Effective action of scalar electrodynamics

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The idea of a critical field traces back to Sauter [1931], where Dirac equation in a linear potential was solved.

1 Classical field

Complex Klein–Gordon action in flat space-time

$$S_{\mathrm{S}}[\phi, \phi^*] \coloneqq \int \mathrm{d}^{d+1}x \left\{-\eta^{\mu\nu}(\partial_\mu\phi)^*(\partial_\nu\phi) - m^2\phi^*\phi\right\}. \tag{1}$$

Interaction terms

$$S_{\mathrm{SM}}[A_\mu, \phi, \phi^*] \coloneqq \int \mathrm{d}^{d+1}x\, \eta^{\mu\nu}\left\{\mathrm{i}eA_\mu(-\phi^*\partial_\nu\phi + \phi\partial_\nu\phi^*) + e^2A_\mu A_\nu\phi^*\phi\right\}. \tag{2}$$

The total action for scalar electrodynamics reads

$$\begin{aligned} S[A_\mu, \phi, \phi^*] &\coloneqq S_{\mathrm{S}} + S_{\mathrm{SM}} + S_{\mathrm{Maxwell}} \\ &= \int \mathrm{d}^{d+1}x \left\{-(\nabla_\mu\phi)^*(\nabla^\mu\phi) - m^2\phi^*\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right\}, \end{aligned} \tag{3}$$

where

$$\nabla_\mu\phi \coloneqq (\partial_\mu + \mathrm{i}eA_\mu)\phi. \tag{4}$$

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2 Functional quantisation

2.1 Wick rotation

Lorentzian generating functional

Wick rotation

$$x_{\mathrm{E}}^4 = \mathrm{i}x^0, \quad A_4 = -\mathrm{i}A_0, \tag{5}$$

so that

$$\partial_{x^0} = \mathrm{i}\partial_{x_{\mathrm{E}}^4}, \quad F_{0i} = \mathrm{i}F_{4i}. \tag{6}$$

The Euclidean action reads

$$S_{\mathrm{E}}[A_I, \phi, \phi^*] = \int \mathrm{d}^D x_{\mathrm{E}} \left(\frac{1}{4} F_{IJ} F^{IJ} + (\nabla_I \phi)^* \left(\nabla^I \phi\right) + m^2 \phi^* \phi \right). \tag{7}$$

Working with the Euclidean signature is easier than in the Lorentzian signature.

2.2 Effective action

Generating functional and the *connected generating functional* (omitting subscript E for Euclidean systematically)

$$\begin{aligned} \mathcal{Z}\left[j^I, J, J^*\right] &:= \int \mathrm{D}A_I\, \mathrm{D}\phi^*\, \mathrm{D}\phi \exp\left\{ -S - \int \mathrm{d}^D x \left(j^I A_I + J^* \phi + \phi^* J \right) \right\} && (8) \\ &=: \exp\left\{ -W\left[j^I, J, J^*\right] \right\}. && (9) \end{aligned}$$

The expectation value of A_I, ϕ etc.

$$\begin{aligned} A_I^{\mathrm{e}}\left[j^I, J, J^*\right] &:= \frac{\delta W}{\delta j^I}, && (10) \\ \phi^{\mathrm{e}}\left[j^I, J, J^*\right] &:= \frac{\delta W}{\delta J^*}. && (11) \end{aligned}$$

One would like to find *an effective action for A*, defined by the Legendre transformation

$$\begin{aligned} \varGamma_A[A_I] + \int \mathrm{d}^D x\, \bar{j}^I A_I &:= W\left[\bar{j}^I, 0, 0\right] \equiv -\ln \mathcal{Z}\left[\bar{j}^I, 0, 0\right], && (12) \\ \left.\frac{\delta W[j, 0, 0]}{\delta j^I}\right|_{j^I = \bar{j}^I} &:= A_I. && (13) \end{aligned}$$

In order to understand $\mathit{\Gamma}_A[A_I]$, take j^I as the fundamental field, and take functional derivative with respect to it in eq. (12), yielding

$$\frac{\delta \mathit{\Gamma}_A}{\delta A_I}\frac{\delta A_I}{\delta j^J} + A_J + j^I \frac{\delta A_I}{\delta j^J} = \frac{\delta W}{\delta j^J} = A_J, \tag{14}$$

or

$$\boxed{\frac{\delta \mathit{\Gamma}_A}{\delta A_I} + j^I = 0.} \tag{15}$$

At the leading order, eq. (15) reduces to the expression for classical action principle with source. One also has

$$\mathit{\Gamma}_A[A_I^{\mathrm{e}}] \equiv -\ln \int \mathrm{D}A_I\, \mathrm{D}\phi^*\, \mathrm{D}\phi \exp\{-S[A_I^{\mathrm{c}} + A_I, \phi, \phi^*]\}; \tag{16}$$

at the leading order, the external field A_I^{e} reduces to the classical field A_I^{c}. See [Schwartz, 2013, ch. 34], and better [Kleinert, 2015, ch. 22].

The derivative term of ϕ can be rearranged

$$(\nabla_I \phi)^* \big(\nabla^I \phi\big) = \partial_I\big(\phi^* \nabla^I \phi\big) - \phi^* \nabla_I \nabla^I \phi. \tag{17}$$

Hence up to boundary terms,

$$S[A_I, \phi, \phi^*] = \int \mathrm{d}^D x\, \frac{1}{4} F_{IJ} F^{IJ} + \int \mathrm{d}^D x\, \mathrm{d}^D y\, \phi^*(x) M[A_I; x-y] \phi(y), \tag{18}$$

where

$$M[A_I; x-y] \coloneqq \Big(-\nabla_{x^I} \nabla^{x^I} + m^2\Big) \delta^{(D)}(x-y), \tag{19}$$

see e.g. [Mosel, 2004, ch. 6] for details. Now the scalar field can formally be integrated, giving the (Euclidean) Euler–Heisenberg "effective action"

$$\mathcal{Z}\big[j^I, 0, 0\big] \eqqcolon \int \mathrm{D}A_I \exp\Big\{-\mathit{\Gamma}_{\mathrm{EH}}[A_I] + \int \mathrm{d}^D x\, j^I A_I\Big\}, \tag{20}$$

$$\begin{aligned}\mathit{\Gamma}_{\mathrm{EH}}[A_I] - \int \mathrm{d}^D x\, \frac{1}{4} F_{IJ} F^{IJ} &= -\ln \int \mathrm{D}\phi^*\, \mathrm{D}\phi \exp\Big\{-\int \mathrm{d}^D x\, \big((\nabla_I \phi)^* \big(\nabla^I \phi\big) + m^2 \phi^* \phi\big)\Big\} \\ &= -\ln \frac{1}{\det M[A_I; x-y]} = \operatorname{tr} \ln M,\end{aligned} \tag{21}$$

where normalisation is implicit in $\det M$. Equation (21) traces back to Heisenberg and Euler [1936], Weisskopf [1936].

Going back to eq. (12), one now has

$$\begin{aligned}\mathcal{Z}\big[j^I, 0, 0\big] &= \int \mathrm{D}A \exp\Big\{-\int \mathrm{d}^D x \left(\frac{1}{4} F_{IJ} F^{IJ} - j^I A_I\right) - \operatorname{tr} \ln M\Big\} \\ &= \mathcal{Z}_A[0] \left\langle j^I \,\middle|\, \exp(-\operatorname{tr} \ln M) \,\middle|\, j^I \right\rangle \equiv \mathcal{Z}_A[0] \left\langle A_I \,\middle|\, \exp(-\operatorname{tr} \ln M) \,\middle|\, A_I \right\rangle,\end{aligned} \tag{22}$$

where the expectation is defined as

$$\langle A_I \,|\, \mathcal{O} \,|\, A_I \rangle \coloneqq \mathcal{Z}_A^{-1}[0] \int \mathrm{D}A \, \mathcal{O} \exp\left\{ - \int \mathrm{d}^D x \left(\frac{1}{4} F_{IJ} F^{IJ} - j^I A_I \right) \right\}, \tag{23}$$

$$\mathcal{Z}_A\left[j^I\right] \coloneqq \int \mathrm{D}A \exp\left\{ - \int \mathrm{d}^D x \left(\frac{1}{4} F_{IJ} F^{IJ} - j^I A_I \right) \right\}. \tag{24}$$

2.3 Certain limit

In a certain limit (**which limit?**), the correlation between $\operatorname{tr} \ln M$ can be omitted, so that eq. (22) goes

$$\mathcal{Z}\left[j^I, 0, 0\right] \approx \mathcal{Z}_A[0] \exp(- \langle A_I \,|\, \operatorname{tr} \ln M \,|\, A_I \rangle), \tag{25}$$

and eq. (12) goes

$$\mathit{\Gamma}_A[A_I] \approx - \ln \mathcal{Z}_A[0] + \langle A_I \,|\, \operatorname{tr} \ln M \,|\, A_I \rangle \,. \tag{26}$$

2.4 World-line formalism

In eqs. (21), (22) and (26), $\operatorname{tr} \ln M$ is crucial.

Using the Schwinger integral representation Schwinger [1951] (up to normalisation, see appendix D)

$$\ln \alpha = - \int_0^{+\infty} \frac{\mathrm{d}s}{s} \, \mathrm{e}^{-\alpha s}, \tag{27}$$

one has

$$- \operatorname{tr} \ln M = \int_0^{+\infty} \frac{\mathrm{d}T}{T} \exp\left(- \frac{m^2 T}{2 \widetilde{m}} \right) \operatorname{tr} \exp\left(- \frac{M}{2 \widetilde{m}} \right), \tag{28}$$

where T has the dimension of time, and $\widetilde{m}$ that of mass, which will both be eliminated later. Introduce the Hamiltonian of a non-relativistic point particle in D spatial dimensions (note originally one works in $d = D - 1$ spatial dimensions) (**check sign!**)

$$H \coloneqq \frac{1}{2 \widetilde{m}} (P_I + e A_I)^2, \tag{29}$$

so that quantisation yields the following representation (**check sign!**)

$$\operatorname{tr} \exp\left(- \frac{M}{2 \widetilde{m}} \right) = \int_{-\infty}^{+\infty} \mathrm{d}x \, \left\langle x \,\middle|\, \mathrm{e}^{-\widehat{H} T} \,\middle|\, x \right\rangle \tag{30}$$

$$= \oint \mathrm{D}x \exp\left\{ - \int_0^T \mathrm{d}T' \left(\frac{\widetilde{m}}{2} \left(\frac{\mathrm{d}x^I}{\mathrm{d}T'} \right)^2 + \mathrm{i} e A_I \frac{\mathrm{d}x^I}{\mathrm{d}T'} \right) \right\}. \tag{31}$$

Rescaling $T' =: \lambda T$ gives

$$-\operatorname{tr}\ln M = \int_0^{+\infty} \frac{\mathrm{d}T}{T} \exp\left(-\frac{m^2 T}{2\widetilde{m}}\right) \oint \mathrm{D}x \exp\left(-\frac{\widetilde{m}}{2T} \int_0^1 \mathrm{d}\lambda\, \dot{x}_I^2 - \mathrm{i}e \oint A_I \,\mathrm{d}x^I\right). \tag{32}$$

2.5 Euler–Heisenberg "effective action"

If the instanton magnetic field in eq. (31) is constant, the path integral can be performed exactly Feynman and Hibbs [1965], so that eq. (21) can be expressed in terms of an integral of T.

It is difficult to obtain a classical solution for the motion of a point particle in a more generic magnetic field. Therefore the generalisation in this direction is limited.

2.6 World-line instanton approximations

In eq. (32), one may also perform the T integral first. Using the integral expression and the asymptotic expansion for a modified Bessel function

$$\mathrm{K}_0(x) = \frac{1}{2} \int_0^{+\infty} \frac{\mathrm{d}t}{t} \exp\left(-t - \frac{x^2}{4t}\right) \tag{33}$$

$$\approx \sqrt{\frac{\pi}{2x}}\, \mathrm{e}^{-x}, \qquad x \gg 1, \tag{34}$$

one has

$$-\operatorname{tr}\ln M = 2 \oint \mathrm{D}x\, \mathrm{K}_0\left(m\sqrt{\int_0^1 \mathrm{d}\lambda\, \dot{x}_I^2}\right) \exp\left(-\mathrm{i}e \oint A_I \,\mathrm{d}x^I\right) \tag{35}$$

$$\approx \sqrt{\frac{2\pi}{m}} \oint \mathrm{D}x \left(\int_0^1 \mathrm{d}\lambda\, \dot{x}_I^2\right)^{-1/4} \exp\left(-m\sqrt{\int_0^1 \mathrm{d}\lambda\, \dot{x}_I^2} - \mathrm{i}e \oint A_I \,\mathrm{d}x^I\right), \tag{36}$$

where eq. (36) works for

$$m\sqrt{\int_0^1 \mathrm{d}\lambda\, \dot{x}_I^2} \gg 1 \qquad \text{or} \qquad \int_0^1 \mathrm{d}\lambda\, \dot{x}_I^2 \gg m^{-2}. \tag{37}$$

This idea traces back to Affleck et al. [1982].

In such a limit, estimate $\operatorname{tr}\ln M$ by the classical trajectory of the non-linear instanton action

$$S_{\mathrm{i}} = -\ln \mathrm{K}_0\left(m\sqrt{\int_0^1 \mathrm{d}\lambda\, \dot{x}_I^2}\right) + \mathrm{i}e \oint A_I\, \mathrm{d}x^I. \tag{38}$$

The action principle gives

$$0 = \frac{\delta S_{\mathrm{i}}}{\delta x_I} = -m \frac{\mathrm{K}_1\left(m\sqrt{\int_0^1 \mathrm{d}\lambda\, \dot{x}_I^2}\right)}{\mathrm{K}_0\left(m\sqrt{\int_0^1 \mathrm{d}\lambda\, \dot{x}_I^2}\right)} \frac{\ddot{x}^I}{\sqrt{\int_0^1 \mathrm{d}\lambda\, \dot{x}_I^2}} - \mathrm{i}e \tag{39}$$

2.7 Application of instanton approximations

Dunne and Schubert [2005]

2.7.1 Constant electric field

3 Flat space-time (Lorentzian signature)

The content below needs revise.

Generating functional

$$\mathcal{Z}[j^\mu, J, J^*] := \int \mathrm{D}A\, \mathrm{D}\phi\, \mathrm{D}\phi^* \exp\left\{\mathrm{i}\left(S_0 + \int \mathrm{d}^{d+1}x\, (j^\mu A_\mu + J^*\phi + \phi^* J)\right)\right\}. \tag{40}$$

Effective action

$$\mathcal{Z}[j^\mu, 0, 0] =: \int \mathrm{D}A \exp\left\{\mathrm{i}\left(S_{\text{Maxwell}} + \mathit{\Gamma}_{\mathrm{W}}[A_\mu] + \int \mathrm{d}^{d+1}x\, j^\mu A_\mu\right)\right\}. \tag{41}$$

In other words,

$$\begin{aligned}\exp\{\mathrm{i}\mathit{\Gamma}_{\mathrm{W}}[A_\mu]\} &:= \int \mathrm{D}\phi\, \mathrm{D}\phi^* \exp\{\mathrm{i}(S_{\text{CKG}} + S_{\text{ICKGM}})\} \\ &\equiv \int \mathrm{D}\phi\, \mathrm{D}\phi^* \exp\left\{\mathrm{i}\int \mathrm{d}^{d+1}x\, \left\{-(\nabla_\mu\phi)^*(\nabla^\mu\phi) - m^2\phi^*\phi\right\}\right\}.\end{aligned} \tag{42}$$

The integral in the exponent can be manipulated; only the first term is essential

$$\int \mathrm{d}^{d+1}x\,(-(\nabla_\mu\phi)^*(\nabla^\mu\phi))$$
$$= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\big(-(\nabla_{x^\mu}\phi(x))^*\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\big), \tag{43}$$

where

$$\begin{aligned}\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y) &= \delta^{d+1}(x-y)\big\{\partial^{y^\mu} + \mathrm{i}eA^\mu(y)\big\}\phi(y)\\ &= \big\{-\big(\nabla^{y^\mu}\big)^*\delta^{d+1}(x-y)\big\}\phi(y) + \partial^{y^\mu}B,\end{aligned} \tag{44}$$

in which

$$B = B(x,y) \coloneqq \delta^{d+1}(x-y)\phi(y); \tag{45}$$

going back to eq. (43),

$$\begin{aligned}&= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\{-\{\{\partial_{x^\mu} + \mathrm{i}eA_\mu(x)\}\phi(x)\}^*\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\}\\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\{-\partial_{x^\mu}C^\mu + \phi^*(x)\nabla_{x^\mu}\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y)\}\\ &= \int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\{-\partial_{x^\mu}C^\mu + \phi^*(x)\big\{-(\nabla_{y^\mu})\big(\nabla^{y^\mu}\big)^*\delta^{d+1}(x-y)\big\}\phi(y) + \partial^{y^\mu}\nabla_{x^\mu}B\big\},\end{aligned} \tag{46}$$

in which

$$C^\mu = C^\mu(x,y) \coloneqq \phi^*(x)\delta^{d+1}(x-y)\nabla^{y^\mu}\phi(y). \tag{47}$$

Now eq. (42) can be written as (dropping the boundary terms)

$$\begin{aligned}&= \int \mathrm{D}\phi\,\mathrm{D}\phi^*\exp\bigg\{-\mathrm{i}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\phi^*(x)D^{-1}(x,y)\phi(y)\bigg\}\\ &= \tilde{\mathcal{N}}\big\{\det\big[D^{-1}(x,y)\big]\big\}^{-1/2},\end{aligned} \tag{48}$$

where

$$D^{-1}(x,y) \coloneqq \mathcal{D}_y^{-1}\delta^{d+1}(x-y), \qquad \mathcal{D}_y^{-1} \coloneqq +(\nabla_{y^\mu})\big(\nabla^{y^\mu}\big)^* + m^2. \tag{49}$$

$$\begin{aligned}\mathit{\Gamma}_{\mathrm{W}}[A_\mu] &\equiv -\mathrm{i}\bigg(\ln\tilde{\mathcal{N}} - \frac{1}{2}\ln\det D^{-1}\bigg)\\ &= \frac{\mathrm{i}}{2}\,\mathrm{tr}_x\ln\big(\mathcal{N}^{-1}D^{-1}\big)\\ &= \frac{\mathrm{i}}{2}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\int \mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\,\delta^{d+1}(x-y)\\ &\quad\cdot\Big\{-\mathrm{e}^{\mathrm{i}s\left(+(\nabla_{y^\mu})\left(\nabla^{y^\mu}\right)^* + m^2 + \mathrm{i}0^+\right)\delta^{d+1}(x-y)} + \mathrm{e}^{\mathrm{i}s\left(\mathcal{N}+\mathrm{i}0^+\right)}\Big\}.\end{aligned} \tag{50}$$

Weisskopf [1936]

A Notions and conventions

The metric convention is mostly positive, i.e. $\eta_{\mu\nu} \coloneqq \mathrm{diag}(-,+,+,\ldots)$

Pauli matrices

$$\sigma^1 \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 \coloneqq \begin{pmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 \coloneqq \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{51}$$

The γ-matrices satisfy [Weinberg, 1995, sec. 5]

$$[\gamma^\mu, \gamma^\nu]_+ \coloneqq 2\eta^{\mu\nu}\mathbf{1}_4. \tag{52}$$

$$\mathscr{J}^{\mu\nu} \coloneqq -\frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]_- \tag{53}$$

$$\sigma^{\mu\nu} \coloneqq \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]_- \equiv -2\mathscr{J}^{\mu\nu}. \tag{54}$$

In $(3+1)$ dimensions, choose the chiral representation

$$\gamma^\mu = -\mathrm{i}\begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \tag{55}$$

where

$$\sigma^\mu \coloneqq (1_2, +\vec{\sigma}), \qquad \bar{\sigma}^\mu \coloneqq (1_2, -\vec{\sigma}). \tag{56}$$

$$\begin{aligned} \sigma^{\mu\nu} &\equiv -\frac{\mathrm{i}}{2}\begin{bmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{bmatrix} \\ &= \begin{cases} 0 & \mu = 0, \nu = 0; \\ \mathrm{i}\begin{bmatrix} +\sigma^j & 0 \\ 0 & -\sigma^j \end{bmatrix} & \mu = 0, \nu = j; \\ \mathrm{i}\begin{bmatrix} -\sigma^i & 0 \\ 0 & +\sigma^i \end{bmatrix} & \mu = i, \nu = 0; \\ \begin{bmatrix} +\epsilon^{ij}{}_k\sigma^k & 0 \\ 0 & +\epsilon^{ij}{}_k\sigma^k \end{bmatrix} & \mu = i, \nu = j. \end{cases} \end{aligned} \tag{57}$$

B Fresnel functional integral

[Mosel, 2004, ch. 10]

C Algebra

$$\left[a\partial_1\partial_2, x^1\right]_- = a\partial_2 \tag{58}$$

central.

Baker–Campbell–Hausdorff formula

$$\mathrm{e}^{+a\partial_1\partial_2} x^1 \mathrm{e}^{-a\partial_1\partial_2} = x^1 + a\partial_2. \tag{59}$$

D Schwinger integral

$$-\int_{\epsilon}^{+\infty} \frac{\mathrm{d}t}{t}\, \mathrm{e}^{-\alpha t} = -\Gamma(0, \alpha\epsilon) = \gamma_{\mathrm{E}} + \ln\alpha + \ln\epsilon + O(\epsilon), \tag{60}$$

where $\Gamma(a, z)$ is the incomplete Gamma function, γ_{E} is Euler's constant.

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