Draft of Notes of Quantum Mechanics of Generalised Squeezed Coherent State Classified for Internal Use Only

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Contents

1 Rotation operator

$$\widehat{R}(\theta) := \exp\mathbb{i}\left\{\theta\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\right\}. \tag{1.1}$$

$$\widehat{R}^{-1}(\theta) \equiv \widehat{R}^\dagger(\theta) \equiv \widehat{R}(-\theta). \tag{1.2}$$

$$\widehat{R}(\theta)\hat{a}\widehat{R}^\dagger(\theta) = \mathbb{e}^{\mathbb{i}\theta}\hat{a}. \tag{1.3}$$

2 Coherent state

Let

$$\hat{b}^- = \hat{a}^- - \alpha, \qquad \hat{b}^+ = \left(\hat{b}^-\right)^\dagger = \hat{a}^+ - \alpha^*. \tag{2.1}$$

One seeks the state $|\alpha\rangle$ satisfying

$$\hat{b}^- |\alpha\rangle = 0. \tag{2.2}$$

Expanding $\hat{b}^-$ in terms of $\hat{x}$ and $\hat{p}$,

$$\hat{b}^- = \sqrt{\frac{m\Omega}{2}}(\hat{x} - x_\alpha) + \frac{\mathbb{i}}{\sqrt{2m\Omega}}(\hat{p} - p_\alpha), \tag{2.3}$$

where

$$x_\alpha = \sqrt{\frac{2}{m\Omega}}\Re\alpha, \qquad p_\alpha = \sqrt{2m\Omega}\Im\alpha. \tag{2.4}$$

One may define

$$\hat{y} := \hat{x} - x_\alpha\hat{1} = \hat{x} - \mathbb{i}x_\alpha[\hat{p},\hat{x}]_- = \mathbb{e}^{-\mathbb{i}x_\alpha\hat{p}}\hat{x}\mathbb{e}^{+\mathbb{i}x_\alpha\hat{p}}, \tag{2.5}$$

$$\hat{k} := \hat{p} - p_\alpha\hat{1} = \hat{p} + \mathbb{i}p_\alpha[\hat{x},\hat{p}]_- = \mathbb{e}^{+\mathbb{i}p_\alpha\hat{x}}\hat{p}\mathbb{e}^{-\mathbb{i}p_\alpha\hat{x}}, \tag{2.6}$$

where in the last steps, Lemma C.2 has been applied. One can verify

$$\left[\hat{y},\hat{k}\right]_- = \mathbb{i}\hat{1}. \tag{2.7}$$

2.1 Displacement operator

Inspired by eqs. (2.5) and (2.6), one attempts

$$\widehat{D}(\alpha)\hat{a}^-\widehat{D}^{-1}(\alpha), \tag{2.8}$$

where

$$\widehat{D}(\alpha) := \mathbb{e}^{+\alpha\hat{a}^+ - \alpha^*\hat{a}^-} \equiv \mathbb{e}^{\mathbb{i}(-x_\alpha\hat{p} + p_\alpha\hat{x})} \tag{2.9}$$

is the *displacement operator*. Note that

$$\widehat{D}^{-1}(\alpha) = \widehat{D}(-\alpha) = \widehat{D}^\dagger(\alpha). \tag{2.10}$$

$$b^- = a^- - \alpha = a^- + \alpha[a^+, a^-]_- = a^- + [\alpha a^+ - \alpha^* a^-, a^-]_-. \tag{2.11}$$

Note that

$$\mathrm{ad}^{(n)}_{\alpha a^+ - \alpha^* a^-}\, a^- \equiv 0, \qquad \forall n \geq 2. \tag{2.12}$$

By Lemma C.2,

$$b^- = \mathbb{e}^{+\alpha a^+ - \alpha^* a^-} a^- \mathbb{e}^{-\alpha a^+ + \alpha^* a^-} = D(\alpha) a^- D(\alpha)^{-1}. \tag{2.13}$$

Now

$$a^- \left|0\right\rangle := 0 =: D^\dagger(\alpha) b^- \left|0\right\rangle = a^- D^\dagger(\alpha) \left|\alpha\right\rangle. \tag{2.14}$$

With $D(\alpha)$ unitary in mind, one finds

$$\left|\alpha\right\rangle = \mathbb{e}^{\mathbb{i}\theta} D(\alpha) \left|0\right\rangle. \tag{2.15}$$

One can fix the phase θ to be zero.

2.2 Time evolution

Compute

$$\widehat{\mathcal{U}}(t) \widehat{D}(\alpha) \widehat{\mathcal{U}}^\dagger(t). \tag{2.16}$$

Equation (2.16) simplifies to

$$\mathbb{e}^{-\mathbb{i}\Omega t a^+ a^-} D(\alpha) \mathbb{e}^{+\mathbb{i}\Omega t a^+ a^-} \tag{2.17}$$

One wishes to interchange $\mathbb{e}^{-\mathbb{i}\Omega t a^+ a^-}$ and $D(\alpha)$. By Theorem C.4, one computes

$$\begin{aligned}
\mathrm{ad}^{(1)}_{-\mathbb{i}\Omega t a^+ a^-} (\alpha a^+ - \alpha^* a^-) &= [-\mathbb{i}\Omega t a^+ a^-, \alpha a^+ - \alpha^* a^-]_- \\
&= -\mathbb{i}\Omega t \Big(\alpha a^+ [a^-, a^+]_- - \alpha^* [a^+, a^-]_- a^- \Big) \\
&= -\mathbb{i}\Omega t (+\alpha a^+ + \alpha^* a^-) \\
&= +\alpha(-\mathbb{i}\Omega t)^1 a^+ - \alpha(+\mathbb{i}\Omega t)^1 a^-;
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
\mathrm{ad}^{(2)}_{-\mathbb{i}\Omega t a^+ a^-} (\alpha a^+ - \alpha^* a^-) &= [-\mathbb{i}\Omega t a^+ a^-, -\mathbb{i}\Omega t(+\alpha a^+ + \alpha^* a^-)]_- \\
&= \Omega^2 t^2 (-\alpha a^+ + \alpha^* a^-) \\
&= +\alpha(-\mathbb{i}\Omega t)^2 a^+ - \alpha(+\mathbb{i}\Omega t)^2 a^-.
\end{aligned} \tag{2.19}$$

By induction one can prove

$$\mathrm{ad}^{(n)}_{-\mathbb{i}\Omega t a^+ a^-} (\alpha a^+ - \alpha^* a^-) = +\alpha(-\mathbb{i}\Omega t)^n a^+ - \alpha(+\mathbb{i}\Omega t)^n a^-, \tag{2.20}$$

so that eq. (2.16) transforms to

$$\exp\big(+\alpha \mathbb{e}^{-\mathbb{i}\Omega t} a^+ - \alpha^* \mathbb{e}^{+\mathbb{i}\Omega t} a^-\big) \mathbb{e}^{-\mathbb{i}\Omega t a^+ a^-} \mathbb{e}^{+\mathbb{i}\Omega t a^+ a^-} \tag{2.21}$$

$$= \exp\big(+\alpha \mathbb{e}^{-\mathbb{i}\Omega t} a^+ - \alpha^* \mathbb{e}^{+\mathbb{i}\Omega t} a^-\big) \tag{2.22}$$

$$\equiv D(\alpha^t). \tag{2.23}$$

by Theorem C.4, where

$$\alpha^t := \alpha \mathbb{e}^{-\mathbb{i}\Omega t}. \tag{2.24}$$

As a result, the time evolution of the coherent ground state is

$$\mathcal{U}(t) \left|\alpha\right\rangle = \mathcal{U}(t) D(\alpha) \left|0\right\rangle = D(\alpha^t) \mathcal{U}(t) \left|0\right\rangle = \mathbb{e}^{-\mathbb{i}\Omega t/2} \left|\alpha^t\right\rangle, \tag{2.25}$$

in terms of the coherent parameter α.

2.3 Particle numbers

(2.1) **Lemma**

$$\widehat{D}(\alpha) = \mathbb{e}^{-|\alpha|^2/2}\mathbb{e}^{+\alpha\hat{a}^+}\mathbb{e}^{-\alpha^*\hat{a}^-}. \tag{2.26}$$

Proof. Use Corollary C.3. □

(2.2) **Particle number representation of $|\alpha\rangle$**

$$\langle n \,|\, \alpha\rangle = \mathbb{e}^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}. \tag{2.27}$$

(2.3) **$\hat{a}$-particle number in $|\alpha\rangle$**

$$\langle \alpha \,|\, \hat{a}^+\hat{a}^- \,|\, \alpha\rangle = |\alpha|^2. \tag{2.28}$$

Proof.

$$\widehat{D}^\dagger(\alpha)\hat{a}^-\widehat{D}(\alpha) = \hat{a}^- + [-\alpha\hat{a}^+, \hat{a}^-]_- = \hat{a}^- + \alpha; \tag{2.29}$$

$$\widehat{D}^\dagger(\alpha)\hat{a}^+\hat{a}^-\widehat{D}(\alpha) = \widehat{D}^\dagger(\alpha)\hat{a}^+\widehat{D}(\alpha)\widehat{D}^\dagger(\alpha)\hat{a}^-\widehat{D}(\alpha) = (\hat{a}^+ + \alpha^*)(\hat{a}^- + \alpha). \tag{2.30}$$

□

(2.4) **$\hat{b}$-particle number in $\hat{a}$-ground state**

$$\left\langle 0 \,\middle|\, \hat{b}^+\hat{b}^- \,\middle|\, 0\right\rangle = |\alpha|^2 \tag{2.31}$$

Proof.

$$\hat{b}^+\hat{b}^- = (\hat{a}^+ - \alpha^*)(\hat{a}^- - \alpha) \tag{2.32}$$

according to 2.1. □

(2.5) *Remark*
Since $|0\rangle$ is stationary, $\left\langle 0 \,\middle|\, \hat{b}^+\hat{b}^- \,\middle|\, 0\right\rangle$ is a constant of motion.

2.4 Wave function

Since $[+\mathbb{i}p_\alpha\hat{x}, -\mathbb{i}x_\alpha\hat{p}]_- = \mathbb{i}p_\alpha x_\alpha\hat{1}$ is central, one has

$$\begin{aligned}\langle x|\,\widehat{D}(\alpha) &= \langle x|\,\mathbb{e}^{\mathbb{i}(+p_\alpha\hat{x}-x_\alpha\hat{p})}\\ &= \langle x|\,\mathbb{e}^{+\mathbb{i}p_\alpha\hat{x}}\mathbb{e}^{-\mathbb{i}x_\alpha\hat{p}}\mathbb{e}^{-[+\mathbb{i}p_\alpha\hat{x},-\mathbb{i}x_\alpha\hat{p}]_-/2}\\ &= \mathbb{e}^{-\mathbb{i}p_\alpha x_\alpha/2}\mathbb{e}^{+\mathbb{i}p_\alpha x}\,\langle x|\,\mathbb{e}^{-\mathbb{i}x_\alpha\hat{p}}\\ &= \mathbb{e}^{-\mathbb{i}p_\alpha x_\alpha/2}\mathbb{e}^{+\mathbb{i}p_\alpha x}\,\langle x - x_\alpha|\,, \end{aligned} \tag{2.33}$$

using eq. (B.6). Similarly

$$\begin{aligned}\langle x|\,\widehat{D}(\alpha) &= \mathbb{e}^{-[-\mathbb{i}x_\alpha\hat{p},+\mathbb{i}p_\alpha\hat{x}]_-/2}\mathbb{e}^{-\mathbb{i}x_\alpha\hat{p}}\mathbb{e}^{+\mathbb{i}p_\alpha\hat{x}}\,|x\rangle\\ &= \mathbb{e}^{+\mathbb{i}p_\alpha x_\alpha/2}\mathbb{e}^{+\mathbb{i}p_\alpha x}\,|x + x_\alpha\rangle\,. \end{aligned} \tag{2.34}$$

Note the extra phase factors compared with eqs. (B.5) and (B.6).

Equation (2.33) solves the wave function of coherent ground state

$$\langle x \,|\, \alpha\rangle = \langle x \,|\, D(\alpha) \,|\, 0\rangle$$
$$= \left(\frac{m\Omega}{\pi}\right)^{1/4} \exp\left\{-\frac{1}{2}m\Omega(x - x_\alpha)^2\right\} \exp i\left\{p_\alpha x - \frac{1}{2}p_\alpha x_\alpha\right\}, \qquad (2.35)$$

as well as its time evolution

$$\left(\frac{m\Omega}{\pi}\right)^{-1/4} \langle x \,|\, \mathcal{U}(t) \,|\, \alpha\rangle$$
$$= e^{-i\Omega t/2} \left\langle x \,\middle|\, D(\alpha e^{-i\Omega t}) \,\middle|\, 0\right\rangle$$
$$= \exp\left\{-\frac{1}{2}m\Omega(x - x_\alpha^t)^2\right\} \exp i\left\{-\frac{1}{2}\Omega t + x p_\alpha^t - \frac{1}{2}p_\alpha^t x_\alpha^t\right\}, \qquad (2.36)$$

where

$$x_\alpha^t := x_{\alpha^t}, \qquad p_\alpha^t := p_{\alpha^t}, \qquad (2.37)$$

or

$$\begin{pmatrix} (m\Omega)^{+1/2} x_\alpha^t \\ (m\Omega)^{-1/2} p_\alpha^t \end{pmatrix} := \begin{pmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{pmatrix} \begin{pmatrix} (m\Omega)^{+1/2} x_\alpha \\ (m\Omega)^{-1/2} p_\alpha \end{pmatrix} \qquad (2.38)$$

2.5 Density matrix and Wigner function

(2.6) **Density matrix in the particle number representation**

$$\langle n_1 \,|\, \hat\rho(t) \,|\, n_2\rangle = e^{-|\alpha|^2} \frac{(\alpha^t)^{n_1} (\alpha^{t*})^{n_2}}{\sqrt{n_1!\, n_2!}} = e^{-|\alpha|^2} \frac{\alpha^{n_1} (\alpha^*)^{n_2}}{\sqrt{n_1!\, n_2!}} e^{-i\Omega(n_1 - n_2)t}. \qquad (2.39)$$

(2.7) **Density matrix in the position representation**

$$\langle x_1 \,|\, \hat\rho(t) \,|\, x_2\rangle$$
$$= \left(\frac{m\Omega}{\pi}\right)^{1/2} \exp\left\{-\frac{m\Omega}{2}\left((x_1 - x_\alpha^t)^2 + (x_1 - x_\alpha^t)^2\right)\right\} e^{+i p_\alpha^t (x_1 - x_2)} \qquad (2.40)$$

(2.8) **Wigner function**

$$W(x, p; t) = \frac{1}{\pi} \exp\left\{-m\Omega(x - x_\alpha^t)^2 - \frac{1}{m\Omega}(p - p_\alpha^t)^2\right\}. \qquad (2.41)$$

3 Single-mode squeezing

Let

$$\hat b^- = \frac{\hat a^- - \beta \hat a^+}{\sqrt{1 - |\beta|^2}}, \qquad \hat b^+ = \left(\hat b^-\right)^\dagger = \frac{\hat a^+ - \beta^* \hat a^-}{\sqrt{1 - |\beta|^2}}, \qquad (3.1)$$

so that

$$\left[\hat b^-, \hat b^+\right]_- = \left(1 - |\beta|^2\right)^{-1} \left([\hat a^-, \hat a^+]_- + \beta\beta^* [\hat a^+, \hat a^-]_-\right) = \hat 1. \qquad (3.2)$$

One seeks the state $|\beta\rangle$ satisfying

$$\hat b^- \,|\beta\rangle = 0. \qquad (3.3)$$

Note that eq. (3.1) is a restricted *Bogolyubov transformation*, in that the coefficient of $\hat a^-$ for $\hat b^-$ is real.

3.1 Single-mode squeeze operator

One attempts
$$\widehat{S}(z)\hat{a}^-\widehat{S}^{-1}(z), \tag{3.4}$$
where
$$\begin{aligned}\widehat{S}(z) &:= \exp\left\{\frac{1}{2}\Big(z(\hat{a}^+)^2 - z^*(\hat{a}^-)^2\Big)\right\} \\ &\equiv \exp\mathrm{i}\left\{\Im z\left(\frac{m\Omega}{2}\hat{x}^2 - \frac{2}{m\Omega}\hat{p}^2\right) - \Re z\frac{\hat{x}\hat{p}+\hat{p}\hat{x}}{2}\right\}\end{aligned} \tag{3.5}$$
is the *single-mode squeeze operator*. Note that
$$\widehat{S}^{-1}(z) = \widehat{S}(-z) = \widehat{S}^\dagger(z). \tag{3.6}$$

The parameterisation $z = r\mathrm{e}^{\mathrm{i}\phi}$ will also be used, where $r = |z|$, $\phi = \arg z$.

(3.1) **Proposition**
Let $X = \Big(z(\hat{a}^+)^2 - z^*(\hat{a}^-)^2\Big)/2$. *Then*
$$\mathrm{ad}_X^{(2n)}\,\hat{a}^- = |z|^{2n}\hat{a}^- = r^{2n}\hat{a}^-, \tag{3.7}$$
$$\mathrm{ad}_X^{(2n+1)}\,\hat{a}^- = -|z|^{2n+1}\frac{z}{|z|}\hat{a}^+ = -r^{2n+1}\mathrm{e}^{\mathrm{i}\phi}\hat{a}^+, \qquad \forall n \geq 0. \tag{3.8}$$

Proof.
$$\mathrm{ad}_X^{(0)}\,\hat{a}^- = \hat{a}^-; \tag{3.9}$$
$$\begin{aligned}\mathrm{ad}_X^{(1)}\,\hat{a}^- &= \left[\frac{z}{2}(\hat{a}^+)^2, \hat{a}^-\right]_- = \frac{z}{2}\Big(\hat{a}^+[\hat{a}^+,\hat{a}^-]_- + [\hat{a}^+,\hat{a}^-]_-\hat{a}^+\Big) = -z\hat{a}^+ \\ &= -\frac{z}{|z|}|z|^1\hat{a}^+ = -\mathrm{e}^{\mathrm{i}\phi}r^1\hat{a}^+,\end{aligned} \tag{3.10}$$
$$\begin{aligned}\mathrm{ad}_X^{(2)}\,\hat{a}^- &= \left[-\frac{z^*}{2}(\hat{a}^-)^2, -z\hat{a}^+\right]_- = \frac{|z|^2}{2}2\hat{a}^- \\ &= |z|^2\hat{a}^- = r^2\hat{a}^-;\end{aligned} \tag{3.11}$$
$$\begin{aligned}\mathrm{ad}_X^{(3)}\,\hat{a}^- &= \left[\frac{z}{2}(\hat{a}^+)^2, |z|^2\hat{a}^-\right]_- = \frac{z|z|^2}{2}(-2)\hat{a}^+ \\ &= -\frac{z}{|z|}|z|^3\hat{a}^+ = -\mathrm{e}^{\mathrm{i}\phi}r^3\hat{a}^+.\end{aligned} \tag{3.12}$$
Prop. (3.1) can then be proved by induction for even and odd integers, respectively. □

(3.2) **Proposition**
$$\begin{aligned}\widehat{S}(z)\hat{a}^-\widehat{S}^\dagger(z) &= \hat{a}^-\cosh|z| - \hat{a}^+\frac{z}{|z|}\sinh|z| \\ &= \hat{a}^-\cosh r - \hat{a}^+\mathrm{e}^{\mathrm{i}\phi}\sinh r,\end{aligned} \tag{3.13}$$
so that z *or* (r,ϕ) *is a parameterisation of* β *in eq.* (3.1) *in that*
$$|\beta| = \tanh|z| = \tanh r, \qquad \arg\beta = \arg z = \phi. \tag{3.14}$$

Proof. By prop. 3.1 and eq. (C.5),

$$\widehat{S}(z)\hat{a}^{-}\widehat{S}^{\dagger}(z) = \sum_{n=0}^{+\infty}\left(\frac{|z|^{2n}}{(2n)!}\hat{a}^{-} - \frac{z}{|z|}\frac{|z|^{2n+1}}{(2n+1)!}\hat{a}^{+}\right) \tag{3.15}$$

which proves eq. (3.13). Comparing eq. (3.15) with eq. (3.1), one gets

$$\begin{cases} \left(1-|\beta|^{2}\right)^{-1/2} = \cosh|z| = \cosh r, \\ \beta(1-|\beta|^{2})^{-1/2} = \frac{z}{|z|}\sinh|z| = \mathbb{e}^{\mathbb{i}\phi}\sinh r, \end{cases} \tag{3.16}$$

which gives eq. (3.14). □

(3.3) **Squeezed ground state**
$|\beta\rangle$ in eq. (3.3) is the β-squeezed ground state, namely

$$|\beta\rangle = \widehat{S}(z)\,|0\rangle\,. \tag{3.17}$$

Proof. Note that

$$a^{-}\,|0\rangle := 0 =: S^{\dagger}(\beta)b^{-}\,|0\rangle = a^{-}S^{\dagger}(\beta)\,|\beta\rangle\,. \tag{3.18}$$

With $S(\alpha)$ unitary in mind, one finds

$$|\beta\rangle = \mathbb{e}^{\mathbb{i}\theta}S(\beta)\,|0\rangle\,. \tag{3.19}$$

One can fix the phase θ to be zero. □

3.2 Time evolution

(3.4) **Proposition**
Let $X = \mathbb{i}\Omega t a^{+}a^{-}$, $Y = \frac{1}{2}\left(z(a^{+})^{2} - z^{*}(a^{-})^{2}\right)$.

$$\mathrm{ad}_{X}^{(n)}\,Y = \frac{1}{2}\left(z(+2\mathbb{i}\Omega t)^{n}(a^{+})^{2} - z^{*}(-2\mathbb{i}\Omega t)^{n}(a^{-})^{2}\right). \tag{3.20}$$

Proof.

$$\mathrm{ad}_{X}^{(0)}\,Y = Y = (2\mathbb{i}t)^{0}Y; \tag{3.21}$$

$$\begin{aligned}\mathrm{ad}_{X}^{(1)}\,Y &= \left[\mathbb{i}\Omega t a^{+}a^{-}, \frac{1}{2}\left(z(a^{+})^{2} - z^{*}(a^{-})^{2}\right)\right]_{-} \\ &= \mathbb{i}\Omega t\left(z(a^{+})^{2} + z^{*}(a^{-})^{2}\right) \\ &= \frac{1}{2}\left(z(+2\mathbb{i}\Omega t)^{1}(a^{+})^{2} - z^{*}(-2\mathbb{i}\Omega t)^{1}(a^{-})^{2}\right);\end{aligned} \tag{3.22}$$

$$\begin{aligned}\mathrm{ad}_{X}^{(2)}\,Y &= \left[\mathbb{i}\Omega t a^{+}a^{-}, \mathbb{i}\Omega t\left(z(a^{+})^{2} + z^{*}(a^{-})^{2}\right)\right]_{-} \\ &= 2(\mathbb{i}\Omega t)^{2}\left(z(a^{+})^{2} - z^{*}(a^{-})^{2}\right) \\ &= \frac{1}{2}\left(z(+2\mathbb{i}\Omega t)^{2}(a^{+})^{2} - z^{*}(-2\mathbb{i}\Omega t)^{2}(a^{-})^{2}\right);\end{aligned} \tag{3.23}$$

Can be proved by induction. □

(3.5) **Proposition**

$$\mathcal{U}(t)S(z) = S(z^t)\mathcal{U}(t), \tag{3.24}$$

where

$$z^t := z\mathbb{e}^{2\mathbb{i}\Omega t}. \tag{3.25}$$

Proof. Sum up. □

Note one also has $\beta^t := \beta\mathbb{e}^{2\mathbb{i}\Omega t}$.

3.3 Particle numbers

(3.6) **Lemma** (Factorising $\widehat{S}(z)$)

$$\begin{aligned}\widehat{S}(z) &= \exp\left\{+\frac{1}{2}(\hat{a}^+)^2\mathbb{e}^{+\mathbb{i}\theta}\tanh r\right\} \\ &\quad\cdot\exp\left\{\left(\frac{1}{2}+\hat{a}^+\hat{a}^-\right)\ln\operatorname{sech} r\right\}\exp\left\{-\frac{1}{2}(\hat{a}^-)^2\mathbb{e}^{-\mathbb{i}\theta}\tanh r\right\} &(3.26)\\ &= \exp\left\{\frac{\beta}{2}(\hat{a}^+)^2\right\}\exp\left\{\left(\frac{1}{2}+\hat{a}^+\hat{a}^-\right)\ln\sqrt{1-|\beta|^2}\right\}\exp\left\{-\frac{\beta^*}{2}(\hat{a}^-)^2\right\}. &(3.27)\end{aligned}$$

(3.7) **Particle number representation of $|\beta\rangle$**

$$\langle 2n \,|\, \beta\rangle = \left(1-|\beta|^2\right)^{1/4}\frac{\sqrt{(2n)!}}{2^n n!}\beta^n, \tag{3.28}$$

$$\langle 2n+1 \,|\, \beta\rangle = 0. \tag{3.29}$$

$$\langle\beta\,|\,\hat{a}^+\hat{a}^-\,|\,\beta\rangle = \langle 0\,|\,S^\dagger(z)\hat{a}^+\hat{a}^-S(z)\,|\,0\rangle = \sinh^2 r = \frac{|\beta|^2}{1-|\beta|^2}. \tag{3.30}$$

$$\left\langle 0\,\middle|\,\hat{b}^+\hat{b}^-\,\middle|\,0\right\rangle = \langle 0\,|\,S(z)\hat{a}^+S^\dagger(z)S(z)\hat{a}^-S^\dagger(z)\,|\,0\rangle = \sinh^2 r = \frac{|\beta|^2}{1-|\beta|^2}. \tag{3.31}$$

3.4 Wave function

(3.8) **Lemma** (Factorising $\widehat{S}(z)$)

$$\widehat{S}(z) = \operatorname{expi}\left(\frac{m\Omega}{2}\sigma_z x^2\right)\operatorname{expi}\left(-\ln\lambda_z\frac{\hat{x}\hat{p}+\hat{p}\hat{x}}{2}\right)\operatorname{expi}\left(-\frac{\sigma_z}{2m\Omega}p^2\right), \tag{3.32}$$

with

$$\lambda_z := \cosh r + \cos\phi\,\sinh r \equiv \mathbb{e}^{+r}\cos^2\frac{\phi}{2} + \mathbb{e}^{-r}\sin^2\frac{\phi}{2}, \tag{3.33}$$

$$\sigma_z := \frac{1}{\cot\phi + \coth r\,\csc\phi} \equiv \frac{\sin\phi\,\sinh r}{\lambda_z}. \tag{3.34}$$

(3.9) **Proposition**

$$\lambda_z^2(1+\sigma_z^2) = \lambda_{2z} \tag{3.35}$$

(3.10) **Wave function of $|\beta\rangle$**

$$\begin{aligned}
&\left\langle x \,\middle|\, \mathbb{e}^{\mathbb{i}b_1\widehat{x}^2}\mathbb{e}^{\mathbb{i}b_2(\widehat{x}\widehat{p}+\widehat{p}\widehat{x})/2}\mathbb{e}^{\mathbb{i}b_3\widehat{p}^2} \,\middle|\, \psi\right\rangle \\
&= \mathbb{e}^{\mathbb{i}b_1x^2}\left\langle x \,\middle|\, \mathbb{e}^{\mathbb{i}b_2(\widehat{x}\widehat{p}+\widehat{p}\widehat{x})/2}\mathbb{e}^{\mathbb{i}b_3\widehat{p}^2} \,\middle|\, \psi\right\rangle \\
&= \mathbb{e}^{\mathbb{i}b_1x^2+b_2/2}\left\langle x\mathbb{e}^{b_2} \,\middle|\, \mathbb{e}^{\mathbb{i}b_3\widehat{p}^2} \,\middle|\, \psi\right\rangle \\
&= \frac{\mathbb{e}^{\mathbb{i}b_1x^2+b_2/2}}{(-4\pi\mathbb{i}b_3)^{1/2}}\int_{-\infty}^{+\infty}\mathbb{d}x'\,\mathrm{expi}\left\{-\frac{(x\mathbb{e}^{b_2}-x')^2}{4b_3}\right\}\langle x' \,|\, \psi\rangle\,.
\end{aligned} \tag{3.36}$$

For $\widehat{S}(z)$, $b_1 = m\Omega\sigma_z/2$, $b_2 = -\ln\lambda_z$, $b_3 = -\sigma_z/2m\Omega$, and $|\psi\rangle = |0\rangle$, so that

$$\begin{aligned}
&\left(\frac{m\Omega}{\pi}\right)^{-1/4}\left\langle x \,\middle|\, \widehat{S}(z) \,\middle|\, 0\right\rangle \\
&= (\lambda_z(1+\mathbb{i}\sigma_z))^{-1/2}\exp\left\{-\frac{m\Omega x^2}{2}\left(\frac{1}{\lambda_z^2(1+\mathbb{i}\sigma_z)}-\mathbb{i}\sigma_z\right)\right\} \\
&= (\lambda_z^2(1+\sigma_z^2))^{-1/4}\exp\left\{-\frac{m\Omega x^2}{2\lambda_z^2(1+\sigma_z^2)}\right\}\mathrm{expi}\left\{\frac{m\Omega x^2\sigma_z}{2\lambda_z^2(1+\sigma_z^2)}-\frac{\arg(1+\mathbb{i}\sigma_z)}{2}\right\} \\
&= \lambda_{2z}^{-1/4}\exp\left\{-\frac{m\Omega}{2\lambda_{2z}}x^2\right\}\mathrm{expi}\left\{\frac{m\Omega\sigma_{2z}}{2}x^2-\frac{1}{2}\arg(1+\mathbb{i}\sigma_z)\right\}.
\end{aligned} \tag{3.37}$$

3.5 Density matrix and Wigner function

(3.11) **Density matrix in the particle number representation**

$$\begin{aligned}
\langle 2n_1 \,|\, \widehat{\rho} \,|\, 2n_2\rangle &= \sqrt{1-|\beta|^2}\,\frac{\sqrt{(2n_1)!\,(2n_2)!}}{2^{n_1+n_2}n_1!\,n_2!}\beta^{n_1}(\beta^*)^{n_2} \\
&= \sqrt{1-|\beta|^2}\,\frac{\sqrt{(2n_1)!\,(2n_2)!}}{2^{n_1+n_2}n_1!\,n_2!}\mathbb{e}^{\mathbb{i}(n1-n2)\phi}(\tanh r)^{n_1+n_2}.
\end{aligned} \tag{3.38}$$

(3.12) **Density matrix in the position representation**

$$\langle x_1 \,|\, \widehat{\rho} \,|\, x_2\rangle = \lambda_{2z}^{-1/2}\exp\left\{-\frac{m\Omega}{2\lambda_{2z}}(x_1^2+x_2^2)\right\}\mathrm{expi}\left\{\frac{m\Omega\sigma_{2z}}{2}(x_1^2-x_2^2)\right\}. \tag{3.39}$$

(3.13) **Wigner function**

$$\begin{aligned}
&W(x,p) \\
&= \pi^{-1}\exp\left\{-m\Omega\lambda_{-2z}x^2+2\lambda_{2z}\sigma_{2z}xp-\lambda_{2z}\frac{p^2}{m\Omega}\right\}
\end{aligned} \tag{3.40}$$

$$\equiv \pi^{-1}\exp\left\{-m\Omega\frac{X^2}{\mathbb{e}^{+2r}}-\frac{1}{m\Omega}\frac{P^2}{\mathbb{e}^{-2r}}\right\}, \tag{3.41}$$

where

$$\begin{pmatrix} (m\Omega)^{+1/2}X \\ (m\Omega)^{-1/2}P \end{pmatrix} := \begin{pmatrix} \cos\phi/2 & \sin\phi/2 \\ -\sin\phi/2 & \cos\phi/2 \end{pmatrix} \begin{pmatrix} (m\Omega)^{+1/2}x \\ (m\Omega)^{-1/2}p \end{pmatrix} \tag{3.42}$$

are principle coordinates of the Wigner ellipse. One sees that $\phi/2$ rotates the canonical coordinates for the Wigner function, while $\mathbb{e}^{+r}$ ($\mathbb{e}^{-r}$) stretches (squeezes) the principle axis of X (P), if $r > 0$.

4 Double-mode squeezing

Let

$$\hat{b}_i^- = \frac{\hat{a}_i^- - \beta\hat{a}_j^+}{\sqrt{1-|\beta|^2}}, \qquad \hat{b}_i^+ = \left(\hat{b}_i^-\right)^\dagger = \frac{\hat{a}_i^+ - \beta^*\hat{a}_j^-}{\sqrt{1-|\beta|^2}}, \tag{4.1}$$

where $\{i,j\} = \{1,2\}$, so that

$$\left[\hat{b}_i^-, \hat{b}_j^+\right]_- = \delta_{ij}\hat{1}. \tag{4.2}$$

One seeks the state $|\beta\rangle$ satisfying

$$\hat{b}^- |\beta\rangle = 0. \tag{4.3}$$

Note that eq. (3.1) is again a restricted *Bogolyubov transformation*, in that the coefficients of $\hat{a}_i^-$ for $\hat{b}_i^-$ are real.

4.1 Double-mode squeeze operator

One attempts

$$\widehat{S}_2(z)\hat{a}^-\widehat{S}_2^{-1}(z), \tag{4.4}$$

where

$$\widehat{S}_2(z) := \exp(z\hat{a}_1^+\hat{a}_2^+ - z^*\hat{a}_1^-\hat{a}_2^-) \tag{4.5}$$

is the *double-mode squeeze operator*. Note that

$$\widehat{S}_2^{-1}(z) = \widehat{S}_2(-z) = \widehat{S}_2^\dagger(z). \tag{4.6}$$

The parameterisation $z = r\mathbb{e}^{\mathbb{i}\phi}$ will also be used, where $r = |z|$, $\phi = \arg z$.

(4.1) **Proposition**
Let $X = z\hat{a}_1^+\hat{a}_2^+ - z^*\hat{a}_1^-\hat{a}_2^-$. Then

$$\mathrm{ad}_X^{(2n)}\,\hat{a}_i^- = |z|^{2n}\hat{a}_i^- = r^{2n}\hat{a}_i^-, \tag{4.7}$$

$$\mathrm{ad}_X^{(2n+1)}\,\hat{a}_i^- = -|z|^{2n+1}\frac{z}{|z|}\hat{a}_j^+ = \mathbb{e}^{\mathbb{i}\phi}r^{2n+1}\mathbb{e}^{\mathbb{i}\phi}\hat{a}_j^+, \qquad \forall n \geq 0. \tag{4.8}$$

(4.2) **Proposition**

$$\begin{aligned}\widehat{S}_2(z)\hat{a}_i^-\widehat{S}_2^\dagger(z) &= \hat{a}_i^-\cosh|z| - \hat{a}_j^+\frac{z}{|z|}\sinh|z| \\ &= \hat{a}_i^-\cosh r - \hat{a}_j^+\mathbb{e}^{\mathbb{i}\phi}\sinh r,\end{aligned} \tag{4.9}$$

so that z or (r,ϕ) is a parameterisation of β in eq. (3.1) *in that*

$$|\beta| = \tanh|z| = \tanh r, \qquad \arg\beta = \arg z = \phi. \tag{4.10}$$

(4.3) **Squeezed ground state**
$|\beta\rangle$ *in eq.* (3.3) *is the* β*-squeezed ground state, namely*

$$|\beta\rangle = \widehat{S}_2(z)\,|0\rangle\,. \tag{4.11}$$

4.2 Time evolution

4.3 Particle numbers

4.4 Wave function

(4.4) **Wave function**

$$\begin{aligned}
&\left(\frac{m\Omega}{2}\right)^{-1/2} \langle x_1, x_2\,|\,\beta\rangle \\
&= \left(\cosh^2 r - \mathrm{e}^{2\mathrm{i}\phi}\sinh^2 r\right)^{-1/2} \\
&\quad\cdot\exp\left\{-\frac{m\Omega}{2}\left(\frac{1+\mathrm{e}^{2\mathrm{i}\phi}\tanh^2 r}{1-\mathrm{e}^{2\mathrm{i}\phi}\tanh^2 r}(x_1^2+x_2^2) - \frac{4\mathrm{e}^{\mathrm{i}\phi}\tanh r}{1-\mathrm{e}^{2\mathrm{i}\phi}\tanh^2 r}x_1x_2\right)\right\} \\
&= \left(\frac{1-|\beta|^2}{1-\beta^2}\right)^{1/2} \exp\left\{-\frac{m\Omega}{2}\left(\frac{1+\beta^2}{1-\beta^2}(x_1^2+x_2^2) - \frac{4\beta}{1-\beta^2}x_1x_2\right)\right\}.
\end{aligned} \tag{4.12}$$

4.5 Density matrix and Wigner function

Did not find any simplification for $\langle x_1, x_2\,|\,\hat{\rho}\,|\,y_1, y_2\rangle \equiv \langle x_1, x_2\,|\,\beta\rangle\,\langle\beta\,|\,y_1, y_2\rangle$.

$$\begin{aligned}
W(x_i; p_i) = \pi^{-2}\exp\Bigg\{&-\left(\frac{x_1^2+x_2^2}{(m\Omega)^{-1}} + \frac{p_1^2+p_2^2}{(m\Omega)^{+1}}\right)\cosh 2r \\
+\ 2&\left(\left(\frac{x_1x_2}{(m\Omega)^{-1}} - \frac{p_1p_2}{(m\Omega)^{+1}}\right)\cos\phi - (x_1p_2 + x_2p_1)\sin\phi\right)\sinh 2r\Bigg\}
\end{aligned} \tag{4.13}$$

Any insight?

A Notes on analytical geometry

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{A.1}$$

$B^2 - 4AC < 0$

$$\Delta := \det\begin{vmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{vmatrix} = \left(AC - \frac{B^2}{4}\right)F + \frac{BED}{4} - \frac{CD^2}{4} - \frac{AE^2}{4} \tag{A.2}$$

$$
\begin{aligned}
A &= a^2 \sin^2\theta + b^2 \cos^2\theta \\
B &= 2(b^2 - a^2)\sin\theta\,\cos\theta \\
C &= a^2\cos^2\theta + b^2\sin^2\theta \\
D &= -2Ax_0 - By_0 \\
E &= -Bx_0 - 2Cy_0 \\
F &= Ax_0^2 + Bx_0y_0 + cy_0^2 - a^2b^2
\end{aligned}
\tag{A.3}
$$

$$
\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1, \tag{A.4}
$$

where

$$
\begin{pmatrix} X \\ Y \end{pmatrix} := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}. \tag{A.5}
$$

B Notes on quantum mechanics

B.1 $\hat{x}$ and $\hat{p}$ acting on $|x\rangle$

(B.1) **Linear hermitian functions**

$$
\hat{p}\,|x\rangle = \left(+\mathbb{i}\frac{\partial}{\partial x}\right)|x\rangle\,, \qquad \langle x|\,\hat{p} = \left(-\mathbb{i}\frac{\partial}{\partial x}\right)\langle x|\,. \tag{B.1}
$$

Proof.

$$
\begin{aligned}
\langle x|\,\hat{p} &= \int_{-\infty}^{+\infty} \mathbb{d}p\, \langle x\,|\,p\rangle\,\langle p|\,\hat{p} = \int_{-\infty}^{+\infty} \mathbb{d}p\,(2\pi)^{-1/2}\mathbb{e}^{\mathbb{i}px}\,\langle p|\,p \\
&= \int_{-\infty}^{+\infty} \mathbb{d}p\left(-\mathbb{i}\frac{\partial}{\partial x}\right)(2\pi)^{-1/2}\mathbb{e}^{\mathbb{i}px}\,\langle p| = \left(-\mathbb{i}\frac{\partial}{\partial x}\right)\int_{-\infty}^{+\infty}\mathbb{d}p\,\langle x\,|\,p\rangle\,\langle p| \\
&= \left(-\mathbb{i}\frac{\partial}{\partial x}\right)\langle x|\,.
\end{aligned}
\tag{B.2}
$$

□

(B.2) **Linear unitary exponentials**

$$
\mathbb{e}^{\mathbb{i}p_0\hat{x}}\,|x\rangle = \mathbb{e}^{\mathbb{i}p_0x}\,|x\rangle\,, \qquad \langle x|\,\mathbb{e}^{\mathbb{i}p_0\hat{x}} = \langle x|\,\mathbb{e}^{\mathbb{i}p_0x}; \tag{B.3}
$$

$$
\mathbb{e}^{\mathbb{i}x_0\hat{p}}\,|x\rangle = |x - x_0\rangle\,, \qquad \langle x|\,\mathbb{e}^{\mathbb{i}x_0\hat{p}} = \langle x + x_0|\,. \tag{B.4}
$$

Proof.

$$
\begin{aligned}
\mathbb{e}^{\mathbb{i}x_0\hat{p}}\,|x\rangle &= \int_{-\infty}^{+\infty}\mathbb{d}p\,\mathbb{e}^{\mathbb{i}x_0\hat{p}}\,|p\rangle\,\langle p\,|\,x\rangle \\
&= \int_{-\infty}^{+\infty}\mathbb{d}p\,\mathbb{e}^{\mathbb{i}x_0p}\,|p\rangle\,(2\pi)^{-1/2}\mathbb{e}^{-\mathbb{i}xp} = \int_{-\infty}^{+\infty}\mathbb{d}p\,(2\pi)^{-1/2}\mathbb{e}^{-\mathbb{i}(x-x_0)p}\,|p\rangle \\
&= \int_{-\infty}^{+\infty}\mathbb{d}p\,|p\rangle\,\langle p\,|\,x - x_0\rangle = |x - x_0\rangle\,;
\end{aligned}
\tag{B.5}
$$

$$
\langle x|\,\mathbb{e}^{\mathbb{i}x_0\hat{p}} = \int_{-\infty}^{+\infty}\mathbb{d}p\,\mathbb{e}^{\mathbb{i}x_0p}\,\langle p|\,(2\pi)^{-1/2}\mathbb{e}^{+\mathbb{i}xp} = \langle x + x_0|\,. \tag{B.6}
$$

□

(B.3) **Quadratic hermitian functions**

$$\widehat{x}^2 \left|x\right\rangle = x^2 \left|x\right\rangle , \qquad \left\langle x\right| \widehat{x}^2 = \left\langle x\right| x^2; \tag{B.7}$$

$$\widehat{p}^2 \left|x\right\rangle = -\frac{\partial^2}{\partial x^2} \left|x\right\rangle , \qquad \left\langle x\right| \widehat{p}^2 = -\frac{\partial^2}{\partial x^2} \left\langle x\right| ; \tag{B.8}$$

$$\frac{\widehat{x}\widehat{p} + \widehat{p}\widehat{x}}{2} \left|x\right\rangle = +\mathbb{i}\left(\frac{1}{2} + x\frac{\partial}{\partial x}\right) \left|x\right\rangle , \qquad \left\langle x\right| \frac{\widehat{x}\widehat{p} + \widehat{p}\widehat{x}}{2} = -\mathbb{i}\left(\frac{1}{2} + x\frac{\partial}{\partial x}\right) \left\langle x\right| . \tag{B.9}$$

Proof. 456 □

(B.4) **Quadratic unitary exponentials**

$$\mathbb{e}^{\mathbb{i}\sigma\widehat{x}^2} \left|x\right\rangle = \mathbb{e}^{\mathbb{i}\sigma x^2} \left|x\right\rangle , \qquad \left\langle x\right| \mathbb{e}^{\mathbb{i}\sigma\widehat{x}^2} = \left\langle x\right| \mathbb{e}^{\mathbb{i}\sigma x^2}; \tag{B.10}$$

$$\begin{aligned} \mathbb{e}^{\mathbb{i}\sigma\widehat{p}^2} \left|x\right\rangle &= (-4\mathbb{\pi}\mathbb{i}\sigma)^{-1/2} \int_{-\infty}^{+\infty} \mathbb{d}x' \, \left|x'\right\rangle \operatorname{expi}\left\{-\frac{(x-x')^2}{4\sigma}\right\}, \\ \left\langle x\right| \mathbb{e}^{\mathbb{i}\sigma\widehat{p}^2} &= (-4\mathbb{\pi}\mathbb{i}\sigma)^{-1/2} \int_{-\infty}^{+\infty} \mathbb{d}x' \, \left\langle x'\right| \operatorname{expi}\left\{-\frac{(x-x')^2}{4\sigma}\right\}; \end{aligned} \tag{B.11}$$

$$\begin{aligned} \mathbb{e}^{\mathbb{i}\lambda(\widehat{x}\widehat{p}+\widehat{p}\widehat{x})/2} \left|x\right\rangle &= \mathbb{e}^{-\lambda/2} \left|x\mathbb{e}^{-\lambda}\right\rangle , \\ \left\langle x\right| \mathbb{e}^{\mathbb{i}\lambda(\widehat{x}\widehat{p}+\widehat{p}\widehat{x})/2} &= \mathbb{e}^{+\lambda/2} \left\langle x\mathbb{e}^{+\lambda}\right| . \end{aligned} \tag{B.12}$$

Proof. 789 □

B.2 Time evolution

$$\widehat{\mathcal{U}}(t;t_0) = \mathbb{e}^{-\mathbb{i}\widehat{H}(t-t_0)} \tag{B.13}$$

is the time-evolution operator in the Schrödinger picture.

B.3 Wigner transformation

Wigner transformation

$$g(x,p) := \frac{1}{2\mathbb{\pi}} \int_{-\infty}^{+\infty} \mathbb{d}s \, \mathbb{e}^{\mathbb{i}ps} \left\langle x - \frac{s}{2} \middle| \widehat{G} \middle| x + \frac{s}{2} \right\rangle . \tag{B.14}$$

$$\int_{\mathbb{R}^2} \mathbb{d}x \, \mathbb{d}p \, g(x,p) \equiv \operatorname{tr} \widehat{G}. \tag{B.15}$$

C Notes on BCH formula

Baker–Campbell–Hausdorff

(C.1) **Linear adjoint endomophism**

$$\mathrm{ad}_X^{(0)} Y := Y, \tag{C.1}$$
$$\mathrm{ad}_X^{(n)} Y := \left[X, \mathrm{ad}_X^{(n-1)} Y\right]_-, \qquad \forall n \geq 1. \tag{C.2}$$
$$\tag{C.3}$$

Specifically,

$$\mathrm{ad}_X Y := \mathrm{ad}_X^{(1)} Y \equiv [X, Y]_-. \tag{C.4}$$

(C.2) **Lemma**

$$\mathrm{e}^X Y \mathrm{e}^{-X} = \sum_{n=0}^{+\infty} \frac{1}{n!} \mathrm{ad}_X^{(n)} Y \tag{C.5}$$

(C.3) **Corollary**
For $[X, Y]_-$ central, i.e. commuting with both X and Y,

$$\mathrm{e}^X \mathrm{e}^Y = \mathrm{e}^{X+Y+[X,Y]_-/2}. \tag{C.6}$$

(C.4) **Theorem** (Braiding identity)

$$\mathrm{e}^X \mathrm{e}^Y = \exp\left(\sum_{n=0}^{+\infty} \frac{1}{n!} \mathrm{ad}_X^{(n)} Y\right) \mathrm{e}^X. \tag{C.7}$$

(C.5) **A factorisation algorithm for quadratic x and p**
Given $a_i \in \mathbb{R}$, solve

$$\begin{aligned} U &= \mathrm{expi}\left\{a_1 x^2 + a_2 \frac{xp + px}{2} + a_3 p^2\right\} \\ &= \mathrm{expi}\left\{b_1 x^2\right\} \mathrm{expi}\left\{b_2 \frac{xp + px}{2}\right\} \mathrm{expi}\left\{b_3 p^2\right\} \end{aligned} \tag{C.8}$$

for b_i, $i = 1, 2, 3$.
Let

$$L(t) := \mathrm{expi}\left\{t\left(a_1 x^2 + a_2 \frac{xp + px}{2} + a_3 p^2\right)\right\}, \tag{C.9}$$
$$R(t) := \mathrm{expi}\left\{c_1(t) x^2\right\} \mathrm{expi}\left\{c_2(t) \frac{xp + px}{2}\right\} \mathrm{expi}\left\{\frac{c_3}{t} p^2\right\}, \tag{C.10}$$

where $0 \leq t \leq 1$. One hopes to solve

$$L(t) = R(t) \tag{C.11}$$

for $c_i(t)$. By setting $t = 0$ and observing $L(1) = R(1) = U$, one recognises the boundary conditions

$$c_i(0) = 0, \qquad c_i(1) = b_i. \tag{C.12}$$

By (C.11) one has

$$\dot{L}L^\dagger = \dot{R}R^\dagger. \tag{C.13}$$

Evaluating the right hand side of (C.13) with the formulas above and comparing the corresponding coefficients, one derives coefficients

$$\begin{cases} a_1 = \dot{c}_1 - 2c_1\dot{c}_2 + 4c_1^2 \mathrm{e}^{-2c_2}\dot{c}_3 \\ a_2 = \dot{c}_2 - 4c_1 \mathrm{e}^{-2c_2}\dot{c}_3 \\ a_3 = \mathrm{e}^{-2c_2}\dot{c}_3, \end{cases} \tag{C.14}$$

which further transforms to

$$\begin{cases} \dot{c}_1 = a_1 + 2a_2c_1 + 4a_3c_1^2 \\ \dot{c}_2 = a_2 + 4a_3c_1 \\ \dot{c}_3 = a_3\mathrm{e}^{2c_2}. \end{cases} \tag{C.15}$$

Equation (C.15) can be integrated on by one.

D Simple harmonic oscillators

D.1 Single oscillator

Let the canonical hamiltonian be

$$H = \frac{p^2}{2m} + \frac{1}{2}m\Omega^2x^2. \tag{D.1}$$

$$\widehat{H} = \Omega\left(\hat{a}^+\hat{a}^- + \frac{1}{2}\right). \tag{D.2}$$

(D.1) **Quantum ground state**

$$\hat{a}^- \left|0\right\rangle = 0; \tag{D.3}$$

$$\langle x \,|\, 0\rangle = N_0\mathrm{e}^{-m\Omega x^2/2}. \tag{D.4}$$

$$N_0 = \left(\frac{m\Omega}{\pi}\right)^{1/4}. \tag{D.5}$$

$$\langle x \,|\, 0\rangle = \left(\frac{m\Omega}{\pi}\right)^{1/4}\mathrm{e}^{-m\Omega x^2/2}. \tag{D.6}$$

(D.2) **Time evolution**

D.2 Double oscillators with identical frequency

$$H = H_1 + H_2, \tag{D.7}$$

$$H_i = \frac{p_i^2}{2m} + \frac{1}{2} m\Omega^2 x_i^2, \qquad i = 1, 2. \tag{D.8}$$

(D.3) **Quantum ground state**

$$\hat{a}_i^- \left|0\right\rangle = 0; \tag{D.9}$$

$$\langle x_1, x_2 \,|\, 0\rangle = \left(\frac{m\Omega}{\pi\hbar}\right)^{1/2} \exp\left\{-\frac{m\Omega}{2}(x_1^2 + x_2^2)\right\}. \tag{D.10}$$

(D.4) **Time evolution**