Euler-Heisenberg Effective Action

immediate

1 Spinor electrodynamics in flat space-time

Maxwell Lagrangian

$$S_{\mathrm{Maxwell}}[A_\mu] \coloneqq \int \mathrm{d}^{d+1}x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \tag{1}$$

Dirac Lagrangian [1, sec. 11]

$$S_{\mathrm{Dirac}}[\psi,\bar{\psi}] \coloneqq \int \mathrm{d}^{d+1}x \left\{-\bar{\psi}(\gamma^\mu\partial_\mu + m)\psi\right\}. \tag{2}$$

Interaction term

$$S_{\mathrm{IDM}}[A_\mu,\psi,\bar{\psi}] \coloneqq \int \mathrm{d}^{d+1}x \left(-\bar{\psi}\gamma^\mu \mathrm{i}eA_\mu\psi\right). \tag{3}$$

The total action for spinor electrodynamics reads

$$\begin{aligned} S_{1/2}[A_\mu,\psi,\bar{\psi}] &\coloneqq S_{\mathrm{Dirac}} + S_{\mathrm{IDM}} + S_{\mathrm{Maxwell}} \\ &= \int \mathrm{d}^{d+1}x \left\{-\bar{\psi}(\gamma^\mu\nabla_\mu + m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right\}, \end{aligned} \tag{4}$$

where

$$\nabla_\mu\psi \coloneqq (\partial_\mu + \mathrm{i}eA_\mu)\psi. \tag{5}$$

Generating functional

$$\mathcal{Z}[j^\mu,\bar{\eta},\eta] \coloneqq \int \mathrm{D}A\,\mathrm{D}\psi\,\mathrm{D}\bar{\psi}\exp\left\{\mathrm{i}\left(S_{1/2} + \int \mathrm{d}^{d+1}x \left(j^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta\right)\right)\right\} \tag{6}$$

Effective action

$$\mathcal{Z}[j^\mu,0,0] =: \int \mathrm{D}A\exp\biggl\{\mathrm{i}\biggl(S_{\text{Maxwell}} + \mathit{\Gamma}_{\text{EH}}[A_\mu] + \int \mathrm{d}^{d+1}x\, j^\mu A_\mu\biggr)\biggr\}. \tag{7}$$

In other words,

$$\begin{aligned}
\exp\{\mathrm{i}\mathit{\Gamma}_{\text{EH}}[A_\mu]\} &\equiv \int \mathrm{D}\psi\,\mathrm{D}\bar{\psi}\exp\{\mathrm{i}(S_{\text{Dirac}} + S_{\text{IDM}})\} \\
&\equiv \int \mathrm{D}\psi\,\mathrm{D}\bar{\psi}\exp\biggl\{\mathrm{i}\int \mathrm{d}^{d+1}x\,\bar{\psi}\bigl(-\partial\!\!\!/ - \mathrm{i}e A\!\!\!/ - m\bigr)\psi\biggr\} \\
&= \int \mathrm{D}\psi\,\mathrm{D}\bar{\psi}\exp\biggl\{\mathrm{i}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\bar{\psi}(x)S^{-1}(x,y)\psi(y)\biggr\} \\
&= \tilde{\mathcal{N}}\det\bigl[-\mathrm{i}S^{-1}(x,y)\bigr],
\end{aligned} \tag{8}$$

where

$$S^{-1}(x,y) := \bigl(+\partial\!\!\!/_y - \mathrm{i}e A\!\!\!/(y) - m\bigr)\delta^{d+1}(x-y). \tag{9}$$

$$\begin{aligned}
\mathit{\Gamma}_{\text{EH}}[A_\mu] &\equiv -\mathrm{i}\Bigl(\ln\tilde{\mathcal{N}} + \ln\det\bigl[-\mathrm{i}S^{-1}\bigr]\Bigr) \\
&= -\mathrm{i}\Bigl(\ln\tilde{\mathcal{N}} + \operatorname{Tr}\ln\bigl[-\mathrm{i}S^{-1}(x,y)\bigr]\Bigr).
\end{aligned} \tag{10}$$

Note that

$$\begin{aligned}
\operatorname{Tr}\ln\bigl[-\mathrm{i}S^{-1}(x,y)\bigr] &\equiv \operatorname{Tr}\ln[-\mathrm{i}M^{\intercal}(x,y)] \\
&= \operatorname{Tr}\ln\bigl[\mathrm{i}\bigl(+\partial\!\!\!/_y^{\intercal} - \mathrm{i}e A\!\!\!/^{\intercal}(y) - m\bigr)\delta^{d+1}(x-y)\bigr] \\
&= \operatorname{Tr}\ln\bigl[\mathrm{i}\bigl(-\mathscr{C}\partial\!\!\!/_y\mathscr{C}^{-1} + \mathrm{i}e\mathscr{C}A\!\!\!/(y)\mathscr{C}^{-1} - \mathscr{C}m\mathscr{C}^{-1}\bigr)\delta^{d+1}(x-y)\bigr] \\
&= \operatorname{Tr}\bigl\{\mathscr{C}\ln\bigl[\mathrm{i}\bigl(-\partial\!\!\!/_y + \mathrm{i}e A\!\!\!/(y) - m\bigr)\delta^{d+1}(x-y)\bigr]\mathscr{C}^{-1}\bigr\} \\
&= \operatorname{Tr}\ln\bigl[\mathrm{i}\bigl(-\partial\!\!\!/_y + \mathrm{i}e A\!\!\!/(y) - m\bigr)\delta^{d+1}(x-y)\bigr].
\end{aligned} \tag{11}$$

where the transpose $^{\intercal}$ is taken in the spinor space. Therefore

$$\operatorname{Tr}\ln\bigl[-\mathrm{i}S^{-1}(x,y)\bigr] = \frac{1}{2}\operatorname{Tr}\ln\bigl[S_2^{-1}(x,y)\bigr], \tag{12}$$

where

$$S_2^{-1}(x,y) := \mathcal{S}_{2y}^{-1}\delta^{d+1}(x-y), \quad \mathcal{S}_{2y}^{-1} := \Bigl(\bigl(\partial\!\!\!/_y - \mathrm{i}e A\!\!\!/(y)\bigr)^2 - m^2\Bigr). \tag{13}$$

One may further simplify eq. (13) by noting (y suppressed)

$$\begin{aligned}\left(\partial\!\!\!/ - \mathrm{i}eA\!\!\!/\right)^2 &= \partial^2 - e^2A_\mu^2 - \mathrm{i}e\frac{1}{2}\left([\gamma^\mu,\gamma^\nu]_- + [\gamma^\mu,\gamma^\nu]_+\right)(\partial_\mu A_\nu + A_\mu\partial_\nu + A_\nu\partial_\mu)\\ &= \partial^2 - e^2A_\mu^2 - \mathrm{i}e(\partial_\mu A^\mu + 2A^\mu\partial_\mu - \mathrm{i}\sigma^{\mu\nu}\partial_\mu A_\nu)\\ &= (\partial_\mu - \mathrm{i}eA_\mu)^2 - e\sigma^{\mu\nu}\partial_{[\mu}A_{\nu]}\\ &= (\partial_\mu - \mathrm{i}eA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu},\end{aligned} \tag{14}$$

so that

$$\mathcal{S}_2^{-1} \equiv \left((\partial_\mu - \mathrm{i}eA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\right), \tag{15}$$

where y is suppressed as well. Now we may write

$$\begin{aligned}\mathit{\Gamma}_{\mathrm{EH}}[A_\mu] &= -\frac{\mathrm{i}}{2}\,\mathrm{Tr}\ln\left\{\mathcal{N}^{-1}S_2^{-1}(x,y)\right\}\\ &= -\frac{\mathrm{i}}{2}\,\mathrm{Tr}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\left\{-\mathrm{e}^{\mathrm{i}s\left(S_2^{-1}(x,y)+\mathrm{i}0^+\right)} + \mathrm{e}^{\mathrm{i}s\left(\mathcal{N}+\mathrm{i}0^+\right)}\right\}\\ &= +\frac{\mathrm{i}}{2}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\delta^{d+1}(x-y)\\ &\quad\cdot\mathrm{tr}\left\{+\mathrm{e}^{\mathrm{i}s\left((\partial_\mu-\mathrm{i}eA_\mu)^2-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}-m^2+\mathrm{i}0^+\right)\delta^{d+1}(x-y)} - \mathrm{e}^{\mathrm{i}s\left(\mathcal{N}+\mathrm{i}0^+\right)}\right\},\end{aligned} \tag{16}$$

where tr takes place in the spinor space.

1.1 Constant background field in $(3+1)$-dimensions

Equation (12) can be solved exactly when $F_{\mu\nu}$ is constant throughout space-time. One has

$$F_{0i} \equiv -F_{i0} = -E_i, \qquad F_{ij} \equiv -F_{ji} = \epsilon_{ijk}B^k, \tag{17}$$

and the spinor part of eq. (15) can be calculated

$$-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} = -e\begin{bmatrix}\left(\vec{B}-\mathrm{i}\vec{E}\right)\cdot\vec{\sigma} & 0\\ 0 & \left(\vec{B}+\mathrm{i}\vec{E}\right)\cdot\vec{\sigma}\end{bmatrix}, \tag{18}$$

so that

$$\begin{aligned}&\mathrm{tr}\,\mathrm{e}^{\mathrm{i}s\left((\partial_\mu-\mathrm{i}eA_\mu)^2-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}-m^2+\mathrm{i}0^+\right)\delta^{d+1}(x-y)}\\ &= \mathrm{e}^{\mathrm{i}s\left((\partial_\mu-\mathrm{i}eA_\mu)^2-m^2+\mathrm{i}0^+\right)\delta^{d+1}(x-y)}\,\mathrm{tr}\,\mathrm{e}^{\mathrm{i}s\left(-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)},\end{aligned} \tag{19}$$

in which

$$\operatorname{tr} \mathrm{e}^{\mathrm{i}s\left(-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)} = 2\left\{\cos\left(se\sqrt{\left(\vec{B}+\mathrm{i}\vec{E}\right)^2}\right) + \cos\left(se\sqrt{\left(\vec{B}-\mathrm{i}\vec{E}\right)^2}\right)\right\} \tag{20}$$

$$= 4\cos(seB)\cosh(seE), \qquad \text{for} \quad \vec{B} = B\hat{x}^3, \quad \vec{E} = E\hat{x}^3. \tag{21}$$

A Landau-like choice of four-potential [3] reads

$$A_\mu \coloneqq \begin{pmatrix}0 & -x^0E_1 + x^3B^2 & -x^0E_2 + x^1B^3 & -x^0E_3 + x^2B^1\end{pmatrix}, \tag{22}$$

applying which leads to eq. (17).

Consider the case where $\vec{B} = B\hat{x}^3$, $\vec{E} = E\hat{x}^3$. The scalar part of eq. (15) simplifies to

$$\begin{aligned}(\partial_\mu - \mathrm{i}eA_\mu)^2 - m^2 &= -\partial_0^2 + \partial_1^2 + \left(\partial_2 - \mathrm{i}ex^1B\right)^2 + \left(\partial_3 + \mathrm{i}ex^0E\right)^2 - m^2 \\ &= U\left\{-\partial_0^2 + \partial_1^2 - (eB)^2\left(x^1\right)^2 - (eE)^2\left(x^0\right)^2\right\}U^\dagger,\end{aligned} \tag{23}$$

where

$$U \coloneqq \mathrm{e}^{+\mathrm{i}(\partial_1\partial_2/eB - \partial_3\partial_0/eE)}. \tag{24}$$

One can first solve the sub-eigensystem

$$\left\{+\partial_1^2 - (eB)^2\left(x^1\right)^2 - \lambda\right\}\tilde{f}_\lambda\left(x^1\right) = 0, \tag{25}$$

and the general solution is

$$\tilde{f}_\lambda(x) = c_1\mathrm{D}_{-\frac{\lambda}{2eB}-\frac{1}{2}}\left(\sqrt{2eB}x\right) + c_2\mathrm{D}_{+\frac{\lambda}{2eB}-\frac{1}{2}}\left(\mathrm{i}\sqrt{2eB}x\right), \tag{26}$$

where $\mathrm{D}_\nu(z)$ is the parabolic cylinder function. For

$$\lambda = -(2n+1)eB, \qquad n = -\frac{1}{2} - \frac{\lambda}{2eB} \tag{27}$$

eq. (26) reduces to a combination of the Hermite polynomials

$$\tilde{f}_n(x) = c_1 2^{-n/2}\mathrm{e}^{-eBx^2/2}\mathrm{H}_n\left(\sqrt{eB}x\right) + c_2 2^{+(n+1)/2}\mathrm{e}^{+eBx^2/2}\mathrm{H}_{-n-1}\left(\mathrm{i}\sqrt{eB}x\right). \tag{28}$$

The first branch constitutes a complete orthogonal set of basis for $L^2(\mathbb{R})$ when $n \in \mathbb{Z}^+$, and its normalised version will be denoted as $\{f_n\}$.

It has been argued that the same solution applies for

$$\left\{-\partial_0^2 - (eE)^2\left(x^0\right)^2 - \lambda\right\}\tilde{g}_\lambda\left(x^0\right) = 0 \tag{29}$$

if one applies $E \to \mathrm{i}B$, so that the eigenfunctions are

$$g_n(x) = f_n(x), \qquad n = -\frac{1}{2} - \frac{\mathrm{i}\lambda}{2eE} \quad \text{or} \quad \lambda = \mathrm{i}(2n+1)eE. \tag{30}$$

However, we know that complex differential equations are subtle!

The eigenfunctions of eq. (23) can now be written down

$$\begin{aligned} &\big\{(\partial_\mu - \mathrm{i}eA_\mu)^2 - m^2\big\}\big\{Uf_n\big(x^1\big)g_k\big(x^0\big)\big\} \\ &= U\Big\{-\partial_0^2 + \partial_1^2 - (eB)^2\big(x^1\big)^2 - (eE)^2\big(x^0\big)^2\Big\}f_n\big(x^1\big)g_k\big(x^0\big) \\ &= -\mathrm{i}(2n+1)(2k+1)e^2BE\big\{Uf_n\big(x^1\big)g_k\big(x^0\big)\big\}, \end{aligned} \tag{31}$$

and they are

$$\phi_{nk}(x) \coloneqq Uf_n\big(x^1\big)g_k\big(x^0\big). \tag{32}$$

$$\begin{aligned} \delta^4(x-y) &= \sum_{n,k} \phi^*_{nk}(x)\phi_{nk}(y) \\ &= \sum_{n,k} \big\{U(x)f_n\big(x^1\big)g_k\big(x^0\big)\big\}^\dagger\big\{U(y)f_n\big(y^1\big)g_k\big(y^0\big)\big\}. \end{aligned} \tag{33}$$

[4]

2 Scalar electrodynamics in flat space-time

Complex Klein–Gordon Lagrangian

$$S_{\mathrm{CKG}}[\phi,\phi^*] \coloneqq \int \mathrm{d}^{d+1}x\,\big\{-\eta^{\mu\nu}(\partial_\mu\phi)^*(\partial_\nu\phi) - m^2\phi^*\phi\big\}. \tag{34}$$

Interaction term

$$S_{\mathrm{ICKGM}}[A_\mu,\phi,\phi^*] \coloneqq \int \mathrm{d}^{d+1}x\,\eta^{\mu\nu}\big\{\mathrm{i}eA_\mu(-\phi^*\partial_\nu\phi + \phi\partial_\nu\phi^*) + e^2A_\mu A_\nu\phi^*\phi\big\}. \tag{35}$$

The total action for scalar electrodynamics reads

$$\begin{aligned} S_0[A_\mu,\phi,\phi^*] &\coloneqq S_{\mathrm{CKG}} + S_{\mathrm{ICKGM}} + S_{\mathrm{Maxwell}} \\ &= \int \mathrm{d}^{d+1}x\,\Big\{-(\nabla_\mu\phi)^*(\nabla^\mu\phi) - m^2\phi^*\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\Big\}, \end{aligned} \tag{36}$$

where

$$\nabla_\mu \phi \coloneqq (\partial_\mu + \mathrm{i}eA_\mu)\phi. \tag{37}$$

Generating functional

$$\mathcal{Z}\big[j^\mu, \bar{J}, J\big] \coloneqq \int \mathrm{D}A\,\mathrm{D}\phi\,\mathrm{D}\phi^* \exp\bigg\{\mathrm{i}\bigg(S_0 + \int \mathrm{d}^{d+1}x\,(j^\mu A_\mu + J^*\psi + \psi^* J)\bigg)\bigg\}. \tag{38}$$

Effective action

$$\mathcal{Z}[j^\mu, 0, 0] \eqqcolon \int \mathrm{D}A \exp\bigg\{\mathrm{i}\bigg(S_{\text{Maxwell}} + \mathit{\Gamma}_{\mathrm{W}}[A_\mu] + \int \mathrm{d}^{d+1}x\, j^\mu A_\mu\bigg)\bigg\}. \tag{39}$$

In other words,

$$\begin{aligned}\exp\{\mathrm{i}\mathit{\Gamma}_{\mathrm{W}}[A_\mu]\} &\coloneqq \int \mathrm{D}\phi\,\mathrm{D}\phi^* \exp\{\mathrm{i}(S_{\text{CKG}} + S_{\text{ICKGM}})\} \\ &\equiv \int \mathrm{D}\phi\,\mathrm{D}\phi^* \exp\bigg\{\mathrm{i}\int \mathrm{d}^{d+1}x\,\{-(\nabla_\mu\phi)^*(\nabla^\mu\phi) - m^2\phi^*\phi\}\bigg\}.\end{aligned} \tag{40}$$

The integral in the exponent can be manipulated; only the first term is essential

$$\begin{aligned}&\int \mathrm{d}^{d+1}x\,(-(\nabla_\mu\phi)^*(\nabla^\mu\phi)) \\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\big(-(\nabla_{x^\mu}\phi(x))^*\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\big),\end{aligned} \tag{41}$$

where

$$\begin{aligned}\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y) &= \delta^{d+1}(x-y)\big\{\partial^{y^\mu} + \mathrm{i}eA^\mu(y)\big\}\phi(y) \\ &= \big\{-\big(\nabla^{y^\mu}\big)^*\delta^{d+1}(x-y)\big\}\phi(y) + \partial^{y^\mu}B,\end{aligned} \tag{42}$$

in which

$$B = B(x,y) \coloneqq \delta^{d+1}(x-y)\phi(y); \tag{43}$$

going back to eq. (41),

$$\begin{aligned}&= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\{-\{\{\partial_{x^\mu} + \mathrm{i}eA_\mu(x)\}\phi(x)\}^*\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\} \\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\{-\partial_{x^\mu}C^\mu + \phi^*(x)\nabla_{x^\mu}\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\} \\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\big\{-\partial_{x^\mu}C^\mu + \phi^*(x)\big\{-(\nabla_{y^\mu})\big(\nabla^{y^\mu}\big)^*\delta^{d+1}(x-y)\big\}\phi(y) + \partial^{y^\mu}\nabla_{x^\mu}B\big\},\end{aligned} \tag{44}$$

in which

$$C^\mu = C^\mu(x,y) \coloneqq \phi^*(x)\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y). \tag{45}$$

Now eq. (40) can be written as (dropping the boundary terms)

$$\begin{aligned}
&= \int \mathrm{D}\phi\,\mathrm{D}\phi^*\exp\biggl\{-\mathrm{i}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\phi^*(x)D^{-1}(x,y)\phi(y)\biggr\}\\
&= \tilde{\mathcal{N}}\bigl\{\det\bigl[D^{-1}(x,y)\bigr]\bigr\}^{-1/2},
\end{aligned} \tag{46}$$

where

$$D^{-1}(x,y) \coloneqq \mathcal{D}_y^{-1}\delta^{d+1}(x-y), \qquad \mathcal{D}_y^{-1} \coloneqq +(\nabla_{y^\mu})\bigl(\nabla^{y^\mu}\bigr)^* + m^2. \tag{47}$$

$$\begin{aligned}
\mathit{\Gamma}_{\mathrm{W}}[A_\mu] &\equiv -\mathrm{i}\biggl(\ln\tilde{\mathcal{N}} - \frac{1}{2}\ln\det D^{-1}\biggr)\\
&= \frac{\mathrm{i}}{2}\,\mathrm{tr}_x\ln\bigl(\mathcal{N}^{-1}D^{-1}\bigr)\\
&= \frac{\mathrm{i}}{2}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\delta^{d+1}(x-y)\\
&\quad\cdot\Bigl\{-\mathrm{e}^{\mathrm{i}s\left(+\left(\nabla_{y^\mu}\right)\left(\nabla^{y^\mu}\right)^*+m^2+\mathrm{i}0^+\right)\delta^{d+1}(x-y)} + \mathrm{e}^{\mathrm{i}s\left(\mathcal{N}+\mathrm{i}0^+\right)}\Bigr\}.
\end{aligned} \tag{48}$$

[5]

A Notions and conventions

The metric convention is mostly positive, i.e. $\eta_{\mu\nu} \coloneqq \mathrm{diag}(-,+,+,\dots)$

Pauli matrices

$$\sigma^1 \coloneqq \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma^2 \coloneqq \begin{pmatrix} 0 & -\mathrm{i}\\ +\mathrm{i} & 0\end{pmatrix}, \quad \sigma^3 \coloneqq \begin{pmatrix} +1 & 0\\ 0 & -1\end{pmatrix}. \tag{49}$$

The γ-matrices satisfy [1, sec. 5]

$$[\gamma^\mu, \gamma^\nu]_+ \coloneqq 2\eta^{\mu\nu}\mathbf{1}_4. \tag{50}$$

$$\mathscr{J}^{\mu\nu} \coloneqq -\frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]_- \tag{51}$$

$$\sigma^{\mu\nu} \coloneqq \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]_- \equiv -2\mathscr{J}^{\mu\nu}. \tag{52}$$

In $(3+1)$ dimensions, choose the chiral representation

$$\gamma^\mu = -\mathrm{i}\begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \tag{53}$$

where

$$\sigma^\mu \coloneqq (1_2, +\vec{\sigma}), \qquad \bar{\sigma}^\mu \coloneqq (1_2, -\vec{\sigma}). \tag{54}$$

$$\begin{aligned}
\sigma^{\mu\nu} &\equiv -\frac{\mathrm{i}}{2}\begin{bmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{bmatrix} \\
&= \begin{cases} 0 & \mu = 0, \nu = 0; \\ \mathrm{i}\begin{bmatrix} +\sigma^j & 0 \\ 0 & -\sigma^j \end{bmatrix} & \mu = 0, \nu = j; \\ \mathrm{i}\begin{bmatrix} -\sigma^i & 0 \\ 0 & +\sigma^i \end{bmatrix} & \mu = i, \nu = 0; \\ \begin{bmatrix} +\epsilon^{ij}{}_k\sigma^k & 0 \\ 0 & +\epsilon^{ij}{}_k\sigma^k \end{bmatrix} & \mu = i, \nu = j. \end{cases}
\end{aligned} \tag{55}$$

B Fresnel functional integral

[6, ch. 10]

C Algebra

$$\left[a\partial_1\partial_2, x^1\right]_- = a\partial_2 \tag{56}$$

central.

Baker–Campbell–Hausdorff formula

$$\mathrm{e}^{+a\partial_1\partial_2}x^1\mathrm{e}^{-a\partial_1\partial_2} = x^1 + a\partial_2. \tag{57}$$

References

[1] Steven Weinberg. *The Quantum Theory of Fields*. Vol. I: Foundations. Cambridge University Press, 1995.

[2] Akaki Rusetsky and Ulf-G. Meißner. *Lecture on Advanced Theoretical Hadron Physics*. URL: https://www.hiskp.uni-bonn.de/index.php?id=239.

[3] Lev Davidovich Landau. "Diamagnetismus der Metalle". In: *Zeitschrift für Physik* 64.9-10 (Sept. 1930), pp. 629–637.

[4] Werner Karl Heisenberg and Hans Heinrich Euler. "Folgerungen aus der Diracschen Theorie des Positrons". In: *Zeitschrift für Physik* 98.11-12 (Nov. 1936), pp. 714–732.

[5] Victor Frederick Weisskopf. "Über die Elektrodynamik des Vakuums auf Grund der Quantentheorie des Elektrons". In: *Kong. Dan. Vid. Sel. Mat. Fys. Med.* 14N6 (1936), pp. 1–39.

[6] Ulrich Mosel. *Path Integrals in Field Theory*. Springer Berlin Heidelberg, 2004.