Effective action of spinor electrodynamics

immediate

1 Spinor electrodynamics in flat space-time

Maxwell Lagrangian

$$S_{\text{Maxwell}}[A_\mu] \coloneqq \int \mathrm{d}^{d+1}x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \tag{1}$$

Dirac Lagrangian [**?** , sec. 11]

$$S_{\text{Dirac}}\big[\psi,\bar{\psi}\big] \coloneqq \int \mathrm{d}^{d+1}x \left\{-\bar{\psi}(\gamma^\mu\partial_\mu + m)\psi\right\}. \tag{2}$$

Interaction term

$$S_{\text{IDM}}\big[A_\mu,\psi,\bar{\psi}\big] \coloneqq \int \mathrm{d}^{d+1}x \left(-\bar{\psi}\gamma^\mu \mathrm{i}eA_\mu\psi\right). \tag{3}$$

The total action for spinor electrodynamics reads

$$\begin{aligned} S_{1/2}\big[A_\mu,\psi,\bar{\psi}\big] &\coloneqq S_{\text{Dirac}} + S_{\text{IDM}} + S_{\text{Maxwell}} \\ &= \int \mathrm{d}^{d+1}x \left\{-\bar{\psi}(\gamma^\mu\nabla_\mu + m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right\}, \end{aligned} \tag{4}$$

where

$$\nabla_\mu\psi \coloneqq (\partial_\mu + \mathrm{i}eA_\mu)\psi. \tag{5}$$

Generating functional

$$\mathcal{Z}[j^\mu,\bar{\eta},\eta] \coloneqq \int \mathrm{D}A\,\mathrm{D}\psi\,\mathrm{D}\bar{\psi}\exp\left\{\mathrm{i}\left(S_{1/2} + \int \mathrm{d}^{d+1}x \left(j^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta\right)\right)\right\} \tag{6}$$

Effective action

$$\mathcal{Z}[j^\mu,0,0] =: \int \mathrm{D}A \exp\biggl\{\mathrm{i}\biggl(S_{\text{Maxwell}} + \mathit{\Gamma}_{\text{EH}}[A_\mu] + \int \mathrm{d}^{d+1}x\, j^\mu A_\mu\biggr)\biggr\}. \tag{7}$$

In other words,

$$\begin{aligned}
\exp\{\mathrm{i}\mathit{\Gamma}_{\text{EH}}[A_\mu]\} &\equiv \int \mathrm{D}\psi\, \mathrm{D}\bar{\psi} \exp\{\mathrm{i}(S_{\text{Dirac}} + S_{\text{IDM}})\} \\
&\equiv \int \mathrm{D}\psi\, \mathrm{D}\bar{\psi} \exp\biggl\{\mathrm{i}\int \mathrm{d}^{d+1}x\, \bar{\psi}\bigl(-\not{\partial} - \mathrm{i}e\not{A} - m\bigr)\psi\biggr\} \\
&= \int \mathrm{D}\psi\, \mathrm{D}\bar{\psi} \exp\biggl\{\mathrm{i}\int \mathrm{d}^{d+1}x\, \mathrm{d}^{d+1}y\, \bar{\psi}(x) S^{-1}(x,y)\psi(y)\biggr\} \\
&= \tilde{\mathcal{N}} \det\bigl[-\mathrm{i}S^{-1}(x,y)\bigr],
\end{aligned} \tag{8}$$

where

$$S^{-1}(x,y) := \bigl(+\not{\partial}_y - \mathrm{i}e\not{A}(y) - m\bigr)\delta^{d+1}(x-y). \tag{9}$$

$$\begin{aligned}
\mathit{\Gamma}_{\text{EH}}[A_\mu] &\equiv -\mathrm{i}\Bigl(\ln\tilde{\mathcal{N}} + \ln\det\bigl[-\mathrm{i}S^{-1}\bigr]\Bigr) \\
&= -\mathrm{i}\Bigl(\ln\tilde{\mathcal{N}} + \operatorname{Tr}\ln\bigl[-\mathrm{i}S^{-1}(x,y)\bigr]\Bigr).
\end{aligned} \tag{10}$$

Note that

$$\begin{aligned}
\operatorname{Tr}\ln\bigl[-\mathrm{i}S^{-1}(x,y)\bigr] &\equiv \operatorname{Tr}\ln[-\mathrm{i}M^{\intercal}(x,y)] \\
&= \operatorname{Tr}\ln\bigl[\mathrm{i}\bigl(+\not{\partial}_y^{\intercal} - \mathrm{i}e\not{A}^{\intercal}(y) - m\bigr)\delta^{d+1}(x-y)\bigr] \\
&= \operatorname{Tr}\ln\bigl[\mathrm{i}\bigl(-\mathscr{C}\not{\partial}_y\mathscr{C}^{-1} + \mathrm{i}e\mathscr{C}\not{A}(y)\mathscr{C}^{-1} - \mathscr{C}m\mathscr{C}^{-1}\bigr)\delta^{d+1}(x-y)\bigr] \\
&= \operatorname{Tr}\bigl\{\mathscr{C}\ln\bigl[\mathrm{i}\bigl(-\not{\partial}_y + \mathrm{i}e\not{A}(y) - m\bigr)\delta^{d+1}(x-y)\bigr]\mathscr{C}^{-1}\bigr\} \\
&= \operatorname{Tr}\ln\bigl[\mathrm{i}\bigl(-\not{\partial}_y + \mathrm{i}e\not{A}(y) - m\bigr)\delta^{d+1}(x-y)\bigr].
\end{aligned} \tag{11}$$

where the transpose $^{\intercal}$ is taken in the spinor space. Therefore

$$\operatorname{Tr}\ln\bigl[-\mathrm{i}S^{-1}(x,y)\bigr] = \frac{1}{2}\operatorname{Tr}\ln\bigl[S_2^{-1}(x,y)\bigr], \tag{12}$$

where

$$S_2^{-1}(x,y) := \mathcal{S}_{2y}^{-1}\delta^{d+1}(x-y), \quad \mathcal{S}_{2y}^{-1} := \Bigl(\bigl(\not{\partial}_y - \mathrm{i}e\not{A}(y)\bigr)^2 - m^2\Bigr). \tag{13}$$

One may further simplify eq. (13) by noting (y suppressed)

$$\begin{aligned}
\left(\not{\partial} - \mathrm{i}e\not{A}\right)^2 &= \partial^2 - e^2 A_\mu^2 - \mathrm{i}e\frac{1}{2}\left([\gamma^\mu,\gamma^\nu]_- + [\gamma^\mu,\gamma^\nu]_+\right)(\partial_\mu A_\nu + A_\mu\partial_\nu + A_\nu\partial_\mu) \\
&= \partial^2 - e^2 A_\mu^2 - \mathrm{i}e(\partial_\mu A^\mu + 2A^\mu\partial_\mu - \mathrm{i}\sigma^{\mu\nu}\partial_\mu A_\nu) \\
&= (\partial_\mu - \mathrm{i}eA_\mu)^2 - e\sigma^{\mu\nu}\partial_{[\mu}A_{\nu]} \\
&= (\partial_\mu - \mathrm{i}eA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}, \qquad (14)
\end{aligned}$$

so that

$$\mathcal{S}_2^{-1} \equiv \left((\partial_\mu - \mathrm{i}eA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\right), \qquad (15)$$

where y is suppressed as well. Now we may write

$$\begin{aligned}
\mathit{\Gamma}_{\mathrm{EH}}[A_\mu] &= -\frac{\mathrm{i}}{2}\,\mathrm{Tr}\ln\left\{\mathcal{N}^{-1}S_2^{-1}(x,y)\right\} \\
&= -\frac{\mathrm{i}}{2}\,\mathrm{Tr}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\left\{-\mathrm{e}^{\mathrm{i}s\left(S_2^{-1}(x,y)+\mathrm{i}0^+\right)} + \mathrm{e}^{\mathrm{i}s\left(\mathcal{N}+\mathrm{i}0^+\right)}\right\} \\
&= +\frac{\mathrm{i}}{2}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\delta^{d+1}(x-y) \\
&\quad \cdot\,\mathrm{tr}\left\{+\mathrm{e}^{\mathrm{i}s\left((\partial_\mu-\mathrm{i}eA_\mu)^2-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}-m^2+\mathrm{i}0^+\right)\delta^{d+1}(x-y)} - \mathrm{e}^{\mathrm{i}s\left(\mathcal{N}+\mathrm{i}0^+\right)}\right\}, \qquad (16)
\end{aligned}$$

where tr takes place in the spinor space.

1.1 Constant background field in $(3+1)$-dimensions

Equation (12) can be solved exactly when $F_{\mu\nu}$ is constant throughout space-time. One has

$$F_{0i} \equiv -F_{i0} = -E_i, \qquad F_{ij} \equiv -F_{ji} = \epsilon_{ijk}B^k, \qquad (17)$$

and the spinor part of eq. (15) can be calculated

$$-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} = -e\begin{bmatrix}\left(\vec{B}-\mathrm{i}\vec{E}\right)\cdot\vec{\sigma} & 0 \\ 0 & \left(\vec{B}+\mathrm{i}\vec{E}\right)\cdot\vec{\sigma}\end{bmatrix}, \qquad (18)$$

so that

$$\begin{aligned}
&\mathrm{tr}\,\mathrm{e}^{\mathrm{i}s\left((\partial_\mu-\mathrm{i}eA_\mu)^2-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}-m^2+\mathrm{i}0^+\right)\delta^{d+1}(x-y)} \\
&= \mathrm{e}^{\mathrm{i}s\left((\partial_\mu-\mathrm{i}eA_\mu)^2-m^2+\mathrm{i}0^+\right)\delta^{d+1}(x-y)}\,\mathrm{tr}\,\mathrm{e}^{\mathrm{i}s\left(-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)}, \qquad (19)
\end{aligned}$$

in which

$$\operatorname{tr} \mathrm{e}^{\mathrm{i}s\left(-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)} = 2\left\{\cos\left(se\sqrt{\left(\vec{B}+\mathrm{i}\vec{E}\right)^2}\right) + \cos\left(se\sqrt{\left(\vec{B}-\mathrm{i}\vec{E}\right)^2}\right)\right\} \tag{20}$$

$$= 4\cos(seB)\cosh(seE), \qquad \text{for} \quad \vec{B} = B\hat{x}^3, \quad \vec{E} = E\hat{x}^3. \tag{21}$$

A Landau-like choice of four-potential [?] reads

$$A_\mu := \begin{pmatrix} 0 & -x^0E_1 + x^3B^2 & -x^0E_2 + x^1B^3 & -x^0E_3 + x^2B^1 \end{pmatrix}, \tag{22}$$

applying which leads to eq. (17).

Consider the case where $\vec{B} = B\hat{x}^3$, $\vec{E} = E\hat{x}^3$. The scalar part of eq. (15) simplifies to

$$\begin{aligned}(\partial_\mu - \mathrm{i}eA_\mu)^2 - m^2 &= -\partial_0^2 + \partial_1^2 + \left(\partial_2 - \mathrm{i}ex^1B\right)^2 + \left(\partial_3 + \mathrm{i}ex^0E\right)^2 - m^2 \\ &= U\left\{-\partial_0^2 + \partial_1^2 - (eB)^2\left(x^1\right)^2 - (eE)^2\left(x^0\right)^2\right\}U^\dagger,\end{aligned} \tag{23}$$

where

$$U := \mathrm{e}^{+\mathrm{i}(\partial_1\partial_2/eB - \partial_3\partial_0/eE)}. \tag{24}$$

One can first solve the sub-eigensystem

$$\left\{+\partial_1^2 - (eB)^2\left(x^1\right)^2 - \lambda\right\}\tilde{f}_\lambda\left(x^1\right) = 0, \tag{25}$$

and the general solution is

$$\tilde{f}_\lambda(x) = c_1\mathrm{D}_{-\frac{\lambda}{2eB}-\frac{1}{2}}\left(\sqrt{2eB}x\right) + c_2\mathrm{D}_{+\frac{\lambda}{2eB}-\frac{1}{2}}\left(\mathrm{i}\sqrt{2eB}x\right), \tag{26}$$

where $\mathrm{D}_\nu(z)$ is the parabolic cylinder function. For

$$\lambda = -(2n+1)eB, \qquad n = -\frac{1}{2} - \frac{\lambda}{2eB} \tag{27}$$

eq. (26) reduces to a combination of the Hermite polynomials

$$\tilde{f}_n(x) = c_1 2^{-n/2}\mathrm{e}^{-eBx^2/2}\mathrm{H}_n\left(\sqrt{eB}x\right) + c_2 2^{+(n+1)/2}\mathrm{e}^{+eBx^2/2}\mathrm{H}_{-n-1}\left(\mathrm{i}\sqrt{eB}x\right). \tag{28}$$

The first branch constitutes a complete orthogonal set of basis for $L^2(\mathbb{R})$ when $n \in \mathbb{Z}^+$, and its normalised version will be denoted as $\{f_n\}$.

It has been argued that the same solution applies for

$$\left\{-\partial_0^2 - (eE)^2\left(x^0\right)^2 - \lambda\right\}\tilde{g}_\lambda\left(x^0\right) = 0 \tag{29}$$

if one applies $E \to \mathrm{i}B$, so that the eigenfunctions are

$$g_n(x) = f_n(x), \qquad n = -\frac{1}{2} - \frac{\mathrm{i}\lambda}{2eE} \quad \text{or} \quad \lambda = \mathrm{i}(2n+1)eE. \tag{30}$$

However, we know that complex differential equations are subtle!

The eigenfunctions of eq. (23) can now be written down

$$\begin{aligned}
&\big\{(\partial_\mu - \mathrm{i}eA_\mu)^2 - m^2\big\}\big\{Uf_n\big(x^1\big)g_k\big(x^0\big)\big\} \\
&= U\Big\{-\partial_0^2 + \partial_1^2 - (eB)^2\big(x^1\big)^2 - (eE)^2\big(x^0\big)^2\Big\}f_n\big(x^1\big)g_k\big(x^0\big) \\
&= -\mathrm{i}(2n+1)(2k+1)e^2BE\big\{Uf_n\big(x^1\big)g_k\big(x^0\big)\big\},
\end{aligned} \tag{31}$$

and they are

$$\phi_{nk}(x) := Uf_n\big(x^1\big)g_k\big(x^0\big). \tag{32}$$

$$\begin{aligned}
\delta^4(x-y) &= \sum_{n,k} \phi^\dagger_{nk}(x)\phi_{nk}(y) \\
&= \sum_{n,k} \big\{U(x)f_n\big(x^1\big)g_k\big(x^0\big)\big\}^\dagger\big\{U(y)f_n\big(y^1\big)g_k\big(y^0\big)\big\}.
\end{aligned} \tag{33}$$

[**?**]

A Notions and conventions

The metric convention is mostly positive, i.e. $\eta_{\mu\nu} := \mathrm{diag}(-,+,+,\dots)$

B Fresnel functional integral

[**?** , ch. 10]

C Algebra

$$\big[a\partial_1\partial_2, x^1\big]_- = a\partial_2 \tag{34}$$

central.

Baker–Campbell–Hausdorff formula

$$\mathrm{e}^{+a\partial_1\partial_2}x^1\mathrm{e}^{-a\partial_1\partial_2} = x^1 + a\partial_2. \tag{35}$$

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