Cosmological Perturbations

Yi-Fan Wang (王 一帆)

April 11, 2018

Most of the conventions and notations in [1, ch. 5] will be followed.

Suppose the metric can be expanded up to the linear order as

$$g = g^{(0)} + \epsilon g^{(1)} + O(\epsilon^2). \tag{1}$$

The background metric $g^{(0)}$ takes the Robertson–Walker form

$$g^{(0)}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -N^2(t)\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\Omega^2_{3\mathrm{F}}, \tag{2}$$

in which $\mathrm{d}\Omega^2_{3\mathrm{F}} = \mathrm{d}\chi^2 + \chi^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big)$ is the dimensionless flat spatial metric. The linear perturbation can be decomposed into scalar, vector and tensor parts

$$g^{(1)}_{00} = -N^2 E, \tag{3}$$

$$g^{(1)}_{i0} = g^{(1)}_{i0} = Na\big(F_{,i} + G_i\big), \tag{4}$$

$$g^{(1)}_{ij} = a^2\big(A\delta_{ij} + B_{,i,j} + C_{i,j} + C_{j,i} + D_{ij}\big). \tag{5}$$

Here one has some weird condition, where the contractions do not follow the one-up-one-down tradition

$$C_{i,i} = G_{i,i} = 0, \quad D_{ij,i} = 0, \quad D_{ii} = 0. \tag{6}$$

1 Transform under diffeomorphism

Consider a diffeomorphism generated by $\xi^\mu = O(\epsilon)$, which can in turn be decomposed into $\xi_0 = \zeta$, $\xi_i = \xi^{\mathrm{S}}_{,i} + \xi^{\mathrm{V}}_i$.

One has here again some weird condition, where the contractions do not follow the one-up-one-down tradition

$$\xi^{\mathrm{V}}_{i,i} = 0. \tag{7}$$

The Lie derivative of the metric $\mathbb{L}_\xi g$ is

$$\big(\mathbb{L}_\xi g\big)_{\mu\nu} = \xi^\lambda g_{\mu\nu,\lambda} + \xi^\lambda{}_{,\mu} g_{\lambda\nu} + \xi^\lambda{}_{,\nu} g_{\mu\lambda}. \tag{8}$$

In components and expansion, these are

$$\left(\mathbb{L}_{\xi} g\right)_{00} = 2\dot{\zeta} - 2\zeta\frac{\dot{N}}{N} + O(\epsilon^2), \tag{9}$$

$$\left(\mathbb{L}_{\xi} g\right)_{i0} = \left(\mathbb{L}_{\xi} g\right)_{0i} =, \tag{10}$$

$$\tag{11}$$

2 Scalar perturbations

3 Vector perturbations

4 Tensor perturbations

References

[1] Steven Weinberg. *Cosmology*. Oxford University Press, 2008. ISBN: 9780198526827. URL: https://global.oup.com/academic/product/cosmology-9780198526827.