Classical string theory

Yi-Fan Wang (王 一帆)

September 24, 2018

1 Bosonic string

1.1 Nambu–Goto action

The action reads [2, 1]

$$S_{\mathrm{NG}} := -T \int \mathrm{d}A = -T \int \mathrm{d}^2\sigma \sqrt{-\tilde{\tilde{\gamma}}}, \tag{1}$$

where

$$\tilde{\tilde{\gamma}} := \det \gamma_{\alpha\beta} \equiv \gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21} \tag{2}$$

is the metric determinant, $\alpha, \beta, \ldots = 1, 2$ are the world-sheet indices, (σ^α) are the world-sheet coordinates,

$$\gamma_{\alpha\beta} := g_{\mu\nu} X^\mu{}_{,\alpha} X^\nu{}_{,\beta}, \tag{3}$$

is the induced metric on the world-sheet, $\rho, \nu, \ldots = 0, 1, \ldots d$ are the target-space indices, $X^\mu = X^\mu(\sigma^\alpha)$ are the world-sheet coordinates. X^μ are the dynamical variables.

The inverse metric can also be expressed in a closed form

$$\begin{aligned}
\gamma^{\alpha\beta} &= \frac{1}{(2-1)!} \tilde{\epsilon}^{\alpha\gamma} \tilde{\epsilon}^{\beta\delta} \tilde{\tilde{\gamma}}^{-1} \gamma_{\gamma\delta} \\
&= \tilde{\epsilon}^{\alpha\gamma} \tilde{\epsilon}^{\beta\delta} \tilde{\tilde{\gamma}}^{-1} \gamma_{\gamma\delta} = \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} \gamma_{\gamma\delta} && (4) \\
&= \tilde{\epsilon}^{\alpha\gamma} \tilde{\epsilon}^{\beta\delta} \tilde{\tilde{\gamma}}^{-1} g_{\rho\sigma} X^\rho{}_{,\gamma} X^\sigma{}_{,\delta} = \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} g_{\rho\sigma} X^\rho{}_{,\gamma} X^\sigma{}_{,\delta} && (5) \\
&\equiv \tilde{\tilde{\gamma}}^{-1} \begin{pmatrix} \gamma_{22} & -\gamma_{12} \\ -\gamma_{21} & \gamma_{11} \end{pmatrix}^{\alpha\beta}. && (6)
\end{aligned}$$

Variation of the induced metric determinant can be expressed in terms of that of the induced metric

$$\delta \tilde{\tilde{\gamma}} = \tilde{\tilde{\gamma}} \gamma^{\alpha\beta}\, \delta\gamma_{\alpha\beta}. \tag{7}$$

Variation of the induced metric in terms of the world-sheet coordinates reads

$$\delta\gamma_{\alpha\beta} = X^{\nu}{}_{,\alpha}\big(2g_{\nu\lambda}\,\delta X^{\lambda}{}_{,\beta} + X^{\rho}{}_{,\beta}g_{\nu\rho,\lambda}\,\delta X^{\lambda}\big). \tag{8}$$

Variation of the area element reads

$$\begin{aligned}
\delta\sqrt{-\tilde{\tilde{\gamma}}} &= \frac{1}{2}\sqrt{-\tilde{\tilde{\gamma}}}\,\gamma^{\alpha\beta}\,\delta\gamma_{\alpha\beta} \\
&= \frac{1}{2}\sqrt{-\tilde{\tilde{\gamma}}}\,\gamma^{\alpha\beta}X^{\nu}{}_{,\alpha}\big(2g_{\nu\lambda}\,\delta X^{\lambda}{}_{,\beta} + X^{\rho}{}_{,\beta}g_{\nu\rho,\lambda}\,\delta X^{\lambda}\big) \\
&= (\ldots)^{\beta}{}_{,\beta} - \left\{\left(\sqrt{-\tilde{\tilde{\gamma}}}\,\gamma^{\alpha\beta}X^{\nu}{}_{,\alpha}g_{\nu\lambda}\right)_{,\beta} + \frac{1}{2}\sqrt{-\tilde{\tilde{\gamma}}}\,\gamma^{\alpha\beta}X^{\nu}{}_{,\alpha}X^{\rho}{}_{,\beta}g_{\nu\rho,\lambda}\right\}\delta X^{\lambda} \\
&= (\ldots)^{\beta}{}_{,\beta} - \sqrt{-\tilde{\tilde{\gamma}}}\,\Big\{\Box_{\sigma}X^{\mu}g_{\mu\lambda} + \gamma^{\alpha\beta}X^{\nu}{}_{,\alpha}g_{\nu\lambda,\rho}X^{\rho}{}_{,\beta} \\
&\qquad\qquad - \frac{1}{2}\gamma^{\alpha\beta}X^{\nu}{}_{,\alpha}X^{\rho}{}_{,\beta}g_{\nu\rho,\lambda}\Big\}\,\delta X^{\lambda} \\
&= (\ldots)^{\beta}{}_{,\beta} - \sqrt{-\tilde{\tilde{\gamma}}}\,\Big\{\Box_{\sigma}X^{\mu}g_{\mu\lambda} \\
&\qquad\qquad + \frac{1}{2}\gamma^{\alpha\beta}X^{\nu}{}_{,\alpha}X^{\rho}{}_{,\beta}\big(-g_{\nu\rho,\lambda} + g_{\rho\lambda,\nu} + g_{\lambda\nu,\rho}\big)\Big\}\,\delta X^{\lambda} \\
&= (\ldots)^{\beta}{}_{,\beta} - \sqrt{-\tilde{\tilde{\gamma}}}\,g_{\mu\lambda}\big\{\Box_{\sigma}X^{\mu} + \Gamma^{\mu}{}_{\nu\rho}\gamma^{\alpha\beta}X^{\nu}{}_{,\alpha}X^{\rho}{}_{,\beta}\big\}\,\delta X^{\lambda},
\end{aligned} \tag{9}$$

where

$$(\ldots)^{\beta} := \sqrt{-\tilde{\tilde{\gamma}}}\,\gamma^{\alpha\beta}X^{\nu}{}_{,\alpha}g_{\nu\lambda}\,\delta X^{\lambda}, \tag{10}$$

$$\Box_{\sigma}X^{\mu} := \frac{1}{\sqrt{-\tilde{\tilde{\gamma}}}}\left(\sqrt{-\tilde{\tilde{\gamma}}}\gamma^{\alpha\beta}X^{\mu}{}_{,\alpha}\right)_{,\beta} \tag{11}$$

References

[1] Tetsuo Gotō. “Relativistic Quantum Mechanics of One-Dimensional Mechanical Continuum and Subsidiary Condition of Dual Resonance Model”. In: *Progress of Theoretical Physics* 46.5 (Nov. 1971), pp. 1560–1569. DOI: `10.1143/ptp.46.1560`.

[2] Yōichirō Nambu. “Duality and Hadrodynamics”. Notes prepared for the Copenhagen High Energy Symposium. Aug. 1970.