Classical p-brane

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1 Point particle: 0-brane

1.1 Linear action

$$S_1[x^M] := -m \int_\gamma \mathrm{d}s = -m \int_\gamma \mathrm{d}\lambda \sqrt{-g_{MN}\dot{x}^M\dot{x}^N}. \tag{1}$$

1.2 Quadratic action

$$S_2[e, x^M] := \frac{1}{2} \int_\gamma \mathrm{d}\lambda\, e(e^{-2} g_{MN}\dot{x}^M\dot{x}^N - m^2). \tag{2}$$

2 Classical bosonic string: 1-brane

[5] contains a

2.1 Nambu–Goto action

The action reads [6, 4]

$$S_{\mathrm{NG}}[X^M] := -T \int_\Sigma \mathrm{d}A = -T \int_\Sigma \mathrm{d}^2\sigma \sqrt{-\tilde{\tilde{\psi}}}, \tag{3}$$

where

$$\tilde{\tilde{\psi}} := \det \psi_{\alpha\beta} \equiv \psi_{11}\psi_{22} - \psi_{12}\psi_{21} \tag{4}$$

is the metric determinant, $\alpha, \beta, \ldots = 1, 2$ are the world-sheet indices, $(\varSigma^\alpha)$ are the world-sheet coordinates,

$$\psi_{\alpha\beta} := g_{MN} X^M{}_{,\alpha} X^N{}_{,\beta}, \tag{5}$$

is the induced metric on the world-sheet, $P, N, \ldots = 0, 1, \ldots d$ are the target-space indices, $X^M = X^M(\varSigma^\alpha)$ are the world-sheet [8] coordinates. The immersion map $X^M(\sigma^\alpha)$ are the dynamical variables.

The inverse metric can also be expressed in a closed form

$$
\begin{aligned}
\psi^{\alpha\beta} &= \frac{1}{(2-1)!}\tilde{\epsilon}^{\alpha\gamma}\tilde{\epsilon}^{\beta\delta}\tilde{\tilde{\psi}}^{-1}\psi_{\gamma\delta} \\
&= \tilde{\epsilon}^{\alpha\gamma}\tilde{\epsilon}^{\beta\delta}\tilde{\tilde{\psi}}^{-1}\psi_{\gamma\delta} = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\psi_{\gamma\delta} && (6)\\
&= \tilde{\epsilon}^{\alpha\gamma}\tilde{\epsilon}^{\beta\delta}\tilde{\tilde{\psi}}^{-1}g_{P\Sigma}X^{P}{}_{,\gamma}X^{\Sigma}{}_{,\delta} = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}g_{P\Sigma}X^{P}{}_{,\gamma}X^{\Sigma}{}_{,\delta} && (7)\\
&\equiv \frac{1}{\tilde{\tilde{\psi}}}\begin{pmatrix}\psi_{22} & -\psi_{12}\\ -\psi_{21} & \psi_{11}\psi\end{pmatrix}^{\alpha\beta}. && (8)
\end{aligned}
$$

Unfortunately this is not useful.

Variation of the induced metric determinant can be expressed in terms of that of the induced metric

$$\delta\tilde{\tilde{\psi}} = \tilde{\tilde{\psi}}\psi^{\alpha\beta}\,\delta\psi_{\alpha\beta}. \tag{9}$$

Variation of the induced metric in terms of the world-sheet coordinates reads

$$\delta\psi_{\alpha\beta} = X^{N}{}_{,\alpha}\big(2g_{N\Lambda}\,\delta X^{\Lambda}{}_{,\beta} + X^{P}{}_{,\beta}g_{NP,\Lambda}\,\delta X^{\Lambda}\big). \tag{10}$$

Variation of the area element reads

$$
\begin{aligned}
\delta\sqrt{-\tilde{\tilde{\psi}}} &= \frac{1}{2}\sqrt{-\tilde{\tilde{\psi}}}\,\psi^{\alpha\beta}\,\delta\psi_{\alpha\beta}\\
&= \frac{1}{2}\sqrt{-\tilde{\tilde{\psi}}}\,\psi^{\alpha\beta}X^{N}{}_{,\alpha}\big(2g_{N\Lambda}\,\delta X^{\Lambda}{}_{,\beta} + X^{P}{}_{,\beta}g_{NP,\Lambda}\,\delta X^{\Lambda}\big)\\
&= -\frac{1}{T}\Big\{(\ldots)^{\beta}_{\mathrm{NG},\beta} + \mathcal{L}_{\mathrm{NG},X^{\Lambda}}\,\delta X^{\Lambda}\Big\}, && (11)
\end{aligned}
$$

where

$$\frac{1}{T}(\ldots)^{\beta}_{\mathrm{NG}} := -\sqrt{-\tilde{\tilde{\psi}}}\,\psi^{\alpha\beta}X^{N}{}_{,\alpha}g_{N\Lambda}\,\delta X^{\Lambda}, \tag{12}$$

is a boundary term,

$$
\begin{aligned}
\frac{\mathcal{L}_{\mathrm{NG},X^{\Lambda}}}{T\sqrt{-\tilde{\tilde{\psi}}}} &= \Box_{\psi}X^{M}g_{M\Lambda} + \psi^{\alpha\beta}X^{N}{}_{,\alpha}g_{N\Lambda,P}X^{P}{}_{,\beta} - \frac{1}{2}\psi^{\alpha\beta}X^{N}{}_{,\alpha}X^{P}{}_{,\beta}g_{NP,\Lambda}\\
&= \Box X^{M}g_{M\Lambda} + \frac{1}{2}\psi^{\alpha\beta}X^{N}{}_{,\alpha}X^{P}{}_{,\beta}\big(-g_{NP,\Lambda} + g_{P\Lambda,N} + g_{\Lambda N,P}\big)\\
&= g_{M\Lambda}\big(\Box_{\psi}X^{M} + \Gamma^{M}{}_{NP}\psi^{\alpha\beta}X^{N}{}_{,\alpha}X^{P}{}_{,\beta}\big) && (13)
\end{aligned}
$$

gives the Euler–Lagrange derivative, in which

$$\Box_{\psi}X^{M} := \frac{1}{\sqrt{-\tilde{\tilde{\psi}}}}\left(\sqrt{-\tilde{\tilde{\psi}}}\,\psi^{\alpha\beta}X^{M}{}_{,\alpha}\right)_{,\beta} \tag{14}$$

is a d'Alembertian on the world-sheet with respect to $\psi_{\alpha\beta}$.

2.2 External symmetry

If ξ^M is a Killing vector, one can show

$$\left(\sqrt{-\tilde{\tilde{\psi}}}\,\psi^{\alpha\beta}X^M{}_{,\alpha}\xi_M\right)_{,\beta} = 0. \tag{15}$$

Geometrically, this corresponds to

$$\operatorname{div}\xi^\top = 0. \tag{16}$$

Since one also has

$$\mathbb{L}_X\omega = (\operatorname{div}X)\omega, \tag{17}$$

where ω is the volume form on Σ, one has

$$\mathbb{L}_{\xi^\top}\omega = 0, \tag{18}$$

i.e. the volume form is invariant under diffeomorphism generated by $\xi^\top$.

2.3 Polyakov action

The action reads [3, 2, 7]

$$S_{\mathrm{P}}\left[h_{\alpha\beta}, X^M\right] = -\frac{T}{2}\int_\Sigma \mathbb{d}^2\sigma\,\sqrt{-\tilde{\tilde{h}}}\,h^{\alpha\beta}\psi_{\alpha\beta}. \tag{19}$$

Variation of the integrand reads

$$\begin{aligned}\delta\left(\sqrt{-\tilde{\tilde{h}}}\,h^{\alpha\beta}\psi_{\alpha\beta}\right) &= \delta_h\left(\sqrt{-\tilde{\tilde{h}}}\,h^{\alpha\beta}\right)\psi_{\alpha\beta} + \left(\sqrt{-\tilde{\tilde{h}}}\,h^{\alpha\beta}\right)\delta_X\psi_{\alpha\beta} \\ &= -\frac{2}{T}\Big\{(\ldots)^\beta_{\mathrm{P},\beta} + \mathcal{L}_{\mathrm{P},h^{\alpha\beta}}\,\delta h^{\alpha\beta} + \mathcal{L}_{\mathrm{P},X^\Lambda}\,\delta X^\Lambda\Big\},\end{aligned} \tag{20}$$

where

$$\frac{2}{T}(\ldots)^\beta_{\mathrm{P},\beta} := -2\sqrt{-\tilde{\tilde{h}}}\,h^{\alpha\beta}X^N{}_{,\alpha}g_{N\Lambda}\,\delta X^\Lambda, \tag{21}$$

is a boundary term,

$$\frac{2\mathcal{L}_{\mathrm{P},h^{\alpha\beta}}}{T\sqrt{-\tilde{\tilde{h}}}} = \left(\frac{1}{2}h^{\gamma\delta}h_{\alpha\beta} - \delta^\gamma{}_\alpha\delta^\delta{}_\beta\right)X^M{}_{,\gamma}X^N{}_{,\delta}g_{MN} \tag{22}$$

gives the Euler–Lagrange derivative with respect to $h^{\alpha\beta}$,

$$\frac{2\mathcal{L}_{\mathrm{P},X^\Lambda}}{T\sqrt{-\tilde{\tilde{h}}}} = 2g_{M\Lambda}(\Box_h X^M + \Gamma^M{}_{NP}X^N{}_{,\alpha}X^P{}_{,\beta}h^{\alpha\beta}) \tag{23}$$

gives the Euler–Lagrange derivative with respect to X^Λ, and

$$\Box_h X^M := \frac{1}{\sqrt{-\tilde{\tilde{h}}}}\left(\sqrt{-\tilde{\tilde{h}}}\,h^{\alpha\beta}X^M{}_{,\alpha}\right)_{,\beta} \tag{24}$$

is another d'Alembertian on the world-sheet with respect to $h_{\alpha\beta}$.

3 *p*-brane

[1]

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