Hawking Radiation

A Comparison of Pure-state and Thermal Description

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Outline

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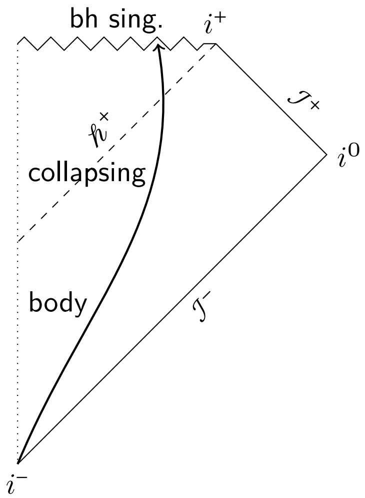
Hawking Radiation

Background space-time: sperically collapsing body



- An early-time vacuum on $\mathscr{I}^-$ in collapsing background

$$\hat{a}(p)\,|h\rangle := 0 \quad \Rightarrow \quad \langle \hat{n}_a(p)\rangle_h \equiv 0 \tag{1}$$

evolves to a late-time state on $\mathscr{I}^+ \cup \mathscr{h}^+$ with particles [Haw74]

$$\langle \hat{n}_b(\omega)\rangle_h \approx \Gamma_\omega \big(\mathrm{e}^{2\pi\omega/\kappa} - 1\big)^{-1}. \tag{2}$$

- Comparing eq. (2) with the Bose–Einstein distribution

$$\langle \hat{n}(\omega)\rangle_{\mathsf{BE}} = \big(\mathrm{e}^{\omega/T} - 1\big)^{-1}, \tag{3}$$

one may conclude that eq. (2) describes a grey-body radiation with the Hawking temperature [Haw75],

$$T_{\mathsf{H}} := \kappa/2\pi. \tag{4}$$

Hawking Radiation
Tension in the interpretation

- The state $|h\rangle$ or its density operator is pure,

$$\hat{\rho}_h = |h\rangle \langle h| \,, \tag{5}$$

whilst the Bose–Einstein state of equilibrium bosonic gas

$$\hat{\rho}_{\mathsf{BE}} = Z^{-1} \mathbb{e}^{-\widehat{H}/T} Z^{-1} \sim Z^{-1} \sum_E \mathbb{e}^{-E/T} |E\rangle \langle E| \tag{6}$$

is thermal and mixed.

- How different are they? [Kie01; HR09]

Outline

Hawking Radiation

Model of Gravitation

Correlator of Field Strength

Distance between Density Operators

$(1+1)$-dimensional Dilaton Gravity Model

Classical theory 1/2 [Cal+92; DK96; APR11]

- The action of the dilaton gravity model reads

$$S = \int \mathrm{d}^2 x \sqrt{-\overline{g}} \left\{ \frac{\mathrm{e}^{-2\overline{\phi}}}{G} \left[\overline{R} + 4\left(\nabla \overline{\phi}\right)^2 + 4\lambda^2 \right] - \frac{1}{2} (\nabla f)^2 \right\}$$

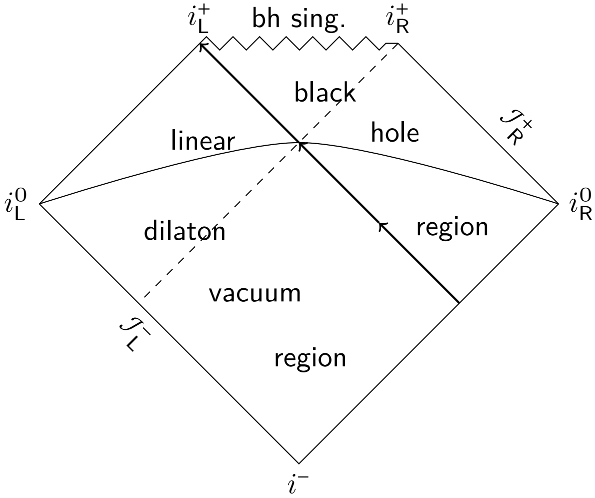
$$= \int \mathrm{d}^2 x \sqrt{-g} \left\{ \frac{1}{G} \left[R\phi + 4\lambda^2 \right] - \frac{1}{2} (\nabla f)^2 \right\}, \tag{7}$$

where in eq. (7), the kinetic term of the dilaton field ϕ is eliminated by substituting $\phi = \mathrm{e}^{-2\overline{\phi}}$ and $g_{\alpha\beta} = \mathrm{e}^{-2\overline{\phi}} \overline{g}_{\alpha\beta}$.

- Has a solution which resembles the collapsing body in $(3+1)$d Einstein gravitation

$(1+1)$-dimensional Dilaton Gravity Model

Classical theory 2/2 [Cal+92; DK96; APR11]



$(1+1)$-dimensional Dilaton Gravity Model

Quantum theory 1/2 [DK96]

- Can formally be canonically quantised as a constraint system
- Apply a Born–Oppenheimer-type approximation to $\Psi[g, \phi, f]$ [Kie12, § 5.4]
  - Separate '*Heavy, slow*' gravity and '*light, fast*' matter
  - Apply WKB approximation to the gravity part
  - Assume disentanglement $\Psi[g, \phi, f] = D[g, \phi]\chi[g, \phi, f]$
- Insert the ansatz $\Psi[g, \phi, f] = \mathrm{e}^{\mathrm{i}(G^{-1} S_0 + S_1 + G S_2 + \ldots)}$
  - Order G^{-1}: Hamilton–Jacobi equation for pure gravity
  - Order G^0: functional Schrödinger equation for matter

$$\mathrm{i}\frac{\partial \chi}{\partial t} = \widehat{H}_{\mathrm{m}} \chi, \tag{7}$$

$$\widehat{H}_{\mathrm{m}} \equiv \frac{1}{2} \int_0^{+\infty} \mathrm{d}k \left(-\frac{\boldsymbol{\delta}^2}{\boldsymbol{\delta} f^2(k)} + k^2 f^2(k) \right). \tag{8}$$

$(1+1)$-dimensional Dilaton Gravity Model

Quantum theory 2/2 [DK96]

- At early time, the ground-state solution to eq. (7) is

$$\chi_0[f] \propto \exp\left\{-\frac{1}{2}\int_0^{+\infty} \mathbb{d}k\, k\, f(k)^2\right\}, \tag{7}$$

while at late time it evolves to

$$\chi_b[g] \propto \exp\left\{-\int_{-\infty}^{+\infty} \mathbb{d}p\, p \coth\left(\frac{\pi p}{2\lambda}\right)|g(p)|^2\right\}, \tag{8}$$

where $f(k)$ and $g(p)$ are the Fourier transform of the matter field at early and late time, respectively.

- At late time, particle-number expectations are

$$\langle \hat{n}_b(p)\rangle_{\chi_b} = \left(\mathbb{e}^{2\pi|p|/\lambda} - 1\right)^{-1}, \tag{9}$$

leading to a Hawking-like black-body temperature

$$\boxed{T_{\mathsf{HD}} := \lambda/2\pi.} \tag{10}$$

Outline

Correlation of Fourier Modes

The discrepancy

The Fourier-mode correlators of different states can be calculated,

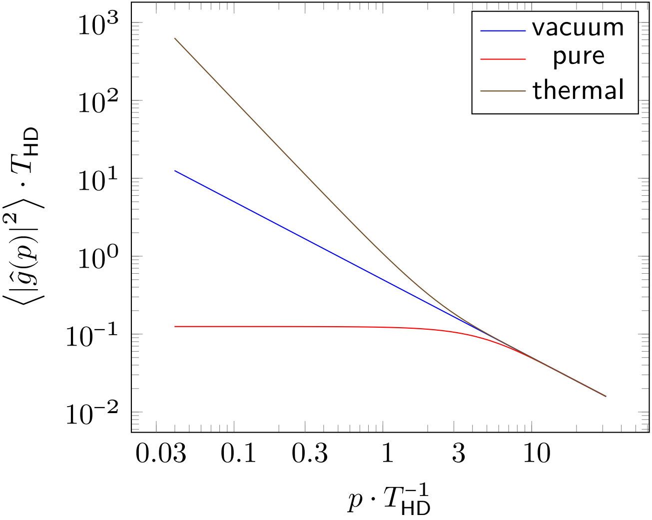
$$\langle \hat{g}^\dagger(p_1)\hat{g}(p_2)\rangle = \frac{1}{T_{\mathsf{HD}}}\delta(p_1 - p_2)\cdot \begin{cases} \frac{1}{2}\frac{1}{q}, & \text{vacuum;} \\ \frac{1}{8}\frac{\tanh\frac{q}{4}}{\frac{q}{4}}, & \chi_b; \\ \frac{1}{4}\frac{\coth\frac{q}{2}}{\frac{q}{2}}, & \hat{\rho}_{\mathsf{BE}}(T_{\mathsf{HD}}), \end{cases} \tag{11}$$

where $q := p_1/T_{\mathsf{HD}}$.
Note that

$$\left\langle (\boldsymbol{\Delta}\hat{g})^2 \right\rangle = \langle \hat{g}^2\rangle - \langle \hat{g}\rangle^2 = \langle \hat{g}^2\rangle. \tag{12}$$

Correlation of Fourier Modes

Fluctuation of the Fourier modes: diagram in log-log scale



Correlation of Fourier Modes

Fluctuation of the Fourier modes: interpretation

- Vacuum fluctuation

$$\left\langle |\hat{g}|^2 \right\rangle_{\text{vac}} \sim \mathrm{O}(q^{-1}) \tag{13}$$

- A black hole does not alter the high-energy processes.

$$\left\langle |\hat{g}|^2 \right\rangle_{\chi_b} \approx \left\langle |\hat{g}|^2 \right\rangle_{\text{th}} \approx \left\langle |\hat{g}|^2 \right\rangle_{\text{vac}} \sim \mathrm{O}(q^{-1}), \quad |p| \gg T_{\text{HD}} \tag{14}$$

- A black hole suppresses low-energy fluctuation of the pure state, while enhancing that of the thermal state.

$$\mathrm{O}(1) \sim \left\langle |\hat{g}|^2 \right\rangle_{\chi_b} \ll \left\langle |\hat{g}|^2 \right\rangle_{\text{vac}} \ll \left\langle |\hat{g}|^2 \right\rangle_{\text{th}} \sim \mathrm{O}(q^{-2}), \quad |p| \ll T_{\text{HD}} \tag{15}$$

- Critical scale $|p| \sim T_{\text{HD}}$

Outline

Another New Foundation of Statistical Physics [PSW06]

Specific and easy version of the construction

- Total isolated system U with energy constraint $\left\langle \widehat{H}_U \right\rangle := E_U$, divided into a (sub)system S and an environment E
- Hilbert spaces $\mathcal{H}_R \supseteq \mathcal{H}_U = \mathcal{H}_S \otimes \mathcal{H}_E$; $\hat{1}_R$ identity on $\mathcal{H}_R$, dimension $d_R := \dim \mathcal{H}_R < +\infty$
- Equiprobable / maximal-ignorant state of U

$$\hat{\mathcal{E}}_R := d_R^{-1} \hat{1}_R \in \mathcal{H}_R \tag{16}$$

- Hamiltonians $\widehat{H}_U = \widehat{H}_S + \widehat{H}_E + \widehat{H}_{\text{int}}$
- Canonical state of S with energy constraint [LL80, § 28]

$$\widehat{\Omega}_S^{(\mathsf{E})} := \operatorname{tr}_E \hat{\mathcal{E}}_R \propto \exp\left(-\widehat{H}_S / T_{\text{th}}\right) \tag{17}$$

- Theorem: $\forall \left|\phi\right\rangle \in \mathcal{H}_R$, the reduced state of S

$$\boxed{\operatorname{tr}_E \left|\phi\right\rangle \left\langle\phi\right| =: \hat{\rho}_S(\phi) \approx \widehat{\Omega}_S^{(\mathsf{E})}.} \tag{18}$$

Another New Foundation of Statistical Physics [PSW06]

Generic and exact version of the construction

- Arbitrary constraint R; study the trace distance T between $\hat{\rho}_S(\phi)$ and $\widehat{\Omega}_S$
- Lemma: average distance is small w.r.t. d_S/d_E^{eff}

$$\left\langle T\left(\hat{\rho}_S(\phi), \widehat{\Omega}_S\right)\right\rangle \leq \tfrac{1}{2}\sqrt{d_S/d_E^{\text{eff}}} \tag{19}$$

- Theorem: probability of large deviation is exponentially small w.r.t. the distance; an easy version

$$\frac{V\left[\left\{|\phi\rangle \;\middle|\; T\left(\hat{\rho}_S(\phi), \widehat{\Omega}_S\right) \geq d_R^{-\frac{1}{3}}\right\}\right]}{V[\{|\phi\rangle\}]} \leq 4\exp\left(-\frac{2d_R^{\frac{1}{3}}}{9\pi^3}\right) \tag{20}$$

- Effective dimension of E: setting $\widehat{\Omega}_E = \operatorname{tr}_S \hat{\mathcal{E}}_R$,

$$d_U/d_S \equiv d_E \geq d_E^{\text{eff}} := \left(\operatorname{tr} \widehat{\Omega}_E^2\right)^{-1} \geq d_R/d_S. \tag{21}$$

Trace Distance [Wil09, ch. 9]

Definitions

- Trace distance between Hermitian operators $\widehat{M}$ and $\widehat{N}$

$$T\left(\widehat{M},\widehat{N}\right) := \frac{1}{2}\,\mathrm{tr}\sqrt{\left(\widehat{M}-\widehat{N}\right)^{\dagger}\left(\widehat{M}-\widehat{N}\right)} \tag{22}$$

  - Positivity, homogeneity, triangle ineq., isometric inv.
- For density operators $\hat{\rho}$ and $\hat{\sigma}$,
  - $0 \leq T(\hat{\rho},\widehat{\sigma}) \leq 1$.
  - Controlled by fidelity in [FG99]

$$1 - F(\hat{\rho},\widehat{\sigma}) \leq T(\hat{\rho},\widehat{\sigma}) \leq \sqrt{1 - F^2(\hat{\rho},\widehat{\sigma})}, \tag{23}$$

  where we only need

$$F(|\alpha\rangle\,,\widehat{\sigma}) := \sqrt{\langle\alpha\,|\,\widehat{\sigma}\,|\,\alpha\rangle}. \tag{24}$$

  - In our application, T is difficult while F can be obtained.

Trace Distance [Wil09, ch. 9]

Interpretation

- Maximal probability-difference obtainable

$$T(\hat{\rho}, \hat{\sigma}) = \max_{0 \leq \widehat{\Lambda} \leq \widehat{1}} \mathrm{tr}\left\{\widehat{\Lambda}(\hat{\rho} - \hat{\sigma})\right\}, \tag{25}$$

  where all eigenvalues of $\widehat{\Lambda}$ are in the range $[0, 1]$
- E.g. $\widehat{\Lambda} := |\alpha\rangle\langle\alpha|$, $|\alpha\rangle$ eigenstate of $\widehat{\mathrm{A}}$ with eigenvalue α
  - $\mathrm{tr}\left\{\widehat{\Lambda}\hat{\rho}\right\}$: the probability of getting α in measuring $\widehat{\mathrm{A}}$
  - $\mathrm{tr}\left\{\widehat{\Lambda}(\hat{\rho} - \widehat{\sigma})\right\}$: the difference of the probability above
  - $T(\hat{\rho}, \widehat{\sigma})$: the maximal value of the difference above

Distances between the Density Operators

Single-mode trace distances

- $\chi[f] \sim S$, $\Psi[g,\phi,f] \sim U$
- The pure wave functional can sloppily be decomposed $\chi_b[g] \sim \sum_p \chi_b^{(p)}(g_p) \equiv \sum_p \left\langle g_p \,\middle|\, \chi_b^{(p)} \right\rangle$, where $g_p := g(p)$
- So does the thermal density operator $\hat{\rho}_{\text{th}}(T) \sim \bigotimes_p \hat{\rho}_{\text{th}}^{(p)}(T)$
- Fidelity in eq. (23) can be factorised as well

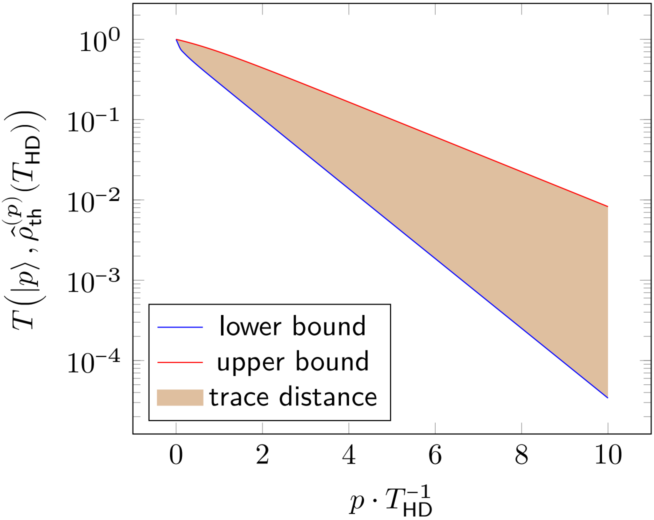
$$F \equiv \langle \chi_b \,|\, \hat{\rho}_{\text{th}} \,|\, \chi_b \rangle^{\frac{1}{2}} \sim \prod_p \left\langle \chi_b^{(p)} \,\middle|\, \hat{\rho}_{\text{th}}^{(p)} \,\middle|\, \chi_b^{(p)} \right\rangle^{\frac{1}{2}} =: \prod_p F^{(p)}; \tag{26}$$

- $F^{(p)}$ can be computed in order to find bounds of T

$$F^{(p)}\left(|p\rangle \,, \hat{\rho}_{\text{th}}^{(p)}(T_{\text{HD}})\right) = \frac{\sqrt{u-1}}{\sqrt[4]{u^2+u+1}}, \quad u := \mathrm{e}^{q} \equiv \mathrm{e}^{|p|/T_{\text{HD}}}. \tag{27}$$

Distances between the Density Operators

Single-mode trace distances, bounds set by fidelity and eq. (23)



Distances between the Density Operators

All-mode trace distance

- 'Go to the continuum limit': Λ dimension regulator

$$\sum_p g(p) \to \frac{1}{2\pi\Lambda} \int \mathrm{d}p\, g(p), \tag{26}$$

- Analogously, to regularising a positive product

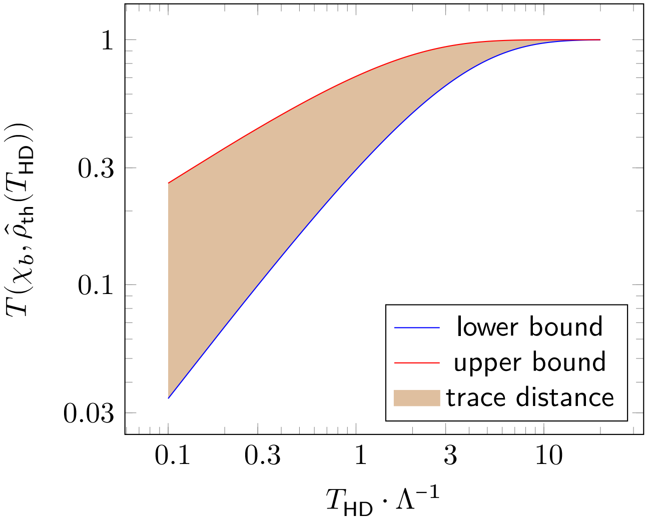
$$\prod_p f(p) \equiv \exp\left\{ \sum_p \ln f(p) \right\} \to \exp\left\{ \frac{1}{2\pi\Lambda} \int \mathrm{d}p\, \ln f(p) \right\} \tag{27}$$

- Regularised F can be calculated in order to set bounds of T

$$F(\chi_b, \hat{\rho}_{\text{th}}) = \exp\left\{ \frac{2}{2\pi\Lambda} \int_0^{+\infty} \mathrm{d}p\, \ln F^{(p)} \right\} = \exp\left(-\frac{\pi}{9} \frac{T_{\text{HD}}}{\Lambda} \right) \tag{28}$$

Distances between the Density Operators

All-mode trace distance, bounds set by fidelity and eq. (23)



Summary

- Compared the pure and the thermal descriptions of the radiation within the solvable dilaton gravity model
- Fourier-mode fluctuation: that of the thermal state diverges faster than the vacuum case does at low energy, while the pure-state fluc. remains finite; at high energies they converge.
- Trace distance: goes exponentially small with black hole temperature going to zero.

- Outlook
  - Understand the dilaton gravity model
  - Understand the Fourier-mode fluctuation
  - Evaluate the real space correlator
  - Understand practical meaning of trace distance
  - Understand the regulator in total trace distance
  - Evaluate the exact trace distance
  - Include the gravitational wave functional

For Further Reading I

Claus Kiefer. *Quantum Gravity*. 3rd ed. Oxford University Press (OUP), Apr. 2012.

Lev D. Landau and E.M. Lifshitz. *Statistical Physics*. 3rd ed. Butterworth-Heinemann, Jan. 1980.

Mark M. Wilde. *Quantum Information Theory*. Cambridge University Press (CUP), 2009.

Abhay Ashtekar, Frans Pretorius, and Fethi M. Ramazanoğlu. "Evaporation of two-dimensional black holes". In: *Phys Rev D* 83.4 (Feb. 2011).

Curtis G. Callan et al. "Evanescent black holes". In: *Phys Rev D* 45.4 (Feb. 1992), R1005–R1009.

Jean-Guy Demers and Claus Kiefer. "Decoherence of black holes by Hawking radiation". In: *Phys Rev D* 53.12 (June 1996), pp. 7050–7061.

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For Further Reading II

- C.A. Fuchs and J. van de Graaf. “Cryptographic distinguishability measures for quantum-mechanical states”. In: *IEEE Trans. Inf. Theory* 45.4 (May 1999), pp. 1216–1227.
- S. W. Hawking. “Black hole explosions?” In: *Nature* 248.5443 (Mar. 1974), pp. 30–31.
- S. W. Hawking. “Particle creation by black holes”. In: *Commun. Math. Phys.* 43.3 (Aug. 1975), pp. 199–220.
- Stephen D. H. Hsu and David Reeb. “Black holes, information, and decoherence”. In: *Phys Rev D* 79.12 (June 2009).
- Claus Kiefer. “Hawking radiation from decoherence”. In: *Classical Quantum Gravity* 18.22 (Nov. 2001), pp. L151–L154.

For Further Reading III

Sandu Popescu, Anthony J. Short, and Andreas Winter. "Entanglement and the foundations of statistical mechanics". In: *Nat. Phys.* 2.11 (Oct. 2006), pp. 754–758.