Classical string theory

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1 Bosonic string

1.1 Nambu–Goto action

The action reads [2, 1]

$$S_{\mathrm{NG}} := -T \int \mathrm{d}A = -T \int \mathrm{d}^2\sigma \sqrt{-\tilde{\tilde{\gamma}}}, \tag{1}$$

where

$$\tilde{\tilde{\gamma}} := \det \gamma_{\alpha\beta} \equiv \gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21} \tag{2}$$

is the metric determinant, $\alpha, \beta, \ldots = 1, 2$ are the world-sheet indices, (σ^α) are the world-sheet coordinates,

$$\gamma_{\alpha\beta} := g_{\mu\nu} X^\mu, \alpha X^\nu{}_{,\beta}, \tag{3}$$

is the induced metric on the world-sheet, $\rho, \nu, \ldots = 0, 1, \ldots d$ are the target-space indices, $X^\mu = X^\mu(\sigma^\alpha)$ are the world-sheet coordinates. X^μ are the dynamical variables.

The inverse metric can also be expressed in a closed form

$$\begin{aligned}
\gamma^{\alpha\beta} &= \frac{1}{(2-1)!} \tilde{\epsilon}^{\alpha\gamma} \tilde{\epsilon}^{\beta\delta} \tilde{\tilde{\gamma}}^{-1} \gamma_{\gamma\delta} \\
&= \tilde{\epsilon}^{\alpha\gamma} \tilde{\epsilon}^{\beta\delta} \tilde{\tilde{\gamma}}^{-1} \gamma_{\gamma\delta} = \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} \gamma_{\gamma\delta} && (4) \\
&= \tilde{\epsilon}^{\alpha\gamma} \tilde{\epsilon}^{\beta\delta} \tilde{\tilde{\gamma}}^{-1} g_{\rho\sigma} X^\rho{}_{,\gamma} X^\sigma{}_{,\delta} = \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} g_{\rho\sigma} X^\rho{}_{,\gamma} X^\sigma{}_{,\delta} && (5) \\
&\equiv \tilde{\tilde{\gamma}}^{-1} \begin{pmatrix} \gamma_{22} & -\gamma_{12} \\ -\gamma_{21} & \gamma_{11} \end{pmatrix}^{\alpha\beta}. && (6)
\end{aligned}$$

Variation of the induced metric determinant can be expressed in terms of that of the induced metric

$$\delta \tilde{\tilde{\gamma}} = \tilde{\tilde{\gamma}} \gamma^{\alpha\beta} \delta\gamma_{\alpha\beta}. \tag{7}$$

Variation of the induced metric in terms of the world-sheet coordinates reads

$$\delta\gamma_{\alpha\beta} = X^{\nu}{}_{,\beta}\big(2g_{\mu\nu}\,\delta X^{\mu}{}_{\alpha} + X^{\rho}{}_{\alpha}g_{\nu\rho,\mu}\,\delta X^{\mu}\big). \tag{8}$$

Variation of the area element reads

$$\begin{aligned}\delta\sqrt{-\tilde{\tilde{\gamma}}} &= \frac{1}{2}\sqrt{-\tilde{\tilde{\gamma}}}\,\gamma^{\alpha\beta}\,\delta\gamma_{\alpha\beta} \\ &= -\tilde{\epsilon}^{\alpha\gamma}\tilde{\epsilon}^{\beta\delta}\big(-\tilde{\tilde{\gamma}}\big)^{-1/2}\gamma_{\gamma\delta}\frac{\partial X^{\nu}}{\partial\sigma^{\beta}}g_{\mu\nu}\,\delta\frac{\partial X^{\mu}}{\partial\sigma^{\alpha}} \\ &= -\tilde{\epsilon}^{\alpha\gamma}\tilde{\epsilon}^{\beta\delta}\Bigg\{\frac{\partial}{\partial\sigma^{\alpha}}\bigg[\big(-\tilde{\tilde{\gamma}}\big)^{-1/2}\gamma_{\gamma\delta}\frac{\partial X^{\nu}}{\partial\sigma^{\beta}}g_{\mu\nu}\,\delta X^{\mu}\bigg] \\ &\qquad - \frac{\partial}{\partial\sigma^{\alpha}}\bigg[\big(-\tilde{\tilde{\gamma}}\big)^{-1/2}\gamma_{\gamma[\delta}\frac{\partial X^{\nu}}{\partial\sigma^{\beta]}}g_{\mu\nu}\bigg]\,\delta X^{\mu}\Bigg\}\end{aligned} \tag{9}$$

References

[1] Tetsuo Gotō. “Relativistic Quantum Mechanics of One-Dimensional Mechanical Continuum and Subsidiary Condition of Dual Resonance Model”. In: *Progress of Theoretical Physics* 46.5 (Nov. 1971), pp. 1560–1569. DOI: `10.1143/ptp.46.1560`.

[2] Yōichirō Nambu. “Duality and Hadrodynamics”. Notes prepared for the Copenhagen High Energy Symposium. Aug. 1970.