Classical string theory

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September 26, 2018

1 Point particle

1.1 Linear action

$$S_1[x^\mu] := -m \int_\gamma \mathbb{d}s = -m \int_\gamma \mathbb{d}\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}. \tag{1}$$

1.2 Quadratic action

$$S_2[x^\mu] := \frac{1}{2} \int_\gamma \mathbb{d}\lambda \, e \big(e^{-2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - m^2\big). \tag{2}$$

2 Bosonic string

2.1 Nambu–Goto action

The action reads [4, 3]

$$S_{\mathrm{NG}}[X^\mu] := -T \int_\Sigma \mathbb{d}A = -T \int_\Sigma \mathbb{d}^2\sigma \sqrt{-\tilde{\tilde{\psi}}}, \tag{3}$$

where

$$\tilde{\tilde{\psi}} := \det \psi_{\alpha\beta} \equiv \psi_{11}\psi_{22} - \psi_{12}\psi_{21} \tag{4}$$

is the metric determinant, $\alpha, \beta, \ldots = 1, 2$ are the world-sheet indices, (σ^α) are the world-sheet coordinates,

$$\psi_{\alpha\beta} := g_{\mu\nu} X^\mu{}_{,\alpha} X^\nu{}_{,\beta}, \tag{5}$$

is the induced metric on the world-sheet, $\rho, \nu, \ldots = 0, 1, \ldots d$ are the target-space indices, $X^\mu = X^\mu(\sigma^\alpha)$ are the world-sheet coordinates. X^μ are the dynamical variables.

The inverse metric can also be expressed in a closed form

$$
\begin{aligned}
\psi^{\alpha\beta} &= \frac{1}{(2-1)!}\tilde{\epsilon}^{\alpha\gamma}\tilde{\epsilon}^{\beta\delta}\tilde{\tilde{\psi}}^{-1}\psi_{\gamma\delta} \\
&= \tilde{\epsilon}^{\alpha\gamma}\tilde{\epsilon}^{\beta\delta}\tilde{\tilde{\psi}}^{-1}\psi_{\gamma\delta} = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\psi_{\gamma\delta} \qquad (6) \\
&= \tilde{\epsilon}^{\alpha\gamma}\tilde{\epsilon}^{\beta\delta}\tilde{\tilde{\psi}}^{-1} g_{\rho\sigma}X^{\rho}{}_{,\gamma}X^{\sigma}{}_{,\delta} = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}g_{\rho\sigma}X^{\rho}{}_{,\gamma}X^{\sigma}{}_{,\delta} \qquad (7) \\
&\equiv \frac{1}{\tilde{\tilde{\psi}}}\begin{pmatrix} \psi_{22} & -\psi_{12} \\ -\psi_{21} & \psi_{11}\psi \end{pmatrix}^{\alpha\beta}. \qquad (8)
\end{aligned}
$$

Unfortunately this is not useful.

Variation of the induced metric determinant can be expressed in terms of that of the induced metric

$$\delta\tilde{\tilde{\psi}} = \tilde{\tilde{\psi}}\psi^{\alpha\beta}\,\delta\psi_{\alpha\beta}. \qquad (9)$$

Variation of the induced metric in terms of the world-sheet coordinates reads

$$\delta\psi_{\alpha\beta} = X^{\nu}{}_{,\alpha}\big(2g_{\nu\lambda}\,\delta X^{\lambda}{}_{,\beta} + X^{\rho}{}_{,\beta}g_{\nu\rho,\lambda}\,\delta X^{\lambda}\big). \qquad (10)$$

Variation of the area element reads

$$
\begin{aligned}
\delta\sqrt{-\tilde{\tilde{\psi}}} &= \frac{1}{2}\sqrt{-\tilde{\tilde{\psi}}}\,\psi^{\alpha\beta}\,\delta\psi_{\alpha\beta} \\
&= \frac{1}{2}\sqrt{-\tilde{\tilde{\psi}}}\,\psi^{\alpha\beta}X^{\nu}{}_{,\alpha}\big(2g_{\nu\lambda}\,\delta X^{\lambda}{}_{,\beta} + X^{\rho}{}_{,\beta}g_{\nu\rho,\lambda}\,\delta X^{\lambda}\big) \\
&= -\frac{1}{T}\Big\{(\ldots)^{\beta}_{\mathrm{NG},\beta} + \mathcal{L}_{\mathrm{NG},X^{\lambda}}\,\delta X^{\lambda}\Big\}, \qquad (11)
\end{aligned}
$$

where

$$\frac{1}{T}(\ldots)^{\beta}_{\mathrm{NG}} := -\sqrt{-\tilde{\tilde{\psi}}}\,\psi^{\alpha\beta}X^{\nu}{}_{,\alpha}g_{\nu\lambda}\,\delta X^{\lambda}, \qquad (12)$$

is a boundary term,

$$
\begin{aligned}
\frac{\mathcal{L}_{\mathrm{NG},X^{\lambda}}}{T\sqrt{-\tilde{\tilde{\psi}}}} &= \Box_{\psi}X^{\mu}g_{\mu\lambda} + \psi^{\alpha\beta}X^{\nu}{}_{,\alpha}g_{\nu\lambda,\rho}X^{\rho}{}_{,\beta} - \frac{1}{2}\psi^{\alpha\beta}X^{\nu}{}_{,\alpha}X^{\rho}{}_{,\beta}g_{\nu\rho,\lambda} \\
&= \Box X^{\mu}g_{\mu\lambda} + \frac{1}{2}\psi^{\alpha\beta}X^{\nu}{}_{,\alpha}X^{\rho}{}_{,\beta}\big(-g_{\nu\rho,\lambda} + g_{\rho\lambda,\nu} + g_{\lambda\nu,\rho}\big) \\
&= g_{\mu\lambda}\big(\Box_{\psi}X^{\mu} + \mathit{\Gamma}^{\mu}{}_{\nu\rho}\psi^{\alpha\beta}X^{\nu}{}_{,\alpha}X^{\rho}{}_{,\beta}\big) \qquad (13)
\end{aligned}
$$

gives the Euler–Lagrange derivative, in which

$$\Box_{\psi}X^{\mu} := \frac{1}{\sqrt{-\tilde{\tilde{\psi}}}}\left(\sqrt{-\tilde{\tilde{\psi}}}\,\psi^{\alpha\beta}X^{\mu}{}_{,\alpha}\right)_{,\beta} \qquad (14)$$

is a d'Alembertian on the world-sheet with respect to $\psi_{\alpha\beta}$.

2.2 Polyakov action

The action reads [2, 1, 5]

$$S_{\mathrm{P}}\left[h_{\alpha\beta}, X^{\mu}\right] = -\frac{T}{2}\int_{\Sigma} \mathrm{d}^2\sigma \sqrt{-\tilde{\tilde{h}}}\, h^{\alpha\beta}\psi_{\alpha\beta}. \tag{15}$$

Variation of the integrand reads

$$\begin{aligned} \delta\left(\sqrt{-\tilde{\tilde{h}}}\, h^{\alpha\beta}\psi_{\alpha\beta}\right) &= \delta_h\left(\sqrt{-\tilde{\tilde{h}}}\, h^{\alpha\beta}\right)\psi_{\alpha\beta} + \left(\sqrt{-\tilde{\tilde{h}}}\, h^{\alpha\beta}\right)\delta_X\psi_{\alpha\beta} \\ &= -\frac{2}{T}\left\{(\ldots)^{\beta}_{\mathrm{P},\beta} + \mathcal{L}_{\mathrm{P},h^{\alpha\beta}}\,\delta h^{\alpha\beta} + \mathcal{L}_{\mathrm{P},X^{\lambda}}\,\delta X^{\lambda}\right\}, \end{aligned} \tag{16}$$

where

$$\frac{2}{T}(\ldots)^{\beta}_{\mathrm{P},\beta} := -2\sqrt{-\tilde{\tilde{h}}}\, h^{\alpha\beta}X^{\nu}{}_{,\alpha}g_{\nu\lambda}\,\delta X^{\lambda}, \tag{17}$$

is a boundary term,

$$\frac{2\mathcal{L}_{\mathrm{P},h^{\alpha\beta}}}{T\sqrt{-\tilde{\tilde{h}}}} = \left(\frac{1}{2}h^{\gamma\delta}h_{\alpha\beta} - \delta^{\gamma}{}_{\alpha}\delta^{\delta}{}_{\beta}\right)X^{\mu}{}_{,\gamma}X^{\nu}{}_{,\delta}g_{\mu\nu} \tag{18}$$

gives the Euler–Lagrange derivative with respect to $h^{\alpha\beta}$,

$$\frac{2\mathcal{L}_{\mathrm{P},X^{\lambda}}}{T\sqrt{-\tilde{\tilde{h}}}} = 2g_{\mu\lambda}(\Box_h X^{\mu} + \Gamma^{\mu}{}_{\nu\rho}X^{\nu}{}_{,\alpha}X^{\rho}{}_{,\beta}h^{\alpha\beta}) \tag{19}$$

gives the Euler–Lagrange derivative with respect to X^{λ}, and

$$\Box_h X^{\mu} := \frac{1}{\sqrt{-\tilde{\tilde{h}}}}\left(\sqrt{-\tilde{\tilde{h}}}\, h^{\alpha\beta}X^{\mu}{}_{,\alpha}\right)_{,\beta} \tag{20}$$

is another d'Alembertian on the world-sheet with respect to $h_{\alpha\beta}$.

References

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