Cosmological Perturbations

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Most of the conventions and notations in [6, ch. 5] will be followed.

Suppose the metric can be expanded up to the linear order as

$$g = g^{(0)} + \epsilon g^{(1)} + O(\epsilon^2). \tag{1}$$

The background metric $g^{(0)}$ takes the Robertson–Walker form

$$g^{(0)}_{\mu\nu} \,\mathrm{d}x^\mu \,\mathrm{d}x^\nu = -N^2(t) \,\mathrm{d}t^2 + a^2(t) \,\mathrm{d}\Omega^2_{3\mathrm{F}}, \tag{2}$$

in which $\mathrm{d}\Omega^2_{3\mathrm{F}} = \mathrm{d}\chi^2 + \chi^2\left(\mathrm{d}\theta^2 + \sin^2\theta \,\mathrm{d}\phi^2\right)$ is the dimensionless flat spatial metric. The linear perturbation can be decomposed into scalar, vector and tensor parts

$$g^{(1)}_{00} = -E, \tag{3}$$

$$g^{(1)}_{i0} = g^{(1)}_{i0} = F_{,i} + G_i, \tag{4}$$

$$g^{(1)}_{ij} = A\delta_{ij} + B_{,i,j} + C_{i,j} + C_{j,i} + D_{ij}. \tag{5}$$

Here one has some weird condition, where the contractions do not follow the one-up-one-down tradition

$$C_{i,i} = G_{i,i} = 0, \quad D_{ij,i} = 0, \quad D_{ii} = 0. \tag{6}$$

1 Metric perturbation under diffeomorphism

Consider a diffeomorphism generated by ξ^μ

$$x^\mu \to \overline{x}^\mu = x^\mu - \epsilon\xi^\mu. \tag{7}$$

The generator ξ^μ can in turn be decomposed into $\xi_0 = \zeta$, $\xi_i = \xi^{\mathrm{S}}_{,i} + \xi^{\mathrm{V}}_i$.

One has here again some weird condition, where the contractions do not follow the one-up-one-down tradition

$$\xi^{\mathrm{V}}_{i,i} = 0. \tag{8}$$

The Lie derivative of the metric $\mathbb{L}_\xi g$ is

$$\left(\mathbb{L}_\xi g\right)_{\mu\nu} = \xi^\lambda g_{\mu\nu,\lambda} + \xi^\lambda{}_{,\mu} g_{\lambda\nu} + \xi^\lambda{}_{,\nu} g_{\mu\lambda}. \tag{9}$$

In components and expansion, these are

$$\left(\mathbb{L}_\xi g\right)_{00} = 2\dot{\zeta} - 2\zeta\frac{\dot{N}}{N} + O(\epsilon), \tag{10}$$

$$\left(\mathbb{L}_\xi g\right)_{i0} = \left(\mathbb{L}_\xi g\right)_{0i} = \left(\zeta - 2\frac{\dot{a}}{a}\xi^{\mathrm{S}} + \dot{\xi}^{\mathrm{S}}\right)_{,i} + \left(-2\frac{\dot{a}}{a}\xi_i^{\mathrm{V}} + \dot{\xi}_i^{\mathrm{V}}\right) + O(\epsilon), \tag{11}$$

$$\left(\mathbb{L}_\xi g\right)_{ji} = \left(\mathbb{L}_\xi g\right)_{ij} = -\frac{2a\dot{a}}{N^2}\zeta\delta_{ij} + 2\xi^{\mathrm{S}}_{,i,j} + \xi^{\mathrm{V}}_{i,j} + \xi^{\mathrm{V}}_{j,i} + O(\epsilon). \tag{12}$$

2 Scalar perturbations

$$-N^2 - \epsilon E + O(\epsilon^2) \to -N^2 - \epsilon E + \epsilon\left(2\dot{\zeta} - 2\zeta\frac{\dot{N}}{N}\right) + O(\epsilon^2), \tag{13}$$

so one can write

$$\mathbb{L}_\xi E = -2\dot{\zeta} + 2\zeta\frac{\dot{N}}{N}. \tag{14}$$

Similarly one can read-off

$$\mathbb{L}_\xi F = \zeta - 2\frac{\dot{a}}{a}\xi^{\mathrm{S}} + \dot{\xi}^{\mathrm{S}}, \tag{15}$$

$$\mathbb{L}_\xi A = -\frac{2a\dot{a}}{N^2}\zeta, \tag{16}$$

$$\mathbb{L}_\xi B = 2\xi^{\mathrm{S}}. \tag{17}$$

The four scalar perturbations are generated by ζ and ξ^{S}, so that only two independent perturbations exists. It is clear that

$$\mathbb{L}_\xi\left(\frac{F}{a} - \frac{\mathbb{d}}{\mathbb{d}t}\frac{B}{2a}\right) = \frac{\zeta}{a}. \tag{18}$$

One can verify that

$$\mathbb{L}_\xi\left\{\frac{E}{2N} + \frac{\mathbb{d}}{\mathbb{d}t}\left[\frac{a}{N}\left(\frac{F}{a} - \frac{\mathbb{d}}{\mathbb{d}t}\frac{B}{2a}\right)\right]\right\} = 0, \tag{19}$$

$$\mathbb{L}_\xi\left\{\frac{A}{2} + \frac{a^2\dot{a}}{N^2}\left(\frac{F}{a} - \frac{\mathbb{d}}{\mathbb{d}t}\frac{B}{2a}\right)\right\} = 0. \tag{20}$$

3 Vector perturbations

4 Tensor perturbations

5 Scalar field perturbation under diffeomorphism

6 Perturbation in Arnowitt–Deser–Misner Hamiltonian formalism

Up to boundary terms, the Hamiltonian action for General Relativity in terms of Arnowitt–Deser–Misner variables is [3, ch.4.2.2]

$$S = \int \mathbb{d}t\,\mathbb{d}x^3 \left\{ \mathfrak{p}^{ij}\dot{h}_{ij} + \mathfrak{P}\dot{N} + \mathfrak{P}^i\dot{N}_i - N\mathfrak{H}^\perp - N_i\mathfrak{H}^i - \mathfrak{P}V - \mathfrak{P}^iV_i \right\}, \tag{21}$$

$$\mathfrak{H}^\perp = 2\varkappa\mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl} - \frac{\sqrt{\mathfrak{h}}}{2\varkappa}R[h] = 2\varkappa\mathfrak{F}^{ijkl}h_{ij}h_{kl} - \frac{\sqrt{\mathfrak{h}}}{2\varkappa}R[h], \tag{22}$$

$$\mathfrak{H}^i = -2\mathfrak{p}^{ij}{}_{|j}, \tag{23}$$

$$\mathfrak{G}_{ijkl} := \frac{1}{2\sqrt{\mathfrak{h}}}\left(h_{ik}h_{jl} + h_{il}h_{kj} - h_{ij}h_{kl}\right), \tag{24}$$

$$\mathfrak{F}^{ijkl} := \frac{1}{2\sqrt{\mathfrak{h}}}\left(\mathfrak{p}^{ik}\mathfrak{p}^{jl} + \mathfrak{p}^{il}\mathfrak{p}^{kj} - \mathfrak{p}^{ij}\mathfrak{p}^{kl}\right), \tag{25}$$

where V and V_i are velocities of N and N_i and play the role of Lagrange multipliers. Technical details about the boundary terms can be found in [5, ch. 4.2] and the references therein. Note that $\left\{N, N_i, h_{ij}; \mathfrak{P}, \mathfrak{P}^i, \mathfrak{p}^{ij}\right\}$ are not the unique choice of canonical variables for General Relativity in Hamiltonian formalism; instead, they are a special parametrisation of the phase space. One can also choose the components of the original four-metric and their conjugate momenta $\left\{g_{\mu\nu}; \mathfrak{p}^{\mu\nu}\right\}$ as canonical variables, as Dirac has done in [2]. The two approaches are different in some subtle aspects; see [4] for a comparison.

Gauge transformations in the Arnowitt–Deser–Misner canonical variables are generated by [1]

$$\begin{aligned} G = -\int \mathbb{d}^3x \Big\{ &\Big[\xi_\perp\Big(\mathfrak{H}^\perp + N_{|i}\mathfrak{P}^i + \left(N\mathfrak{P}^i\right)_{|i} + \left(N_i\mathfrak{P}\right)^{|i}\Big) + \dot{\xi}_\perp\mathfrak{P}\Big] \\ &+ \Big[\xi_i\Big(\mathfrak{H}^i + N_j{}^{|i}\mathfrak{P}^j + \left(N_j\mathfrak{P}^i\right)^{|j} + N^{|i}\mathfrak{P}\Big) + \dot{\xi}_i\mathfrak{P}^i\Big]\Big\}. \end{aligned} \tag{26}$$

Possible boundary terms have not been discussed so far. The infinitesimal gauge transformation of $\{N, N_i\}$ is

$$\delta N = [N, G]_{\mathrm{P}} = \xi_\perp{}^{|i}N_zi - \dot{\xi}_\perp - \xi_iN^{|i}, \tag{27}$$

$$\delta N_i = -\xi_\perp N_{|i} + \xi_{\perp|i}N - \xi_jN_i{}^{|j} + \xi_i{}^{|j}N_j - \dot{\xi}_i, \tag{28}$$

which can be found in [4]. Transformations for g_{ij} and the momenta have to be worked out as

$$\delta\mathfrak{P} = -(\xi_\perp\mathfrak{P}^i)_{|i} - \xi_{\perp|i}\mathfrak{P}^i - (\xi_i\mathfrak{P})^{|i}, \tag{29}$$

$$\delta\mathfrak{P}^i = -\xi_{\perp|i}\mathfrak{P} - \left(\xi_j\mathfrak{P}^i\right)^{|j} - \xi_j{}^{|i}\mathfrak{P}^j, \tag{30}$$

where only the primary constraints are involved;

$$\begin{aligned}
\delta g_{ij} &= -\frac{\partial}{\partial\mathfrak{p}^{ij}}(\xi_\perp\mathfrak{H}^\perp + \xi_i\mathfrak{H}^i) \\
&= -\xi^\perp\frac{2\varkappa}{\sqrt{\mathfrak{h}}}\big(h_{ik}h_{jl} + h_{il}h_{kj} - h_{ij}h_{kl}\big)\mathfrak{p}^{kl} - \xi_{i|j} - \xi_{j|i},
\end{aligned} \tag{31}$$

$$\begin{aligned}
\delta\mathfrak{p}^{ij} &= \frac{\partial}{\partial g_{ij}}(\xi_\perp\mathfrak{H}^\perp + \xi_i\mathfrak{H}^i) \\
&= \xi_\perp\Bigg\{\frac{\varkappa}{\sqrt{\mathfrak{h}}}\Bigg[(4\delta^i{}_k\delta^j{}_m - h^{ij}h_{km})h_{ln} \\
&\qquad - \frac{1}{2}(4\delta^i{}_k\delta^j{}_l - h^{ij}h_{kl})h_{mn}\Bigg]\mathfrak{p}^{kl}\mathfrak{p}^{mn} \\
&\qquad + \frac{\sqrt{\mathfrak{h}}}{2\varkappa}G^{ij}[h]\Bigg\} + \frac{\sqrt{\mathfrak{h}}}{2\varkappa}\left(\xi_{\perp|k}{}^{|k}h^{ij} - \xi_\perp^{|(i|j)}\right) \\
&\quad - \left(\xi_k\mathfrak{p}^{ij}\right)^{|k} + 2\xi_{k|l}h^{k(i}\mathfrak{p}^{j)l},
\end{aligned} \tag{32}$$

where $G^{ij}[h] = R^{ij}[h] - h^{ij}R[h]/2$, and only the secondary constraints are involved. In eq. (32), the first two lines come from the variation of the 'kinetic' term in $\mathfrak{H}^\perp$, the third comes from the 'potential' term in $\mathfrak{H}^\perp$, and the last line from $\mathfrak{H}^i$. The results can be checked with [5, p. 4.2.7].

6.1 Expansion of the action with fluctuations

Some useful results

First variations

First variation of h^{ij}

$$\delta h^{ij} = -h^{ik}h^{jl}\,\delta h_{kl} = -h^{i(k}h^{l)j}\,\delta h_{kl}. \tag{33}$$

First variation of $\mathfrak{h} = \det h_{ij}$

$$\delta\mathfrak{h} = \mathfrak{h}h^{ij}\,\delta h_{ij}. \tag{34}$$

First variation of $\Gamma^i{}_{jk}$

$$\begin{aligned}
\delta\Gamma^i{}_{jk} &= \frac{1}{2}h^{il}\Big\{-\big(\delta h_{jk}\big)_{|l} + (\delta h_{kl})_{|j} + \big(\delta h_{lj}\big)_{|k}\Big\} \qquad (35)\\
&= \frac{1}{2}\big\{-h^{il}\delta^m{}_j\delta^n{}_k + h^{in}\delta^l{}_j\delta^m{}_k + h^{im}\delta^n{}_j\delta^l{}_k\big\}\,(\delta h_{mn})_{|l}. \qquad (36)
\end{aligned}$$

First variation of $R_{ij}[h]$ and $R^{ij}[h]$

$$\delta R_{ij}[h] = \left(\delta\Gamma^k{}_{ij}\right)_{|k} - \left(\delta\Gamma^k{}_{ik}\right)_{|j}, \tag{37}$$

$$\delta R^{ij}[h] = -2R^{ki}h^{jl}\,\delta h_{kl} + h^{ik}\,\bar{\delta}u^{jl}{}_{k|l}, \tag{38}$$

where

$$\bar{\delta}u^{ij}{}_k := h^{il}\,\delta\Gamma^j{}_{kl} - h^{ij}\,\delta\Gamma^l{}_{kl}. \tag{39}$$

Equation (37) can be obtained by using normal coordinates.

First variation of $\sqrt{\mathfrak{h}}\,R[h]$

$$\delta\left(\sqrt{\mathfrak{h}}R[h]\right) = \sqrt{\mathfrak{h}}\left\{-G^{ij}[h]\,\delta h_{ij} + \bar{\delta}u^{ji}{}_{j|i}\right\}. \tag{40}$$

First variation of $\mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl} \equiv \mathfrak{F}^{ijkl}h_{ij}h_{kl}$

$$\begin{aligned}&\delta\left(\mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl}\right) \equiv \delta\left(\mathfrak{F}^{ijkl}h_{ij}h_{kl}\right)\\ &= \delta h_{ij}\left(-\frac{1}{2}h^{ij}\mathfrak{F}^{klmn}h_{mn} + 2\mathfrak{F}^{ijkl}\right)h_{kl} + \delta\mathfrak{p}^{ij}\,2\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}.\end{aligned} \tag{41}$$

First variation of $\mathfrak{H}^\perp$

$$\begin{aligned}\delta\mathfrak{H}^\perp = \delta h_{ij}&\left\{2\varkappa\left(-\frac{1}{2}h^{ij}\mathfrak{F}^{klmn}h_{mn} + 2\mathfrak{F}^{ijkl}\right)h_{kl}\right.\\ &\left.+\frac{\sqrt{\mathfrak{h}}}{2\varkappa}\left(G^{ij}[h] - \bar{\delta}u^{ji}{}_{j|i}\right)\right\} + \delta\mathfrak{p}^{ij}\,4\varkappa\mathfrak{G}_{ijkl}\mathfrak{p}^{kl},\end{aligned} \tag{42}$$

where the term with a $\sqrt{\mathfrak{h}}/2\varkappa$ factor comes from variation of the three Ricci scalar.

First variation of $\mathfrak{H}^i$

$$\delta\mathfrak{H}^i = \left(\delta p^{ij}\right)_{|j} + \delta\Gamma^i{}_{jk}\,p^{jk}. \tag{43}$$

First variation of $\mathfrak{F}^{ijkl}h_{kl}$

$$\begin{aligned}\delta(\mathfrak{F}^{ijkl}h_{kl}) = \delta h_{kl}&\left(-\frac{1}{2}\mathfrak{F}^{ijmn}h^{kl}h_{mn} + \mathfrak{F}^{ijkl}\right)\\ &+\delta\mathfrak{p}^{kl}\left(\delta^i{}_k\mathfrak{G}^j{}_{lmn} + \delta^i{}_m\mathfrak{G}^j{}_{nkl}\right)\mathfrak{p}^{mn}.\end{aligned} \tag{44}$$

First variation of $\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}$

$$\begin{aligned}\delta\left(\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}\right) = \delta h_{kl}&\left(-\frac{1}{2}\mathfrak{G}_{ijmn}h^{kl} + \delta^k{}_i\mathfrak{G}^l{}_{jmn} + \delta^k{}_m\mathfrak{G}^l{}_{nij}\right)\mathfrak{p}^{mn}\\ &+\delta p^{kl}\,\mathfrak{G}_{ijkl}.\end{aligned} \tag{45}$$

First variation of $\sqrt{\mathfrak{h}}\,G^{ij}[h]$

$$
\begin{aligned}
\delta\Big(\sqrt{\mathfrak{h}}\,G^{ij}[h]\Big) = \sqrt{\mathfrak{h}}\Bigg\{&\delta h_{kl}\cdot \\
&\left(-\frac{1}{2}\right)\left(R^{ik}h^{lj} + R^{il}h^{kj} - R^{ij}h^{kl} + h^{ik}G^{lj} + h^{il}G^{kj} - h^{ij}G^{kl}\right) \\
&+ \left(h^{il}\,\bar{\delta}u^{jk}{}_{l} - \frac{1}{2}h^{ij}\,\bar{\delta}u^{lk}{}_{l}\right)_{|k}\Bigg\}.
\end{aligned}
\tag{46}
$$

Second variations

Second variation of $\mathfrak{H}^{\perp}$

$$
\begin{aligned}
&\delta^2(\mathfrak{H}^{\perp}) \\
&= \delta h_{ij}\,\delta h_{kl}\,\Bigg\{2\varkappa\Bigg[\frac{1}{4}(h^{ik}h^{lj} + h^{il}h^{kj} + h^{ij}h^{kl})\mathfrak{F}^{mnrs}h_{mn}h_{rs} \\
&\qquad\qquad - (h^{ij}\mathfrak{F}^{klmn} + \mathfrak{F}^{ijmn}h^{kl})h_{mn} + \mathfrak{F}^{ijkl}\Bigg] \\
&\qquad - \frac{\sqrt{\mathfrak{h}}}{4\varkappa}(R^{ik}h^{lj} + R^{il}h^{kj} - R^{ij}h^{kl} + h^{ik}G^{lj} + h^{il}G^{kj} - h^{ij}G^{kl})\Bigg\} \\
&\quad + \delta h_{ij}\,\frac{\sqrt{\mathfrak{h}}}{2\varkappa}\left(h^{il}\,\bar{\delta}u^{jk}{}_{l} - \frac{1}{2}h^{ij}\,\bar{\delta}u^{lk}{}_{l}\right)_{|k} \\
&\quad + \delta h_{ij}\,\delta\mathfrak{p}^{kl}\,4\varkappa\{-h^{ij}\mathfrak{G}_{klmn} + 2(\delta^{i}{}_{k}\mathfrak{G}^{j}{}_{lmn} + \delta^{i}{}_{m}\mathfrak{G}^{j}{}_{nkl})\}\mathfrak{p}^{mn} \\
&\quad + \delta p^{ij}\,\delta p^{kl}\,4\varkappa\mathfrak{G}_{ijkl} \\
&\quad - \delta h_{ij}\,\delta\left(\frac{\sqrt{\mathfrak{h}}}{2\varkappa}\,\bar{\delta}u^{lk}{}_{l|k}\right).
\end{aligned}
\tag{47}
$$

Second variation of $\Gamma^{i}{}_{jk}$

$$
\delta^2\Gamma^{i}{}_{jk} = -h^{im}\,\delta\Gamma^{l}{}_{jk}\,\delta h_{lm}. \tag{48}
$$

Second variation of $\mathfrak{H}^{i}$

$$
\delta^2(\mathfrak{H}^{i}) = -h^{im}p^{jk}\,\delta\Gamma^{l}{}_{jk}\,\delta h_{lm} + 2\,\delta\Gamma^{i}{}_{jk}\,\delta p^{jk}. \tag{49}
$$

Other second variations

Second variation of h^{ij}

$$
\delta^2 h^{ij} = (h^{im}h^{jl}h^{kn} + h^{ik}h^{jm}h^{ln})\,\delta h_{kl}\,\delta h_{mn} \tag{50}
$$

Second variation of $\mathfrak{h} = \det h_{ij}$

$$
\delta^2\mathfrak{h} = -\frac{1}{4}\mathfrak{h}(h^{ik}h^{jl} + h^{il}h^{kj} - h^{ij}h^{kl})\,\delta h_{ij}\,\delta h_{kl}. \tag{51}
$$

First variation of $\left(\delta h_{ij}\right)_{|k}$

$$\delta\left\{\left(\delta h_{ij}\right)_{|k}\right\} = -2\delta\Gamma^{l}{}_{k(i}\,\delta h_{j)l}. \tag{52}$$

In a general background, the second variations of the quantities are much more tedious.

In Robertson–Walker background,

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