Cosmological Perturbations

Yi-Fan Wang (王 一帆)

May 7, 2018

Most of the conventions and notations in [6, ch. 5] will be followed.

Suppose the metric can be expanded up to the linear order as

$$g = g^{(0)} + \epsilon g^{(1)} + O(\epsilon^2). \tag{1}$$

The background metric $g^{(0)}$ takes the Robertson–Walker form

$$g^{(0)}_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu = -N^2(t) \, \mathrm{d}t^2 + a^2(t) \, \mathrm{d}\Omega^2_{3\mathrm{F}}, \tag{2}$$

in which $\mathrm{d}\Omega^2_{3\mathrm{F}} = \mathrm{d}\chi^2 + \chi^2\left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2\right)$ is the dimensionless flat spatial metric. The linear perturbation can be decomposed into scalar, vector and tensor parts

$$g^{(1)}_{00} = -E, \tag{3}$$
$$g^{(1)}_{i0} = g^{(1)}_{i0} = F_{,i} + G_i, \tag{4}$$
$$g^{(1)}_{ij} = A\delta_{ij} + B_{,i,j} + C_{i,j} + C_{j,i} + D_{ij}. \tag{5}$$

Here one has some weird condition, where the contractions do not follow the one-up-one-down tradition

$$C_{i,i} = G_{i,i} = 0, \quad D_{ij,i} = 0, \quad D_{ii} = 0. \tag{6}$$

1 Metric perturbation under diffeomorphism

Consider a diffeomorphism generated by ξ^μ

$$x^\mu \to \overline{x}^\mu = x^\mu - \epsilon\xi^\mu. \tag{7}$$

The generator ξ^μ can in turn be decomposed into $\xi_0 = \zeta$, $\xi_i = \xi^{\mathrm{S}}_{,i} + \xi^{\mathrm{V}}_i$.

One has here again some weird condition, where the contractions do not follow the one-up-one-down tradition

$$\xi^{\mathrm{V}}_{i,i} = 0. \tag{8}$$

The Lie derivative of the metric $\mathbb{L}_\xi g$ is

$$\left(\mathbb{L}_\xi g\right)_{\mu\nu} = \xi^\lambda g_{\mu\nu,\lambda} + \xi^\lambda{}_{,\mu} g_{\lambda\nu} + \xi^\lambda{}_{,\nu} g_{\mu\lambda}. \tag{9}$$

In components and expansion, these are

$$\left(\mathbb{L}_\xi g\right)_{00} = 2\dot{\zeta} - 2\zeta\frac{\dot{N}}{N} + O(\epsilon), \tag{10}$$

$$\left(\mathbb{L}_\xi g\right)_{i0} = \left(\mathbb{L}_\xi g\right)_{0i} = \left(\zeta - 2\frac{\dot{a}}{a}\xi^{\mathrm{S}} + \dot{\xi}^{\mathrm{S}}\right)_{,i} + \left(-2\frac{\dot{a}}{a}\xi^{\mathrm{V}}_i + \dot{\xi}^{\mathrm{V}}_i\right) + O(\epsilon), \tag{11}$$

$$\left(\mathbb{L}_\xi g\right)_{ji} = \left(\mathbb{L}_\xi g\right)_{ij} = -\frac{2a\dot{a}}{N^2}\zeta\delta_{ij} + 2\xi^{\mathrm{S}}_{,i,j} + \xi^{\mathrm{V}}_{i,j} + \xi^{\mathrm{V}}_{j,i} + O(\epsilon). \tag{12}$$

2 Scalar perturbations

$$-N^2 - \epsilon E + O(\epsilon^2) \to -N^2 - \epsilon E + \epsilon\left(2\dot{\zeta} - 2\zeta\frac{\dot{N}}{N}\right) + O(\epsilon^2), \tag{13}$$

so one can write

$$\mathbb{L}_\xi E = -2\dot{\zeta} + 2\zeta\frac{\dot{N}}{N}. \tag{14}$$

Similarly one can read-off

$$\mathbb{L}_\xi F = \zeta - 2\frac{\dot{a}}{a}\xi^{\mathrm{S}} + \dot{\xi}^{\mathrm{S}}, \tag{15}$$

$$\mathbb{L}_\xi A = -\frac{2a\dot{a}}{N^2}\zeta, \tag{16}$$

$$\mathbb{L}_\xi B = 2\xi^{\mathrm{S}}. \tag{17}$$

The four scalar perturbations are generated by ζ and ξ^{S}, so that only two independent perturbations exists. It is clear that

$$\mathbb{L}_\xi\left(\frac{F}{a} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{B}{2a}\right) = \frac{\zeta}{a}. \tag{18}$$

One can verify that

$$\mathbb{L}_\xi\left\{\frac{E}{2N} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{a}{N}\left(\frac{F}{a} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{B}{2a}\right)\right]\right\} = 0, \tag{19}$$

$$\mathbb{L}_\xi\left\{\frac{A}{2} + \frac{a^2\dot{a}}{N^2}\left(\frac{F}{a} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{B}{2a}\right)\right\} = 0. \tag{20}$$

3 Vector perturbations

4 Tensor perturbations

5 Scalar field perturbation under diffeomorphism

6 Perturbation in Arnowitt–Deser–Misner Hamiltonian formalism

Up to boundary terms, the Hamiltonian action for General Relativity in terms of Arnowitt–Deser–Misner variables is [3, ch.4.2.2]

$$S = \int \mathbb{d}t\, \mathbb{d}x^3 \left\{ \mathfrak{p}^{ij}\dot{h}_{ij} + \mathfrak{P}\dot{N} + \mathfrak{P}^i\dot{N}_i - N\mathfrak{H}^\perp - N_i\mathfrak{H}^i - \mathfrak{P}V - \mathfrak{P}^iV_i \right\}, \tag{21}$$

$$\mathfrak{H}^\perp = 2\mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl} - \frac{\sqrt{\mathfrak{h}}}{2\varkappa}R[h] = 2\mathfrak{H}^{ijkl}h_{ij}h_{kl} - \frac{\sqrt{\mathfrak{h}}}{2\varkappa}R[h], \tag{22}$$

$$\mathfrak{H}^i = -2\mathfrak{p}^{ij}{}_{|j}, \mathfrak{G}_{ijkl} \qquad := \frac{1}{2\sqrt{\mathfrak{h}}}\left(h_{ik}h_{jl} + h_{il}h_{kj} - \right. \tag{23}$$

$$\mathfrak{H}^{ijkl} := \frac{1}{2\sqrt{\mathfrak{h}}}\left(\mathfrak{p}^{ik}\mathfrak{p}^{jl} + \mathfrak{p}^{il}\mathfrak{p}^{kj} - \mathfrak{p}^{ij}\mathfrak{p}^{kl}\right), \tag{24}$$

where $\{V, V_i\}$ are velocity of N and N_i and play the role of Lagrange multipliers. Technical details about the boundary terms can be found in [5, ch. 4.2] and the references therein. Note that $\left\{N, N_i, h_{ij}; \mathfrak{P}, \mathfrak{P}^i, \mathfrak{p}^{ij}\right\}$ are not the unique choice of canonical variables for General Relativity in Hamiltonian formalism; instead, they are a special parametrisation of the phase space. One can also choose the components of the original four-metric and their conjugate momenta $\left\{g_{\mu\nu}; \mathfrak{p}^{\mu\nu}\right\}$ as canonical variables, as Dirac has done in [2]. The two approaches are different in some subtle aspects; see [4] for a comparison.

Gauge transformations in the Arnowitt–Deser–Misner canonical variables are generated by [1]

$$\begin{aligned} G = -\int \mathbb{d}^3x \Big\{ &\left[\xi_\perp\left(\mathfrak{H}^\perp + N_{|i}\mathfrak{P}^i + \left(N\mathfrak{P}^i\right)_{|i} + \left(N_i\mathfrak{P}\right)^{|i}\right) + \dot{\xi}_\perp\mathfrak{P}\right] \\ &+ \left[\xi_i\left(\mathfrak{H}^i + N_j{}^{|i}\mathfrak{P}^j + \left(N_j\mathfrak{P}^i\right)^{|j} + N^{|i}\mathfrak{P}\right) + \dot{\xi}_i\mathfrak{P}^i\right]\Big\}. \end{aligned} \tag{25}$$

Possible boundary terms have not been discussed so far. The infinitesimal gauge transformation of $\{N, N_i\}$ is

$$\delta N = [N, G]_{\mathrm{P}} = \xi_\perp{}^{|i}N_z i - \dot{\xi}_\perp - \xi_i N^{|i}, \tag{26}$$

$$\delta N_i = -\xi_\perp N_{|i} + \xi_{\perp|i}N - \xi_j N_i{}^{|j} + \xi_i{}^{|j}N_j - \dot{\xi}_i, \tag{27}$$

which can be found in [4]. Transformations for g_{ij} and the momenta have to be worked out as

$$\delta\mathfrak{P} = -\left(\xi_\perp \mathfrak{P}^i\right)_{|i} - \xi_{\perp|i}\mathfrak{P}^i - (\xi_i \mathfrak{P})^{|i}, \tag{28}$$

$$\delta\mathfrak{P}^i = -\xi_{\perp|i}\mathfrak{P} - \left(\xi_j \mathfrak{P}^i\right)^{|j} - \xi_j{}^{|i}\mathfrak{P}^j, \tag{29}$$

where only the primary constraints are involved;

$$\begin{aligned}
\delta g_{ij} &= -\frac{\partial}{\partial \mathfrak{p}^{ij}}(\xi_\perp \mathfrak{H}^\perp + \xi_i \mathfrak{H}^i) \\
&= -\xi^\perp \frac{2\varkappa}{\sqrt{\mathfrak{h}}}\left(h_{ik}h_{jl} + h_{il}h_{kj} - h_{ij}h_{kl}\right)\mathfrak{p}^{kl} - \xi_{i|j} - \xi_{j|i},
\end{aligned} \tag{30}$$

$$\begin{aligned}
\delta \mathfrak{p}^{ij} &= \frac{\partial}{\partial g_{ij}}(\xi_\perp \mathfrak{H}^\perp + \xi_i \mathfrak{H}^i) \\
&= \xi_\perp \Bigg\{ \frac{\varkappa}{\sqrt{\mathfrak{h}}}\Bigg[(4\delta^i{}_k \delta^j{}_m - h^{ij}h_{km})h_{ln} \\
&\qquad - \frac{1}{2}(4\delta^i{}_k\delta^j{}_l - h^{ij}h_{kl})h_{mn}\Bigg]\mathfrak{p}^{kl}\mathfrak{p}^{mn} \\
&\qquad + \frac{\sqrt{\mathfrak{h}}}{2\varkappa}G^{ij}[h]\Bigg\} + \frac{\sqrt{\mathfrak{h}}}{2\varkappa}\left(\xi_{\perp|k}{}^{|k}h^{ij} - \xi_\perp^{|(i|j)}\right) \\
&\quad - \left(\xi_k \mathfrak{p}^{ij}\right)^{|k} + 2\xi_{k|l}h^{k(i}\mathfrak{p}^{j)l},
\end{aligned} \tag{31}$$

where $G^{ij}[h] = R^{ij}[h] - h^{ij}R[h]/2$, and only the secondary constraints are involved. In eq. (31), the first two lines come from the variation of the 'kinetic' term in $\mathfrak{H}^\perp$, the third comes from the 'potential' term in $\mathfrak{H}^\perp$, and the last line from $\mathfrak{H}^i$. The results can be checked with [5, p. 4.2.7].

6.1 Expansion of the action with fluctuations

Some useful results

First variations

First variation of h^{ij}

$$\delta h^{ij} = -h^{ik}h^{jl}\,\delta h_{kl} = -h^{i(k}h^{l)j}\,\delta h_{kl}. \tag{32}$$

First variation of $\mathfrak{h} = \det h_{ij}$

$$\delta\mathfrak{h} = \mathfrak{h}h^{ij}\,\delta h_{ij}. \tag{33}$$

First variation of $\Gamma^i{}_{jk}$

$$\delta\Gamma^i{}_{jk} = \frac{1}{2}h^{il}\Big\{-\left(\delta h_{jk}\right)_{|l} + \left(\delta h_{kl}\right)_{|j} + \left(\delta h_{lj}\right)_{|k}\Big\}. \tag{34}$$

First variation of $R_{ij}[h]$ and $R^{ij}[h]$

$$\delta R_{ij}[h] = \left(\delta\Gamma^k{}_{ij}\right)_{|k} - \left(\delta\Gamma^k{}_{ik}\right)_{|j}, \tag{35}$$

$$\delta R^{ij}[h] = -2R^{ki}h^{jl}\,\delta h_{kl} + h^{ik}\,\bar{\delta}u^{jl}{}_{k|l}, \tag{36}$$

where

$$\bar{\delta}u^{ij}{}_k := h^{il}\,\delta\Gamma^j{}_{kl} - h^{ij}\,\delta\Gamma^l{}_{kl}. \tag{37}$$

Equation (35) can be obtained by using normal coordinates.

First variation of $\sqrt{\mathfrak{h}}\,R[h]$

$$\delta\left(\sqrt{\mathfrak{h}}R[h]\right) = \sqrt{\mathfrak{h}}\left\{-G^{ij}[h]\,\delta h_{ij} + \bar{\delta}u^{ji}{}_{j|i}\right\}. \tag{38}$$

First variation of $\mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl} \equiv \mathfrak{H}^{ijkl}h_{ij}h_{kl}$

$$\delta\left(\mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl}\right) = \delta h_{ij}\left(-\frac{1}{2}h^{ij}\mathfrak{H}^{klmn}h_{kl}h_{mn} + 2\mathfrak{H}^{ijkl}h_{kl}\right) + \delta\mathfrak{p}^{ij}\,2\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}. \tag{39}$$

First variation of $\mathfrak{H}^{\perp}$

$$\delta\mathfrak{H}^{\perp} = \delta h_{ij}\left\{2\varkappa\left[-\frac{1}{2}h^{ij}\mathfrak{H}^{klmn}h_{kl}h_{mn} + 2\mathfrak{H}^{ijkl}h_{kl}\right] + \frac{\sqrt{\mathfrak{h}}}{2\varkappa}\left(G^{ij}[h] - \bar{\delta}u^{ji}{}_{j|i}\right)\right\} + \delta\mathfrak{p}^{ij}\,4\varkappa\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}, \tag{40}$$

where the second line comes from variation of the three Ricci scalar. The last term in the second line is expected to be cancelled by the boundary terms.

First variation of $\mathfrak{H}^{ijkl}h_{kl}$

$$\delta\left(\mathfrak{H}^{ijkl}h_{kl}\right) = \delta h_{kl}\left(-\frac{1}{2}\mathfrak{H}^{ijmn}h^{kl}h_{mn} + \mathfrak{H}^{ijkl}\right) + \delta\mathfrak{p}^{kl}\left(\delta^i{}_k\mathfrak{G}^j{}_{lmn} + \delta^i{}_m\mathfrak{G}^j{}_{nkl}\right)\mathfrak{p}^{mn}. \tag{41}$$

First variation of $\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}$

$$\delta\left(\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}\right) = \delta h_{kl}\left(-\frac{1}{2}\mathfrak{G}_{ijmk}h^{kl} + \delta^l{}_i\mathfrak{G}^k{}_{jmn} + \delta^l{}_m\mathfrak{G}^k{}_{nij}\right)\mathfrak{p}^{mn} + \delta p^{kl}\,\mathfrak{G}_{ijkl} \tag{42}$$

First variation of $\sqrt{\mathfrak{h}}\,G^{ij}[h]$

$$\delta\left(\sqrt{\mathfrak{h}}\,G^{ij}[h]\right) = \sqrt{\mathfrak{h}}\left\{\delta h_{kl}\cdot \left(-\frac{1}{2}\right)\left(R^{ik}h^{lj} + R^{il}h^{kj} - R^{ij}h^{kl} + h^{ik}G^{lj} + h^{il}G^{kj} - h^{ij}G^{kl}\right) + \left(h^{il}\,\bar{\delta}u^{jk}{}_l - \frac{1}{2}h^{ij}\,\bar{\delta}u^{lk}{}_l\right)_{|k}\right\}. \tag{43}$$

Second variations

Second variation of h^{ij}

$$\delta^2 h^{ij} = \left(h^{im} h^{jl} h^{kn} + h^{ik} h^{jm} h^{ln}\right) \delta h_{kl}\, \delta h_{mn} \tag{44}$$

Second variation of $\mathfrak{h} = \det h_{ij}$

$$\delta^2 \mathfrak{h} = -\frac{1}{4} \mathfrak{h} \left(h^{ik} h^{jl} + h^{il} h^{kj} - h^{ij} h^{kl}\right) \delta h_{ij}\, \delta h_{kl}. \tag{45}$$

First variation of $\left(\delta h_{ij}\right)_{|k}$

$$\delta \left\{ \left(\delta h_{ij}\right)_{|k} \right\} = -2 \delta \Gamma^l{}_{k(i}\, \delta h_{j)l}. \tag{46}$$

Second variation of $\Gamma^i{}_{jk}$

$$\delta^2 \Gamma^i{}_{jk} = -\frac{1}{2} h^{im}\, \delta \Gamma^l{}_{jk}\, \delta h_{lm}. \tag{47}$$

Second variation of $\sqrt{\mathfrak{h}} R[h]$

$$\begin{aligned} \delta^2 \left(\sqrt{\mathfrak{h}} R\right) = \sqrt{\mathfrak{h}} \Bigg\{ & \delta h_{ij}\, \delta h_{kl} \left[\frac{1}{4} R \left(h^{ik} h^{jl} + h^{il} h^{jk} - h^{ij} h^{kl}\right) \right. \\ & \left. + \frac{1}{2} \left(R^{ij} h^{kl} + R^{kl} h^{ij} - R^{ik} h^{jl} - R^{il} h^{jk} - R^{jk} h^{il} - R^{il} h^{kj}\right) \right] \\ & + h^{ik}\, \bar{\delta} u^{jl}{}_{k|l} + \delta \left(\bar{\delta} u^{ji}{}_{j|i} \right) \Bigg\} \end{aligned} \tag{48}$$

In a general background, the second variations of the quantities are much more tedious.

In Robertson–Walker background,

References

[1] Leonardo Castellani. "Symmetries in constrained Hamiltonian systems". In: *Annals of Physics* 143.2 (Oct. 1982), pp. 357–371. DOI: `10.1016/0003-4916(82)90031-8`.

[2] Paul A. M. Dirac. "The Theory of Gravitation in Hamiltonian Form". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 246.1246 (Aug. 1958), pp. 333–343. DOI: `10.1098/rspa.1958.0142`.

[3] Claus Kiefer. *Quantum Gravity*. 3rd ed. Oxford University Press, Apr. 2012. DOI: `10.1093/acprof:oso/9780199585205.001.0001`.

[4] Natalia Kiriushcheva and Sergei Kuzmin. “The Hamiltonian formulation of General Relativity: myths and reality”. In: *Central Eur.J.Phys.9:576-615,2011* 9.3 (Aug. 31, 2008). DOI: 10.2478/s11534-010-0072-2. arXiv: 0809.0097 [gr-qc].

[5] Eric Poisson. *A Relativist's Toolkit*. Cambridge University Press, 2004. DOI: 10.1017/cbo9780511606601.

[6] Steven Weinberg. *Cosmology*. Oxford University Press, 2008. ISBN: 9780198526827. URL: https://global.oup.com/academic/product/cosmology-9780198526827.