Effective action of scalar electrodynamics

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1 Wick rotation

Complex Klein–Gordon action in flat space-time

$$S_{\mathrm{S}}[\phi,\phi^*] \coloneqq \int \mathrm{d}^{d+1}x \left\{-\eta^{\mu\nu}(\partial_\mu\phi)^*(\partial_\nu\phi) - m^2\phi^*\phi\right\}. \tag{1}$$

Interaction terms

$$S_{\mathrm{SM}}[A_\mu,\phi,\phi^*] \coloneqq \int \mathrm{d}^{d+1}x\, \eta^{\mu\nu}\left\{\mathrm{i}eA_\mu(-\phi^*\partial_\nu\phi + \phi\partial_\nu\phi^*) + e^2A_\mu A_\nu\phi^*\phi\right\}. \tag{2}$$

The total action for scalar electrodynamics reads

$$\begin{aligned} S[A_\mu,\phi,\phi^*] &\coloneqq S_{\mathrm{S}} + S_{\mathrm{SM}} + S_{\mathrm{Maxwell}} \\ &= \int \mathrm{d}^{d+1}x \left\{-(\nabla_\mu\phi)^*(\nabla^\mu\phi) - m^2\phi^*\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right\}, \end{aligned} \tag{3}$$

where

$$\nabla_\mu\phi \coloneqq (\partial_\mu + \mathrm{i}eA_\mu)\phi. \tag{4}$$

Wick rotation

$$x_{\mathrm{E}}^4 = \mathrm{i}x^0, \quad A_4 = -\mathrm{i}A_0, \tag{5}$$

so that

$$\partial_{x^0} = \mathrm{i}\partial_{x_{\mathrm{E}}^4}, \quad F_{0i} = \mathrm{i}F_{4i}. \tag{6}$$

The Euclidean action reads

$$S_{\mathrm{E}}[A_I,\phi,\phi^*] = \int \mathrm{d}^D x_{\mathrm{E}} \left(\frac{1}{4}F_{IJ}F^{IJ} + (\nabla_I\phi)^*\left(\nabla^I\phi\right) + m^2\phi^*\phi\right). \tag{7}$$

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2 Euclidean signature

Working with the Euclidean signature is much easier than in the Lorentzian signature.

2.1 Effective action

Generating functional (omitting subscript E for Euclidean systematically)

$$\mathcal{Z}\left[j^I, J, J^*\right] \coloneqq \int \mathrm{D}A\, \mathrm{D}\phi^*\, \mathrm{D}\phi \exp\left\{-\left(S + \int \mathrm{d}^D x \left(j^I A_I + J^*\phi + \phi^* J\right)\right)\right\}. \tag{8}$$

The scalar fields are to be integrated out. The derivative term can be rearranged

$$(\nabla_I \phi)^* \left(\nabla^I \phi\right) = \partial_I \left(\phi^* \nabla^I \phi\right) - \phi^* \nabla_I \nabla^I \phi. \tag{9}$$

Hence up to boundary terms,

$$S = \int \mathrm{d}^D x \, \frac{1}{4} F_{IJ} F^{IJ} + \int \mathrm{d}^D x \, \mathrm{d}^D y \, \phi^*(x) M[A_I; x-y) \phi(y), \tag{10}$$

where

$$M[A_I; x-y) \coloneqq \left(-\nabla_{x^I} \nabla^{x^I} + m^2\right) \delta^{(D)}(x-y), \tag{11}$$

see [Mosel, 2004, ch. 6] for details. Now the scalar field can formally be integrated

$$\begin{aligned}
\exp\{-\mathcal{W}_{\mathrm{S}}[A_I; J^*, J]\} &\coloneqq \int \mathrm{D}\phi^*\, \mathrm{D}\phi \exp\left\{-\int \mathrm{d}^D x \left((\nabla_I \phi)^* \left(\nabla^I \phi\right) + m^2 \phi^* \phi + J^* \phi + \phi^* J\right)\right\} \\
&= \frac{1}{\det M[A_I; x-y)} \exp\left\{-\int \mathrm{d}^D x \, \mathrm{d}^D y \, J^*(x) D[A_I; x-y) J(y)\right\} \quad (12) \\
&\eqqcolon \frac{1}{\det M[A_I; x-y)} \exp\{-\mathcal{W}_{\mathrm{SJ}}[J^*, J]\}, \quad (13)
\end{aligned}$$

where the Euclidean Green's function

$$D[A_I; x-y) \coloneqq M^{-1}[A_I; x, y); \tag{14}$$

for vanishing A_I,

$$D[0; x-y) = \frac{1}{(2\pi)^{D/2}} \left(\frac{m}{x}\right)^{\frac{D}{2}-1} \mathrm{K}_{\frac{D}{2}-1}(m(x-y)), \tag{15}$$

calculated by Chao-Ming Jian (Everett You's notes). (**Check!**)

The connected generating functional of the scalar fields reads

$$\mathcal{W}_{\mathrm{S}}[A_I; J^*, J] = \mathcal{W}_{\mathrm{S}}[A_I; 0, 0] + \mathcal{W}_{\mathrm{SJ}}[A_I; J^*, J], \tag{16}$$

$$\mathcal{W}_{\mathrm{SE}}[A_I; 0, 0] = -\ln\left[(\det M)^{-1}\right] = \operatorname{tr}\ln M, \tag{17}$$

which traces back to Heisenberg and Euler [1936], Weisskopf [1936]. Setting the scalar current to zero, the Euclidean generating functional now reads

$$\begin{aligned}\mathcal{Z}\left[j^I, 0, 0\right] &= \int \mathrm{D}A \exp\left\{-\int \mathrm{d}^D x \left(\frac{1}{4} F_{IJ} F^{IJ} + j^I A_I\right) - \operatorname{tr}\ln M\right\} \\ &=: \mathcal{Z}_{\mathrm{A}}[0] \langle \exp(-\operatorname{tr}\ln M)\rangle_{j^I},\end{aligned} \tag{18}$$

where the average is defined as

$$\langle \mathcal{O}\rangle_{j^I} := \mathcal{Z}_{\mathrm{A}}^{-1}[0] \int \mathrm{D}A\, \mathcal{O} \exp\left\{-\int \mathrm{d}^D x \left(\frac{1}{4} F_{IJ} F^{IJ} + j^I A_I\right)\right\}, \tag{19}$$

$$\mathcal{Z}_{\mathrm{A}}\left[j^I\right] := \int \mathrm{D}A \exp\left\{-\int \mathrm{d}^D x \left(\frac{1}{4} F_{IJ} F^{IJ} + j^I A_I\right)\right\}. \tag{20}$$

The connected generating functional

$$\mathcal{W}\left[j^I, 0, 0\right] := -\ln \mathcal{Z}\left[j^I, 0, 0\right] = -\ln \mathcal{Z}_{\mathrm{A}}[0] - \ln \langle \exp(-\operatorname{tr}\ln M)\rangle_{j^I}. \tag{21}$$

2.2 World-line formalism

In eq. (18), $\operatorname{tr}\ln M$ is crucial. Using the Schwinger integral representation Schwinger [1951] (up to normalisation)

$$\ln \alpha = -\int_0^{+\infty} \frac{\mathrm{d}s}{s}\, \mathrm{e}^{-\alpha s}, \tag{22}$$

one has (up to normalisation)

$$-\operatorname{tr}\ln M = \int_0^{+\infty} \frac{\mathrm{d}T}{T} \exp\left(-\frac{m^2 T}{2M}\right) \operatorname{tr}\exp\left(-\frac{M}{2M}\right), \tag{23}$$

where T has the dimension of time, and M that of mass, which will both be eliminated later. Introduce the Hamiltonian of a non-relativistic point particle (**check sign!**)

$$H := \frac{1}{2M}(P_I + eA_I)^2, \tag{24}$$

so that quantisation yields the following representation (**check sign!**)

$$\operatorname{tr}\exp\left(-\frac{M}{2M}\right) = \int_{-\infty}^{+\infty} \mathrm{d}x \, \left\langle x \,\middle|\, \mathrm{e}^{-\widehat{H}T} \,\middle|\, x \right\rangle \tag{25}$$

$$= \oint \mathrm{D}x \exp\left\{-\int_0^T \mathrm{d}T' \left(\frac{M}{2}\left(\frac{\mathrm{d}x^I}{\mathrm{d}T'}\right)^2 + \mathrm{i}eA_I \frac{\mathrm{d}x^I}{\mathrm{d}T'}\right)\right\}. \tag{26}$$

Rescaling $T' =: \lambda T$ gives

$$-\operatorname{tr}\ln M = \int_0^{+\infty} \frac{\mathrm{d}T}{T} \exp\left(-\frac{m^2 T}{2M}\right) \oint \mathrm{D}x \exp\left(-\frac{M}{2T}\int_0^1 \mathrm{d}\lambda \, \dot{x}_I^2 - \mathrm{i}e \oint A_I \, \mathrm{d}x^I\right). \tag{27}$$

2.3 Euler–Heisenberg effective Lagrangian

If the instanton magnetic field in eq. (26) is constant, the path integral can be performed exactly Feynman and Hibbs [1965]. The result is the Euler–Heisenberg effective Lagrangian.

It is difficult to obtain a classical solution for the motion of a point particle in a more generic magnetic field, e.g. Kondo and Toshioka [1964]. Therefore the generalisation in this direction is limited.

2.4 World-line instanton approximations

In eq. (27), the T integral can be performed first. Using the integral expression and the asymptotic expansion for a modified Bessel function

$$\mathrm{K}_0(x) = \frac{1}{2}\int_0^{+\infty} \frac{\mathrm{d}\mathrm{d}t}{\mathrm{d}t} \exp\left(-t - \frac{x^2}{4t}\right) \tag{28}$$

$$\approx \sqrt{\frac{\pi}{2x}}\, \mathrm{e}^{-x} \qquad x \gg 1, \tag{29}$$

one has

$$-\operatorname{tr}\ln M = 2\oint \mathrm{D}x \, \mathrm{K}_0\left(m\sqrt{\int_0^1 \mathrm{d}\lambda \, \dot{x}_I^2}\right) \exp\left(-\mathrm{i}e \oint A_I \, \mathrm{d}x^I\right) \tag{30}$$

$$\approx \sqrt{\frac{2\pi}{m}} \oint \mathrm{D}x \left(\int_0^1 \mathrm{d}\lambda \, \dot{x}_I^2\right)^{-1/4} \exp\left(-m\sqrt{\int_0^1 \mathrm{d}\lambda \, \dot{x}_I^2} - \mathrm{i}e \oint A_I \, \mathrm{d}x^I\right), \tag{31}$$

where eq. (31) works for

$$m\sqrt{\int_0^1 \mathrm{d}\lambda\,\dot{x}_I^2} \gg 1 \quad \text{or} \quad \int_0^1 \mathrm{d}\lambda\,\dot{x}_I^2 \gg m^{-2}. \tag{32}$$

This idea traces back to Affleck et al. [1982]

Another loop-based approximation:

$$\langle \exp(-\operatorname{tr}\ln M)\rangle_{j^I} \approx \exp\Bigl(-\langle \operatorname{tr}\ln M\rangle_{j^I}\Bigr). \tag{33}$$

2.5 Application of instanton approximation

Dunne and Schubert [2005]

3 Flat space-time (Lorentzian signature)

The content below needs revise.

Generating functional

$$\mathcal{Z}[j^\mu, J^*, J] := \int \mathrm{D}A\,\mathrm{D}\phi\,\mathrm{D}\phi^* \exp\biggl\{\mathrm{i}\biggl(S_0 + \int \mathrm{d}^{d+1}x\,(j^\mu A_\mu + J^*\phi + \phi^* J)\biggr)\biggr\}. \tag{34}$$

Effective action

$$\mathcal{Z}[j^\mu, 0, 0] =: \int \mathrm{D}A \exp\biggl\{\mathrm{i}\biggl(S_{\mathrm{Maxwell}} + \mathit{\Gamma}_{\mathrm{W}}[A_\mu] + \int \mathrm{d}^{d+1}x\, j^\mu A_\mu\biggr)\biggr\}. \tag{35}$$

In other words,

$$\begin{aligned}\exp\{\mathrm{i}\mathit{\Gamma}_{\mathrm{W}}[A_\mu]\} &:= \int \mathrm{D}\phi\,\mathrm{D}\phi^* \exp\{\mathrm{i}(S_{\mathrm{CKG}} + S_{\mathrm{ICKGM}})\} \\ &\equiv \int \mathrm{D}\phi\,\mathrm{D}\phi^* \exp\biggl\{\mathrm{i}\int \mathrm{d}^{d+1}x\,\bigl\{-(\nabla_\mu\phi)^*(\nabla^\mu\phi) - m^2\phi^*\phi\bigr\}\biggr\}.\end{aligned} \tag{36}$$

The integral in the exponent can be manipulated; only the first term is essential

$$\begin{aligned}&\int \mathrm{d}^{d+1}x\,(-(\nabla_\mu\phi)^*(\nabla^\mu\phi)) \\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\bigl(-(\nabla_{x^\mu}\phi(x))^*\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\bigr),\end{aligned} \tag{37}$$

where

$$\begin{aligned}\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y) &= \delta^{d+1}(x-y)\big\{\partial^{y^\mu} + \mathrm{i}eA^\mu(y)\big\}\phi(y)\\ &= \big\{-\big(\nabla^{y^\mu}\big)^*\delta^{d+1}(x-y)\big\}\phi(y) + \partial^{y^\mu}B,\end{aligned} \tag{38}$$

in which

$$B = B(x,y) \coloneqq \delta^{d+1}(x-y)\phi(y); \tag{39}$$

going back to eq. (37),

$$\begin{aligned}&= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\big\{-\{\{\partial_{x^\mu} + \mathrm{i}eA_\mu(x)\}\phi(x)\}^*\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\big\}\\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\big\{-\partial_{x^\mu}C^\mu + \phi^*(x)\nabla_{x^\mu}\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\big\}\\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\big\{-\partial_{x^\mu}C^\mu + \phi^*(x)\big\{-(\nabla_{y^\mu})\big(\nabla^{y^\mu}\big)^*\delta^{d+1}(x-y)\big\}\phi(y) + \partial^{y^\mu}\nabla_{x^\mu}B\big\},\end{aligned} \tag{40}$$

in which

$$C^\mu = C^\mu(x,y) \coloneqq \phi^*(x)\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y). \tag{41}$$

Now eq. (36) can be written as (dropping the boundary terms)

$$\begin{aligned}&= \int \mathrm{D}\phi\,\mathrm{D}\phi^*\exp\Big\{-\mathrm{i}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\phi^*(x)D^{-1}(x,y)\phi(y)\Big\}\\ &= \tilde{\mathcal{N}}\big\{\det\big[D^{-1}(x,y)\big]\big\}^{-1/2},\end{aligned} \tag{42}$$

where

$$D^{-1}(x,y) \coloneqq \mathcal{D}_y^{-1}\delta^{d+1}(x-y), \qquad \mathcal{D}_y^{-1} \coloneqq +(\nabla_{y^\mu})\big(\nabla^{y^\mu}\big)^* + m^2. \tag{43}$$

$$\begin{aligned}\mathit{\Gamma}_{\mathrm{W}}[A_\mu] &\equiv -\mathrm{i}\Big(\ln\tilde{\mathcal{N}} - \frac{1}{2}\ln\det D^{-1}\Big)\\ &= \frac{\mathrm{i}}{2}\,\mathrm{tr}_x\ln\big(\mathcal{N}^{-1}D^{-1}\big)\\ &= \frac{\mathrm{i}}{2}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\delta^{d+1}(x-y)\\ &\quad\cdot\Big\{-\mathrm{e}^{\mathrm{i}s\left(+(\nabla_{y^\mu})\left(\nabla^{y^\mu}\right)^* + m^2 + \mathrm{i}0^+\right)\delta^{d+1}(x-y)} + \mathrm{e}^{\mathrm{i}s\left(\mathcal{N}+\mathrm{i}0^+\right)}\Big\}.\end{aligned} \tag{44}$$

Weisskopf [1936]

A Notions and conventions

The metric convention is mostly positive, i.e. $\eta_{\mu\nu} \coloneqq \mathrm{diag}(-,+,+,\ldots)$

Pauli matrices

$$\sigma^1 \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 \coloneqq \begin{pmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 \coloneqq \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{45}$$

The γ-matrices satisfy [Weinberg, 1995, sec. 5]

$$[\gamma^\mu, \gamma^\nu]_+ \coloneqq 2\eta^{\mu\nu}\mathbf{1}_4. \tag{46}$$

$$\mathscr{J}^{\mu\nu} \coloneqq -\frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]_- \tag{47}$$

$$\sigma^{\mu\nu} \coloneqq \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]_- \equiv -2\mathscr{J}^{\mu\nu}. \tag{48}$$

In $(3+1)$ dimensions, choose the chiral representation

$$\gamma^\mu = -\mathrm{i}\begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \tag{49}$$

where

$$\sigma^\mu \coloneqq (1_2, +\vec{\sigma}), \qquad \bar{\sigma}^\mu \coloneqq (1_2, -\vec{\sigma}). \tag{50}$$

$$\begin{aligned} \sigma^{\mu\nu} &\equiv -\frac{\mathrm{i}}{2}\begin{bmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{bmatrix} \\ &= \begin{cases} 0 & \mu = 0, \nu = 0; \\ \mathrm{i}\begin{bmatrix} +\sigma^j & 0 \\ 0 & -\sigma^j \end{bmatrix} & \mu = 0, \nu = j; \\ \mathrm{i}\begin{bmatrix} -\sigma^i & 0 \\ 0 & +\sigma^i \end{bmatrix} & \mu = i, \nu = 0; \\ \begin{bmatrix} +\epsilon^{ij}{}_k\sigma^k & 0 \\ 0 & +\epsilon^{ij}{}_k\sigma^k \end{bmatrix} & \mu = i, \nu = j. \end{cases} \end{aligned} \tag{51}$$

B Fresnel functional integral

[Mosel, 2004, ch. 10]

C Algebra

$$\left[a\partial_1\partial_2, x^1\right]_- = a\partial_2 \tag{52}$$

central.

Baker–Campbell–Hausdorff formula

$$\mathrm{e}^{+a\partial_1\partial_2} x^1 \mathrm{e}^{-a\partial_1\partial_2} = x^1 + a\partial_2. \tag{53}$$

D Schwinger integral

$$-\int_\epsilon^{+\infty} \frac{\mathrm{d}t}{t}\, \mathrm{e}^{-\alpha t} = -\Gamma(0, \alpha\epsilon) = \gamma_\mathrm{E} + \ln\alpha + \ln\epsilon + O(\epsilon), \tag{54}$$

where $\Gamma(a, z)$ is the incomplete Gamma function, γ_E is Euler's constant.

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