Cosmological Perturbations

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Most of the conventions and notations in [6, ch. 5] will be followed. Suppose the metric can be expanded up to the linear order as

$$g = g^{(0)} + \epsilon g^{(1)} + O(\epsilon^2). \tag{1}$$

The background metric $g^{(0)}$ takes the Robertson–Walker form

$$g^{(0)}_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = -N^2(t)\, \mathrm{d}t^2 + a^2(t)\, \mathrm{d}\Omega^2_{3\mathrm{F}}, \tag{2}$$

in which $\mathrm{d}\Omega^2_{3\mathrm{F}} = \mathrm{d}\chi^2 + \chi^2\big(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\big)$ is the dimensionless flat spatial metric. The linear perturbation can be decomposed into scalar, vector and tensor parts

$$g^{(1)}_{00} = -E, \tag{3}$$

$$g^{(1)}_{i0} = g^{(1)}_{i0} = F_{,i} + G_i, \tag{4}$$

$$g^{(1)}_{ij} = A\delta_{ij} + B_{,i,j} + C_{i,j} + C_{j,i} + D_{ij}. \tag{5}$$

Here one has some weird condition, where the contractions do not follow the one-up-one-down tradition

$$C_{i,i} = G_{i,i} = 0, \quad D_{ij,i} = 0, \quad D_{ii} = 0. \tag{6}$$

1 Metric perturbation under diffeomorphism

Consider a diffeomorphism generated by ξ^μ

$$x^\mu \to \overline{x}^\mu = x^\mu - \epsilon \xi^\mu. \tag{7}$$

The generator ξ^μ can in turn be decomposed into $\xi_0 = \zeta$, $\xi_i = \xi^{\mathrm{S}}_{,i} + \xi^{\mathrm{V}}_i$.

One has here again some weird condition, where the contractions do not follow the one-up-one-down tradition

$$\xi^{\mathrm{V}}_{i,i} = 0. \tag{8}$$

1

The Lie derivative of the metric $\mathbb{L}_\xi g$ is

$$\left(\mathbb{L}_\xi g\right)_{\mu\nu} = \xi^\lambda g_{\mu\nu,\lambda} + \xi^\lambda{}_{,\mu} g_{\lambda\nu} + \xi^\lambda{}_{,\nu} g_{\mu\lambda}. \tag{9}$$

In components and expansion, these are

$$\left(\mathbb{L}_\xi g\right)_{00} = 2\dot{\zeta} - 2\zeta\frac{\dot{N}}{N} + O(\epsilon), \tag{10}$$

$$\left(\mathbb{L}_\xi g\right)_{i0} = \left(\mathbb{L}_\xi g\right)_{0i} = \left(\zeta - 2\frac{\dot{a}}{a}\xi^{\mathrm{S}} + \dot{\xi}^{\mathrm{S}}\right)_{,i} + \left(-2\frac{\dot{a}}{a}\xi_i^{\mathrm{V}} + \dot{\xi}_i^{\mathrm{V}}\right) + O(\epsilon), \tag{11}$$

$$\left(\mathbb{L}_\xi g\right)_{ji} = \left(\mathbb{L}_\xi g\right)_{ij} = -\frac{2a\dot{a}}{N^2}\zeta\delta_{ij} + 2\xi_{,i,j}^{\mathrm{S}} + \xi_{i,j}^{\mathrm{V}} + \xi_{j,i}^{\mathrm{V}} + O(\epsilon). \tag{12}$$

2 Scalar perturbations

$$-N^2 - \epsilon E + O(\epsilon^2) \to -N^2 - \epsilon E + \epsilon\left(2\dot{\zeta} - 2\zeta\frac{\dot{N}}{N}\right) + O(\epsilon^2), \tag{13}$$

so one can write

$$\mathbb{L}_\xi E = -2\dot{\zeta} + 2\zeta\frac{\dot{N}}{N}. \tag{14}$$

Similarly one can read-off

$$\mathbb{L}_\xi F = \zeta - 2\frac{\dot{a}}{a}\xi^{\mathrm{S}} + \dot{\xi}^{\mathrm{S}}, \tag{15}$$

$$\mathbb{L}_\xi A = -\frac{2a\dot{a}}{N^2}\zeta, \tag{16}$$

$$\mathbb{L}_\xi B = 2\xi^{\mathrm{S}}. \tag{17}$$

The four scalar perturbations are generated by ζ and ξ^{S}, so that only two independent perturbations exists. It is clear that

$$\mathbb{L}_\xi\left(\frac{F}{a} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{B}{2a}\right) = \frac{\zeta}{a}. \tag{18}$$

One can verify that

$$\mathbb{L}_\xi\left\{\frac{E}{2N} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{a}{N}\left(\frac{F}{a} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{B}{2a}\right)\right]\right\} = 0, \tag{19}$$

$$\mathbb{L}_\xi\left\{\frac{A}{2} + \frac{a^2\dot{a}}{N^2}\left(\frac{F}{a} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{B}{2a}\right)\right\} = 0. \tag{20}$$

3   Vector perturbations

4   Tensor perturbations

5   Scalar field perturbation under diffeomorphism

6   Perturbation in Arnowitt–Deser–Misner Hamiltonian formalism

The Hamiltonian action for General Relativity in terms of Arnowitt–Deser–Misner variables is [3, ch.4.2.2]

$$S = \int \mathrm{d}t\, \mathrm{d}x^3 \left\{ \mathfrak{p}^{ij}\dot{h}_{ij} + \mathfrak{P}\dot{N} + \mathfrak{P}^i \dot{N}_i - N\mathfrak{H}^\perp - N_i \mathfrak{H}^i - \mathfrak{P}V - \mathfrak{P}^i V_i \right\} \qquad \text{+boundary terms}$$
(21)

where

$$\mathfrak{H}^\perp = 2\varkappa \mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl} - \frac{\sqrt{\mathfrak{h}}}{2\varkappa}R[h] \equiv 2\varkappa \mathfrak{F}^{ijkl}h_{ij}h_{kl} - \frac{\sqrt{\mathfrak{h}}}{2\varkappa}R[h],$$
(22)

$$\mathfrak{H}^i = -2\mathfrak{p}^{ij}{}_{|j},$$
(23)

are the secondary constraints,

$$\mathfrak{G}_{ijkl} := \frac{1}{2\mathfrak{h}^{1/2}}\left(h_{ik}h_{jl} + h_{il}h_{kj} - h_{ij}h_{kl} \right) \equiv -\frac{\delta\big(\mathfrak{h}^{-1/2}h_{ij}\big)}{\delta h^{kl}},$$
(24)

$$\mathfrak{F}^{ijkl} := \frac{1}{2\mathfrak{h}^{1/2}}\left(\mathfrak{p}^{ik}\mathfrak{p}^{jl} + \mathfrak{p}^{il}\mathfrak{p}^{kj} - \mathfrak{p}^{ij}\mathfrak{p}^{kl} \right)$$
(25)

are convenient notations. In eq. (21), V and V_i are velocities of N and N_i and play the role of Lagrange multipliers. Technical details about the boundary terms can be found in [5, ch. 4.2] and the references therein. Note that $\big\{ N, N_i, h_{ij}; \mathfrak{P}, \mathfrak{P}^i, \mathfrak{p}^{ij} \big\}$ are not the unique choice of canonical variables for General Relativity in Hamiltonian formalism; instead, they are a special parametrisation of the phase space. One can also choose the components of the original four-metric and their conjugate momenta $\big\{ g_{\mu\nu}; \mathfrak{p}^{\mu\nu} \big\}$ as canonical variables, as Dirac has done in [2]. The two approaches are different in some subtle aspects; see [4] for a comparison.

Gauge transformations in the Arnowitt–Deser–Misner canonical variables are generated by [1]

$$G = -\int \mathrm{d}^3 x \left\{ \left[\xi_\perp \left(\mathfrak{H}^\perp + N_{|i}\mathfrak{P}^i + (N\mathfrak{P}^i)_{|i} + (N_i\mathfrak{P})^{|i} \right) + \dot{\xi}_\perp \mathfrak{P} \right] \right.$$
$$\left. + \left[\xi_i \left(\mathfrak{H}^i + N_j{}^{|i}\mathfrak{P}^j + (N_j\mathfrak{P}^i)^{|j} + N^{|i}\mathfrak{P} \right) + \dot{\xi}_i\mathfrak{P}^i \right] \right\}.$$
(26)

Possible boundary terms have not been discussed so far. The infinitesimal gauge transformation of $\{N, N_i\}$ is

$$\delta N = [N, G]_{\mathrm{P}} = \xi_\perp{}^{|i} N_i - \dot{\xi}_\perp - \xi_i N^{|i}, \tag{27}$$

$$\delta N_i = -\xi_\perp N_{|i} + \xi_{\perp|i} N - \xi_j N_i{}^{|j} + \xi_i{}^{|j} N_j - \dot{\xi}_i, \tag{28}$$

which can be found in [4]. Transformations for g_{ij} and the momenta have to be worked out as

$$\delta \mathfrak{P} = -\left(\xi_\perp \mathfrak{P}^i\right)_{|i} - \xi_{\perp|i} \mathfrak{P}^i - (\xi_i \mathfrak{P})^{|i}, \tag{29}$$

$$\delta \mathfrak{P}^i = -\xi_{\perp|i} \mathfrak{P} - \left(\xi_j \mathfrak{P}^i\right)^{|j} - \xi_j{}^{|i} \mathfrak{P}^j, \tag{30}$$

where only the primary constraints are involved;

$$
\begin{aligned}
\delta h_{ij} &= -\frac{\partial}{\partial \mathfrak{p}^{ij}}\left(\xi_\perp \mathfrak{H}^\perp + \xi_i \mathfrak{H}^i\right) \\
&= -\xi^\perp 4\varkappa\, \mathfrak{G}_{ijkl}\mathfrak{p}^{kl} - \xi_{i|j} - \xi_{j|i},
\end{aligned}\tag{31}
$$

$$
\begin{aligned}
\delta \mathfrak{p}^{ij} &= \frac{\partial}{\partial h_{ij}}\left(\xi_\perp \mathfrak{H}^\perp + \xi_i \mathfrak{H}^i\right) \\
&= 2\varkappa\, \xi_\perp \left(-\frac{1}{2} h^{ij}\mathfrak{F}^{klmn} h_{mn} + 2\mathfrak{F}^{ijkl}\right) h_{kl} \\
&\quad + \frac{1}{2\varkappa}\left(\sqrt{\mathfrak{h}}\, \xi_\perp G^{ij}[h] - \mathfrak{G}^{ijkl}(\xi_\perp)_{|k|l}\right) \\
&\quad + \left\{\left(\mathfrak{p}^{il} h^{kj} + \mathfrak{p}^{jl} h^{ki} - \mathfrak{p}^{ij} h^{kl}\right)\xi_k\right\}_{|l},
\end{aligned}\tag{32}
$$

where $G^{ij}[h] = R^{ij}[h] - h^{ij} R[h]/2$, and only the secondary constraints are involved. The results can be checked with [5, p. 4.2.7].

6.1   Expansion of the action with fluctuations

Some useful results

First variations

The first variation of the inverse metric h^{ij} reads

$$\delta h^{ij} = -h^{ik} h^{jl}\, \delta h_{kl} = -h^{i(k} h^{l)j}\, \delta h_{kl}. \tag{33}$$

The first variation of $\mathfrak{h} = \det h_{ij}$ reads

$$\delta \mathfrak{h} = \mathfrak{h} h^{ij}\, \delta h_{ij}. \tag{34}$$

The first variation of $\Gamma^i{}_{jk}$ can be obtained in normal coordinates, which reads

$$\delta \Gamma^i{}_{jk} = \frac{1}{2} h^{il}\left\{-\left(\delta h_{jk}\right)_{|l} + \left(\delta h_{kl}\right)_{|j} + \left(\delta h_{lj}\right)_{|k}\right\} \tag{35}$$

$$= \frac{1}{2}\left\{-h^{il}\delta^m{}_j \delta^n{}_k + h^{in}\delta^l{}_j \delta^m{}_k + h^{im}\delta^n{}_j \delta^l{}_k\right\}\left(\delta h_{mn}\right)_{|l}. \tag{36}$$

The first variation of $R_{ij}[h]$ and $R^{ij}[h]$

$$\delta R_{ij}[h] = \left(\delta\Gamma^k{}_{ji}\right)_{|k} - \left(\delta\Gamma^k{}_{ki}\right)_{|j}, \tag{37}$$

$$\delta R^{ij}[h] = -2R^{k(i}h^{j)l}\,\delta h_{kl} + h^{k(i}\,\bar{\delta}u^{j)l}{}_{k|l}, \tag{38}$$

where

$$\bar{\delta}u^{ij}{}_k := h^{il}\,\delta\Gamma^j{}_{lk} - h^{ij}\,\delta\Gamma^l{}_{lk} \tag{39}$$

is related to the boundary terms. Equation (37) can be obtained in normal coordinates.

For the first variation of the constraints, one also needs

$$\bar{\delta}u^{ji}{}_j = (\delta h_{kl})_{|j}\left(h^{i(k}h^{l)j} - h^{ij}h^{kl}\right) = \mathfrak{h}^{-1/2}\mathfrak{G}^{ijkl}\,(\delta h_{kl})_{|j}; \tag{40}$$

therefore,

$$\sqrt{\mathfrak{h}}\,N\,\bar{\delta}u^{ji}{}_{j|i} = \delta h_{ij}\,\mathfrak{G}^{ijkl}N_{|k|l} + \left\{\mathfrak{G}^{ijkl}\left(N\,(\delta h_{kl})_{|j} - N_{|j}\,\delta h_{kl}\right)\right\}_{|i}. \tag{41}$$

In the Hamiltonian constraint, the first variation of the 'kinetic term' $\mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl} \equiv \mathfrak{F}^{ijkl}h_{ij}h_{kl}$ reads

$$\delta\left(\mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl}\right) \equiv \delta\left(\mathfrak{F}^{ijkl}h_{ij}h_{kl}\right)$$
$$= \delta h_{ij}\left(-\frac{1}{2}h^{ij}\mathfrak{F}^{klmn}h_{mn} + 2\mathfrak{F}^{ijkl}\right)h_{kl} + \delta\mathfrak{p}^{ij}\,2\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}. \tag{42}$$

Equipped with eqs. (38) and (40), the first variation of the 'potential' $\sqrt{\mathfrak{h}}\,R[h]$ reads

$$\delta\left(\sqrt{\mathfrak{h}}R[h]\right) = \sqrt{\mathfrak{h}}\left\{-G^{ij}[h]\,\delta h_{ij} + \bar{\delta}u^{ji}{}_{j|i}\right\}$$
$$= -\sqrt{\mathfrak{h}}G^{ij}[h]\,\delta h_{ij} + \mathfrak{G}^{ijkl}\,(\delta h_{kl})_{|j|i}. \tag{43}$$

One can now write down the first variation of $N\mathfrak{H}^{\perp}$ with respect to $\left\{h_{ij}, \mathfrak{p}^{ij}\right\}$,

$$N\,\delta\mathfrak{H}^{\perp} = \delta h_{ij}\left\{2\varkappa\,N\left(-\frac{1}{2}h^{ij}\mathfrak{F}^{klmn}h_{mn} + 2\mathfrak{F}^{ijkl}\right)h_{kl}\right.$$
$$\left. + \frac{1}{2\varkappa}\left(\sqrt{\mathfrak{h}}\,NG^{ij}[h] - \mathfrak{G}^{ijkl}N_{|k|l}\right)\right\}$$
$$+ \delta\mathfrak{p}^{ij}\,4\varkappa\,N\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}$$
$$- \frac{1}{2\varkappa}\left\{\mathfrak{G}^{ijkl}\left(N\,(\delta h_{kl})_{|j} - N_{|j}\,\delta h_{kl}\right)\right\}_{|i}, \tag{44}$$

where the terms in the last line will be cancelled by the boundary term. Furthermore, the first variation of $N_i \mathfrak{H}^i$ with respect to $\{h_{ij}, \mathfrak{p}^{ij}\}$ is easier,

$$
\begin{aligned}
N_i \, \delta \mathfrak{H}^i = \delta h_{ij} \Big\{ &(\mathfrak{p}^{il} h^{kj} + \mathfrak{p}^{jl} h^{ki} - \mathfrak{p}^{ij} h^{kl}) N_k \Big\}_{|l} + \delta \mathfrak{p}^{ij} \, 2 N_{(i|j)} \\
&- \big(-h^{il} \delta^m{}_j \delta^n{}_k + h^{in} \delta^l{}_j \delta^m{}_k + h^{im} \delta^n{}_j \delta^l{}_k \big)\big(N_i \mathfrak{p}^{jk} \, \delta h_{mn}\big)_{|l} \\
&- 2\big(\delta p^{ij} \, N_{(i}\big)_{|j)},
\end{aligned}
\tag{45}
$$

where the last two lines are to be cancelled by the boundary term.

Second variations

First variation of $\mathfrak{F}^{ijkl} h_{kl}$

$$
\begin{aligned}
\delta(\mathfrak{F}^{ijkl} h_{kl}) = \delta h_{kl} &\left(-\frac{1}{2}\mathfrak{F}^{ijmn} h^{kl} h_{mn} + \mathfrak{F}^{ijkl}\right) \\
&+ \delta \mathfrak{p}^{kl} \left(\delta^i{}_k \mathfrak{G}^j{}_{lmn} + \delta^i{}_m \mathfrak{G}^j{}_{nkl}\right) \mathfrak{p}^{mn}.
\end{aligned}
\tag{46}
$$

First variation of $\mathfrak{G}_{ijkl} \mathfrak{p}^{kl}$

$$
\begin{aligned}
\delta\big(\mathfrak{G}_{ijkl} \mathfrak{p}^{kl}\big) = \delta h_{kl} &\left(-\frac{1}{2}\mathfrak{G}_{ijmn} h^{kl} + \delta^k{}_i \mathfrak{G}^l{}_{jmn} + \delta^k{}_m \mathfrak{G}^l{}_{nij}\right) \mathfrak{p}^{mn} \\
&+ \delta p^{kl} \, \mathfrak{G}_{ijkl}.
\end{aligned}
\tag{47}
$$

First variation of $\sqrt{\mathfrak{h}} \, G^{ij}[h]$

$$
\begin{aligned}
\delta\big(\sqrt{\mathfrak{h}} \, G^{ij}[h]\big) = \sqrt{\mathfrak{h}} \Big\{ \delta h_{kl} \cdot & \\
\left(-\frac{1}{2}\right)\big(R^{ik} h^{lj} + R^{il} h^{kj} - R^{ij} h^{kl} &+ h^{ik} G^{lj} + h^{il} G^{kj} - h^{ij} G^{kl}\big) \\
+ \left(h^{il} \, \bar{\delta} u^{jk}{}_l - \frac{1}{2} h^{ij} \, \bar{\delta} u^{lk}{}_l\right)_{|k} &\Big\}.
\end{aligned}
\tag{48}
$$

For the following calculation, one also needs

$$
h^{k(i} \, \bar{\delta} u^{j)l}{}_{k|l} =
\tag{49}
$$

About $\bar{\delta} u^{ij}{}_k$, the identity

$$
\sqrt{\mathfrak{h}} \, N h^{k(i} \, \bar{\delta} u^{j)l}{}_{k|l} =
\tag{50}
$$

is also useful.

Second variation of $N\mathfrak{H}^\perp$

$$\delta^2(N\mathfrak{H}^\perp)$$
$$= \delta h_{ij}\,\delta h_{kl}\left\{2\varkappa\left[\frac{1}{4}(h^{ik}h^{lj}+h^{il}h^{kj}+h^{ij}h^{kl})\mathfrak{F}^{mnrs}h_{mn}h_{rs}\right.\right.$$
$$\left.-\,(h^{ij}\mathfrak{F}^{klmn}+\mathfrak{F}^{ijmn}h^{kl})h_{mn}+\mathfrak{F}^{ijkl}\right]$$
$$\left.-\,\frac{\sqrt{\mathfrak{h}}}{4\varkappa}(R^{ik}h^{lj}+R^{il}h^{kj}-R^{ij}h^{kl}+h^{ik}G^{lj}+h^{il}G^{kj}-h^{ij}G^{kl})\right\}$$
$$+\,\delta h_{ij}\frac{\sqrt{\mathfrak{h}}}{2\varkappa}\left(h^{il}\,\bar\delta u^{jk}{}_{l}-\frac{1}{2}h^{ij}\,\bar\delta u^{lk}{}_{l}\right)_{|k}$$
$$+\,\delta h_{ij}\,\delta\mathfrak{p}^{kl}\,4\varkappa\{-h^{ij}\mathfrak{G}_{klmn}+2(\delta^i{}_k\mathfrak{G}^j{}_{lmn}+\delta^i{}_m\mathfrak{G}^j{}_{nkl})\}\mathfrak{p}^{mn}$$
$$+\,\delta p^{ij}\,\delta p^{kl}\,4\varkappa\mathfrak{G}_{ijkl}$$
$$-\,\delta h_{ij}\,\delta\left(\frac{\sqrt{\mathfrak{h}}}{2\varkappa}\,\bar\delta u^{lk}{}_{l|k}\right). \tag{51}$$

Second variation of $\Gamma^i{}_{jk}$

$$\delta^2\Gamma^i{}_{jk}=-h^{im}\,\delta\Gamma^l{}_{jk}\,\delta h_{lm}. \tag{52}$$

Second variation of $\mathfrak{H}^i$

$$\delta^2(\mathfrak{H}^i)=-h^{im}p^{jk}\,\delta\Gamma^l{}_{jk}\,\delta h_{lm}+2\,\delta\Gamma^i{}_{jk}\,\delta p^{jk}. \tag{53}$$

Other second variations

Second variation of h^{ij}

$$\delta^2 h^{ij}=(h^{im}h^{jl}h^{kn}+h^{ik}h^{jm}h^{ln})\,\delta h_{kl}\,\delta h_{mn} \tag{54}$$

Second variation of $\mathfrak{h}=\det h_{ij}$

$$\delta^2\mathfrak{h}=-\frac{1}{4}\mathfrak{h}\big(h^{ik}h^{jl}+h^{il}h^{kj}-h^{ij}h^{kl}\big)\,\delta h_{ij}\,\delta h_{kl}. \tag{55}$$

First variation of $\big(\delta h_{ij}\big)_{|k}$

$$\delta\left\{\big(\delta h_{ij}\big)_{|k}\right\}=-2\delta\Gamma^l{}_{k(i}\,\delta h_{j)l}. \tag{56}$$

In a general background, the second variations of the quantities are much more tedious.

In Robertson–Walker background,

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