Euler-Heisenberg Effective Action

immediate

1 Spinor electrodynamics in flat space-time

Maxwell Lagrangian

$$S_{\mathrm{Maxwell}}[A_\mu] \coloneqq \int \mathrm{d}^{d+1}x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \tag{1}$$

Dirac Lagrangian [1, sec. 11]

$$S_{\mathrm{Dirac}}\big[\psi,\bar{\psi}\big] \coloneqq \int \mathrm{d}^{d+1}x \left\{-\bar{\psi}(\gamma^\mu\partial_\mu + m)\psi\right\}. \tag{2}$$

Interaction term

$$S_{\mathrm{IDM}}\big[A_\mu,\psi,\bar{\psi}\big] \coloneqq \int \mathrm{d}^{d+1}x \left(-\bar{\psi}\gamma^\mu \mathrm{i}eA_\mu\psi\right). \tag{3}$$

The total action for spinor electrodynamics reads

$$\begin{aligned} S_{1/2}\big[A_\mu,\psi,\bar{\psi}\big] &\coloneqq S_{\mathrm{Dirac}} + S_{\mathrm{IDM}} + S_{\mathrm{Maxwell}} \\ &= \int \mathrm{d}^{d+1}x \left\{-\bar{\psi}(\gamma^\mu\nabla_\mu + m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right\}, \end{aligned} \tag{4}$$

where

$$\nabla_\mu\psi \coloneqq (\partial_\mu + \mathrm{i}eA_\mu)\psi. \tag{5}$$

Generating functional

$$\mathcal{Z}[j^\mu,\bar{\eta},\eta] \coloneqq \int \mathrm{D}A\,\mathrm{D}\psi\,\mathrm{D}\bar{\psi}\exp\left\{\mathrm{i}\left(S_{1/2} + \int \mathrm{d}^{d+1}x\left(j^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta\right)\right)\right\} \tag{6}$$

Effective action

$$\mathcal{Z}[j^\mu,0,0] =: \int \mathrm{D}A \exp\biggl\{\mathrm{i}\Bigl(S_{\text{Maxwell}} + \Gamma[A_\mu] + \int \mathrm{d}^{d+1}x\, j^\mu A_\mu\Bigr)\biggr\}. \tag{7}$$

In other words,

$$\begin{aligned}
\exp\{\mathrm{i}\Gamma[A_\mu]\} &\equiv \int \mathrm{D}\psi\,\mathrm{D}\bar\psi \exp\{\mathrm{i}(S_{\text{Dirac}} + S_{\text{IDM}})\} \\
&\equiv \int \mathrm{D}\psi\,\mathrm{D}\bar\psi \exp\biggl\{\mathrm{i}\int \mathrm{d}^{d+1}x\,\bar\psi\bigl(-\not{\partial} - \mathrm{i}e\not{A} - m\bigr)\psi\biggr\} \\
&= \int \mathrm{D}\psi\,\mathrm{D}\bar\psi \exp\biggl\{\mathrm{i}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\bar\psi(x)M(x,y)\psi(y)\biggr\} \\
&= \mathcal{N}\det[-\mathrm{i}M(x,y)],
\end{aligned} \tag{8}$$

where

$$M(x,y) \coloneqq \bigl(+\not{\partial}_y - \mathrm{i}e\not{A}(y) - m\bigr)\delta^{d+1}(x-y). \tag{9}$$

$$\begin{aligned}
\Gamma[A_\mu] &\equiv -\mathrm{i}(\ln\mathcal{N} + \ln\det[-\mathrm{i}M]) \\
&= -\mathrm{i}(\ln\mathcal{N} + \operatorname{Tr}\ln[-\mathrm{i}M(x,y)]).
\end{aligned} \tag{10}$$

Note that

$$\begin{aligned}
\operatorname{Tr}\ln[-\mathrm{i}M(x,y)] &\equiv \operatorname{Tr}\ln[-\mathrm{i}M^\intercal(x,y)] \\
&= \operatorname{Tr}\ln\bigl[\mathrm{i}\bigl(+\not{\partial}_y^\intercal - \mathrm{i}e\not{A}^\intercal(y) - m\bigr)\delta^{d+1}(x-y)\bigr] \\
&= \operatorname{Tr}\ln\bigl[\mathrm{i}\bigl(-\mathscr{C}\not{\partial}_y\mathscr{C}^{-1} + \mathrm{i}e\mathscr{C}\not{A}(y)\mathscr{C}^{-1} - \mathscr{C}m\mathscr{C}^{-1}\bigr)\delta^{d+1}(x-y)\bigr] \\
&= \operatorname{Tr}\bigl\{\mathscr{C}\ln\bigl[\mathrm{i}\bigl(-\not{\partial}_y + \mathrm{i}e\not{A}(y) - m\bigr)\delta^{d+1}(x-y)\bigr]\mathscr{C}^{-1}\bigr\} \\
&= \operatorname{Tr}\ln\bigl[\mathrm{i}\bigl(-\not{\partial}_y + \mathrm{i}e\not{A}(y) - m\bigr)\delta^{d+1}(x-y)\bigr].
\end{aligned} \tag{11}$$

where the transpose $^\intercal$ is taken in the spinor space. Therefore

$$\operatorname{Tr}\ln[-\mathrm{i}M(x,y)] = \frac{1}{2}\operatorname{Tr}\ln[M_2(x,y)], \tag{12}$$

where

$$M_2(x,y) \coloneqq \mathcal{M}_y\delta^{d+1}(x-y), \quad \mathcal{M}_y \coloneqq \Bigl(\bigl(\not{\partial}_y - \mathrm{i}e\not{A}(y)\bigr)^2 - m^2\Bigr). \tag{13}$$

One may further simplify eq. (13) by noting (y suppressed)

$$\begin{aligned}
\bigl(\not{\partial} - \mathrm{i}e\not{A}\bigr)^2 &= \partial^2 - e^2A_\mu^2 - \mathrm{i}e\frac{1}{2}\bigl([\gamma^\mu,\gamma^\nu]_- + [\gamma^\mu,\gamma^\nu]_+\bigr)(\partial_\mu A_\nu + A_\mu\partial_\nu + A_\nu\partial_\mu) \\
&= \partial^2 - e^2A_\mu^2 - \mathrm{i}e(\partial_\mu A^\mu + 2A^\mu\partial_\mu - \mathrm{i}\sigma^{\mu\nu}\partial_\mu A_\nu) \\
&= (\partial_\mu - \mathrm{i}eA_\mu)^2 - e\sigma^{\mu\nu}\partial_{[\mu}A_{\nu]} \\
&= (\partial_\mu - \mathrm{i}eA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu},
\end{aligned} \tag{14}$$

so that

$$\mathcal{M} \equiv \Big((\partial_\mu - \mathrm{i}eA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\Big), \tag{15}$$

where y is suppressed as well.

1.1 Constant background field in $(3+1)$-dimensions

Equation (12) can be solved exactly when $F_{\mu\nu}$ is constant throughout space-time. Take the case [2] where $\vec{E} \parallel \vec{B}$ and, without loss of generality, $\vec{B} \parallel \vec{z}$. One has

$$F_{30} \equiv -F_{03} = E_3 =: E, \qquad F_{12} \equiv -F_{21} = B_3 =: B. \tag{16}$$

A Landau-like choice of four-potential [3]

$$A_\mu := \begin{pmatrix} 0 & -Bx_2 & 0 & Ex_0 \end{pmatrix} \tag{17}$$

can be applied which leads to eq. (16). In this choice the matrix reduces to

$$\mathcal{M} = \tag{18}$$

[4]

2 Scalar electrodynamics in flat space-time

Complex Klein–Gordon Lagrangian

$$S_{\mathrm{CKG}}[\phi, \phi^*] := \int \mathrm{d}^{d+1}x \left\{-\eta^{\mu\nu}(\partial_\mu\phi)^*(\partial_\nu\phi) - m^2\phi^*\phi\right\}. \tag{19}$$

Interaction term

$$S_{\mathrm{ICKGM}}[A_\mu, \phi, \phi^*] := \int \mathrm{d}^{d+1}x\, \eta^{\mu\nu}\left\{\mathrm{i}eA_\mu(-\phi^*\partial_\nu\phi + \phi\partial_\nu\phi^*) + e^2A_\mu A_\nu\phi^*\phi\right\}. \tag{20}$$

The total action for scalar electrodynamics reads

$$\begin{aligned} S_0[A_\mu, \phi, \phi^*] &:= S_{\mathrm{CKG}} + S_{\mathrm{ICKGM}} + S_{\mathrm{Maxwell}} \\ &= \int \mathrm{d}^{d+1}x \left\{-(\nabla_\mu\phi)^*(\nabla^\mu\phi) - m^2\phi^*\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right\}, \end{aligned} \tag{21}$$

where

$$\nabla_\mu \phi \coloneqq (\partial_\mu + \mathrm{i}eA_\mu)\phi. \tag{22}$$

Generating functional

$$\mathcal{Z}\big[j^\mu, \bar{J}, J\big] \coloneqq \int \mathrm{D}A\,\mathrm{D}\phi\,\mathrm{D}\phi^* \exp\bigg\{\mathrm{i}\bigg(S_0 + \int \mathrm{d}^{d+1}x\,(j^\mu A_\mu + J^*\psi + \psi^* J)\bigg)\bigg\}. \tag{23}$$

Effective action

$$\mathcal{Z}[j^\mu, 0, 0] \eqqcolon \int \mathrm{D}A \exp\bigg\{\mathrm{i}\bigg(S_{\mathrm{Maxwell}} + \mathit{\Gamma}[A_\mu] + \int \mathrm{d}^{d+1}x\, j^\mu A_\mu\bigg)\bigg\}. \tag{24}$$

In other words,

$$\begin{aligned}\exp\{\mathrm{i}\mathit{\Gamma}[A_\mu]\} &\coloneqq \int \mathrm{D}\phi\,\mathrm{D}\phi^* \exp\{\mathrm{i}(S_{\mathrm{CKG}} + S_{\mathrm{ICKGM}})\}\\ &\equiv \int \mathrm{D}\phi\,\mathrm{D}\phi^* \exp\bigg\{\mathrm{i}\int \mathrm{d}^{d+1}x\,\{-(\nabla_\mu\phi)^*(\nabla^\mu\phi) - m^2\phi^*\phi\}\bigg\}.\end{aligned} \tag{25}$$

The integral in the exponent can be manipulated; only the first term is essential

$$\begin{aligned}&\int \mathrm{d}^{d+1}x\,(-(\nabla_\mu\phi)^*(\nabla^\mu\phi))\\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\big(-(\nabla_{x^\mu}\phi(x))^*\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\big),\end{aligned} \tag{26}$$

where

$$\begin{aligned}\delta^{d+1}(x-y)\big(\nabla^{y^\mu}\phi(y)\big) &= \delta^{d+1}(x-y)\big(\partial^{y^\mu} + \mathrm{i}eA^\mu(y)\big)\phi(y)\\ &= \big\{-\big(\nabla^{y^\mu}\big)^*\delta^{d+1}(x-y)\big\}\phi(y) + \partial^{y^\mu}B,\end{aligned} \tag{27}$$

in which

$$B = B(x,y) \coloneqq \delta^{d+1}(x-y)\phi(y); \tag{28}$$

going back to eq. (26),

$$\begin{aligned}&= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\big\{-\{(\partial_{x^\mu} + \mathrm{i}eA_\mu(x))\phi(x)\}^*\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\big\}\\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\big\{-\partial_{x^\mu}C^\mu + \phi^*(x)\nabla_{x^\mu}\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\big\}\\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\big\{-\partial_{x^\mu}C^\mu + \phi^*(x)\big(-(\nabla_{y^\mu})\big(\nabla^{y^\mu}\big)^*\delta^{d+1}(x-y)\big)\phi(y) + \partial^{y^\mu}\nabla_{x^\mu}B\big\},\end{aligned} \tag{29}$$

in which

$$C^\mu = C^\mu(x,y) \coloneqq \phi^*(x)\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y). \tag{30}$$

Now eq. (25) can be written as (dropping boundary terms)

$$\begin{aligned} &= \int \mathrm{D}\phi\,\mathrm{D}\phi^* \exp\left\{\mathrm{i}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\phi^*(x)M(x,y)\phi(y)\right\} \\ &= \mathcal{N}\{\det[-\mathrm{i}M(x,y)]\}^{-1/2}, \end{aligned} \tag{31}$$

where

$$M(x,y) \coloneqq \mathcal{M}_y\delta^{d+1}(x-y), \qquad \mathcal{M}_y \coloneqq -(\nabla_{y^\mu})\big(\nabla^{y^\mu}\big)^* - m^2. \tag{32}$$

[5]

A Notions and conventions

$\eta_{\mu\nu} \coloneqq \mathrm{diag}(-,+,+,+)$

Pauli matrices

$$\sigma^1 \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 \coloneqq \begin{pmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 \coloneqq \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{33}$$

[1, sec. 5]

$$[\gamma^\mu, \gamma^\nu]_+ \coloneqq 2\eta^{\mu\nu} \tag{34}$$

$$\mathscr{J}^{\mu\nu} \coloneqq -\frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]_- \tag{35}$$

$$\sigma^{\mu\nu} \coloneqq \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]_- \equiv -2\mathscr{J}^{\mu\nu}. \tag{36}$$

Choose the chiral representation

$$\gamma^\mu = -\mathrm{i}\begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \tag{37}$$

where

$$\sigma^\mu \coloneqq (1_2, +\vec{\sigma}), \qquad \bar{\sigma}^\mu \coloneqq (1_2, -\vec{\sigma}). \tag{38}$$

$$\sigma^{\mu\nu} \equiv -\frac{\mathrm{i}}{2}\begin{bmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{bmatrix}$$
$$= \begin{cases} 0 & \mu = 0, \nu = 0; \\ \mathrm{i}\begin{bmatrix} -\sigma^j & 0 \\ 0 & +\sigma^j \end{bmatrix} & \mu = 0, \nu = j; \\ \mathrm{i}\begin{bmatrix} +\sigma^i & 0 \\ 0 & -\sigma^i \end{bmatrix} & \mu = i, \nu = 0; \\ \begin{bmatrix} -\epsilon^{ij}{}_k\sigma^k & 0 \\ 0 & +\epsilon^{ij}{}_k\sigma^k \end{bmatrix} & \mu = i, \nu = j. \end{cases} \tag{39}$$

B Fresnel functional integral

[6, ch. 10]

References

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