Cheat Sheet for Pseudo-Riemannian Geometry

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0.1 Levi-Civita connection

An affine connection ∇ is called a *Levi-Civita connection* if

- it preserves the metric, i.e. $\nabla g = 0$.
- it is torsion-free, i.e. for any vector fields X and Y we have $\nabla_X Y - \nabla_Y X = [X, Y]$, where $[X, Y]$ is the Lie bracket of the vector fields X and Y.

  Condition 1 above is sometimes referred to as compatibility with the metric, and condition 2 is sometimes called symmetry, c.f. Do Carmo's text.

  If a Levi-Civita connection exists, it is uniquely determined. Using conditions 1 and the symmetry of the metric tensor g we find:

$$\begin{aligned} & X\big(g(Y,Z)\big) + Y\big(g(Z,X)\big) - Z\big(g(Y,X)\big) \\ &= g(\nabla_X Y + \nabla_Y X, Z) + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_Y Z - \nabla_Z Y, X). \end{aligned} \tag{1}$$

By condition 2, the right hand side is equal to

$$2g(\nabla_X Y, Z) - g\big([X,Y],Z\big) + g\big([X,Z],Y\big) + g\big([Y,Z],X\big), \tag{2}$$

so we find the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X\big(g(Y,Z)\big) + Y\big(g(Z,X)\big) - Z\big(g(X,Y)\big) \\ &\quad + g\big([X,Y],Z\big) - g\big([Y,Z],X\big) - g\big([X,Z],Y\big) \\ &= \mathbb{L}_Y g(X,Z) + \big(\mathbb{d}Y^\flat\big)(X,Z). \end{aligned} \tag{3}$$

Since Z is arbitrary, this uniquely determines $\nabla_X Y$. Conversely, using the last line as a definition one shows that the expression so defined is a connection compatible with the metric, i.e. is a Levi-Civita connection.