Masterthesis

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2.4 Thermodynamic laws

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3.1 Fields in Schwarzschild space-time

3.1.1 Linear, massless scalar field

[3]

(3.1) **Complete exterior modes**

$$u^{\rightarrow}_{\omega lm}(x) = (4\pi\omega)^{-\frac{1}{2}} e^{-i\omega t} R^{\rightarrow}_{l}(\omega|r) Y_{lm}(\theta,\phi), \tag{3.1}$$
$$u^{\leftarrow}_{\omega lm}(x) = (4\pi\omega)^{-\frac{1}{2}} e^{-i\omega t} R^{\leftarrow}_{l}(\omega|r) Y_{lm}(\theta,\phi), \tag{3.2}$$

where the in- and outgoing radial functions $R^{\leftrightarrow}_{l}(\omega|r)$ has the asymptotic forms

$$R^{\rightarrow}_{l}(\omega|r) \sim r^{-1} \cdot \begin{cases} e^{i\omega r_*} + A^{\rightarrow}_{l}(\omega) e^{-i\omega r_*}, & r \to R_S^+ \\ B_l(\omega) e^{+i\omega r_*}, & r \to +\infty \end{cases} \tag{3.3}$$
$$R^{\leftarrow}_{l}(\omega|r) \sim r^{-1} \cdot \begin{cases} B_l(\omega) e^{i\omega r_*}, & r \to R_S^+ \\ e^{-i\omega r_*} + A^{\leftarrow}_{l}(\omega) e^{+i\omega r_*} & r \to +\infty \end{cases} \tag{3.4}$$

in which the relation of r_* and r is given by eq. (A.8).

(3.2) **The Boulware state**
$|B\rangle$ [2] retarded propagator is given by

$$\begin{aligned} E^+_{B}(x,x') = i \sum_{lm} \int_0^{+\infty} \frac{d\omega}{4\pi\omega} e^{-i\omega(t-t')} Y_{lm}(\theta,\phi) {Y_{lm}}^*(\theta',\phi') \\ \cdot \left(R^{\rightarrow}_{l}(\omega|r) {R^{\rightarrow}_{l}}^*(\omega|r') + R^{\leftarrow}_{l}(\omega|r) {R^{\leftarrow}_{l}}^*(\omega|r')\right) \end{aligned} \tag{3.5}$$

(3.3) **The Unruh state**
$|U\rangle$ [20] retarded propagator is given by

$$E_{\mathrm{U}}^{+}(x,x') = \mathbb{i}\sum_{lm}\int_{-\infty}^{+\infty}\frac{\mathbb{d}\omega}{4\pi\omega}\mathbb{e}^{-\mathbb{i}\omega(t-t')}Y_{lm}(\theta,\phi){Y_{lm}}^{*}(\theta',\phi') \\ \left(\frac{R_l^{\rightarrow}(\omega|r)R_l^{\rightarrow *}(\omega|r')}{1-\mathbb{e}^{-2\pi\omega/\kappa}} + \theta(\omega)R_l^{\leftarrow}(\omega|r)R_l^{\leftarrow *}(\omega|r')\right) \tag{3.6}$$

(3.4) **The Hartle-Hawking-Israel state**
$|H\rangle$ [10, 15] retarded propagator is given by

$$E_{\mathrm{H}}^{+}(x,x') = \mathbb{i}\sum_{lm}\int_{-\infty}^{+\infty}\frac{\mathbb{d}\omega}{4\pi\omega} \\ \cdot\left(\mathbb{e}^{-\mathbb{i}\omega(t-t')}Y_{lm}(\theta,\phi){Y_{lm}}^{*}(\theta',\phi')\frac{R_l^{\rightarrow}(\omega|r)R_l^{\rightarrow *}(\omega|r')}{1-\mathbb{e}^{-2\pi\omega/\kappa}}\right. \\ \left.+\mathbb{e}^{+\mathbb{i}\omega(t-t')}{Y_{lm}}^{*}(\theta,\phi)Y_{lm}(\theta',\phi')\frac{R_l^{\leftarrow *}(\omega|r)R_l^{\leftarrow}(\omega|r')}{\mathbb{e}^{+2\pi\omega/\kappa}-1}\right) \tag{3.7}$$

Property 3.1. The symptotic values of $\langle\phi^2(x)\rangle_{\mathrm{ren}}$ [3]
$-\frac{1}{2\pi^2(1-R_{\mathrm{S}}/r)}\int_0^{+\infty}\frac{\mathbb{d}\omega\,\omega}{\mathbb{e}^{2\pi\omega/\kappa}-1}$, $\frac{1}{48\pi^2R_{\mathrm{S}}^2}-\frac{1}{8\pi^2R_{\mathrm{S}}^2}\int_0^{+\infty}\frac{\mathbb{d}\omega\,\sum(2l+1)|B_l(\omega)|^2}{\omega(\mathbb{e}^{2\pi\omega/\kappa}-1)}$, $\frac{1}{48\pi^2R_{\mathrm{S}}^2}$; 0, $\frac{1}{2\pi^2r^2}\int_0^{+\infty}\frac{\mathbb{d}\omega\,\sum(2l+1)|B_l(\omega)|^2}{\omega(\mathbb{e}^{2\pi\omega/\kappa}-1)}$, $\frac{1}{2\pi^2}\int_0^{+\infty}\frac{\mathbb{d}\omega\,\omega}{\mathbb{e}^{2\pi\omega/\kappa}-1}$.

Stress-momentum tensors [5, 3]

3.2 Fields in space-time of collapsing body

[11, 13]
[17]

$$_{+}\langle n_{\omega}, m_{\omega'}\,|\,\Omega\rangle_{-} = \delta_{nm}\delta_{\omega\omega'}\mathbb{e}^{-n\pi\omega/\kappa}\,_{+}\langle\Omega\,|\,\Omega\rangle_{-} \tag{3.8}$$

A Aspects of space-time geometry

A.1 Aspects of the fundamentals

Cauchy surface
Globally hyperbolic
Surface gravity
[4]
[8]

A.2 Maximally symmetric space-times

(A.1) **Unit 2-sphere**

$$\mathbb{d}\Omega^2 = \mathbb{d}\theta^2 + \sin^2\theta\,\mathbb{d}\phi^2 \tag{A.1}$$

is the metric of unit 2-sphere S^2.

(A.2) **Spherically symmetric space-times**
All spherically symmetric space-times in this work possess coordinates in which the metric takes the form

$$\mathbb{d}s^2 = -f(r)\,\mathbb{d}t^2 + f(r)^{-1}\mathbb{d}r^2 + r^2\,\mathbb{d}\Omega^2, \tag{A.2}$$

which is manifest in its $\mathsf{SO}\,(3)$ symmetry.

(A.3) **Spherical coordinates of maximally symmetric space-times**

$$f(r) = 1 - \frac{\mathsf{G}\Lambda}{3}r^2. \tag{A.3}$$

$$\frac{t+r}{R_0} = \tan\frac{\eta+\chi}{2}, \qquad \frac{t-r}{R_0} = \tan\frac{\eta-\chi}{2}. \tag{A.4}$$

(A.4) **Conformal coordinates**

$$\mathbb{d}s^2 = \frac{R_0^2}{(\cos\eta + \cos\chi)^2}\big(-\mathbb{d}\eta^2 + \mathbb{d}\chi^2 + \sin^2\chi\,\mathbb{d}\Omega^2\big) \tag{A.5}$$

A.3 Schwarzschild space-time

A.3.1 Coordinates

(A.5) **Schwarzschild coordinates**
The metric of Schwarzschild space-time is given by eq. (A.2) with

$$f(r) = \left(1 - \frac{R_\mathrm{S}}{r}\right), \tag{A.6}$$

where $R_\mathrm{S} = 2\mathsf{G}M$ is the Schwarzschild radius.

To exploit the Weyl-flatness of $\mathcal{M}/S^2$, one may transform the holonomic co-frame by

$$\mathbb{d}r^* = \left(1 - \frac{R_\mathrm{S}}{r}\right)^{-1}\mathbb{d}r, \tag{A.7}$$

which can be integrated to

$$\mathbb{e}^{r_*/R_\mathrm{S}-1} = (r/R_\mathrm{S} - 1)\mathbb{e}^{r/R_\mathrm{S}-1}, \tag{A.8}$$

generating the transformations

$$r_* = r + R_\mathrm{S}\ln\left(\frac{r}{R_\mathrm{S}} - 1\right) \quad \text{and} \quad r = R_\mathrm{S}\big(1 + W\big(\mathbb{e}^{r_*/R_\mathrm{S}-1}\big)\big),$$

where $W(x)$ is known as the Lambert W-function [21], satisfying

$$W(x)\mathbb{e}^{W(x)} = x. \tag{A.9}$$

Transformation of eq. (A.6) by eq. (A.7) gives

(A.6) **Regge-Wheeler tortoise coordinates**
The metric of Schwarzschild space-time is given by

$$\mathbb{d}s^2 = \left(1 - \frac{R_\mathrm{S}}{r}\right)\big(-\,\mathbb{d}t^2 + \mathbb{d}{r_*}^2\big) + r(r_*)^2\,\mathbb{d}\Omega^2. \tag{A.10}$$

In order to keep the causal structure in further coordinate transformations, switch first to the *light-cone* coordinates

$$u = t - r_* \qquad v = t + r_*. \tag{A.11}$$

Using one of them to replace t in A.5 gives

(A.7) **Eddington-Finklestein coordinates**
The advanced (using v) and retarded (u) Eddington-Finklestein coordinates give the forms

$$\begin{aligned} \mathbb{d}s^2 &= -\left(1 - \frac{R_\mathrm{S}}{r}\right) \mathbb{d}v^2 + 2\,\mathbb{d}v\,\mathbb{d}r + r^2\,\mathbb{d}\Omega^2 && \text{(A.12)} \\ &= -\left(1 - \frac{R_\mathrm{S}}{r}\right) \mathbb{d}u^2 - 2\,\mathbb{d}u\,\mathbb{d}r + r^2\,\mathbb{d}\Omega^2 && \text{(A.13)} \end{aligned}$$

to the metric, respectively.

Note that the coordinate singularity at $r = R_\mathrm{S}$ in A.5 has been eliminated. whereas the components of the metric are now

$$\mathbb{d}s^2 = -\left(1 - \frac{R_\mathrm{S}}{r(u,v)}\right) \mathbb{d}u\,\mathbb{d}v + r(u,v)^2\,\mathbb{d}\Omega^2. \tag{A.14}$$

To bypass the coordinate singularity at $r = R_\mathrm{S}$, using eqs. (A.8) and (A.11) to get

$$1 - \frac{R_\mathrm{S}}{r} = \frac{R_\mathrm{S}}{r} \exp\left(-\frac{r}{R_\mathrm{S}}\right) \exp\left(\frac{-u+v}{2R_\mathrm{S}}\right), \tag{A.15}$$

so eq. (A.14) becomes

$$\mathbb{d}s^2 = -\frac{R_\mathrm{S}}{r} \exp\left(-\frac{r}{R_\mathrm{S}}\right) \exp\left(\frac{-u+v}{2R_\mathrm{S}}\right) \mathbb{d}u\,\mathbb{d}v + r^2\,\mathbb{d}\Omega^2. \tag{A.16}$$

Absorbing the corresponding exponentials by introducing the *Kruskal-Szekeres* light-cone coordinates,

$$U = -R_\mathrm{S} \exp\left(-\frac{u}{2R_\mathrm{S}}\right) \quad \text{and} \quad V = +R_\mathrm{S} \exp\left(+\frac{v}{2R_\mathrm{S}}\right), \tag{A.17}$$

leads to the new components

$$\mathbb{d}s^2 = -\frac{4R_\mathrm{S}}{r(U,V)} \exp\left(-\frac{r(U,V)}{R_\mathrm{S}}\right) \mathbb{d}U\,\mathbb{d}V + r(U,V)^2\,\mathbb{d}\Omega^2. \tag{A.18}$$

By further rotating the light-cone coordinates into time- and space-like ones,

$$U = T - R \quad \text{and} \quad V = T + R, \tag{A.19}$$

one can derive the Kruskal-Szekeres coordinates, the metric in which takes the form

$$\mathbb{d}s^2 = \frac{16R_\mathrm{S}}{r(T,R)} \exp\left(-\frac{r(T,R)}{R_\mathrm{S}}\right) (-\mathbb{d}T^2 + \mathbb{d}R^2) + r(T,R)^2\,\mathbb{d}\Omega^2. \tag{A.20}$$

Finally, in order to compactify the coordinates T and R into finite range while keeping the causal structure, one may introducing the *conformal* coordinates of light-cone and space-time

$$\mu = \arctan \frac{U}{R_{\mathrm{S}}} = \eta - \chi \quad \text{and} \quad \nu = \arctan \frac{V}{R_{\mathrm{S}}} = \eta + \chi, \tag{A.21}$$

leading to the components

$$\mathbb{d}s^2 = -\frac{4R_{\mathrm{S}}^3}{r} \mathbb{e}^{-r/R_{\mathrm{S}}} \sec^2 \mu \sec^2 \nu \, \mathbb{d}\mu \, \mathbb{d}\nu + r^2 \, \mathbb{d}\Omega^2 \tag{A.22}$$

$$= \frac{16R_{\mathrm{S}}^3}{r} \mathbb{e}^{-r/R_{\mathrm{S}}} \sec^2(\eta - \chi) \sec^2(\eta + \chi)(-\mathbb{d}\eta^2 + \mathbb{d}\chi^2) + r^2 \, \mathbb{d}\Omega^2. \tag{A.23}$$

A.3.2 Properties

(A.8) Surface gravity
The surface gravity of Schwarzschild black hole is

$$\kappa_{Sch} = \frac{1}{2R_S}. \tag{A.24}$$

Proof. $\partial_t \equiv (\partial_t)_{\mathrm{Sch}}$

$$\begin{aligned}
\nabla_{\partial_t} \partial_t &= (\partial_v)^{\mu'} \left(\nabla_{\mu'} (\partial_v)^{\nu'} \right) \partial_{\nu'} \\
&= \delta_v^{\nu'} \left(\partial_{\nu'} \delta_v^{\mu'} + \Gamma^{\mu'}{}_{\nu'\rho'} \delta_v^{\rho'} \right) \partial_{\nu'} \\
&= \Gamma^{\mu'}{}_{vv} \partial_{\mu'} = \frac{R_{\mathrm{S}}}{2r^2} \left(\partial_v + \left(1 - \frac{R_{\mathrm{S}}}{r} \right) \partial_r \right).
\end{aligned}$$

$$\kappa_{\mathrm{Sch}} \, \partial_t \big|_{h^+} \equiv \nabla_{\partial_t} \partial_t \Big|_{h^+} = \frac{1}{2R_{\mathrm{S}}} \, \partial_t \big|_{h^+}. \tag{A.25}$$

□

A.4 Other space-times of black hole

A.4.1 Spherically symmetric space-times

(A.9) **Schwarzschild space-time with cosmological constant**

$$f(r) = 1 - \frac{2\mathsf{G}M}{r} - \frac{\mathsf{G}\Lambda}{3} r^2. \tag{A.26}$$

Volume
Reissner–Nordström space-time

A.4.2 Axially symmetric space-times

(A.10) **Kerr–Newman space-times with cosmological constant**

$$\begin{aligned}
\mathbb{d}s^2 = -\frac{\Delta_r}{\Xi^2 \varrho^2} \left(\mathbb{d}t^2 - a \sin^2 \theta \, \mathbb{d}\theta \right)^2 + \frac{\varrho^2}{\Delta_r} \, \mathbb{d}r^2 + \frac{\varrho^2}{\Delta_\theta} \, \mathbb{d}\theta^2 \\
+ \frac{\Delta_\theta \sin^2 \theta}{\Xi^2 \varrho^2} \left(a \, \mathbb{d}t^2 - (r^2 + a^2) \, \mathbb{d}\theta \right)^2,
\end{aligned} \tag{A.27}$$

where

$$
\begin{aligned}
\varrho^2 &= r^2 + a^2 \cos^2\theta, \\
\Delta_r &= (r^2 + a^2)\left(1 - \frac{\mathsf{G}\Lambda}{3} r^2\right) - 2\mathsf{G}Mr + \mathsf{G}e^2, \\
\Delta_\theta &= 1 + \frac{\mathsf{G}\Lambda}{3} a^2 \cos^2\theta, \\
\Xi &= 1 + \frac{\mathsf{G}\Lambda}{3} a^2,
\end{aligned}
$$

and the potential one-form for the electromagnetic field is

$$
\mathrm{A} = -\frac{er}{\varrho^2}(\mathbb{d}r - a \sin^2\theta\, \mathbb{d}\phi). \tag{A.28}
$$

B Quantum field theory

B.1 Linear scalar fields: canonical quantisation

B.2 Linear scalar fields: algebraic quantisation

B.3 Fields with higher spin and interactions

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