Hawking Radiation

A Comparison of Pure-state and Thermal Descriptions

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Outline

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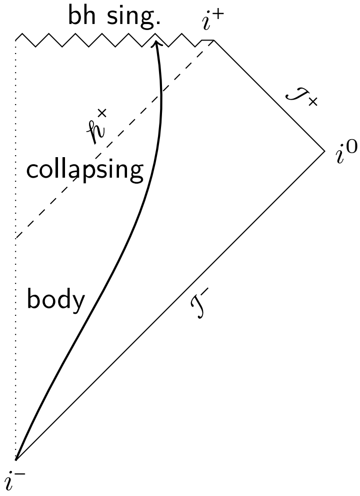
Overview

- **Topic**: to compare the pure state of the radiation field used by Hawking, and the thermal equilibrium state of the field, if it were at Hawking temperature
- **Importance**: to understand Hawking temperature from a new perspective, which plays a central role in BH physics and QG
  - information loss, origin of BH entropy, etc.
- **Methods**: based on a dilaton gravity model, which is more solvable than Einstein–Hilbert; using methods in traditional QFT and QIT
- **Results**: the difference exists and is quantitatively visualised

Outline

Background space-time: spherically collapsing body

Schematic conformal (Penrose–Carter) diagram



- QFT on fixed background
- Each point represents an S^2
- $i^{\mp}$: past / future time-like infinities
- i^0: space-like infinity
- $\mathscr{I}^{\mp}$: past / future null inf.
- $\mathscr{h}^+$: future event horizon
- **Thick** line: boundary of the collapsing body

Hawking Radiation

Results and interpretation [Haw74; Haw75]

- An early-time vacuum on $\mathscr{I}^-$ in collapsing background

$$\hat{a}(p)\,|h\rangle := 0 \quad \Rightarrow \quad \langle h\,|\,\hat{n}_a(p)\,|\,h\rangle =: \langle \hat{n}_a(p)\rangle_h \equiv 0 \tag{1}$$

evolves to a late-time state on $\mathscr{I}^+ \cup \mathscr{h}^+$ with particles on $\mathscr{I}^+$

$$\langle \hat{n}_{\text{on }\mathscr{I}^+}(\omega)\rangle_h =: \langle \hat{n}_b(\omega)\rangle_h \approx \Gamma_\omega\big(\mathrm{e}^{2\pi\omega/\kappa} - 1\big)^{-1}. \tag{2}$$

- Comparing eq. (2) with the Bose–Einstein distribution

$$\langle \hat{n}(\omega)\rangle_{\text{BE}} = \big(\mathrm{e}^{\omega/T} - 1\big)^{-1}, \tag{3}$$

one may conclude that eq. (2) describes a grey-body radiation with the Hawking temperature,

$$T_{\text{H}} := \kappa/2\pi \equiv \hbar/ck \cdot \kappa/2\pi. \tag{4}$$

Hawking Radiation

Tension in the interpretation

- The state $|h\rangle$ or its density operator is pure,

$$\hat{\rho}_h = |h\rangle \langle h| \,, \tag{5}$$

whilst the of equilibrium bosonic ideal gas

$$\hat{\rho}_{\mathsf{BE}}(T) = Z^{-1} \mathrm{e}^{-\widehat{H}/T} \sim Z^{-1} \sum_E \mathrm{e}^{-E/T} |E\rangle \langle E| \tag{6}$$

is thermal and mixed.

- How different are they? [Kie01; HR09]

Outline

Classical theory of $(1+1)$d Dilaton Gravity Model I

[Cal+92; DK96; APR11]

- The action of the dilaton gravity model reads

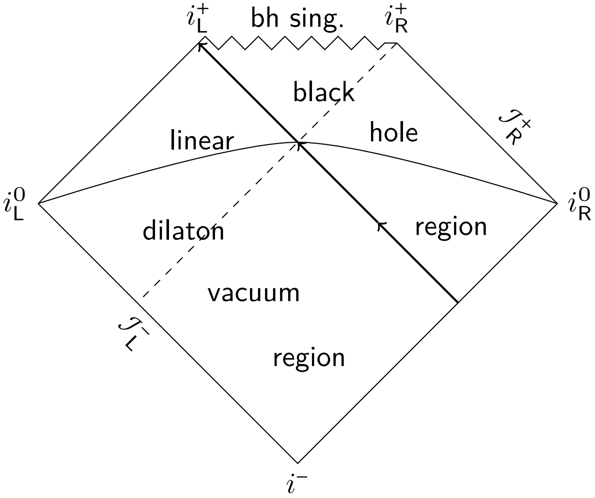
$$S = \int \mathbb{d}^2 x \sqrt{-g} \left\{ \frac{\mathbb{e}^{-2\phi}}{G} \left[R + 4(\nabla\phi)^2 + 4\lambda^2 \right] - \frac{1}{2}(\nabla f)^2 \right\}, \tag{7}$$

  - $\phi(x)$ the dilaton field, without which topological
  - $f(x)$ a massless neutral scalar field representing matter
  - $\lambda > 0$ the cosmological constant

- Has a solution which resembles the collapsing body in $(3+1)$-dimensional Einstein gravitation

Classical theory of $(1+1)$d Dilaton Gravity Model II

[Cal+92; DK96; APR11]



- Each point represents a point
- **Thick** line: null matter shell

Quantum theory of $(1+1)$d Dilaton Gravity Model I
[DK96]

- Constraint system: Schrödinger quantisation does not apply
- (Formally) Dirac quantisation

$$\widehat{\mathcal{H}}_{\parallel}\Psi[g,\phi,f]=0, \qquad \widehat{\mathcal{H}}_{\perp}\Psi[g,\phi,f]=0. \tag{8}$$

- Semi-classical approximation: $\Psi = \mathbb{e}^{\mathbb{i}(G^{-1}S_0+S_1+GS_2+\dots)}$
  - $\mathrm{O}(G^{-1})$: Hamilton–Jacobi equation for pure gravity
  - $\mathrm{O}(G^{0})$: $\Psi = D[g,\phi]\chi[g,\phi,f]$; functional Schrödinger equation for matter $\mathbb{i}\partial_t\chi[f]=\widehat{H}_{\mathrm{m}}\chi[f]$, where

$$\widehat{H}_{\mathrm{m}} = \frac{1}{2}\int_0^{+\infty}\mathbb{d}k\left(-\frac{\boldsymbol{\delta}^2}{\boldsymbol{\delta}f^2(k)}+k^2f^2(k)\right). \tag{9}$$

- A quantum field theory in curved space-time can be derived!

Quantum theory of $(1+1)$d Dilaton Gravity Model II

[DK96]

- At early time, the vacuum wave functional is

$$\chi_0[f_{\mathsf{e}}] \propto \exp\left\{-\frac{1}{2}\int_{\mathbb{R}^+} \mathbb{d}k\, k\, f_{\mathsf{e}}^2(k)\right\} \sim \prod_k \exp\left\{\frac{1}{2}\frac{k}{\Lambda} f_{\mathsf{e}}^2\right\}, \quad (10)$$

while at late time it evolves to

$$\chi_b[f_{\mathsf{l}}] \propto \exp\left\{-\int_{\mathbb{R}} \mathbb{d}p\, p \coth\left(\frac{\pi p}{2\lambda}\right)|f_{\mathsf{l}}(p)|^2\right\} \sim \prod_p \mathbb{e}^{\cdots}, \quad (11)$$

where $f_{\mathsf{e}}(k)$ and $f_{\mathsf{l}}(p)$ are the Fourier transform of the matter field at early and late time, respectively.

- At late time, particle-number expectations are

$$\langle \hat{n}_b(p)\rangle_{\chi_b} = \left(\mathbb{e}^{2\pi|p|/\lambda} - 1\right)^{-1}, \quad (12)$$

leading to a Hawking-like black-body temperature

$$\boxed{T_{\mathsf{HD}} := \lambda/2\pi.} \quad (13)$$

Outline

Correlation of Fourier Modes

The discrepancy

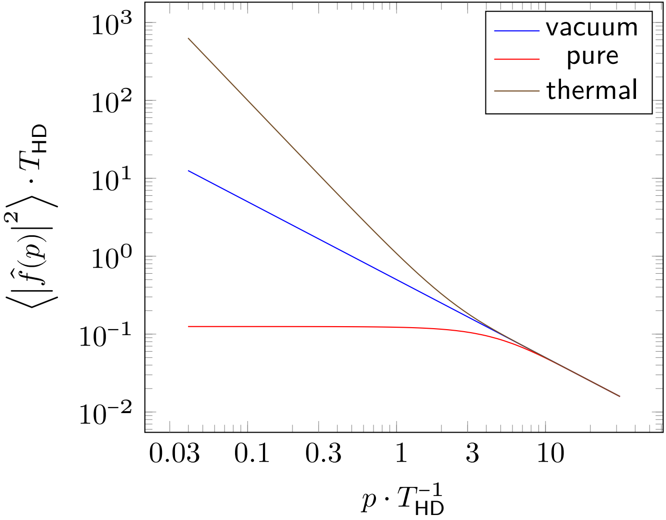
- The Fourier-mode correlators can be calculated,

$$\left\langle \hat{f}^{\dagger}(p_1)\hat{f}(p_2)\right\rangle = \frac{1}{T_{\mathsf{HD}}}\delta(p_1 - p_2)\cdot\begin{cases}\frac{1}{2}\frac{1}{q}, & \text{vacuum;}\\ \frac{1}{8}\frac{\tanh\frac{q}{4}}{\frac{q}{4}}, & \chi_b;\\ \frac{1}{4}\frac{\coth\frac{q}{2}}{\frac{q}{2}}, & \hat{\rho}_{\mathsf{BE}}(T_{\mathsf{HD}}),\end{cases} \tag{14}$$

  where $q := p_1/T_{\mathsf{HD}}$
- Diagonal elements (fluctuations) are plotted

Correlation of Fourier Modes

Fluctuation of the Fourier modes: diagram in log-log scale



Correlation of Fourier Modes

Fluctuation of the Fourier modes: interpretation

- Vacuum fluctuation

$$\left\langle \left| \hat{f} \right|^2 \right\rangle_{\text{vac}} = \mathrm{O}(q^{-1}) \tag{15}$$

- A black hole does not alter the high-energy processes

$$\left\langle \left| \hat{f} \right|^2 \right\rangle_{\chi_b} \approx \left\langle \left| \hat{f} \right|^2 \right\rangle_{\text{th}} \approx \left\langle \left| \hat{f} \right|^2 \right\rangle_{\text{vac}} = \mathrm{O}(q^{-1}), \quad |p| \gg T_{\text{HD}} \tag{16}$$

- A black hole suppresses low-energy fluctuation of the pure state, while enhancing that of the thermal state

$$\mathrm{O}(1) \sim \left\langle \left| \hat{f} \right|^2 \right\rangle_{\chi_b} \ll \left\langle \left| \hat{f} \right|^2 \right\rangle_{\text{vac}} \ll \left\langle \left| \hat{f} \right|^2 \right\rangle_{\text{th}} \sim \mathrm{O}(q^{-2}), \quad |p| \ll T_{\text{HD}} \tag{17}$$

- Critical scale $|p| \sim T_{\text{HD}}$

Outline

Trace Distance [Wil09, ch. 9]

Definitions

- Trace distance between Hermitian operators $\widehat{M}$ and $\widehat{N}$

$$T\left(\widehat{M}, \widehat{N}\right) := \frac{1}{2} \operatorname{tr} \sqrt{\left(\widehat{M} - \widehat{N}\right)^{\dagger} \left(\widehat{M} - \widehat{N}\right)} \tag{18}$$

- For density operators $\hat{\rho}$ and $\hat{\sigma}$,
  - $0 \leq T(\hat{\rho}, \widehat{\sigma}) \leq 1$;
  - Controlled by fidelity in [FG99]

$$\boxed{1 - F(\hat{\rho}, \widehat{\sigma}) \leq T(\hat{\rho}, \widehat{\sigma}) \leq \sqrt{1 - F^2(\hat{\rho}, \widehat{\sigma})},} \tag{19}$$

  where we only need

$$F(|\alpha\rangle, \widehat{\sigma}) := \langle \alpha \,|\, \widehat{\sigma} \,|\, \alpha \rangle^{\frac{1}{2}} \,. \tag{20}$$

  - In our application, T is difficult while F can be obtained.

Single-mode Distances between the Density Operators I

Bounds set by fidelity and eq. (19)

- The pure state can be decomposed upon discretisation $\chi_b[f] \sim \sum_p \chi_b^{(p)}(f_p) \equiv \sum_p \left\langle f_p \,\middle|\, \chi_b^{(p)} \right\rangle$, where $f_p := f(p)$
- So is the thermal density operator $\hat{\rho}_{\text{th}}(T) \sim \bigotimes_p \hat{\rho}_{\text{th}}^{(p)}(T)$
- Fidelity (in eq. (19)) can be factorised as well

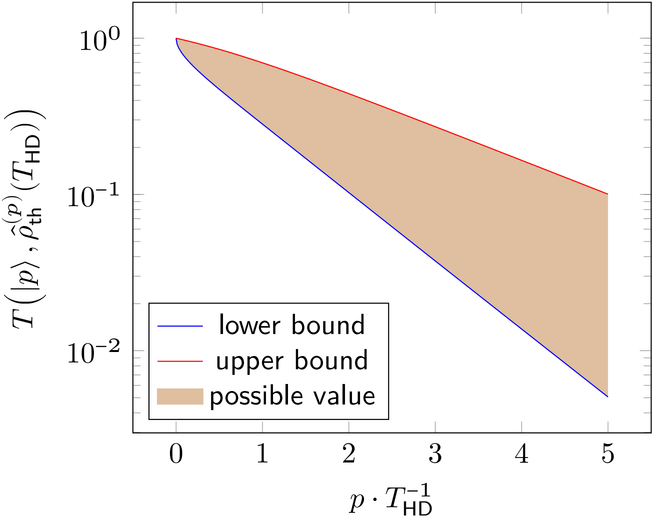
$$F \equiv \langle \chi_b \,|\, \hat{\rho}_{\text{th}} \,|\, \chi_b \rangle^{\frac{1}{2}} \sim \prod_p \left\langle \chi_b^{(p)} \,\middle|\, \hat{\rho}_{\text{th}}^{(p)} \,\middle|\, \chi_b^{(p)} \right\rangle^{\frac{1}{2}} =: \prod_p F^{(p)}; \tag{21}$$

- $F^{(p)}$ can be computed in order to find bounds of T

$$F^{(p)}\left(|p\rangle\,, \hat{\rho}_{\text{th}}^{(p)}(T_{\text{HD}})\right) = \frac{\sqrt{u-1}}{\sqrt[4]{u^2+u+1}}, \quad u := \mathrm{e}^{q} \equiv \mathrm{e}^{|p|/T_{\text{HD}}}. \tag{22}$$

Single-mode Distances between the Density Operators II

Bounds set by fidelity and eq. (19)



All-modes Distances between the Density Operators I

Bounds set by fidelity and eq. (19)

- 'Go to the continuum limit': Λ dimension regulator

$$\sum_p g(p) \to \frac{1}{2\pi\Lambda} \int \mathrm{d}p\, g(p), \tag{23}$$

- Analogously, to regularise a product

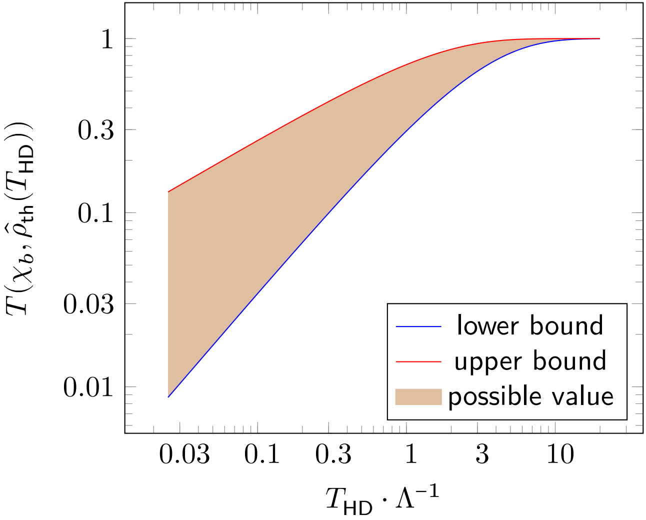
$$\prod_p f(p) \equiv \exp\left\{ \sum_p \ln f(p) \right\} \to \exp\left\{ \frac{1}{2\pi\Lambda} \int \mathrm{d}p \, \ln f(p) \right\} \tag{24}$$

- Regularised F can be calculated in order to set bounds of T

$$F(\chi_b, \hat{\rho}_{\mathsf{th}}) = \exp\left\{ \frac{2}{2\pi\Lambda} \int_0^{+\infty} \mathrm{d}p \, \ln F^{(p)} \right\} = \exp\left(-\frac{\pi}{9} \frac{T_{\mathsf{HD}}}{\Lambda} \right) \tag{25}$$

All-modes Distances between the Density Operators II

Bounds set by fidelity and eq. (19)



Summary

- Compared the pure and the thermal descriptions of the radiation within the solvable dilaton gravity model
- Fourier-mode fluctuation: that of the thermal state diverges faster than the vacuum case does at low energy, while the pure-state fluc. remains finite; at high energies they converge.
- Trace distance: goes exponentially small with black hole temperature going to zero.

- Outlook in the proposed PhD study
  - Further discussion within full CGHS, BTZ, etc.
  - Make use of the decoherence theory

  - Nature of the microscopic degrees of freedom for BH entropy
  - Breakdown of the semi-classical approximations in QG

For Further Reading I

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- Jean-Guy Demers and Claus Kiefer. "Decoherence of black holes by Hawking radiation". In: *Phys Rev D* 53.12 (June 1996), pp. 7050–7061.

For Further Reading II

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- Stephen D. H. Hsu and David Reeb. "Black holes, information, and decoherence". In: *Phys Rev D* 79.12 (June 2009).
- Claus Kiefer. "Hawking radiation from decoherence". In: *Classical Quantum Gravity* 18.22 (Nov. 2001), pp. L151–L154.

For Further Reading III

Sandu Popescu, Anthony J. Short, and Andreas Winter. "Entanglement and the foundations of statistical mechanics". In: *Nat. Phys.* 2.11 (Oct. 2006), pp. 754–758.

More on Trace Distance [Wil09, ch. 9]

Interpretation

- Maximal probability-difference obtainable

$$T(\hat{\rho}, \hat{\sigma}) = \max_{0 \leq \widehat{\Lambda} \leq \widehat{1}} \mathrm{tr}\left\{\widehat{\Lambda}(\hat{\rho} - \hat{\sigma})\right\}, \tag{26}$$

  where all eigenvalues of $\widehat{\Lambda}$ are in the range $[0, 1]$

- E.g. $\widehat{\Lambda} := |\alpha\rangle\langle\alpha|$, $|\alpha\rangle$ eigenstate of $\widehat{\mathrm{A}}$ with eigenvalue α
  - $\mathrm{tr}\left\{\widehat{\Lambda}\hat{\rho}\right\}$: the probability of getting α in measuring $\widehat{\mathrm{A}}$
  - $\mathrm{tr}\left\{\widehat{\Lambda}(\hat{\rho} - \widehat{\sigma})\right\}$: the difference of the probability above
  - $T(\hat{\rho}, \widehat{\sigma})$: the maximal value of the difference above

More on Fidelity

- General definition [Pet08, ch. 6]

$$F(|\alpha\rangle, |\beta\rangle) = |\langle\alpha\,|\,\beta\rangle| \tag{27}$$

$$F(|\alpha\rangle, \hat{\sigma}) = \sqrt{\langle\alpha\,|\,\hat{\sigma}\,|\,\alpha\rangle} \tag{20 rev.}$$

$$F(\hat{\rho}, \hat{\sigma}) = \operatorname{tr}\sqrt{\hat{\rho}^{\frac{1}{2}}\hat{\sigma}\hat{\rho}^{\frac{1}{2}}} \tag{28}$$

- Intepretation: faithfulness

$$F(|\alpha\rangle, |\alpha\rangle) = 1 \tag{29}$$

Another New Foundation of Statistical Physics [PSW06]

Specific and easy version of the construction

- Total isolated system U with energy constraint $\left\langle \widehat{H}_U \right\rangle := E_U$, divided into a (sub)system S and an environment E
- Hilbert spaces $\mathcal{H}_R \supseteq \mathcal{H}_U = \mathcal{H}_S \otimes \mathcal{H}_E$; $\hat{1}_R$ identity on $\mathcal{H}_R$, dimension $d_R := \dim \mathcal{H}_R < +\infty$
- Equiprobable / maximal-ignorant state of U

$$\hat{\mathcal{E}}_R := d_R^{-1} \hat{1}_R \in \mathcal{H}_R \tag{30}$$

- Hamiltonians $\widehat{H}_U = \widehat{H}_S + \widehat{H}_E + \widehat{H}_{\text{int}}$
- Canonical state of S with energy constraint [LL80, § 28]

$$\widehat{\Omega}_S^{(\mathsf{E})} := \operatorname{tr}_E \hat{\mathcal{E}}_R \propto \exp\left(-\widehat{H}_S / T_{\text{th}}\right) \tag{31}$$

- Theorem: $\forall \, |\phi\rangle \in \mathcal{H}_R$, the reduced state of S

$$\boxed{\operatorname{tr}_E |\phi\rangle \langle\phi| =: \hat{\rho}_S(\phi) \approx \widehat{\Omega}_S^{(\mathsf{E})}.} \tag{32}$$

Another New Foundation of Statistical Physics [PSW06]

Generic and exact version of the construction

- Arbitrary constraint R; study the trace distance T between $\hat{\rho}_S(\phi)$ and $\widehat{\Omega}_S$
- Lemma: average distance is small w.r.t. d_S/d_E^{eff}

$$\left\langle T\left(\hat{\rho}_S(\phi), \widehat{\Omega}_S\right)\right\rangle \leq \tfrac{1}{2}\sqrt{d_S/d_E^{\text{eff}}} \tag{33}$$

- Theorem: probability of large deviation is exponentially small w.r.t. the distance; an easy version

$$\frac{V\left[\left\{|\phi\rangle \;\middle|\; T\left(\hat{\rho}_S(\phi), \widehat{\Omega}_S\right) \geq d_R^{-\frac{1}{3}}\right\}\right]}{V[\{|\phi\rangle\}]} \leq 4\exp\left(-\frac{2d_R^{\frac{1}{3}}}{9\pi^3}\right) \tag{34}$$

- Effective dimension of E: setting $\widehat{\Omega}_E = \operatorname{tr}_S \hat{\mathcal{E}}_R$,

$$d_U/d_S \equiv d_E \geq d_E^{\text{eff}} := \left(\operatorname{tr} \widehat{\Omega}_E^2\right)^{-1} \geq d_R/d_S. \tag{35}$$