

# Cosmological Perturbations

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May 3, 2018

Most of the conventions and notations in [5, ch. 5] will be followed.  
Suppose the metric can be expanded up to the linear order as

$$g = g^{(0)} + \epsilon g^{(1)} + O(\epsilon^2). \quad (1)$$

The background metric  $g^{(0)}$  takes the Robertson–Walker form

$$g_{\mu\nu}^{(0)} dx^\mu dx^\nu = -N^2(t) dt^2 + a^2(t) d\Omega_{3F}^2, \quad (2)$$

in which  $d\Omega_{3F}^2 = d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)$  is the dimensionless flat spatial metric. The linear perturbation can be decomposed into scalar, vector and tensor parts

$$g_{00}^{(1)} = -E, \quad (3)$$

$$g_{i0}^{(1)} = g_{0i}^{(1)} = F_{,i} + G_i, \quad (4)$$

$$g_{ij}^{(1)} = A\delta_{ij} + B_{,i,j} + C_{i,j} + C_{j,i} + D_{ij}. \quad (5)$$

Here one has some weird condition, where the contractions do not follow the one-up-one-down tradition

$$C_{i,i} = G_{i,i} = 0, \quad D_{ij,i} = 0, \quad D_{ii} = 0. \quad (6)$$

## 1 Metric perturbation under diffeomorphism

Consider a diffeomorphism generated by  $\xi^\mu$

$$x^\mu \rightarrow \bar{x}^\mu = x^\mu - \epsilon \xi^\mu. \quad (7)$$

The generator  $\xi^\mu$  can in turn be decomposed into  $\xi_0 = \zeta$ ,  $\xi_i = \xi_{,i}^S + \xi_i^V$ .

One has here again some weird condition, where the contractions do not follow the one-up-one-down tradition

$$\xi_{i,i}^V = 0. \quad (8)$$

The Lie derivative of the metric  $\mathbb{L}_\xi g$  is

$$(\mathbb{L}_\xi g)_{\mu\nu} = \xi^\lambda g_{\mu\nu,\lambda} + \xi^\lambda_{,\mu} g_{\lambda\nu} + \xi^\lambda_{,\nu} g_{\mu\lambda}. \quad (9)$$

In components and expansion, these are

$$(\mathbb{L}_\xi g)_{00} = 2\dot{\zeta} - 2\zeta \frac{\dot{N}}{N} + O(\epsilon), \quad (10)$$

$$(\mathbb{L}_\xi g)_{i0} = (\mathbb{L}_\xi g)_{0i} = \left( \zeta - 2\frac{\dot{a}}{a} \xi^S + \dot{\xi}^S \right)_{,i} + \left( -2\frac{\dot{a}}{a} \xi_i^V + \dot{\xi}_i^V \right) + O(\epsilon), \quad (11)$$

$$(\mathbb{L}_\xi g)_{ji} = (\mathbb{L}_\xi g)_{ij} = -\frac{2a\dot{a}}{N^2} \zeta \delta_{ij} + 2\xi_{,i,j}^S + \xi_{i,j}^V + \xi_{j,i}^V + O(\epsilon). \quad (12)$$

## 2 Scalar perturbations

$$-N^2 - \epsilon E + O(\epsilon^2) \rightarrow -N^2 - \epsilon E + \epsilon \left( 2\dot{\zeta} - 2\zeta \frac{\dot{N}}{N} \right) + O(\epsilon^2), \quad (13)$$

so one can write

$$\mathbb{L}_\xi E = -2\dot{\zeta} + 2\zeta \frac{\dot{N}}{N}. \quad (14)$$

Similarly one can read-off

$$\mathbb{L}_\xi F = \zeta - 2\frac{\dot{a}}{a} \xi^S + \dot{\xi}^S, \quad (15)$$

$$\mathbb{L}_\xi A = -\frac{2a\dot{a}}{N^2} \zeta, \quad (16)$$

$$\mathbb{L}_\xi B = 2\xi^S. \quad (17)$$

The four scalar perturbations are generated by  $\zeta$  and  $\xi^S$ , so that only two independent perturbations exists. It is clear that

$$\mathbb{L}_\xi \left( \frac{F}{a} - \frac{\mathbb{D}}{\mathbb{D}t} \frac{B}{2a} \right) = \frac{\zeta}{a}. \quad (18)$$

One can verify that

$$\mathbb{L}_\xi \left\{ \frac{E}{2N} + \frac{\mathbb{D}}{\mathbb{D}t} \left[ \frac{a}{N} \left( \frac{F}{a} - \frac{\mathbb{D}}{\mathbb{D}t} \frac{B}{2a} \right) \right] \right\} = 0, \quad (19)$$

$$\mathbb{L}_\xi \left\{ \frac{A}{2} + \frac{a^2 \dot{a}}{N^2} \left( \frac{F}{a} - \frac{\mathbb{D}}{\mathbb{D}t} \frac{B}{2a} \right) \right\} = 0. \quad (20)$$

### 3 Vector perturbations

### 4 Tensor perturbations

### 5 Scalar field perturbation under diffeomorphism

### 6 Perturbation of Arnowitt–Deser–Misner Hamiltonian formalism

The well known Arnowitt–Deser–Misner’s Hamiltonian action for gravitation is [3, ch.4.2.2]

$$S = \int dt dx^3 \left\{ \mathfrak{p}^{ij} \dot{h}_{ij} + \mathfrak{P} \dot{N} + \mathfrak{P}^i \dot{N}_i - N \mathfrak{H}^\perp - N_i \mathfrak{H}^i - \mathfrak{P} V - \mathfrak{P}^i V_i \right\}, \quad (21)$$

$$\mathfrak{H}^\perp = \frac{\kappa}{\sqrt{\mathfrak{h}}} (h_{ik} h_{jl} + h_{il} h_{kj} - h_{ij} h_{kl}) \mathfrak{p}^{ij} \mathfrak{p}^{kl} - \frac{\sqrt{\mathfrak{h}}}{2\kappa} [R] h, \quad (22)$$

$$\mathfrak{H}^i = -2\mathfrak{p}^{ij}{}_{|j}, \quad (23)$$

where  $\{V, V_i\}$  are velocity of  $N$  and  $N_i$  and play the role of Lagrange multipliers. Note that  $\{N, N_i, h_{ij}; \mathfrak{P}, \mathfrak{P}^i, \mathfrak{p}^{ij}\}$  are not the unique choice of canonical variables for General Relativity in Hamiltonian formalism; instead, they are a special parametrisation of the phase space. One can also choose the components of the original four-metric and their conjugate momenta  $\{g_{\mu\nu}; \mathfrak{p}^{\mu\nu}\}$  as canonical variables, as Dirac has done [2]. The two approaches are different in some subtle aspects; see [4] for a comparison.

Gauge transformations in the Arnowitt–Deser–Misner variables are generated by [1]

$$G = - \int \mathfrak{d}^3 x \left\{ \left[ \xi_\perp \left( \mathfrak{H}^\perp + N_{|i} \mathfrak{P}^i + (N \mathfrak{P}^i)_{|i} + (N_i \mathfrak{P})^{|i} \right) + \dot{\xi}_\perp \mathfrak{P} \right] \right. \\ \left. + \left[ \xi_i \left( \mathfrak{H}^i + N_j{}^{|i} \mathfrak{P}^j + (N_j \mathfrak{P}^i)^{|j} + N^{|i} \mathfrak{P} \right) + \dot{\xi}_i \mathfrak{P}^i \right] \right\}, \quad (24)$$

and the infinitesimal gauge transformation of  $N$  is

$$\delta N = [N, G]_P = \xi_{\perp|i} N^i - \dot{\xi}_\perp - \xi_i N^{|i}, \quad (25)$$

$$\delta N_i = -\xi_\perp N_{|i} + \xi_{\perp|i} N - \xi_j N_i{}^{|j} + \xi_i{}^{|j} N_j - \dot{\xi}_i, \quad (26)$$

which can be found in [4]. Transformations for  $g_{ij}$  and the momenta have to be worked out as

$$\delta \mathfrak{P} = -(\xi_\perp \mathfrak{P}^i)_{|i} - \xi_{\perp|i} \mathfrak{P}^i - (\xi_i \mathfrak{P})^{|i}, \quad (27)$$

$$\delta \mathfrak{P}^i = -\xi_{\perp|i} \mathfrak{P} - (\xi_j \mathfrak{P}^i)^{|j} - \xi_j{}^{|i} \mathfrak{P}^j; \quad (28)$$

$$\begin{aligned}
\delta g_{ij} &= -\frac{\partial}{\partial \mathbf{p}^{ij}} (\xi_{\perp} \mathfrak{H}^{\perp} + \xi_i \mathfrak{H}^i) \\
&= -\xi_{\perp}^{\perp} \frac{2\mathfrak{N}}{\sqrt{\mathfrak{h}}} (h_{ik} h_{jl} + h_{il} h_{kj} - h_{ij} h_{kl}) \mathbf{p}^{kl} - \xi_{i|j} - \xi_{j|i}, \quad (29) \\
\delta \mathbf{p}^{ij} &= \frac{\partial}{\partial g_{ij}} (\xi_{\perp} \mathfrak{H}^{\perp} + \xi_i \mathfrak{H}^i) \\
&= \xi_{\perp} \left\{ \frac{\mathfrak{N}}{\sqrt{\mathfrak{h}}} \left[ (4\delta^i_k \delta^j_m h_{ln} - h^{ij} h_{km} h_{ln}) \right. \right. \\
&\quad \left. \left. - \frac{1}{2} (4\delta^i_k \delta^j_m h_{ln} - h^{ij} h_{km} h_{ln}) \right] \mathbf{p}^{kl} \mathbf{p}^{mn} \right. \\
&\quad \left. + \frac{\sqrt{\mathfrak{h}}}{2\mathfrak{N}} G^{ij}[h] \right\} \\
&\quad + \xi_i \quad (30)
\end{aligned}$$

## References

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