An Introduction to [WZU17]

An Attempt to Avoid the Old Cosmological Constant Problem

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Outline

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Overview

The Old Cosmological Constant Problem

Matter and space-time models

Solving $a(t, \vec{x})$: parametric oscillator

Solving $a(t, \vec{x})$: adiabatic approximation

Distance between Density Operators

Overview

- If contributed by vacuum energy, the cosmological constant could differ 120 order-of-magnitude from observation.
- Wang, Zhu, and Unruh consider vacuum energy density instead, which is subject to fluctuation, and their result moderates the problem.
- They assumed a localised RW metric and obtained FL-like equations for $a(t, \vec{x})$, whose coefficient contains quantum fluctuation.
- The equations leads to local Hubble parameters which fluctuate, but a global average could be defined.
- The solution to the equations is evaluated with the help of theories of parametric oscillator and adiabatic invariance.
- The result is qualitatively supported by numerical calculation and further quantitative discussions.

Outline

Cosmological constant and vacuum energy

- GR + vacuum QFT

$$G_{\mu\nu} + \lambda_{\mathrm{b}} g_{\mu\nu} = 8\pi G T^{\mathrm{vac}}_{\mu\nu} \tag{1}$$

- In Minkowski space-time, Lorentz invariance requires

$$T^{\mathrm{vac}}_{\mu\nu} = -\rho^{\mathrm{vac}}_{\mu\nu} \eta_{\mu\nu}, \tag{2}$$

which generalises to curved space-time as

$$T^{\mathrm{vac}}_{\mu\nu} = -\rho^{\mathrm{vac}}_{\mu\nu} g_{\mu\nu}. \tag{3}$$

- Effectively, eq. (1) can be written as

$$G_{\mu\nu} + \lambda_{\mathrm{eff}} g_{\mu\nu} = 0, \qquad \lambda_{\mathrm{eff}} = \lambda_{\mathrm{b}} + 8\pi G \rho^{\mathrm{vac}}; \tag{4}$$

$$G_{\mu\nu} = -8\pi G \rho^{\mathrm{vac}}_{\mathrm{eff}} g_{\mu\nu}, \qquad \rho^{\mathrm{vac}}_{\mathrm{eff}} = \rho^{\mathrm{vac}} + \frac{\lambda_{\mathrm{b}}}{8\pi G}. \tag{5}$$

Hubble parameter and cosmological constant

- Homogeneity and isotropy: Robertson–Walker metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{6}$$

- Hubble parameter / expansion rate $H := \dot{a}/a$; eqs. (4) and (5) take the corresponding Friedmann–Lemaître form

$$3H^2 = \lambda_{\text{eff}} = 8\pi G\rho^{\text{vac}}_{\text{eff}}, \tag{7}$$

$$3\ddot{a} = \lambda_{\text{eff}} a = 8\pi G\rho^{\text{vac}}_{\text{eff}} a. \tag{8}$$

- Solution to eq. (8)

$$a(t) = a(t_0)\,\mathrm{e}^{H(t-t_0)} \tag{9}$$

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The Old Cosmological Problem

- Contributions to $\rho_{\text{eff}}^{\text{vac}}$ or λ_{eff}: vacuum fluctuation of all quantum fields, Electroweak phase transition, etc.
- λ_{eff} by vacuum fluctuation evaluated in Minkowski space: taking a scalar field and using sharp-momentum cut-off (also see below),

$$\langle \rho^{\text{vac}} \rangle = \frac{\Lambda^4}{16\pi^2}. \tag{10}$$

  Setting $\Lambda = E_{\text{P}}$ results in a surpass of the observed value of λ_{eff} by $\sim 10^{120}$.

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Matter: field and quantisation

- Matter model

$$S_{\mathsf{m}}[\phi] = \int \mathrm{d}^4 x \left[-\frac{1}{2} \eta^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) \right] \tag{11}$$

- Canonical quantisation

$$\phi(t, \vec{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} \left(a_{\vec{k}}\, \mathrm{e}^{-\mathrm{i}(\omega t - \vec{k}\cdot\vec{x})} + \text{h.c.} \right), \tag{12}$$

so that

$$T_{00} = \frac{1}{2}\left(\dot{\phi}^2 + (\nabla\phi)^2 \right) \tag{13}$$

$$= \int \mathrm{d}^3 k_1\, \mathrm{d}^3 k_2\, f\left(a_{\vec{k}_1} a^\dagger_{\vec{k}_2}, a^\dagger_{\vec{k}_1} a_{\vec{k}_2}, a_{\vec{k}_1} a_{\vec{k}_2}, a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} \right) \tag{14}$$

Matter: fluctuating vacuum energy density

- $|0\rangle$ is an eigenstate of $H = \int \mathrm{d}^3 x T_{00}$, because

$$a_{\vec{k}} |0\rangle := 0, \quad \forall \vec{k} \tag{15}$$

- This is not the case for T_{00}, and

$$\left\langle (\Delta T_{00})^2 \right\rangle = \frac{2}{3} \langle T_{00} \rangle^2, \qquad \langle T_{00} \rangle = \frac{\Lambda^4}{16\pi^2}. \tag{16}$$

Space-time: localised Robertson–Walker metric

- Inhomogeneity and isotropy

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t, \vec{x})\, \delta_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j. \tag{17}$$

- Generalising the proper distance and Hubble parameter

$$L(1 \to 2; t) = \int_{\vec{x}_1}^{\vec{x}_2} \sqrt{a^2(t, \vec{x})}\, \mathrm{d}l, \quad H(1 \to 2; t) = \frac{\dot{L}(1 \to 2)}{L(1 \to 2)} \tag{18}$$

- Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{19}$$

where $T_{\mu\nu}$'s are viewed as (random c-)numbers (and subject to fluctuation)

Space-time: stochastic oscillator equation for a

- Taking the trace of eq. (19) leads to

$$\ddot{a} + \Omega^2(t, \vec{x})\, a = 0, \tag{20}$$

  where

$$\Omega^2 = \frac{4\pi G}{3} g^{\mu\nu} T_{\mu\nu}, \tag{21}$$

  which is viewed as a (random c-)number (and subject to fluctuation)
- $T_{\mu\nu}$ given by the quantised scalar field eqs. (11) and (12); evaluate Hubble parameter by solving eq. (20)

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Parametric oscillator

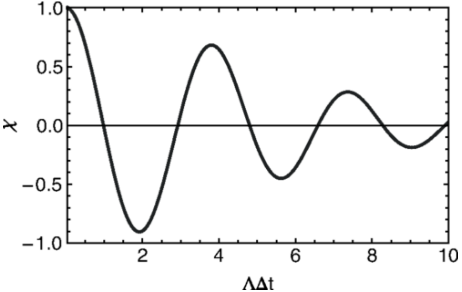
- If $\varOmega$ were definite and periodic in t, eq. (20) would be a Hill's equation, describing parametric oscillation (see e.g. [LL76, §27] and Wikipedia)
- General solution

$$a(t,\vec{x}) = c_1 \mathbb{e}^{+H_{\vec{x}}\,t} P_1(t,\vec{x}) + c_2 \mathbb{e}^{-H_{\vec{x}}\,t} P_2(t,\vec{x}) \tag{22}$$

  where $H_{\vec{x}} > 0$ (thus second term exponentially suppressed at late time) and P_i's have the same period as $\varOmega$.
- **Claim**: $\varOmega$ behaves quasi-periodically, because of $\mathrm{Cov}\left[\varOmega^2(t_1,\vec{x}), \varOmega^2(t_2,\vec{x})\right]$ (see below)

Quasi-periodicity

- Covariance of $\Omega^2(t)$, taken from [WZU17]



- The solution is modified (at late time) to be

$$a(t,\vec{x}) \approx e^{\int_0^t H_{\vec{x}}(t')\,dt'} P(t,\vec{x}), \tag{23}$$

where $P(t,\vec{x})$ behaves also only quasi-periodically (see below).

Proper distance and Hubble parameter revisited

- The expressions for proper distance and Hubble parameter in terms of eq. (23) are

$$L(1 \to 2; t) = L(0)\, e^{Ht}, \tag{24}$$

$$L(1 \to 2; 0) = \int_{\vec{x}_1}^{\vec{x}_2} \sqrt{P^2(t, \vec{x})}\, dt, \tag{25}$$

$$H = \frac{1}{t} \int_0^t H_{\vec{x}}(t')\, dt'. \tag{26}$$

- It looks like that the author assumed averaging $H_{\vec{x}}$ over a long time smooths out the difference between different $\vec{x}$.

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Trace Distance

Definitions

- Trace distance between Hermitian operators $\widehat{M}$ and $\widehat{N}$

$$T\left(\widehat{M}, \widehat{N}\right) := \frac{1}{2} \operatorname{tr} \sqrt{\left(\widehat{M} - \widehat{N}\right)^{\dagger} \left(\widehat{M} - \widehat{N}\right)} \tag{27}$$

- For density operators $\hat{\rho}$ and $\hat{\sigma}$,
  - $0 \leq T(\hat{\rho}, \widehat{\sigma}) \leq 1$;
  - Controlled by fidelity in

$$\boxed{1 - F(\hat{\rho}, \widehat{\sigma}) \leq T(\hat{\rho}, \widehat{\sigma}) \leq \sqrt{1 - F^2(\hat{\rho}, \widehat{\sigma})},} \tag{28}$$

  where we only need

$$F(|\alpha\rangle, \widehat{\sigma}) := \langle \alpha \,|\, \widehat{\sigma} \,|\, \alpha \rangle^{\frac{1}{2}} . \tag{29}$$

  - In our application, T is difficult while F can be obtained.

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Single-mode Distances between the Density Operators I

Bounds set by fidelity and eq. (28)

- The pure state can be decomposed upon discretisation $\chi_b[f] \sim \sum_p \chi_b^{(p)}(f_p) \equiv \sum_p \left\langle f_p \,\middle|\, \chi_b^{(p)} \right\rangle$, where $f_p := f(p)$
- So is the thermal density operator $\hat{\rho}_{\text{th}}(T) \sim \bigotimes_p \hat{\rho}_{\text{th}}^{(p)}(T)$
- Fidelity (in eq. (28)) can be factorised as well

$$F \equiv \langle \chi_b \,|\, \hat{\rho}_{\text{th}} \,|\, \chi_b \rangle^{\frac{1}{2}} \sim \prod_p \left\langle \chi_b^{(p)} \,\middle|\, \hat{\rho}_{\text{th}}^{(p)} \,\middle|\, \chi_b^{(p)} \right\rangle^{\frac{1}{2}} =: \prod_p F^{(p)}; \tag{30}$$

- $F^{(p)}$ can be computed in order to find bounds of T

$$F^{(p)}\left(|p\rangle \,, \hat{\rho}_{\text{th}}^{(p)}(T_{\text{HD}})\right) = \frac{\sqrt{u-1}}{\sqrt[4]{u^2+u+1}}, \quad u := \mathbb{e}^{q} \equiv \mathbb{e}^{|p|/T_{\text{HD}}}. \tag{31}$$

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All-modes Distances between the Density Operators I

Bounds set by fidelity and eq. (28)

- 'Go to the continuum limit': Λ dimension regulator

$$\sum_p g(p) \to \frac{1}{2\pi\Lambda} \int \mathrm{d}p\, g(p), \tag{32}$$

- Analogously, to regularise a product

$$\prod_p f(p) \equiv \exp\left\{ \sum_p \ln f(p) \right\} \to \exp\left\{ \frac{1}{2\pi\Lambda} \int \mathrm{d}p \, \ln f(p) \right\} \tag{33}$$

- Regularised F can be calculated in order to set bounds of T

$$F(\chi_b, \hat{\rho}_{\mathsf{th}}) = \exp\left\{ \frac{2}{2\pi\Lambda} \int_0^{+\infty} \mathrm{d}p \, \ln F^{(p)} \right\} = \exp\left(-\frac{\pi}{9} \frac{T_{\mathsf{HD}}}{\Lambda} \right) \tag{34}$$

Summary

- Compared the pure and the thermal descriptions of the radiation within the solvable dilaton gravity model
- Fourier-mode fluctuation: that of the thermal state diverges faster than the vacuum case does at low energy, while the pure-state fluc. remains finite; at high energies they converge.
- Trace distance: goes exponentially small with black hole temperature going to zero.

- Outlook in the proposed PhD study
  - Further discussion within full CGHS, BTZ, etc.
  - Make use of the decoherence theory

  - Nature of the microscopic degrees of freedom for BH entropy
  - Breakdown of the semi-classical approximations in QG

For Further Reading I

L.D. Landau and E.M. Lifshitz. *Mechanics*. 3rd. Vol. 1. Course of Theoretical Physics. Elsevier, 1976.

Qingdi Wang, Zhen Zhu, and William G. Unruh. "How the huge energy of quantum vacuum gravitates to drive the slow accelerating expansion of the Universe". In: *Physical Review D* 95.10 (May 2017).

More on Trace Distance
Interpretation

- Maximal probability-difference obtainable

$$T(\hat{\rho}, \hat{\sigma}) = \max_{0 \leq \widehat{\Lambda} \leq \widehat{1}} \mathrm{tr}\left\{\widehat{\Lambda}(\hat{\rho} - \hat{\sigma})\right\}, \tag{35}$$

  where all eigenvalues of $\widehat{\Lambda}$ are in the range $[0, 1]$
- E.g. $\widehat{\Lambda} := |\alpha\rangle \langle\alpha|$, $|\alpha\rangle$ eigenstate of $\widehat{\mathrm{A}}$ with eigenvalue α
  - $\mathrm{tr}\left\{\widehat{\Lambda}\hat{\rho}\right\}$: the probability of getting α in measuring $\widehat{\mathrm{A}}$
  - $\mathrm{tr}\left\{\widehat{\Lambda}(\hat{\rho} - \widehat{\sigma})\right\}$: the difference of the probability above
  - $T(\hat{\rho}, \widehat{\sigma})$: the maximal value of the difference above

More on Fidelity

- General definition

$$F(|\alpha\rangle, |\beta\rangle) = |\langle\alpha \,|\, \beta\rangle| \tag{36}$$

$$F(|\alpha\rangle, \hat{\sigma}) = \sqrt{\langle\alpha \,|\, \hat{\sigma} \,|\, \alpha\rangle} \tag{29 rev.}$$

$$F(\hat{\rho}, \hat{\sigma}) = \mathrm{tr}\,\sqrt{\hat{\rho}^{\frac{1}{2}}\hat{\sigma}\hat{\rho}^{\frac{1}{2}}} \tag{37}$$

- Intepretation: faithfulness

$$F(|\alpha\rangle, |\alpha\rangle) = 1 \tag{38}$$

Another New Foundation of Statistical Physics

Specific and easy version of the construction

- Total isolated system U with energy constraint $\left\langle \widehat{H}_U \right\rangle := E_U$, divided into a (sub)system S and an environment E
- Hilbert spaces $\mathcal{H}_R \supseteq \mathcal{H}_U = \mathcal{H}_S \otimes \mathcal{H}_E$; $\hat{1}_R$ identity on $\mathcal{H}_R$, dimension $d_R := \dim \mathcal{H}_R < +\infty$
- Equiprobable / maximal-ignorant state of U

$$\hat{\mathcal{E}}_R := d_R^{-1} \hat{1}_R \in \mathcal{H}_R \tag{39}$$

- Hamiltonians $\widehat{H}_U = \widehat{H}_S + \widehat{H}_E + \widehat{H}_{\text{int}}$
- Canonical state of S with energy constraint

$$\widehat{\Omega}_S^{(\mathsf{E})} := \operatorname{tr}_E \hat{\mathcal{E}}_R \propto \exp\left(-\widehat{H}_S / T_{\text{th}}\right) \tag{40}$$

- Theorem: $\forall \left|\phi\right\rangle \in \mathcal{H}_R$, the reduced state of S

$$\boxed{\operatorname{tr}_E \left|\phi\right\rangle \left\langle\phi\right| =: \hat{\rho}_S(\phi) \approx \widehat{\Omega}_S^{(\mathsf{E})}.} \tag{41}$$

Another New Foundation of Statistical Physics

Generic and exact version of the construction

- Arbitrary constraint R; study the trace distance T between $\hat{\rho}_S(\phi)$ and $\widehat{\Omega}_S$
- Lemma: average distance is small w.r.t. d_S/d_E^{eff}

$$\left\langle T\left(\hat{\rho}_S(\phi), \widehat{\Omega}_S\right)\right\rangle \leq \tfrac{1}{2}\sqrt{d_S/d_E^{\text{eff}}} \tag{42}$$

- Theorem: probability of large deviation is exponentially small w.r.t. the distance; an easy version

$$\frac{V\left[\left\{|\phi\rangle \;\middle|\; T\left(\hat{\rho}_S(\phi), \widehat{\Omega}_S\right) \geq d_R^{-\frac{1}{3}}\right\}\right]}{V[\{|\phi\rangle\}]} \leq 4\exp\left(-\frac{2d_R^{\frac{1}{3}}}{9\pi^3}\right) \tag{43}$$

- Effective dimension of E: setting $\widehat{\Omega}_E = \operatorname{tr}_S \hat{\mathcal{E}}_R$,

$$d_U/d_S \equiv d_E \geq d_E^{\text{eff}} := \left(\operatorname{tr}\widehat{\Omega}_E^2\right)^{-1} \geq d_R/d_S. \tag{44}$$