Effective action of scalar electrodynamics

Yi-Fan Wang*[3]

[3]Institut für Theoretische Physik, Universität zu Köln, Zülpicher Straße 77, 50937 Köln, Germany

1 Wick rotation

Complex Klein–Gordon action in flat space-time

$$S_{\mathrm{S}}[\phi,\phi^*] \coloneqq \int \mathrm{d}^{d+1}x \left\{-\eta^{\mu\nu}(\partial_\mu\phi)^*(\partial_\nu\phi) - m^2\phi^*\phi\right\}. \tag{1}$$

Interaction terms

$$S_{\mathrm{SM}}[A_\mu,\phi,\phi^*] \coloneqq \int \mathrm{d}^{d+1}x\, \eta^{\mu\nu}\left\{\mathrm{i}eA_\mu(-\phi^*\partial_\nu\phi + \phi\partial_\nu\phi^*) + e^2A_\mu A_\nu\phi^*\phi\right\}. \tag{2}$$

The total action for scalar electrodynamics reads

$$\begin{aligned} S[A_\mu,\phi,\phi^*] &\coloneqq S_{\mathrm{S}} + S_{\mathrm{SM}} + S_{\mathrm{Maxwell}} \\ &= \int \mathrm{d}^{d+1}x \left\{-(\nabla_\mu\phi)^*(\nabla^\mu\phi) - m^2\phi^*\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right\}, \end{aligned} \tag{3}$$

where

$$\nabla_\mu\phi \coloneqq (\partial_\mu + \mathrm{i}eA_\mu)\phi. \tag{4}$$

Wick rotation

$$x^4_{\mathrm{E}} = \mathrm{i}x^0, \quad A_4 = -\mathrm{i}A_0, \tag{5}$$

so that

$$\partial_{x^0} = \mathrm{i}\partial_{x^4_{\mathrm{E}}}, \quad F_{0i} = \mathrm{i}F_{4i}. \tag{6}$$

The Euclidean action reads

$$S_{\mathrm{E}}[A_I,\phi,\phi^*] = \int \mathrm{d}^D x_{\mathrm{E}} \left(\frac{1}{4}F_{IJ}F^{IJ} + (\nabla_I\phi)^*\left(\nabla^I\phi\right) + m^2\phi^*\phi\right). \tag{7}$$

*yfwang@thp.uni-koeln.de

2 Euclidean signature

Working with the Euclidean signature is much easier than in the Lorentzian signature.

2.1 Effective action

Generating functional and the *connected generating functional* (omitting subscript E for Euclidean systematically)

$$\mathcal{Z}\big[j^I, J, J^*\big] := \int \mathrm{D}A_I\,\mathrm{D}\phi^*\,\mathrm{D}\phi \exp\Big\{-S + \int \mathrm{d}^D x\,\big(j^I A_I + J^*\phi + \phi^* J\big)\Big\} \tag{8}$$

$$=: \exp\big\{-W\big[j^I, J, J^*\big]\big\}. \tag{9}$$

One would like to find an *effective action* for A

$$\mathit{\Gamma}_A[A^{\mathrm{c}}_I] + \int \mathrm{d}^D x\, j^I_{\mathrm{c}} A^{\mathrm{c}}_I := W\big[j^I_{\mathrm{c}}, 0, 0\big] \equiv -\ln \mathcal{Z}\big[j^I_{\mathrm{c}}, 0, 0\big], \tag{10}$$

$$\left.\frac{\updelta W[j,0,0]}{\updelta j^I}\right|_{j^I = j^I_{\mathrm{c}}} := A^{\mathrm{c}}_I. \tag{11}$$

One also has

$$\mathit{\Gamma}_A[A^{\mathrm{c}}_I] \equiv -\ln \int \mathrm{D}A_I\,\mathrm{D}\phi^*\,\mathrm{D}\phi \exp\{-S[A^{\mathrm{c}}_I + A_I, \phi, \phi^*]\}. \tag{12}$$

For a discussion, see e.g. [Schwartz, 2013, ch. 34].

The derivative term of ϕ can be rearranged

$$(\nabla_I\phi)^*\big(\nabla^I\phi\big) = \partial_I\big(\phi^*\nabla^I\phi\big) - \phi^*\nabla_I\nabla^I\phi. \tag{13}$$

Hence up to boundary terms,

$$S[A_I, \phi, \phi^*] = \int \mathrm{d}^D x\, \frac{1}{4} F_{IJ}F^{IJ} + \int \mathrm{d}^D x\,\mathrm{d}^D y\, \phi^*(x) M[A_I; x-y]\phi(y), \tag{14}$$

where

$$M[A_I; x-y] := \Big(-\nabla_{x^I}\nabla^{x^I} + m^2\Big)\delta^{(D)}(x-y), \tag{15}$$

see e.g. [Mosel, 2004, ch. 6] for details. Now the scalar field can formally be integrated, giving the (Euclidean) Euler–Heisenberg "effective action"

$$\mathcal{Z}\left[j^I,0,0\right] =: \int \mathrm{D}A_I \exp\left\{-\mathit{\Gamma}_{\mathrm{EH}}[A_I] + \int \mathrm{d}^D x\, j^I A_I\right\}, \tag{16}$$

$$\begin{aligned}\mathit{\Gamma}_{\mathrm{EH}}[A_I] - \int \mathrm{d}^D x\, \frac{1}{4} F_{IJ}F^{IJ} &= -\ln \int \mathrm{D}\phi^*\, \mathrm{D}\phi \exp\left\{-\int \mathrm{d}^D x \left((\nabla_I\phi)^* \left(\nabla^I\phi\right) + m^2\phi^*\phi\right)\right\} \\ &= -\ln \frac{1}{\det M[A_I; x-y)} = \operatorname{tr}\ln M,\end{aligned} \tag{17}$$

where normalisation is implicit in $\det M$. Equation (17) traces back to Heisenberg and Euler [1936], Weisskopf [1936].

Going back to eq. (10), one now has

$$\begin{aligned}\mathcal{Z}\left[j^I,0,0\right] &= \int \mathrm{D}A \exp\left\{-\int \mathrm{d}^D x \left(\frac{1}{4}F_{IJ}F^{IJ} - j^I A_I\right) - \operatorname{tr}\ln M\right\} \\ &= \mathcal{Z}_A[0] \left\langle j^I \,\middle|\, \exp(-\operatorname{tr}\ln M) \,\middle|\, j^I\right\rangle \equiv \mathcal{Z}_A[0] \left\langle A_I \,|\, \exp(-\operatorname{tr}\ln M) \,|\, A_I\right\rangle,\end{aligned} \tag{18}$$

where the expectation is defined as

$$\langle A_I \,|\, \mathcal{O} \,|\, A_I\rangle := \mathcal{Z}_A^{-1}[0] \int \mathrm{D}A\, \mathcal{O} \exp\left\{-\int \mathrm{d}^D x \left(\frac{1}{4}F_{IJ}F^{IJ} - j^I A_I\right)\right\}, \tag{19}$$

$$\mathcal{Z}_A\left[j^I\right] := \int \mathrm{D}A \exp\left\{-\int \mathrm{d}^D x \left(\frac{1}{4}F_{IJ}F^{IJ} - j^I A_I\right)\right\}. \tag{20}$$

2.2 Certain limit

In a certain limit (**which limit?**), the correlation between $\operatorname{tr}\ln M$ can be omitted, so that eq. (18) goes

$$\mathcal{Z}\left[j^I,0,0\right] \approx \mathcal{Z}_A[0]\exp(-\langle A_I \,|\, \operatorname{tr}\ln M \,|\, A_I\rangle), \tag{21}$$

and eq. (10) goes

$$\mathit{\Gamma}_A[A_I] \approx -\ln \mathcal{Z}_A[0] + \langle A_I \,|\, \operatorname{tr}\ln M \,|\, A_I\rangle\,. \tag{22}$$

2.3 World-line formalism

In eqs. (17), (18) and (22), $\operatorname{tr}\ln M$ is crucial.

Using the Schwinger integral representation Schwinger [1951] (up to normalisation)

$$\ln\alpha = -\int_0^{+\infty} \frac{\mathrm{d}s}{s}\,\mathrm{e}^{-\alpha s}, \tag{23}$$

one has (up to normalisation)

$$-\operatorname{tr}\ln M = \int_0^{+\infty} \frac{\mathrm{d}T}{T}\exp\left(-\frac{m^2T}{2M}\right)\operatorname{tr}\exp\left(-\frac{M}{2M}\right), \tag{24}$$

where T has the dimension of time, and M that of mass, which will both be eliminated later. Introduce the Hamiltonian of a non-relativistic point particle (**check sign!**)

$$H \coloneqq \frac{1}{2M}(P_I + eA_I)^2, \tag{25}$$

so that quantisation yields the following representation (**check sign!**)

$$\operatorname{tr}\exp\left(-\frac{M}{2M}\right) = \int_{-\infty}^{+\infty}\mathrm{d}x\,\left\langle x\,\middle|\,\mathrm{e}^{-\widehat{H}T}\,\middle|\,x\right\rangle \tag{26}$$

$$= \oint \mathrm{D}x\exp\left\{-\int_0^T \mathrm{d}T'\left(\frac{M}{2}\left(\frac{\mathrm{d}x^I}{\mathrm{d}T'}\right)^2 + \mathrm{i}eA_I\frac{\mathrm{d}x^I}{\mathrm{d}T'}\right)\right\}. \tag{27}$$

Rescaling $T' \eqqcolon \lambda T$ gives

$$-\operatorname{tr}\ln M = \int_0^{+\infty}\frac{\mathrm{d}T}{T}\exp\left(-\frac{m^2T}{2M}\right)\oint\mathrm{D}x\exp\left(-\frac{M}{2T}\int_0^1\mathrm{d}\lambda\,\dot{x}_I^2 - \mathrm{i}e\oint A_I\,\mathrm{d}x^I\right). \tag{28}$$

2.4 Euler–Heisenberg "effective action"

If the instanton magnetic field in eq. (27) is constant, the path integral can be performed exactly Feynman and Hibbs [1965], so that eq. (17) can be expressed in terms of an integral of T.

It is difficult to obtain a classical solution for the motion of a point particle in a more generic magnetic field, e.g. Kondo and Toshioka [1964]. Therefore the generalisation in this direction is limited.

2.5 World-line instanton approximations

In eq. (28), one may also perform the T integral first. Using the integral expression and the asymptotic expansion for a modified Bessel function

$$\mathrm{K}_0(x) = \frac{1}{2}\int_0^{+\infty} \frac{\mathrm{d}\mathrm{d}t}{\mathrm{d}t}\exp\left(-t - \frac{x^2}{4t}\right) \tag{29}$$

$$\approx \sqrt{\frac{\pi}{2x}}\,\mathrm{e}^{-x}, \qquad x \gg 1, \tag{30}$$

one has

$$-\operatorname{tr}\ln M = 2\oint \mathrm{D}x\,\mathrm{K}_0\left(m\sqrt{\int_0^1 \mathrm{d}\lambda\,\dot{x}_I^2}\right)\exp\left(-\mathrm{i}e\oint A_I\,\mathrm{d}x^I\right) \tag{31}$$

$$\approx \sqrt{\frac{2\pi}{m}}\oint \mathrm{D}x\left(\int_0^1 \mathrm{d}\lambda\,\dot{x}_I^2\right)^{-1/4}\exp\left(-m\sqrt{\int_0^1 \mathrm{d}\lambda\,\dot{x}_I^2} - \mathrm{i}e\oint A_I\,\mathrm{d}x^I\right), \tag{32}$$

where eq. (32) works for

$$m\sqrt{\int_0^1 \mathrm{d}\lambda\,\dot{x}_I^2} \gg 1 \qquad \text{or} \qquad \int_0^1 \mathrm{d}\lambda\,\dot{x}_I^2 \gg m^{-2}. \tag{33}$$

This idea traces back to Affleck et al. [1982]

2.6 Application of instanton approximation

Dunne and Schubert [2005]

3 Flat space-time (Lorentzian signature)

The content below needs revise.

Generating functional

$$\mathcal{Z}[j^\mu, J, J^*] := \int \mathrm{D}A\,\mathrm{D}\phi\,\mathrm{D}\phi^*\exp\left\{\mathrm{i}\left(S_0 + \int \mathrm{d}^{d+1}x\,(j^\mu A_\mu + J^*\phi + \phi^* J)\right)\right\}. \tag{34}$$

Effective action

$$\mathcal{Z}[j^\mu, 0, 0] =: \int \mathrm{D}A\exp\left\{\mathrm{i}\left(S_{\text{Maxwell}} + \varGamma_{\mathrm{W}}[A_\mu] + \int \mathrm{d}^{d+1}x\,j^\mu A_\mu\right)\right\}. \tag{35}$$

In other words,

$$\begin{aligned}\exp\{\mathrm{i}\mathit{\Gamma}_{\mathrm{W}}[A_\mu]\} &\coloneqq \int \mathrm{D}\phi\,\mathrm{D}\phi^*\exp\{\mathrm{i}(S_{\mathrm{CKG}}+S_{\mathrm{ICKGM}})\}\\ &\equiv \int \mathrm{D}\phi\,\mathrm{D}\phi^*\exp\Big\{\mathrm{i}\int \mathrm{d}^{d+1}x\,\{-(\nabla_\mu\phi)^*(\nabla^\mu\phi)-m^2\phi^*\phi\}\Big\}. \end{aligned}\tag{36}$$

The integral in the exponent can be manipulated; only the first term is essential

$$\begin{aligned}&\int \mathrm{d}^{d+1}x\,(-(\nabla_\mu\phi)^*(\nabla^\mu\phi))\\ &=\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\big(-(\nabla_{x^\mu}\phi(x))^*\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\big),\end{aligned}\tag{37}$$

where

$$\begin{aligned}\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y) &= \delta^{d+1}(x-y)\big\{\partial^{y^\mu}+\mathrm{i}eA^\mu(y)\big\}\phi(y)\\ &= \big\{-\big(\nabla^{y^\mu}\big)^*\delta^{d+1}(x-y)\big\}\phi(y)+\partial^{y^\mu}B,\end{aligned}\tag{38}$$

in which

$$B=B(x,y)\coloneqq \delta^{d+1}(x-y)\phi(y);\tag{39}$$

going back to eq. (37),

$$\begin{aligned}&=\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\{-\{\{\partial_{x^\mu}+\mathrm{i}eA_\mu(x)\}\phi(x)\}^*\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\}\\ &=\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\{-\partial_{x^\mu}C^\mu+\phi^*(x)\nabla_{x^\mu}\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\}\\ &=\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\{-\partial_{x^\mu}C^\mu+\phi^*(x)\big\{-(\nabla_{y^\mu})\big(\nabla^{y^\mu}\big)^*\delta^{d+1}(x-y)\big\}\phi(y)+\partial^{y^\mu}\nabla_{x^\mu}B\big\},\end{aligned}\tag{40}$$

in which

$$C^\mu=C^\mu(x,y)\coloneqq \phi^*(x)\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y).\tag{41}$$

Now eq. (36) can be written as (dropping the boundary terms)

$$\begin{aligned}&=\int \mathrm{D}\phi\,\mathrm{D}\phi^*\exp\Big\{-\mathrm{i}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\phi^*(x)D^{-1}(x,y)\phi(y)\Big\}\\ &=\tilde{\mathcal{N}}\big\{\det\big[D^{-1}(x,y)\big]\big\}^{-1/2},\end{aligned}\tag{42}$$

where

$$D^{-1}(x,y)\coloneqq \mathcal{D}_y^{-1}\delta^{d+1}(x-y),\qquad \mathcal{D}_y^{-1}\coloneqq +(\nabla_{y^\mu})\big(\nabla^{y^\mu}\big)^*+m^2.\tag{43}$$

$$
\begin{aligned}
\mathit{\Gamma}_{\mathrm{W}}[A_\mu] &\equiv -\mathrm{i}\left(\ln\tilde{\mathcal{N}} - \frac{1}{2}\ln\det D^{-1}\right)\\
&= \frac{\mathrm{i}}{2}\,\mathrm{tr}_x \ln\!\left(\mathcal{N}^{-1}D^{-1}\right)\\
&= \frac{\mathrm{i}}{2}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\delta^{d+1}(x-y)\\
&\quad\cdot\left\{-\mathrm{e}^{\mathrm{i}s\left(+\left(\nabla_{y^\mu}\right)\left(\nabla^{y^\mu}\right)^*+m^2+\mathrm{i}0^+\right)\delta^{d+1}(x-y)} + \mathrm{e}^{\mathrm{i}s\left(\mathcal{N}+\mathrm{i}0^+\right)}\right\}.
\end{aligned}
\tag{44}
$$

Weisskopf [1936]

A Notions and conventions

The metric convention is mostly positive, i.e. $\eta_{\mu\nu} \coloneqq \mathrm{diag}(-,+,+,\ldots)$

Pauli matrices

$$
\sigma^1 \coloneqq \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma^2 \coloneqq \begin{pmatrix}0 & -\mathrm{i}\\ +\mathrm{i} & 0\end{pmatrix},\quad \sigma^3 \coloneqq \begin{pmatrix}+1 & 0\\ 0 & -1\end{pmatrix}. \tag{45}
$$

The γ-matrices satisfy [Weinberg, 1995, sec. 5]

$$
[\gamma^\mu,\gamma^\nu]_+ \coloneqq 2\eta^{\mu\nu}\mathbf{1}_4. \tag{46}
$$

$$
\mathscr{J}^{\mu\nu} \coloneqq -\frac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]_- \tag{47}
$$

$$
\sigma^{\mu\nu} \coloneqq \frac{\mathrm{i}}{2}[\gamma^\mu,\gamma^\nu]_- \equiv -2\mathscr{J}^{\mu\nu}. \tag{48}
$$

In $(3+1)$ dimensions, choose the chiral representation

$$
\gamma^\mu = -\mathrm{i}\begin{bmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{bmatrix}, \tag{49}
$$

where

$$
\sigma^\mu \coloneqq (1_2, +\vec{\sigma}),\qquad \bar{\sigma}^\mu \coloneqq (1_2, -\vec{\sigma}). \tag{50}
$$

$$
\sigma^{\mu\nu} \equiv -\frac{\mathrm{i}}{2}\begin{bmatrix}\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\end{bmatrix}
$$

$$
= \begin{cases} 0 & \mu = 0, \nu = 0; \\ \mathrm{i}\begin{bmatrix} +\sigma^j & 0 \\ 0 & -\sigma^j \end{bmatrix} & \mu = 0, \nu = j; \\ \mathrm{i}\begin{bmatrix} -\sigma^i & 0 \\ 0 & +\sigma^i \end{bmatrix} & \mu = i, \nu = 0; \\ \begin{bmatrix} +\epsilon^{ij}{}_k\sigma^k & 0 \\ 0 & +\epsilon^{ij}{}_k\sigma^k \end{bmatrix} & \mu = i, \nu = j. \end{cases} \tag{51}
$$

B Fresnel functional integral

[Mosel, 2004, ch. 10]

C Algebra

$$
\left[a\partial_1\partial_2, x^1\right]_- = a\partial_2 \tag{52}
$$

central.

Baker–Campbell–Hausdorff formula

$$
\mathrm{e}^{+a\partial_1\partial_2}x^1\mathrm{e}^{-a\partial_1\partial_2} = x^1 + a\partial_2. \tag{53}
$$

D Schwinger integral

$$
-\int_\epsilon^{+\infty}\frac{\mathrm{d}t}{t}\,\mathrm{e}^{-\alpha t} = -\Gamma(0,\alpha\epsilon) = \gamma_\mathrm{E} + \ln\alpha + \ln\epsilon + O(\epsilon), \tag{54}
$$

where $\Gamma(a,z)$ is the incomplete Gamma function, γ_E is Euler's constant.

References

Ian K. Affleck, Orlando Alvarez, and Nicholas S. Manton. Pair production at strong coupling in weak external fields. *Nuclear Physics B*, 197(3):509–519, apr 1982. doi: 10.1016/0550-3213(82)90455-2.

Gerald V. Dunne and Christian Schubert. Worldline instantons and pair production in inhomogenous fields. *Physical Review D*, 72(10), nov 2005. doi: 10.1103/PhysRevD.72.105004.

Richard P. Feynman and Albert R. Hibbs. *Quantum Mechanics and Path Integrals*. McGraw-Hill College, 1965. ISBN 9780070206502.

Werner Karl Heisenberg and Hans Heinrich Euler. Folgerungen aus der diracschen theorie des positrons. *Zeitschrift für Physik*, 98(11-12):714–732, nov 1936. doi: 10.1007/bf01343663. URL https://doi.org/10.1007%2Fbf01343663.

Hiromichi Kondo and Katsushi Toshioka. Motion of a charged particle in inhomogeneous magnetic field. *Journal of the Physical Society of Japan*, 19(9):1731–1733, sep 1964. doi: 10.1143/jpsj.19.1731.

Ulrich Mosel. *Path Integrals in Field Theory*. Springer Berlin Heidelberg, 2004. doi: 10.1007/978-3-642-18797-1. URL https://doi.org/10.1007/978-3-642-18797-1.

Matthew D. Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, 2013. ISBN 9781107034730. URL www.cambridge.org/knowledge/isbn/item7298695/.

Julian Schwinger. On gauge invariance and vacuum polarization. *Physical Review*, 82(5):664–679, jun 1951. doi: 10.1103/PhysRev.82.664.

Steven Weinberg. *The Quantum Theory of Fields*, volume I: Foundations. Cambridge University Press, 1995. ISBN 978-0521550017. doi: 10.1017/cbo9781139644167. URL https://doi.org/10.1017%2Fcbo9781139644167.

Victor Frederick Weisskopf. Über die elektrodynamik des vakuums auf grund der quantentheorie des elektrons. *Kongelige Danske Videnskabernes Selskab Matematisk-Fysiske Meddelelser*, 14N6:1–39, 1936.