

Cosmological Perturbations

Yi-Fan Wang (王一帆)

May 1, 2018

Most of the conventions and notations in [5, ch. 5] will be followed.
Suppose the metric can be expanded up to the linear order as

$$g = g^{(0)} + \epsilon g^{(1)} + O(\epsilon^2). \quad (1)$$

The background metric $g^{(0)}$ takes the Robertson–Walker form

$$g_{\mu\nu}^{(0)} dx^\mu dx^\nu = -N^2(t) dt^2 + a^2(t) d\Omega_{3F}^2, \quad (2)$$

in which $d\Omega_{3F}^2 = d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)$ is the dimensionless flat spatial metric. The linear perturbation can be decomposed into scalar, vector and tensor parts

$$g_{00}^{(1)} = -E, \quad (3)$$

$$g_{i0}^{(1)} = g_{0i}^{(1)} = F_{,i} + G_i, \quad (4)$$

$$g_{ij}^{(1)} = A\delta_{ij} + B_{,i,j} + C_{i,j} + C_{j,i} + D_{ij}. \quad (5)$$

Here one has some weird condition, where the contractions do not follow the one-up-one-down tradition

$$C_{i,i} = G_{i,i} = 0, \quad D_{ij,i} = 0, \quad D_{ii} = 0. \quad (6)$$

1 Metric perturbation under diffeomorphism

Consider a diffeomorphism generated by ξ^μ

$$x^\mu \rightarrow \bar{x}^\mu = x^\mu - \epsilon \xi^\mu. \quad (7)$$

The generator ξ^μ can in turn be decomposed into $\xi_0 = \zeta$, $\xi_i = \xi_{,i}^S + \xi_i^V$.

One has here again some weird condition, where the contractions do not follow the one-up-one-down tradition

$$\xi_{i,i}^V = 0. \quad (8)$$

The Lie derivative of the metric $\mathbb{L}_\xi g$ is

$$(\mathbb{L}_\xi g)_{\mu\nu} = \xi^\lambda g_{\mu\nu,\lambda} + \xi^\lambda_{,\mu} g_{\lambda\nu} + \xi^\lambda_{,\nu} g_{\mu\lambda}. \quad (9)$$

In components and expansion, these are

$$(\mathbb{L}_\xi g)_{00} = 2\dot{\zeta} - 2\zeta \frac{\dot{N}}{N} + O(\epsilon), \quad (10)$$

$$(\mathbb{L}_\xi g)_{i0} = (\mathbb{L}_\xi g)_{0i} = \left(\zeta - 2\frac{\dot{a}}{a} \xi^S + \dot{\xi}^S \right)_{,i} + \left(-2\frac{\dot{a}}{a} \xi_i^V + \dot{\xi}_i^V \right) + O(\epsilon), \quad (11)$$

$$(\mathbb{L}_\xi g)_{ji} = (\mathbb{L}_\xi g)_{ij} = -\frac{2a\dot{a}}{N^2} \zeta \delta_{ij} + 2\xi_{,i,j}^S + \xi_{i,j}^V + \xi_{j,i}^V + O(\epsilon). \quad (12)$$

2 Scalar perturbations

$$-N^2 - \epsilon E + O(\epsilon^2) \rightarrow -N^2 - \epsilon E + \epsilon \left(2\dot{\zeta} - 2\zeta \frac{\dot{N}}{N} \right) + O(\epsilon^2), \quad (13)$$

so one can write

$$\mathbb{L}_\xi E = -2\dot{\zeta} + 2\zeta \frac{\dot{N}}{N}. \quad (14)$$

Similarly one can read-off

$$\mathbb{L}_\xi F = \zeta - 2\frac{\dot{a}}{a} \xi^S + \dot{\xi}^S, \quad (15)$$

$$\mathbb{L}_\xi A = -\frac{2a\dot{a}}{N^2} \zeta, \quad (16)$$

$$\mathbb{L}_\xi B = 2\xi^S. \quad (17)$$

The four scalar perturbations are generated by ζ and ξ^S , so that only two independent perturbations exists. It is clear that

$$\mathbb{L}_\xi \left(\frac{F}{a} - \frac{\mathbb{D}}{\mathbb{D}t} \frac{B}{2a} \right) = \frac{\zeta}{a}. \quad (18)$$

One can verify that

$$\mathbb{L}_\xi \left\{ \frac{E}{2N} + \frac{\mathbb{D}}{\mathbb{D}t} \left[\frac{a}{N} \left(\frac{F}{a} - \frac{\mathbb{D}}{\mathbb{D}t} \frac{B}{2a} \right) \right] \right\} = 0, \quad (19)$$

$$\mathbb{L}_\xi \left\{ \frac{A}{2} + \frac{a^2 \dot{a}}{N^2} \left(\frac{F}{a} - \frac{\mathbb{D}}{\mathbb{D}t} \frac{B}{2a} \right) \right\} = 0. \quad (20)$$

3 Vector perturbations

4 Tensor perturbations

5 Scalar field perturbation under diffeomorphism

6 Perturbation of Arnowitt–Deser–Misner Hamiltonian formalism

The well known Arnowitt–Deser–Misner’s Hamiltonian action for gravitation is [3, ch.4.2.2]

$$S = \int dt dx^3 \left\{ \mathfrak{p}^{ij} \dot{h}_{ij} + \mathfrak{P} \dot{N} + \mathfrak{P}^i \dot{N}_i - N \mathfrak{H}^\perp - N_i \mathfrak{H}^i - \mathfrak{P} V - \mathfrak{P}^i V_i \right\}, \quad (21)$$

$$\mathfrak{H}^\perp = \frac{\varkappa}{\sqrt{\mathfrak{h}}} (h_{ik} h_{jl} + h_{il} h_{kj} - h_{ij} h_{kl}) \mathfrak{p}^{ij} \mathfrak{p}^{kl} - \frac{\sqrt{\mathfrak{h}}}{2\varkappa} {}^{(3)}R, \quad (22)$$

$$\mathfrak{H}^i = -2\mathfrak{p}^{ij}{}_{|j}, \quad (23)$$

where $\{V, V_i\}$ are velocity of N and N_i and play the role of Lagrange multipliers. Note that $\{N, N_i, h_{ij}; \mathfrak{P}, \mathfrak{P}^i, \mathfrak{p}^{ij}\}$ are not the unique choice of canonical variables for General Relativity in Hamiltonian formalism; instead, they are a special parametrisation of the phase space. One can also choose the components of the original four-metric and their conjugate momenta $\{g_{\mu\nu}; \mathfrak{p}^{\mu\nu}\}$ as canonical variables, as Dirac has done [2]. The two approaches are different in some subtle aspects; see [4] for a comparison.

Gauge transformations in the Arnowitt–Deser–Misner variables are generated by [1]

$$G = - \int \mathfrak{d}^3x \left\{ \left[\xi_\perp \left(\mathfrak{H}^\perp + N_{|i} \mathfrak{P}^i + (N \mathfrak{P}^i)_{|i} + (N^i \mathfrak{P})_{|i} \right) + \dot{\xi}_\perp \mathfrak{P} \right] \right. \\ \left. + \left[\xi_i \left(\mathfrak{H}^i + N_j{}^{|i} \mathfrak{P}^j + (N_j \mathfrak{P}^i)^{|j} + N^{|i} \mathfrak{P} \right) + \dot{\xi}_i \mathfrak{P}^i \right] \right\}, \quad (24)$$

and the infinitesimal gauge transformation of N is

$$\delta N = [N, G]_P = \xi_\perp{}_{|j} N^j - \dot{\xi}_\perp - \xi_i N^{i|}, \quad (25)$$

$$\delta N_i = -\xi_\perp N_{|i} + \xi_\perp{}_{|i} N - \xi_j N_i{}^{j|} + \xi_i{}^{j|} N_j - \dot{\xi}_i, \quad (26)$$

which can be found in [4]. The transformation for g_{ij} and the momenta

have to be worked out as

$$\delta g_{ij} = -\xi^\perp \frac{2\mathcal{N}}{\sqrt{h}} (h_{ik}h_{jl} + h_{il}h_{kj} - h_{ij}h_{kl})\mathfrak{p}^{kl} - \xi_{i|j} - \xi_{j|i}, \quad (27)$$

$$\delta \mathfrak{P} =, \quad (28)$$

$$\delta \mathfrak{P}^i =, \quad (29)$$

$$\delta \mathfrak{p}^{ij} =. \quad (30)$$

$$(31)$$

References

- [1] Leonardo Castellani. “Symmetries in constrained Hamiltonian systems”. In: *Annals of Physics* 143.2 (Oct. 1982), pp. 357–371. DOI: 10.1016/0003-4916(82)90031-8.
- [2] Paul A. M. Dirac. “The Theory of Gravitation in Hamiltonian Form”. In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 246.1246 (Aug. 1958), pp. 333–343. DOI: 10.1098/rspa.1958.0142.
- [3] Claus Kiefer. *Quantum Gravity*. 3rd ed. Oxford University Press, Apr. 2012. DOI: 10.1093/acprof:oso/9780199585205.001.0001.
- [4] Natalia Kiriushcheva and Sergei Kuzmin. “The Hamiltonian formulation of General Relativity: myths and reality”. In: *Central Eur.J.Phys.* 9:576-615,2011 9.3 (Aug. 31, 2008). DOI: 10.2478/s11534-010-0072-2. arXiv: 0809.0097 [gr-qc].
- [5] Steven Weinberg. *Cosmology*. Oxford University Press, 2008. ISBN: 9780198526827. URL: <https://global.oup.com/academic/product/cosmology-9780198526827>.