Effective action of scalar electrodynamics

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1 Wick rotation

Complex Klein–Gordon action

$$S_{\mathrm{CKG}}[\phi,\phi^*] \coloneqq \int \mathrm{d}^{d+1}x\,\left\{-\eta^{\mu\nu}(\partial_\mu\phi)^*(\partial_\nu\phi) - m^2\phi^*\phi\right\}. \tag{1}$$

Interaction term

$$S_{\mathrm{ICKGM}}[A_\mu,\phi,\phi^*] \coloneqq \int \mathrm{d}^{d+1}x\,\eta^{\mu\nu}\left\{\mathrm{i}eA_\mu(-\phi^*\partial_\nu\phi + \phi\partial_\nu\phi^*) + e^2A_\mu A_\nu\phi^*\phi\right\}. \tag{2}$$

The total action for scalar electrodynamics reads

$$\begin{aligned} S[A_\mu,\phi,\phi^*] &\coloneqq S_{\mathrm{CKG}} + S_{\mathrm{ICKGM}} + S_{\mathrm{Maxwell}} \\ &= \int \mathrm{d}^{d+1}x\,\left\{-(\nabla_\mu\phi)^*(\nabla^\mu\phi) - m^2\phi^*\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right\}, \end{aligned} \tag{3}$$

where

$$\nabla_\mu\phi \coloneqq (\partial_\mu + \mathrm{i}eA_\mu)\phi. \tag{4}$$

Wick rotation

$$x_{\mathrm{E}}^4 = \mathrm{i}x^0, \quad A_4 = -\mathrm{i}A_0, \tag{5}$$

so that

$$\partial_{x^0} = \mathrm{i}\partial_{x_{\mathrm{E}}^4}, \quad F_{0i} = \mathrm{i}F_{4i}. \tag{6}$$

The Euclidean action reads

$$S_{\mathrm{E}}[A_I,\phi,\phi^*] = \int \mathrm{d}^D x_{\mathrm{E}}\,\left(\frac{1}{4}F_{IJ}F^{IJ} + (\nabla_I\phi)^*\left(\nabla^I\phi\right) + m^2\phi^*\phi\right). \tag{7}$$

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2 Euclidean signature

2.1 Weisskopf action

Current-free generating functional

$$\mathcal{Z}_{\mathrm{E}}\left[j^I, J, J^*\right] \coloneqq \int \mathrm{D}A\,\mathrm{D}\phi^*\,\mathrm{D}\phi \exp\left\{-\left(S_{\mathrm{E}} + \int \mathrm{d}^D x\,\left(j^I A_I + J^*\phi + \phi^* J\right)\right)\right\}. \tag{8}$$

The scalar fields are to be integrated out. The derivative term can be rearranged

$$(\nabla_I \phi)^* \left(\nabla^I \phi\right) = \partial_I\left(\phi^* \nabla^I \phi\right) - \phi^* \nabla_I \nabla^I \phi. \tag{9}$$

Hence

$$S_{\mathrm{E}} = \int \mathrm{d}^D x\,\frac{1}{4} F_{IJ} F^{IJ} + \int \mathrm{d}^D x\,\mathrm{d}^D y\,\phi^*(x) M(x,y) \phi(y), \tag{10}$$

where

$$M_{\mathrm{E}}(x,y) \coloneqq \left(-\nabla_{x^I} \nabla^{x^I} + m^2\right) \delta^{(D)}(x-y), \tag{11}$$

see [3, ch. 6] for details. Now the scalar field can formally be integrated

$$\begin{aligned}\exp[-\mathit{\Gamma}_{\mathrm{SE}}[J^*, J]] &\coloneqq \int \mathrm{D}\phi^*\,\mathrm{D}\phi \exp\left\{-\int \mathrm{d}^D x\,\left((\nabla_I \phi)^* \left(\nabla^I \phi\right) + m^2 \phi^* \phi + J^* \phi + \phi^* J\right)\right\} \\ &= \left[\det M_{\mathrm{E}}(x,y)\right]^{-1} \exp\left\{-\int \mathrm{d}^D x\,\mathrm{d}^D y\,J^*(x) M_{\mathrm{E}}^{-1}(x,y) J(y)\right\},\end{aligned} \tag{12}$$

so that

$$\mathit{\Gamma}_{\mathrm{SE}}[J^*, J] = \mathit{\Gamma}_{\mathrm{SE}}[0,0] + \int \mathrm{d}^D x\,\mathrm{d}^D y\,J^*(x) M_{\mathrm{E}}^{-1}(x,y) J(y), \tag{13}$$

$$\mathit{\Gamma}_{\mathrm{SE}}[0,0] = -\ln\left[(\det M_{\mathrm{E}})^{-1}\right] = \operatorname{tr}\ln M_{\mathrm{E}}, \tag{14}$$

which traces back to [6]. The Euclidean generating functional now reads

$$\begin{aligned}\mathcal{Z}_{\mathrm{E}}\left[j^I, J, J^*\right] \coloneqq \int \mathrm{D}A \exp\Bigg\{ &-\int \mathrm{d}^D x\left(\frac{1}{4} F_{IJ} F^{IJ}\right) - \operatorname{tr}\ln M_{\mathrm{E}} \\ &+ \int \mathrm{d}^D x\,j^I A_I - \int \mathrm{d}^D x\,\mathrm{d}^D y\,J^*(x) M_{\mathrm{E}}^{-1}(x,y) J(y)\Bigg\}.\end{aligned} \tag{15}$$

2.2 World-line formalism

In eq. (15), $\operatorname{tr}\ln M_{\mathrm{E}}$ is crucial. Using the Schwinger integral representation [4]

$$\ln\alpha = -\int_0^{+\infty}\frac{\mathrm{d}s}{s}\,\mathrm{e}^{-\alpha s}, \tag{16}$$

one has

$$-\operatorname{tr}\ln M_{\mathrm{E}} = \int_0^{+\infty}\frac{\mathrm{d}T}{T}\exp\left(-\frac{m^2T}{2M}\right)\operatorname{tr}\exp\left(-\frac{M_{\mathrm{E}}}{2M}\right), \tag{17}$$

where T has the dimension of time, and M that of mass. Introduce the Hamiltonian of a non-relativistic point particle (**check sign!**)

$$H := \frac{1}{2M}(P_I + eA_I)^2, \tag{18}$$

so that quantisation yields the following representation (**check sign!**)

$$\operatorname{tr}\exp\left(-\frac{M_{\mathrm{E}}}{2M}\right) = \int_{-\infty}^{+\infty}\mathrm{d}x\,\left\langle x\,\middle|\,\mathrm{e}^{-\widehat{H}T}\,\middle|\,x\right\rangle \tag{19}$$

$$= \oint \mathrm{D}x\exp\left\{-\int_0^T \mathrm{d}T'\left(\frac{M}{2}\left(\frac{\mathrm{d}x^I}{\mathrm{d}T'}\right)^2 + \mathrm{i}eA_I\frac{\mathrm{d}x^I}{\mathrm{d}T'}\right)\right\}. \tag{20}$$

Rescaling $T' =: \lambda T$ gives

$$-\operatorname{tr}\ln M_{\mathrm{E}} = \int_0^{+\infty}\frac{\mathrm{d}T}{T}\exp\left(-\frac{m^2T}{2M}\right)\oint\mathrm{D}x\exp\left\{-\frac{M}{2T}\int_0^1\dot{x}_I^2 - \mathrm{i}e\oint A_I\,\mathrm{d}x^I\right\}. \tag{21}$$

2.3 World-line instanton approximation

[1]

2.4 Application of instanton approximation

[2]

3 Flat space-time (Lorentzian signature)

Generating functional

$$\mathcal{Z}[j^\mu, J^*, J] := \int \mathrm{D}A\,\mathrm{D}\phi\,\mathrm{D}\phi^* \exp\biggl\{\mathrm{i}\biggl(S_0 + \int \mathrm{d}^{d+1}x\,(j^\mu A_\mu + J^*\phi + \phi^* J)\biggr)\biggr\}. \tag{22}$$

Effective action

$$\mathcal{Z}[j^\mu, 0, 0] =: \int \mathrm{D}A \exp\biggl\{\mathrm{i}\biggl(S_{\text{Maxwell}} + \mathit{\Gamma}_{\mathrm{W}}[A_\mu] + \int \mathrm{d}^{d+1}x\, j^\mu A_\mu\biggr)\biggr\}. \tag{23}$$

In other words,

$$\begin{aligned}\exp\{\mathrm{i}\mathit{\Gamma}_{\mathrm{W}}[A_\mu]\} &:= \int \mathrm{D}\phi\,\mathrm{D}\phi^* \exp\{\mathrm{i}(S_{\text{CKG}} + S_{\text{ICKGM}})\}\\ &\equiv \int \mathrm{D}\phi\,\mathrm{D}\phi^* \exp\biggl\{\mathrm{i}\int \mathrm{d}^{d+1}x\,\{-(\nabla_\mu\phi)^*(\nabla^\mu\phi) - m^2\phi^*\phi\}\biggr\}.\end{aligned} \tag{24}$$

The integral in the exponent can be manipulated; only the first term is essential

$$\begin{aligned}&\int \mathrm{d}^{d+1}x\,(-(\nabla_\mu\phi)^*(\nabla^\mu\phi))\\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\bigl(-(\nabla_{x^\mu}\phi(x))^*\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\bigr),\end{aligned} \tag{25}$$

where

$$\begin{aligned}\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y) &= \delta^{d+1}(x-y)\bigl\{\partial^{y^\mu} + \mathrm{i}eA^\mu(y)\bigr\}\phi(y)\\ &= \bigl\{-\bigl(\nabla^{y^\mu}\bigr)^*\delta^{d+1}(x-y)\bigr\}\phi(y) + \partial^{y^\mu}B,\end{aligned} \tag{26}$$

in which

$$B = B(x,y) := \delta^{d+1}(x-y)\phi(y); \tag{27}$$

going back to eq. (25),

$$\begin{aligned}&= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\bigl\{-\{\{\partial_{x^\mu} + \mathrm{i}eA_\mu(x)\}\phi(x)\}^*\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\bigr\}\\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\bigl\{-\partial_{x^\mu}C^\mu + \phi^*(x)\nabla_{x^\mu}\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\bigr\}\\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\bigl\{-\partial_{x^\mu}C^\mu + \phi^*(x)\bigl\{-(\nabla_{y^\mu})\bigl(\nabla^{y^\mu}\bigr)^*\delta^{d+1}(x-y)\bigr\}\phi(y) + \partial^{y^\mu}\nabla_{x^\mu}B\bigr\},\end{aligned} \tag{28}$$

in which

$$C^{\mu} = C^{\mu}(x,y) \coloneqq \phi^*(x)\delta^{d+1}(x-y)\nabla^{y^{\mu}}\phi(y). \tag{29}$$

Now eq. (24) can be written as (dropping the boundary terms)

$$\begin{aligned}
&= \int \mathrm{D}\phi\,\mathrm{D}\phi^* \exp\Big\{-\mathrm{i}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\phi^*(x)D^{-1}(x,y)\phi(y)\Big\}\\
&= \tilde{\mathcal{N}}\big\{\det\big[D^{-1}(x,y)\big]\big\}^{-1/2},
\end{aligned} \tag{30}$$

where

$$D^{-1}(x,y) \coloneqq \mathcal{D}_y^{-1}\delta^{d+1}(x-y), \qquad \mathcal{D}_y^{-1} \coloneqq +(\nabla_{y^{\mu}})\big(\nabla^{y^{\mu}}\big)^* + m^2. \tag{31}$$

$$\begin{aligned}
\mathit{\Gamma}_{\mathrm{W}}[A_{\mu}] &\equiv -\mathrm{i}\Big(\ln\tilde{\mathcal{N}} - \frac{1}{2}\ln\det D^{-1}\Big)\\
&= \frac{\mathrm{i}}{2}\,\mathrm{tr}_x \ln\big(\mathcal{N}^{-1}D^{-1}\big)\\
&= \frac{\mathrm{i}}{2}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\delta^{d+1}(x-y)\\
&\quad\cdot\Big\{-\mathrm{e}^{\mathrm{i}s\left(+\left(\nabla_{y^{\mu}}\right)\left(\nabla^{y^{\mu}}\right)^*+m^2+\mathrm{i}0^+\right)\delta^{d+1}(x-y)} + \mathrm{e}^{\mathrm{i}s\left(\mathcal{N}+\mathrm{i}0^+\right)}\Big\}.
\end{aligned} \tag{32}$$

[6]

A Notions and conventions

The metric convention is mostly positive, i.e. $\eta_{\mu\nu} \coloneqq \mathrm{diag}(-,+,+,\dots)$

Pauli matrices

$$\sigma^1 \coloneqq \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma^2 \coloneqq \begin{pmatrix} 0 & -\mathrm{i}\\ +\mathrm{i} & 0\end{pmatrix}, \quad \sigma^3 \coloneqq \begin{pmatrix} +1 & 0\\ 0 & -1\end{pmatrix}. \tag{33}$$

The γ-matrices satisfy [5, sec. 5]

$$[\gamma^{\mu},\gamma^{\nu}]_+ \coloneqq 2\eta^{\mu\nu}\mathbf{1}_4. \tag{34}$$

$$\mathscr{J}^{\mu\nu} \coloneqq -\frac{\mathrm{i}}{4}[\gamma^{\mu},\gamma^{\nu}]_- \tag{35}$$

$$\sigma^{\mu\nu} := \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]_- \equiv -2\mathscr{J}^{\mu\nu}. \tag{36}$$

In $(3+1)$ dimensions, choose the chiral representation

$$\gamma^\mu = -\mathrm{i}\begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \tag{37}$$

where

$$\sigma^\mu := (1_2, +\vec{\sigma}), \qquad \bar{\sigma}^\mu := (1_2, -\vec{\sigma}). \tag{38}$$

$$\begin{aligned}\sigma^{\mu\nu} &\equiv -\frac{\mathrm{i}}{2}\begin{bmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{bmatrix} \\ &= \begin{cases} 0 & \mu = 0, \nu = 0; \\ \mathrm{i}\begin{bmatrix} +\sigma^j & 0 \\ 0 & -\sigma^j \end{bmatrix} & \mu = 0, \nu = j; \\ \mathrm{i}\begin{bmatrix} -\sigma^i & 0 \\ 0 & +\sigma^i \end{bmatrix} & \mu = i, \nu = 0; \\ \begin{bmatrix} +\epsilon^{ij}{}_k\sigma^k & 0 \\ 0 & +\epsilon^{ij}{}_k\sigma^k \end{bmatrix} & \mu = i, \nu = j. \end{cases}\end{aligned} \tag{39}$$

B Fresnel functional integral

[3, ch. 10]

C Algebra

$$\left[a\partial_1\partial_2, x^1\right]_- = a\partial_2 \tag{40}$$

central.

Baker–Campbell–Hausdorff formula

$$\mathrm{e}^{+a\partial_1\partial_2}x^1\mathrm{e}^{-a\partial_1\partial_2} = x^1 + a\partial_2. \tag{41}$$

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