Complex Harmonic Oscillators and How to Squeeze Them

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Contents

1 Single complex oscillator

(1.1) **Classical Hamiltonian in canonical coordinates**
Complex phase space

$$H = \frac{1}{2}\pi^+\pi^- + \frac{\Omega^2}{2}\phi^+\phi^-, \tag{1.1}$$

The corresponding Poisson brackets read

$$\left[f(\eta^C), g(\eta^C)\right]_{\mathrm{P}} = \sum_C \left(\frac{\partial f}{\partial \phi^C}\frac{\partial g}{\partial \pi^C} - \frac{\partial f}{\partial \pi^C}\frac{\partial g}{\partial \phi^C} \right), \tag{1.2}$$

where $C \in \{+,-\}$, $\eta^+ = (\eta^-)^*$ and $\eta \in \{\pi, \phi\}$, so that

$$\left[\phi^{C_1}, \pi^{C_2}\right]_{\mathrm{P}} = \delta^{C_1C_2}, \qquad \left[\phi^{C_1}, \phi^{C_2}\right]_{\mathrm{P}} = \left[\pi^{C_1}, \pi^{C_2}\right]_{\mathrm{P}} = 0. \tag{1.3}$$

Real phase space

$$\phi^C = \frac{1}{\sqrt{2}}(\phi_{\mathfrak{R}} - C\mathring{\imath}\phi_{\mathfrak{I}}), \qquad \pi^C = \frac{1}{\sqrt{2}}(\pi_{\mathfrak{R}} + C\mathring{\imath}\pi_{\mathfrak{I}}). \tag{1.4}$$

Inverse transformation

$$\begin{aligned} \phi_{\mathfrak{R}} &= \frac{1}{\sqrt{2}}(\phi^+ + \phi^-), \quad \phi_{\mathfrak{I}} = \frac{\mathring{\imath}}{\sqrt{2}}(\phi^+ - \phi^-), \\ \pi_{\mathfrak{R}} &= \frac{1}{\sqrt{2}}(\pi^- + \pi^+), \quad \pi_{\mathfrak{I}} = \frac{\mathring{\imath}}{\sqrt{2}}(\pi^- - \pi^+). \end{aligned} \tag{1.5}$$

One can verify

$$H = \sum_F \frac{1}{2}\pi_F^2 + \frac{\Omega^2}{2}\phi_F^2, \tag{1.6}$$

where $F \in \{\mathfrak{R}, \mathfrak{I}\}$, and

$$[f(\eta_F), g(\eta_F)]_{\mathrm{P}} = \sum_F \frac{\partial f}{\partial \phi_F}\frac{\partial g}{\partial \pi_F} - \frac{\partial f}{\partial \pi_F}\frac{\partial g}{\partial \phi_F}, \tag{1.7}$$

so that

$$\left[\phi_{F_1}, \pi_{F_2}\right]_{\mathrm{P}} = \delta_{F_1 F_2}, \qquad \left[\phi_{F_1}, \phi_{F_2}\right]_{\mathrm{P}} = \left[\pi_{F_1}, \pi_{F_2}\right]_{\mathrm{P}} = 0. \tag{1.8}$$

hold as well.

(1.2) **Ladder coordinates**
Ladder coordinates (ladder 'numbers', later to be quantised) in complex phase space

$$a_\phi^C = \frac{1}{\sqrt{2}}\left(\Omega^{+\frac{1}{2}}\phi^C - C\mathring{\imath}\Omega^{-\frac{1}{2}}\pi^{-C}\right), \tag{1.9}$$

$$a_\pi^C = \frac{1}{\sqrt{2}}\left(\Omega^{+\frac{1}{2}}\phi^{-C} - C\mathring{\imath}\Omega^{-\frac{1}{2}}\pi^C\right), \tag{1.10}$$

where $-- = +$, $-+ = -$. Poisson brackets? Inverse transformation

$$\phi^- = \frac{\Omega^{-\frac{1}{2}}}{\sqrt{2}}\left(a_\pi^+ + a_\phi^-\right), \qquad \pi^- = \frac{\mathring{\imath}\Omega^{+\frac{1}{2}}}{\sqrt{2}}\left(a_\phi^+ - a_\pi^-\right). \tag{1.11}$$

Ladder coordinates in real phase space

$$a_F^C = \frac{1}{\sqrt{2}}\left(\Omega^{+\frac{1}{2}}\phi_F - C\mathring{\imath}\Omega^{-\frac{1}{2}}\pi_F\right). \tag{1.12}$$

Poisson brackets? Inverse transformation

$$\phi_F = \frac{\Omega^{-\frac{1}{2}}}{\sqrt{2}}(a_F^+ + a_F^-), \qquad \pi_F = \frac{\mathring{\imath}\Omega^{+\frac{1}{2}}}{\sqrt{2}}(a_F^+ - a_F^-). \tag{1.13}$$

One can check that

$$a_\phi^C = \frac{1}{\sqrt{2}}(a_{\mathfrak{R}}^C - C\mathring{\imath}a_{\mathfrak{I}}^C), \qquad a_\pi^C = \frac{1}{\sqrt{2}}(a_{\mathfrak{R}}^C + C\mathring{\imath}a_{\mathfrak{I}}^C). \tag{1.14}$$

(1.3) **Quantisation**
Quantisation in complex canonical coordinates

$$f \mapsto \hat{f}; \qquad [f,g]_{\mathrm{P}} \mapsto \left[\hat{f}, \hat{g}\right]_- = \mathring{\imath}\widehat{[f,g]}_{\mathrm{P}}. \tag{1.15}$$

All classical equations listed above can be immediately quantised, since no product of non-commuting operators appears.

Commutators of the ladder operators

$$\left[\hat{a}_{\eta_1}^{-C_1}, \hat{a}_{\eta_2}^{C_2}\right]_- = \delta_{\eta_1\eta_2}\delta^{C_1C_2}\hat{1}; \tag{1.16}$$

$$\left[\hat{a}_{F_1}^{-C_1}, \hat{a}_{F_2}^{C_2}\right]_- = \delta_{F_1F_2}\delta^{C_1C_2}\hat{1}. \tag{1.17}$$

$$\tag{1.18}$$

Number operators

$$\hat{n}_\eta := \hat{a}^+_\eta \hat{a}^-_\eta, \qquad \hat{n}_F := \hat{a}^+_F \hat{a}^-_F. \tag{1.19}$$

Angular momentum operator

$$\begin{aligned}
\widehat{L} &:= \hat{\phi}_{\mathfrak{R}} \hat{\pi}_{\mathfrak{I}} - \hat{\phi}_{\mathfrak{I}} \hat{\pi}_{\mathfrak{R}} \\
&= \mathbb{i}(\hat{a}^+_{\mathfrak{I}} \hat{a}^-_{\mathfrak{R}} - \hat{a}^+_{\mathfrak{R}} \hat{a}^-_{\mathfrak{I}}) \\
&= \mathbb{i}\Big(\hat{\phi}^- \hat{\pi}^- - \hat{\phi}^+ \hat{\pi}^+\Big) = \mathbb{i}\Big(\hat{\pi}^- \hat{\phi}^- - \hat{\pi}^+ \hat{\phi}^+\Big) \\
&= \hat{n}_\pi - \hat{n}_\phi.
\end{aligned} \tag{1.20}$$

$\widehat{L} = \widehat{L}^\dagger$.

$$\begin{aligned}
\hat{n}_\phi &= \frac{1}{2}\Big(\Omega^{+\frac{1}{2}} \hat{\phi}^+ - \mathbb{i}\Omega^{-\frac{1}{2}} \hat{\pi}^-\Big)\Big(\Omega^{+\frac{1}{2}} \hat{\phi}^- + \mathbb{i}\Omega^{-\frac{1}{2}} \hat{\pi}^+\Big) \\
&= \frac{1}{2}\Big(\Omega^{+1} \hat{\phi}^+ \hat{\phi}^- + \mathbb{i}\Big(\hat{\phi}^+ \hat{\pi}^+ - \hat{\pi}^- \hat{\phi}^-\Big) + \Omega^{-1} \hat{\pi}^- \hat{\pi}^+\Big) \\
&= \Omega^{-1} \widehat{H} - \frac{1}{2}\Big(1 + \widehat{L}\Big).
\end{aligned} \tag{1.21}$$

Substituting (1.20) yields the quantum Hamiltonian

$$\boxed{\widehat{H} = \frac{\Omega}{2}\big(\hat{n}_\phi + \hat{n}_\pi + 1\big).} \tag{1.22}$$

Luckily,

$$\Big[\widehat{L}, \widehat{H}\Big]_- = 0 \tag{1.23}$$

so that

$$\Omega\Big[\hat{n}, \widehat{L}\Big]_- = \Big[\hat{n}, \widehat{H}\Big]_- = 0 \tag{1.24}$$

as well.

(1.4) **Wave function**
One may choose the

2 Rotating

3 Cohering

4 Single–mode squeezing

5 Double–mode squeezing