Hamiltonian Dynamics of Maxwell–Proca theory in $(d+1)$-dimensions

Yi-Fan Wang (王 一帆)

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1 Maxwell–Proca theory in flat space-time

Consider a Maxwell–Proca theory with source

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + A_\mu J^\mu, \tag{1.1}$$

where $m > 0$ corresponds to the Proca theory [1, sec. 2.3], and $m = 0$ the Maxwell theory [2, sec. 3.3.3], [1, sec. 2.4]. In $(3+1)$ dimensions, the electric and magnetic fields are

$$-F_{0i} = F_{i0} = E_i = \partial_i A_0 - \partial_0 A_i, \tag{1.2}$$

$$F_{ij} = \epsilon_{ijk}B^k, \qquad B^i = \frac{1}{2}\epsilon^{ijk}F_{ij}. \tag{1.3}$$

The action with velocity is

$$S^{\mathrm{v}}[A, \Pi, V] := \int \mathrm{d}t \int \mathrm{d}^d x \Big(\mathcal{L}^{\mathrm{v}} + \Pi^\mu\big(\dot{A}_\mu - V_\mu\big)\Big), \tag{1.4}$$

where the Lagrangian density with velocity reads

$$\mathcal{L}^{\mathrm{v}} = \frac{1}{2}(V_i - \partial_i A_0)^2 - \frac{1}{4}F_{ij}^2 + \frac{m^2}{2}(A_0^2 - A_i^2) + A_0 J^0 + A_i J^i. \tag{1.5}$$

In $(3+1)$ dimensions, eq. (1.5) can also be written as

$$\mathcal{L}^{\mathrm{v}} = \frac{1}{2}\Big(\vec{E}^2 - \vec{B}^2\Big) + \frac{m^2}{2}\Big(\Phi^2 - \vec{A}^2\Big) - \rho\Phi + \vec{A}\cdot\vec{J}. \tag{1.6}$$

On the velocity shell, the canonical momenta densities are

$$\Pi^0 := \frac{\partial\mathcal{L}^{\mathrm{v}}}{\partial V_0} = 0, \qquad \Pi^i := \frac{\partial\mathcal{L}^{\mathrm{v}}}{\partial V_i} = V^i - \partial^i A_0. \tag{1.7}$$

The fundamental Poisson brackets are

$$\left[A_\mu(\vec{x}_1), \Pi^\nu(\vec{x}_2)\right]_{\mathrm{P}} = \delta_\mu{}^\nu \delta^d(\vec{x}_1 - \vec{x}_2). \tag{1.8}$$

Brining the V_i's on shell, the primary action reads

$$S^{\mathrm{p}}[A, \Pi, V_0] = \int \mathrm{d}t \int \mathrm{d}^d x \Big(\mathcal{H}^{\mathrm{p}} + \Pi^\mu \dot{A}_\mu + \partial_i\big(\Pi^i A_0\big)\Big), \tag{1.9}$$

in which the primary Hamiltonian is

$$\begin{aligned} \mathcal{H}^{\mathrm{p}} = \frac{1}{2}(\Pi^i)^2 + \frac{1}{4}F_{ij}^2 + \frac{m^2}{2}(-A_0^2 + A_i^2) - A_i J^i \\ + V_0 \Pi^0 - A_0(\partial_i \Pi^i + J^0), \end{aligned} \tag{1.10}$$

and

$$\Phi_1 := \Pi^0 \tag{1.11}$$

is the only primary constraint.

1.1 Constraint algebra

The Poisson bracket of Φ_1 and $\mathcal{H}^{\mathrm{p}}$ is

$$\begin{aligned} [\Phi_1(\vec{x}_1), \mathcal{H}^{\mathrm{p}}(\vec{x}_2)]_{\mathrm{P}} &= \left[\left(\Pi^0\right)_1, -\frac{m^2}{2}A_0^2 - A_0(\partial_i \Pi^i + J^0)\right]_{\mathrm{P}} \\ &= \left(-m^2 A^0 + \partial_i \Pi^i + J^0\right)_2 \delta(\vec{x}_1 - \vec{x}_2). \end{aligned} \tag{1.12}$$

Integration with $\mathrm{d}^d x_2$ yields the secondary constraint

$$[\Phi_1, H^{\mathrm{p}}]_{\mathrm{P}} = -m^2 A^0 + \partial_i \Pi^i + J^0 =: \Phi_2, \tag{1.13}$$

so that

$$[\Phi_1(\vec{x}_1), \Phi_2(\vec{x}_2)]_{\mathrm{P}} = -m^2 \delta(\vec{x}_1 - \vec{x}_2). \tag{1.14}$$

One may further compute

$$\begin{aligned} &[\Phi_2(\vec{x}_1), \mathcal{H}^{\mathrm{p}}(\vec{x}_2)]_{\mathrm{P}} \\ &= \left[\left(\partial_i \Pi^i\right)_1, \left(\frac{1}{4}F_{jk}^2 + \frac{m^2}{2}A_j^2 - A_j J^j\right)_2\right]_{\mathrm{P}} + \left[-m^2\left(A^0\right)_1, \left(V_0 \Pi^0\right)_2\right]_{\mathrm{P}}, \end{aligned} \tag{1.15}$$

in which

$$\begin{aligned} \left[\left(\partial_i \Pi^i\right)_1, \left(\frac{1}{4}F_{jk}^2\right)_2\right]_{\mathrm{P}} &= \left(\partial_j A_k - \partial_k A_j\right)_2 (\partial_i)_1 \left[\left(\Pi^i\right)_1, \left(\partial^j A^k\right)_2\right]_{\mathrm{P}} \\ &= -\left(F^{ij}\partial_j\right)_2 (\partial_i)_1 \delta(\vec{x}_1 - \vec{x}_2). \end{aligned} \tag{1.16}$$

The Poisson bracket can be evaluated as

$$[\Phi_2(\vec{x}_1), \mathcal{H}^{\mathrm{p}}(\vec{x}_2)]_{\mathrm{P}} = \Big(-\big(F^{ij}\partial_j + m^2 A^i + J^i\big)_2 (\partial_i)_1 + m^2 (V_0)_2\Big)\delta(\vec{x}_1 - \vec{x}_2). \tag{1.17}$$

Integration with $\mathbb{d}^d x_2$ yields

$$[\Phi_2, H^{\mathrm{p}}]_{\mathrm{P}} = -\partial_i(m^2 A^i + J^i) + m^2 V_0. \tag{1.18}$$

1.2 Proca theory

For Proca theory $m > 0$, then the algorithm terminates, and one obtains a pure second-class system.

$$\mathbf{Q} = \begin{pmatrix} 0 & -m^2 \\ +m^2 & 0 \end{pmatrix}, \qquad \mathbf{Q}^{-1} = \begin{pmatrix} 0 & +m^{-2} \\ -m^{-2} & 0 \end{pmatrix}. \tag{1.19}$$

Dirac bracket

$$\begin{aligned} [f(\vec{x}_1), g(\vec{x}_2)]_{\mathrm{D}} &= \big[(f)_1, (g)_2\big]_{\mathrm{P}} \\ + \int \mathbb{d}^d x_3 \Big(&- \big[(f)_1, (\Pi^0)_3\big]_{\mathrm{P}} \big[(m^{-2}\partial_i \Pi^i + A_0)_3, (g)_2\big]_{\mathrm{P}} \\ &+ \big[(f)_1, (m^{-2}\partial_i \Pi^i + A_0)_3\big]_{\mathrm{P}} \big[(\Pi^0)_3, (g)_2\big]_{\mathrm{P}}\Big). \end{aligned} \tag{1.20}$$

The fundamental ones different from Poisson brackets are

$$[A_0(\vec{x}_1), A_i(\vec{x}_2)]_{\mathrm{D}} = m^{-2}(\partial_i)_1 \delta(\vec{x}_1 - \vec{x}_2), \qquad [A_0(\vec{x}_1), \Pi^0(\vec{x}_2)]_{\mathrm{D}} = 0. \tag{1.21}$$

1.2.1 Physical coordinates

Introducing the regularising coordinates

$$\alpha_i = A_i + m^{-2}(\partial_i \Pi^0 - J_i), \qquad \beta^i = \Pi^i; \tag{1.22}$$

$$\alpha_0 = A_0 + m^{-2}(\partial_i \Pi^i - J_0), \qquad \beta^0 = \Pi^0. \tag{1.23}$$

It is easy[1] to show that

$$\big[\alpha_i(\vec{x}_1), \beta^j(\vec{x}_2)\big]_{\mathrm{D}} = \delta^i{}_j \delta(\vec{x}_1, \vec{x}_2), \tag{1.24}$$

$$\big[\alpha_i(\vec{x}_1), \alpha_j(\vec{x}_2)\big]_{\mathrm{D}} = 0 = [\beta^i(\vec{x}_1), \beta^j(\vec{x}_2)]_{\mathrm{D}}. \tag{1.25}$$

Furthermore, one has

$$\mathcal{H}^{\mathrm{p}} = \mathcal{H}^{\mathrm{phy}} + \mathcal{H}^{\mathrm{con}} + \mathcal{H}^{\mathrm{irr}}, \tag{1.26}$$

[1] Really? Have I done it?

where

$$\begin{aligned}\mathcal{H}^{\text{phy}} &= \frac{1}{2}\left(\beta^i\right)^2 + \frac{m^2}{2}\alpha_i^2 + \frac{1}{4}\left(\partial_i\alpha_j - \partial_j\alpha_i\right)^2 + \frac{1}{2m^2}\left(\partial_i\beta^i\right)^2 \\ &\quad + \frac{1}{m^2}J^0\partial_i\beta^i,\end{aligned} \tag{1.27}$$

$$\mathcal{H}^{\text{con}} = -\frac{m^2}{2}\alpha_0^2 - \frac{1}{2m^2}\left(\partial_i\beta^0\right)^2, \tag{1.28}$$

$$\begin{aligned}\mathcal{H}^{\text{irr}} &= \partial_i\left(\alpha_0\beta^i - \beta^0\alpha_i + \frac{1}{m^2}\left(\beta^0\partial_i\beta^0 - \beta^i\partial_j\beta^j - J^0\beta^i\right)\right) \\ &\quad + \frac{1}{2m^2}\left(\left(J^0\right)^2 - \left(J^i\right)^2\right).\end{aligned} \tag{1.29}$$

Further more,

$$\Phi_1 = \beta^0, \qquad \Phi_2 = m^2\alpha_0 \propto \alpha_0. \tag{1.30}$$

Thus the $\left(\alpha_i, \beta^i\right)$ are regular pairs of canonical variables, whereas $\left(\alpha_0, \beta^0\right)$ are the singular variables as constraints. The canonical dynamics of the physical $\left(\alpha_i, \beta^i\right)$'s are determined by $\mathcal{H}^{\text{phy}}$ as a regular system.

1.3 Free Maxwell theory

For Maxwell theory $m = 0$. The primary Hamiltonian in eq. (1.10) takes the form

$$\mathcal{H}^{\text{p}} = \frac{1}{2}\left(\Pi^i\right)^2 + \frac{1}{4}F_{ij}^2 - A_iJ^i + V_0\Pi^0 - A_0\left(\partial_i\Pi^i + J^0\right), \tag{1.31}$$

the secondary constraint Φ_2 in eq. (1.13) now reads

$$\Phi_2 = \partial_i\Pi^i + J^0. \tag{1.32}$$

In $(3+1)$ dimensions, the first two terms in eq. (1.31) reads

$$\frac{1}{2}\left(\Pi^i\right)^2 + \frac{1}{4}F_{ij}^2 = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right). \tag{1.33}$$

Since the Poisson bracket of Φ_2 and H^{p}

$$\left[\Phi_2, H^{\text{p}}\right]_{\text{P}} = -\partial_iJ^i \tag{1.34}$$

contains now no canonical variable, the algorithm terminates. Furthermore, the constraint algebra is commutative, hence the system is a purely first-class one.

Persistence condition on Φ_2 requires

$$\partial_iJ^i = 0, \tag{1.35}$$

which is confusing.

References

[1] Dmitriy M. Gitman and Igor V. Tyutin. *Quantization of Fields with Constraints*. Springer Series in Nuclear and Particle Physics. Springer Berlin Heidelberg, 1990. ISBN: http://id.crossref.org/isbn/978-3-642-83938-2. DOI: 10.1007/978-3-642-83938-2. URL: http://dx.doi.org/10.1007/978-3-642-83938-2. Дмитрий Максимович Гитман and Игорь Викторович Тютин. *Каноническое квантование полей со связями*. 030077 г. 11овосибирск-77, Станиславского, 25: Наука, 1986.

[2] Heinz J Rothe and Klaus D Rothe. *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*. World Scientific Lecture Notes in Physics. World Scientific, Apr. 2010. ISBN: http://id.crossref.org/isbn/978-981-4299-65-7. DOI: 10.1142/7689. URL: http://dx.doi.org/10.1142/7689.