Hamiltonian Dynamics of Maxwell–Proca Theory

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1 Maxwell–Proca theory in flat space-time

Consider a Maxwell–Proca theory in Minkowski space-time with source

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + A_\mu J^\mu, \tag{1}$$

where $m > 0$ corresponds to the Proca theory [1, sec. 2.3], and $m = 0$ the Maxwell theory [2, sec. 3.3.3], [1, sec. 2.4].

Since the source is external and not dynamical, it seems necessary to impose $\partial_\mu J^\mu = 0$ by hand.

The action with velocity is

$$S^{\mathrm{v}}[A, \varPi, V] := \int \mathbb{d}t \int \mathbb{d}^d x \left\{ \mathcal{L}^{\mathrm{v}} + \varPi^\mu \left(\dot{A}_\mu - V_\mu \right) \right\}, \tag{2}$$

where the Lagrangian density with velocity reads

$$\mathcal{L}^{\mathrm{v}} = \frac{1}{2}(V_i - \partial_i A_0)^2 - \frac{1}{4}F_{ij}^2 + \frac{m^2}{2}(A_0^2 - A_i^2) + A_0 J^0 + A_i J^i. \tag{3}$$

On the velocity shell, the canonical momenta densities are

$$\varPi^0 := \frac{\partial \mathcal{L}^{\mathrm{v}}}{\partial V_0} = 0, \qquad \varPi^i := \frac{\partial \mathcal{L}^{\mathrm{v}}}{\partial V_i} = V^i - \partial^i A_0. \tag{4}$$

The fundamental Poisson brackets are

$$\left[A_\mu(\vec{x}_1), \varPi^\nu(\vec{x}_2) \right]_{\mathrm{P}} = \delta_\mu{}^\nu \delta^d(\vec{x}_1 - \vec{x}_2). \tag{5}$$

Brining the V_i's on shell, the primary action reads

$$S^{\mathrm{p}}[A, \varPi, V_0] = \int \mathbb{d}t \int \mathbb{d}^d x \Big(\mathcal{H}^{\mathrm{p}} + \varPi^\mu \dot{A}_\mu + \partial_i (\varPi^i A_0) \Big), \tag{6}$$

in which the primary Hamiltonian is

$$\begin{aligned} \mathcal{H}^{\mathrm{p}} = {} & \frac{1}{2}(\varPi^i)^2 + \frac{1}{4}F_{ij}^2 + \frac{m^2}{2}(-A_0^2 + A_i^2) - A_i J^i \\ & + V_0 \varPi^0 - A_0(\partial_i \varPi^i + J^0), \end{aligned} \tag{7}$$

and

$$\varPhi_1 := \varPi^0 \tag{8}$$

is the only primary constraint.

In $(3+1)$-dimensions, the electromagnetic potentials, fields and current are

$$\varPhi = -A_0, \qquad \vec{A}^i = A^i; \tag{9}$$

$$E_i = -F_{0i} = F_{i0} = \partial_i A_0 - \partial_0 A_i = \partial_i A_0 - V_i = -\varPi_i; \tag{10}$$

$$B^i = \frac{1}{2}\epsilon^{ijk} F_{ij}, \qquad F_{ij} = \epsilon_{ijk} B^k; \tag{11}$$

$$\rho = J^0, \qquad \vec{J}^i = J^i. \tag{12}$$

Equations (3) and (7) can also be written as

$$\mathcal{L}^{\mathrm{v}} = \frac{1}{2}\Big(\vec{E}^2 - \vec{B}^2\Big) + \frac{m^2}{2}\Big(\varPhi^2 - \vec{A}^2\Big) - \rho\varPhi + \vec{A}\cdot\vec{J}, \tag{13}$$

$$\mathcal{H}^{\mathrm{p}} = \frac{1}{2}\Big(\vec{E}^2 + \vec{B}^2\Big) - \frac{m^2}{2}\Big(\varPhi^2 - \vec{A}^2\Big) - \vec{A}\cdot J + V_0\varPi^0 + \varPhi(-\nabla\cdot E + \rho) \tag{14}$$

1.1 Constraint algebra

The Poisson bracket of $\varPhi_1$ and $\mathcal{H}^{\mathrm{p}}$ is

$$\begin{aligned}[\varPhi_1(\vec{x}_1), \mathcal{H}^{\mathrm{p}}(\vec{x}_2)]_{\mathrm{P}} &= \left[\big(\varPi^0\big)_1, -\frac{m^2}{2}A_0^2 - A_0(\partial_i\varPi^i + J^0)\right]_{\mathrm{P}} \\ &= \big(-m^2A^0 + \partial_i\varPi^i + J^0\big)_2\,\delta(\vec{x}_1 - \vec{x}_2).\end{aligned} \tag{15}$$

Integration with $\mathrm{d}^d x_2$ yields the secondary constraint

$$[\varPhi_1, H^{\mathrm{p}}]_{\mathrm{P}} = -m^2A^0 + \partial_i\varPi^i + J^0 =: \varPhi_2, \tag{16}$$

so that

$$[\varPhi_1(\vec{x}_1), \varPhi_2(\vec{x}_2)]_{\mathrm{P}} = -m^2\delta(\vec{x}_1 - \vec{x}_2). \tag{17}$$

One may further compute

$$\begin{aligned}&[\varPhi_2(\vec{x}_1), \mathcal{H}^{\mathrm{p}}(\vec{x}_2)]_{\mathrm{P}} \\ &= \left[\big(\partial_i\varPi^i\big)_1, \left(\frac{1}{4}F_{jk}^2 + \frac{m^2}{2}A_j^2 - A_jJ^j\right)_2\right]_{\mathrm{P}} + \Big[-m^2\big(A^0\big)_1, \big(V_0\varPi^0\big)_2\Big]_{\mathrm{P}},\end{aligned} \tag{18}$$

in which

$$\begin{aligned}\left[\big(\partial_i\varPi^i\big)_1, \left(\frac{1}{4}F_{jk}^2\right)_2\right]_{\mathrm{P}} &= \big(\partial_jA_k - \partial_kA_j\big)_2(\partial_i)_1\Big[\big(\varPi^i\big)_1, \big(\partial^jA^k\big)_2\Big]_{\mathrm{P}} \\ &= -\big(F^{ij}\partial_j\big)_2(\partial_i)_1\delta(\vec{x}_1 - \vec{x}_2).\end{aligned} \tag{19}$$

The Poisson bracket can be evaluated as

$$[\Phi_2(\vec{x}_1), \mathcal{H}^{\mathrm{p}}(\vec{x}_2)]_{\mathrm{P}} \tag{20}$$

$$= \Big(-\big(F^{ij}\partial_j + m^2 A^i + J^i\big)_2 (\partial_i)_1 + m^2 (V_0)_2\Big)\delta(\vec{x}_1 - \vec{x}_2). \tag{21}$$

Integration with $\mathbb{d}^d x_2$ yields

$$[\Phi_2, H^{\mathrm{p}}]_{\mathrm{P}} = -\partial_i(m^2 A^i + J^i) + m^2 V_0. \tag{22}$$

1.2 Proca theory

For Proca theory $m > 0$, then the algorithm terminates, and one obtains a pure second-class system.

$$\mathbf{Q} = \begin{pmatrix} 0 & -m^2 \\ +m^2 & 0 \end{pmatrix}, \qquad \mathbf{Q}^{-1} = \begin{pmatrix} 0 & +m^{-2} \\ -m^{-2} & 0 \end{pmatrix}. \tag{23}$$

Dirac bracket

$$\begin{aligned} [f(\vec{x}_1), g(\vec{x}_2)]_{\mathrm{D}} &= \big[(f)_1, (g)_2\big]_{\mathrm{P}} \\ + \int \mathbb{d}^d x_3 \Big(&- \big[(f)_1, (\Pi^0)_3\big]_{\mathrm{P}} \big[(m^{-2}\partial_i \Pi^i + A_0)_3, (g)_2\big]_{\mathrm{P}} \\ &+ \big[(f)_1, (m^{-2}\partial_i \Pi^i + A_0)_3\big]_{\mathrm{P}} \big[(\Pi^0)_3, (g)_2\big]_{\mathrm{P}} \Big). \end{aligned} \tag{24}$$

The fundamental Dirac brackets, which are different from Poisson brackets, are

$$\begin{aligned} [A_0(\vec{x}_1), A_i(\vec{x}_2)]_{\mathrm{D}} &= m^{-2}(\partial_i)_1 \delta(\vec{x}_1 - \vec{x}_2), \\ \big[A_0(\vec{x}_1), \Pi^0(\vec{x}_2)\big]_{\mathrm{D}} &= 0. \end{aligned} \tag{25}$$

1.2.1 Physical coordinates

Introducing the regularising coordinates

$$\alpha_i = A_i + m^{-2}(\partial_i \Pi^0 - J_i), \qquad \beta^i = \Pi^i; \tag{26}$$

$$\alpha_0 = A_0 + m^{-2}(\partial_i \Pi^i - J_0), \qquad \beta^0 = \Pi^0. \tag{27}$$

It is easy[1] to show that

$$\big[\alpha_i(\vec{x}_1), \beta^j(\vec{x}_2)\big]_{\mathrm{D}} = \delta^i{}_j \delta(\vec{x}_1, \vec{x}_2), \tag{28}$$

$$\big[\alpha_i(\vec{x}_1), \alpha_j(\vec{x}_2)\big]_{\mathrm{D}} = 0 = [\beta^i(\vec{x}_1), \beta^j(\vec{x}_2)]_{\mathrm{D}}. \tag{29}$$

Furthermore, one has

$$\mathcal{H}^{\mathrm{p}} = \mathcal{H}^{\mathrm{phy}} + \mathcal{H}^{\mathrm{con}} + \mathcal{H}^{\mathrm{irr}}, \tag{30}$$

[1] Really? Have I done it?

where[2]

$$\begin{aligned}\mathcal{H}^{\text{phy}} &= \frac{1}{2}\big(\beta^i\big)^2 + \frac{m^2}{2}\alpha_i^2 + \frac{1}{4}\big(\partial_i\alpha_j - \partial_j\alpha_i\big)^2 + \frac{1}{2m^2}\big(\partial_i\beta^i\big)^2 \\ &\quad + \frac{1}{m^2}J^0\partial_i\beta^i,\end{aligned} \tag{31}$$

$$\mathcal{H}^{\text{con}} = -\frac{m^2}{2}\alpha_0^2 - \frac{1}{2m^2}\big(\partial_i\beta^0\big)^2, \tag{32}$$

$$\begin{aligned}\mathcal{H}^{\text{irr}} &= \partial_i\Big(\alpha_0\beta^i - \beta^0\alpha_i + \frac{1}{m^2}\big(\beta^0\partial_i\beta^0 - \beta^i\partial_j\beta^j - J^0\beta^i\big)\Big) \\ &\quad + \frac{1}{2m^2}\Big(\big(J^0\big)^2 - \big(J^i\big)^2\Big).\end{aligned} \tag{33}$$

Further more,

$$\varPhi_1 = \beta^0, \qquad \varPhi_2 = m^2\alpha_0 \propto \alpha_0. \tag{34}$$

Thus the (α_i, β^i) are regular pairs of canonical variables, whereas (α_0, β^0) are the singular variables as constraints. The canonical dynamics of the physical (α_i, β^i)'s are determined by $\mathcal{H}^{\text{phy}}$ as a regular system.

1.3 Free Maxwell theory

For Maxwell theory $m = 0$. The primary Hamiltonian in eq. (7) takes the form

$$\mathcal{H}^{\text{p}} = \frac{1}{2}\big(\varPi^i\big)^2 + \frac{1}{4}F_{ij}^2 - A_iJ^i + V_0\varPi^0 - A_0(\partial_i\varPi^i + J^0), \tag{35}$$

the secondary constraint $\varPhi_2$ in eq. (16) now reads

$$\varPhi_2 = \partial_i\varPi^i + J^0. \tag{36}$$

In $(3+1)$ dimensions, the first two terms in eq. (35) reads

$$\frac{1}{2}\big(\varPi^i\big)^2 + \frac{1}{4}F_{ij}^2 = \frac{1}{2}\big(\vec{E}^2 + \vec{B}^2\big). \tag{37}$$

Since the Poisson bracket of $\varPhi_2$ and H^{p}

$$[\varPhi_2, H^{\text{p}}]_{\text{P}} = -\partial_iJ^i \tag{38}$$

contains now no canonical variable, the algorithm terminates. Furthermore, the constraint algebra is commutative, hence the system is a purely first-class one.

Persistence condition on $\varPhi_2$ requires

$$\partial_iJ^i = 0, \tag{39}$$

which is confusing.

[2]This is to be re-calculated, since a boundary term has been split at the beginning.

1.3.1 Gauge transformation

1.3.2 Physical coordinates

References

[1] Dmitriy M. Gitman and Igor V. Tyutin. *Quantization of Fields with Constraints*. Springer Series in Nuclear and Particle Physics. Springer Berlin Heidelberg, 1990. ISBN: http://id.crossref.org/isbn/978-3-642-83938-2. DOI: `10.1007/978-3-642-83938-2`. URL: `http://dx.doi.org/10.1007/978-3-642-83938-2`. and . . 030077 . 11 -77, , 25: , 1986.

[2] Heinz J Rothe and Klaus D Rothe. *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*. World Scientific Lecture Notes in Physics. World Scientific, Apr. 2010. ISBN: http://id.crossref.org/isbn/978-981-4299-65-7. DOI: `10.1142/7689`. URL: `http://dx.doi.org/10.1142/7689`.