Perturbation of the ADM Hamiltonian Action

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Gauge structure is one of the corner stones in modern theoretical physics. In particle physics, where fundamental interactions are modelled by Yang–Mills theories, gauge structures are input of the theoretical description, and mathematical physicists use redundant variables in the configuration space such that the gauge invariance is manifest.

In gravitational physics, things are less clear compared to the former case. On the one hand, if one follows the way by which a Yang–Mills theory would be constructed, one derives Poincaré gauge theories, which is much more generic than the acknowledged General Relativity. On the other hand, even if one starts with General Relativity and call the reparametrisation invariance as a gauge invariance, one still faces another challenge, that in many applications, the theory is not written down with those variables where the gauge invariance is most manifest and the gauge transformation is simplest. The usage of the Arnowitt–Deser–Misner variables is such an example.

To be more specific, if one uses components of the metric $\{g_{\mu\nu}\}$ as the (superficial) degrees of freedom, the gauge transformation of which is related to the infinitesimal general coordinate transformation

$$x^\mu \mapsto x^\mu - \eta^\mu(x) + O(\eta^2) \tag{1}$$

by

$$\begin{aligned} \delta g_{\mu\nu} &= \mathbb{L}_\eta g_{\mu\nu} + O(\eta^2), \\ \mathbb{L}_\eta g_{\mu\nu} &= \eta_{\mu;\nu} + \eta_{\nu;\mu} = \eta^\lambda g_{\mu\nu,\lambda} + \eta^\lambda{}_{,\mu} g_{\lambda\nu} + \eta^\lambda{}_{,\nu} g_{\mu\lambda}. \end{aligned} \tag{2}$$

Here the generator of the general coordinate transformation $\eta^\mu(x)$ plays the role of the gauge generator.

Rewrite the text below …

Upon canonical quantisation, one would like to apply eq. (2) to the canonical variables. However, the common practice is to use the Arnowitt–Deser–Misner variables $\{N, N_i, h_{ij}\}$ in the canonical formalism, instead of $\{g_{\mu\nu}\}$. The former ones are an alternative parametrisation of the latter, and they are related by

$$g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = \mathrm{d}s^2 = -N^2\, \mathrm{d}t^2 + h_{ij}(N^i\, \mathrm{d}t + \mathrm{d}x^i)(N^j\, \mathrm{d}t + \mathrm{d}x^j), \tag{3}$$

where N^i is understood as $h^{ij}N_j$, and h^{ij} are implicit functions of h_{ij}. It is less well-known that one can also consistently derive a Hamiltonian action by using $\{g_{\mu\nu}\}$ and their conjugate momenta $\{\boldsymbol{\rho}^{\mu\nu}\}$, as Dirac has done in [2]. The Dirac and the Arnowitt–Deser–Misner approaches have been compared in [4], where some subtle differences have been shown; in particular, the gauge transformation in the former case (eq. (2)) is different than the latter one (eqs. (11) to (16)). This difference can be quantified, in the sense that the relation between $\{\eta_\mu\}$ and $\{\xi_\perp, \xi_i\}$, which are the gauge generator of the Dirac and Arnowitt–Deser–Misner approaches respectively, can be derived explicitly.

We would like to see if this difference has any consequence on the cosmological perturbation theory; in particular, we would like to see if the gauge-invariant variables could be changed from this perspective. In order to do this, we start from the Hamiltonian formalism in the Arnowitt–Deser–Misner variables and derive the gauge transformations of all the canonical variables; we then perturb the Hamiltonian action up to the second order on an arbitrary background with those variables directly; and finally we fix a Robertson–Walker background metric and discuss the gauge transformation of the perturbative coordinates and momenta.

1 Gauge transformation of the ADM canonical variables

The Hamiltonian action for General Relativity in terms of Arnowitt–Deser–Misner variables is [3, ch. 4.2.2]

$$\begin{aligned} S = \int \mathrm{d}t\, \mathrm{d}x^3 \Big\{ &\mathfrak{p}^{ij}\dot{h}_{ij} + \mathfrak{P}\dot{N} + \mathfrak{P}^i\dot{N}_i - N\mathfrak{H}^\perp - N_i\mathfrak{H}^i \\ &- \mathfrak{P}V - \mathfrak{P}^iV_i \Big\} + \text{boundary terms}, \end{aligned} \tag{4}$$

where

$$\mathfrak{H}^\perp = 2\varkappa\, \mathfrak{G}_{ijkl}\mathfrak{p}^{ij}\mathfrak{p}^{kl} - \frac{\sqrt{\mathfrak{h}}}{2\varkappa}R[h] \equiv 2\varkappa\, \mathfrak{F}^{ijkl}h_{ij}h_{kl} - \frac{\sqrt{\mathfrak{h}}}{2\varkappa}R[h], \tag{5}$$

$$\mathfrak{H}^i = -2\mathfrak{p}^{ij}{}_{|j} \tag{6}$$

are the secondary constraints,

$$\mathfrak{G}_{ijkl} := \frac{1}{2\mathfrak{h}^{1/2}}\left(h_{ik}h_{lj} + h_{il}h_{kj} - h_{ij}h_{kl}\right) \equiv -\frac{\delta\left(\mathfrak{h}^{-1/2}h_{ij}\right)}{\delta h^{kl}}, \tag{7}$$

$$\mathfrak{G}^{ijkl} := \frac{\mathfrak{h}^{1/2}}{2}\left(h^{ik}h^{lj} + h^{il}h^{kj} - 2h^{ij}h^{kl}\right) \equiv -\mathfrak{h}^{-1/2}\frac{\delta\left(\mathfrak{h}h^{ij}\right)}{\delta h_{kl}} \tag{8}$$

are the DeWitt metric and its inverse, and

$$\mathfrak{F}^{ijkl} := \frac{1}{2\mathfrak{h}^{1/2}}(\mathfrak{p}^{ik}\mathfrak{p}^{jl} + \mathfrak{p}^{il}\mathfrak{p}^{kj} - \mathfrak{p}^{ij}\mathfrak{p}^{kl}) \tag{9}$$

is a convenient notation. In eq. (4), V and V_i are velocities of N and N_i and play the role of Lagrange multipliers. Technical details about the boundary terms can be found in [5, ch. 4.2] and the references therein.

Gauge transformations in the Arnowitt–Deser–Misner canonical variables are generated by [1]

$$\begin{aligned} G = -\int \mathrm{d}^3x \Big\{ &\Big[\xi_\perp \Big(\mathfrak{H}^\perp + N_{|i}\mathfrak{P}^i + (N\mathfrak{P}^i)_{|i} + (N_i\mathfrak{P})^{|i}\Big) + \dot{\xi}_\perp \mathfrak{P}\Big] \\ &+ \Big[\xi_i \Big(\mathfrak{H}^i + N_j{}^{|i}\mathfrak{P}^j + \big(N_j\mathfrak{P}^i\big)^{|j} + N^{|i}\mathfrak{P}\Big) + \dot{\xi}_i\mathfrak{P}^i\Big]\Big\} \\ + \text{ boundary terms}, \end{aligned} \tag{10}$$

where the boundary terms have probably not been discussed so far, but are surely needed for non-compact spatial topologies in order to cancel the second spatial derivatives in the potential term in $\mathfrak{H}^\perp$. The infinitesimal gauge transformations of $\{N, N_i; \mathfrak{P}, \mathfrak{P}^i\}$ are [4]

$$\delta N = [N, G]_{\mathrm{P}} = \xi_\perp{}^{|i}N_i - \dot{\xi}_\perp - \xi_i N^{|i}, \tag{11}$$

$$\delta N_i = -\xi_\perp N_{|i} + \xi_{\perp|i}N - \xi_j N_i{}^{|j} + \xi_i{}^{|j}N_j - \dot{\xi}_i; \tag{12}$$

$$\delta\mathfrak{P} = -(\xi_\perp\mathfrak{P}^i)_{|i} - \xi_{\perp|i}\mathfrak{P}^i - (\xi_i\mathfrak{P})^{|i}, \tag{13}$$

$$\delta\mathfrak{P}^i = -\xi_\perp{}^{|i}\mathfrak{P} - \big(\xi_j\mathfrak{P}^i\big)^{|j} - \xi_j{}^{|i}\mathfrak{P}^j, \tag{14}$$

where only the primary constraints are involved; transformations of $\{g_{ij}; \mathfrak{p}^{ij}\}$ are

$$\begin{aligned} \delta h_{ij} &= -\frac{\partial}{\partial\mathfrak{p}^{ij}}(\xi_\perp\mathfrak{H}^\perp + \xi_i\mathfrak{H}^i) \\ &= -\xi^\perp 4\varkappa\,\mathfrak{G}_{ijkl}\mathfrak{p}^{kl} - \xi_{i|j} - \xi_{j|i}, \end{aligned} \tag{15}$$

$$\begin{aligned} \delta\mathfrak{p}^{ij} &= \frac{\partial}{\partial h_{ij}}(\xi_\perp\mathfrak{H}^\perp + \xi_i\mathfrak{H}^i) \\ &= 2\varkappa\,\xi_\perp\Big(-\frac{1}{2}h^{ij}\mathfrak{F}^{klmn}h_{mn} + 2\mathfrak{F}^{ijkl}\Big)h_{kl} \\ &\quad + \frac{1}{2\varkappa}\Big(\sqrt{\mathfrak{h}}\,\xi_\perp G^{ij}[h] - \mathfrak{G}^{ijkl}(\xi_\perp)_{|k|l}\Big) \\ &\quad + \big\{(\mathfrak{p}^{il}h^{kj} + \mathfrak{p}^{jl}h^{ki} - \mathfrak{p}^{ij}h^{kl})\xi_k\big\}_{|l}, \end{aligned} \tag{16}$$

where only the secondary constraints are involved.

Equation (15) could be different than what one would expect from eq. (2). In section 2 we will try to understand this discrepancy.

2 The Dirac and the ADM canonical variables

From eq. (3) one can read off the relations between $\{g_{\mu\nu}\}$ and $\{N, N_i, h_{ij}\}$, motivating one to use the generating functional of the third type $F_3 = F_3[\boldsymbol{\rho}^{\mu\nu}; N, N_i, h_{ij}]$. Then the relations can be written as

$$g_{00} = -N^2 + h^{ij}N_iN_j = -\frac{\delta F_3}{\delta \boldsymbol{\rho}^{00}}, \tag{17}$$

$$g_{i0} = N_i \qquad = -\frac{\delta F_3}{\delta \boldsymbol{\rho}^{i0}}, \quad g_{0i} = -\frac{\delta F_3}{\delta \boldsymbol{\rho}^{0i}}, \tag{18}$$

$$g_{ji} = h_{ji} \qquad = -\frac{\delta F_3}{\delta \boldsymbol{\rho}^{ji}}, \quad g_{ij} = -\frac{\delta F_3}{\delta \boldsymbol{\rho}^{ij}}; \tag{19}$$

the inverse of eqs. (17) to (19) reads[1]

$$N = \frac{1}{(-g^{00})^{1/2}}, \qquad N_i = \frac{g_{i0} + g_{0i}}{2}, \qquad h_{ij} = g_{ij}. \tag{20}$$

It is also useful to note that the transformation of the inverse metric is

$$g^{00} = -\frac{1}{N^2}, \qquad g^{0i} = g^{i0} = \frac{N^i}{N^2}, \qquad g^{ij} = h^{ij} - \frac{N^iN^j}{N^2}, \tag{21}$$

and the variation of N with respect to $\{g_{\mu\nu}\}$ reads

$$\delta N = \frac{1}{2}\left(-\sqrt{-g^{00}}\,\delta g_{00} + \frac{g^{0i}\,\delta g_{0i} + g^{i0}\,\delta g_{i0}}{(-g^{00})^{1/2}} - \frac{g^{0i}g^{0j}}{(-g^{00})^{3/2}}\,\delta g_{ij}\right). \tag{22}$$

Equations (17) to (19) can be formally integrated to get

$$F_3 = -\int \mathbb{d}^3x\,\{\boldsymbol{\rho}^{00}(-N^2 + h^{ij}N_iN_j) + (\boldsymbol{\rho}^{0i} + \boldsymbol{\rho}^{i0})N_i + \boldsymbol{\rho}^{ij}h_{ij}\}. \tag{23}$$

Calculation shows that the momenta transform as

$$\mathfrak{P} = -\frac{\delta F_3}{\delta N} = -2N\boldsymbol{\rho}^{00} \qquad = -\frac{2}{(-g^{00})^{1/2}}\boldsymbol{\rho}^{00}, \tag{24}$$

$$\mathfrak{P}^i = -\frac{\delta F_3}{\delta N_i} = +2N^i\boldsymbol{\rho}^{00} + \boldsymbol{\rho}^{0i} + \boldsymbol{\rho}^{i0} = -\frac{g^{0i} + g^{i0}}{g^{00}}\boldsymbol{\rho}^{00} + \boldsymbol{\rho}^{0i} + \boldsymbol{\rho}^{i0}, \tag{25}$$

$$\mathfrak{p}^{ij} = -\frac{\delta F_3}{\delta h_{ij}} = -N^iN^j\boldsymbol{\rho}^{00} + \boldsymbol{\rho}^{ij} \qquad = -\frac{g^{0i}g^{0j}}{(g^{00})^2}\boldsymbol{\rho}^{00} + \boldsymbol{\rho}^{ij}. \tag{26}$$

The inverse transformations of eqs. (24) to (26) are

$$\boldsymbol{\rho}^{00} = -\frac{\mathfrak{P}}{2N}, \quad \boldsymbol{\rho}^{0i} = \boldsymbol{\rho}^{i0} = \frac{\mathfrak{P}^i}{2} + N^i\frac{\mathfrak{P}}{2N}, \quad \boldsymbol{\rho}^{ij} = \mathfrak{p}^{ij} - N^iN^j\frac{\mathfrak{P}}{2N}. \tag{27}$$

[1] We take $N > 0$.

One can verify that

$$\mathfrak{P}\,\delta N + \mathfrak{P}^i\,\delta N_i + \mathfrak{p}^{ij}\,\delta h_{ij} = \boldsymbol{\rho}^{\mu\nu}\,\delta g_{\mu\nu} \tag{28}$$

holds, which is a criterion of time-independent canonical transformation; one may further verify that the fundamental Poisson brackets[2]

$$\big[g_{\mu\nu}(x_1), \boldsymbol{\rho}^{\rho\sigma}(x_2)\big]_{\mathrm{P}} = \frac{1}{2}\big(\delta^\rho{}_\mu\delta^\sigma{}_\nu + \delta^\sigma{}_\mu\delta^\rho{}_\nu\big)\boldsymbol{\delta}^{(3)}(x_1, x_2); \tag{29}$$

$$\begin{aligned}
[N(x_1), \mathfrak{P}(x_2)]_{\mathrm{P}} &= \boldsymbol{\delta}^{(3)}(x_1, x_2),\\
[N_i(x_1), \mathfrak{P}^j(x_2)]_{\mathrm{P}} &= \delta^j{}_i\boldsymbol{\delta}^{(3)}(x_1, x_2),\\
\big[h_{ij}(x_1), \mathfrak{p}^{kl}(x_2)\big]_{\mathrm{P}} &= \frac{1}{2}\big(\delta^k{}_i\delta^l{}_j + \delta^l{}_i\delta^k{}_j\big)\boldsymbol{\delta}^{(3)}(x_1, x_2);\\
[N(x_1), \mathfrak{P}^i(x_2)] &= [N(x_1), \mathfrak{p}^{ij}(x_2)] = [N^i(x_1), \mathfrak{p}^{jk}(x_2)] = 0
\end{aligned} \tag{30}$$

are invariant.

3 Perturbative expansion of the ADM Hamiltonian action

$$\begin{aligned}
\delta S = \int \mathrm{d}t\,\mathrm{d}^3x\,\Big\{&\mathcal{P}^{ij}\,\delta h_{ij} + \mathcal{Q}_{ij}\,\delta\mathfrak{p}^{ij} + \mathcal{C}\,\delta N + \mathcal{C}^i\,\delta N_i\\
&+ \mathcal{D}\,\delta\mathcal{P} + \mathcal{D}_i\,\delta\mathcal{P}^i + \mathscr{E}\,\delta V + \mathscr{E}^i\,\delta V_i\Big\}
\end{aligned} \tag{31}$$

No further boundary terms!

Some useful results

First variations

The first variation of the inverse metric h^{ij} reads

$$\delta h^{ij} = -h^{ik}h^{jl}\,\delta h_{kl} = -h^{i(k}h^{l)j}\,\delta h_{kl}. \tag{32}$$

The first variation of $\mathfrak{h} = \det h_{ij}$ reads

$$\delta\mathfrak{h} = \mathfrak{h}h^{ij}\,\delta h_{ij}. \tag{33}$$

The first variation of $\Gamma^i{}_{jk}$ can be obtained in normal coordinates, which reads

$$\delta\Gamma^i{}_{jk} = \frac{1}{2}h^{il}\Big\{-\big(\delta h_{jk}\big)_{|l} + (\delta h_{kl})_{|j} + \big(\delta h_{lj}\big)_{|k}\Big\} \tag{34}$$

$$= \frac{1}{2}\big\{-h^{il}\delta^m{}_j\delta^n{}_k + h^{in}\delta^l{}_j\delta^m{}_k + h^{im}\delta^n{}_j\delta^l{}_k\big\}\,(\delta h_{mn})_{|l}. \tag{35}$$

[2]Note that $[h_{12}, \mathfrak{p}^{12}] \propto 1/2$, not 1!

The first variation of $R_{ij}[h]$ and $R^{ij}[h]$

$$\delta R_{ij}[h] = \left(\delta \Gamma^k{}_{ji}\right)_{|k} - \left(\delta \Gamma^k{}_{ki}\right)_{|j}, \tag{36}$$

$$\delta R^{ij}[h] = -2R^{k(i} h^{j)l}\, \delta h_{kl} + h^{k(i}\, \bar{\delta} u^{j)l}{}_{k|l}, \tag{37}$$

where

$$\bar{\delta} u^{ij}{}_k := h^{il}\, \delta \Gamma^j{}_{lk} - h^{ij}\, \delta \Gamma^l{}_{lk} \tag{38}$$

is related to the boundary terms. Equation (36) can be obtained in normal coordinates.

For the first variation of the constraints, one also needs

$$\bar{\delta} u^{ji}{}_j = \left(\delta h_{kl}\right)_{|j} \left(h^{i(k} h^{l)j} - h^{ij} h^{kl}\right) = \mathfrak{h}^{-1/2} \mathfrak{G}^{ijkl} \left(\delta h_{kl}\right)_{|j}; \tag{39}$$

therefore,

$$\sqrt{\mathfrak{h}}\, N\, \bar{\delta} u^{ji}{}_{j|i} = \delta h_{ij}\, \mathfrak{G}^{ijkl} N_{|k|l} + \left\{ \mathfrak{G}^{ijkl} \left(N \left(\delta h_{kl}\right)_{|j} - N_{|j}\, \delta h_{kl} \right) \right\}_{|i}. \tag{40}$$

In the Hamiltonian constraint, the first variation of the ‘kinetic term’ $\mathfrak{G}_{ijkl} \mathfrak{p}^{ij} \mathfrak{p}^{kl} \equiv \mathfrak{F}^{ijkl} h_{ij} h_{kl}$ reads

$$\begin{aligned} &\delta\left(\mathfrak{G}_{ijkl} \mathfrak{p}^{ij} \mathfrak{p}^{kl}\right) \equiv \delta\left(\mathfrak{F}^{ijkl} h_{ij} h_{kl}\right) \\ &= \delta h_{ij} \left(-\frac{1}{2} h^{ij} \mathfrak{F}^{klmn} h_{mn} + 2\mathfrak{F}^{ijkl} \right) h_{kl} + \delta \mathfrak{p}^{ij}\, 2\mathfrak{G}_{ijkl} \mathfrak{p}^{kl}. \end{aligned} \tag{41}$$

Equipped with eqs. (37) and (39), the first variation of the ‘potential’ $\sqrt{\mathfrak{h}}\, R[h]$ reads

$$\begin{aligned} \delta\left(\sqrt{\mathfrak{h}} R[h]\right) &= \sqrt{\mathfrak{h}} \left\{ -G^{ij}[h]\, \delta h_{ij} + \bar{\delta} u^{ji}{}_{j|i} \right\} \\ &= -\sqrt{\mathfrak{h}}\, G^{ij}[h]\, \delta h_{ij} + \mathfrak{G}^{ijkl} \left(\delta h_{kl}\right)_{|j|i}. \end{aligned} \tag{42}$$

One can now write down the first variation of $N\mathfrak{H}^{\perp}$ with respect to $\left\{h_{ij}, \mathfrak{p}^{ij}\right\}$,

$$\begin{aligned} N\, \delta \mathfrak{H}^{\perp} = \delta h_{ij} &\left\{ 2\varkappa N \left(-\frac{1}{2} h^{ij} \mathfrak{F}^{klmn} h_{mn} + 2\mathfrak{F}^{ijkl} \right) h_{kl} \right. \\ &\left. + \frac{1}{2\varkappa} \left(\sqrt{\mathfrak{h}}\, N G^{ij}[h] - \mathfrak{G}^{ijkl} N_{|k|l} \right) \right\} \\ &+ \delta \mathfrak{p}^{ij}\, 4\varkappa N \mathfrak{G}_{ijkl} \mathfrak{p}^{kl} \\ &- \frac{1}{2\varkappa} \left\{ \mathfrak{G}^{ijkl} \left(N \left(\delta h_{kl}\right)_{|j} - N_{|j}\, \delta h_{kl} \right) \right\}_{|i}, \end{aligned} \tag{43}$$

where the terms in the last line will be pushed to the spatial boundary $\partial\Sigma$; the second term vanishes by $\delta h_{ij}\big|_{\partial\Sigma} = 0$, whereas the first one is cancelled by the boundary term.

Finally, the first variation of $N_i\mathfrak{H}^i$ with respect to $\{h_{ij}, \mathfrak{p}^{ij}\}$ is easier,

$$
\begin{aligned}
N_i\,\delta\mathfrak{H}^i &= \delta h_{ij}\left\{(\mathfrak{p}^{il}h^{kj} + \mathfrak{p}^{jl}h^{ki} - \mathfrak{p}^{ij}h^{kl})N_k\right\}_{|l} + \delta\mathfrak{p}^{ij}\,2N_{(i|j)} \\
&\quad - \left(-h^{il}\delta^m{}_j\delta^n{}_k + h^{in}\delta^l{}_j\delta^m{}_k + h^{im}\delta^n{}_j\delta^l{}_k\right)\left(N_i\mathfrak{p}^{jk}\,\delta h_{mn}\right)_{|l} \\
&\quad - 2\left(\delta\mathfrak{p}^{ij}\,N_j\right)_{|i},
\end{aligned}
\tag{44}
$$

where the last two lines will be pushed to the spatial boundary and vanish by $\delta h_{ij}\big|_{\partial\Sigma} = 0 = \delta\mathfrak{p}^{ij}\big|_{\partial\Sigma}$. The results can be checked with [5, ch. 4.2.7].

Second variations

First variation of $\mathfrak{F}^{ijkl}h_{kl}$

$$
\begin{aligned}
\delta(\mathfrak{F}^{ijkl}h_{kl}) = \delta h_{kl}&\left(-\frac{1}{2}\mathfrak{F}^{ijmn}h^{kl}h_{mn} + \mathfrak{F}^{ijkl}\right) \\
&+ \delta\mathfrak{p}^{kl}\left(\delta^i{}_k\mathfrak{G}^j{}_{lmn} + \delta^i{}_m\mathfrak{G}^j{}_{nkl}\right)\mathfrak{p}^{mn}.
\end{aligned}
\tag{45}
$$

First variation of $\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}$

$$
\begin{aligned}
\delta\left(\mathfrak{G}_{ijkl}\mathfrak{p}^{kl}\right) = \delta h_{kl}&\left(-\frac{1}{2}\mathfrak{G}_{ijmn}h^{kl} + \delta^k{}_i\mathfrak{G}^l{}_{jmn} + \delta^k{}_m\mathfrak{G}^l{}_{nij}\right)\mathfrak{p}^{mn} \\
&+ \delta p^{kl}\,\mathfrak{G}_{ijkl}.
\end{aligned}
\tag{46}
$$

First variation of $\sqrt{\mathfrak{h}}\,G^{ij}[h]$

$$
\begin{aligned}
\delta\left(\sqrt{\mathfrak{h}}\,G^{ij}[h]\right) = \sqrt{\mathfrak{h}}\Bigg\{&\delta h_{kl}\cdot \\
&\left(-\frac{1}{2}\right)\left(R^{ik}h^{lj} + R^{il}h^{kj} - R^{ij}h^{kl} + h^{ik}G^{lj} + h^{il}G^{kj} - h^{ij}G^{kl}\right) \\
&+ \left(h^{il}\,\bar{\delta}u^{jk}{}_l - \frac{1}{2}h^{ij}\,\bar{\delta}u^{lk}{}_l\right)_{|k}\Bigg\}.
\end{aligned}
\tag{47}
$$

For the following calculation, one also needs

$$
h^{k(i}\,\bar{\delta}u^{j)l}{}_{k|l} =
\tag{48}
$$

About $\bar{\delta}u^{ij}{}_k$, the identity

$$
\sqrt{\mathfrak{h}}\,Nh^{k(i}\,\bar{\delta}u^{j)l}{}_{k|l} =
\tag{49}
$$

is also useful.

Second variation of $N\mathfrak{H}^{\perp}$

$$
\begin{aligned}
&\delta^2(N\mathfrak{H}^{\perp})\\
&= \delta h_{ij}\,\delta h_{kl}\left\{2\varkappa\left[\frac{1}{4}(h^{ik}h^{lj}+h^{il}h^{kj}+h^{ij}h^{kl})\mathfrak{F}^{mnrs}h_{mn}h_{rs}\right.\right.\\
&\qquad\qquad\left. -(h^{ij}\mathfrak{F}^{klmn}+\mathfrak{F}^{ijmn}h^{kl})h_{mn}+\mathfrak{F}^{ijkl}\right]\\
&\qquad\left. -\frac{\sqrt{\mathfrak{h}}}{4\varkappa}(R^{ik}h^{lj}+R^{il}h^{kj}-R^{ij}h^{kl}+h^{ik}G^{lj}+h^{il}G^{kj}-h^{ij}G^{kl})\right\}\\
&\quad +\delta h_{ij}\,\frac{\sqrt{\mathfrak{h}}}{2\varkappa}\left(h^{il}\,\bar{\delta}u^{jk}{}_{l}-\frac{1}{2}h^{ij}\,\bar{\delta}u^{lk}{}_{l}\right)_{|k}\\
&\quad +\delta h_{ij}\,\delta\mathfrak{p}^{kl}\,4\varkappa\{-h^{ij}\mathfrak{G}_{klmn}+2(\delta^{i}{}_{k}\mathfrak{G}^{j}{}_{lmn}+\delta^{i}{}_{m}\mathfrak{G}^{j}{}_{nkl})\}\mathfrak{p}^{mn}\\
&\quad +\delta p^{ij}\,\delta p^{kl}\,4\varkappa\mathfrak{G}_{ijkl}\\
&\quad -\delta h_{ij}\,\delta\left(\frac{\sqrt{\mathfrak{h}}}{2\varkappa}\,\bar{\delta}u^{lk}{}_{l|k}\right).
\end{aligned}
\tag{50}
$$

Second variation of $\Gamma^{i}{}_{jk}$

$$\delta^2\Gamma^{i}{}_{jk} = -h^{im}\,\delta\Gamma^{l}{}_{jk}\,\delta h_{lm}. \tag{51}$$

Second variation of $\mathfrak{H}^{i}$

$$\delta^2(\mathfrak{H}^{i}) = -h^{im}p^{jk}\,\delta\Gamma^{l}{}_{jk}\,\delta h_{lm}+2\,\delta\Gamma^{i}{}_{jk}\,\delta p^{jk}. \tag{52}$$

Other second variations

Second variation of h^{ij}

$$\delta^2 h^{ij} = (h^{im}h^{jl}h^{kn}+h^{ik}h^{jm}h^{ln})\,\delta h_{kl}\,\delta h_{mn} \tag{53}$$

Second variation of $\mathfrak{h}=\det h_{ij}$

$$\delta^2\mathfrak{h} = -\frac{1}{4}\mathfrak{h}(h^{ik}h^{jl}+h^{il}h^{kj}-h^{ij}h^{kl})\,\delta h_{ij}\,\delta h_{kl}. \tag{54}$$

First variation of $\left(\delta h_{ij}\right)_{|k}$

$$\delta\left\{\left(\delta h_{ij}\right)_{|k}\right\} = -2\delta\Gamma^{l}{}_{k(i}\,\delta h_{j)l}. \tag{55}$$

In a general background, the second variations of the quantities are much more tedious.

In Robertson–Walker background,

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