Integrable Liouville Cosmological Models

The self-adjointness of Hamiltonian and the Semi-classical Wave Functions

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Outline

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- Dynamical Dark Energy has been modelled by quintessence[1] and phantom[2] matter, which can be realised by minimally-coupled real scalar fields with $\ell = \pm 1$[3]

$$\mathcal{S}_{\mathsf{L}} = \int \mathrm{d}^4 x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \tag{1}$$

- The Liouville field[4] $\mathcal{V}(\phi) = V \mathrm{e}^{\lambda\phi}$ is of interest, where $\lambda, V \in \mathbb{R}$.
- Assume a flat Robertson–Walker metric $g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -N^2(t)\,\mathrm{d}t^2 + \varkappa \mathrm{e}^{2\alpha(t)}\,\mathrm{d}\Omega^2_{3\mathsf{F}}$, where $\varkappa := 8\pi G$, $\mathrm{d}\Omega^2_{3\mathsf{F}} := \mathrm{d}\chi^2 + \chi^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)$, and N lapse function.
- The total action reads $\mathcal{S} := \mathcal{S}_{\mathsf{EH}} + \mathcal{S}_{\mathsf{GHY}} + \mathcal{S}_{\mathsf{L}} = \int \mathrm{d}\Omega^2_{3\mathsf{F}} \int \mathrm{d}t\, L$, where

$$L := \varkappa^{3/2} N \mathrm{e}^{3\alpha} \left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{N^2} + \ell \frac{\dot{\phi}^2}{2N^2} - V \mathrm{e}^{\lambda\phi} \right). \tag{2}$$

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[1] R. R. Caldwell *et al.*, *Phys. Rev. Lett.* **80**, 1582–1585 (Feb. 1998).

[2] R. R. Caldwell, *Phys. Lett. B* **545**, 23–29 (Oct. 2002).

[3] The signature of metric is mostly positive.

[4] Y. Nakayama, *Int. J. Mod. Phys. A* **19**, 2771–2930 (July 2004).

Integrability

Implicit trajectories can be obtained explicitly; the minisuperspace Wheeler–DeWitt equation can be integrated exactly.

Self-adjointness

The Hamiltonian can be asymmetric; imposing self-adjointness leads to removal of degeneracy and discretisation of levels.

Semi-classical wave functions

Semi-classical wave functions can be constructed by the principle of constructive interference, the Wentzel–Kramers–Brillouin approximation and numerical methods.

Outline

Decoupling the variables

Via orthogonal transformation

- By rescaling $\overline{N} := N\mathrm{e}^{-3\alpha}$ and the orthogonal transformation

$$\begin{pmatrix} \alpha(t) \\ \phi(t) \end{pmatrix} = \frac{\varsigma}{g} \begin{pmatrix} \lambda & -\ell\varkappa \\ -6 & \lambda \end{pmatrix} \begin{pmatrix} \varsigma_\beta \cdot \beta(t) \\ \varsigma_\chi \cdot \chi(t) \end{pmatrix} \qquad \text{where } \varsigma_\beta, \varsigma_\chi = \pm 1, \tag{3}$$

the effective Lagrangian eq. (2) can be decoupled ($\varsigma_\beta = \varsigma_\chi = +1$ from now on)

$$L = \varkappa^{3/2}\overline{N}\left(-\varsigma\frac{3}{\varkappa}\frac{\dot{\beta}^2}{\overline{N}^2} + \ell\varsigma\frac{\dot{\chi}^2}{2\overline{N}^2} - V\mathrm{e}^{g\chi}\right), \tag{4}$$

where $\Delta := \lambda^2 - 6\ell\varkappa$, $\varsigma := \operatorname{sgn}\Delta$ and $g := \varsigma\sqrt{|\Delta|} \equiv \varsigma\sqrt{\varsigma\Delta}$.

- The Euler–Lagrange equations with respect to $\overline{N}$, β and χ will be called the modified first, second Friedmann equations and the Klein–Gordon equation, respectively.
- Since $\beta(t)$ is cyclic, its conjugate momentum p_β is conserved[5], and the modified second Friedmann equation can be readily integrated.

[5] The same first integral has been found in C. Lan, PhD thesis, Saint Petersburg State University, 2016, https://search.rsl.ru/ru/record/01006663434, A. A. Andrianov *et al.*, *Theor. Math. Phys.* **184**, 1224–1233 (Sept. 2015), in canonical formalism.

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Integration and the implicit trajectories

- For $p_\beta \neq 0$, fixing the *implicit gauge* $\overline{N} = -6\jmath\sqrt{\varkappa}\dot{\beta}/p_\beta$, the modified first Friedmann equation can be integrated, yielding the implicit trajectory

$$\mathbb{e}^{g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|} s^2\left(\sqrt{\frac{3}{2\varkappa}}\, g(\beta-\beta_0)\right), \tag{5}$$

in which $\upsilon := \operatorname{sgn} V$, $(\operatorname{sgn}, \operatorname{sgn})$ means $(\ell, \jmath\upsilon)$, and

$$\begin{aligned} s_{++}(\gamma) &:= \operatorname{sech}(\gamma), & s_{+-}(\gamma) &:= \operatorname{csch}(\gamma), \\ s_{-+}(\gamma) &:= \sec(\gamma), & s_{--}(\gamma) &:= \mathbb{i}\csc(\gamma). \end{aligned} \tag{6}$$

  - The integral for the phantom model with $(-,+)$ is periodic for β with $T_\beta = \sqrt{\frac{2\varkappa}{3}}\frac{2\pi}{g}$.
  - The integral for the phantom model with $(-,-)$ is not real.
  - The trajectories can be parametrised by β, inspiring recognising β as a 'time variable'.

- For $p_\beta = 0$, integrating the modified second Friedmann equation yields $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell\lambda\alpha/\varkappa$[6], which is the well-known power-law special solution[7].

[6] It only works for $(+,-)$ or $(-,+)$ if one checks consistency with the modified first Friedmann equation.

[7] For instance, A. R. Liddle, D. H. Lyth, *Cosmological Inflation and Large-Scale Structure*, (Cambridge University, 2000) , ch. 3.

Outline

Dirac quantisation

- The primary Hamiltonian and the Hamiltonian constraint[8] reads

$$H^{\mathrm{p}} = \overline{N} H_{\perp} + v^{\overline{N}} p_{\overline{N}}, \tag{7}$$

$$H_{\perp} := -\jmath \frac{p_{\beta}^2}{12\varkappa^{1/2}} + \ell \jmath \frac{p_{\chi}^2}{2\varkappa^{3/2}} + \varkappa^{3/2} V \mathrm{e}^{g\chi}. \tag{8}$$

- Applying the Dirac quantisation rules with the Laplace–Beltrami operator ordering[9], one gets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(\jmath \frac{\hbar^2}{12\varkappa^{1/2}} \partial_{\beta}^2 - \ell \jmath \frac{\hbar^2}{2\varkappa^{3/2}} \partial_{\chi}^2 + \varkappa^{3/2} V \mathrm{e}^{g\chi} \right) \Psi. \tag{9}$$

- Equation (9) is Klein–Gordon-like, hyperbolic for $\ell = +1$ and *elliptic* for $\ell = -1$.

[8] D. M. Gitman, I. V. Tyutin, *Quantization of Fields with Constraints*, (Springer, 1990) , H. J. Rothe, K. D. Rothe, *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*, (World Scientific, Apr. 2010) .

[9] C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012) , ch. 8.

- Fourier transforming β to $k_\beta \in \mathbb{R}$ yields the time-independent Schrödinger equation

$$E\psi(x) := \ell \frac{\hbar^2 \tilde{g}^2 \nu^2}{8M_\mathsf{P}} \psi = \widehat{H}_\mathsf{eff}\psi := \left(-\frac{\hbar^2}{2M_\mathsf{P}}\partial_x^2 + \widehat{V}_\mathsf{eff}(x)\right)\psi, \quad V_\mathsf{eff} := \ell\,\mathfrak{s}\,\mathfrak{v}\widetilde{V}\mathbb{e}^{\tilde{g}x}, \quad (10)$$

in which $M_\mathsf{P} := \sqrt{\frac{\hbar}{\varkappa}}$, $x := \sqrt{\hbar\varkappa}\cdot\chi$, $\widetilde{V} := M_\mathsf{P}^{-3}|V| \geq 0$, and $\nu = \sqrt{\frac{2\varkappa}{3}}\frac{|k_\beta|}{g} \geq 0$.

  - V_eff is a special case of the Morse potential[10].

- Let $\mathbb{e}^{2\widetilde{x}} = 8M_\mathsf{P}\widetilde{V}/\hbar^2\tilde{g}^2 \cdot \mathbb{e}^{\tilde{g}x}$

	$E \geq 0$	$E < 0$
$V_\mathsf{eff} > 0$	$(+,+)$; $\mathrm{K}_{\mathbb{i}\nu}(\mathbb{e}^{\widetilde{x}})$	$(-,-)$; none
$V_\mathsf{eff} < 0$	$(+,-)$; $\mathrm{F}_{\mathbb{i}\nu}(\mathbb{e}^{\widetilde{x}})$, $\mathrm{G}_{\mathbb{i}\nu}(\mathbb{e}^{\widetilde{x}})$	$(-,+)$; $\mathrm{J}_\nu(\mathbb{e}^{\widetilde{x}})$

[10] P. M. Morse, *Phys. Rev.* **34**, 57–64 (July 1929), D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) .

- Even in quantum geometrodynamics, the self-adjointness of the matter Hamiltonian is desirable, which is typically an unbounded operator in the Hilbert space $\mathbf{F}$ endowed with the Schrödinger inner product $(\cdot,\cdot)_{\mathrm{S}}$.
- Mathematically, an *unbounded* operator H is characterised not only by its action on a vector, but also by its domain $\mathrm{Dom}(H) \subsetneq \mathbf{F}$[11].
- In addition to the *symmetricity* $(H^\dagger\phi_1, \phi_2) \equiv (\phi_1, H\phi_2)$, the self-adjointness of an unbounded operator also requires $\mathrm{Dom}(H^\dagger) = \mathrm{Dom}(H)$.
- For quantum systems not bounded below, not only the self-adjointness, but also the symmetricity of the Hamiltonian is not guaranteed automatically.
- Even when one could find a $\mathrm{Dom}_0(H)$ such that H is symmetric, one would still be left with $\mathrm{Dom}_0(H^\dagger) \supseteq \mathrm{Dom}_0(H)$ in general.
- Sloppily speaking, the process of extending $\mathrm{Dom}(H)$ such that $\mathrm{Dom}(H^\dagger) = \mathrm{Dom}(H)$ is called self-adjoint extension[12]; if the extension is unique, the operator is called *essentially self-adjoint*.

[11] B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267 , ch. 9.

[12] D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) , G. Bonneau *et al.*, *Am. J. Phys* **69**, 322–331 (Mar. 28, 2001), V. S. Araujo *et al.*, *Am. J. Phys* **72**, 203–213 (Feb. 2004), A. M. Essin, D. J. Griffiths, *Am. J. Phys* **74**, 109–117 (Feb. 2006).

The self-adjointness of the Hamiltonian

Results for the Liouville potential[16]

- For $V_{\text{eff}} > 0$: $\widehat{H}_{\text{eff}}$ is essentially self-adjoint[13].
  - For the quintessence model $(+,+)$, nothing special.
- For $V_{\text{eff}} < 0$: $\widehat{H}_{\text{eff}}$ admits a family of self-adjoint extensions.
  - For the quintessence model $(+,-)$, the generalised eigenfunction is a linear combination of F and G, and the degeneracy is removed.
  - For the phantom model $(-,+)$, the eigenfunctions are discretised

$$\psi_{-+,\nu} \propto \mathrm{J}_\nu, \qquad \nu = 2n + a_<,\ n \in \mathbb{Z}_\geq, \tag{11}$$

    and the corresponding full wave-functions are Bloch-periodic $\Psi_{-+,n}(\beta,\chi) = \mathrm{e}^{\mathrm{i}\pi a_<} \Psi_{-+,n}(\beta + T_\beta, \chi)$. Imposing (anti-)periodic boundary condition would fix $a_< = 0$ (1).
- The self-adjointness has been discussed in quantum cosmology in e.g.[14] (in the operator ordering scenario) and more recently in[15].

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[13] See also B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267 , sec. 9.9.
[14] C. R. Almeida *et al.*, *Gravitation Cosmol.* **21**, 191–199 (July 2015).
[15] S. Gryb, K. P. Y. Thébault, arXiv: 1801.05789 (gr-qc) (Jan. 17, 2018).
[16] D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) , sec. 8.5.

Outline

Matching quantum number with classical first integral

Principle of constructive interference; $(+,+)$ as the example

- Write the mode function in the Wentzel–Kramers–Brillouin form $\Psi_\nu(\beta,\chi) \sim \sqrt{\rho(\beta,\chi)}\exp\{\frac{i}{\hbar}S(\beta,\chi)\}$. For $S/\hbar \gg 1$ and $\nu \gg 1$, $S(\beta,\chi)$ becomes the Hamilton principal function in the leading-order approximation, which is stationary with respect to the variation of integral constants $\partial S/\partial\nu = 0$.

- Fixing $\nu/\sigma > 1$,

$$\mathrm{K}_{i\nu}(\sigma) \sim \frac{\sqrt{2\pi}\cos\left(\sqrt{\nu^2-\sigma^2} - \nu\arccos\frac{\nu}{\sigma} - \frac{\pi}{4}\right)}{\left(\nu^2-\sigma^2\right)^{1/4} e^{\pi\nu/2}} + O\left(\frac{1}{\sigma}\right). \tag{12}$$

- Applying *the principle* to the phase of $\Psi_\nu(\sigma)$, one has $\sigma/\nu = \operatorname{sech}(s_\beta\gamma)$, which matches the trajectory if $\hbar k_\beta \equiv \hbar\sqrt{\frac{3}{2\varkappa}}g\nu = p_\beta$.
- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; it is not within the Wentzel–Kramers–Brillouin regime.

- The conclusions also hold for $\mathrm{F}_{i\nu}(\sigma)$, $\mathrm{G}_{i\nu}(\sigma)$ for $(+,-)$, and $\mathrm{J}_\nu(\sigma)$ for $(-,+)$.

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Inner products for wave functions

Schrödinger and Mostafazadeh inner product

- A common starting point is the *Schrödinger product*[17]

$$(\Psi_1, \Psi_2)_{\mathrm{S}} := \int \mathrm{d}\chi\, \Psi_1^*(\beta, \chi)\Psi_2(\beta, \chi); \tag{13}$$

  - $(\Psi, \Psi)_{\mathrm{S}}$ is positive-definite, and the integrand $\rho_{\mathrm{S}}(\beta, \chi)$ is non-negative
  - The corresponding Schrödinger current is real but does not satisfy continuity equation

- Mostafazadeh[18] found an inner product *for self-adjoint* $\mathbb{D}$ *with positive spectrum*:

$$(\Psi_1, \Psi_2)^{\kappa}_{\mathrm{M}} := \kappa\Big\{\big(\Psi_1, \mathbb{D}^{+1/2}\Psi_2\big)_{\mathrm{S}} + \big(\partial_\beta\Psi_1, \mathbb{D}^{-1/2}\partial_\beta\Psi_2\big)_{\mathrm{S}}\Big\}, \qquad \kappa > 0. \tag{14}$$

  - Real power of $\mathbb{D}$ is defined by spectral decomposition
  - It can be shown[19] that *the density*

$$\varrho^{\kappa}_{\mathrm{M}} := \kappa\Big\{\big|\mathbb{D}^{+1/4}\Psi\big|^2 + \big|\mathbb{D}^{-1/4}\partial_\beta\Psi\big|^2\Big\} \tag{15}$$

    - is equivalent to the integrand $\rho^{\kappa}_{\mathrm{M}}$ up to a boundary term $\int \mathrm{d}\chi\, \varrho^{\kappa}_{\mathrm{M}} = \int \mathrm{d}\chi\, \rho^{\kappa}_{\mathrm{M}} \equiv (\Psi_1, \Psi_2)^{\kappa}_{\mathrm{M}}$
    - is non-negative
    - The corresponding current $\vec{\mathcal{J}}^{\kappa}_{\mathrm{M}}$ is real but not conserved[20].

[17] C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012) , ch. 5.
[18] A. Mostafazadeh, *Class. Quantum Grav.* **20**, 155–171 (Dec. 2002).
[19] A. Mostafazadeh, F. Zamani, *Ann. Phys.* **321**, 2183–2209 (Sept. 2006).
[20] B. Rosenstein, L. P. Horwitz, *J. Phys. A: Math. Gen.* **18**, 2115–2121 (Aug. 1985).

Wave packets of Gaussian amplitude for continuous spectrum

Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu;\overline{\nu},\sigma) := \left(\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(\nu-\overline{\nu})^2}{2\sigma^2}\right)\right)^{1/2} \tag{16}$$

- In[21], $A\left(\nu;\overline{\nu},\sigma/\sqrt{2}\right)$ was chosen.

[21] M. P. Dąbrowski *et al.*, *Phys. Rev. D* **74**, arXiv: `hep-th/0605229 (hep-th)` (Aug. 2006).

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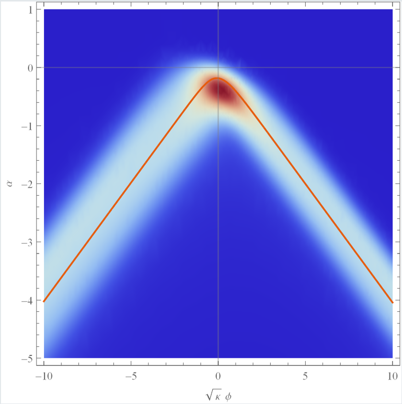
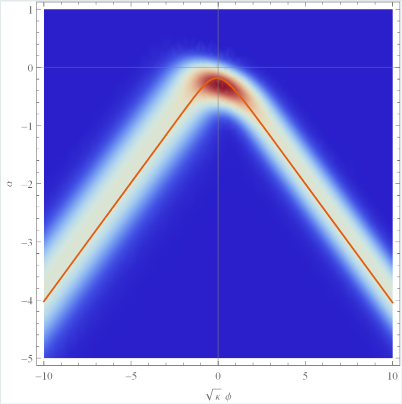
Wave packets of Gaussian amplitude for quintessence model $(+,+)$

$\mathrm{K}_{i\nu}$, **with** $\lambda = \varkappa^{1/2}/2$, $V = -\varkappa^{-2}$, $\bar{k}_\beta = -2$ **and** $\sigma_\beta = 5/4$

Schrödinger



Mostafazadeh



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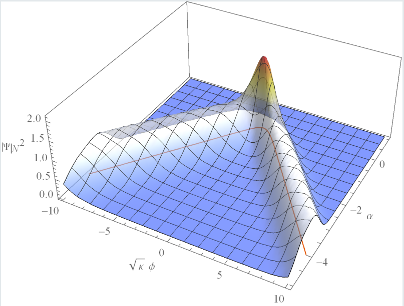

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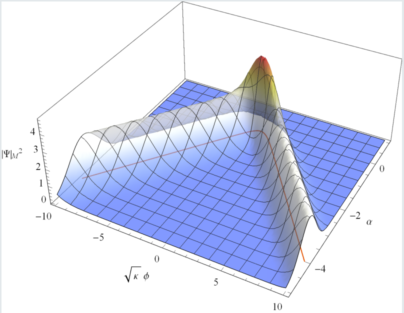

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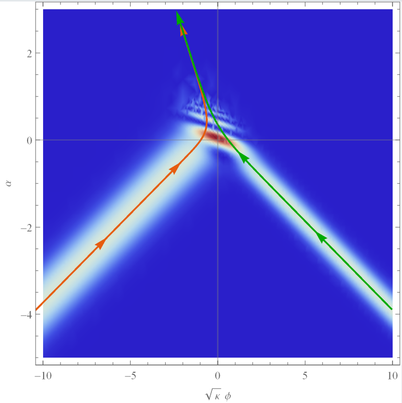
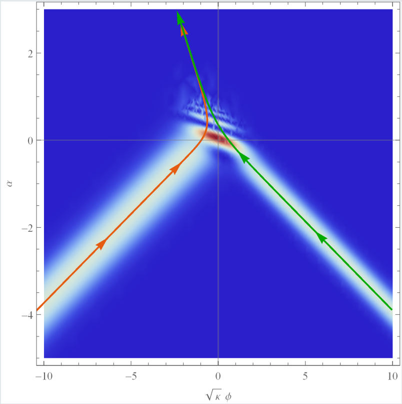
Wave packets of Gaussian amplitude for quintessence model $(+,-)$

$\mathrm{F}_{\mathring{\imath}\nu}$, **with** $\lambda = 4\varkappa^{1/2}/5$, $V = +\varkappa^{-2}$, $\bar{k}_\beta = -7/2$ **and** $\sigma_\beta = 7/5$

Schrödinger



Mostafazadeh



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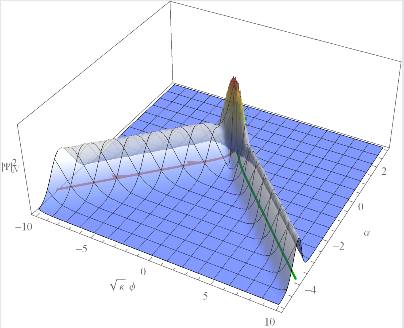
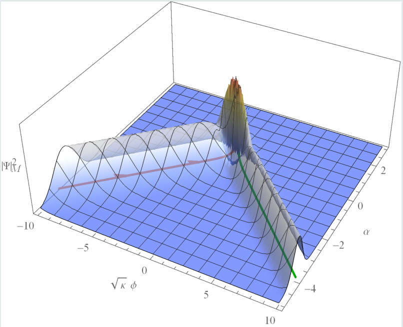


Mostafazadeh



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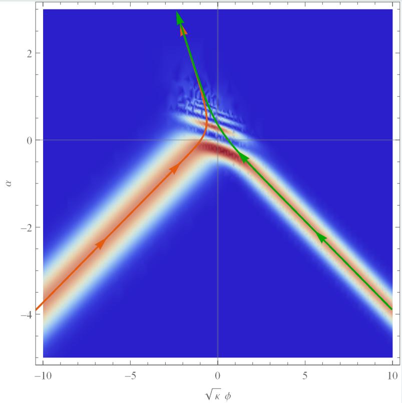
Wave packets of Gaussian amplitude for quintessence model $(+,-)$

$\mathrm{G}_{\mathbb{i}\nu}$, with $\lambda = 4\varkappa^{1/2}/5$, $V = +\varkappa^{-2}$, $\overline{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



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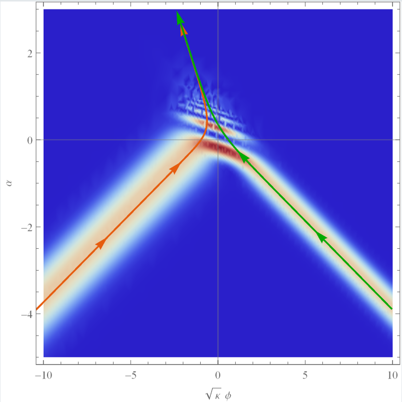

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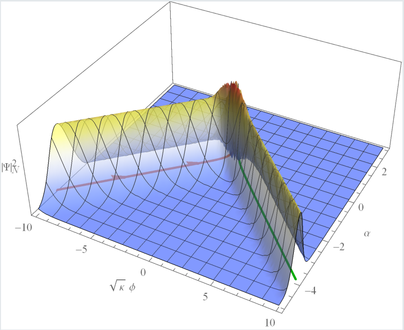

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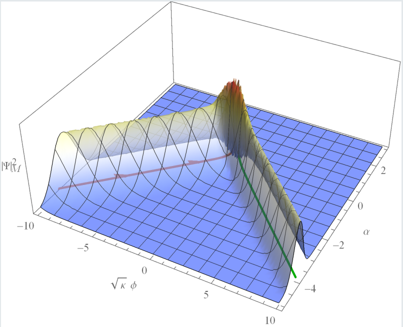


Wave packets with Poissonian amplitude for discrete spectrum

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\overline{n}) := \left(\mathrm{e}^{-\overline{n}}\frac{\overline{n}^n}{n!}\right)^{1/2} \tag{17}$$

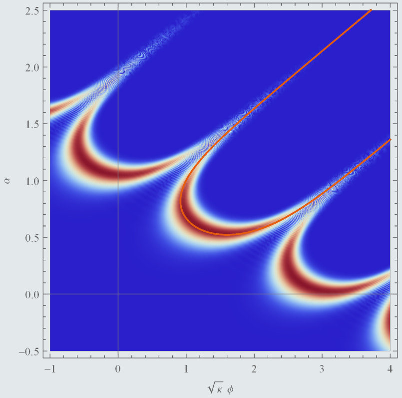
- In[22], $A_n\left(\overline{n}/\sqrt{2}\right)$ was chosen.

[22] C. Kiefer, *Nucl. Phys. B* **341**, 273–293 (Sept. 1990).

Wave packets of Poissonian amplitude for phantom model

J_{2n+1}, **with** $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$ **and** $\bar{k}_\beta = 8$

Schrödinger



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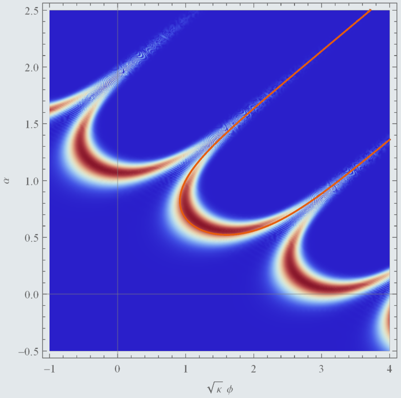

Wave packets of Poissonian amplitude for phantom model

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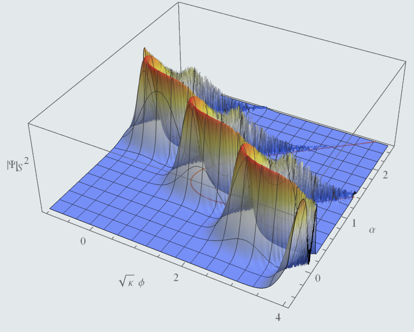
Schrödinger



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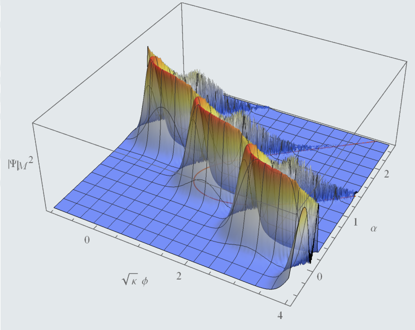

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Outline

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Highlights
Revisited

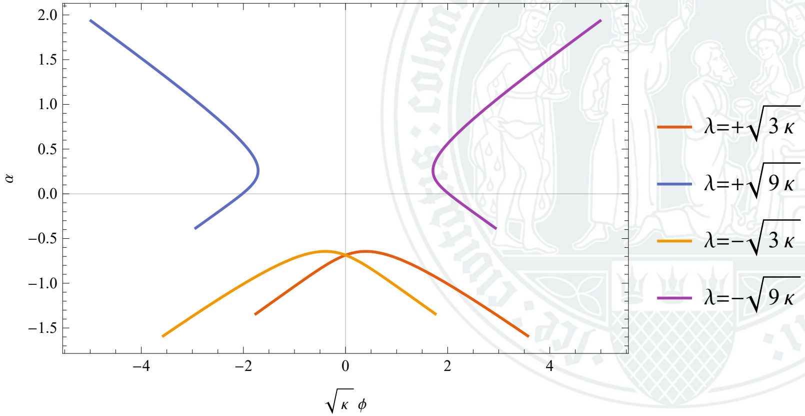
- An integral of motion was found for the Liouville cosmological model.
  - Implicit trajectories in minisuperspace were obtained explicitly.
  - The minisuperspace Wheeler–Dewitt equation was integrated exactly.
- The self-adjointness of matter Hamiltonian was found to be non-trivial.
  - The degeneracy of quintessence model $(+,-)$ was removed.
  - The levels of phantom Liouville model were discretised.
- The semi-classical wave functions were compared with the classical trajectory.
  - Mode functions in the Wentzel–Kramers–Brillouin approximation were matched with the classical Hamilton principal function.
  - Semi-classical wave packets were constructed and compared with the classical trajectory under the Schrödinger and Mostafazadeh inner product.

Outlook

- Beyond isotropy: generalise to Bianchi models
- Beyond homogeneity: cosmological perturbation
- Beyond single Liouville field: multiple potential / multiple potential
- Beyond classical matter and Schrödinger inner product: PT field

Trajectories for quintessence model $(+,+)$

sech, **with** $\beta_0 = 0$, $|V| = \varkappa^{-2}$ **and** $p_\beta^2 = \varkappa^2\sqrt{|V|}$; **varying** λ



- has two asymptotes $\chi \propto \pm\beta$

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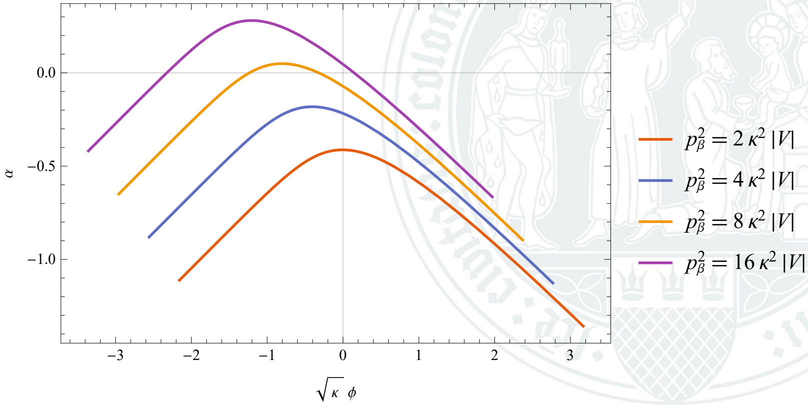
Trajectories for quintessence model $(+,+)$

sech, **with** $\beta_0 = 0$, $|V| = \varkappa^{-2}$ **and** $\lambda^2 = 3\varkappa$; **varying** p_β



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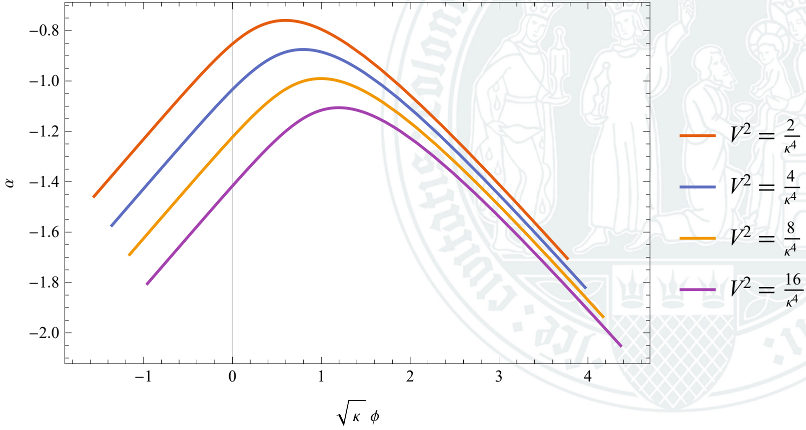
Trajectories for quintessence model $(+,+)$

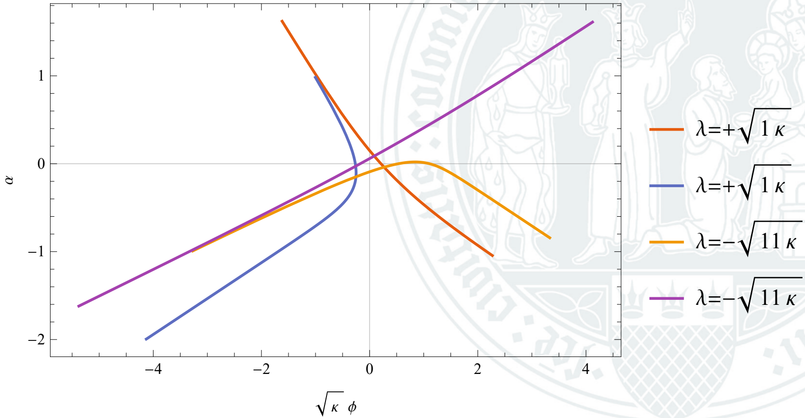
sech, **with** $\beta_0 = 0$, $\lambda^2 = 3\varkappa$ **and** $p_\beta^2 = \varkappa^2\sqrt{|V|}$; **varying** V



- has two asymptotes $\chi \propto \pm\beta$

Trajectories for quintessence model $(+,-)$

csch, **with** $\beta_0 = 0$, $|V| = \varkappa^{-2}$ **and** $p_\beta^2 = \varkappa^2 \sqrt{|V|}$; **varying** λ



- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$

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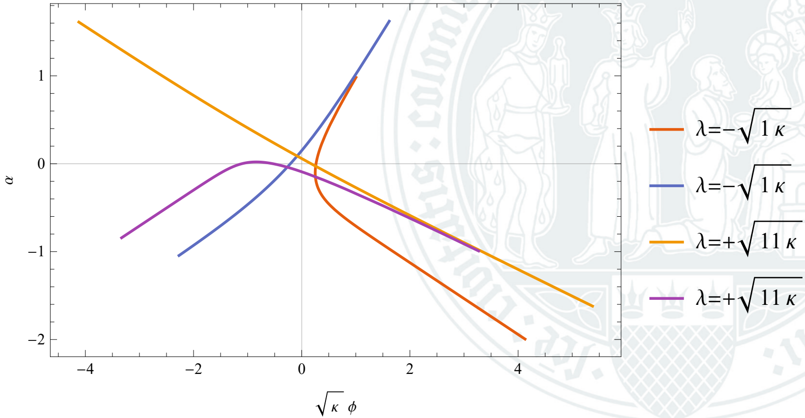
Trajectories for quintessence model $(+,-)$: csch

csch, with $\beta_0 = 0$, $|V| = \varkappa^{-2}$ and $p_\beta^2 = \varkappa^2\sqrt{|V|}$; varying λ



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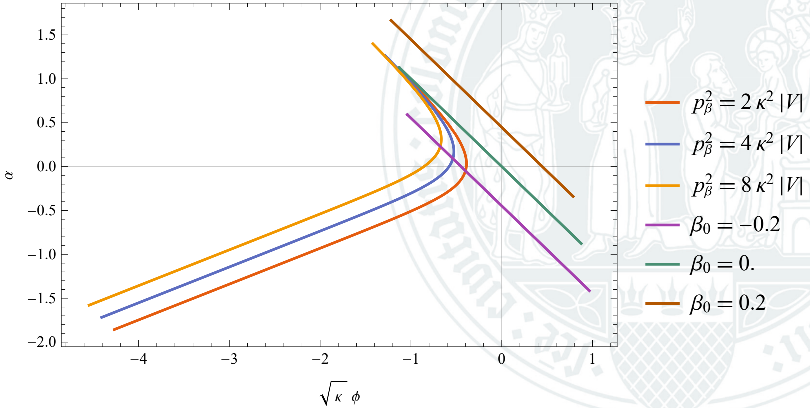
Trajectories for quintessence model $(+,-)$: csch

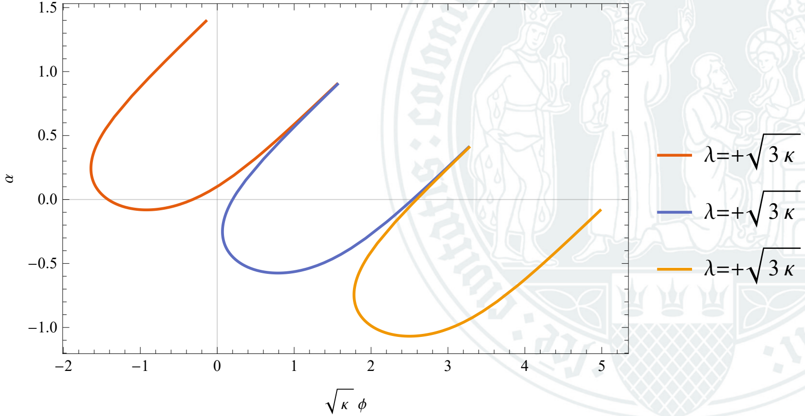
csch, **with** $V = \varkappa^{-2}$ **and** $\lambda^2 = \varkappa$; **varying** p_β



- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$

Trajectories for phantom model $(-,+)$

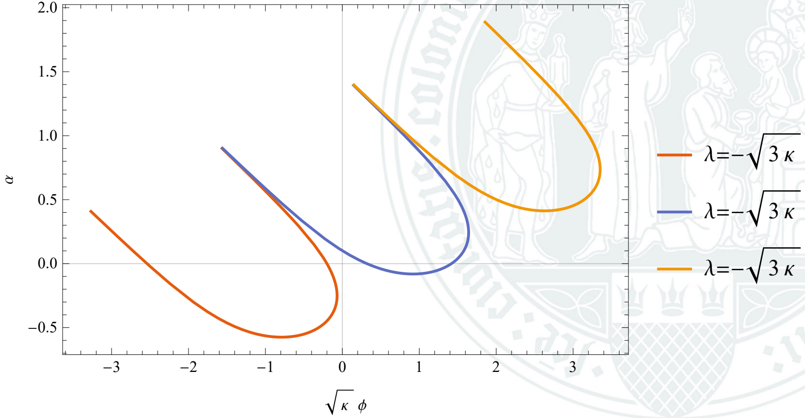
csc, **with** $V = \varkappa^{-2}$ **and** $p_\beta^2 = 3\varkappa^2\sqrt{|V|}$



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$

Trajectories for phantom model $(-,+)$

csc, **with** $V = \varkappa^{-2}$ **and** $p_\beta^2 = 3\varkappa^2\sqrt{|V|}$



- contains infinite distinct solutions
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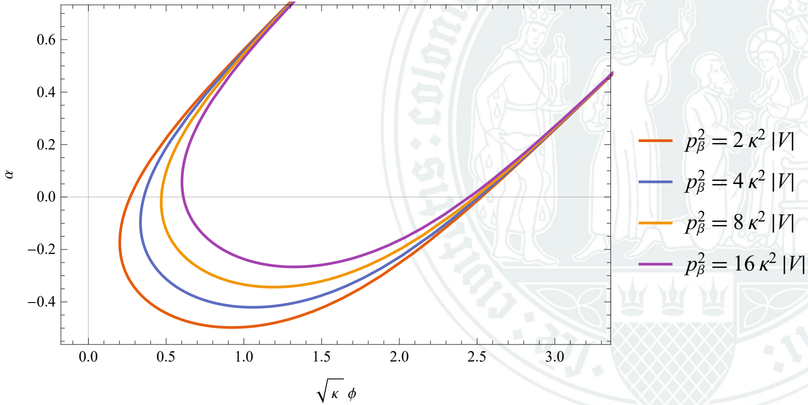
Trajectories for phantom model $(-,+)$

csc, with $V = \varkappa^{-2}$ and $\lambda^2 = 3\varkappa$; varying p_β



- contains infinite distinct solutions
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Integration of the transformed first Friedmann equation

General integral for $p_\beta \neq 0$

In order to integrate the equation under the implicit gauge

$$\jmath \frac{p_\beta^2}{12}\left(-\ell \frac{\varkappa^{1/2}}{6}\left(\frac{\mathrm{d}\chi}{\mathrm{d}\beta}\right)^2 + \varkappa^{-1/2}\right) - \varkappa^{3/2} V \mathrm{e}^{g\chi} = 0, \tag{18}$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa}} g\beta, \qquad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} \mathrm{e}^{-g\chi}, \tag{19}$$

to get

$$\left(\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\gamma}\right)^2 + \ell(\jmath v - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-\jmath v + \tilde{\sigma}^2)}}, \tag{20}$$

which is of the standard inverse hyperbolic / trigonometric form except for $(-,-)$.

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Integration of the separated minisuperspace WDW equation

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \qquad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \mathscr{s}\mathscr{v} \frac{12\varkappa^2 |V|}{\hbar^2}, \tag{21}$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{k_\beta}{g}, \qquad \sigma^2 := \frac{8\varkappa^3 |V| \mathrm{e}^{g\chi}}{\hbar^2 g^2}, \tag{22}$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma \psi'(\sigma) + \ell (\nu^2 - \mathscr{s}\mathscr{v}\sigma^2) \psi(\sigma) = 0, \tag{23}$$

which is of the standard Besselian form.

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