# Non-relativistic Particle in an Exponential Potential

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Consider the one-dimensional motion of a non-relativistic particle in an exponential potential, the motion of which can be described by the Lagrangian action

$$S := \int \mathrm{d}t \left\{ \frac{m}{2}\dot{x}^2 - V\mathrm{e}^{gx} \right\}, \tag{1}$$

where $g$ and $V$ are real quantities. One sees that when $V > 0$ ($< 0$), the potential is bounded below (above), and the second case is potentially problematic.

## 1 Canonical formalism

The canonical Hamiltonian of the particle reads

$$H = \frac{p^2}{2m} + V\mathrm{e}^{gx}. \tag{2}$$

## 2 Canonical quantisation

Using the Laplace–Beltrami operator, the Hamiltonian "operator" reads

$$\widehat{H} = -\frac{\hbar^2}{2m}\partial_x^2 + V\mathrm{e}^{gx}. \tag{3}$$

Note that the domain of the unbounded operator has not been specified; hence comes the quotation marks. In [1, ch. 4], it was suggested that one could use *operation* instead of "operator" to distinguish the case, where only the action of an operator is described, whereas the domain is not.

### 2.1 Spectrum and generalised eigenfunctions of the Hamiltonian

The eigenvalue equation of the Hamiltonian, or the time-independent Schrödinger equation, reads

$$-\frac{\hbar^2}{2m}\partial_x^2\psi(x) + V\mathrm{e}^{gx}\psi(x) = E\psi(x). \tag{4}$$

| Sign | Solution 1 | Solution 2 |
|---|---|---|
| $(+,+)$ | $\mathrm{K}_{\mathbb{i}\nu}(\xi)$, D | $\mathrm{I}_{\mathbb{i}\nu}(\xi)$, U |
| $(-,-)$ | $\mathrm{K}_{\nu}(\xi)$, U | $\mathrm{I}_{\nu}(\xi)$, U |
| $(+,-)$ | $\mathrm{F}_{\mathbb{i}\nu}(\xi)$, D | $\mathrm{G}_{\mathbb{i}\nu}(\xi)$, D |
| $(-,+)$ | $\mathrm{J}_{\nu}(\xi)$, N | $\mathrm{Y}_{\nu}(\xi)$, U |

Table 1: Local solutions of eq. (8) and their normalisability, where sign means $(v, e)$, N denotes normalisable, D $\delta$-normalisable, and U unnormalisable.

In order to solve eq. (4), define

$$\nu := \frac{\sqrt{8m|E|}}{g\hbar}, \tag{5}$$

and transform the coordinate

$$\xi := \frac{\sqrt{8m|V|\mathbb{e}^{gx}}}{g\hbar}, \tag{6}$$

so that the Hamiltonian "operator" reads

$$\widehat{H} = \frac{g^2\hbar^2}{8m}\big(-\xi^2\partial_\xi^2 - \xi\partial_\xi + v\xi^2\big), \qquad v := \operatorname{sgn} V, \tag{7}$$

and eq. (4) transforms into the standard Besselian form

$$\xi^2\psi''(\xi) + \xi^1\psi'(\xi) + (-v\xi^2 + e\nu^2)\psi(\xi) = 0, \qquad e := \operatorname{sgn} E. \tag{8}$$

The solutions of eq. (8) can be classified into four cases according to $(v, e)$ and are listed in table 1. Because of the transformation in eq. (6), the corresponding representation space $\mathbf{F}_\xi$ of the state vectors [3, ch. 5.1] consists of the square-integrable functions on $(0, +\infty)$ endowed with the inner product

$$(\psi, \phi)_\xi := \int_0^{+\infty} \frac{\mathbb{d}\xi}{\xi}\,\psi^*(\xi)\phi(\xi). \tag{9}$$

Transforming

$$\mathbb{e}^y := \xi = \frac{\sqrt{8m|V|\mathbb{e}^{gx}}}{g\hbar} \tag{10}$$

yields the Hamiltonian "operator" in terms of an alternative dimensionless form

$$\widehat{H} = \frac{g^2\hbar^2}{8m}\big(-\partial_y^2 + v\mathbb{e}^{2y}\big). \tag{11}$$

The solutions of the eigenvalue equation for eq. (11) are the ones listed in table 1 with $\xi$ replaced by $\mathbb{e}^y$. The representation space $\mathbf{F}_y$ is comprised of the square-integrable functions on $(-\infty, +\infty)$ endowed with the inner product

$$(\psi, \phi)_y := \int_{-\infty}^{+\infty} \mathbb{d}y\, \psi^*(y)\phi(y). \tag{12}$$

### 2.2 Problem of self-adjointness

On a Hilbert space $\mathbf{H}$ endowed with the inner product $(\cdot, \cdot)$, An operator $A$ is characterised by its *domain* $\mathrm{Dom}(A)$ and the operation on a vector in $\mathrm{Dom}(A)$. Physicists often skip the discussion about the domain, which proves to be problematic in the current case.

To be more specific, the following definitions are needed. $A$ is called *symmetric* if $(f, Ag) \equiv (Af, g)$, $\forall f, g \in \mathrm{Dom}(A)$. The *adjoint* of $A$ is denoted as $A^\dagger$ and satisfies $(A^\dagger f, g) \equiv (f, Ag)$, $\forall g \in \mathrm{Dom}(A)$. Finally, $A$ is *self-adjoint* if $A^\dagger = A$, which implies the identical operation $A^\dagger f = Af$, $\forall f \in \mathrm{Dom}(A)$ and the identical domain, $\mathrm{Dom}(A^\dagger) = \mathrm{Dom}(A)$.

Note that in infinite dimensions, for an unbounded $A$, $\mathbf{H} \supsetneq \mathrm{Dom}(A^\dagger) \supseteq \mathrm{Dom}(A)$ [2, ch. 9], which is the main difference from the case in finite dimensions, where $\mathbf{H} = \mathrm{Dom}(A^\dagger) = \mathrm{Dom}(A)$.

The Hamiltonian in eq. (11) is manifestly not self-adjoint for some of the cases. If it were always self-adjoint, the (generalised) eigenfunctions of different eigenvalues are necessarily orthogonal. This is obviously not the case for $(-, +)$. One can check that

$$\left(J_{\nu_1}, J_{\nu_2}\right)_y = \tag{13}$$

# References


[1] Dmitri Maximovitch Gitman, Igor Viktorovich Tyutin, and B. L. Voronov. *Self-adjoint Extensions in Quantum Mechanics. General Theory and Applications to Schrödinger and Dirac Equations with Singular Potentials.* Birkhäuser Boston, 2012. DOI: 10.1007/978-0-8176-4662-2.

[2] Brian C. Hall. *Quantum Theory for Mathematicians.* Springer New York, 2013. DOI: 10.1007/978-1-4614-7116-5.

[3] Claus Kiefer. *Quantum Gravity.* 3rd ed. Oxford University Press, Apr. 2012. DOI: 10.1093/acprof:oso/9780199585205.001.0001.