Effective action of spinor electrodynamics

immediate

1 Spinor electrodynamics in flat space-time

Maxwell Lagrangian

$$S_{\text{Maxwell}}[A_\mu] := \int \mathrm{d}^{d+1}x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) \tag{1}$$

Dirac Lagrangian [**?**, sec. 11]

$$S_{\text{Dirac}}[\psi, \bar{\psi}] := \int \mathrm{d}^{d+1}x \left\{ -\bar{\psi}(\gamma^\mu \partial_\mu + m)\psi \right\}. \tag{2}$$

Interaction term

$$S_{\text{IDM}}[A_\mu, \psi, \bar{\psi}] := \int \mathrm{d}^{d+1}x \left(-\bar{\psi}\gamma^\mu \mathrm{i}e A_\mu \psi \right). \tag{3}$$

The total action for spinor electrodynamics reads

$$S_{1/2}[A_\mu, \psi, \bar{\psi}] := S_{\text{Dirac}} + S_{\text{IDM}} + S_{\text{Maxwell}}$$

$$= \int \mathrm{d}^{d+1}x \left\{ -\bar{\psi}(\gamma^\mu \nabla_\mu + m)\psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right\}, \tag{4}$$

where

$$\nabla_\mu \psi := (\partial_\mu + \mathrm{i}e A_\mu)\psi. \tag{5}$$

Generating functional

$$\mathcal{Z}[j^\mu, \bar{\eta}, \eta] := \int \mathrm{D}A\, \mathrm{D}\psi\, \mathrm{D}\bar{\psi} \exp\left\{ \mathrm{i} \left(S_{1/2} + \int \mathrm{d}^{d+1}x \left(j^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta \right) \right) \right\} \tag{6}$$

Effective action

$$\mathcal{Z}[j^\mu, 0, 0] =: \int \mathrm{D}A \exp\left\{ \mathrm{i}\left(S_{\text{Maxwell}} + \Gamma_{\text{EH}}[A_\mu] + \int \mathrm{d}^{d+1}x\, j^\mu A_\mu \right) \right\}. \tag{7}$$

In other words,

$$
\begin{aligned}
\exp\{\mathrm{i}\Gamma_{\text{EH}}[A_\mu]\} &\equiv \int \mathrm{D}\psi\, \mathrm{D}\bar{\psi} \exp\{\mathrm{i}(S_{\text{Dirac}} + S_{\text{IDM}})\} \\
&\equiv \int \mathrm{D}\psi\, \mathrm{D}\bar{\psi} \exp\left\{ \mathrm{i} \int \mathrm{d}^{d+1}x\, \bar{\psi}\left(-\slashed{\partial} - \mathrm{i}e\slashed{A} - m\right)\psi \right\} \\
&= \int \mathrm{D}\psi\, \mathrm{D}\bar{\psi} \exp\left\{ \mathrm{i} \int \mathrm{d}^{d+1}x\, \mathrm{d}^{d+1}y\, \bar{\psi}(x)S^{-1}(x,y)\psi(y) \right\} \\
&= \tilde{\mathcal{N}} \det\left[-\mathrm{i}S^{-1}(x,y)\right],
\end{aligned}
\tag{8}
$$

where

$$S^{-1}(x,y) := \left(+\slashed{\partial}_y - \mathrm{i}e\slashed{A}(y) - m\right)\delta^{d+1}(x-y). \tag{9}$$

$$
\begin{aligned}
\Gamma_{\text{EH}}[A_\mu] &\equiv -\mathrm{i}\left(\ln\tilde{\mathcal{N}} + \ln\det\left[-\mathrm{i}S^{-1}\right]\right) \\
&= -\mathrm{i}\left(\ln\tilde{\mathcal{N}} + \operatorname{Tr}\ln\left[-\mathrm{i}S^{-1}(x,y)\right]\right).
\end{aligned}
\tag{10}
$$

Note that

$$
\begin{aligned}
\operatorname{Tr}\ln\left[-\mathrm{i}S^{-1}(x,y)\right] &\equiv \operatorname{Tr}\ln[-\mathrm{i}M^\mathsf{T}(x,y)] \\
&= \operatorname{Tr}\ln\left[\mathrm{i}\left(+\slashed{\partial}_y^\mathsf{T} - \mathrm{i}e\slashed{A}^\mathsf{T}(y) - m\right)\delta^{d+1}(x-y)\right] \\
&= \operatorname{Tr}\ln\left[\mathrm{i}\left(-\mathscr{C}\slashed{\partial}_y\mathscr{C}^{-1} + \mathrm{i}e\mathscr{C}\slashed{A}(y)\mathscr{C}^{-1} - \mathscr{C}m\mathscr{C}^{-1}\right)\delta^{d+1}(x-y)\right] \\
&= \operatorname{Tr}\left\{\mathscr{C}\ln\left[\mathrm{i}\left(-\slashed{\partial}_y + \mathrm{i}e\slashed{A}(y) - m\right)\delta^{d+1}(x-y)\right]\mathscr{C}^{-1}\right\} \\
&= \operatorname{Tr}\ln\left[\mathrm{i}\left(-\slashed{\partial}_y + \mathrm{i}e\slashed{A}(y) - m\right)\delta^{d+1}(x-y)\right].
\end{aligned}
\tag{11}
$$

where the transpose $^\mathsf{T}$ is taken in the spinor space. Therefore

$$\operatorname{Tr}\ln\left[-\mathrm{i}S^{-1}(x,y)\right] = \frac{1}{2}\operatorname{Tr}\ln\left[S_2^{-1}(x,y)\right], \tag{12}$$

where

$$S_2^{-1}(x,y) := \mathcal{S}_{2y}^{-1}\delta^{d+1}(x-y), \quad \mathcal{S}_{2y}^{-1} := \left(\left(\slashed{\partial}_y - \mathrm{i}e\slashed{A}(y)\right)^2 - m^2\right). \tag{13}$$

One may further simplify eq. (13) by noting (y suppressed)

$$
\begin{aligned}
\left(\slashed{\partial} - \mathrm{i}e\slashed{A}\right)^2 &= \partial^2 - e^2 A_\mu^2 - \mathrm{i}e\frac{1}{2}\left([\gamma^\mu, \gamma^\nu]_- + [\gamma^\mu, \gamma^\nu]_+\right)\left(\partial_\mu A_\nu + A_\mu \partial_\nu + A_\nu \partial_\mu\right) \\
&= \partial^2 - e^2 A_\mu^2 - \mathrm{i}e\left(\partial_\mu A^\mu + 2A^\mu \partial_\mu - \mathrm{i}\sigma^{\mu\nu}\partial_\mu A_\nu\right) \\
&= \left(\partial_\mu - \mathrm{i}eA_\mu\right)^2 - e\sigma^{\mu\nu}\partial_{[\mu}A_{\nu]} \\
&= \left(\partial_\mu - \mathrm{i}eA_\mu\right)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu},
\end{aligned}
\tag{14}
$$

so that

$$
\mathcal{S}_2^{-1} \equiv \left(\left(\partial_\mu - \mathrm{i}eA_\mu\right)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\right),
\tag{15}
$$

where y is suppressed as well. Now we may write

$$
\begin{aligned}
\Gamma_{\mathrm{EH}}[A_\mu] &= -\frac{\mathrm{i}}{2}\,\mathrm{Tr}\ln\left\{\mathcal{N}^{-1}S_2^{-1}(x,y)\right\} \\
&= -\frac{\mathrm{i}}{2}\,\mathrm{Tr}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\left\{-\mathrm{e}^{\mathrm{i}s\left(S_2^{-1}(x,y)+\mathrm{i}0^+\right)} + \mathrm{e}^{\mathrm{i}s\left(\mathcal{N}+\mathrm{i}0^+\right)}\right\} \\
&= +\frac{\mathrm{i}}{2}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\delta^{d+1}(x-y) \\
&\quad \cdot \mathrm{tr}\left\{+\mathrm{e}^{\mathrm{i}s\left((\partial_\mu - \mathrm{i}eA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2 + \mathrm{i}0^+\right)\delta^{d+1}(x-y)} - \mathrm{e}^{\mathrm{i}s\left(\mathcal{N}+\mathrm{i}0^+\right)}\right\},
\end{aligned}
\tag{16}
$$

where tr takes place in the spinor space.

1.1   Constant background field in $(3+1)$-dimensions

Equation (12) can be solved exactly when $F_{\mu\nu}$ is constant throughout space-time. One has

$$
F_{0i} \equiv -F_{i0} = -E_i, \qquad F_{ij} \equiv -F_{ji} = \epsilon_{ijk}B^k,
\tag{17}
$$

and the spinor part of eq. (15) can be calculated

$$
-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} = -e\begin{bmatrix} \left(\vec{B} - \mathrm{i}\vec{E}\right)\cdot\vec{\sigma} & 0 \\ 0 & \left(\vec{B} + \mathrm{i}\vec{E}\right)\cdot\vec{\sigma} \end{bmatrix},
\tag{18}
$$

so that

$$
\begin{aligned}
&\mathrm{tr}\,\mathrm{e}^{\mathrm{i}s\left((\partial_\mu - \mathrm{i}eA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2 + \mathrm{i}0^+\right)\delta^{d+1}(x-y)} \\
&= \mathrm{e}^{\mathrm{i}s\left((\partial_\mu - \mathrm{i}eA_\mu)^2 - m^2 + \mathrm{i}0^+\right)\delta^{d+1}(x-y)}\,\mathrm{tr}\,\mathrm{e}^{\mathrm{i}s\left(-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)},
\end{aligned}
\tag{19}
$$

in which

$$\mathrm{tr}\, \mathrm{e}^{\mathrm{i}s\left(-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)} = 2\left\{\cos\left(se\sqrt{\left(\vec{B}+\mathrm{i}\vec{E}\right)^2}\right) + \cos\left(se\sqrt{\left(\vec{B}-\mathrm{i}\vec{E}\right)^2}\right)\right\} \tag{20}$$

$$= 4\cos(seB)\cosh(seE), \qquad \text{for} \quad \vec{B} = B\hat{x}^3, \quad \vec{E} = E\hat{x}^3. \tag{21}$$

A Landau-like choice of four-potential [?] reads

$$A_\mu := \begin{pmatrix} 0 & -x^0 E_1 + x^3 B^2 & -x^0 E_2 + x^1 B^3 & -x^0 E_3 + x^2 B^1 \end{pmatrix}, \tag{22}$$

applying which leads to eq. (17).

Consider the case where $\vec{B} = B\hat{x}^3$, $\vec{E} = E\hat{x}^3$. The scalar part of eq. (15) simplifies to

$$(\partial_\mu - \mathrm{i}eA_\mu)^2 - m^2 = -\partial_0^2 + \partial_1^2 + \left(\partial_2 - \mathrm{i}ex^1 B\right)^2 + \left(\partial_3 + \mathrm{i}ex^0 E\right)^2 - m^2$$
$$= U\left\{-\partial_0^2 + \partial_1^2 - (eB)^2\left(x^1\right)^2 - (eE)^2\left(x^0\right)^2\right\}U^\dagger, \tag{23}$$

where

$$U := \mathrm{e}^{+\mathrm{i}(\partial_1\partial_2/eB - \partial_3\partial_0/eE)}. \tag{24}$$

One can first solve the sub-eigensystem

$$\left\{+\partial_1^2 - (eB)^2\left(x^1\right)^2 - \lambda\right\}\tilde{f}_\lambda(x^1) = 0, \tag{25}$$

and the general solution is

$$\tilde{f}_\lambda(x) = c_1 \mathrm{D}_{-\frac{\lambda}{2eB}-\frac{1}{2}}\left(\sqrt{2eB}x\right) + c_2 \mathrm{D}_{+\frac{\lambda}{2eB}-\frac{1}{2}}\left(\mathrm{i}\sqrt{2eB}x\right), \tag{26}$$

where $\mathrm{D}_\nu(z)$ is the parabolic cylinder function. For

$$\lambda = -(2n+1)eB, \qquad n = -\frac{1}{2} - \frac{\lambda}{2eB} \tag{27}$$

eq. (26) reduces to a combination of the Hermite polynomials

$$\tilde{f}_n(x) = c_1 2^{-n/2}\mathrm{e}^{-eBx^2/2}\mathrm{H}_n\left(\sqrt{eB}x\right) + c_2 2^{+(n+1)/2}\mathrm{e}^{+eBx^2/2}\mathrm{H}_{-n-1}\left(\mathrm{i}\sqrt{eB}x\right). \tag{28}$$

The first branch constitutes a complete orthogonal set of basis for $L^2(\mathbb{R})$ when $n \in \mathbb{Z}^+$, and its normalised version will be denoted as $\{f_n\}$.

It has been argued that the same solution applies for

$$\left\{-\partial_0^2 - (eE)^2\left(x^0\right)^2 - \lambda\right\}\tilde{g}_\lambda(x^0) = 0 \tag{29}$$

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if one applies $E \to iB$, so that the eigenfunctions are

$$g_n(x) = f_n(x), \qquad n = -\frac{1}{2} - \frac{i\lambda}{2eE} \quad \text{or} \quad \lambda = i(2n+1)eE. \tag{30}$$

However, we know that complex differential equations are subtle!

The eigenfunctions of eq. (23) can now be written down

$$\begin{aligned}
&\left\{ (\partial_\mu - ieA_\mu)^2 - m^2 \right\} \left\{ U f_n(x^1) g_k(x^0) \right\} \\
&= U \left\{ -\partial_0^2 + \partial_1^2 - (eB)^2 (x^1)^2 - (eE)^2 (x^0)^2 \right\} f_n(x^1) g_k(x^0) \\
&= -i(2n+1)(2k+1)e^2 BE \left\{ U f_n(x^1) g_k(x^0) \right\},
\end{aligned} \tag{31}$$

and they are

$$\phi_{nk}(x) := U f_n(x^1) g_k(x^0). \tag{32}$$

$$\begin{aligned}
\delta^4(x-y) &= \sum_{n,k} \phi_{nk}^\dagger(x) \phi_{nk}(y) \\
&= \sum_{n,k} \left\{ U(x) f_n(x^1) g_k(x^0) \right\}^\dagger \left\{ U(y) f_n(y^1) g_k(y^0) \right\}.
\end{aligned} \tag{33}$$

?

A   Notions and conventions

The metric convention is mostly positive, i.e. $\eta_{\mu\nu} := \mathrm{diag}(-,+,+,\ldots)$

B   Fresnel functional integral

[?, ch. 10]

C   Algebra

$$\left[a\partial_1 \partial_2, x^1 \right]_- = a\partial_2 \tag{34}$$

central.

Baker–Campbell–Hausdorff formula

$$e^{+a\partial_1\partial_2}x^1 e^{-a\partial_1\partial_2} = x^1 + a\partial_2. \tag{35}$$

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