Cosmological Perturbations

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Most of the conventions and notations in [6, ch. 5] will be followed.

Suppose the metric can be expanded up to the linear order as

$$g = g^{(0)} + \epsilon g^{(1)} + O(\epsilon^2). \tag{1}$$

The background metric $g^{(0)}$ takes the Robertson–Walker form

$$g^{(0)}_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu = -N^2(t) \, \mathrm{d}t^2 + a^2(t) \, \mathrm{d}\Omega^2_{3\mathrm{F}}, \tag{2}$$

in which $\mathrm{d}\Omega^2_{3\mathrm{F}} = \mathrm{d}\chi^2 + \chi^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2\right)$ is the dimensionless flat spatial metric. The linear perturbation can be decomposed into scalar, vector and tensor parts

$$g^{(1)}_{00} = -E, \tag{3}$$

$$g^{(1)}_{i0} = g^{(1)}_{i0} = F_{,i} + G_i, \tag{4}$$

$$g^{(1)}_{ij} = A\delta_{ij} + B_{,i,j} + C_{i,j} + C_{j,i} + D_{ij}. \tag{5}$$

Here one has some weird condition, where the contractions do not follow the one-up-one-down tradition

$$C_{i,i} = G_{i,i} = 0, \quad D_{ij,i} = 0, \quad D_{ii} = 0. \tag{6}$$

1 Metric perturbation under diffeomorphism

Consider a diffeomorphism generated by ξ^μ

$$x^\mu \to \overline{x}^\mu = x^\mu - \epsilon \xi^\mu. \tag{7}$$

The generator ξ^μ can in turn be decomposed into $\xi_0 = \zeta$, $\xi_i = \xi^{\mathrm{S}}_{,i} + \xi^{\mathrm{V}}_i$.

One has here again some weird condition, where the contractions do not follow the one-up-one-down tradition

$$\xi^{\mathrm{V}}_{i,i} = 0. \tag{8}$$

The Lie derivative of the metric $\mathbb{L}_\xi g$ is

$$\left(\mathbb{L}_\xi g\right)_{\mu\nu} = \xi^\lambda g_{\mu\nu,\lambda} + \xi^\lambda{}_{,\mu} g_{\lambda\nu} + \xi^\lambda{}_{,\nu} g_{\mu\lambda}. \tag{9}$$

In components and expansion, these are

$$\left(\mathbb{L}_\xi g\right)_{00} = 2\dot\zeta - 2\zeta\frac{\dot N}{N} + O(\epsilon), \tag{10}$$

$$\left(\mathbb{L}_\xi g\right)_{i0} = \left(\mathbb{L}_\xi g\right)_{0i} = \left(\zeta - 2\frac{\dot a}{a}\xi^{\mathrm{S}} + \dot\xi^{\mathrm{S}}\right)_{,i} + \left(-2\frac{\dot a}{a}\xi^{\mathrm{V}}_i + \dot\xi^{\mathrm{V}}_i\right) + O(\epsilon), \tag{11}$$

$$\left(\mathbb{L}_\xi g\right)_{ji} = \left(\mathbb{L}_\xi g\right)_{ij} = -\frac{2a\dot a}{N^2}\zeta\delta_{ij} + 2\xi^{\mathrm{S}}_{,i,j} + \xi^{\mathrm{V}}_{i,j} + \xi^{\mathrm{V}}_{j,i} + O(\epsilon). \tag{12}$$

2 Scalar perturbations

$$-N^2 - \epsilon E + O(\epsilon^2) \to -N^2 - \epsilon E + \epsilon\left(2\dot\zeta - 2\zeta\frac{\dot N}{N}\right) + O(\epsilon^2), \tag{13}$$

so one can write

$$\mathbb{L}_\xi E = -2\dot\zeta + 2\zeta\frac{\dot N}{N}. \tag{14}$$

Similarly one can read-off

$$\mathbb{L}_\xi F = \zeta - 2\frac{\dot a}{a}\xi^{\mathrm{S}} + \dot\xi^{\mathrm{S}}, \tag{15}$$

$$\mathbb{L}_\xi A = -\frac{2a\dot a}{N^2}\zeta, \tag{16}$$

$$\mathbb{L}_\xi B = 2\xi^{\mathrm{S}}. \tag{17}$$

The four scalar perturbations are generated by ζ and ξ^{S}, so that only two independent perturbations exists. It is clear that

$$\mathbb{L}_\xi\left(\frac{F}{a} - \frac{\mathbb{d}}{\mathbb{d}t}\frac{B}{2a}\right) = \frac{\zeta}{a}. \tag{18}$$

One can verify that

$$\mathbb{L}_\xi\left\{\frac{E}{2N} + \frac{\mathbb{d}}{\mathbb{d}t}\left[\frac{a}{N}\left(\frac{F}{a} - \frac{\mathbb{d}}{\mathbb{d}t}\frac{B}{2a}\right)\right]\right\} = 0, \tag{19}$$

$$\mathbb{L}_\xi\left\{\frac{A}{2} + \frac{a^2\dot a}{N^2}\left(\frac{F}{a} - \frac{\mathbb{d}}{\mathbb{d}t}\frac{B}{2a}\right)\right\} = 0. \tag{20}$$

3 Vector perturbations

4 Tensor perturbations

5 Scalar field perturbation under diffeomorphism

6 Perturbation of Arnowitt–Deser–Misner Hamiltonian formalism

The well known Arnowitt–Deser–Misner's Hamiltonian action for gravitation is [3, ch.4.2.2]

$$S = \int \mathrm{d}t\, \mathrm{d}x^3 \left\{ \mathfrak{p}^{ij} \dot{h}_{ij} + \mathfrak{P}\dot{N} + \mathfrak{P}^i \dot{N}_i - N\mathfrak{H}^\perp - N_i \mathfrak{H}^i - \mathfrak{P}V - \mathfrak{P}^i V_i \right\}, \quad (21)$$

$$\mathfrak{H}^\perp = \frac{\varkappa}{\sqrt{\mathfrak{h}}} (h_{ik}h_{jl} + h_{il}h_{kj} - h_{ij}h_{kl}) \mathfrak{p}^{ij}\mathfrak{p}^{kl} - \frac{\sqrt{\mathfrak{h}}}{2\varkappa} R[h], \quad (22)$$

$$\mathfrak{H}^i = -2\mathfrak{p}^{ij}{}_{|j}, \quad (23)$$

where $\{V, V_i\}$ are velocity of N and N_i and play the role of Lagrange multipliers. Note that $\left\{N, N_i, h_{ij}; \mathfrak{P}, \mathfrak{P}^i, \mathfrak{p}^{ij}\right\}$ are not the unique choice of canonical variables for General Relativity in Hamiltonian formalism; instead, they are a special parametrisation of the phase space. One can also choose the components of the original four-metric and their conjugate momenta $\left\{g_{\mu\nu}; \mathfrak{p}^{\mu\nu}\right\}$ as canonical variables, as Dirac has done [2]. The two approaches are different in some subtle aspects; see [4] for a comparison.

Gauge transformations in the Arnowitt–Deser–Misner variables are generated by [1]

$$\begin{aligned} G = -\int \mathrm{d}^3x \bigg\{ &\left[\xi_\perp \left(\mathfrak{H}^\perp + N_{|i}\mathfrak{P}^i + \left(N\mathfrak{P}^i\right)_{|i} + (N_i \mathfrak{P})^{|i} \right) + \dot{\xi}_\perp \mathfrak{P} \right] \\ &+ \left[\xi_i \left(\mathfrak{H}^i + N_j{}^{|i}\mathfrak{P}^j + \left(N_j \mathfrak{P}^i\right)^{|j} + N^{|i}\mathfrak{P} \right) + \dot{\xi}_i \mathfrak{P}^i \right] \bigg\}, \end{aligned} \quad (24)$$

and the infinitesimal gauge transformation of N is

$$\delta N = [N, G]_{\mathrm{P}} = \xi_{\perp|i} N^i - \dot{\xi}_\perp - \xi_i N^{|i}, \quad (25)$$

$$\delta N_i = -\xi_\perp N_{|i} + \xi_{\perp|i} N - \xi_j N_i{}^{|j} + \xi_i{}^{|j} N_j - \dot{\xi}_i, \quad (26)$$

which can be found in [4]. Transformations for g_{ij} and the momenta have to be worked out as

$$\delta\mathfrak{P} = -\left(\xi_\perp \mathfrak{P}^i\right)_{|i} - \xi_{\perp|i}\mathfrak{P}^i - (\xi_i \mathfrak{P})^{|i}, \quad (27)$$

$$\delta\mathfrak{P}^i = -\xi_{\perp|i}\mathfrak{P} - \left(\xi_j \mathfrak{P}^i\right)^{|j} - \xi_j{}^{|i}\mathfrak{P}^j, \quad (28)$$

where only the primary constraints are involved;

$$\begin{aligned}
\delta g_{ij} &= -\frac{\partial}{\partial \mathfrak{p}^{ij}}(\xi_\perp \mathfrak{H}^\perp + \xi_i \mathfrak{H}^i) \\
&= -\xi^\perp \frac{2\varkappa}{\sqrt{\mathfrak{h}}}(h_{ik}h_{jl} + h_{il}h_{kj} - h_{ij}h_{kl})\mathfrak{p}^{kl} - \xi_{i|j} - \xi_{j|i},
\end{aligned} \tag{29}$$

$$\begin{aligned}
\delta \mathfrak{p}^{ij} &= \frac{\partial}{\partial g_{ij}}(\xi_\perp \mathfrak{H}^\perp + \xi_i \mathfrak{H}^i) \\
&= \xi_\perp \Bigg\{ \frac{\varkappa}{\sqrt{\mathfrak{h}}} \Bigg[(4\delta^i{}_k \delta^j{}_m h_{ln} - h^{ij}h_{km}h_{ln}) \\
&\qquad - \frac{1}{2}(4\delta^i{}_k \delta^j{}_m h_{ln} - h^{ij}h_{km}h_{ln}) \Bigg] \mathfrak{p}^{kl}\mathfrak{p}^{mn} \\
&\quad + \frac{\sqrt{\mathfrak{h}}}{2\varkappa} G^{ij}[h] \Bigg\} + \frac{\sqrt{\mathfrak{h}}}{2\varkappa}\left(\xi_{\perp|k}{}^{|k} h^{ij} - \xi_\perp^{|(i|j)}\right) \\
&\quad - \xi_k{}^{|k}\mathfrak{p}^{ij} + 2\xi_{k|l}h^{k(i}p^{j)l},
\end{aligned} \tag{30}$$

where $G^{ij}[h] = R^{ij}[h] - h^{ij}R[h]/2$, and only the secondary constraints are involved. In eq. (30), the first two lines come from the variation of the 'kinetic' term in $\mathfrak{H}^\perp$, the third comes from the 'potential' term in $\mathfrak{H}^\perp$, and the last line from $\mathfrak{H}^i$. The results can be checked with [5, p. 4.2.7].

References

[1] Leonardo Castellani. "Symmetries in constrained Hamiltonian systems". In: *Annals of Physics* 143.2 (Oct. 1982), pp. 357–371. DOI: 10.1016/0003-4916(82)90031-8.

[2] Paul A. M. Dirac. "The Theory of Gravitation in Hamiltonian Form". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 246.1246 (Aug. 1958), pp. 333–343. DOI: 10.1098/rspa.1958.0142.

[3] Claus Kiefer. *Quantum Gravity*. 3rd ed. Oxford University Press, Apr. 2012. DOI: 10.1093/acprof:oso/9780199585205.001.0001.

[4] Natalia Kiriushcheva and Sergei Kuzmin. "The Hamiltonian formulation of General Relativity: myths and reality". In: *Central Eur.J.Phys.9:576-615,2011* 9.3 (Aug. 31, 2008). DOI: 10.2478/s11534-010-0072-2. arXiv: 0809.0097 [gr-qc].

[5] Eric Poisson. *A Relativist's Toolkit*. Cambridge University Press, 2004. DOI: 10.1017/cbo9780511606601.

[6] Steven Weinberg. *Cosmology*. Oxford University Press, 2008. ISBN: 9780198526827. URL: https://global.oup.com/academic/product/cosmology-9780198526827.