An Introduction to [WZU17]

An Attempt to Avoid the Old Cosmological Constant Problem

Yi-Fan Wang (王 一帆)

Institut für Theoretische Physik
Universität zu Köln

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Outline

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Overview

- If contributed by vacuum energy, the cosmological constant could differ 120 order-of-magnitude from observation.
- Wang, Zhu, and Unruh consider vacuum energy density instead, which is subject to fluctuation, and their result moderates the problem.
- They assumed a localised RW metric and obtained FL-like equations for $a(t, \vec{x})$, whose coefficient contains quantum fluctuation.
- The equations leads to local Hubble parameters which fluctuate, but a global average could be defined.
- The solution to the equations is evaluated with the help of theories of parametric oscillator and adiabatic invariance.
- The result is qualitatively supported by numerical calculation and further quantitative discussions.

Outline

Cosmological constant and vacuum energy

- GR + vacuum QFT

$$G_{\mu\nu} + \lambda_{\mathrm{b}} g_{\mu\nu} = 8\pi G T^{\mathrm{vac}}_{\mu\nu} \tag{1}$$

- In Minkowski space-time, Lorentz invariance requires

$$T^{\mathrm{vac}}_{\mu\nu} = -\rho^{\mathrm{vac}}_{\mu\nu} \eta_{\mu\nu}, \tag{2}$$

which generalises to curved space-time as

$$T^{\mathrm{vac}}_{\mu\nu} = -\rho^{\mathrm{vac}}_{\mu\nu} g_{\mu\nu}. \tag{3}$$

- Effectively, eq. (1) can be written as

$$G_{\mu\nu} + \lambda_{\mathrm{eff}} g_{\mu\nu} = 0, \qquad \lambda_{\mathrm{eff}} = \lambda_{\mathrm{b}} + 8\pi G \rho^{\mathrm{vac}}; \tag{4}$$

$$G_{\mu\nu} = -8\pi G \rho^{\mathrm{vac}}_{\mathrm{eff}} g_{\mu\nu}, \qquad \rho^{\mathrm{vac}}_{\mathrm{eff}} = \rho^{\mathrm{vac}} + \frac{\lambda_{\mathrm{b}}}{8\pi G}. \tag{5}$$

Hubble parameter and cosmological constant

- Homogeneity and isotropy: Robertson–Walker metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{6}$$

- Hubble parameter / expansion rate $H := \dot{a}/a$; eqs. (4) and (5) take the corresponding Friedmann–Lemaître form

$$3H^2 = \lambda_{\text{eff}} = 8\pi G \rho_{\text{eff}}^{\text{vac}}, \tag{7}$$

$$3\ddot{a} = \lambda_{\text{eff}} a = 8\pi G \rho_{\text{eff}}^{\text{vac}} a. \tag{8}$$

- Solution to eq. (8)

$$a(t) = a(t_0)\,\mathrm{e}^{H(t-t_0)} \tag{9}$$

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The Old Cosmological Problem

- Contributions to $\rho_{\text{eff}}^{\text{vac}}$ or λ_{eff}: vacuum fluctuation of all quantum fields, Electroweak phase transition, etc.
- λ_{eff} by vacuum fluctuation evaluated in Minkowski space: taking a scalar field and using sharp-momentum cut-off (also see below),

$$\langle \rho^{\text{vac}} \rangle = \frac{\Lambda^4}{16\pi^2}. \tag{10}$$

Setting $\Lambda = E_{\text{P}}$ results in a surpass of the observed value of λ_{eff} by $\sim 10^{120}$.

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Matter: field and quantisation

- Matter model

$$S_{\mathsf{m}}[\phi] = \int \mathbb{d}^4x \left[-\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)\right] \tag{11}$$

- Canonical quantisation

$$\phi(t,\vec{x}) = \int \frac{\mathbb{d}^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} \left(a_{\vec{k}}\, \mathbb{e}^{-\mathbb{i}\left(\omega t - \vec{k}\cdot\vec{x}\right)} + \text{h.c.}\right), \tag{12}$$

so that

$$T_{00} = \frac{1}{2}\left(\dot{\phi}^2 + (\nabla\phi)^2\right) \tag{13}$$

$$= \int \mathbb{d}^3k_1\, \mathbb{d}^3k_2\, f\left(a_{\vec{k}_1} a^\dagger_{\vec{k}_2}, a^\dagger_{\vec{k}_1} a_{\vec{k}_2}, {\color{red} a_{\vec{k}_1} a_{\vec{k}_2}, a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2}}\right) \tag{14}$$

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Matter: fluctuating vacuum energy density

- $|0\rangle$ is an eigenstate of $H = \int \mathrm{d}^3 x T_{00}$, because

$$a_{\vec{k}} |0\rangle := 0, \quad \forall \vec{k} \tag{15}$$

- This is not the case for T_{00}, and

$$\left\langle (\Delta T_{00})^2 \right\rangle = \frac{2}{3} \langle T_{00} \rangle^2, \qquad \langle T_{00} \rangle = \frac{\Lambda^4}{16\pi^2}. \tag{16}$$

Space-time: localised Robertson–Walker metric

- Inhomogeneity and isotropy

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t,\vec{x})\,\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j.} \tag{17}$$

- Generalising the proper distance and Hubble parameter

$$L_{1\to 2}(t) = \int_{\vec{x}_1}^{\vec{x}_2} \sqrt{a^2(t,\vec{x})}\,\mathrm{d}l, \quad H_{1\to 2}(t) = \frac{\dot{L}_{1\to 2}}{L_{1\to 2}} \tag{18}$$

- Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{19}$$

where $T_{\mu\nu}$'s are viewed as (random c-)numbers (and subject to fluctuation)

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Space-time: stochastic oscillator equation for a

- Taking the trace of eq. (19) leads to

$$\boxed{\ddot{a} + \Omega^2(t, \vec{x})\, a = 0,} \tag{20}$$

  where

$$\Omega^2 = \frac{4\pi G}{3} g^{\mu\nu} T_{\mu\nu}, \tag{21}$$

  which is viewed as a (random c-)number (and subject to fluctuation)
- $T_{\mu\nu}$ given by the quantised scalar field eqs. (11) and (12); evaluate Hubble parameter by solving eq. (20)

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Parametric oscillator

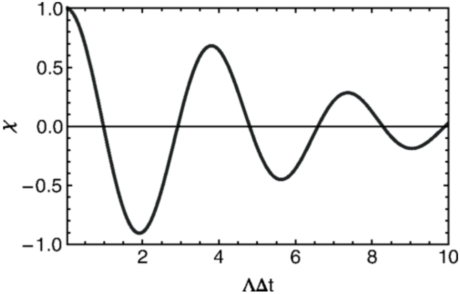
- If Ω were definite and periodic in t, eq. (20) would be a Hill's equation, describing parametric oscillation (see e.g. [LL76, §27] and Wikipedia)
- General solution

$$a(t,\vec{x}) = c_1 \mathbb{e}^{+H_{\vec{x}}t} P_1(t,\vec{x}) + c_2 \mathbb{e}^{-H_{\vec{x}}t} P_2(t,\vec{x}) \tag{22}$$

  where $H_{\vec{x}} > 0$ (thus second term exponentially suppressed at late time) and P_i's have the same period as Ω.
- **Claim**: Ω behaves quasi-periodically, because of $\mathrm{Cov}[\Omega^2(t_1,\vec{x}),\Omega^2(t_2,\vec{x})]$ (see below)

Quasi-periodicity

- Covariance of $\Omega^2(t)$, taken from [WZU17]



- The solution is modified (at late time) to be

$$\boxed{a(t,\vec{x}) \approx \mathrm{e}^{\int_0^t H_{\vec{x}}(t')\,\mathrm{d}t'} P(t,\vec{x}),} \tag{23}$$

where $P(t,\vec{x})$ behaves also only quasi-periodically (see below).

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Proper distance and Hubble parameter revisited

- The expressions for proper distance and Hubble parameter in terms of eq. (23) are

$$L_{1\to2}(t) = L(0)\,\mathbb{e}^{Ht}, \quad L_{1\to2}(0) = \int_{\vec{x}_1}^{\vec{x}_2} \sqrt{P^2(t,\vec{x})}\,\mathbb{d}t, \tag{24}$$

$$\boxed{H = \frac{1}{t}\int_0^t H_{\vec{x}}(t')\,\mathbb{d}t'.} \tag{25}$$

- It looks like that the authors assumed averaging $H_{\vec{x}}$ over a long time smooths out the difference between different $\vec{x}$.
- It was claimed that WKB gives

$$a(t,\vec{x}) \approx \frac{A_0\mathbb{e}^{\int_0^t H_{\vec{x}}(t')\,\mathbb{d}t'}}{\sqrt{\Omega}}\cos\left(\int_0^t \Omega(t',\vec{x})\,\mathbb{d}t' + \theta_{\vec{x}}\right). \tag{26}$$

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Review: canonical transformation

- Introducing the (type-2) generating function $G_2(q, P, t)$

$$S = \int p\,\mathrm{d}q - H\,\mathrm{d}t \equiv \int P\,\mathrm{d}Q - K\,\mathrm{d}t + \mathrm{d}(-QP + G_2) \quad (27)$$

where $H = H(q, p, t)$, $K = K(Q, P, t)$, so that

$$\int (p - \partial_q G_2)\,\mathrm{d}q + (Q - \partial_P G_2)\,\mathrm{d}P + (K - H + \partial_t G_2)\,\mathrm{d}t \equiv 0. \quad (28)$$

- Arbitrariness of the track in phase space requires

$$p = \partial_q G_2, \quad Q = \partial_P G_2, \quad K = H - \partial_t G_2. \quad (29)$$

- It can be shown that G_2 generates a Possion-bracket-keeping transformation from $(q, p; H)$ to $(Q, P; K)$.

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Action-angle variables

- Consider a $G_2(q, J)$ which brings $(q, p; H(q, p))$ to $(\varphi, I; K(I))$; note the time-independence
- Dynamics in the new variables reads

$$\begin{aligned} \dot{\varphi} &= \partial_I K, & \dot{I} &= 0; \qquad (30) \\ \Rightarrow \varphi &= (\partial_I K)\, t + \varphi_0, & I &\equiv I_0 \qquad (31) \end{aligned}$$

- G_2 can be solved by the Hamilton–Jacobi-like equation

$$H(q, \partial_q G_2) = K(J). \qquad (32)$$

- If the motion has a period T and it requires $\Delta\varphi := (\partial_I K)\, T \equiv 2\pi$ (angle variable), it can be shown that

$$I = \frac{1}{2\pi} \oint p \, \mathrm{d}q, \qquad (33)$$

thus the name action-angle variable comes.

Adiabatic approximation

See e.g. [Hen93]

- The variables above can be extended to $H = H(q, p; \lambda)$;, where a perturbative expansion is made in terms of $\epsilon = \mathrm{d}\lambda/\mathrm{d}t$.
- It can be shown that the leading-order expansion gives

$$I(t) = I(0) + O(\epsilon), \quad 0 \leq t \leq \epsilon^{-1} \tag{34}$$

- Let ω be the angular frequency of the unperturbed system. If

$$\lambda^{-1}\epsilon = \frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t} \ll \omega, \tag{35}$$

then I varies only slightly during one quasi-period $2\pi/\omega$ and is thus a adiabatic invariance.

Adiabaticity of eq. (20)

- Using sharp-momentum cut-off $\varLambda$,

$$\langle \Omega^2 \rangle = \frac{8\pi G}{3} \langle \dot{\phi}^2 \rangle = \frac{G}{6\pi} \varLambda^4, \tag{36}$$

$$\langle \dot{\Omega}^2 \rangle = \frac{8\pi G}{3} \langle \ddot{\phi}^2 \rangle = \frac{G}{9\pi} \varLambda^6. \tag{37}$$

- Evaluating eq. (35):

$$\frac{\dot{\Omega}}{\Omega} \sim \sqrt{\frac{\langle \dot{\Omega} \rangle^2}{\langle \Omega^2 \rangle}} = \frac{2}{3} \varLambda, \quad \Omega \sim \sqrt{\langle \Omega^2 \rangle} = \frac{1}{\sqrt{6\pi}} \sqrt{G} \Lambda^2, \tag{38}$$

so the adiabatic condition is satisfied if $\sqrt{G}\varLambda \to +\infty$.

- Cut-off much higher than the Planck scale?

Action-angle variables of eq. (20)

- For fixed Ω,

$$I = \frac{E}{\Omega} = \frac{1}{2} A_0^2, \tag{39}$$

where A is the amplitude. It was argued that one can replace $A_0 \to A_0 \mathrm{e}^{\int_0^t H_{\vec{x}}(t')\,\mathrm{d}t'}$.

- Transforming $(a, \dot{a})$ to (φ, I)

$$a = \sqrt{2I/\Omega}\sin\varphi, \quad \dot{a} = \sqrt{2I\Omega}\cos\varphi, \tag{40}$$

so the canonical equations are

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -I\frac{\dot{\Omega}}{\Omega}\cos 2\varphi, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \Omega + \frac{\dot{\Omega}}{2\Omega}\sin 2\varphi. \tag{41}$$

- Integration yields

$$I(t) = I(0)\exp\left(-\int_0^t \frac{\dot{\Omega}}{2\Omega}\cos 2\varphi\,\mathrm{d}t\right) \tag{42}$$

Evaluation of the global Hubble parameter

- Combining eqs. (39) and (42) gives $H_{\vec{x}} = -\frac{\dot{\Omega}}{2\Omega}\cos 2\varphi$.
- The global Hubble parameter thus reads

$$H = -\frac{1}{t}\Re\int_0^t \frac{\dot{\Omega}}{2\Omega}\mathbb{e}^{2\mathbb{i}\varphi}\,\mathbb{d}t' = -\frac{1}{t}\Re\int_{\varphi_0}^{\varphi} \frac{\dot{\Omega}}{2\Omega}\mathbb{e}^{2\mathbb{i}\varphi'}\frac{\mathbb{d}t'}{\mathbb{d}\varphi'}\,\mathbb{d}\varphi. \quad (43)$$

- Completing the complex integration contour and picking only the main pole suggest $H \sim \Lambda\,\mathbb{e}^{-2\Im\varphi_{(0)}} = \Lambda\,\mathbb{e}^{-2\Im\varphi\left(t_{(0)}\right)}$.
- It was argued that $\varphi\left(t_{(0)}\right) \sim \Omega t_{(0)}$, where $\Omega \sim \sqrt{G}\Lambda$, $t_{(0)} \sim \Lambda^{-1}$.
- The final expression of the global Hubble parameter reads

$$\boxed{H \sim \alpha\Lambda\,\mathbb{e}^{-\beta\sqrt{G}\Lambda}.} \quad (44)$$

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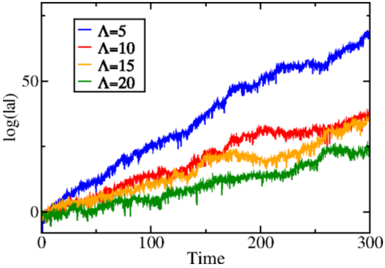
Summary

- Making of the oscillator equation eq. (20):
  1. Quantised Klein–Gordon field in flat space-time
  2. Classical, localised Robertson–Walker metric
- Solving the oscillator equation eq. (20):
  1. Ignoring the stochastic frequency in most cases
  2. Parametric oscillation
  3. Adiabatic approximation
- Validation of the various approximation: numerics [WZU17]



- Summarised up to eq. (75); The rest (eq. (214); 3 appd.) contains details and discussions

Comments

- Differential equation with fluctuating coefficients: stochastic differential equation [Kam07]
  - Famous case: Langevin equation for Brownian motion
- Harmonic oscillator with stochastic frequency: studied, e.g. [BFP73; Kam76]
- Applying quantum fluctuation to semi-classical Einstein equation: studied, e.g. [HV08]

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For Further Reading I

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For Further Reading II

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