$$I(t) = I(0)\exp\left(-\int_0^t \frac{\dot{\Omega}}{2\Omega}\cos 2\varphi\,\mathrm{d}t\right). \tag{43}$$

Evaluation of the global Hubble parameter

- Combining eqs. (40) and (43) gives $H_{\vec{x}} = -\frac{\dot{\Omega}}{2\Omega}\cos 2\varphi$.
- The global Hubble parameter thus reads

$$H = -\frac{1}{t}\Re\int_0^t \frac{\dot{\Omega}}{2\Omega}\mathrm{e}^{2\mathrm{i}\varphi}\,\mathrm{d}t' = -\frac{1}{t}\Re\int_{\varphi_0}^{\varphi}\frac{\dot{\Omega}}{2\Omega}\mathrm{e}^{2\mathrm{i}\varphi'}\frac{\mathrm{d}t'}{\mathrm{d}\varphi'}\,\mathrm{d}\varphi. \quad (44)$$

- Completing the complex integration contour and picking only the main pole suggest $H \sim \Lambda\,\mathrm{e}^{-2\Im\varphi_{(0)}} = \Lambda\,\mathrm{e}^{-2\Im\varphi\left(t_{(0)}\right)}$.
- It was argued that $\varphi\left(t_{(0)}\right) \sim \Omega t_{(0)}$, where $\Omega \sim \sqrt{G}\Lambda$, $t_{(0)} \sim \Lambda^{-1}$.
- The final expression of the global Hubble parameter reads

$$\boxed{H \sim \alpha\Lambda\,\mathrm{e}^{-\beta\sqrt{G}\Lambda}.} \quad (45)$$

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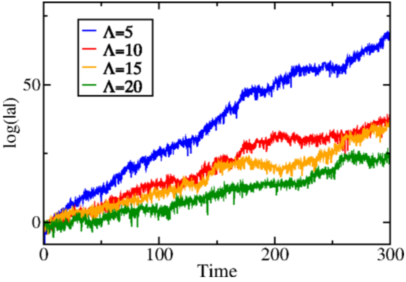
Numerical simulation

Figures from [WZU17]

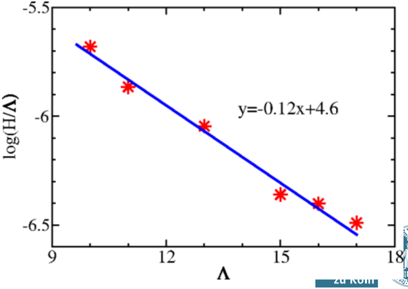
- Discretise to get $\{\phi_{\vec{k}}\}$
- Ground-state field is described by a positive Wigner function

$$W(\{\phi_{\vec{k}}\},\{\pi_{\vec{k}}\},t)=\prod_{\vec{k}}\frac{1}{\pi}\exp\left(-\frac{\pi_{\vec{k}}^2}{\omega}-\phi_{\vec{k}}^2\omega\right). \tag{46}$$

Higher cut-off gives lower H



Fitting $\ln(H/\Lambda)$ gives α and β



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Summary

- Making of the oscillator equation eq. (21):
  1. Quantised Klein–Gordon field in flat space-time
  2. Classical, localised Robertson–Walker metric
- Solving the oscillator equation eq. (21):
  1. Ignoring the Stochasticity in most calculation
  2. Parametric oscillation
  3. WKB approximation
  4. Adiabatic approximation
- Validation of the various approximation: numerics
- Summarised up to eq. (75); The rest (up to eq. (214); 3 appendixes) contains more details and discussions

Comments

- Differential equation with fluctuating coefficients: stochastic differential equation [Kam07]
  - Famous case: Langevin equation for Brownian motion

$$m\dot{\vec{v}} + \lambda\vec{v} = \vec{\eta}(t) \tag{47}$$

- Harmonic oscillator with stochastic frequency: studied, e.g. [BFP73; Kam76]
- Applying quantum fluctuation to semi-classical Einstein equation: studied, e.g. [HV08]

For Further Reading I

N.G. van Kampen. *Stochastic Processes in Physics and Chemistry*. 3rd. Elsevier, 2007.

L.D. Landau and E.M. Lifshitz. *Mechanics*. 3rd. Vol. 1. Course of Theoretical Physics. Elsevier, 1976.

R.C. Bourret, U. Frisch, and A. Pouquet. "Brownian motion of harmonic oscillator with stochastic frequency". In: *Physica* 65.2 (Apr. 1973), pp. 303–320.

J. Henrard. "The Adiabatic Invariant in Classical Mechanics". In: *Dynamics Reported*. Springer, 1993, pp. 117–235.

Bei Lok Hu and Enric Verdaguer. "Stochastic Gravity: Theory and Applications". In: *Living Reviews in Relativity* 11.1 (May 2008).

N.G. van Kampen. "Stochastic differential equations". In: *Physics Reports* 24.3 (Mar. 1976), pp. 171–228.

For Further Reading II

Qingdi Wang, Zhen Zhu, and William G. Unruh. "How the huge energy of quantum vacuum gravitates to drive the slow accelerating expansion of the Universe". In: *Physical Review D* 95.10 (May 2017).