Hamiltonian Dynamics of Maxwell–Proca theory in $(d + 1)$-dimensions

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1 Maxwell–Proca theory in flat space-time

Consider a Maxwell–Proca theory with source

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + A_\mu J^\mu, \tag{1.1}$$

where $m > 0$ corresponds to the Proca theory [1, sec. 2.3], and $m = 0$ the Maxwell theory [2, sec. 3.3.3], [1, sec. 2.4].

The Lagrangian density with velocity reads

$$\mathcal{L}^{\mathrm{v}} = \frac{1}{2}(V_i - \partial_i A_0)^2 - \frac{1}{4}F_{ij}^2 + \frac{m^2}{2}(A_0^2 - A_i^2) + A_0 J^0 + A_i J^i; \tag{1.2}$$

the canonical momenta densities are

$$\mathit{\Pi}^0 := \frac{\partial \mathcal{L}^{\mathrm{v}}}{\partial V_0} = 0, \qquad \mathit{\Pi}^i := \frac{\partial \mathcal{L}^{\mathrm{v}}}{\partial V_i} = V^i - \partial^i A_0. \tag{1.3}$$

The fundamental Poisson brackets are

$$\left[A_i(\vec{x}_1), \mathit{\Pi}^j(\vec{x}_2)\right]_{\mathrm{p}} = \delta^i{}_j \delta^d(\vec{x}_1 - \vec{x}_2). \tag{1.4}$$

The Hamiltonian with primary constraint, as well as the canonical Hamiltonian, are

$$\mathcal{H}^{\mathrm{p}} = \mathcal{H}^{\mathrm{c}} + V_0 \mathit{\Phi}_1, \tag{1.5}$$

$$\mathcal{H}^{\mathrm{c}} = \frac{1}{2}(\mathit{\Pi}^i)^2 + \mathit{\Pi}^i \partial_i A_0 + \frac{1}{4}F_{ij}^2 + \frac{m^2}{2}(-A_0^2 + A_i^2) - A_0 J^0 - A_i J^i, \tag{1.6}$$

$$\tag{1.7}$$

where

$$\mathit{\Phi}_1 := \mathit{\Pi}^0 \tag{1.8}$$

is the only primary constraint.

1.1 Constraint algebra

The Poisson bracket of $\varPhi_1$ and $\mathcal{H}^{\mathrm{p}}$ is

$$
\begin{aligned}
[\varPhi_1(\vec{x}_1), \mathcal{H}^{\mathrm{p}}(\vec{x}_2)]_{\mathrm{P}} &= \left[\left(\varPi^0\right)_1, \left(\varPi^i \partial_i A_0 - \frac{m^2}{2} A_0^2 - A_0 J^0 \right)_2 \right]_{\mathrm{P}} \\
&= \left(-\varPi^i \partial_i + m^2 A_0 - J_0\right)_2 \delta(\vec{x}_1 - \vec{x}_2),
\end{aligned} \tag{1.9}
$$

where $J_0 = -J^0$. Integration with $\mathbb{d}^d x_2$ yields the secondary constraint

$$
[\varPhi_1, H^{\mathrm{p}}]_{\mathrm{P}} = \partial_i \varPi^i + m^2 A_0 - J_0 =: \varPhi_2, \tag{1.10}
$$

so that

$$
[\varPhi_1(\vec{x}_1), \varPhi_2(\vec{x}_2)]_{\mathrm{P}} = -m^2 \delta(\vec{x}_1 - \vec{x}_2). \tag{1.11}
$$

One may further compute

$$
[\varPhi_2(\vec{x}_1), \mathcal{H}^{\mathrm{c}}(\vec{x}_2)]_{\mathrm{P}} = \left[\left(\partial_i \varPi^i\right)_1, \left(\frac{1}{4} F_{jk}^2 + \frac{m^2}{2} A_j^2 - A_j J^j \right)_2 \right]_{\mathrm{P}}, \tag{1.12}
$$

in which

$$
\begin{aligned}
\left[\left(\partial_i \varPi^i\right)_1, \left(\frac{1}{4} F_{jk}^2 \right)_2 \right]_{\mathrm{P}} &= \left(\partial_j A_k - \partial_k A_j\right)_2 (\partial_i)_1 \left[\left(\varPi^i\right)_1, \left(\partial^j A^k\right)_2 \right]_{\mathrm{P}} \\
&= -\left(F^{ij} \partial_j\right)_2 \left(\partial_i\right)_1 \delta(\vec{x}_1 - \vec{x}_2).
\end{aligned} \tag{1.13}
$$

The Poisson bracket can be evaluated to be

$$
[\varPhi_2(\vec{x}_1), \mathcal{H}^{\mathrm{c}}(\vec{x}_2)]_{\mathrm{P}} = -\left(F^{ij} \partial_j + m^2 A^i + J^i\right)_2 \left(\partial_i\right)_1 \delta(\vec{x}_1 - \vec{x}_2). \tag{1.14}
$$

Integration with $\mathbb{d}^d x_2$ yields

$$
[\varPhi_2, H^{\mathrm{c}}]_{\mathrm{P}} = -\partial_i \left(m^2 A^i + J^i\right). \tag{1.15}
$$

1.2 Free Maxwell theory

For Maxwell theory $m = 0$. Persistence condition on eq. (1.15) requires $\partial_i J = 0$. The canonical Hamiltonian in eq. (1.6) takes the form

$$
\mathcal{H}^{\mathrm{c}} = \frac{1}{2} \left(\varPi^i\right)^2 + \varPi^i \partial_i A_0 + \frac{1}{4} F_{ij}^2 - A_0 J^0 - A_i J^i, \tag{1.16}
$$

the secondary constraint $\varPhi_2$ in eq. (1.10) now reads

$$
\varPhi_2 = \partial_i \varPi^i - J_0. \tag{1.17}
$$

Since the Poisson bracket of $\varPhi_2$ and H^{p}

$$
[\varPhi_2, H^{\mathrm{p}}]_{\mathrm{P}} = -\partial_i J^i \tag{1.18}
$$

contains now no canonical variable, the algorithm terminates. Furthermore, the constraint algebra is commutative, hence the system is a purely first-class one.

1.3 Proca theory

For Proca theory $m > 0$, then the algorithm terminates, and one obtains a pure second-class system.

$$\mathbf{Q} = \begin{pmatrix} 0 & -m^2 \\ +m^2 & 0 \end{pmatrix}, \qquad \mathbf{Q}^{-1} = \begin{pmatrix} 0 & +m^{-2} \\ -m^{-2} & 0 \end{pmatrix}. \tag{1.19}$$

Dirac bracket

$$\begin{aligned} [f(\vec{x}_1), g(\vec{x}_2)]_{\mathrm{D}} = \big[(f)_1, (g)_2\big]_{\mathrm{P}} + \int \mathbb{d}^d x_3 \\ \Big(- \big[(f)_1, (\Pi^0)_3\big]_{\mathrm{P}} \big[(m^{-2}\partial_i \Pi^i + A_0)_3, (g)_2\big]_{\mathrm{P}} \\ + \big[(f)_1, (m^{-2}\partial_i \Pi^i + A_0)_3\big]_{\mathrm{P}} \big[(\Pi^0)_3, (g)_2\big]_{\mathrm{P}} \Big). \end{aligned} \tag{1.20}$$

The fundamental ones different from Poisson brackets are

$$[A_0(\vec{x}_1), A_i(\vec{x}_2)]_{\mathrm{D}} = m^{-2} (\partial_i)_1 \delta(\vec{x}_1 - \vec{x}_2), \qquad [A_0(\vec{x}_1), \Pi^0(\vec{x}_2)]_{\mathrm{D}} = 0. \tag{1.21}$$

Introducing the regularising coordinates

$$\alpha_i = A_i + m^{-2}(\partial_i \Pi^0 - J_i), \qquad \beta^i = \Pi^i; \tag{1.22}$$

$$\alpha_0 = A_0 + m^{-2}(\partial_i \Pi^i - J_0), \qquad \beta^0 = \Pi^0. \tag{1.23}$$

Note that $J^i = J_i$. It is easy to show that

$$\big[\alpha_i(\vec{x}_1), \beta^j(\vec{x}_2)\big]_{\mathrm{D}} = \delta^i_j \delta(\vec{x}_1, \vec{x}_2), \tag{1.24}$$

$$\big[\alpha_i(\vec{x}_1), \alpha_j(\vec{x}_2)\big]_{\mathrm{D}} = 0 = [\beta^i(\vec{x}_1), \beta^j(\vec{x}_2)]_{\mathrm{D}}. \tag{1.25}$$

Furthermore, one has

$$\mathcal{H}^{\mathrm{p}} = \mathcal{H}^{\mathrm{phy}} + \mathcal{H}^{\mathrm{con}} + \mathcal{H}^{\mathrm{irr}}, \tag{1.26}$$

where

$$\begin{aligned} \mathcal{H}^{\mathrm{phy}} = \frac{1}{2}(\beta^i)^2 + \frac{m^2}{2}\alpha_i^2 + \frac{1}{4}(\partial_i \alpha_j - \partial_j \alpha_i)^2 + \frac{1}{2m^2}(\partial_i \beta^i)^2 \\ + \frac{1}{m^2} J^0 \partial_i \beta^i, \end{aligned} \tag{1.27}$$

$$\mathcal{H}^{\mathrm{con}} = -\frac{m^2}{2}\alpha_0^2 - \frac{1}{2m^2}(\partial_i \beta^0)^2, \tag{1.28}$$

$$\begin{aligned} \mathcal{H}^{\mathrm{irr}} = \partial_i \Big(\alpha_0 \beta^i - \beta^0 \alpha_i + \frac{1}{m^2}\big(\beta^0 \partial_i \beta^0 - \beta^i \partial_j \beta^j - J^0 \beta^i\big) \Big) \\ + \frac{1}{2m^2}\big((J^0)^2 - (J^i)^2\big). \end{aligned} \tag{1.29}$$

Further more,

$$\Phi_1 = \beta^0, \qquad \Phi_2 = m^2 \alpha_0 \propto \alpha_0. \tag{1.30}$$

Thus the (α_i, β^i) are regular pairs of canonical variables, whereas (α_0, β^0) are the singular variables as constraints. The canonical dynamics of the physical (α_i, β^i)'s are determined by $\mathscr{H}^{\text{phy}}$ as a regular system.

Should one compute $\mathscr{H}^{\text{a}}$ here?

References

[1] Dmitriy M. Gitman and Igor V. Tyutin. *Quantization of Fields with Constraints.* Springer Series in Nuclear and Particle Physics. Springer Berlin Heidelberg, 1990. ISBN: http://id.crossref.org/isbn/978-3-642-83938-2. DOI: `10.1007/978-3-642-83938-2`. URL: `http://dx.doi.org/10.1007/978-3-642-83938-2`. Дмитрий Максимович Гитман and Игорь Викторович Тютин. *Каноническое квантование полей со связями.* 030077 г. 11овосибирск-77, Станиславского, 25: Наука, 1986.

[2] Heinz J Rothe and Klaus D Rothe. *Classical and Quantum Dynamics of Constrained Hamiltonian Systems.* World Scientific Lecture Notes in Physics. World Scientific, Apr. 2010. ISBN: http://id.crossref.org/isbn/978-981-4299-65-7. DOI: `10.1142/7689`. URL: `http://dx.doi.org/10.1142/7689`.