Classical p-brane

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1 Point particle: 0-brane

1.1 Linear action

Arc-length term

$$S_1[x^M] := -m \int_\gamma \mathrm{d}s = -m \int_\gamma \mathrm{d}\lambda \sqrt{|g_{MN}\dot{x}^M \dot{x}^N|}. \tag{1}$$

Scalar term
1-form term

1.2 Quadratic action

Auxiliary term

$$S_2[e, x^M] := \frac{1}{2} \int_\gamma \mathrm{d}\lambda \, e(e^{-2} g_{MN} \dot{x}^M \dot{x}^N - m^2). \tag{2}$$

2 Classical bosonic string: 1-brane

[5] contains a

2.1 Nambu–Goto action

The action reads [6, 4]

$$S_{\mathrm{NG}}[X^M] := -T \int_\Sigma \mathrm{d}A = -T \int_\Sigma \mathrm{d}^2\sigma \sqrt{\left|\tilde{\tilde{\psi}}\right|}, \tag{3}$$

where

$$\tilde{\tilde{\psi}} := \det \psi_{\alpha\beta} \equiv \psi_{11}\psi_{22} - \psi_{12}\psi_{21} \tag{4}$$

is the metric determinant, $\alpha, \beta, \ldots = 1, 2$ are the world-sheet indices, (Σ^α) are the world-sheet coordinates,

$$\psi_{\alpha\beta} := g_{MN} X^M{}_{,\alpha} X^N{}_{,\beta}, \tag{5}$$

is the induced metric on the world-sheet, $P, N, \ldots = 0, 1, \ldots d$ are the target-space indices, $X^M = X^M(\Sigma^\alpha)$ are the world-sheet [8] coordinates. The immersion map $X^M(\sigma^\alpha)$ are the dynamical variables.

The inverse metric can also be expressed in a closed form

$$\begin{aligned}
\psi^{\alpha\beta} &= \frac{1}{(2-1)!}\tilde{\epsilon}^{\alpha\gamma}\tilde{\epsilon}^{\beta\delta}\tilde{\tilde{\psi}}^{-1}\psi_{\gamma\delta} \\
&= \tilde{\epsilon}^{\alpha\gamma}\tilde{\epsilon}^{\beta\delta}\tilde{\tilde{\psi}}^{-1}\psi_{\gamma\delta} = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\psi_{\gamma\delta} &(6)\\
&= \tilde{\epsilon}^{\alpha\gamma}\tilde{\epsilon}^{\beta\delta}\tilde{\tilde{\psi}}^{-1} g_{P\Sigma} X^P{}_{,\gamma} X^\Sigma{}_{,\delta} = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta} g_{P\Sigma} X^P{}_{,\gamma} X^\Sigma{}_{,\delta} &(7)\\
&\equiv \frac{1}{\tilde{\tilde{\psi}}}\begin{pmatrix} \psi_{22} & -\psi_{12} \\ -\psi_{21} & \psi_{11}\psi \end{pmatrix}^{\alpha\beta}. &(8)
\end{aligned}$$

Unfortunately this is not useful.

Variation of the induced metric determinant can be expressed in terms of that of the induced metric

$$\delta\tilde{\tilde{\psi}} = \tilde{\tilde{\psi}}\psi^{\alpha\beta}\,\delta\psi_{\alpha\beta}. \tag{9}$$

Variation of the induced metric in terms of the world-sheet coordinates reads

$$\delta\psi_{\alpha\beta} = X^N{}_{,\alpha}\big(2 g_{N\Lambda}\,\delta X^\Lambda{}_{,\beta} + X^P{}_{,\beta} g_{NP,\Lambda}\,\delta X^\Lambda\big). \tag{10}$$

Variation of the area element reads

$$\begin{aligned}
\delta\sqrt{\left|\tilde{\tilde{\psi}}\right|} &= \frac{1}{2}\sqrt{\left|\tilde{\tilde{\psi}}\right|}\,\psi^{\alpha\beta}\,\delta\psi_{\alpha\beta} \\
&= \frac{1}{2}\sqrt{\left|\tilde{\tilde{\psi}}\right|}\,\psi^{\alpha\beta} X^N{}_{,\alpha}\big(2 g_{N\Lambda}\,\delta X^\Lambda{}_{,\beta} + X^P{}_{,\beta} g_{NP,\Lambda}\,\delta X^\Lambda\big) \\
&= -\frac{1}{T}\Big\{(\ldots)^\beta_{\mathrm{NG},\beta} + \mathcal{L}_{\mathrm{NG},X^\Lambda}\,\delta X^\Lambda\Big\}, &(11)
\end{aligned}$$

where

$$\frac{1}{T}(\ldots)^\beta_{\mathrm{NG}} := -\sqrt{\left|\tilde{\tilde{\psi}}\right|}\,\psi^{\alpha\beta} X^N{}_{,\alpha} g_{N\Lambda}\,\delta X^\Lambda, \tag{12}$$

is a boundary term,

$$\begin{aligned}
\frac{\mathcal{L}_{\mathrm{NG},X^\Lambda}}{T\sqrt{\left|\tilde{\tilde{\psi}}\right|}} &= \Box_\psi X^M g_{M\Lambda} + \psi^{\alpha\beta} X^N{}_{,\alpha} g_{N\Lambda,P} X^P{}_{,\beta} - \frac{1}{2}\psi^{\alpha\beta} X^N{}_{,\alpha} X^P{}_{,\beta} g_{NP,\Lambda} \\
&= \Box X^M g_{M\Lambda} + \frac{1}{2}\psi^{\alpha\beta} X^N{}_{,\alpha} X^P{}_{,\beta}\big(-g_{NP,\Lambda} + g_{P\Lambda,N} + g_{\Lambda N,P}\big) \\
&= g_{M\Lambda}\big(\Box_\psi X^M + \Gamma^M{}_{NP}\psi^{\alpha\beta} X^N{}_{,\alpha} X^P{}_{,\beta}\big) &(13)
\end{aligned}$$

gives the Euler–Lagrange derivative, in which

$$\Box_\psi X^M := \left|\tilde{\tilde{\psi}}\right|^{-1/2} \left(\sqrt{\left|\tilde{\tilde{\psi}}\right|}\, \psi^{\alpha\beta} X^M{}_{,\alpha} \right)_{,\beta} \tag{14}$$

is a d'Alembertian on the world-sheet with respect to $\psi_{\alpha\beta}$.

2.2 External symmetry

If ξ^M is a Killing vector, one can show

$$\left(\sqrt{\left|\tilde{\tilde{\psi}}\right|}\, \psi^{\alpha\beta} X^M{}_{,\alpha} \xi_M \right)_{,\beta} = 0. \tag{15}$$

Geometrically, this corresponds to

$$\operatorname{div} \xi^\top = 0. \tag{16}$$

Since one also has

$$\mathbb{L}_X \omega = (\operatorname{div} X)\omega, \tag{17}$$

where ω is the volume form on Σ, one has

$$\mathbb{L}_{\xi^\top} \omega = 0, \tag{18}$$

i.e. the volume form is invariant under diffeomorphism generated by $\xi^\top$.

2.3 Polyakov action

The action reads [3, 2, 7]

$$S_{\mathrm{P}}\left[h_{\alpha\beta}, X^M\right] = -\frac{T}{2} \int_\Sigma \mathbb{d}^2\sigma \sqrt{\left|\tilde{\tilde{h}}\right|}\, h^{\alpha\beta} \psi_{\alpha\beta}. \tag{19}$$

Variation of the integrand reads

$$\begin{aligned} \delta\left(\sqrt{\left|\tilde{\tilde{h}}\right|}\, h^{\alpha\beta} \psi_{\alpha\beta} \right) &= \delta_h \left(\sqrt{\left|\tilde{\tilde{h}}\right|}\, h^{\alpha\beta} \right) \psi_{\alpha\beta} + \left(\sqrt{\left|\tilde{\tilde{h}}\right|}\, h^{\alpha\beta} \right) \delta_X \psi_{\alpha\beta} \\ &= -\frac{2}{T} \left\{ (\ldots)^\beta_{\mathrm{P},\beta} + \mathcal{L}_{\mathrm{P},h^{\alpha\beta}}\, \delta h^{\alpha\beta} + \mathcal{L}_{\mathrm{P},X^\Lambda}\, \delta X^\Lambda \right\}, \end{aligned} \tag{20}$$

where

$$\frac{2}{T} (\ldots)^\beta_{\mathrm{P},\beta} := -2 \sqrt{\left|\tilde{\tilde{h}}\right|}\, h^{\alpha\beta} X^N{}_{,\alpha} g_{N\Lambda}\, \delta X^\Lambda, \tag{21}$$

is a boundary term,

$$\frac{2\mathcal{L}_{\mathrm{P},h^{\alpha\beta}}}{T\sqrt{\left|\tilde{\tilde{h}}\right|}} = \left(\frac{1}{2}h^{\gamma\delta}h_{\alpha\beta} - \delta^{\gamma}{}_{\alpha}\delta^{\delta}{}_{\beta}\right)X^{M}{}_{,\gamma}X^{N}{}_{,\delta}g_{MN} \tag{22}$$

gives the Euler–Lagrange derivative with respect to $h^{\alpha\beta}$,

$$\frac{2\mathcal{L}_{\mathrm{P},X^{\Lambda}}}{T\sqrt{\left|\tilde{\tilde{h}}\right|}} = 2g_{M\Lambda}\left(\Box_{h}X^{M} + \Gamma^{M}{}_{NP}X^{N}{}_{,\alpha}X^{P}{}_{,\beta}h^{\alpha\beta}\right) \tag{23}$$

gives the Euler–Lagrange derivative with respect to X^{Λ}, and

$$\Box_{h}X^{M} := \left|\tilde{\tilde{h}}\right|^{-1/2}\left(\sqrt{\left|\tilde{\tilde{h}}\right|}\,h^{\alpha\beta}X^{M}{}_{,\alpha}\right)_{,\beta} \tag{24}$$

is another d'Alembertian on the world-sheet with respect to $h_{\alpha\beta}$.

3 p-brane

[1]

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