

Cosmological Perturbations

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Most of the conventions and notations in [6, ch. 5] will be followed.
Suppose the metric can be expanded up to the linear order as

$$g = g^{(0)} + \epsilon g^{(1)} + O(\epsilon^2). \quad (1)$$

The background metric $g^{(0)}$ takes the Robertson–Walker form

$$g_{\mu\nu}^{(0)} dx^\mu dx^\nu = -N^2(t) dt^2 + a^2(t) d\Omega_{3F}^2, \quad (2)$$

in which $d\Omega_{3F}^2 = d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)$ is the dimensionless flat spatial metric. The linear perturbation can be decomposed into scalar, vector and tensor parts

$$g_{00}^{(1)} = -E, \quad (3)$$

$$g_{i0}^{(1)} = g_{0i}^{(1)} = F_{,i} + G_i, \quad (4)$$

$$g_{ij}^{(1)} = A\delta_{ij} + B_{,i,j} + C_{i,j} + C_{j,i} + D_{ij}. \quad (5)$$

Here one has some weird condition, where the contractions do not follow the one-up-one-down tradition

$$C_{i,i} = G_{i,i} = 0, \quad D_{ij,i} = 0, \quad D_{ii} = 0. \quad (6)$$

1 Metric perturbation under diffeomorphism

Consider a diffeomorphism generated by ξ^μ

$$x^\mu \rightarrow \bar{x}^\mu = x^\mu - \epsilon \xi^\mu. \quad (7)$$

The generator ξ^μ can in turn be decomposed into $\xi_0 = \zeta$, $\xi_i = \xi_{,i}^S + \xi_i^V$.

One has here again some weird condition, where the contractions do not follow the one-up-one-down tradition

$$\xi_{i,i}^V = 0. \quad (8)$$

The Lie derivative of the metric $\mathbb{L}_\xi g$ is

$$(\mathbb{L}_\xi g)_{\mu\nu} = \xi^\lambda g_{\mu\nu,\lambda} + \xi^\lambda_{,\mu} g_{\lambda\nu} + \xi^\lambda_{,\nu} g_{\mu\lambda}. \quad (9)$$

In components and expansion, these are

$$(\mathbb{L}_\xi g)_{00} = 2\dot{\zeta} - 2\zeta \frac{\dot{N}}{N} + O(\epsilon), \quad (10)$$

$$(\mathbb{L}_\xi g)_{i0} = (\mathbb{L}_\xi g)_{0i} = \left(\zeta - 2\frac{\dot{a}}{a} \xi^S + \dot{\xi}^S \right)_{,i} + \left(-2\frac{\dot{a}}{a} \xi_i^V + \dot{\xi}_i^V \right) + O(\epsilon), \quad (11)$$

$$(\mathbb{L}_\xi g)_{ji} = (\mathbb{L}_\xi g)_{ij} = -\frac{2a\dot{a}}{N^2} \zeta \delta_{ij} + 2\xi_{i,j}^S + \xi_{i,j}^V + \xi_{j,i}^V + O(\epsilon). \quad (12)$$

2 Scalar perturbations

$$-N^2 - \epsilon E + O(\epsilon^2) \rightarrow -N^2 - \epsilon E + \epsilon \left(2\dot{\zeta} - 2\zeta \frac{\dot{N}}{N} \right) + O(\epsilon^2), \quad (13)$$

so one can write

$$\mathbb{L}_\xi E = -2\dot{\zeta} + 2\zeta \frac{\dot{N}}{N}. \quad (14)$$

Similarly one can read-off

$$\mathbb{L}_\xi F = \zeta - 2\frac{\dot{a}}{a} \xi^S + \dot{\xi}^S, \quad (15)$$

$$\mathbb{L}_\xi A = -\frac{2a\dot{a}}{N^2} \zeta, \quad (16)$$

$$\mathbb{L}_\xi B = 2\xi^S. \quad (17)$$

The four scalar perturbations are generated by ζ and ξ^S , so that only two independent perturbations exists. It is clear that

$$\mathbb{L}_\xi \left(\frac{F}{a} - \frac{\mathbb{D}}{\mathbb{D}t} \frac{B}{2a} \right) = \frac{\zeta}{a}. \quad (18)$$

One can verify that

$$\mathbb{L}_\xi \left\{ \frac{E}{2N} + \frac{\mathbb{D}}{\mathbb{D}t} \left[\frac{a}{N} \left(\frac{F}{a} - \frac{\mathbb{D}}{\mathbb{D}t} \frac{B}{2a} \right) \right] \right\} = 0, \quad (19)$$

$$\mathbb{L}_\xi \left\{ \frac{A}{2} + \frac{a^2 \dot{a}}{N^2} \left(\frac{F}{a} - \frac{\mathbb{D}}{\mathbb{D}t} \frac{B}{2a} \right) \right\} = 0. \quad (20)$$

3 Vector perturbations

4 Tensor perturbations

5 Scalar field perturbation under diffeomorphism

6 Perturbation in Arnowitt–Deser–Misner Hamiltonian formalism

Up to boundary terms, the Hamiltonian action for General Relativity in terms of Arnowitt–Deser–Misner variables is [3, ch.4.2.2]

$$S = \int dt dx^3 \left\{ \mathfrak{p}^{ij} \dot{h}_{ij} + \mathfrak{P} \dot{N} + \mathfrak{P}^i \dot{N}_i - N \mathfrak{H}^\perp - N_i \mathfrak{H}^i - \mathfrak{P} V - \mathfrak{P}^i V_i \right\}, \quad (21)$$

$$\mathfrak{H}^\perp = \frac{\mathscr{N}}{\sqrt{\mathfrak{h}}} (h_{ik} h_{jl} + h_{il} h_{kj} - h_{ij} h_{kl}) \mathfrak{p}^{ij} \mathfrak{p}^{kl} - \frac{\sqrt{\mathfrak{h}}}{2\mathscr{N}} R[h], \quad (22)$$

$$\mathfrak{H}^i = -2\mathfrak{p}^{ij}{}_{|j}, \quad (23)$$

where $\{V, V_i\}$ are velocity of N and N_i and play the role of Lagrange multipliers. Technical details about the boundary terms can be found in [5, ch. 4.2] and the references therein. Note that $\{N, N_i, h_{ij}, \mathfrak{P}, \mathfrak{P}^i, \mathfrak{p}^{ij}\}$ are not the unique choice of canonical variables for General Relativity in Hamiltonian formalism; instead, they are a special parametrisation of the phase space. One can also choose the components of the original four-metric and their conjugate momenta $\{g_{\mu\nu}, \mathfrak{p}^{\mu\nu}\}$ as canonical variables, as Dirac has done in [2]. The two approaches are different in some subtle aspects; see [4] for a comparison.

Gauge transformations in the Arnowitt–Deser–Misner canonical variables are generated by [1]

$$G = - \int \mathfrak{d}^3x \left\{ \left[\xi_\perp \left(\mathfrak{H}^\perp + N_{|i} \mathfrak{P}^i + (N \mathfrak{P}^i)_{|i} + (N_i \mathfrak{P})^{|i} \right) + \dot{\xi}_\perp \mathfrak{P} \right] \right. \\ \left. + \left[\xi_i \left(\mathfrak{H}^i + N_j{}^{|i} \mathfrak{P}^j + (N_j \mathfrak{P}^i)^{|j} + N^{|i} \mathfrak{P} \right) + \dot{\xi}_i \mathfrak{P}^i \right] \right\}. \quad (24)$$

Possible boundary terms have not been discussed so far. The infinitesimal gauge transformation of $\{N, N_i\}$ is

$$\delta N = [N, G]_P = \xi_\perp{}^{|i} N_{z|i} - \dot{\xi}_\perp - \xi_i N^{|i}, \quad (25)$$

$$\delta N_i = -\xi_\perp N_{|i} + \xi_\perp{}_{|i} N - \xi_j N_i{}^{|j} + \xi_i{}^{|j} N_j - \dot{\xi}_i, \quad (26)$$

which can be found in [4]. Transformations for g_{ij} and the momenta have to be worked out as

$$\delta \mathfrak{P} = -(\xi_\perp \mathfrak{P}^i)_{|i} - \xi_\perp{}_{|i} \mathfrak{P}^i - (\xi_i \mathfrak{P})^{|i}, \quad (27)$$

$$\delta \mathfrak{P}^i = -\xi_\perp{}_{|i} \mathfrak{P} - (\xi_j \mathfrak{P}^i)^{|j} - \xi_j{}^{|i} \mathfrak{P}^j, \quad (28)$$

where only the primary constraints are involved;

$$\begin{aligned}
\delta g_{ij} &= -\frac{\partial}{\partial \mathbf{p}^{ij}}(\xi_{\perp} \mathfrak{H}^{\perp} + \xi_i \mathfrak{H}^i) \\
&= -\xi_{\perp}^{\perp} \frac{2\mathcal{N}}{\sqrt{\mathfrak{h}}} (h_{ik} h_{jl} + h_{il} h_{kj} - h_{ij} h_{kl}) \mathbf{p}^{kl} - \xi_{i|j} - \xi_{j|i}, \quad (29) \\
\delta \mathbf{p}^{ij} &= \frac{\partial}{\partial g_{ij}}(\xi_{\perp} \mathfrak{H}^{\perp} + \xi_i \mathfrak{H}^i) \\
&= \xi_{\perp} \left\{ \frac{\mathcal{N}}{\sqrt{\mathfrak{h}}} \left[(4\delta^i_k \delta^j_m - h^{ij} h_{km}) h_{ln} \right. \right. \\
&\quad \left. \left. - \frac{1}{2} (4\delta^i_k \delta^j_l - h^{ij} h_{kl}) h_{mn} \right] \mathbf{p}^{kl} \mathbf{p}^{mn} \right. \\
&\quad \left. + \frac{\sqrt{\mathfrak{h}}}{2\mathcal{N}} G^{ij}[h] \right\} + \frac{\sqrt{\mathfrak{h}}}{2\mathcal{N}} (\xi_{\perp|k} h^{ij} - \xi_{\perp}^{(i|j)}) \\
&\quad - (\xi_k \mathbf{p}^{ij})^{|k} + 2\xi_{k|l} h^{k(i} \mathbf{p}^{j)l}, \quad (30)
\end{aligned}$$

where $G^{ij}[h] = R^{ij}[h] - h^{ij} R[h]/2$, and only the secondary constraints are involved. In eq. (30), the first two lines come from the variation of the ‘kinetic’ term in \mathfrak{H}^{\perp} , the third comes from the ‘potential’ term in \mathfrak{H}^{\perp} , and the last line from \mathfrak{H}^i . The results can be checked with [5, p. 4.2.7].

6.1 Expansion of the action with fluctuations

Some useful results

First variations

First variation of h^{ij}

$$\delta h^{ij} = -h^{ik} h^{jl} \delta h_{kl} = -h^{i(k} h^{l)j} \delta h_{kl}. \quad (31)$$

First variation of $\mathfrak{h} = \det h_{ij}$

$$\delta \mathfrak{h} = \frac{1}{2} \mathfrak{h} h^{ij} \delta h_{ij}. \quad (32)$$

First variation of Γ^i_{jk}

$$\delta \Gamma^i_{jk} = \frac{1}{2} h^{il} \left\{ -(\delta h_{jk})_{|l} + (\delta h_{kl})_{|j} + (\delta h_{lj})_{|k} \right\}. \quad (33)$$

First variation of $R_{ij}[h]$ and $R^{ij}[h]$

$$\delta R_{ij}[h] = (\delta \Gamma^k_{ij})_{|k} - (\delta \Gamma^k_{ik})_{|j}, \quad (34)$$

$$\delta R^{ij}[h] = -2R^{ki} h^{jl} \delta h_{kl} + h^{ik} \bar{\delta} u^{jl}_{k|l}, \quad (35)$$

where

$$\bar{\delta}u^{ij}{}_k := h^{il} \delta\Gamma^j{}_{kl} - h^{ij} \delta\Gamma^l{}_{kl}. \quad (36)$$

Equation (34) can be obtained by using normal coordinates.

First variation of $\sqrt{\mathfrak{h}}R[h]$

$$\delta(\sqrt{\mathfrak{h}}R[h]) = \sqrt{\mathfrak{h}}\{-G^{ij}[h]\delta h_{ij} + \bar{\delta}u^{ji}{}_{j|i}\}. \quad (37)$$

Second variations

Second variation of h^{ij}

$$\delta^2 h^{ij} = (h^{im}h^{jl}h^{kn} + h^{ik}h^{jm}h^{ln})\delta h_{kl}\delta h_{mn} \quad (38)$$

Second variation of $\mathfrak{h} = \det h_{ij}$

$$\delta^2 \mathfrak{h} = -\frac{1}{4}\mathfrak{h}(h^{ik}h^{jl} + h^{il}h^{kj} - h^{ij}h^{kl})\delta h_{ij}\delta h_{kl}. \quad (39)$$

First variation of $(\delta h_{ij})|_k$

$$\delta\left\{(\delta h_{ij})|_k\right\} = -2\delta\Gamma^l{}_{k(i}\delta h_{j)l}. \quad (40)$$

Second variation of $\Gamma^i{}_{jk}$

$$\delta^2 \Gamma^i{}_{jk} = -\frac{1}{2}h^{im}\delta\Gamma^l{}_{jk}\delta h_{lm}. \quad (41)$$

Second variation of $\sqrt{\mathfrak{h}}R[h]$

$$\begin{aligned} \delta^2(\sqrt{\mathfrak{h}}R) = & \sqrt{\mathfrak{h}}\left\{\delta h_{ij}\delta h_{kl}\left[\frac{1}{4}R(h^{ik}h^{jl} + h^{il}h^{jk} - h^{ij}h^{kl})\right.\right. \\ & \left.+\frac{1}{2}(R^{ij}h^{kl} + R^{kl}h^{ij} - R^{ik}h^{jl} - R^{il}h^{jk} - R^{jk}h^{il} - R^{il}h^{kj})\right] \\ & \left.+h^{ik}\bar{\delta}u^{jl}{}_{k|l} + \delta(\bar{\delta}u^{ji}{}_{j|i})\right\} \end{aligned} \quad (42)$$

In a general background, the second variations of the quantities are much more tedious.

In Robertson–Walker background,

References

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