

Effective action of scalar electrodynamics

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The idea of a critical field traces back to [Sauter \[1931\]](#), where Dirac equation in a linear potential was solved.

1 Classical field

Complex Klein–Gordon action in flat space-time

$$S_S[\phi, \phi^*] := \int d^{d+1}x \{ -\eta^{\mu\nu} (\partial_\mu \phi)^* (\partial_\nu \phi) - m^2 \phi^* \phi \}. \quad (1)$$

Interaction terms

$$S_{SM}[A_\mu, \phi, \phi^*] := \int d^{d+1}x \eta^{\mu\nu} \{ ie A_\mu (-\phi^* \partial_\nu \phi + \phi \partial_\nu \phi^*) + e^2 A_\mu A_\nu \phi^* \phi \}. \quad (2)$$

The total action for scalar electrodynamics reads

$$\begin{aligned} S[A_\mu, \phi, \phi^*] &:= S_S + S_{SM} + S_{\text{Maxwell}} \\ &= \int d^{d+1}x \left\{ -(\nabla_\mu \phi)^* (\nabla^\mu \phi) - m^2 \phi^* \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right\}, \end{aligned} \quad (3)$$

where

$$\nabla_\mu \phi := (\partial_\mu + ie A_\mu) \phi. \quad (4)$$

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2 Functional quantisation

2.1 Wick rotation

Lorentzian generating functional

Wick rotation

$$x_{\text{E}}^4 = ix^0, \quad A_4 = -iA_0, \quad (5)$$

so that

$$\partial_{x^0} = i\partial_{x_{\text{E}}^4}, \quad F_{0i} = iF_{4i}. \quad (6)$$

The Euclidean action reads

$$S_{\text{E}}[A_I, \phi, \phi^*] = \int d^D x_{\text{E}} \left(\frac{1}{4} F_{IJ} F^{IJ} + (\nabla_I \phi)^* (\nabla^I \phi) + m^2 \phi^* \phi \right). \quad (7)$$

Working with the Euclidean signature is easier than in the Lorentzian signature.

2.2 Effective action

Generating functional and the *connected generating functional* (omitting subscript E for Euclidean systematically)

$$\mathcal{Z}[j^I, J, J^*] := \int DA_I D\phi^* D\phi \exp \left\{ -S - \int d^D x (j^I A_I + J^* \phi + \phi^* J) \right\} \quad (8)$$

$$=: \exp \{ -W[j^I, J, J^*] \}. \quad (9)$$

The expectation value of A_I , ϕ etc.

$$A_I^{\text{e}}[j^I, J, J^*] := \frac{\delta W}{\delta j^I}, \quad (10)$$

$$\phi^{\text{e}}[j^I, J, J^*] := \frac{\delta W}{\delta J^*}. \quad (11)$$

One would like to find *an effective action for A*, defined by the Legendre transformation

$$\Gamma_A[A_I] + \int d^D x \bar{j}^I A_I := W[\bar{j}^I, 0, 0] \equiv -\ln \mathcal{Z}[\bar{j}^I, 0, 0], \quad (12)$$

$$\left. \frac{\delta W[j, 0, 0]}{\delta j^I} \right|_{j^I = \bar{j}^I} := A_I. \quad (13)$$

In order to understand $\Gamma_A[A_I]$, take j^I as the fundamental field, and take functional derivative with respect to it in eq. (12), yielding

$$\frac{\delta \Gamma_A}{\delta A_I} \frac{\delta A_I}{\delta j^J} + A_J + j^I \frac{\delta A_I}{\delta j^J} = \frac{\delta W}{\delta j^J} = A_J, \quad (14)$$

or

$$\boxed{\frac{\delta \Gamma_A}{\delta A_I} + j^I = 0.} \quad (15)$$

At the leading order, eq. (15) reduces to the expression for classical action principle with source. One also has

$$\Gamma_A[A_I^c] \equiv -\ln \int \mathrm{D}A_I \mathrm{D}\phi^* \mathrm{D}\phi \exp\{-S[A_I^c + A_I, \phi, \phi^*]\}; \quad (16)$$

at the leading order, the external field A_I^c reduces to the classical field A_I^c . See [Schwartz, 2013, ch. 34], and better [Kleinert, 2015, ch. 22].

The derivative term of ϕ can be rearranged

$$(\nabla_I \phi)^* (\nabla^I \phi) = \partial_I (\phi^* \nabla^I \phi) - \phi^* \nabla_I \nabla^I \phi. \quad (17)$$

Hence up to boundary terms,

$$S[A_I, \phi, \phi^*] = \int \mathrm{d}^D x \frac{1}{4} F_{IJ} F^{IJ} + \int \mathrm{d}^D x \mathrm{d}^D y \phi^*(x) M[A_I; x - y] \phi(y), \quad (18)$$

where

$$M[A_I; x - y] := \left(-\nabla_{x^I} \nabla^{x^I} + m^2 \right) \delta^{(D)}(x - y), \quad (19)$$

see e.g. [Mosel, 2004, ch. 6] for details. Now the scalar field can formally be integrated, giving the (Euclidean) Euler–Heisenberg “effective action”

$$\mathcal{Z}[j^I, 0, 0] = \int \mathrm{D}A_I \exp \left\{ -\Gamma_{\text{EH}}[A_I] + \int \mathrm{d}^D x j^I A_I \right\}, \quad (20)$$

$$\begin{aligned} \Gamma_{\text{EH}}[A_I] - \int \mathrm{d}^D x \frac{1}{4} F_{IJ} F^{IJ} &= -\ln \int \mathrm{D}\phi^* \mathrm{D}\phi \exp \left\{ -\int \mathrm{d}^D x ((\nabla_I \phi)^* (\nabla^I \phi) + m^2 \phi^* \phi) \right\} \\ &= -\ln \frac{1}{\det M[A_I; x - y]} = \text{tr} \ln M, \end{aligned} \quad (21)$$

where normalisation is implicit in $\det M$. Equation (21) traces back to Heisenberg and Euler [1936], Weisskopf [1936].

Going back to eq. (12), one now has

$$\begin{aligned} \mathcal{Z}[j^I, 0, 0] &= \int \mathrm{D}A \exp \left\{ -\int \mathrm{d}^D x \left(\frac{1}{4} F_{IJ} F^{IJ} - j^I A_I \right) - \text{tr} \ln M \right\} \\ &= \mathcal{Z}_A[0] \langle j^I | \exp(-\text{tr} \ln M) | j^I \rangle \equiv \mathcal{Z}_A[0] \langle A_I | \exp(-\text{tr} \ln M) | A_I \rangle, \end{aligned} \quad (22)$$

where the expectation is defined as

$$\langle A_I | \mathcal{O} | A_I \rangle := \mathcal{Z}_A^{-1}[0] \int \mathrm{D}A \mathcal{O} \exp \left\{ - \int \mathrm{d}^D x \left(\frac{1}{4} F_{IJ} F^{IJ} - j^I A_I \right) \right\}, \quad (23)$$

$$\mathcal{Z}_A[j^I] := \int \mathrm{D}A \exp \left\{ - \int \mathrm{d}^D x \left(\frac{1}{4} F_{IJ} F^{IJ} - j^I A_I \right) \right\}. \quad (24)$$

2.3 Certain limit

In a certain limit (**which limit?**), the correlation between $\mathrm{tr} \ln M$ can be omitted, so that eq. (22) goes

$$\mathcal{Z}[j^I, 0, 0] \approx \mathcal{Z}_A[0] \exp(-\langle A_I | \mathrm{tr} \ln M | A_I \rangle), \quad (25)$$

and eq. (12) goes

$$\Gamma_A[A_I] \approx -\ln \mathcal{Z}_A[0] + \langle A_I | \mathrm{tr} \ln M | A_I \rangle. \quad (26)$$

2.4 World-line formalism

In eqs. (21), (22) and (26), $\mathrm{tr} \ln M$ is crucial.

Using the Schwinger integral representation [Schwinger \[1951\]](#) (up to normalisation, see appendix D)

$$\ln \alpha = - \int_0^{+\infty} \frac{\mathrm{d}s}{s} e^{-\alpha s}, \quad (27)$$

one has

$$-\mathrm{tr} \ln M = \int_0^{+\infty} \frac{\mathrm{d}T}{T} \exp \left(-\frac{m^2 T}{2\tilde{m}} \right) \mathrm{tr} \exp \left(-\frac{M}{2\tilde{m}} \right), \quad (28)$$

where T has the dimension of time, and \tilde{m} that of mass, which will both be eliminated later. Introduce the Hamiltonian of a non-relativistic point particle in D spatial dimensions (note originally one works in $d = D - 1$ spatial dimensions) (**check sign!**)

$$H := \frac{1}{2\tilde{m}} (P_I + eA_I)^2, \quad (29)$$

so that quantisation yields the following representation (**check sign!**)

$$\mathrm{tr} \exp \left(-\frac{M}{2\tilde{m}} \right) = \int_{-\infty}^{+\infty} \mathrm{d}x \left\langle x \left| e^{-\hat{H}T} \right| x \right\rangle \quad (30)$$

$$= \oint \mathrm{D}x \exp \left\{ - \int_0^T \mathrm{d}T' \left(\frac{\tilde{m}}{2} \left(\frac{\mathrm{d}x^I}{\mathrm{d}T'} \right)^2 + ieA_I \frac{\mathrm{d}x^I}{\mathrm{d}T'} \right) \right\}. \quad (31)$$

Rescaling $T' =: \lambda T$ gives

$$-\text{tr} \ln M = \int_0^{+\infty} \frac{dT}{T} \exp\left(-\frac{m^2 T}{2\tilde{m}}\right) \oint Dx \exp\left(-\frac{\tilde{m}}{2T} \int_0^1 d\lambda \dot{x}_I^2 - ie \oint A_I dx^I\right). \quad (32)$$

2.5 Euler–Heisenberg “effective action”

If the instanton magnetic field in eq. (31) is constant, the path integral can be performed exactly Feynman and Hibbs [1965], so that eq. (21) can be expressed in terms of an integral of T .

It is difficult to obtain a classical solution for the motion of a point particle in a more generic magnetic field. Therefore the generalisation in this direction is limited.

2.6 World-line instanton approximations

In eq. (32), one may also perform the T integral first. Using the integral expression and the asymptotic expansion for a modified Bessel function

$$K_0(x) = \frac{1}{2} \int_0^{+\infty} \frac{dt}{t} \exp\left(-t - \frac{x^2}{4t}\right) \quad (33)$$

$$\approx \sqrt{\frac{\pi}{2x}} e^{-x}, \quad x \gg 1, \quad (34)$$

one has

$$-\text{tr} \ln M = 2 \oint Dx K_0\left(m \sqrt{\int_0^1 d\lambda \dot{x}_I^2}\right) \exp\left(-ie \oint A_I dx^I\right) \quad (35)$$

$$\approx \sqrt{\frac{2\pi}{m}} \oint Dx \left(\int_0^1 d\lambda \dot{x}_I^2\right)^{-1/4} \exp\left(-m \sqrt{\int_0^1 d\lambda \dot{x}_I^2} - ie \oint A_I dx^I\right), \quad (36)$$

where eq. (36) works for

$$m \sqrt{\int_0^1 d\lambda \dot{x}_I^2} \gg 1 \quad \text{or} \quad \int_0^1 d\lambda \dot{x}_I^2 \gg m^{-2}. \quad (37)$$

This idea traces back to Affleck et al. [1982].

In such a limit, one may estimate $\text{tr} \ln M$ by the classical trajectory of the non-linear instanton action

$$S_i = -\ln K_0 \left(m \sqrt{\int_0^1 d\lambda \dot{x}_I^2} \right) + ie \oint A_I dx^I. \quad (38)$$

The action principle gives an integro-differential equation

$$0 = \frac{\delta S_i}{\delta x_I} = -m \frac{K_1 \left(m \sqrt{\int_0^1 d\lambda \dot{x}_I^2} \right)}{K_0 \left(m \sqrt{\int_0^1 d\lambda \dot{x}_I^2} \right)} \frac{\ddot{x}^I}{\sqrt{\int_0^1 d\lambda \dot{x}_I^2}} + ie F^{IJ} \dot{x}_J. \quad (39)$$

Contracting eq. (39) with \dot{x}_I yields

$$\frac{d\dot{x}_I^2}{d\lambda} = 0, \quad \dot{x}_I^2 =: a^2, \quad a > 0. \quad (40)$$

Hence eq. (39) turns into

$$mK(a)\ddot{x}^I = ieF^{IJ}\dot{x}_J, \quad K(a) := \frac{1}{a} \frac{K_1(ma)}{K_0(ma)}. \quad (41)$$

In Euclidean signature, eq. (41) is intrinsically complex [Dumlu and Dunne \[2011\]](#).

2.7 Application of instanton approximations

[Dunne and Schubert \[2005\]](#)

2.7.1 Constant electric field

2.8 Duality

Idea [Semenoff and Zarembo \[2011\]](#); book [\[Casalderrey-Solana et al., 2014, ch. 5\]](#); review [Kawai et al. \[2015\]](#);

3 Flat space-time (Lorentzian signature)

The content below needs revise.

Generating functional

$$\mathcal{Z}[j^\mu, J, J^*] := \int \mathrm{D}A \mathrm{D}\phi \mathrm{D}\phi^* \exp \left\{ i \left(S_0 + \int \mathrm{d}^{d+1}x (j^\mu A_\mu + J^* \phi + \phi^* J) \right) \right\}. \quad (42)$$

Effective action

$$\mathcal{Z}[j^\mu, 0, 0] = \int \mathrm{D}A \exp \left\{ i \left(S_{\text{Maxwell}} + \Gamma_{\text{W}}[A_\mu] + \int \mathrm{d}^{d+1}x j^\mu A_\mu \right) \right\}. \quad (43)$$

In other words,

$$\begin{aligned} \exp \{ i \Gamma_{\text{W}}[A_\mu] \} &:= \int \mathrm{D}\phi \mathrm{D}\phi^* \exp \{ i (S_{\text{CKG}} + S_{\text{ICKGM}}) \} \\ &\equiv \int \mathrm{D}\phi \mathrm{D}\phi^* \exp \left\{ i \int \mathrm{d}^{d+1}x \{ -(\nabla_\mu \phi)^* (\nabla^\mu \phi) - m^2 \phi^* \phi \} \right\}. \end{aligned} \quad (44)$$

The integral in the exponent can be manipulated; only the first term is essential

$$\begin{aligned} &\int \mathrm{d}^{d+1}x (-(\nabla_\mu \phi)^* (\nabla^\mu \phi)) \\ &= \int \mathrm{d}^{d+1}x \mathrm{d}^{d+1}y (-(\nabla_{x^\mu} \phi(x))^* \delta^{d+1}(x-y) \nabla^{y^\mu} \phi(y)), \end{aligned} \quad (45)$$

where

$$\begin{aligned} \delta^{d+1}(x-y) \nabla^{y^\mu} \phi(y) &= \delta^{d+1}(x-y) \{ \partial^{y^\mu} + i e A^\mu(y) \} \phi(y) \\ &= \{ -(\nabla^{y^\mu})^* \delta^{d+1}(x-y) \} \phi(y) + \partial^{y^\mu} B, \end{aligned} \quad (46)$$

in which

$$B = B(x, y) := \delta^{d+1}(x-y) \phi(y); \quad (47)$$

going back to eq. (45),

$$\begin{aligned} &= \int \mathrm{d}^{d+1}x \mathrm{d}^{d+1}y \{ -\{ \{ \partial_{x^\mu} + i e A_\mu(x) \} \phi(x) \}^* \delta^{d+1}(x-y) \nabla^{y^\mu} \phi(y) \} \\ &= \int \mathrm{d}^{d+1}x \mathrm{d}^{d+1}y \{ -\partial_{x^\mu} C^\mu + \phi^*(x) \nabla_{x^\mu} \delta^{d+1}(x-y) \nabla^{y^\mu} \phi(y) \} \\ &= \int \mathrm{d}^{d+1}x \mathrm{d}^{d+1}y \{ -\partial_{x^\mu} C^\mu + \phi^*(x) \{ -(\nabla_{y^\mu}) (\nabla^{y^\mu})^* \delta^{d+1}(x-y) \} \phi(y) + \partial^{y^\mu} \nabla_{x^\mu} B \}, \end{aligned} \quad (48)$$

in which

$$C^\mu = C^\mu(x, y) := \phi^*(x) \delta^{d+1}(x-y) \nabla^{y^\mu} \phi(y). \quad (49)$$

Now eq. (44) can be written as (dropping the boundary terms)

$$\begin{aligned}
&= \int D\phi D\phi^* \exp \left\{ -i \int d^{d+1}x d^{d+1}y \phi^*(x) D^{-1}(x, y) \phi(y) \right\} \\
&= \tilde{\mathcal{N}} \{ \det [D^{-1}(x, y)] \}^{-1/2},
\end{aligned} \tag{50}$$

where

$$D^{-1}(x, y) := \mathcal{D}_y^{-1} \delta^{d+1}(x - y), \quad \mathcal{D}_y^{-1} := +(\nabla_{y^\mu})(\nabla^{y^\mu})^* + m^2. \tag{51}$$

$$\begin{aligned}
\Gamma_W[A_\mu] &\equiv -i \left(\ln \tilde{\mathcal{N}} - \frac{1}{2} \ln \det D^{-1} \right) \\
&= \frac{i}{2} \text{tr}_x \ln (\mathcal{N}^{-1} D^{-1}) \\
&= \frac{i}{2} \int_0^{+\infty} \frac{ds}{s} \int d^{d+1}x d^{d+1}y \delta^{d+1}(x - y) \\
&\quad \cdot \left\{ -e^{is(+(\nabla_{y^\mu})(\nabla^{y^\mu})^* + m^2 + i0^+) \delta^{d+1}(x-y)} + e^{is(\mathcal{N} + i0^+)} \right\}.
\end{aligned} \tag{52}$$

Weisskopf [1936]

A Notions and conventions

The metric convention is mostly positive, i.e. $\eta_{\mu\nu} := \text{diag}(-, +, +, \dots)$

Pauli matrices

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma^3 := \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{53}$$

The γ -matrices satisfy [Weinberg, 1995, sec. 5]

$$[\gamma^\mu, \gamma^\nu]_+ := 2\eta^{\mu\nu} \mathbf{1}_4. \tag{54}$$

$$\mathcal{J}^{\mu\nu} := -\frac{i}{4} [\gamma^\mu, \gamma^\nu]_- \tag{55}$$

$$\sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu]_- \equiv -2 \not{\sigma}^{\mu\nu}. \quad (56)$$

In $(3 + 1)$ dimensions, choose the chiral representation

$$\gamma^\mu = -i \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \quad (57)$$

where

$$\sigma^\mu := (1_2, +\vec{\sigma}), \quad \bar{\sigma}^\mu := (1_2, -\vec{\sigma}). \quad (58)$$

$$\begin{aligned} \sigma^{\mu\nu} &\equiv -\frac{i}{2} \begin{bmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \end{bmatrix} \\ &= \begin{cases} 0 & \mu = 0, \nu = 0; \\ i \begin{bmatrix} +\sigma^j & 0 \\ 0 & -\sigma^j \end{bmatrix} & \mu = 0, \nu = j; \\ i \begin{bmatrix} -\sigma^i & 0 \\ 0 & +\sigma^i \end{bmatrix} & \mu = i, \nu = 0; \\ \begin{bmatrix} +\epsilon^{ij}_k \sigma^k & 0 \\ 0 & +\epsilon^{ij}_k \sigma^k \end{bmatrix} & \mu = i, \nu = j. \end{cases} \quad (59) \end{aligned}$$

B Fresnel functional integral

[Mosel, 2004, ch. 10]

C Algebra

$$[a\partial_1\partial_2, x^1]_- = a\partial_2 \quad (60)$$

central.

Baker–Campbell–Hausdorff formula

$$e^{+a\partial_1\partial_2} x^1 e^{-a\partial_1\partial_2} = x^1 + a\partial_2. \quad (61)$$

D Schwinger integral

$$-\int_{\epsilon}^{+\infty} \frac{dt}{t} e^{-\alpha t} = -\Gamma(0, \alpha\epsilon) = \gamma_E + \ln \alpha + \ln \epsilon + O(\epsilon), \quad (62)$$

where $\Gamma(a, z)$ is the incomplete Gamma function, γ_E is Euler's constant.

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