Non-relativistic Particle in an Exponential Potential

Yi-Fan Wang (王 一帆)

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Consider the one-dimensional motion of a non-relativistic particle in an exponential potential, the motion of which can be described by the Lagrangian action

$$S := \int \mathbb{d}t \left\{ \frac{m}{2}\dot{x}^2 - V\mathbb{e}^{gx} \right\}, \tag{1}$$

where g and V are real quantities. One sees that when $V > 0$ (< 0), the potential is bounded below (above), and the second case is potentially problematic.

1 Canonical formalism

The canonical Hamiltonian of the particle reads

$$H = \frac{p^2}{2m} + V\mathbb{e}^{gx}. \tag{2}$$

2 Canonical quantisation

Using the Laplace–Beltrami operator, the Hamiltonian "operator" reads

$$\widehat{H} = -\frac{\hbar^2}{2m}\partial_x^2 + V\mathbb{e}^{gx}. \tag{3}$$

Note that the domain of the unbounded operator has not been specified; hence comes the quotation marks. In [1, ch. 4], it was suggested that one could use *operation* instead of "operator" to distinguish the case, where only the action of an operator is described, whereas the domain is not.

2.1 Spectrum and generalised eigenfunctions of the Hamiltonian

The eigenvalue equation of the Hamiltonian, or the time-independent Schrödinger equation, reads

$$-\frac{\hbar^2}{2m}\partial_x^2\psi(x) + V\mathbb{e}^{gx}\psi(x) = E\psi(x). \tag{4}$$

Defining

$$v := \frac{\sqrt{8m|E|}}{g\hbar}, \tag{5}$$

and transforming the coordinate

$$\xi := \frac{\sqrt{8m|V|\mathbb{e}^{gx}}}{g\hbar} \tag{6}$$

yield the Hamiltonian “operator” in terms of dimensionless variables

$$\widehat{H} = \frac{g^2\hbar^2}{8m}\Big(-\xi^2\partial_\xi^2 - \xi\partial_\xi + \mathscr{v}\xi^2\Big), \qquad \mathscr{v} := \operatorname{sgn} V \tag{7}$$

and the corresponding eigenvalue equation in the standard Besselian form

$$\xi^2\psi''(\xi) + \xi^1\psi'(\xi) + \big(-\mathscr{v}\xi^2 + \mathscr{e}v^2\big)\psi(\xi) = 0, \qquad \mathscr{e} := \operatorname{sgn} E. \tag{8}$$

Transforming

$$\mathbb{e}^y := \xi = \frac{\sqrt{8m|V|\mathbb{e}^{gx}}}{g\hbar} \tag{9}$$

yields the Hamiltonian “operator” in terms of an alternative dimensionless form

$$\widehat{H} = \frac{g^2\hbar^2}{8m}\big(-\partial_y^2 + \mathscr{v}\mathbb{e}^{2y}\big). \tag{10}$$

2.2

References

[1] Dmitri Maximovitch Gitman, Igor Viktorovich Tyutin, and B.L. Voronov. *Self-adjoint Extensions in Quantum Mechanics. General Theory and Applications to Schrödinger and Dirac Equations with Singular Potentials.* Birkhäuser Boston, 2012. DOI: 10.1007/978-0-8176-4662-2.