

Euler-Heisenberg Effective Action

immediate

1 Spinor electrodynamics in flat space-time

Maxwell Lagrangian

$$S_{\text{Maxwell}}[A_\mu] := \int d^{d+1}x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) \quad (1)$$

Dirac Lagrangian [1, sec. 11]

$$S_{\text{Dirac}}[\psi, \bar{\psi}] := \int d^{d+1}x \left\{ -\bar{\psi}(\gamma^\mu \partial_\mu + m)\psi \right\}. \quad (2)$$

Interaction term

$$S_{\text{IDM}}[A_\mu, \psi, \bar{\psi}] := \int d^{d+1}x \left(-\bar{\psi} \gamma^\mu i e A_\mu \psi \right). \quad (3)$$

The total action for spinor electrodynamics reads

$$\begin{aligned} S_{1/2}[A_\mu, \psi, \bar{\psi}] &:= S_{\text{Dirac}} + S_{\text{IDM}} + S_{\text{Maxwell}} \\ &= \int d^{d+1}x \left\{ -\bar{\psi}(\gamma^\mu \nabla_\mu + m)\psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right\}, \end{aligned} \quad (4)$$

where

$$\nabla_\mu \psi := (\partial_\mu + i e A_\mu) \psi. \quad (5)$$

Generating functional

$$\mathcal{Z}[j^\mu, \bar{\eta}, \eta] := \int D A D \psi D \bar{\psi} \exp \left\{ i \left(S_{1/2} + \int d^{d+1}x (j^\mu A_\mu + \bar{\eta} \psi + \bar{\psi} \eta) \right) \right\} \quad (6)$$

Effective action

$$\mathcal{Z}[j^\mu, 0, 0] =: \int \mathcal{D}A \exp \left\{ i \left(S_{\text{Maxwell}} + \Gamma_{\text{EH}}[A_\mu] + \int d^{d+1}x j^\mu A_\mu \right) \right\}. \quad (7)$$

In other words,

$$\begin{aligned} \exp\{i\Gamma_{\text{EH}}[A_\mu]\} &\equiv \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\{i(S_{\text{Dirac}} + S_{\text{IDM}})\} \\ &\equiv \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left\{ i \int d^{d+1}x \bar{\psi} (-\not{\partial} - ie\mathcal{A} - m) \psi \right\} \\ &= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left\{ i \int d^{d+1}x d^{d+1}y \bar{\psi}(x) S^{-1}(x, y) \psi(y) \right\} \\ &= \mathcal{N} \det[-iS^{-1}(x, y)], \end{aligned} \quad (8)$$

where

$$S^{-1}(x, y) := (+\not{\partial}_y - ie\mathcal{A}(y) - m) \delta^{d+1}(x - y). \quad (9)$$

$$\begin{aligned} \Gamma_{\text{EH}}[A_\mu] &\equiv -i(\ln \mathcal{N} + \ln \det[-iS^{-1}]) \\ &= -i(\ln \mathcal{N} + \text{Tr} \ln[-iS^{-1}(x, y)]). \end{aligned} \quad (10)$$

Note that

$$\begin{aligned} \text{Tr} \ln[-iS^{-1}(x, y)] &\equiv \text{Tr} \ln[-iM^\top(x, y)] \\ &= \text{Tr} \ln[i(+\not{\partial}_y^\top - ie\mathcal{A}^\top(y) - m) \delta^{d+1}(x - y)] \\ &= \text{Tr} \ln[i(-\mathcal{C}\not{\partial}_y\mathcal{C}^{-1} + ie\mathcal{C}\mathcal{A}(y)\mathcal{C}^{-1} - \mathcal{C}m\mathcal{C}^{-1}) \delta^{d+1}(x - y)] \\ &= \text{Tr} \{ \mathcal{C} \ln[i(-\not{\partial}_y + ie\mathcal{A}(y) - m) \delta^{d+1}(x - y)] \mathcal{C}^{-1} \} \\ &= \text{Tr} \ln[i(-\not{\partial}_y + ie\mathcal{A}(y) - m) \delta^{d+1}(x - y)]. \end{aligned} \quad (11)$$

where the transpose $^\top$ is taken in the spinor space. Therefore

$$\text{Tr} \ln[-iS^{-1}(x, y)] = \frac{1}{2} \text{Tr} \ln[M_2(x, y)], \quad (12)$$

where

$$M_2(x, y) := \mathcal{S}_y^{-1} \delta^{d+1}(x - y), \quad \mathcal{S}_y^{-1} := \left((\not{\partial}_y - ie\mathcal{A}(y))^2 - m^2 \right). \quad (13)$$

One may further simplify eq. (13) by noting (y suppressed)

$$\begin{aligned} (\not{\partial} - ie\mathcal{A})^2 &= \partial^2 - e^2 A_\mu^2 - ie \frac{1}{2} ([\gamma^\mu, \gamma^\nu]_- + [\gamma^\mu, \gamma^\nu]_+) (\partial_\mu A_\nu + A_\mu \partial_\nu + A_\nu \partial_\mu) \\ &= \partial^2 - e^2 A_\mu^2 - ie(\partial_\mu A^\mu + 2A^\mu \partial_\mu - i\sigma^{\mu\nu} \partial_\mu A_\nu) \\ &= (\partial_\mu - ieA_\mu)^2 - e\sigma^{\mu\nu} \partial_{[\mu} A_{\nu]} \\ &= (\partial_\mu - ieA_\mu)^2 - \frac{e}{2} \sigma^{\mu\nu} F_{\mu\nu}, \end{aligned} \quad (14)$$

so that

$$\mathcal{S}^{-1} \equiv \left((\partial_\mu - ieA_\mu)^2 - \frac{e}{2} \sigma^{\mu\nu} F_{\mu\nu} - m^2 \right), \quad (15)$$

where y is suppressed as well.

1.1 Constant background field in $(3+1)$ -dimensions

Equation (12) can be solved exactly when $F_{\mu\nu}$ is constant throughout space-time. Take the case [2] where $\vec{E} \parallel \vec{B}$ and, without loss of generality, $\vec{B} \parallel \vec{z}$. One has

$$F_{30} \equiv -F_{03} = E_3 =: E, \quad F_{12} \equiv -F_{21} = B_3 =: B. \quad (16)$$

A Landau-like choice of four-potential [3]

$$A_\mu := (0 \quad -Bx_2 \quad 0 \quad Ex_0) \quad (17)$$

can be applied which leads to eq. (16). In this choice the matrix reduces to

$$\mathcal{S}^{-1} = \quad (18)$$

[4]

2 Scalar electrodynamics in flat space-time

Complex Klein–Gordon Lagrangian

$$S_{\text{CKG}}[\phi, \phi^*] := \int d^{d+1}x \left\{ -\eta^{\mu\nu} (\partial_\mu \phi)^* (\partial_\nu \phi) - m^2 \phi^* \phi \right\}. \quad (19)$$

Interaction term

$$S_{\text{ICKGM}}[A_\mu, \phi, \phi^*] := \int d^{d+1}x \eta^{\mu\nu} \left\{ ieA_\mu (-\phi^* \partial_\nu \phi + \phi \partial_\nu \phi^*) + e^2 A_\mu A_\nu \phi^* \phi \right\}. \quad (20)$$

The total action for scalar electrodynamics reads

$$\begin{aligned} S_0[A_\mu, \phi, \phi^*] &:= S_{\text{CKG}} + S_{\text{ICKGM}} + S_{\text{Maxwell}} \\ &= \int d^{d+1}x \left\{ -(\nabla_\mu \phi)^* (\nabla^\mu \phi) - m^2 \phi^* \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right\}, \end{aligned} \quad (21)$$

where

$$\nabla_\mu \phi := (\partial_\mu + ieA_\mu)\phi. \quad (22)$$

Generating functional

$$\mathcal{Z}[j^\mu, \bar{J}, J] := \int DA D\phi D\phi^* \exp \left\{ i \left(S_0 + \int d^{d+1}x (j^\mu A_\mu + J^* \psi + \psi^* J) \right) \right\}. \quad (23)$$

Effective action

$$\mathcal{Z}[j^\mu, 0, 0] = \int DA \exp \left\{ i \left(S_{\text{Maxwell}} + \Gamma_W[A_\mu] + \int d^{d+1}x j^\mu A_\mu \right) \right\}. \quad (24)$$

In other words,

$$\begin{aligned} \exp\{i\Gamma_W[A_\mu]\} &:= \int D\phi D\phi^* \exp\{i(S_{\text{CKG}} + S_{\text{ICKGM}})\} \\ &\equiv \int D\phi D\phi^* \exp \left\{ i \int d^{d+1}x \left\{ -(\nabla_\mu \phi)^* (\nabla^\mu \phi) - m^2 \phi^* \phi \right\} \right\}. \end{aligned} \quad (25)$$

The integral in the exponent can be manipulated; only the first term is essential

$$\begin{aligned} &\int d^{d+1}x \left(-(\nabla_\mu \phi)^* (\nabla^\mu \phi) \right) \\ &= \int d^{d+1}x d^{d+1}y \left(-(\nabla_{x^\mu} \phi(x))^* \delta^{d+1}(x-y) \nabla^{y^\mu} \phi(y) \right), \end{aligned} \quad (26)$$

where

$$\begin{aligned} \delta^{d+1}(x-y) \nabla^{y^\mu} \phi(y) &= \delta^{d+1}(x-y) \left\{ \partial^{y^\mu} + ieA^\mu(y) \right\} \phi(y) \\ &= \left\{ -(\nabla^{y^\mu})^* \delta^{d+1}(x-y) \right\} \phi(y) + \partial^{y^\mu} B, \end{aligned} \quad (27)$$

in which

$$B = B(x, y) := \delta^{d+1}(x-y) \phi(y); \quad (28)$$

going back to eq. (26),

$$\begin{aligned} &= \int d^{d+1}x d^{d+1}y \left\{ -\left\{ \partial_{x^\mu} + ieA_\mu(x) \right\} \phi(x) \right\}^* \delta^{d+1}(x-y) \nabla^{y^\mu} \phi(y) \\ &= \int d^{d+1}x d^{d+1}y \left\{ -\partial_{x^\mu} C^\mu + \phi^*(x) \nabla_{x^\mu} \delta^{d+1}(x-y) \nabla^{y^\mu} \phi(y) \right\} \\ &= \int d^{d+1}x d^{d+1}y \left\{ -\partial_{x^\mu} C^\mu + \phi^*(x) \left\{ -(\nabla_{y^\mu}) (\nabla^{y^\mu})^* \delta^{d+1}(x-y) \right\} \phi(y) + \partial^{y^\mu} \nabla_{x^\mu} B \right\}, \end{aligned} \quad (29)$$

in which

$$C^\mu = C^\mu(x, y) := \phi^*(x) \delta^{d+1}(x - y) \nabla^{y^\mu} \phi(y). \quad (30)$$

Now eq. (25) can be written as (dropping the boundary terms)

$$\begin{aligned} &= \int D\phi D\phi^* \exp \left\{ -i \int d^{d+1}x d^{d+1}y \phi^*(x) D^{-1}(x, y) \phi(y) \right\} \\ &= \mathcal{N} \{ \det [D^{-1}(x, y)] \}^{-1/2}, \end{aligned} \quad (31)$$

where

$$D^{-1}(x, y) := \mathcal{D}_y^{-1} \delta^{d+1}(x - y), \quad \mathcal{D}_y^{-1} := +(\nabla_{y^\mu})(\nabla^{y^\mu})^* + m^2. \quad (32)$$

$$\begin{aligned} \Gamma_W[A_\mu] &\equiv -i \left(\ln \mathcal{N} - \frac{1}{2} \ln \det D^{-1} \right) \\ &= -i \ln \mathcal{N} + \frac{i}{2} \text{tr} \ln D^{-1}. \end{aligned} \quad (33)$$

[5]

A Notions and conventions

$$\eta_{\mu\nu} := \text{diag}(-, +, +, +)$$

Pauli matrices

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma^3 := \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (34)$$

[1, sec. 5]

$$[\gamma^\mu, \gamma^\nu]_+ := 2\eta^{\mu\nu} \quad (35)$$

$$\mathcal{J}^{\mu\nu} := -\frac{i}{4} [\gamma^\mu, \gamma^\nu]_- \quad (36)$$

$$\sigma^{\mu\nu} := \frac{i}{2} [\gamma^\mu, \gamma^\nu]_- \equiv -2\mathcal{J}^{\mu\nu}. \quad (37)$$

Choose the chiral representation

$$\gamma^\mu = -i \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \quad (38)$$

where

$$\sigma^\mu := (1_2, +\vec{\sigma}), \quad \bar{\sigma}^\mu := (1_2, -\vec{\sigma}). \quad (39)$$

$$\begin{aligned} \sigma^{\mu\nu} &\equiv -\frac{i}{2} \begin{bmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \end{bmatrix} \\ &= \begin{cases} 0 & \mu = 0, \nu = 0; \\ i \begin{bmatrix} -\sigma^j & 0 \\ 0 & +\sigma^j \end{bmatrix} & \mu = 0, \nu = j; \\ i \begin{bmatrix} +\sigma^i & 0 \\ 0 & -\sigma^i \end{bmatrix} & \mu = i, \nu = 0; \\ \begin{bmatrix} -\epsilon^{ij}_k \sigma^k & 0 \\ 0 & +\epsilon^{ij}_k \sigma^k \end{bmatrix} & \mu = i, \nu = j. \end{cases} \quad (40) \end{aligned}$$

B Fresnel functional integral

[6, ch. 10]

References

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