

Hawking Radiation

A Comparison of Pure-state and Thermal Descriptions

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Outline

Hawking Radiation

Model of Gravitation

Correlator of Field Strength

Distance between Density Operators



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Model of Gravitation

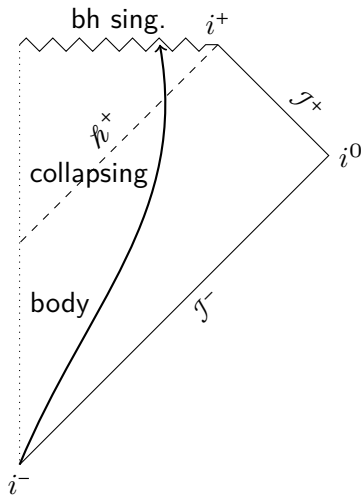
Correlator of Field Strength

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Background space-time: spherically collapsing body

Schematic conformal (Penrose–Carter) diagram



- ▶ QFT on fixed background
- ▶ Each point represents an S^2
- ▶ i^\mp : past / future time-like infinities
- ▶ i^0 : space-like infinity
- ▶ \mathcal{I}^\mp : past / future null inf.
- ▶ \mathcal{H}^+ : future event horizon
- ▶ **Thick** line: **boundary** of the collapsing body



Hawking Radiation

Results and interpretation [Haw74; Haw75]

- ▶ An early-time **vacuum** on \mathcal{I}^- in collapsing background

$$\hat{a}(p) |h\rangle := 0 \quad \Rightarrow \quad \langle h | \hat{n}_a(p) | h \rangle =: \langle \hat{n}_a(p) \rangle_h \equiv 0 \quad (1)$$

evolves to a late-time state on $\mathcal{I}^+ \cup \mathcal{H}^+$ **with particles** on \mathcal{I}^+

$$\langle \hat{n}_{\text{on } \mathcal{I}^+}(\omega) \rangle_h =: \langle \hat{n}_b(\omega) \rangle_h \approx \Gamma_\omega (e^{2\pi\omega/\kappa} - 1)^{-1}. \quad (2)$$

- ▶ Comparing eq. (2) with the Bose–Einstein distribution

$$\langle \hat{n}(\omega) \rangle_{\text{BE}} = (e^{\omega/T} - 1)^{-1}, \quad (3)$$

one may conclude that eq. (2) describes a grey-body radiation with the **Hawking temperature**,

$$T_{\text{H}} := \kappa/2\pi \equiv \hbar/c k \cdot \kappa/2\pi.$$



Hawking Radiation

Tension in the interpretation

- ▶ The state $|h\rangle$ or its density operator is **pure**,

$$\hat{\rho}_h = |h\rangle \langle h|, \quad (5)$$

whilst the of equilibrium bosonic ideal gas

$$\hat{\rho}_{\text{BE}}(T) = Z^{-1} e^{-\widehat{H}/T} \sim Z^{-1} \sum_E e^{-E/T} |E\rangle \langle E| \quad (6)$$

is **thermal** and **mixed**.

- ▶ How different are they? [Kie01; HR09]



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Classical theory of (1 + 1)d Dilaton Gravity Model I

[Cal+92; DK96; APR11]

- ▶ The action of the dilaton gravity model reads

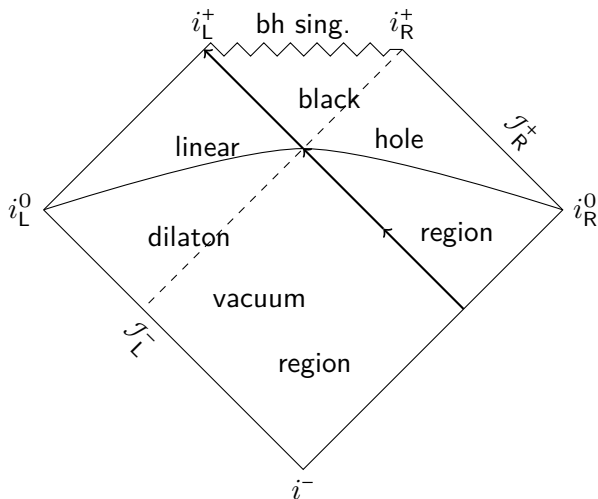
$$S = \int \mathrm{d}^2x \sqrt{-g} \left\{ \frac{\mathrm{e}^{-2\phi}}{G} [R + 4(\nabla\phi)^2 + 4\lambda^2] - \frac{1}{2}(\nabla f)^2 \right\}, \quad (7)$$

- ▶ $\phi(x)$ the dilaton field, without which topological
 - ▶ $f(x)$ a massless neutral scalar field representing matter
 - ▶ $\lambda > 0$ the cosmological constant
- ▶ Has a solution which **resembles** the collapsing body in (3 + 1)-dimensional Einstein gravitation



Classical theory of (1 + 1)d Dilaton Gravity Model II

[Cal+92; DK96; APR11]



- Each point represents a **point**
- **Thick** line: null matter **shell**



Quantum theory of (1 + 1)d Dilaton Gravity Model I

[DK96]

- ▶ Constraint system: Schrödinger quantisation does not apply
- ▶ (Formally) Dirac quantisation

$$\widehat{\mathcal{H}}_{\parallel} \Psi[g, \phi, f] = 0, \quad \widehat{\mathcal{H}}_{\perp} \Psi[g, \phi, f] = 0. \quad (8)$$

- ▶ Semi-classical approximation: $\Psi = e^{\mathfrak{i}(G^{-1}S_0 + S_1 + GS_2 + \dots)}$
 - ▶ $O(G^{-1})$: Hamilton–Jacobi equation for pure gravity
 - ▶ $O(G^0)$: $\Psi = D[g, \phi] \chi[g, \phi, f]$; functional **Schrödinger equation for matter** $\mathfrak{i} \partial_t \chi[f] = \widehat{H}_m \chi[f]$, where

$$\widehat{H}_m = \frac{1}{2} \int_0^{+\infty} \mathfrak{d}k \left(-\frac{\delta^2}{\delta f^2(k)} + k^2 f^2(k) \right). \quad (9)$$

- ▶ A quantum field theory in curved space-time can be **derived!**



Quantum theory of (1 + 1)d Dilaton Gravity Model II

[DK96]

- At early time, the **vacuum** wave functional is

$$\chi_0[f_e] \propto \exp\left\{-\frac{1}{2} \int_{\mathbb{R}^+} dk k f_e^2(k)\right\} \sim \prod_k \exp\left\{\frac{1}{2} \frac{k}{\Lambda} f_e^2\right\}, \quad (10)$$

while at late time it evolves to

$$\chi_b[f_l] \propto \exp\left\{-\int_{\mathbb{R}} dp p \coth\left(\frac{\pi p}{2\lambda}\right) |f_l(p)|^2\right\} \sim \prod_p e^{\dots}, \quad (11)$$

where $f_e(k)$ and $f_l(p)$ are the Fourier transform of the matter field at early and late time, respectively.

- At late time, **particle-number expectations** are

$$\langle \hat{n}_b(p) \rangle_{\chi_b} = \left(e^{2\pi|p|/\lambda} - 1 \right)^{-1}, \quad (12)$$

leading to a Hawking-like **black-body temperature**

$$T_{\text{HD}} := \lambda/2\pi.$$



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Correlation of Fourier Modes

The discrepancy

- ▶ The Fourier-mode correlators can be calculated,

$$\langle \hat{f}^\dagger(p_1) \hat{f}(p_2) \rangle = \frac{1}{T_{\text{HD}}} \delta(p_1 - p_2) \cdot \begin{cases} \frac{1}{2} \frac{1}{q}, & \text{vacuum;} \\ \frac{1}{8} \frac{\tanh \frac{q}{4}}{\frac{q}{4}}, & \chi_b; \\ \frac{1}{4} \frac{\coth \frac{q}{2}}{\frac{q}{2}}, & \hat{\rho}_{\text{BE}}(T_{\text{HD}}), \end{cases} \quad (14)$$

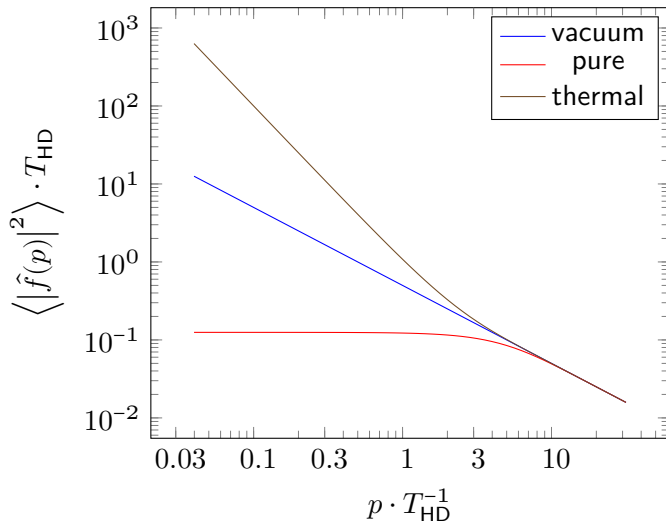
where $q := p_1/T_{\text{HD}}$

- ▶ Diagonal elements (fluctuations) are plotted



Correlation of Fourier Modes

Fluctuation of the Fourier modes: diagram in log-log scale



Correlation of Fourier Modes

Fluctuation of the Fourier modes: interpretation

- Vacuum fluctuation

$$\langle |\hat{f}|^2 \rangle_{\text{vac}} = O(q^{-1}) \quad (15)$$

- A black hole does **not** alter the **high**-energy processes

$$\langle |\hat{f}|^2 \rangle_{\chi_b} \approx \langle |\hat{f}|^2 \rangle_{\text{th}} \approx \langle |\hat{f}|^2 \rangle_{\text{vac}} = O(q^{-1}), \quad |p| \gg T_{\text{HD}} \quad (16)$$

- A black hole **suppresses** low-energy fluctuation of the **pure** state, while **enhancing** that of the **thermal** state

$$O(1) \sim \langle |\hat{f}|^2 \rangle_{\chi_b} \ll \langle |\hat{f}|^2 \rangle_{\text{vac}} \ll \langle |\hat{f}|^2 \rangle_{\text{th}} \sim O(q^{-2}), \quad |p| \ll T_{\text{HD}} \quad (17)$$

- Critical scale $|p| \sim T_{\text{HD}}$



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Trace Distance [Wil09, ch. 9]

Definitions

- ▶ Trace distance between Hermitian operators \widehat{M} and \widehat{N}

$$T(\widehat{M}, \widehat{N}) := \frac{1}{2} \operatorname{tr} \sqrt{(\widehat{M} - \widehat{N})^\dagger (\widehat{M} - \widehat{N})} \quad (18)$$

- ▶ For **density operators** $\hat{\rho}$ and $\hat{\sigma}$,
 - ▶ $0 \leq T(\hat{\rho}, \hat{\sigma}) \leq 1$;
 - ▶ Controlled by **fidelity** in [FG99]

$$\boxed{1 - F(\hat{\rho}, \hat{\sigma}) \leq T(\hat{\rho}, \hat{\sigma}) \leq \sqrt{1 - F^2(\hat{\rho}, \hat{\sigma})},} \quad (19)$$

where we only need

$$F(|\alpha\rangle, \hat{\sigma}) := \langle \alpha | \hat{\sigma} | \alpha \rangle^{\frac{1}{2}}. \quad (20)$$

- ▶ In our application, T is difficult while F can be obtained.



Single-mode Distances between the Density Operators I

Bounds set by fidelity and eq. (19)

- ▶ The pure state can be decomposed upon discretisation $\chi_b[f] \sim \sum_p \chi_b^{(p)}(f_p) \equiv \sum_p \langle f_p | \chi_b^{(p)} \rangle$, where $f_p := f(p)$
- ▶ So is the thermal density operator $\hat{\rho}_{\text{th}}(T) \sim \bigotimes_p \hat{\rho}_{\text{th}}^{(p)}(T)$
- ▶ Fidelity (in eq. (19)) can be factorised as well

$$F \equiv \langle \chi_b | \hat{\rho}_{\text{th}} | \chi_b \rangle^{\frac{1}{2}} \sim \prod_p \langle \chi_b^{(p)} | \hat{\rho}_{\text{th}}^{(p)} | \chi_b^{(p)} \rangle^{\frac{1}{2}} =: \prod_p F^{(p)}; \quad (21)$$

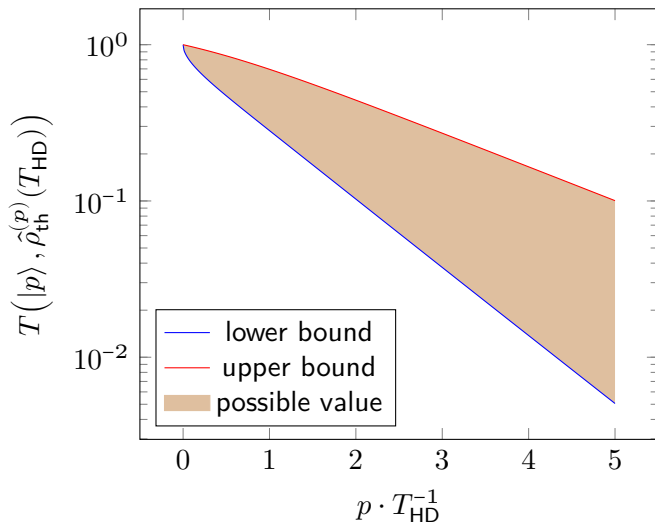
- ▶ $F^{(p)}$ can be computed in order to find bounds of T

$$F^{(p)}(|p\rangle, \hat{\rho}_{\text{th}}^{(p)}(T_{\text{HD}})) = \frac{\sqrt{u-1}}{\sqrt[4]{u^2+u+1}}, \quad u := e^q \equiv e^{|p|/T_{\text{HD}}}. \quad (22)$$



Single-mode Distances between the Density Operators II

Bounds set by fidelity and eq. (19)



All-modes Distances between the Density Operators I

Bounds set by fidelity and eq. (19)

- ▶ 'Go to the continuum limit': Λ dimension regulator

$$\sum_p g(p) \rightarrow \frac{1}{2\pi\Lambda} \int \mathbb{d}p g(p), \quad (23)$$

- ▶ Analogously, to regularise a product

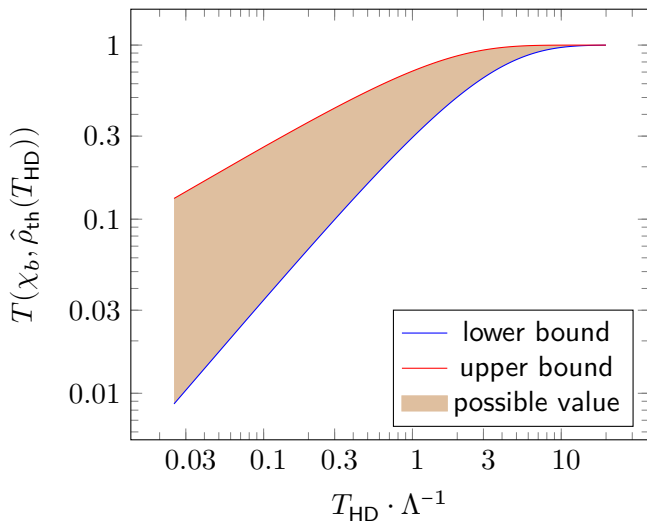
$$\prod_p f(p) \equiv \exp \left\{ \sum_p \ln f(p) \right\} \rightarrow \exp \left\{ \frac{1}{2\pi\Lambda} \int \mathbb{d}p \ln f(p) \right\} \quad (24)$$

- ▶ Regularised F can be calculated **in order to set bounds of T**

$$F(\chi_b, \hat{\rho}_{\text{th}}) = \exp \left\{ \frac{2}{2\pi\Lambda} \int_0^{+\infty} \mathbb{d}p \ln F(p) \right\} = \exp \left(-\frac{\pi}{9} \frac{T_{\text{HD}}}{\Lambda} \right). \quad (25)$$

All-modes Distances between the Density Operators II

Bounds set by fidelity and eq. (19)



Summary

- ▶ Compared the **pure** and the **thermal** descriptions of the radiation within the solvable dilaton gravity model
- ▶ Fourier-mode fluctuation: that of the thermal state **diverges** faster than the vacuum case does at low energy, while the pure-state fluc. remains **finite**; at high energies they **converge**.
- ▶ Trace distance: goes exponentially **small** with black hole **temperature** going to zero.
- ▶ Outlook
 - ▶ Understand the Fourier-mode fluctuation
 - ▶ Evaluate the real space correlator
 - ▶ Understand the regulator in total trace distance
 - ▶ Evaluate the exact trace distance



For Further Reading I



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For Further Reading III



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More on Trace Distance [Wil09, ch. 9]

Interpretation

- ▶ Maximal probability-difference obtainable

$$T(\hat{\rho}, \hat{\sigma}) = \max_{0 \leq \hat{\Lambda} \leq \hat{1}} \text{tr}\{\hat{\Lambda}(\hat{\rho} - \hat{\sigma})\}, \quad (26)$$

where all eigenvalues of $\hat{\Lambda}$ are in the range $[0, 1]$

- ▶ E.g. $\hat{\Lambda} := |\alpha\rangle\langle\alpha|$, $|\alpha\rangle$ eigenstate of \hat{A} with eigenvalue α
 - ▶ $\text{tr}\{\hat{\Lambda}\hat{\rho}\}$: the probability of getting α in measuring \hat{A}
 - ▶ $\text{tr}\{\hat{\Lambda}(\hat{\rho} - \hat{\sigma})\}$: the difference of the probability above
 - ▶ $T(\hat{\rho}, \hat{\sigma})$: the maximal value of the difference above



More on Fidelity

- General definition [Pet08, ch. 6]

$$F(|\alpha\rangle, |\beta\rangle) = |\langle\alpha|\beta\rangle| \quad (27)$$

$$F(|\alpha\rangle, \hat{\sigma}) = \sqrt{\langle\alpha|\hat{\sigma}|\alpha\rangle} \quad (20 \text{ rev.})$$

$$F(\hat{\rho}, \hat{\sigma}) = \text{tr} \sqrt{\hat{\rho}^{\frac{1}{2}} \hat{\sigma} \hat{\rho}^{\frac{1}{2}}} \quad (28)$$

- Interpretation: faithfulness

$$F(|\alpha\rangle, |\alpha\rangle) = 1 \quad (29)$$



Another New Foundation of Statistical Physics [PSW06]

Specific and easy version of the construction

- ▶ Total isolated system U with **energy** constraint $\langle \widehat{H}_U \rangle := E_U$, divided into a **(sub)system** S and an environment E
- ▶ Hilbert spaces $\mathcal{H}_R \supseteq \mathcal{H}_U = \mathcal{H}_S \otimes \mathcal{H}_E$; $\hat{1}_R$ identity on \mathcal{H}_R , dimension $d_R := \dim \mathcal{H}_R < +\infty$
- ▶ Equiprobable / maximal-ignorant state of U

$$\hat{\mathcal{E}}_R := d_R^{-1} \hat{1}_R \in \mathcal{H}_R \quad (30)$$

- ▶ Hamiltonians $\widehat{H}_U = \widehat{H}_S + \widehat{H}_E + \widehat{H}_{\text{int}}$
- ▶ **Canonical state** of S with energy constraint [LL80, § 28]

$$\widehat{\Omega}_S^{(\text{E})} := \text{tr}_E \hat{\mathcal{E}}_R \propto \exp(-\widehat{H}_S/T_{\text{th}}) \quad (31)$$

- ▶ **Theorem:** $\forall |\phi\rangle \in \mathcal{H}_R$, the reduced state of S

$\text{tr}_E |\phi\rangle \langle \phi| =: \hat{\rho}_S(\phi) \approx \widehat{\Omega}_S^{(\text{E})}.$



Another New Foundation of Statistical Physics [PSW06]

Generic and exact version of the construction

- ▶ **Arbitrary** constraint R ; study the **trace distance** T between $\hat{\rho}_S(\phi)$ and $\hat{\Omega}_S$
- ▶ Lemma: **average distance** is small w.r.t. d_S/d_E^{eff}

$$\langle T(\hat{\rho}_S(\phi), \hat{\Omega}_S) \rangle \leq \frac{1}{2} \sqrt{d_S/d_E^{\text{eff}}} \quad (33)$$

- ▶ Theorem: **probability of large deviation** is exponentially small w.r.t. the distance; an easy version

$$\frac{V\left[\left\{|\phi\rangle \mid T(\hat{\rho}_S(\phi), \hat{\Omega}_S) \geq d_R^{-\frac{1}{3}}\right\}\right]}{V[\{|\phi\rangle\}]} \leq 4 \exp\left(-\frac{2d_R^{\frac{1}{3}}}{9\mathfrak{M}^3}\right) \quad (34)$$

- ▶ Effective dimension of E : setting $\hat{\Omega}_E = \text{tr}_S \hat{\mathcal{E}}_R$,

$$d_U/d_S \equiv d_E \geq d_E^{\text{eff}} := \left(\text{tr} \hat{\Omega}_E^2\right)^{-1} \geq d_R/d_S. \quad (35)$$

