Euler-Heisenberg Effective Action

immediate

1 Spinor electrodynamics in flat space-time

Maxwell lagrangian

$$S_{\mathrm{Maxwell}}[A_\mu] \coloneqq \int \mathrm{d}^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \tag{1}$$

Dirac lagrangian [1, sec. 11]

$$S_{\mathrm{Dirac}}[\psi,\bar{\psi}] \coloneqq \int \mathrm{d}^4x \,\{-\bar{\psi}(\gamma^\mu\partial_\mu + m)\psi\}. \tag{2}$$

Interacting term

$$S_{\mathrm{IMD}} \coloneqq \int \mathrm{d}^4x \,(-\bar{\psi}\gamma^\mu \mathrm{i}eA_\mu\psi). \tag{3}$$

$$S_{1/2}[A_\mu,\psi,\bar{\psi}] \coloneqq S_{\mathrm{Maxwell}} + S_{\mathrm{Dirac}} + S_{\mathrm{IMD}}. \tag{4}$$

Generating functional

$$\mathcal{Z}[j^\mu,\bar{\eta},\eta] \coloneqq \int \mathrm{D}A\,\mathrm{D}\psi\,\mathrm{D}\bar{\psi}\exp\left\{\mathrm{i}\left(S_{1/2} + \int \mathrm{d}^4x\,(j^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta)\right)\right\} \tag{5}$$

Effective action

$$\mathcal{Z}[j^\mu,0,0] \eqqcolon \int \mathrm{D}A\exp\left\{\mathrm{i}\left(S_{\mathrm{Maxwell}} + \varGamma[A_\mu] + \int \mathrm{d}^4x\,j^\mu A_\mu\right)\right\}. \tag{6}$$

In other words,

$$
\begin{aligned}
\exp\{\mathrm{i}\mathit{\Gamma}[A_\mu]\} &\equiv \int \mathrm{D}\psi\,\mathrm{D}\bar{\psi}\,\exp\{\mathrm{i}(S_{\text{Dirac}}+S_{\text{IMD}})\} \\
&\equiv \int \mathrm{D}\psi\,\mathrm{D}\bar{\psi}\,\exp\bigg\{\mathrm{i}\int \mathrm{d}^4x\bar{\psi}\big(-\partial\!\!\!/ - \mathrm{i}eA\!\!\!/ - m\big)\psi\bigg\} \\
&= \int \mathrm{D}\psi\,\mathrm{D}\bar{\psi}\,\exp\bigg\{\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{\psi}(x)M(x,y)\psi(y)\bigg\} \\
&= \mathcal{N}\det[-\mathrm{i}M(x,y)],
\end{aligned}
\tag{7}
$$

where

$$
M(x,y) \coloneqq \big(+\partial\!\!\!/_y - \mathrm{i}eA\!\!\!/(y) - m\big)\delta^4(x-y). \tag{8}
$$

$$
\begin{aligned}
\mathit{\Gamma}[A_\mu] &\equiv -\mathrm{i}(\ln\mathcal{N} + \ln\det[-\mathrm{i}M]) \\
&= -\mathrm{i}(\ln\mathcal{N} + \operatorname{Tr}\ln[-\mathrm{i}M(x,y)]).
\end{aligned}
\tag{9}
$$

Note that

$$
\begin{aligned}
\operatorname{Tr}\ln[-\mathrm{i}M(x,y)] &\equiv \operatorname{Tr}\ln[-\mathrm{i}M^{\intercal}(x,y)] \\
&= \operatorname{Tr}\ln\big[\mathrm{i}\big(+\partial\!\!\!/_y^{\intercal} - \mathrm{i}eA\!\!\!/^{\intercal}(y) - m\big)\delta^4(x-y)\big] \\
&= \operatorname{Tr}\ln\big[\mathrm{i}\big(-\mathscr{C}\partial\!\!\!/_y\mathscr{C}^{-1} + \mathrm{i}e\mathscr{C}A\!\!\!/(y)\mathscr{C}^{-1} - \mathscr{C}m\mathscr{C}^{-1}\big)\delta^4(x-y)\big] \\
&= \operatorname{Tr}\big\{\mathscr{C}\ln\big[\mathrm{i}\big(-\partial\!\!\!/_y + \mathrm{i}eA\!\!\!/(y) - m\big)\delta^4(x-y)\big]\mathscr{C}^{-1}\big\} \\
&= \operatorname{Tr}\ln\big[\mathrm{i}\big(-\partial\!\!\!/_y + \mathrm{i}eA\!\!\!/(y) - m\big)\delta^4(x-y)\big].
\end{aligned}
\tag{10}
$$

where the transpose $^{\intercal}$ is taken in the spinor space. Therefore

$$
\operatorname{Tr}\ln[-\mathrm{i}M(x,y)] = \frac{1}{2}\operatorname{Tr}\ln[M_2(x,y)], \tag{11}
$$

where

$$
M_2(x,y) \coloneqq \mathcal{M}_y\delta^4(x-y), \quad \mathcal{M}_y \coloneqq \Big(\big(\partial\!\!\!/_y - \mathrm{i}eA\!\!\!/(y)\big)^2 - m^2\Big). \tag{12}
$$

One may further simplify eq. (12) by noting (y suppressed)

$$
\begin{aligned}
\big(\partial\!\!\!/ - \mathrm{i}eA\!\!\!/\big)^2 &= \partial^2 - e^2A_\mu^2 - \mathrm{i}e\frac{1}{2}\big([\gamma^\mu,\gamma^\nu]_- + [\gamma^\mu,\gamma^\nu]_+\big)(\partial_\mu A_\nu + A_\mu\partial_\nu + A_\nu\partial_\mu) \\
&= \partial^2 - e^2A_\mu^2 - \mathrm{i}e(\partial_\mu A^\mu + 2A^\mu\partial_\mu - \mathrm{i}\sigma^{\mu\nu}\partial_\mu A_\nu) \\
&= (\partial_\mu - \mathrm{i}eA_\mu)^2 - e\sigma^{\mu\nu}\partial_{[\mu}A_{\nu]} \\
&= (\partial_\mu - \mathrm{i}eA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu},
\end{aligned}
\tag{13}
$$

so that

$$
\mathcal{M} \equiv \Big((\partial_\mu - \mathrm{i}eA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\Big), \tag{14}
$$

where y is suppressed as well.

1.1 Constant background field

Equation (11) can be solved exactly when $F_{\mu\nu}$ is constant throughout space-time. Take the case [2] where $\vec{E} \parallel \vec{B}$ and, without loss of generality, $\vec{B} \parallel \vec{z}$. One has

$$F_{03} \equiv -F_{30} = E_3 \coloneqq E, \quad F_{12} \equiv -F_{21} = B_3 \coloneqq B. \tag{15}$$

A Landau-like choice of four-potential [3]

$$A_\mu \coloneqq \begin{pmatrix} 0 & -Bx_2 & 0 & Ex_0 \end{pmatrix} \tag{16}$$

can be applied which leads to eq. (15). In this choice the matrix reduces to

$$\mathcal{M} = \tag{17}$$

2 Scalar electrodynamics in flat space-time

A Notions and conventions

$\eta_{\mu\nu} \coloneqq \operatorname{diag}(-,+,+,+)$

Pauli matrices

$$\sigma^1 \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 \coloneqq \begin{pmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 \coloneqq \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{18}$$

[1, sec. 5]

$$[\gamma^\mu, \gamma^\nu]_+ \coloneqq 2\eta^{\mu\nu} \tag{19}$$

$$\mathscr{J}^{\mu\nu} \coloneqq -\frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]_- \tag{20}$$

$$\sigma^{\mu\nu} \coloneqq \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]_- \equiv -2\mathscr{J}^{\mu\nu}. \tag{21}$$

Choose the chiral representation

$$\gamma^\mu = -\mathrm{i}\begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \tag{22}$$

where

$$\sigma^\mu \coloneqq (1_2, +\vec{\sigma}), \qquad \bar{\sigma}^\mu \coloneqq (1_2, -\vec{\sigma}). \tag{23}$$

$$\begin{aligned}
\sigma^{\mu\nu} &\equiv -\frac{\mathrm{i}}{2}\begin{bmatrix}\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\end{bmatrix} \\
&= \begin{cases} 0 & \mu = 0, \nu = 0; \\ \mathrm{i}\begin{bmatrix}-\sigma^j & 0 \\ 0 & +\sigma^j\end{bmatrix} & \mu = 0, \nu = j; \\ \mathrm{i}\begin{bmatrix}+\sigma^i & 0 \\ 0 & -\sigma^i\end{bmatrix} & \mu = i, \nu = 0; \\ \begin{bmatrix}-\epsilon^{ij}{}_k\sigma^k & 0 \\ 0 & +\epsilon^{ij}{}_k\sigma^k\end{bmatrix} & \mu = i, \nu = j. \end{cases}
\end{aligned} \tag{24}$$

B Fresnel functional integral

[4, ch. 10]

References

[1] Steven Weinberg. *The Quantum Theory of Fields*. Vol. I: Foundations. Cambridge University Press, 1995.

[2] Akaki Rusetsky and Ulf-G. Meißner. *Lecture on Advanced Theoretical Hadron Physics*. URL: https://www.hiskp.uni-bonn.de/index.php?id=239.

[3] Lev Davidovich Landau. "Diamagnetismus der Metalle". In: *Zeitschrift für Physik* 64.9-10 (Sept. 1930), pp. 629–637.

[4] Ulrich Mosel. *Path Integrals in Field Theory*. Springer Berlin Heidelberg, 2004.