Euler-Heisenberg Effective Action

immediate

1 Spinor electrodynamics in flat space-time

Maxwell Lagrangian

$$S_{\mathrm{Maxwell}}[A_\mu] \coloneqq \int \mathrm{d}^{d+1}x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \tag{1}$$

Dirac Lagrangian [1, sec. 11]

$$S_{\mathrm{Dirac}}\big[\psi,\bar{\psi}\big] \coloneqq \int \mathrm{d}^{d+1}x \left\{-\bar{\psi}(\gamma^\mu\partial_\mu + m)\psi\right\}. \tag{2}$$

Interaction term

$$S_{\mathrm{IDM}}\big[A_\mu,\psi,\bar{\psi}\big] \coloneqq \int \mathrm{d}^{d+1}x \left(-\bar{\psi}\gamma^\mu \mathrm{i}eA_\mu\psi\right). \tag{3}$$

The total action for spinor electrodynamics reads

$$\begin{aligned} S_{1/2}\big[A_\mu,\psi,\bar{\psi}\big] &\coloneqq S_{\mathrm{Dirac}} + S_{\mathrm{IDM}} + S_{\mathrm{Maxwell}} \\ &= \int \mathrm{d}^{d+1}x \left\{-\bar{\psi}(\gamma^\mu\nabla_\mu + m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right\}, \end{aligned} \tag{4}$$

where

$$\nabla_\mu\psi \coloneqq (\partial_\mu + \mathrm{i}eA_\mu)\psi. \tag{5}$$

Generating functional

$$\mathcal{Z}[j^\mu,\bar{\eta},\eta] \coloneqq \int \mathrm{D}A\,\mathrm{D}\psi\,\mathrm{D}\bar{\psi}\exp\left\{\mathrm{i}\left(S_{1/2} + \int \mathrm{d}^{d+1}x\left(j^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta\right)\right)\right\} \tag{6}$$

Effective action

$$\mathcal{Z}[j^\mu,0,0] =: \int \mathrm{D}A \exp\left\{\mathrm{i}\left(S_{\text{Maxwell}} + \mathit{\Gamma}_{\text{EH}}[A_\mu] + \int \mathrm{d}^{d+1}x\, j^\mu A_\mu\right)\right\}. \tag{7}$$

In other words,

$$\begin{aligned}
\exp\{\mathrm{i}\mathit{\Gamma}_{\text{EH}}[A_\mu]\} &\equiv \int \mathrm{D}\psi\, \mathrm{D}\bar\psi \exp\{\mathrm{i}(S_{\text{Dirac}} + S_{\text{IDM}})\} \\
&\equiv \int \mathrm{D}\psi\, \mathrm{D}\bar\psi \exp\left\{\mathrm{i}\int \mathrm{d}^{d+1}x\, \bar\psi\big(-\partial\!\!\!/ - \mathrm{i}eA\!\!\!/ - m\big)\psi\right\} \\
&= \int \mathrm{D}\psi\, \mathrm{D}\bar\psi \exp\left\{\mathrm{i}\int \mathrm{d}^{d+1}x\, \mathrm{d}^{d+1}y\, \bar\psi(x) S^{-1}(x,y)\psi(y)\right\} \\
&= \tilde{\mathcal{N}} \det\big[-\mathrm{i}S^{-1}(x,y)\big],
\end{aligned} \tag{8}$$

where

$$S^{-1}(x,y) := \big(+\partial\!\!\!/_y - \mathrm{i}eA\!\!\!/(y) - m\big)\delta^{d+1}(x-y). \tag{9}$$

$$\begin{aligned}
\mathit{\Gamma}_{\text{EH}}[A_\mu] &\equiv -\mathrm{i}\Big(\ln\tilde{\mathcal{N}} + \ln\det\big[-\mathrm{i}S^{-1}\big]\Big) \\
&= -\mathrm{i}\Big(\ln\tilde{\mathcal{N}} + \mathrm{Tr}\ln\big[-\mathrm{i}S^{-1}(x,y)\big]\Big).
\end{aligned} \tag{10}$$

Note that

$$\begin{aligned}
\mathrm{Tr}\ln\big[-\mathrm{i}S^{-1}(x,y)\big] &\equiv \mathrm{Tr}\ln[-\mathrm{i}M^\intercal(x,y)] \\
&= \mathrm{Tr}\ln\big[\mathrm{i}\big(+\partial\!\!\!/_y^\intercal - \mathrm{i}eA\!\!\!/^\intercal(y) - m\big)\delta^{d+1}(x-y)\big] \\
&= \mathrm{Tr}\ln\big[\mathrm{i}\big(-\mathscr{C}\partial\!\!\!/_y\mathscr{C}^{-1} + \mathrm{i}e\mathscr{C}A\!\!\!/(y)\mathscr{C}^{-1} - \mathscr{C}m\mathscr{C}^{-1}\big)\delta^{d+1}(x-y)\big] \\
&= \mathrm{Tr}\big\{\mathscr{C}\ln\big[\mathrm{i}\big(-\partial\!\!\!/_y + \mathrm{i}eA\!\!\!/(y) - m\big)\delta^{d+1}(x-y)\big]\mathscr{C}^{-1}\big\} \\
&= \mathrm{Tr}\ln\big[\mathrm{i}\big(-\partial\!\!\!/_y + \mathrm{i}eA\!\!\!/(y) - m\big)\delta^{d+1}(x-y)\big].
\end{aligned} \tag{11}$$

where the transpose $^\intercal$ is taken in the spinor space. Therefore

$$\mathrm{Tr}\ln\big[-\mathrm{i}S^{-1}(x,y)\big] = \frac{1}{2}\,\mathrm{Tr}\ln\big[S_2^{-1}(x,y)\big], \tag{12}$$

where

$$S_2^{-1}(x,y) := \mathcal{S}_{2y}^{-1}\delta^{d+1}(x-y), \quad \mathcal{S}_{2y}^{-1} := \Big(\big(\partial\!\!\!/_y - \mathrm{i}eA\!\!\!/(y)\big)^2 - m^2\Big). \tag{13}$$

One may further simplify eq. (13) by noting (y suppressed)

$$\begin{aligned}
\big(\partial\!\!\!/ - \mathrm{i}eA\!\!\!/\big)^2 &= \partial^2 - e^2 A_\mu^2 - \mathrm{i}e\frac{1}{2}\big([\gamma^\mu,\gamma^\nu]_- + [\gamma^\mu,\gamma^\nu]_+\big)(\partial_\mu A_\nu + A_\mu\partial_\nu + A_\nu\partial_\mu)\\
&= \partial^2 - e^2 A_\mu^2 - \mathrm{i}e(\partial_\mu A^\mu + 2A^\mu\partial_\mu - \mathrm{i}\sigma^{\mu\nu}\partial_\mu A_\nu)\\
&= (\partial_\mu - \mathrm{i}eA_\mu)^2 - e\sigma^{\mu\nu}\partial_{[\mu}A_{\nu]}\\
&= (\partial_\mu - \mathrm{i}eA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu},
\end{aligned} \tag{14}$$

so that

$$\mathcal{S}_2^{-1} \equiv \Big((\partial_\mu - \mathrm{i}eA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\Big), \tag{15}$$

where y is suppressed as well. Now we may write

$$\begin{aligned}
\mathit{\Gamma}_{\mathrm{EH}}[A_\mu] &= -\frac{\mathrm{i}}{2}\,\mathrm{Tr}\ln\big\{\mathcal{N}^{-1}S_2^{-1}(x,y)\big\}\\
&= -\frac{\mathrm{i}}{2}\,\mathrm{Tr}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\Big\{-\mathrm{e}^{\mathrm{i}s\left(S_2^{-1}(x,y)+\mathrm{i}0^+\right)} + \mathrm{e}^{\mathrm{i}s\left(\mathcal{N}+\mathrm{i}0^+\right)}\Big\}\\
&= +\frac{\mathrm{i}}{2}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\delta^{d+1}(x-y)\\
&\quad\cdot\,\mathrm{tr}\Big\{+\mathrm{e}^{\mathrm{i}s\left((\partial_\mu-\mathrm{i}eA_\mu)^2-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}-m^2+\mathrm{i}0^+\right)\delta^{d+1}(x-y)} - \mathrm{e}^{\mathrm{i}s\left(\mathcal{N}+\mathrm{i}0^+\right)}\Big\},
\end{aligned} \tag{16}$$

where tr takes place in the spinor space.

1.1 Constant background field in $(3+1)$-dimensions

Equation (12) can be solved exactly when $F_{\mu\nu}$ is constant throughout space-time. One has

$$F_{0i} \equiv -F_{i0} = -E_i, \qquad F_{ij} \equiv -F_{ji} = \epsilon_{ijk}B^k, \tag{17}$$

and the spinor part of eq. (15) can be calculated

$$-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} = -e\begin{bmatrix}\left(\vec{B}-\mathrm{i}\vec{E}\right)\cdot\vec{\sigma} & 0\\ 0 & \left(\vec{B}+\mathrm{i}\vec{E}\right)\cdot\vec{\sigma}\end{bmatrix}, \tag{18}$$

so that

$$\begin{aligned}
&\mathrm{tr}\,\mathrm{e}^{\mathrm{i}s\left((\partial_\mu-\mathrm{i}eA_\mu)^2-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}-m^2+\mathrm{i}0^+\right)\delta^{d+1}(x-y)}\\
&= \mathrm{e}^{\mathrm{i}s\left((\partial_\mu-\mathrm{i}eA_\mu)^2-m^2+\mathrm{i}0^+\right)\delta^{d+1}(x-y)}\,\mathrm{tr}\,\mathrm{e}^{\mathrm{i}s\left(-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)},
\end{aligned} \tag{19}$$

in which

$$\operatorname{tr}\mathrm{e}^{\mathrm{i}s\left(-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)} = \cos\left(se\sqrt{\left(\vec{B}+\mathrm{i}\vec{E}\right)^2}\right) + \cos\left(se\sqrt{\left(\vec{B}-\mathrm{i}\vec{E}\right)^2}\right) \tag{20}$$

$$= 4\cos(seB)\cosh(seE) \qquad \vec{B} = B\hat{x}^3, \quad \vec{E} = E\hat{x}^3. \tag{21}$$

A Landau-like choice of four-potential [3] reads

$$\begin{aligned} A_\mu &:= \begin{pmatrix} 0 & -x^0E_1 + x^3B^2 & -x^0E_2 + x^1B^3 & -x^0E_3 + x^2B^1 \end{pmatrix} \\ &= \left(0, -x^0E_i + \epsilon_{ijk}B^jx^k\right). \end{aligned} \tag{22}$$

applying which leads to eq. (17). In this choice the scalar part of eq. (15) is

$$(\partial_\mu - \mathrm{i}eA_\mu)^2 - m^2 = -\partial_0^2 + \left\{\partial_i - \mathrm{i}e\left(-x^0E_i + \epsilon_{ijk}B^jx^k\right)\right\}^2 - m^2. \tag{23}$$

[4]

2 Scalar electrodynamics in flat space-time

Complex Klein–Gordon Lagrangian

$$S_{\mathrm{CKG}}[\phi,\phi^*] := \int \mathrm{d}^{d+1}x \left\{-\eta^{\mu\nu}(\partial_\mu\phi)^*(\partial_\nu\phi) - m^2\phi^*\phi\right\}. \tag{24}$$

Interaction term

$$S_{\mathrm{ICKGM}}[A_\mu,\phi,\phi^*] := \int \mathrm{d}^{d+1}x\, \eta^{\mu\nu}\left\{\mathrm{i}eA_\mu(-\phi^*\partial_\nu\phi + \phi\partial_\nu\phi^*) + e^2A_\mu A_\nu\phi^*\phi\right\}. \tag{25}$$

The total action for scalar electrodynamics reads

$$\begin{aligned} S_0[A_\mu,\phi,\phi^*] &:= S_{\mathrm{CKG}} + S_{\mathrm{ICKGM}} + S_{\mathrm{Maxwell}} \\ &= \int \mathrm{d}^{d+1}x \left\{-(\nabla_\mu\phi)^*(\nabla^\mu\phi) - m^2\phi^*\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right\}, \end{aligned} \tag{26}$$

where

$$\nabla_\mu\phi := (\partial_\mu + \mathrm{i}eA_\mu)\phi. \tag{27}$$

Generating functional

$$\mathcal{Z}\left[j^\mu, \bar{J}, J\right] := \int \mathrm{D}A\,\mathrm{D}\phi\,\mathrm{D}\phi^* \exp\left\{\mathrm{i}\left(S_0 + \int \mathrm{d}^{d+1}x\,(j^\mu A_\mu + J^*\psi + \psi^*J)\right)\right\}. \tag{28}$$

Effective action

$$\mathcal{Z}[j^\mu,0,0] =: \int \mathrm{D}A\exp\Big\{\mathrm{i}\Big(S_{\text{Maxwell}} + \mathit{\Gamma}_{\mathrm{W}}[A_\mu] + \int \mathrm{d}^{d+1}x\, j^\mu A_\mu\Big)\Big\}. \tag{29}$$

In other words,

$$\begin{aligned}\exp\{\mathrm{i}\mathit{\Gamma}_{\mathrm{W}}[A_\mu]\} &\coloneqq \int \mathrm{D}\phi\,\mathrm{D}\phi^*\exp\{\mathrm{i}(S_{\text{CKG}} + S_{\text{ICKGM}})\}\\ &\equiv \int \mathrm{D}\phi\,\mathrm{D}\phi^*\exp\Big\{\mathrm{i}\int \mathrm{d}^{d+1}x\,\{-(\nabla_\mu\phi)^*(\nabla^\mu\phi) - m^2\phi^*\phi\}\Big\}.\end{aligned} \tag{30}$$

The integral in the exponent can be manipulated; only the first term is essential

$$\begin{aligned}&\int \mathrm{d}^{d+1}x\,(-(\nabla_\mu\phi)^*(\nabla^\mu\phi))\\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\big(-(\nabla_{x^\mu}\phi(x))^*\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\big),\end{aligned} \tag{31}$$

where

$$\begin{aligned}\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y) &= \delta^{d+1}(x-y)\big\{\partial^{y^\mu} + \mathrm{i}eA^\mu(y)\big\}\phi(y)\\ &= \big\{-\big(\nabla^{y^\mu}\big)^*\delta^{d+1}(x-y)\big\}\phi(y) + \partial^{y^\mu}B,\end{aligned} \tag{32}$$

in which

$$B = B(x,y) \coloneqq \delta^{d+1}(x-y)\phi(y); \tag{33}$$

going back to eq. (31),

$$\begin{aligned}&= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\{-\{\{\partial_{x^\mu} + \mathrm{i}eA_\mu(x)\}\phi(x)\}^*\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\}\\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\{-\partial_{x^\mu}C^\mu + \phi^*(x)\nabla_{x^\mu}\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\}\\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\{-\partial_{x^\mu}C^\mu + \phi^*(x)\big\{-(\nabla_{y^\mu})\big(\nabla^{y^\mu}\big)^*\delta^{d+1}(x-y)\big\}\phi(y) + \partial^{y^\mu}\nabla_{x^\mu}B\big\},\end{aligned} \tag{34}$$

in which

$$C^\mu = C^\mu(x,y) \coloneqq \phi^*(x)\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y). \tag{35}$$

Now eq. (30) can be written as (dropping the boundary terms)

$$\begin{aligned}&= \int \mathrm{D}\phi\,\mathrm{D}\phi^*\exp\Big\{-\mathrm{i}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\phi^*(x)D^{-1}(x,y)\phi(y)\Big\}\\ &= \tilde{\mathcal{N}}\big\{\det\big[D^{-1}(x,y)\big]\big\}^{-1/2},\end{aligned} \tag{36}$$

where

$$D^{-1}(x,y) \coloneqq \mathcal{D}_y^{-1}\delta^{d+1}(x-y), \qquad \mathcal{D}_y^{-1} \coloneqq +(\nabla_{y^\mu})\big(\nabla^{y^\mu}\big)^* + m^2. \tag{37}$$

$$\begin{aligned}
\mathit{\Gamma}_{\mathrm{W}}[A_\mu] &\equiv -\mathrm{i}\bigg(\ln\tilde{\mathcal{N}} - \frac{1}{2}\ln\det D^{-1}\bigg) \\
&= \frac{\mathrm{i}}{2}\operatorname{tr}_x \ln\big(\mathcal{N}^{-1}D^{-1}\big) \\
&= \frac{\mathrm{i}}{2}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\delta^{d+1}(x-y) \\
&\quad\cdot\Big\{-\mathrm{e}^{\mathrm{i}s\left(+(\nabla_{y^\mu})\left(\nabla^{y^\mu}\right)^*+m^2+\mathrm{i}0^+\right)\delta^{d+1}(x-y)} + \mathrm{e}^{\mathrm{i}s\left(\mathcal{N}+\mathrm{i}0^+\right)}\Big\}.
\end{aligned} \tag{38}$$

[5]

A Notions and conventions

The metric convention is mostly positive, i.e. $\eta_{\mu\nu} \coloneqq \operatorname{diag}(-,+,+,\ldots)$

Pauli matrices

$$\sigma^1 \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 \coloneqq \begin{pmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 \coloneqq \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{39}$$

The γ-matrices satisfy [1, sec. 5]

$$[\gamma^\mu, \gamma^\nu]_+ \coloneqq 2\eta^{\mu\nu}\mathbf{1}_4. \tag{40}$$

$$\mathscr{J}^{\mu\nu} \coloneqq -\frac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]_- \tag{41}$$

$$\sigma^{\mu\nu} \coloneqq \frac{\mathrm{i}}{2}[\gamma^\mu,\gamma^\nu]_- \equiv -2\mathscr{J}^{\mu\nu}. \tag{42}$$

In $(3+1)$ dimensions, choose the chiral representation

$$\gamma^\mu = -\mathrm{i}\begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \tag{43}$$

where

$$\sigma^\mu := (1_2, +\vec{\sigma}), \qquad \bar{\sigma}^\mu := (1_2, -\vec{\sigma}). \tag{44}$$

$$\begin{aligned}
\sigma^{\mu\nu} &\equiv -\frac{\mathrm{i}}{2}\begin{bmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{bmatrix} \\
&= \begin{cases} 0 & \mu = 0, \nu = 0; \\ \mathrm{i}\begin{bmatrix} +\sigma^j & 0 \\ 0 & -\sigma^j \end{bmatrix} & \mu = 0, \nu = j; \\ \mathrm{i}\begin{bmatrix} -\sigma^i & 0 \\ 0 & +\sigma^i \end{bmatrix} & \mu = i, \nu = 0; \\ \begin{bmatrix} +\epsilon^{ij}{}_k\sigma^k & 0 \\ 0 & +\epsilon^{ij}{}_k\sigma^k \end{bmatrix} & \mu = i, \nu = j. \end{cases}
\end{aligned} \tag{45}$$

B Fresnel functional integral

[6, ch. 10]

References

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