

# Euler-Heisenberg Effective Action

immediate

## 1 Spinor electrodynamics in flat space-time

Maxwell lagrangian

$$S_{\text{Maxwell}}[A_\mu] := \int d^4x \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) \quad (1)$$

Dirac lagrangian [1, sec. 11]

$$S_{\text{Dirac}}[\psi, \bar{\psi}] := \int d^4x \{ -\bar{\psi}(\gamma^\mu \partial_\mu + m)\psi \}. \quad (2)$$

Interacting term

$$S_{\text{IMD}} := \int d^4x \left( -\bar{\psi} \gamma^\mu i e A_\mu \psi \right). \quad (3)$$

$$S_{1/2}[A_\mu, \psi, \bar{\psi}] := S_{\text{Maxwell}} + S_{\text{Dirac}} + S_{\text{IMD}}. \quad (4)$$

Generating functional

$$\mathcal{Z}[j^\mu, \bar{\eta}, \eta] := \int DA D\psi D\bar{\psi} \exp \left\{ i \left( S_{1/2} + \int d^4x (j^\mu A_\mu + \bar{\eta} \psi + \bar{\psi} \eta) \right) \right\} \quad (5)$$

Effective action

$$\mathcal{Z}[j^\mu, 0, 0] =: \int DA \exp \left\{ i \left( S_{\text{Maxwell}} + \Gamma[A_\mu] + \int d^4x j^\mu A_\mu \right) \right\}. \quad (6)$$

In other words,

$$\begin{aligned}
\exp\{i\Gamma[A_\mu]\} &\equiv \int D\psi D\bar{\psi} \exp\{i(S_{\text{Dirac}} + S_{\text{IMD}})\} \\
&\equiv \int D\psi D\bar{\psi} \exp\left\{i \int d^4x \bar{\psi}(-\not{\partial} - ie\mathcal{A} - m)\psi\right\} \\
&= \int D\psi D\bar{\psi} \exp\left\{i \int d^4x d^4y \bar{\psi}(x)M(x,y)\psi(y)\right\} \\
&= \mathcal{N} \det[-iM(x,y)],
\end{aligned} \tag{7}$$

where

$$M(x,y) := (+\not{\partial}_y - ie\mathcal{A}(y) - m)\delta^4(x-y). \tag{8}$$

$$\begin{aligned}
\Gamma[A_\mu] &\equiv -i(\ln \mathcal{N} + \ln \det[-iM]) \\
&= -i(\ln \mathcal{N} + \text{Tr} \ln[-iM(x,y)]).
\end{aligned} \tag{9}$$

Note that

$$\begin{aligned}
\text{Tr} \ln[-iM(x,y)] &\equiv \text{Tr} \ln[-iM^\top(x,y)] \\
&= \text{Tr} \ln[i(+\not{\partial}_y^\top - ie\mathcal{A}^\top(y) - m)\delta^4(x-y)] \\
&= \text{Tr} \ln[i(-\mathcal{C}\not{\partial}_y\mathcal{C}^{-1} + ie\mathcal{C}\mathcal{A}(y)\mathcal{C}^{-1} - \mathcal{C}m\mathcal{C}^{-1})\delta^4(x-y)] \\
&= \text{Tr}\{\mathcal{C} \ln[i(-\not{\partial}_y + ie\mathcal{A}(y) - m)\delta^4(x-y)]\mathcal{C}^{-1}\} \\
&= \text{Tr} \ln[i(-\not{\partial}_y + ie\mathcal{A}(y) - m)\delta^4(x-y)].
\end{aligned} \tag{10}$$

where the transpose  $^\top$  is taken in the spinor space. Therefore

$$\text{Tr} \ln[-iM(x,y)] = \frac{1}{2} \text{Tr} \ln[M_2(x,y)], \tag{11}$$

where

$$M_2(x,y) := \mathcal{M}_y \delta^4(x-y), \quad \mathcal{M}_y := \left((\not{\partial}_y - ie\mathcal{A}(y))^2 - m^2\right). \tag{12}$$

One may further simplify eq. (12) by noting ( $y$  suppressed)

$$\begin{aligned}
(\not{\partial} - ie\mathcal{A})^2 &= \partial^2 - e^2 A_\mu^2 - ie \frac{1}{2}([\gamma^\mu, \gamma^\nu]_- + [\gamma^\mu, \gamma^\nu]_+)(\partial_\mu A_\nu + A_\mu \partial_\nu + A_\nu \partial_\mu) \\
&= \partial^2 - e^2 A_\mu^2 - ie(\partial_\mu A^\mu + 2A^\mu \partial_\mu - i\sigma^{\mu\nu} \partial_\mu A_\nu) \\
&= (\partial_\mu - ieA_\mu)^2 - e\sigma^{\mu\nu} \partial_{[\mu} A_{\nu]} \\
&= (\partial_\mu - ieA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu} F_{\mu\nu},
\end{aligned} \tag{13}$$

so that

$$\mathcal{M} \equiv \left((\partial_\mu - ieA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu} F_{\mu\nu} - m^2\right), \tag{14}$$

where  $y$  is suppressed as well.

## 1.1 Constant background field in $(3 + 1)$ -dimensions

Equation (11) can be solved exactly when  $F_{\mu\nu}$  is constant throughout space-time. Take the case [2] where  $\vec{E} \parallel \vec{B}$  and, without loss of generality,  $\vec{B} \parallel \vec{z}$ . One has

$$F_{30} \equiv -F_{03} = E_3 =: E, \quad F_{12} \equiv -F_{21} = B_3 =: B. \quad (15)$$

A Landau-like choice of four-potential [3]

$$A_\mu := (0 \quad -Bx_2 \quad 0 \quad Ex_0) \quad (16)$$

can be applied which leads to eq. (15). In this choice the matrix reduces to

$$\mathcal{M} = \quad (17)$$

## 2 Scalar electrodynamics in flat space-time

### A Notions and conventions

$$\eta_{\mu\nu} := \text{diag}(-, +, +, +)$$

Pauli matrices

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma^3 := \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (18)$$

[1, sec. 5]

$$[\gamma^\mu, \gamma^\nu]_+ := 2\eta^{\mu\nu} \quad (19)$$

$$\mathcal{J}^{\mu\nu} := -\frac{i}{4}[\gamma^\mu, \gamma^\nu]_- \quad (20)$$

$$\sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu]_- \equiv -2\mathcal{J}^{\mu\nu}. \quad (21)$$

Choose the chiral representation

$$\gamma^\mu = -i \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \quad (22)$$

where

$$\sigma^\mu := (1_2, +\vec{\sigma}), \quad \bar{\sigma}^\mu := (1_2, -\vec{\sigma}). \quad (23)$$

$$\begin{aligned} \sigma^{\mu\nu} &\equiv -\frac{i}{2} \begin{bmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \end{bmatrix} \\ &= \begin{cases} 0 & \mu = 0, \nu = 0; \\ i \begin{bmatrix} -\sigma^j & 0 \\ 0 & +\sigma^j \end{bmatrix} & \mu = 0, \nu = j; \\ i \begin{bmatrix} +\sigma^i & 0 \\ 0 & -\sigma^i \end{bmatrix} & \mu = i, \nu = 0; \\ \begin{bmatrix} -\epsilon^{ij}_k \sigma^k & 0 \\ 0 & +\epsilon^{ij}_k \sigma^k \end{bmatrix} & \mu = i, \nu = j. \end{cases} \quad (24) \end{aligned}$$

## B Fresnel functional integral

[4, ch. 10]

## References

- [1] Steven Weinberg. *The Quantum Theory of Fields*. Vol. I: Foundations. Cambridge University Press, 1995.
- [2] Akaki Rusetsky and Ulf-G. Meißner. *Lecture on Advanced Theoretical Hadron Physics*. URL: <https://www.hiskp.uni-bonn.de/index.php?id=239>.
- [3] Lev Davidovich Landau. “Diamagnetismus der Metalle”. In: *Zeitschrift für Physik* 64:9-10 (Sept. 1930), pp. 629–637.
- [4] Ulrich Mosel. *Path Integrals in Field Theory*. Springer Berlin Heidelberg, 2004.