Perturbation of the ADM Hamiltonian Action

Yi-Fan Wang (王 一帆)

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In General Relativity, if one uses components of the metric $\{g_{\mu\nu}\}$ as the (superficial) degrees of freedom, one obtains a reparametrisation-invariant theory, and the gauge transformation of which is related to the infinitesimal general coordinate transformation

$$x^\mu \mapsto x^\mu - \eta^\mu(x) + O(\eta^2) \tag{1}$$

by

$$\delta g_{\mu\nu} = \mathbb{L}_\eta g_{\mu\nu} + O(\eta^2), \tag{2}$$

$$\mathbb{L}_\eta g_{\mu\nu} = \eta_{\mu;\nu} + \eta_{\nu;\mu} = \eta^\lambda g_{\mu\nu,\lambda} + \eta^\lambda{}_{,\mu} g_{\lambda\nu} + \eta^\lambda{}_{,\nu} g_{\mu\lambda}. \tag{3}$$

Here the generator of the general coordinate transformation $\eta^\mu(x)$ plays the role of gauge generator. Upon canonical quantisation, one would like to apply eq. (3) to the canonical variables. However, the common practice is to use the Arnowitt–Deser–Misner variables $\{N, N_i, h_{ij}\}$ in the canonical formalism, instead of $\{g_{\mu\nu}\}$. The former ones are an alternative parametrisation of the latter, and they are related by

$$g_{\mu\nu}\,\mathbb{d}x^\mu\,\mathbb{d}x^\nu = \mathbb{d}s^2 = -N^2\,\mathbb{d}t^2 + h_{ij}(N^i\,\mathbb{d}t + \mathbb{d}x^i)(N^j\,\mathbb{d}t + \mathbb{d}x^j). \tag{4}$$

It is less well-known that one can also consistently derive a Hamiltonian action by using $\{g_{\mu\nu}\}$ and their conjugate momenta $\{\pi^{\mu\nu}\}$, as Dirac has done in [2]. The Dirac and the Arnowitt–Deser–Misner approaches have been compared in [4], where some subtle differences have been shown; in particular, the gauge transformation in the former case (eq. (3)) is different than the latter one (eqs. (12) to (17)). This difference can be quantified, in the sense that the relation between $\{\eta_\mu\}$ and $\{\xi_\perp, \xi_i\}$, which are the gauge generator of the Dirac and Arnowitt–Deser–Misner approaches respectively, can be derived explicitly.

We would like to see if this difference has any consequence on the cosmological perturbation theory; in particular, we would like to see if the gauge-invariant variables could be changed from this perspective. In order to do this, we start from the Hamiltonian formalism in the Arnowitt–Deser–Misner variables and derive the gauge transformations of all the canonical variables; we then perturb the Hamiltonian action up to the second order on an arbitrary background with those variables directly; and finally we fix a Robertson–Walker background metric and discuss the gauge transformation of the perturbative coordinates and momenta.

1 Gauge transformation of the ADM canonical variables

The Hamiltonian action for General Relativity in terms of Arnowitt–Deser–Misner variables is [3, ch. 4.2.2]

$$S = \int \mathrm{d}t\, \mathrm{d}x^3 \Big\{ \mathfrak{p}^{ij}\dot{h}_{ij} + \mathfrak{P}\dot{N} + \mathfrak{P}^i\dot{N}_i - N\mathfrak{H}^\perp - N_i\mathfrak{H}^i - \mathfrak{P}V - \mathfrak{P}^iV_i \Big\} + \text{ boundary terms} \tag{5}$$

where

$$\mathfrak{H}^\perp = 2\varkappa\,\mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl} - \frac{\sqrt{\mathfrak{h}}}{2\varkappa}R[h] \equiv 2\varkappa\,\mathfrak{F}^{ijkl}h_{ij}h_{kl} - \frac{\sqrt{\mathfrak{h}}}{2\varkappa}R[h], \tag{6}$$

$$\mathfrak{H}^i = -2\mathfrak{p}^{ij}{}_{|j} \tag{7}$$

are the secondary constraints,

$$\mathfrak{G}_{ijkl} := \frac{1}{2\mathfrak{h}^{1/2}}\left(h_{ik}h_{lj} + h_{il}h_{kj} - h_{ij}h_{kl}\right) \equiv -\frac{\delta\left(\mathfrak{h}^{-1/2}h_{ij}\right)}{\delta h^{kl}}, \tag{8}$$

$$\mathfrak{G}^{ijkl} := \frac{\mathfrak{h}^{1/2}}{2}\left(h^{ik}h^{lj} + h^{il}h^{kj} - 2h^{ij}h^{kl}\right) \equiv -\mathfrak{h}^{-1/2}\frac{\delta\left(\mathfrak{h}h^{ij}\right)}{\delta h_{kl}} \tag{9}$$

$$\mathfrak{F}^{ijkl} := \frac{1}{2\mathfrak{h}^{1/2}}\left(\mathfrak{p}^{ik}\mathfrak{p}^{jl} + \mathfrak{p}^{il}\mathfrak{p}^{kj} - \mathfrak{p}^{ij}\mathfrak{p}^{kl}\right) \tag{10}$$

are convenient notations. In eq. (5), V and V_i are velocities of N and N_i and play the role of Lagrange multipliers. Technical details about the boundary terms can be found in [5, ch. 4.2] and the references therein.

Gauge transformations in the Arnowitt–Deser–Misner canonical variables are generated by [1]

$$G = -\int \mathrm{d}^3x \Big\{ \Big[\xi_\perp\Big(\mathfrak{H}^\perp + N_{|i}\mathfrak{P}^i + \left(N\mathfrak{P}^i\right)_{|i} + \left(N_i\mathfrak{P}\right)^{|i}\Big) + \dot{\xi}_\perp\mathfrak{P}\Big] + \Big[\xi_i\Big(\mathfrak{H}^i + N_j{}^{|i}\mathfrak{P}^j + \left(N_j\mathfrak{P}^i\right)^{|j} + N^{|i}\mathfrak{P}\Big) + \dot{\xi}_i\mathfrak{P}^i\Big]\Big\} + \text{ boundary terms,} \tag{11}$$

where the boundary terms have probably not been discussed so far, but are surely needed for non-compact spatial topologies in order to cancel the second spatial derivatives in the potential term in $\mathfrak{H}^\perp$. The infinitesimal gauge transformations of $\{N, N_i; \mathfrak{P}, \mathfrak{P}^i\}$ are [4]

$$\delta N = [N, G]_{\mathrm{P}} = \xi_\perp{}^{|i} N_i - \dot{\xi}_\perp - \xi_i N^{|i}, \tag{12}$$

$$\delta N_i = -\xi_\perp N_{|i} + \xi_{\perp|i} N - \xi_j N_i{}^{|j} + \xi_i{}^{|j} N_j - \dot{\xi}_i; \tag{13}$$

$$\delta \mathfrak{P} = -\left(\xi_\perp \mathfrak{P}^i\right)_{|i} - \xi_{\perp|i} \mathfrak{P}^i - \left(\xi_i \mathfrak{P}\right)^{|i}, \tag{14}$$

$$\delta \mathfrak{P}^i = -\xi_\perp{}^{|i} \mathfrak{P} - \left(\xi_j \mathfrak{P}^i\right)^{|j} - \xi_j{}^{|i} \mathfrak{P}^j, \tag{15}$$

where only the primary constraints are involved; transformations of $\left\{g_{ij}; \mathfrak{p}^{ij}\right\}$ are

$$\begin{aligned} \delta h_{ij} &= -\frac{\partial}{\partial \mathfrak{p}^{ij}} (\xi_\perp \mathfrak{H}^\perp + \xi_i \mathfrak{H}^i) \\ &= -\xi^\perp 4\varkappa \mathfrak{G}_{ijkl} \mathfrak{p}^{kl} - \xi_{i|j} - \xi_{j|i}, \end{aligned} \tag{16}$$

$$\begin{aligned} \delta \mathfrak{p}^{ij} &= \frac{\partial}{\partial h_{ij}} (\xi_\perp \mathfrak{H}^\perp + \xi_i \mathfrak{H}^i) \\ &= 2\varkappa \, \xi_\perp \left(-\frac{1}{2} h^{ij} \mathfrak{F}^{klmn} h_{mn} + 2 \mathfrak{F}^{ijkl}\right) h_{kl} \\ &\quad + \frac{1}{2\varkappa} \left(\sqrt{\mathfrak{h}}\, \xi_\perp G^{ij}[h] - \mathfrak{G}^{ijkl} (\xi_\perp)_{|k|l}\right) \\ &\quad + \left\{(\mathfrak{p}^{il} h^{kj} + \mathfrak{p}^{jl} h^{ki} - \mathfrak{p}^{ij} h^{kl}) \xi_k\right\}_{|l}, \end{aligned} \tag{17}$$

where only the secondary constraints are involved. The results can also be checked with [5, ch. 4.2.7].

2 Perturbative expansion of the ADM Hamiltonian action

Some useful results

First variations

The first variation of the inverse metric h^{ij} reads

$$\delta h^{ij} = -h^{ik} h^{jl} \, \delta h_{kl} = -h^{i(k} h^{l)j} \, \delta h_{kl}. \tag{18}$$

The first variation of $\mathfrak{h} = \det h_{ij}$ reads

$$\delta \mathfrak{h} = \mathfrak{h} h^{ij} \, \delta h_{ij}. \tag{19}$$

The first variation of $\Gamma^i{}_{jk}$ can be obtained in normal coordinates, which reads

$$\delta \Gamma^i{}_{jk} = \frac{1}{2} h^{il} \left\{-\left(\delta h_{jk}\right)_{|l} + \left(\delta h_{kl}\right)_{|j} + \left(\delta h_{lj}\right)_{|k}\right\} \tag{20}$$

$$= \frac{1}{2} \left\{-h^{il} \delta^m{}_j \delta^n{}_k + h^{in} \delta^l{}_j \delta^m{}_k + h^{im} \delta^n{}_j \delta^l{}_k\right\} \left(\delta h_{mn}\right)_{|l}. \tag{21}$$

The first variation of $R_{ij}[h]$ and $R^{ij}[h]$

$$\delta R_{ij}[h] = \left(\delta\Gamma^k{}_{ji}\right)_{|k} - \left(\delta\Gamma^k{}_{ki}\right)_{|j}, \tag{22}$$

$$\delta R^{ij}[h] = -2R^{k(i}h^{j)l}\,\delta h_{kl} + h^{k(i}\,\bar{\delta}u^{j)l}{}_{k|l}, \tag{23}$$

where

$$\bar{\delta}u^{ij}{}_k := h^{il}\,\delta\Gamma^j{}_{lk} - h^{ij}\,\delta\Gamma^l{}_{lk} \tag{24}$$

is related to the boundary terms. Equation (22) can be obtained in normal coordinates.

For the first variation of the constraints, one also needs

$$\bar{\delta}u^{ji}{}_j = \left(\delta h_{kl}\right)_{|j}\left(h^{i(k}h^{l)j} - h^{ij}h^{kl}\right) = \mathfrak{h}^{-1/2}\mathfrak{G}^{ijkl}\left(\delta h_{kl}\right)_{|j}; \tag{25}$$

therefore,

$$\sqrt{\mathfrak{h}}\,N\,\bar{\delta}u^{ji}{}_{j|i} = \delta h_{ij}\,\mathfrak{G}^{ijkl}N_{|k|l} + \left\{\mathfrak{G}^{ijkl}\left(N\left(\delta h_{kl}\right)_{|j} - N_{|j}\,\delta h_{kl}\right)\right\}_{|i}. \tag{26}$$

In the Hamiltonian constraint, the first variation of the 'kinetic term' $\mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl} \equiv \mathfrak{F}^{ijkl}h_{ij}h_{kl}$ reads

$$\begin{aligned}&\delta\left(\mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl}\right) \equiv \delta\left(\mathfrak{F}^{ijkl}h_{ij}h_{kl}\right)\\ &= \delta h_{ij}\left(-\frac{1}{2}h^{ij}\mathfrak{F}^{klmn}h_{mn} + 2\mathfrak{F}^{ijkl}\right)h_{kl} + \delta\mathfrak{p}^{ij}\,2\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}.\end{aligned} \tag{27}$$

Equipped with eqs. (23) and (25), the first variation of the 'potential' $\sqrt{\mathfrak{h}}\,R[h]$ reads

$$\begin{aligned}\delta\left(\sqrt{\mathfrak{h}}R[h]\right) &= \sqrt{\mathfrak{h}}\left\{-G^{ij}[h]\,\delta h_{ij} + \bar{\delta}u^{ji}{}_{j|i}\right\}\\ &= -\sqrt{\mathfrak{h}}\,G^{ij}[h]\,\delta h_{ij} + \mathfrak{G}^{ijkl}\left(\delta h_{kl}\right)_{|j|i}.\end{aligned} \tag{28}$$

One can now write down the first variation of $N\mathfrak{H}^{\perp}$ with respect to $\left\{h_{ij}, \mathfrak{p}^{ij}\right\}$,

$$\begin{aligned}N\,\delta\mathfrak{H}^{\perp} = \delta h_{ij}&\left\{2\varkappa N\left(-\frac{1}{2}h^{ij}\mathfrak{F}^{klmn}h_{mn} + 2\mathfrak{F}^{ijkl}\right)h_{kl}\right.\\ &\left.+ \frac{1}{2\varkappa}\left(\sqrt{\mathfrak{h}}\,NG^{ij}[h] - \mathfrak{G}^{ijkl}N_{|k|l}\right)\right\}\\ &+ \delta\mathfrak{p}^{ij}\,4\varkappa N\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}\\ &- \frac{1}{2\varkappa}\left\{\mathfrak{G}^{ijkl}\left(N\left(\delta h_{kl}\right)_{|j} - N_{|j}\,\delta h_{kl}\right)\right\}_{|i},\end{aligned} \tag{29}$$

where the terms in the last line will be pushed to the spatial boundary $\partial\Sigma$; the second term vanishes by $\delta h_{ij}\big|_{\partial\Sigma} = 0$, whereas the first one is cancelled by the boundary term.

Finally, the first variation of $N_i \mathfrak{H}^i$ with respect to $\{h_{ij}, \mathfrak{p}^{ij}\}$ is easier,

$$\begin{aligned} N_i\, \delta\mathfrak{H}^i &= \delta h_{ij} \left\{ (\mathfrak{p}^{il} h^{kj} + \mathfrak{p}^{jl} h^{ki} - \mathfrak{p}^{ij} h^{kl}) N_k \right\}_{|l} + \delta\mathfrak{p}^{ij}\, 2N_{(i|j)} \\ &\quad - \left(-h^{il}\delta^m{}_j \delta^n{}_k + h^{in}\delta^l{}_j \delta^m{}_k + h^{im}\delta^n{}_j \delta^l{}_k\right)\left(N_i \mathfrak{p}^{jk}\, \delta h_{mn}\right)_{|l} \\ &\quad - 2\left(\delta\mathfrak{p}^{ij}\, N_j\right)_{|i}, \end{aligned} \tag{30}$$

where the last two lines will be pushed to the spatial boundary and vanish by $\delta h_{ij}\big|_{\partial\Sigma} = 0 = \delta\mathfrak{p}^{ij}\big|_{\partial\Sigma}$.

Second variations

First variation of $\mathfrak{F}^{ijkl} h_{kl}$

$$\begin{aligned} \delta(\mathfrak{F}^{ijkl} h_{kl}) = \delta h_{kl} & \left(-\frac{1}{2} \mathfrak{F}^{ijmn} h^{kl} h_{mn} + \mathfrak{F}^{ijkl} \right) \\ & + \delta\mathfrak{p}^{kl}\, (\delta^i{}_k \mathfrak{G}^j{}_{lmn} + \delta^i{}_m \mathfrak{G}^j{}_{nkl}) \mathfrak{p}^{mn}. \end{aligned} \tag{31}$$

First variation of $\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}$

$$\begin{aligned} \delta\left(\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}\right) = \delta h_{kl} & \left(-\frac{1}{2} \mathfrak{G}_{ijmn} h^{kl} + \delta^k{}_i \mathfrak{G}^l{}_{jmn} + \delta^k{}_m \mathfrak{G}^l{}_{nij} \right) \mathfrak{p}^{mn} \\ & + \delta p^{kl}\, \mathfrak{G}_{ijkl}. \end{aligned} \tag{32}$$

First variation of $\sqrt{\mathfrak{h}}\, G^{ij}[h]$

$$\begin{aligned} \delta\left(\sqrt{\mathfrak{h}}\, G^{ij}[h]\right) = \sqrt{\mathfrak{h}} \Big\{ & \delta h_{kl} \cdot \\ & \left(-\frac{1}{2}\right) (R^{ik} h^{lj} + R^{il} h^{kj} - R^{ij} h^{kl} + h^{ik} G^{lj} + h^{il} G^{kj} - h^{ij} G^{kl}) \\ & + \left(h^{il}\, \bar{\delta} u^{jk}{}_l - \frac{1}{2} h^{ij}\, \bar{\delta} u^{lk}{}_l \right)_{|k} \Big\}. \end{aligned} \tag{33}$$

For the following calculation, one also needs

$$h^{k(i}\, \bar{\delta} u^{j)l}{}_{k|l} = \tag{34}$$

About $\bar{\delta} u^{ij}{}_k$, the identity

$$\sqrt{\mathfrak{h}}\, N h^{k(i}\, \bar{\delta} u^{j)l}{}_{k|l} = \tag{35}$$

is also useful.

Second variation of $N\mathfrak{H}^{\perp}$

$$
\begin{aligned}
&\delta^2(N\mathfrak{H}^{\perp}) \\
&= \delta h_{ij}\,\delta h_{kl}\biggl\{2\varkappa\biggl[\frac{1}{4}(h^{ik}h^{lj}+h^{il}h^{kj}+h^{ij}h^{kl})\mathfrak{F}^{mnrs}h_{mn}h_{rs} \\
&\qquad\qquad -(h^{ij}\mathfrak{F}^{klmn}+\mathfrak{F}^{ijmn}h^{kl})h_{mn}+\mathfrak{F}^{ijkl}\biggr] \\
&\qquad -\frac{\sqrt{\mathfrak{h}}}{4\varkappa}(R^{ik}h^{lj}+R^{il}h^{kj}-R^{ij}h^{kl}+h^{ik}G^{lj}+h^{il}G^{kj}-h^{ij}G^{kl})\biggr\} \\
&\quad +\delta h_{ij}\,\frac{\sqrt{\mathfrak{h}}}{2\varkappa}\left(h^{il}\,\bar{\delta}u^{jk}{}_{l}-\frac{1}{2}h^{ij}\,\bar{\delta}u^{lk}{}_{l}\right)_{|k} \\
&\quad +\delta h_{ij}\,\delta\mathfrak{p}^{kl}\,4\varkappa\{-h^{ij}\mathfrak{G}_{klmn}+2(\delta^{i}{}_{k}\mathfrak{G}^{j}{}_{lmn}+\delta^{i}{}_{m}\mathfrak{G}^{j}{}_{nkl})\}\mathfrak{p}^{mn} \\
&\quad +\delta p^{ij}\,\delta p^{kl}\,4\varkappa\mathfrak{G}_{ijkl} \\
&\quad -\delta h_{ij}\,\delta\left(\frac{\sqrt{\mathfrak{h}}}{2\varkappa}\,\bar{\delta}u^{lk}{}_{l|k}\right).
\end{aligned}
\tag{36}
$$

Second variation of $\Gamma^{i}{}_{jk}$

$$\delta^2\Gamma^{i}{}_{jk}=-h^{im}\,\delta\Gamma^{l}{}_{jk}\,\delta h_{lm}. \tag{37}$$

Second variation of $\mathfrak{H}^{i}$

$$\delta^2(\mathfrak{H}^{i})=-h^{im}p^{jk}\,\delta\Gamma^{l}{}_{jk}\,\delta h_{lm}+2\,\delta\Gamma^{i}{}_{jk}\,\delta p^{jk}. \tag{38}$$

Other second variations

Second variation of h^{ij}

$$\delta^2 h^{ij}=(h^{im}h^{jl}h^{kn}+h^{ik}h^{jm}h^{ln})\,\delta h_{kl}\,\delta h_{mn} \tag{39}$$

Second variation of $\mathfrak{h}=\det h_{ij}$

$$\delta^2\mathfrak{h}=-\frac{1}{4}\mathfrak{h}(h^{ik}h^{jl}+h^{il}h^{kj}-h^{ij}h^{kl})\,\delta h_{ij}\,\delta h_{kl}. \tag{40}$$

First variation of $\left(\delta h_{ij}\right)_{|k}$

$$\delta\Bigl\{\left(\delta h_{ij}\right)_{|k}\Bigr\}=-2\delta\Gamma^{l}{}_{k(i}\,\delta h_{j)l}. \tag{41}$$

In a general background, the second variations of the quantities are much more tedious.

In Robertson–Walker background,

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