Gauge Transformation and Perturbation of Canonical General Relativity

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Gauge structure is one of the corner stones in modern theoretical physics. In particle physics, where fundamental interactions are modelled by Yang–Mills theories, gauge structure is an *input* of the theoretical description, and mathematical physicists use redundant variables in the configuration space so that the gauge invariance is manifest.

In gravitational physics, things are less clear compared to the former case. On the one hand, if one follows the way by which a Yang–Mills theory would be constructed, one derives Poincaré gauge theories, which is much more generic than the acknowledged General Relativity. On the other hand, even if one starts with General Relativity, and call the reparametrisation invariance[1] as a gauge invariance, one still faces another challenge, that in many applications, the theory is not written down with those variables where the gauge invariance is most manifest and the gauge transformation is simplest. The usage of the Arnowitt–Deser–Misner variables is such an example.

Let us be more specific. If one uses components of the metric $\left\{g_{\mu\nu}\right\}$ as the (superficial) degrees of freedom, where $\mu, \nu, \rho, \ldots$ runs from 0 to d and d is the *spatial* dimension, the gauge transformation of those configuration-space variables is related to the infinitesimal general coordinate transformation

$$x^\mu \mapsto x^\mu - \eta^\mu(x) + O(\eta^2) \tag{1}$$

by

$$\begin{aligned} \delta g_{\mu\nu} &= \mathbb{L}_\eta g_{\mu\nu} + O(\eta^2), \\ \mathbb{L}_\eta g_{\mu\nu} &= \eta_{\mu;\nu} + \eta_{\nu;\mu} = \eta^\lambda g_{\mu\nu,\lambda} + \eta^\lambda{}_{,\mu} g_{\lambda\nu} + \eta^\lambda{}_{,\nu} g_{\mu\lambda}. \end{aligned} \tag{2}$$

Here the generator of the general coordinate transformation $\eta^\mu(x)$ plays the role of gauge generator.

In the canonical formalism of General Relativity, the common practice is to use the Arnowitt–Deser–Misner variables $\left\{N, N_i, h_{ij}\right\}$ in configuration

[1] Also known as diffeomorphism invariance or general covariance.

space, instead of $\{g_{\mu\nu}\}$. The former ones are related to the latter by

$$g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}s^2 = -N^2\,\mathrm{d}t^2 + h_{ij}(N^i\,\mathrm{d}t + \mathrm{d}x^i)(N^j\,\mathrm{d}t + \mathrm{d}x^j), \qquad (3)$$

where N^i is understood as $h^{ij}N_j$, h^{ij} are implicit functions of h_{ij}, and $i, j, k, \ldots$ runs from 1 to d. One expects that $\{N, N_i, h_{ij}\}$ and $\{g_{\mu\nu}\}$ share the same gauge structure; however, the gauge transformation of the two variables looks quite different, as will become clear in our work.

In this work, it is the *Hamiltonian* or *canonical* formalism which is interested in, and the gauge structure will be described by gauge transformation in *phase space*, instead of in configuration space. It is less well-known that one can also derive a canonical action consistently by using $\{g_{\mu\nu}\}$ and their conjugate momenta $\{\boldsymbol{\rho}^{\mu\nu}\}$, as Dirac has done in [2]. The Dirac and the Arnowitt–Deser–Misner approaches have been compared, for instance, in [5], where some subtle differences have been shown; in particular, the gauge transformation (in the phase space) in the former case is drastically different than the latter one. This difference can be quantified, in the sense that the relation between $\{\eta_\mu\}$ and $\{\xi_\perp, \xi_i\}$, which are the gauge generator of the Dirac and Arnowitt–Deser–Misner approaches respectively, can be derived explicitly.

Here it goes with the plan. We would like to see if this difference has any consequence on the perturbation theory of General Relativity, for example, in the context of cosmological and black-hole perturbations. In particular, we would like to see if the gauge-invariant variables, for instance, the Mukhanov–Sasaki variables, could be changed from this perspective.

In order to do this, we start from the canonical formalism of General Relativity in the Arnowitt–Deser–Misner variables. We derive the gauge transformations of all the canonical variables by using the gauge generator derived by other authors (done), and construct the correspondence to the Dirac variables (not yet done, but others have done it).

Here is what is going on next. We then perturb the Hamiltonian action up to the second order on an arbitrary background with those variables directly. For the application in cosmology, we fix a Robertson–Walker background metric and discuss the gauge transformation of the perturbative coordinates and momenta.

If there are any positive outcomes, we could further investigate the application in the context of black-hole space-times.

1 Canonical transformation between the Dirac and ADM variables

In this section we will see that the Dirac and the canonical Arnowitt–Deser–Misner variables, $\{g_{\mu\nu}; \boldsymbol{\rho}^{\mu\nu}\}$ and $\{N, N_i, h_{ij}; \mathfrak{P}, \mathfrak{P}^i, \mathfrak{p}^{ij}\}$, are related by a canonical transformation, where in the momentum sector, coordinates and momenta *mix up*.

This could be surprising, since one might expect that those canonical variables are related by a *point transformation*. Since the transformation in configuration space is non-linear, one actually should expect a non-trivial transformation in the momentum sector.

From eq. (3), one can read off the relations between $\{g_{\mu\nu}\}$ and $\{N, N_i, h_{ij}\}$, motivating one to use the generating functional of the third type $F_3 = F_3\left[\boldsymbol{\rho}^{\mu\nu}; N, N_i, h_{ij}\right]$. Then the relations can be written as

$$-N^2 + h^{ij} N_i N_j = g_{00} = -\frac{\delta F_3}{\delta \boldsymbol{\rho}^{00}}, \tag{4}$$

$$N_i = g_{i0} = -\frac{\delta F_3}{\delta \boldsymbol{\rho}^{i0}}, \qquad g_{0i} = -\frac{\delta F_3}{\delta \boldsymbol{\rho}^{0i}}, \tag{5}$$

$$h_{ji} = g_{ji} = -\frac{\delta F_3}{\delta \boldsymbol{\rho}^{ji}}, \qquad g_{ij} = -\frac{\delta F_3}{\delta \boldsymbol{\rho}^{ij}}; \tag{6}$$

the inverse of eqs. (4) to (6) reads[2]

$$N = \left(-g^{00}\right)^{-1/2}, \qquad N_i = g_{(i0)}, \qquad h_{ij} = g_{ij}. \tag{7}$$

It is also useful to note that the transformations of the inverse metric is

$$g^{00} = -\frac{1}{N^2}, \qquad g^{0i} = g^{i0} = \frac{N^i}{N^2}, \qquad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}; \tag{8}$$

$$N^i = -\frac{g^{(0i)}}{g^{00}}, \qquad h^{ij} = g^{ij} - \frac{g^{(0i)} g^{(0j)}}{g^{00}}. \tag{9}$$

Equations (4) to (6) can be formally integrated to get

$$F_3 = -\int \mathrm{d}^3 x \left\{\boldsymbol{\rho}^{00}\left(-N^2 + h^{ij} N_i N_j\right) + \left(\boldsymbol{\rho}^{0i} + \boldsymbol{\rho}^{i0}\right) N_i + \boldsymbol{\rho}^{ij} h_{ij}\right\}. \tag{10}$$

Calculation shows that the momenta transform as

$$\mathfrak{P} = -\frac{\delta F_3}{\delta N} = -2N\boldsymbol{\rho}^{00} \qquad = -\frac{2}{\left(-g^{00}\right)^{1/2}} \boldsymbol{\rho}^{00}, \tag{11}$$

$$\mathfrak{P}^i = -\frac{\delta F_3}{\delta N_i} = +2N^i \boldsymbol{\rho}^{00} + \boldsymbol{\rho}^{0i} + \boldsymbol{\rho}^{i0} = -\frac{g^{0i} + g^{i0}}{g^{00}} \boldsymbol{\rho}^{00} + \boldsymbol{\rho}^{0i} + \boldsymbol{\rho}^{i0}, \tag{12}$$

$$\mathfrak{p}^{ij} = -\frac{\delta F_3}{\delta h_{ij}} = -N^i N^j \boldsymbol{\rho}^{00} + \boldsymbol{\rho}^{ij} \qquad = -\frac{g^{0i} g^{0j}}{\left(g^{00}\right)^2} \boldsymbol{\rho}^{00} + \boldsymbol{\rho}^{ij}. \tag{13}$$

The inverse transformations of eqs. (11) to (13) are

$$\boldsymbol{\rho}^{00} = -\frac{\mathfrak{P}}{2N}, \quad \boldsymbol{\rho}^{0i} = \boldsymbol{\rho}^{i0} = \frac{\mathfrak{P}^i}{2} + N^i \frac{\mathfrak{P}}{2N}, \quad \boldsymbol{\rho}^{ij} = \mathfrak{p}^{ij} - N^i N^j \frac{\mathfrak{P}}{2N}. \tag{14}$$

[2]We take $N > 0$; this has not been considered seriously.

One can verify that

$$\mathfrak{P}\,\delta N + \mathfrak{P}^i\,\delta N_i + \mathfrak{p}^{ij}\,\delta h_{ij} = \boldsymbol{\rho}^{\mu\nu}\,\delta g_{\mu\nu} \tag{15}$$

holds, which is a criterion for the canonicality of a time-independent transformation; one may further verify that the fundamental Poisson brackets in the old[3]

$$\big[g_{\mu\nu}(x_1), \boldsymbol{\rho}^{\rho\sigma}(x_2)\big]_{\mathrm{P}} = \frac{1}{2}\big(\delta^\rho{}_\mu\delta^\sigma{}_\nu + \delta^\sigma{}_\mu\delta^\rho{}_\nu\big)\boldsymbol{\delta}^{(3)}(x_1, x_2); \tag{16}$$

and new variables

$$\begin{aligned}
\big[N(x_1), \mathfrak{P}(x_2)\big]_{\mathrm{P}} &= \boldsymbol{\delta}^{(3)}(x_1, x_2),\\
\big[N_i(x_1), \mathfrak{P}^j(x_2)\big]_{\mathrm{P}} &= \delta^j{}_i\boldsymbol{\delta}^{(3)}(x_1, x_2),\\
\big[h_{ij}(x_1), \mathfrak{p}^{kl}(x_2)\big]_{\mathrm{P}} &= \frac{1}{2}\big(\delta^k{}_i\delta^l{}_j + \delta^l{}_i\delta^k{}_j\big)\boldsymbol{\delta}^{(3)}(x_1, x_2);\\
\big[N(x_1), \mathfrak{P}^i(x_2)\big]_{\mathrm{P}} &= \big[N(x_1), \mathfrak{p}^{ij}(x_2)\big]_{\mathrm{P}} = \big[N^i(x_1), \mathfrak{p}^{jk}(x_2)\big]_{\mathrm{P}} = 0
\end{aligned} \tag{17}$$

are invariant.

To conclude, since the transformation is complicated in the momentum sector, one expects that the gauge transformations would also take a different form in the different variables of phase space. This will be investigated in section 3.

2 Canonical actions and their first variations

The Lagrangian action for General Relativity in terms of the Arnowitt–Deser–Misner variables is [4, ch. 4.2.2]

$$S = \frac{1}{2\varkappa}\int \mathrm{d}t\,\mathrm{d}^d x\, N\Big\{\mathfrak{G}^{ijkl}K_{ij}K_{kl} + \sqrt{\mathfrak{h}}\,R[h]\Big\} + \text{surface terms}, \tag{18}$$

where

$$\mathfrak{G}_{ijkl} := \frac{1}{2\mathfrak{h}^{1/2}}\big(h_{ik}h_{lj} + h_{il}h_{kj} - h_{ij}h_{kl}\big) \equiv -\frac{\delta\big(\mathfrak{h}^{-1/2}h_{ij}\big)}{\delta h^{kl}}, \tag{19}$$

$$\mathfrak{G}^{ijkl} := \frac{\mathfrak{h}^{1/2}}{2}\big(h^{ik}h^{lj} + h^{il}h^{kj} - 2h^{ij}h^{kl}\big) \equiv -\mathfrak{h}^{-1/2}\frac{\delta\big(\mathfrak{h}h^{ij}\big)}{\delta h_{kl}} \tag{20}$$

are the DeWitt metric and its inverse. We emphasise that eq. (18) is expected to be *identical* to the Einstein–Hilbert action with proper boundary terms of the Gibbons–Hawking–York type.

[3] Note that $\big[g_{12}, \boldsymbol{\rho}^{12}\big]_{\mathrm{P}} \propto 1/2$, not 1!

Equation (18) leads to the canonical action

$$S=\int \mathrm{d}t\,\mathrm{d}x^d\Big\{\mathfrak{p}^{ij}\dot{h}_{ij}+\mathfrak{P}\dot{N}+\mathfrak{P}^i\dot{N}_i-N\mathfrak{H}_{\mathrm{G}}^{\perp}-N_i\mathfrak{H}_{\mathrm{G}}^{i}$$
$$-\mathfrak{P}V-\mathfrak{P}^iV_i\Big\}+\text{surface terms,} \tag{21}$$

where $\mathfrak{P}$ and $\mathfrak{P}^i$ are the primary constraints,

$$\mathfrak{H}_{\mathrm{G}}^{\perp}=2\varkappa\,\mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl}-\frac{\sqrt{\mathfrak{h}}}{2\varkappa}R[h]\equiv 2\varkappa\,\mathfrak{F}^{ijkl}h_{ij}h_{kl}-\frac{\sqrt{\mathfrak{h}}}{2\varkappa}R[h], \tag{22}$$
$$\mathfrak{H}_{\mathrm{G}}^{i}=-2\mathfrak{p}^{ij}{}_{|j} \tag{23}$$

are the secondary constraints, and

$$\mathfrak{F}^{ijkl}:=\frac{1}{2\mathfrak{h}^{1/2}}(\mathfrak{p}^{ik}\mathfrak{p}^{jl}+\mathfrak{p}^{il}\mathfrak{p}^{kj}-\mathfrak{p}^{ij}\mathfrak{p}^{kl}) \tag{24}$$

is a convenient notation. In eq. (21), V and V_i are velocities of N and N_i and play the role of Lagrange multipliers. Details about the action with velocities can be found in [3]; those about the surface terms can be found in [6, ch. 4.2] and the references therein.

As for the canonical action in the Dirac variables, it can be derived in two ways. One is to start from eq. (18) and substitute the coordinate-space transformation in eq. (7); the other is to start from eq. (21) and use the canonical transformation in eq. (7) and eqs. (11) to (13). Kiriushcheva and Kuzmin claimed [5] that the two approaches give different results.

Using the results in appendix A.1, the first variation of eq. (21) can be derived to be

$$\delta S=\int \mathrm{d}t\,\mathrm{d}^3x\Big\{-(\dot{\mathfrak{p}}^{ij}-\mathcal{P}^{ij}(N,N_i))\,\delta h_{ij}+\big(\dot{h}_{ij}-\mathcal{H}_{ij}(N,N_i)\big)\,\delta\mathfrak{p}^{ij}$$
$$-\big(\dot{\mathfrak{P}}+\mathfrak{H}_{\mathrm{G}}^{\perp}\big)\,\delta N-\big(\dot{\mathfrak{P}}^i+\mathfrak{H}_{\mathrm{G}}^{i}\big)\,\delta N_i+\big(\dot{N}-V\big)\,\delta\mathfrak{P}+\big(\dot{N}_i-V_i\big)\,\delta\mathfrak{P}^i$$
$$-\mathfrak{P}\,\delta V-\mathfrak{P}^i\,\delta V_i\Big\}, \tag{25}$$

where

$$\mathcal{H}_{ij}(N,N_i):=2\big(2\varkappa\,N\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}+N_{(i|j)}\big), \tag{26}$$
$$\mathcal{P}^{ij}(N,N_i):=2\varkappa\,N\Big(\frac{1}{2}h^{ij}\mathfrak{F}^{klmn}h_{mn}-2\mathfrak{F}^{ijkl}\Big)h_{kl}$$
$$+\frac{1}{2\varkappa}\big(-N\sqrt{\mathfrak{h}}\,G^{ij}[h]+\mathfrak{G}^{ijkl}N_{|k|l}\big)$$
$$-\big\{(h^{ki}\mathfrak{p}^{jl}+h^{kj}\mathfrak{p}^{il}-h^{kl}\mathfrak{p}^{ij})N_k\big\}_{|l}. \tag{27}$$

Also notes that in eq. (25) no further boundary term is present; those in eq. (21) have all been used to eliminate the boundary integral with first derivatives.

If one would like to find the canonical action in the Dirac variables, one could begin with the Lagrangian action in eq. (18), where the variables with time derivative has been separated from those without, and transform the Arnowitt–Deser–Misner variables to the Dirac ones by using eqs. (4) to (6). In contrast with the original approach by Dirac, which starts from the Einstein–Hilbert action, this approach starts with something where the dynamical and non-dynamical variables have already been separated explicitly, so that the calculation is expected to be simpler.

(I probably need to derive this)

3 Infinitesimal canonical gauge transformation

In this section the infinitesimal gauge transformation of General Relativity will be constructed in terms of the canonical variables, namely in phase space. The results with the Arnowitt–Deser–Misner variables and the Dirac variables will be worked out separately. In the former case, one sees that the lapse and shift functions as well as their conjugate momenta are also involved in the gauge transformation, and the action has to contain them in order to be invariant under the transformation.

3.1 The Arnowitt–Deser–Misner canonical variables

(I need to derive this)

The gauge transformations of the canonical Arnowitt–Deser–Misner variables are generated by [1]

$$\begin{aligned} G = -\int \mathbb{d}^3x \Big\{ & \Big[\xi_\perp \Big(\mathfrak{H}_{\mathrm{G}}^{\perp} + N_{|i}\mathfrak{P}^i + \big(N\mathfrak{P}^i\big)_{|i} + (N_i\mathfrak{P})^{|i}\Big) + \dot{\xi}_\perp \mathfrak{P}\Big] \\ & + \Big[\xi_i \Big(\mathfrak{H}_{\mathrm{G}}^{i} + N_j{}^{|i}\mathfrak{P}^j + \big(N_j\mathfrak{P}^i\big)^{|j} + N^{|i}\mathfrak{P}\Big) + \dot{\xi}_i \mathfrak{P}^i\Big]\Big\} \\ & + \text{boundary terms}, \end{aligned} \tag{28}$$

where the boundary terms have probably not been discussed so far, but are surely needed for non-compact spatial topologies in order to cancel the second spatial derivatives in the potential term in $\mathfrak{H}_{\mathrm{G}}^{\perp}$. The infinitesimal gauge transformations of $\{N, N_i; \mathfrak{P}, \mathfrak{P}^i\}$ are [5]

$$\delta N = [N, G]_{\mathrm{P}} = \xi_\perp{}^{|i} N_i - \dot{\xi}_\perp - \xi_i N^{|i}, \tag{29}$$

$$\delta N_i = -\xi_\perp N_{|i} + \xi_{\perp|i} N - \xi_j N_i{}^{|j} + \xi_i{}^{|j} N_j - \dot{\xi}_i; \tag{30}$$

$$\delta \mathfrak{P} = -\big(\xi_\perp \mathfrak{P}^i\big)_{|i} - \xi_{\perp|i}\mathfrak{P}^i - (\xi_i \mathfrak{P})^{|i}, \tag{31}$$

$$\delta \mathfrak{P}^i = -\xi_\perp{}^{|i}\mathfrak{P} - \big(\xi_j \mathfrak{P}^i\big)^{|j} - \xi_j{}^{|i}\mathfrak{P}^j, \tag{32}$$

where only the primary constraints are involved; transformations of $\{g_{ij};\mathfrak{p}^{ij}\}$ are

$$\delta h_{ij} = -\frac{\partial}{\partial\mathfrak{p}^{ij}}(\xi_\perp\mathfrak{H}_{\mathrm{G}}^\perp + \xi_i\mathfrak{H}_{\mathrm{G}}^i) = -\mathcal{H}_{ij}(\xi_\perp,\xi_i), \tag{33}$$

$$\delta\mathfrak{p}^{ij} = \frac{\partial}{\partial h_{ij}}(\xi_\perp\mathfrak{H}_{\mathrm{G}}^\perp + \xi_i\mathfrak{H}_{\mathrm{G}}^i) = -\mathcal{P}^{ij}(\xi_\perp,\xi_i), \tag{34}$$

where only the secondary constraints are involved, and surface contribution has not been written down explicitly.

3.2 The canonical Dirac variables

(I probably need to work out the generator as well)

The infinitesimal gauge transformations, parametrised by η_μ, of the Dirac canonical variables are

$$\delta g_{\mu\nu} = \eta_{\mu;\nu} + \eta_{\nu;\mu} = \eta^\lambda g_{\mu\nu,\lambda} + \eta^\lambda{}_{,\mu}g_{\lambda\nu} + \eta^\lambda{}_{,\nu}g_{\mu\lambda}, \tag{2 rev.}$$

$$\delta\boldsymbol{\rho}^{\mu\nu} = (\eta^\lambda\boldsymbol{\rho}^{\mu\nu})_{,\lambda} - \eta^\mu{}_{,\lambda}\boldsymbol{\rho}^{\lambda\nu} - \eta^\nu{}_{,\lambda}\boldsymbol{\rho}^{\mu\lambda}, \tag{35}$$

which have a manifest geometric meaning of a infinitesimal coordinate transformation. How to proceed with these? It seems

$$\delta\big(\boldsymbol{\rho}^{\mu\nu}\dot{g}_{\mu\nu}\big) = \big(\eta^\lambda\boldsymbol{\rho}^{\mu\nu}\dot{g}_{\mu\nu}\big)_{,\lambda} + \boldsymbol{\rho}^{\mu\nu}\big(\dot{\eta}_{\mu;\nu} + \dot{\eta}_{\nu;\mu}\big), \tag{36}$$

assuming ∂_t and δ commute.

The relation between η_μ and $\{\xi_\perp,\xi_i\}$ are

$$\eta_0 =, \tag{37}$$

$$\eta_i = . \tag{38}$$

4 Second variation of the canonical ADM action

$$\delta^2 S = \tag{39}$$

5 Canonical perturbation of the Klein–Gordon field in curved space-time

$$\mathfrak{L} = \sqrt{-\mathfrak{g}}\Big(-\frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V(\phi)\Big). \tag{40}$$

[4, ch. 4.2.2]

$$\mathfrak{H}_{\mathrm{KG}}^\perp = \frac{\boldsymbol{\pi}^2}{2\sqrt{\mathfrak{h}}} + \sqrt{\mathfrak{h}}\Big(\frac{1}{2}h^{ij}\phi_{|i}\phi_{|j} + V(\phi)\Big), \tag{41}$$

$$\mathfrak{H}_{\mathrm{KG}}^i = \boldsymbol{\pi}\phi^{|i}. \tag{42}$$

6 Canonical perturbation of the Friedmann–Lemaître–Klein–Gordon model

$$\mathrm{d}s^2 = -N^2(t)\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\vec{x}^2 \tag{43}$$

$h_{ij}(\vec{x}) = a^2(t)\delta_{ij}\boldsymbol{\delta}^3(\vec{x}-\vec{x}')$.

Acknowledgement

A Some useful results

Here some useful intermediate results are collected.

A.1 For the first variations

The variation of N with respect to $\{g_{\mu\nu}\}$ reads

$$\delta N = \frac{1}{2}\left(-g^{00}\right)^{-3/2}\delta g^{00} \tag{44}$$

$$= \frac{1}{2}\left(-\sqrt{-g^{00}}\,\delta g_{00} + \frac{g^{0i}\,\delta g_{0i} + g^{i0}\,\delta g_{i0}}{\left(-g^{00}\right)^{1/2}} - \frac{g^{0i}g^{0j}}{\left(-g^{00}\right)^{3/2}}\,\delta g_{ij}\right). \tag{45}$$

The first variation of the inverse metric h^{ij} reads

$$\delta h^{ij} = -h^{ik}h^{jl}\,\delta h_{kl} = -h^{i(k}h^{l)j}\,\delta h_{kl}. \tag{46}$$

The first variation of $\mathfrak{h} = \det h_{ij}$ reads

$$\delta\mathfrak{h} = \mathfrak{h}h^{ij}\,\delta h_{ij}. \tag{47}$$

The first variation of $\Gamma^i{}_{jk}$ can be obtained in normal coordinates, which reads

$$\delta\Gamma^i{}_{jk} = \frac{1}{2}h^{il}\left\{-\left(\delta h_{jk}\right)_{|l} + \left(\delta h_{kl}\right)_{|j} + \left(\delta h_{lj}\right)_{|k}\right\} \tag{48}$$

$$= \frac{1}{2}\left\{-h^{il}\delta^m{}_j\delta^n{}_k + h^{in}\delta^l{}_j\delta^m{}_k + h^{im}\delta^n{}_j\delta^l{}_k\right\}\left(\delta h_{mn}\right)_{|l}. \tag{49}$$

The first variations of $R_{ij}[h]$ and $R^{ij}[h]$

$$\delta R_{ij}[h] = \left(\delta\Gamma^k{}_{ji}\right)_{|k} - \left(\delta\Gamma^k{}_{ki}\right)_{|j}, \tag{50}$$

$$\delta R^{ij}[h] = -2R^{k(i}h^{j)l}\,\delta h_{kl} + h^{k(i}\,\bar{\delta}u^{j)l}{}_{k|l} \tag{51}$$

$$= -2R^{k(i}h^{j)l}\,\delta h_{kl} + \left(h^{l(i}h^{j)[m}h^{k]n} + \frac{1}{2\mathfrak{h}^{1/2}}h^{m(i}\mathfrak{G}^{j)nkl}\right)\left(\delta h_{kl}\right)_{|m|n}, \tag{52}$$

where

$$\bar{\delta}u^{ij}{}_{k} := h^{il}\,\delta\Gamma^{j}{}_{lk} - h^{ij}\,\delta\Gamma^{l}{}_{lk} \tag{53}$$

is related to the boundary terms, and eq. (64) has been used to obtain eq. (52). Equation (50) can be obtained in normal coordinates.

For the first variation of the constraints, one also needs

$$\bar{\delta}u^{ji}{}_{j} = (\delta h_{kl})_{|j}\big(h^{i(k}h^{l)j} - h^{ij}h^{kl}\big) = \mathfrak{h}^{-1/2}\mathfrak{G}^{ijkl}\,(\delta h_{kl})_{|j}. \tag{54}$$

Equation (54) is consistent with eq. (63). Therefore,

$$\sqrt{\mathfrak{h}}\,N\,\bar{\delta}u^{ji}{}_{j|i} = \delta h_{ij}\,\mathfrak{G}^{ijkl}N_{|k|l} + \Big\{\mathfrak{G}^{ijkl}\Big(N\,(\delta h_{kl})_{|j} - N_{|j}\,\delta h_{kl}\Big)\Big\}_{|i}. \tag{55}$$

In the Hamiltonian constraint, the first variation of the ‘kinetic term’ $\mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl} \equiv \mathfrak{F}^{ijkl}h_{ij}h_{kl}$ reads

$$\delta\big(\mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl}\big) \equiv \delta\big(\mathfrak{F}^{ijkl}h_{ij}h_{kl}\big) \tag{56}$$

$$= \delta h_{ij}\left(-\frac{1}{2}h^{ij}\mathfrak{F}^{klmn}h_{mn} + 2\mathfrak{F}^{ijkl}\right)h_{kl} + \delta\mathfrak{p}^{ij}\,2\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}. \tag{57}$$

Equation (57) is consistent with eq. (62).

Equipped with eqs. (51) and (54), the first variation of the ‘potential’ $\sqrt{\mathfrak{h}}\,R[h]$ reads

$$\begin{aligned}\delta\Big(\sqrt{\mathfrak{h}}R[h]\Big) &= \sqrt{\mathfrak{h}}\,\Big\{-G^{ij}[h]\,\delta h_{ij} + \bar{\delta}u^{ji}{}_{j|i}\Big\}\\ &= -\sqrt{\mathfrak{h}}\,G^{ij}[h]\,\delta h_{ij} + \mathfrak{G}^{ijkl}\,(\delta h_{kl})_{|j|i}.\end{aligned} \tag{58}$$

One can now write down the first variation of $N\mathfrak{H}^{\perp}_{\mathrm{G}}$ with respect to $\big\{h_{ij}, \mathfrak{p}^{ij}\big\}$,

$$\begin{aligned}N\,\delta\mathfrak{H}^{\perp}_{\mathrm{G}} = \delta h_{ij}\Bigg\{&2\varkappa N\left(-\frac{1}{2}h^{ij}\mathfrak{F}^{klmn}h_{mn} + 2\mathfrak{F}^{ijkl}\right)h_{kl}\\ &+ \frac{1}{2\varkappa}\Big(\sqrt{\mathfrak{h}}\,NG^{ij}[h] - \mathfrak{G}^{ijkl}N_{|k|l}\Big)\Bigg\}\\ &+ \delta\mathfrak{p}^{ij}\,4\varkappa\,N\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}\\ &- \frac{1}{2\varkappa}\Big\{\mathfrak{G}^{ijkl}\Big(N\,(\delta h_{kl})_{|j} - N_{|j}\,\delta h_{kl}\Big)\Big\}_{|i},\end{aligned} \tag{59}$$

where the terms in the last line will be pushed to the spatial boundary $\partial\Sigma$; the second term vanishes by $\delta h_{ij}\big|_{\partial\Sigma} = 0$, whereas the first one is cancelled by the boundary term.

Finally, the first variation of $N_i \mathfrak{H}^i_\mathrm{G}$ with respect to $\{h_{ij}, \mathfrak{p}^{ij}\}$ is easier,

$$
\begin{aligned}
N_i\, \delta\mathfrak{H}^i_\mathrm{G} &= \delta h_{ij} \left\{ (h^{ki}\mathfrak{p}^{jl} + h^{kj}\mathfrak{p}^{il} - h^{kl}\mathfrak{p}^{ij}) N_k \right\}_{|l} + \delta\mathfrak{p}^{ij}\, 2N_{(i|j)} \\
&\quad - \left(-h^{il}\delta^m{}_j\delta^n{}_k + h^{in}\delta^l{}_j\delta^m{}_k + h^{im}\delta^n{}_j\delta^l{}_k\right)\left(N_i\mathfrak{p}^{jk}\,\delta h_{mn}\right)_{|l} \\
&\quad - 2\left(\delta\mathfrak{p}^{ij}\, N_j\right)_{|i},
\end{aligned}
\tag{60}
$$

where the last two lines will be pushed to the spatial boundary and vanish by $\delta h_{ij}\big|_{\partial\Sigma} = 0 = \delta\mathfrak{p}^{ij}\big|_{\partial\Sigma}$. The results can be checked with [6, ch. 4.2.7].

A.2 For the second variations

From eqs. (25) to (27) one sees that in order to calculate the second variation of the canonical Arnowitt–Deser–Misner action, the following variations needs to be calculated, in addition to those calculated in appendix A.1.

First variation of $\mathfrak{G}^{ijkl}$

$$
\delta\mathfrak{G}^{ijkl} = \left(\frac{1}{2}\mathfrak{G}^{ijkl}h^{mn} - h^{m(i}\mathfrak{G}^{j)nkl} - \mathfrak{G}^{ijk(l}h^{k)m}\right)\delta h_{mn}. \tag{61}
$$

(check!)

First variation of $\mathfrak{F}^{ijkl}h_{kl}$

$$
\begin{aligned}
\delta(\mathfrak{F}^{ijkl}h_{kl}) &= \delta h_{kl}\left(-\frac{1}{2}\mathfrak{F}^{ijmn}h^{kl}h_{mn} + \mathfrak{F}^{ijkl}\right) \\
&\qquad + \delta\mathfrak{p}^{kl}\, h^{i(p}h^{q)j}\left(h_{kp}\mathfrak{G}_{qlmn} + \mathfrak{G}_{klmp}h_{qn}\right)\mathfrak{p}^{mn}.
\end{aligned}
\tag{62}
$$

Equation (62) is consistent with eq. (57).

Concerning the derivative of $\bar{\delta}u^{ij}{}_k$, one needs

$$
\begin{aligned}
&\bar{\delta}u^{ik}{}_{j|k} \\
&= (\delta h_{kl})_{|m|n}\,\frac{1}{2}\left(h^{i[m}h^{k]n}\delta^l{}_j + h^{i[m}h^{l]n}\delta^k{}_j + \frac{1}{\sqrt{\mathfrak{h}}}\mathfrak{G}^{inkl}\delta^m{}_j\right).
\end{aligned}
\tag{63}
$$

Equation (63) is consistent with eq. (54), and leads to.

$$
h^{k(i}\,\bar{\delta}u^{j)l}{}_{k|l} = (\delta h_{kl})_{|m|n}\left(h^{l(i}h^{j)[m}h^{k]n} + \frac{1}{2\mathfrak{h}^{1/2}}h^{m(i}\mathfrak{G}^{j)nkl}\right). \tag{64}
$$

The following results are not yet solid.

First variation of $\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}$

$$
\begin{aligned}
\delta\left(\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}\right) &= \delta h_{kl}\left(-\frac{1}{2}\mathfrak{G}_{ijmn}h^{kl} + \delta^k{}_i\mathfrak{G}^l{}_{jmn} + \delta^k{}_m\mathfrak{G}^l{}_{nij}\right)\mathfrak{p}^{mn} \\
&\qquad + \delta p^{kl}\,\mathfrak{G}_{ijkl}.
\end{aligned}
\tag{65}
$$

First variation of $\sqrt{\mathfrak{h}}\,G^{ij}[h]$

$$\delta\Big(\sqrt{\mathfrak{h}}\,G^{ij}[h]\Big) = \sqrt{\mathfrak{h}}\bigg\{\delta h_{kl}\cdot \\ \Big(-\frac{1}{2}\Big)(R^{ik}h^{lj} + R^{il}h^{kj} - R^{ij}h^{kl} + h^{ik}G^{lj} + h^{il}G^{kj} - h^{ij}G^{kl}) \\ + \Big(h^{il}\,\bar{\delta}u^{jk}{}_{l} - \frac{1}{2}h^{ij}\,\bar{\delta}u^{lk}{}_{l}\Big)_{|k}\bigg\}. \tag{66}$$

About $\bar{\delta}u^{ij}{}_{k}$, the identity

$$\sqrt{\mathfrak{h}}\,Nh^{k(i}\,\bar{\delta}u^{j)l}{}_{k|l} = \tag{67}$$

is also useful.

Second variation of $N\mathfrak{H}_{\mathrm{G}}^{\perp}$

$$\delta^2(N\mathfrak{H}_{\mathrm{G}}^{\perp}) \\ = \delta h_{ij}\,\delta h_{kl}\bigg\{2\varkappa\bigg[\frac{1}{4}(h^{ik}h^{lj} + h^{il}h^{kj} + h^{ij}h^{kl})\mathfrak{F}^{mnrs}h_{mn}h_{rs} \\ - (h^{ij}\mathfrak{F}^{klmn} + \mathfrak{F}^{ijmn}h^{kl})h_{mn} + \mathfrak{F}^{ijkl}\bigg] \\ - \frac{\sqrt{\mathfrak{h}}}{4\varkappa}(R^{ik}h^{lj} + R^{il}h^{kj} - R^{ij}h^{kl} + h^{ik}G^{lj} + h^{il}G^{kj} - h^{ij}G^{kl})\bigg\} \\ + \delta h_{ij}\,\frac{\sqrt{\mathfrak{h}}}{2\varkappa}\Big(h^{il}\,\bar{\delta}u^{jk}{}_{l} - \frac{1}{2}h^{ij}\,\bar{\delta}u^{lk}{}_{l}\Big)_{|k} \\ + \delta h_{ij}\,\delta\mathfrak{p}^{kl}\,4\varkappa\{-h^{ij}\mathfrak{G}_{klmn} + 2(\delta^{i}{}_{k}\mathfrak{G}^{j}{}_{lmn} + \delta^{i}{}_{m}\mathfrak{G}^{j}{}_{nkl})\}\mathfrak{p}^{mn} \\ + \delta p^{ij}\,\delta p^{kl}\,4\varkappa\mathfrak{G}_{ijkl} \\ - \delta h_{ij}\,\delta\Big(\frac{\sqrt{\mathfrak{h}}}{2\varkappa}\,\bar{\delta}u^{lk}{}_{l|k}\Big). \tag{68}$$

Second variation of $\Gamma^{i}{}_{jk}$

$$\delta^2\Gamma^{i}{}_{jk} = -h^{im}\,\delta\Gamma^{l}{}_{jk}\,\delta h_{lm}. \tag{69}$$

Second variation of $\mathfrak{H}_{\mathrm{G}}^{i}$

$$\delta^2(\mathfrak{H}_{\mathrm{G}}^{i}) = -h^{im}p^{jk}\,\delta\Gamma^{l}{}_{jk}\,\delta h_{lm} + 2\,\delta\Gamma^{i}{}_{jk}\,\delta p^{jk}. \tag{70}$$

Other second variations

Second variation of h^{ij}

$$\delta^2 h^{ij} = (h^{im}h^{jl}h^{kn} + h^{ik}h^{jm}h^{ln})\,\delta h_{kl}\,\delta h_{mn} \tag{71}$$

Second variation of $\mathfrak{h} = \det h_{ij}$

$$\delta^2 \mathfrak{h} = -\frac{1}{4} \mathfrak{h} \left(h^{ik} h^{jl} + h^{il} h^{kj} - h^{ij} h^{kl} \right) \delta h_{ij} \, \delta h_{kl}. \tag{72}$$

First variation of $\left(\delta h_{ij} \right)_{|k}$

$$\delta \left\{ \left(\delta h_{ij} \right)_{|k} \right\} = -2 \delta \Gamma^{l}{}_{k(i} \, \delta h_{j)l}. \tag{73}$$

In a general background, the second variations of the quantities are much more tedious.

In Robertson–Walker background,

References

[1] Leonardo Castellani. “Symmetries in constrained Hamiltonian systems”. In: *Annals of Physics* 143.2 (Oct. 1982), pp. 357–371. DOI: 10.1016/0003-4916(82)90031-8.

[2] Paul A. M. Dirac. “The Theory of Gravitation in Hamiltonian Form”. In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 246.1246 (Aug. 1958), pp. 333–343. DOI: 10.1098/rspa.1958.0142.

[3] Dmitry Maximovitch Gitman and Igor V. Tyutin. *Quantization of Fields with Constraints*. Springer Berlin Heidelberg, 1990. DOI: 10.1007/978-3-642-83938-2.

[4] Claus Kiefer. *Quantum Gravity*. 3rd ed. Oxford University Press, Apr. 2012. DOI: 10.1093/acprof:oso/9780199585205.001.0001.

[5] Natalia Kiriushcheva and Sergei Kuzmin. “The Hamiltonian formulation of General Relativity: myths and reality”. In: *Central Eur.J.Phys.9:576-615,2011* 9.3 (Aug. 31, 2008). DOI: 10.2478/s11534-010-0072-2. arXiv: 0809.0097 [gr-qc].

[6] Eric Poisson. *A Relativist’s Toolkit*. Cambridge University Press, 2004. DOI: 10.1017/cbo9780511606601.