

Effective action of scalar electrodynamics

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1 Wick rotation

Complex Klein–Gordon action in flat space-time

$$S_{\text{CKG}}[\phi, \phi^*] := \int d^{d+1}x \{ -\eta^{\mu\nu} (\partial_\mu \phi)^* (\partial_\nu \phi) - m^2 \phi^* \phi \}. \quad (1)$$

Interaction terms

$$S_{\text{ICKGM}}[A_\mu, \phi, \phi^*] := \int d^{d+1}x \eta^{\mu\nu} \{ ie A_\mu (-\phi^* \partial_\nu \phi + \phi \partial_\nu \phi^*) + e^2 A_\mu A_\nu \phi^* \phi \}. \quad (2)$$

The total action for scalar electrodynamics reads

$$\begin{aligned} S[A_\mu, \phi, \phi^*] &:= S_{\text{CKG}} + S_{\text{ICKGM}} + S_{\text{Maxwell}} \\ &= \int d^{d+1}x \left\{ -(\nabla_\mu \phi)^* (\nabla^\mu \phi) - m^2 \phi^* \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right\}, \end{aligned} \quad (3)$$

where

$$\nabla_\mu \phi := (\partial_\mu + ie A_\mu) \phi. \quad (4)$$

Wick rotation

$$x_{\text{E}}^4 = ix^0, \quad A_4 = -iA_0, \quad (5)$$

so that

$$\partial_{x^0} = i\partial_{x_{\text{E}}^4}, \quad F_{0i} = iF_{4i}. \quad (6)$$

The Euclidean action reads

$$S_{\text{E}}[A_I, \phi, \phi^*] = \int d^D x_{\text{E}} \left(\frac{1}{4} F_{IJ} F^{IJ} + (\nabla_I \phi)^* (\nabla^I \phi) + m^2 \phi^* \phi \right). \quad (7)$$

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2 Euclidean signature

Working with the Euclidean signature is much easier than in the Lorentzian signature.

2.1 Effective action

Generating functional

$$\mathcal{Z}_E[j^I, J, J^*] := \int \mathcal{D}A \mathcal{D}\phi^* \mathcal{D}\phi \exp \left\{ - \left(S_E + \int d^D x (j^I A_I + J^* \phi + \phi^* J) \right) \right\}. \quad (8)$$

The scalar fields are to be integrated out. The derivative term can be rearranged

$$(\nabla_I \phi)^* (\nabla^I \phi) = \partial_I (\phi^* \nabla^I \phi) - \phi^* \nabla_I \nabla^I \phi. \quad (9)$$

Hence

$$S_E = \int d^D x \frac{1}{4} F_{IJ} F^{IJ} + \int d^D x d^D y \phi^*(x) M_E(x, y) \phi(y), \quad (10)$$

where

$$M_E(x, y) := \left(-\nabla_{x^I} \nabla^{x^I} + m^2 \right) \delta^{(D)}(x - y), \quad (11)$$

see [Mosel, 2004, ch. 6] for details. Now the scalar field can formally be integrated

$$\begin{aligned} \exp[-\Gamma_{SE}[J^*, J]] &:= \int \mathcal{D}\phi^* \mathcal{D}\phi \exp \left\{ - \int d^D x ((\nabla_I \phi)^* (\nabla^I \phi) + m^2 \phi^* \phi + J^* \phi + \phi^* J) \right\} \\ &= \frac{1}{\det M_E(x, y)} \exp \left\{ - \int d^D x d^D y J^*(x) D_E(x - y) J(y) \right\}, \end{aligned} \quad (12)$$

where

$$D_E(x - y) := M_E^{-1}(x, y) = \frac{1}{(2\pi)^{D/2}} \left(\frac{m}{x} \right)^{\frac{D}{2}-1} K_{\frac{D}{2}-1}(m(x - y)) \quad (13)$$

is the Euclidean Green's function, calculated by Chao-Ming Jian (Everett You's notes). **(Check!)**

The effective action of the scalar fields reads

$$\Gamma_{SE}[J^*, J] = \Gamma_{SE}[0, 0] + \int d^D x d^D y J^*(x) D_E(x - y) J(y), \quad (14)$$

$$\Gamma_{SE}[0, 0] = -\ln[(\det M_E)^{-1}] = \text{tr} \ln M_E, \quad (15)$$

which traces back to [Heisenberg and Euler \[1936\]](#), [Weisskopf \[1936\]](#). The Euclidean generating functional now reads

$$\begin{aligned}\mathcal{Z}_E[j^I, J, J^*] &= \exp\left\{-\int d^D x d^D y J^*(x) D_E(x, y) J(y)\right\} \\ &\quad \cdot \int DA \exp\left\{-\int d^D x \left(\frac{1}{4} F_{IJ} F^{IJ} + j^I A_I\right) - \text{tr} \ln M_E\right\} \\ &=: \exp\left\{-\int d^D x d^D y J^*(x) D_E(x, y) J(y)\right\} \mathcal{Z}_{AE}[0] \langle \exp(-\text{tr} \ln M_E) \rangle_{j^I},\end{aligned}\quad (16)$$

where the average is defined as

$$\langle \mathcal{O} \rangle_{j^I} := \mathcal{Z}_{AE}^{-1}[0] \int DA \mathcal{O} \exp\left\{-\int d^D x \left(\frac{1}{4} F_{IJ} F^{IJ} + j^I A_I\right)\right\}, \quad (17)$$

$$\mathcal{Z}_{AE}[j^I] := \int DA \exp\left\{-\int d^D x \left(\frac{1}{4} F_{IJ} F^{IJ} + j^I A_I\right)\right\}. \quad (18)$$

2.2 World-line formalism

In eq. (16), $\text{tr} \ln M_E$ is crucial. Using the Schwinger integral representation [Schwinger \[1951\]](#) (up to normalisation)

$$\ln \alpha = - \int_0^{+\infty} \frac{ds}{s} e^{-\alpha s}, \quad (19)$$

one has (up to normalisation)

$$-\text{tr} \ln M_E = \int_0^{+\infty} \frac{dT}{T} \exp\left(-\frac{m^2 T}{2M}\right) \text{tr} \exp\left(-\frac{M_E}{2M}\right), \quad (20)$$

where T has the dimension of time, and M that of mass, which will both be eliminated later. Introduce the Hamiltonian of a non-relativistic point particle (**check sign!**)

$$H := \frac{1}{2M} (P_I + e A_I)^2, \quad (21)$$

so that quantisation yields the following representation (**check sign!**)

$$\text{tr} \exp\left(-\frac{M_E}{2M}\right) = \int_{-\infty}^{+\infty} dx \left\langle x \left| e^{-\hat{H}T} \right| x \right\rangle \quad (22)$$

$$= \oint Dx \exp\left\{-\int_0^T dT' \left(\frac{M}{2} \left(\frac{dx^I}{dT'}\right)^2 + ie A_I \frac{dx^I}{dT'}\right)\right\}. \quad (23)$$

Rescaling $T' =: \lambda T$ gives

$$-\text{tr} \ln M_E = \int_0^{+\infty} \frac{dT}{T} \exp\left(-\frac{m^2 T}{2M}\right) \oint Dx \exp\left(-\frac{M}{2T} \int_0^1 d\lambda \dot{x}_I^2 - ie \oint A_I dx^I\right). \quad (24)$$

2.3 Euler–Heisenberg effective Lagrangian

If the instanton magnetic field in eq. (23) is constant, the path integral can be performed exactly Feynman and Hibbs [1965]. The result is the Euler–Heisenberg effective Lagrangian.

It is difficult to obtain a classical solution for the motion of a point particle in a more generic magnetic field, e.g. Kondo and Toshioka [1964]. Therefore the generalisation in this direction is limited.

2.4 World-line instanton approximation

In eq. (24), the T integral can be performed first. Using the integral expression and the asymptotic expansion for a modified Bessel function

$$K_0(x) = \frac{1}{2} \int_0^{+\infty} \frac{dt}{t} \exp\left(-t - \frac{x^2}{4t}\right) \quad (25)$$

$$\approx \sqrt{\frac{\pi}{2x}} e^{-x} \quad x \gg 1, \quad (26)$$

one has

$$-\text{tr} \ln M_E = 2 \oint Dx K_0 \left(m \sqrt{\int_0^1 d\lambda \dot{x}_I^2} \right) \exp \left(-ie \oint A_I dx^I \right) \quad (27)$$

$$\approx \sqrt{\frac{2\pi}{m}} \oint Dx \left(\int_0^1 d\lambda \dot{x}_I^2 \right)^{-1/4} \exp \left(-m \sqrt{\int_0^1 d\lambda \dot{x}_I^2} - ie \oint A_I dx^I \right), \quad (28)$$

where eq. (28) works for

$$m \sqrt{\int_0^1 d\lambda \dot{x}_I^2} \gg 1 \quad \text{or} \quad \int_0^1 d\lambda \dot{x}_I^2 \gg m^{-2}. \quad (29)$$

This idea traces back to Affleck et al. [1982]

2.5 Another loop-based approximation

$$\langle \exp(-\text{tr} \ln M_E) \rangle_{j^I} \approx \exp \left(-\langle \text{tr} \ln M_E \rangle_{j^I} \right). \quad (30)$$

2.6 Application of instanton approximation

Dunne and Schubert [2005]

3 Flat space-time (Lorentzian signature)

Generating functional

$$\mathcal{Z}[j^\mu, J^*, J] := \int DA D\phi D\phi^* \exp \left\{ i \left(S_0 + \int d^{d+1}x (j^\mu A_\mu + J^* \phi + \phi^* J) \right) \right\}. \quad (31)$$

Effective action

$$\mathcal{Z}[j^\mu, 0, 0] =: \int DA \exp \left\{ i \left(S_{\text{Maxwell}} + \Gamma_W[A_\mu] + \int d^{d+1}x j^\mu A_\mu \right) \right\}. \quad (32)$$

In other words,

$$\begin{aligned} \exp \{ i \Gamma_W[A_\mu] \} &:= \int D\phi D\phi^* \exp \{ i (S_{\text{CKG}} + S_{\text{ICKGM}}) \} \\ &\equiv \int D\phi D\phi^* \exp \left\{ i \int d^{d+1}x \{ -(\nabla_\mu \phi)^* (\nabla^\mu \phi) - m^2 \phi^* \phi \} \right\}. \end{aligned} \quad (33)$$

The integral in the exponent can be manipulated; only the first term is essential

$$\begin{aligned} &\int d^{d+1}x (-(\nabla_\mu \phi)^* (\nabla^\mu \phi)) \\ &= \int d^{d+1}x d^{d+1}y (-(\nabla_{x^\mu} \phi(x))^* \delta^{d+1}(x-y) \nabla^{y^\mu} \phi(y)), \end{aligned} \quad (34)$$

where

$$\begin{aligned} \delta^{d+1}(x-y) \nabla^{y^\mu} \phi(y) &= \delta^{d+1}(x-y) \{ \partial^{y^\mu} + i e A^\mu(y) \} \phi(y) \\ &= \{ -(\nabla^{y^\mu})^* \delta^{d+1}(x-y) \} \phi(y) + \partial^{y^\mu} B, \end{aligned} \quad (35)$$

in which

$$B = B(x, y) := \delta^{d+1}(x-y) \phi(y); \quad (36)$$

going back to eq. (34),

$$\begin{aligned} &= \int d^{d+1}x d^{d+1}y \{ -\{ \partial_{x^\mu} + i e A_\mu(x) \} \phi(x) \}^* \delta^{d+1}(x-y) \nabla^{y^\mu} \phi(y) \} \\ &= \int d^{d+1}x d^{d+1}y \{ -\partial_{x^\mu} C^\mu + \phi^*(x) \nabla_{x^\mu} \delta^{d+1}(x-y) \nabla^{y^\mu} \phi(y) \} \\ &= \int d^{d+1}x d^{d+1}y \{ -\partial_{x^\mu} C^\mu + \phi^*(x) \{ -(\nabla_{y^\mu}) (\nabla^{y^\mu})^* \delta^{d+1}(x-y) \} \phi(y) + \partial^{y^\mu} \nabla_{x^\mu} B \}, \end{aligned} \quad (37)$$

in which

$$C^\mu = C^\mu(x, y) := \phi^*(x) \delta^{d+1}(x - y) \nabla^{y^\mu} \phi(y). \quad (38)$$

Now eq. (33) can be written as (dropping the boundary terms)

$$\begin{aligned} &= \int D\phi D\phi^* \exp \left\{ -i \int d^{d+1}x d^{d+1}y \phi^*(x) D^{-1}(x, y) \phi(y) \right\} \\ &= \tilde{\mathcal{N}} \{ \det [D^{-1}(x, y)] \}^{-1/2}, \end{aligned} \quad (39)$$

where

$$D^{-1}(x, y) := \mathcal{D}_y^{-1} \delta^{d+1}(x - y), \quad \mathcal{D}_y^{-1} := +(\nabla_{y^\mu})(\nabla^{y^\mu})^* + m^2. \quad (40)$$

$$\begin{aligned} \Gamma_W[A_\mu] &\equiv -i \left(\ln \tilde{\mathcal{N}} - \frac{1}{2} \ln \det D^{-1} \right) \\ &= \frac{i}{2} \text{tr}_x \ln (\mathcal{N}^{-1} D^{-1}) \\ &= \frac{i}{2} \int_0^{+\infty} \frac{ds}{s} \int d^{d+1}x d^{d+1}y \delta^{d+1}(x - y) \\ &\quad \cdot \left\{ -e^{is \left(+(\nabla_{y^\mu})(\nabla^{y^\mu})^* + m^2 + i0^+ \right) \delta^{d+1}(x-y)} + e^{is(\mathcal{N} + i0^+)} \right\}. \end{aligned} \quad (41)$$

Weisskopf [1936]

A Notions and conventions

The metric convention is mostly positive, i.e. $\eta_{\mu\nu} := \text{diag}(-, +, +, \dots)$

Pauli matrices

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma^3 := \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (42)$$

The γ -matrices satisfy [Weinberg, 1995, sec. 5]

$$[\gamma^\mu, \gamma^\nu]_+ := 2\eta^{\mu\nu} \mathbf{1}_4. \quad (43)$$

$$\mathcal{J}^{\mu\nu} := -\frac{i}{4} [\gamma^\mu, \gamma^\nu]_- \quad (44)$$

$$\sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu]_- \equiv -2 \not{\sigma}^{\mu\nu}. \quad (45)$$

In $(3+1)$ dimensions, choose the chiral representation

$$\gamma^\mu = -i \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \quad (46)$$

where

$$\sigma^\mu := (1_2, +\vec{\sigma}), \quad \bar{\sigma}^\mu := (1_2, -\vec{\sigma}). \quad (47)$$

$$\begin{aligned} \sigma^{\mu\nu} &\equiv -\frac{i}{2} \begin{bmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \end{bmatrix} \\ &= \begin{cases} 0 & \mu = 0, \nu = 0; \\ i \begin{bmatrix} +\sigma^j & 0 \\ 0 & -\sigma^j \end{bmatrix} & \mu = 0, \nu = j; \\ i \begin{bmatrix} -\sigma^i & 0 \\ 0 & +\sigma^i \end{bmatrix} & \mu = i, \nu = 0; \\ \begin{bmatrix} +\epsilon^{ij}_k \sigma^k & 0 \\ 0 & +\epsilon^{ij}_k \sigma^k \end{bmatrix} & \mu = i, \nu = j. \end{cases} \quad (48) \end{aligned}$$

B Fresnel functional integral

[Mosel, 2004, ch. 10]

C Algebra

$$[a\partial_1\partial_2, x^1]_- = a\partial_2 \quad (49)$$

central.

Baker–Campbell–Hausdorff formula

$$e^{+a\partial_1\partial_2} x^1 e^{-a\partial_1\partial_2} = x^1 + a\partial_2. \quad (50)$$

D Schwinger integral

$$-\int_{\epsilon}^{+\infty} \frac{dt}{t} e^{-\alpha t} = -\Gamma(0, \alpha\epsilon) = \gamma_E + \ln \alpha + \ln \epsilon + O(\epsilon), \quad (51)$$

where $\Gamma(a, z)$ is the incomplete Gamma function, γ_E is Euler's constant.

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