Integrable Cosmological Models with Liouville Fields

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Outline

Introduction

Introduction

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- Flat Robertson–Walker metric $\mathrm{d}s^2 = -N^2(t)\,\mathrm{d}t^2 + \varkappa^{-1}\mathrm{e}^{2\alpha(t)}\,\mathrm{d}\Omega_3^2$
  - $\varkappa := 8\pi G$; $\mathrm{d}\Omega_3^2$ dimensionless spacial metric
- Real Klein–Gordon field with potential $V\mathrm{e}^{\lambda\phi}$ (Liouville), where $\lambda, V \in \mathbb{R}$, and kinetic term with sign $\ell = \pm 1$ (quintessence / phantom model).
- Total action $\mathcal{S} := S_{\mathsf{EH}} + S_{\mathsf{GHY}} + S_{\mathsf{L}} = \int \mathrm{d}\Omega_3^2 \int \mathrm{d}t\, L$,

$$L := \varkappa^{3/2} N \mathrm{e}^{3\alpha}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{N^2} + \ell\,\frac{\dot{\phi}^2}{2N^2} - V\mathrm{e}^{\lambda\phi}\right), \tag{1}$$

in which dot means $\mathrm{d}/\mathrm{d}t$ and $\ell = \pm 1$.

Decoupling the variables

Via *rescaled* special orthogonal transformation

- Setting $\overline{N} := N\mathrm{e}^{-3\alpha}$, eq. (1) transforms to

$$L = \varkappa^{3/2}\overline{N}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{\overline{N}^2} + \ell\frac{\dot{\phi}^2}{2\overline{N}^2} - V\mathrm{e}^{\lambda\phi+6\alpha}\right) \tag{2}$$

- Defining $\varDelta := \lambda^2 - 6\ell\varkappa$, $\jmath := \operatorname{sgn}\varDelta$ and $g := \jmath\sqrt{|\varDelta|} \equiv \jmath\sqrt{\jmath\varDelta}$, the *rescaled* special orthogonal transformation

$$\begin{pmatrix}\alpha\\ \phi\end{pmatrix} = \frac{\jmath}{g}\begin{pmatrix}\lambda & -\ell\kappa\\ -6 & \lambda\end{pmatrix}\begin{pmatrix}\jmath_\beta\beta\\ \jmath_\chi\chi\end{pmatrix} \qquad \text{where } \jmath_\beta, \jmath_\chi = \pm 1 \tag{3}$$

gives the decoupled Lagrangian

$$L = \varkappa^{3/2}\overline{N}\left(-\jmath\frac{3}{\varkappa}\frac{\dot{\beta}^2}{\overline{N}^2} + \ell\jmath\frac{\dot{\chi}^2}{2\overline{N}^2} - V\mathrm{e}^{\jmath_\chi g\chi}\right). \tag{4}$$

- The Euler–Lagrange equations w.r.t. $\overline{N}$, β and χ will be called the trsfed. 1st, 2nd Friedmann eqs. and the Klein–Gordon eq., respectively.

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Possible generalisation of the decoupling technique

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- Beyond isotropy
  - Bianchi Type-I: under investigation
- Beyond one Liouville field
  - Two exponential potentials: $V_1 = V_2$ and special λ_i
  - Multiple Liouville fields: mixing kinetic terms[1]
- Beyond classic matter
  - Non-Hermitian, PT-symmetric Liouville fields[2]: may avoid big-rip etc.

[1] A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theoretical and Mathematical Physics* 184.3 (2015), pp. 1224–1233.

[2] A. A. Andrianov et al. In: *International Journal of Modern Physics D* 19.01 (2010), pp. 97–111, A. A. Andrianov, C. Lan, and O. O. Novikov. In: *Springer Proceedings in Physics*. Springer International Publishing, 2016, pp. 29–44.

Implicitised integration

General integral for $p_\beta \neq 0$

- Since β is cyclic in eq. (4), the trsfed. 2nd Friedmann eq. can be integrated[3]

$$\text{const.} \equiv p_\beta := \frac{\partial L}{\partial \dot{\beta}} = -6\jmath\varkappa^{1/2}\frac{\dot{\beta}}{\overline{N}} \equiv -6\jmath\jmath_\beta \frac{\varkappa^{1/2}}{g}\frac{\lambda\dot{\alpha} + \ell\varkappa\dot{\phi}}{\overline{N}}. \tag{5}$$

- For $p_\beta \neq 0$, fixing the *implicitising gauge* $\overline{N} = -6\jmath\sqrt{\varkappa}\dot{\beta}/p_\beta$, the trsfed. 1st Friedmann equation can be integrated

$$\mathrm{e}^{\jmath_\chi g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|} S^2\left(\jmath_\beta\sqrt{\frac{3}{2\varkappa}}\, g\beta + C\right), \tag{6}$$

in which $v := \operatorname{sgn} V$, $(\operatorname{sgn}, \operatorname{sgn})$ means $(\ell, \jmath v)$, and

$$\begin{aligned} {}_{(+,+)}S(\gamma) &:= \operatorname{sech}(\gamma), & {}_{(+,-)}S(\gamma) &:= \operatorname{csch}(\gamma), \\ {}_{(-,+)}S(\gamma) &:= \sec(\gamma), & {}_{(-,-)}S(\gamma) &:= \mathbb{i}\csc(\gamma). \end{aligned} \tag{7}$$

[3] The same first integral in Hamiltonian formalism has been found in C. Lan. PhD thesis. Saint Petersburg State University, 2016, A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theoretical and Mathematical Physics* 184.3 (2015), pp. 1224–1233.

Trajectories

Quintessence model $(+,+)$

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Trajectories

Quintessence model $(+,-)$

Trajectories

Phantom model $(-,+)$

Integration

Discussions

- The integrals are consistent with the trsfed. Klein–Gordon equation.
- The integral for $(+,+)$
  - has two asymptotes
- The implicitised integral for $(+,-)$
  - contains two distinct solutions
  - has three asymptotes
- The implicitised integral for $(-,+)$
  - is β-even for $C=0$
  - contains infinite distinct solutions
  - has infinite asymptotes, which are pairwise parallel
- The integral for $(-,-)$
  - is not real
- The implicitised integral enables one to compare trajectories with wave functions, see below.

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Implicitised integration

Specific integral for $p_\beta = 0$

- For $p_\beta = 0$, one has $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell\,\lambda\alpha/\kappa$, which is the familiar power-law special solution[4].
- Further integrating the first Friedmann equation demands $(+,-)$ or $(-,+)$ to guarantee $\overline{N} > 0$, and the result is automatically consistent with the trsfed. Klein–Gordon equation.
- Fixing $\overline{N} = (2\varkappa^2|V|)^{-1/2}$ yields

$$\mathrm{e}^{g\jmath_\chi\chi} = \left(\frac{2\kappa}{g(t-t_0)}\right)^2. \tag{8}$$

[4] M. P. Dąbrowski, C. Kiefer, and B. Sandhöfer. In: *Physical Review D* 74.4 (2006).

Trajectories

Quintessence model $(+, +)$

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Trajectories

Quintessence model $(+,-)$

Trajectories

Phantom model $(-, +)$

Dirac quantisation and the mode functions
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- The primary Hamiltonian and the Hamiltonian constraint reads

$$H^{\mathrm{p}} = \overline{N} H_{\perp} + v^{\overline{N}} p_{\overline{N}}, \tag{9}$$

$$H_{\perp} := -\jmath \frac{p_{\beta}^2}{12\varkappa^{1/2}} + \ell \jmath \frac{p_{\chi}^2}{2\varkappa^{3/2}} + \varkappa^{3/2} V \mathrm{e}^{g\jmath_{\chi}\chi}. \tag{10}$$

- Applying the Dirac quantisation rules with Laplace–Beltrami opeator, one gets the mss. Wheeler–DeWitt eq. with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(\jmath \frac{\hbar^2}{12\varkappa^{1/2}} \partial_{\beta}^2 - \ell \jmath \frac{\hbar^2}{2\varkappa^{3/2}} \partial_{\chi}^2 + \varkappa^{3/2} V \mathrm{e}^{g\jmath_{\chi}\chi} \right) \Psi. \tag{11}$$

- Equation (11) is KG-like, hyperbolic for $\ell = +1$ and *elliptic* for $\ell = -1$.

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Separation of the variables and mode functions
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- Writing $\Psi(\beta,\chi)=\varphi(\beta)\psi(\chi)$, eq. (11) can be separated into

$$\partial_\beta^2\varphi(\beta)=k_\beta^2\varphi(\beta); \tag{12}$$

$$\mathbb{D}\psi(\chi)=k_\beta^2\psi(\chi), \qquad \mathbb{D}:=-\ell\frac{6}{\varkappa}\partial_\chi^2+\jmath\upsilon\frac{12\varkappa^2|V|}{\hbar^2}. \tag{13}$$

- Equation (13) turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta,\chi):=\sum_{i=1}^{2}c_i\varphi_\nu^{(i)}(\gamma)\sum_{j=1}^{2}a_j\mathrm{B}_\nu^{(j)}(\sigma), \quad \nu\geq 0; \tag{14}$$

$$\nu:=\sqrt{\frac{2\varkappa}{3}}\frac{1}{g}k_\beta, \quad \gamma:=\sqrt{\frac{3}{2\varkappa}}\frac{g}{1}\beta, \qquad \sigma^2:=\frac{8\varkappa^3|V|\mathrm{e}^{g\jmath_\chi\chi}}{\hbar^2g^2}, \tag{15}$$

$${}_{(+,+)}\mathrm{B}_\nu^{(i)}(\sigma):=\mathrm{K}\text{ and }\mathrm{I}_{\mathbb{i}\nu}(\sigma), \qquad {}_{(+,-)}\mathrm{B}_\nu^{(i)}(\sigma):=\mathrm{F}\text{ and }\mathrm{G}_{\mathbb{i}\nu}(\sigma),$$

$${}_{(-,+)}\mathrm{B}_\nu^{(i)}(\sigma):=\mathrm{J}\text{ and }\mathrm{Y}_\nu(\sigma), \qquad {}_{(-,-)}\mathrm{B}_\nu^{(i)}(\sigma):=\mathrm{K}\text{ and }\mathrm{I}_\nu(\sigma).$$

- Adapted to imaginary order, $\mathrm{F}_\nu(\sigma)$ and $\mathrm{G}_\nu(\sigma)$ are defined in[5]; also see below.

[5] T. M. Dunster. In: *SIAM Journal on Mathematical Analysis* 21.4 (1990), pp. 995–1018.

Physical mode functions

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- Physical mode functions are expected to be regular on the boundary.
- $(+,+)$: $|\mathrm{I}_{\mathbb{i}\nu}(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- $(-,+)$:
  - $\forall n \in \mathbb{N}$, $|\mathrm{Y}_n(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
  - $\forall \nu \in \mathbb{R}^+ \backslash \mathbb{N}$, choose $\mathrm{J}_{-\nu}$ instead of Y_ν, since $\mathrm{J}_{\pm\nu}$ are also linearly independent.
  - $\forall \nu \in \mathbb{R}^+ \backslash \mathbb{N}$, $|\mathrm{J}_{-\nu}(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
- $(-,-)$: $|\mathrm{K}_\nu(\sigma)| \to +\infty$ as $\alpha \to -\infty$; $|\mathrm{I}_\nu(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- These are not to be included in the space of physical wave functions.
  $\forall \nu \geq 0$,
  - $(+,+)$: $\mathrm{K}_{\mathbb{i}\nu}(\sigma)$ survives
  - $(+,-)$: F and $\mathrm{G}_{\mathbb{i}\nu}(\sigma)$ survives
  - $(-,+)$: $\mathrm{J}_\nu(\sigma)$ survives
  - $(-,-)$: drops out

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Matching quantum number with classical first integral

Principle of constructive interference

- Baustelle
- In order to match the quantum number k_β (or *linearly*, ν) with the classical first integral p_β, one may apply the *principle of constructive interference*[6].
- Writing the mode function in the WKB form

$$\Psi_{k_\beta}(\beta, \chi) \sim \sqrt{\rho}\, \mathbb{e}^{\mathbb{i}S/\hbar}, \qquad S/\hbar \gg 1 \text{ and } k_\beta \gg 1, \tag{16}$$

the principle demands that $\partial S/\partial k_\beta = 0$ be equivalent to the classical trajectory.

[6] C. Kiefer. In: *Nuclear Physics B* 341.1 (1990), pp. 273–293.

Matching quantum number with classical first integral

$(+,+)$ as exemplar

- Fixing $\nu/\sigma > 1$, the asymptotic expansion reads

$$\mathrm{K}_{\mathbb{i}\nu}(\sigma) \sim \frac{\sqrt{2\mathbb{\pi}}\cos\left(\sqrt{\nu^2-\sigma^2}-\nu\arccos\frac{\nu}{\sigma}-\frac{\mathbb{\pi}}{4}\right)}{\left(\nu^2-\sigma^2\right)^{1/4}\mathbb{e}^{\mathbb{\pi}\nu/2}} + O\left(\frac{1}{\sigma}\right). \tag{17}$$

  - There are two phases with opposite signs. Assuming c_i, a_j's are real and applying *the principle* to $\Psi_\nu(\sigma)$, one has$\sigma/\nu = \operatorname{sech}(s_\beta\gamma)$, which matches the trajectory with $C = 0$ if

$$\hbar k_\beta \equiv \hbar\sqrt{\frac{3}{2\varkappa}g\hbar\nu} = p_\beta, \tag{18}$$

  - Non-vanishing C can be compensated by the phase of c_i and a_j's.

- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; it is not within the WKB regime.

- The conclusions also hold for $\mathrm{F}_{\mathbb{i}\nu}(\sigma)$, $\mathrm{G}_{\mathbb{i}\nu}(\sigma)$ for $(+,-)$, and $\mathrm{J}_\nu(\sigma)$ for $(-,+)$.

- To make sense of amplitude and wave packet, an inner product is necessary
- A common starting point is the *Schrödinger product*

$$(\Psi_1, \Psi_2)_{\mathrm{S}} := \int \mathrm{d}\chi\, \Psi_1^*(\beta,\chi)\Psi_2(\beta,\chi); \tag{19}$$

- $(\Psi,\Psi)_{\mathrm{S}}$ is positive-definite, and the integrand $\rho_{\mathrm{S}}(\beta,\chi)$ is non-negative
- The corresponding Schrödinger current is real but does not satisfy continuity equation $\dot{\rho}_{\mathrm{S}} + \nabla\cdot\vec{j}_{\mathrm{S}} = 0$, because eq. (11) is KG-like.
- $\mathrm{K}_{\mathring{\imath}\nu}$[7] for $(+,+)$, $\mathrm{F}_{\mathring{\imath}\nu}$ and $\mathrm{G}_{\mathring{\imath}\nu}$ for $(+,-)$ can be proved to be orthogonal and complete among themselves, as well as can be normalised.
  - $\mathrm{J}_{\mathring{\imath}\nu}$'s for $(+,-)$ are not orthogonal

[7] S. B. Yakubovich. In: *Opuscula Math.* 26.1 (2006), pp. 161–172, A. Passian et al. In: *Journal of Mathematical Analysis and Applications* 360.2 (2009), pp. 380–390, R. Szmytkowski and S. Bielski. In: *Journal of Mathematical Analysis and Applications* 365.1 (2010), pp. 195–197.

Peculiarity of the phantom model $(-,+)$

Orthogonality for mode functions; Hermiticity for operators

- $\mathrm{J}_\nu(\sigma)$'s are not orthogonal under the Schrödinger product

$$(\mathrm{J}_\nu, \mathrm{J}_{\tilde{\nu}})_{\mathsf{S}} \propto \int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{J}_\nu^*(\mathrm{e}^x)\mathrm{J}_{\tilde{\nu}}(\mathrm{e}^x) = \frac{2\sin(\pi(\nu-\tilde{\nu})/2)}{\pi(\nu^2-\tilde{\nu}^2)}, \tag{20}$$

  therefore $\mathbb{D}$ in eq. (13) is not Hermitian (though we do not need it so far)
- $\hat{p}_\chi^2$ is not Hermitian for $\{\mathrm{J}_\nu(\sigma)\}$ under the Schrödinger product

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{J}_\nu^*(-\partial_x^2 \mathrm{J}_{\tilde{\nu}}) - \int_{-\infty}^{+\infty} \mathrm{d}x \, (-\partial_x^2 \mathrm{J}_\nu)^* \mathrm{J}_{\tilde{\nu}} = \frac{2}{\pi} \sin \frac{\pi(\nu-\tilde{\nu})}{2}. \tag{21}$$

- In order to save Hermiticity for p_χ^2 and $\mathbb{D}$ and orthogonality of the modes under Schrödinger product, one can restrict

$$\nu = 2n + \nu_0, \qquad n \in \mathbb{N}, \quad \nu_0 \in [0,2). \tag{22}$$

- The classical trajectory is β-even; imposing the same condition fixes $\nu_0 = 1$.

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Discritisation of the phantom model $(-,+)$
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Levels of the phantom model are discretised

...if one imposes Hermiticity of squared momenta under the Schrödinger product

- This kind of subtlety on Hermiticity is named as *self-adjoint extension*, which arises already for infinite square well[8].
- It also applies to x^{-2} potentials[9], which is of cosmological relevance[10].

8 G. Bonneau, J. Faraut, and G. Valent. In: *American Journal of Physics* 69.3 (2001), pp. 322–331.

9 A. M. Essin and D. J. Griffiths. In: *American Journal of Physics* 74.2 (2006), pp. 109–117, V. S. Araujo, F. A. B. Coutinho, and J. F. Perez. In: *American Journal of Physics* 72.2 (2004), pp. 203–213.

10 M. Bouhmadi-López et al. In: *Physical Review D* 79.12 (2009).

Further inner products for wave functions

Klein–Gordon and Mostafazadeh product

- Since eq. (11) is KG-like, another popular choice is the KG product

$$(\Psi_1, \Psi_2)^g_{\mathsf{KG}} := \mathrm{i} g\Big\{\big(\Psi_1, \dot{\Psi}_2\big)_{\mathsf{S}} - \big(\dot{\Psi}_1, \Psi_2\big)_{\mathsf{S}}\Big\}, \qquad g > 0. \tag{23}$$

  - $(\Psi, \Psi)^g_{\mathsf{KG}}$ is real but not positive-definite, so does the integrand ρ_{KG};
  - The corresponding $\vec{J}_{\mathsf{KG}}$ is conserved $\dot{\rho}_{\mathsf{KG}} + \nabla \cdot \vec{J}_{\mathsf{KG}} = 0$ and real.

- Mostafazadeh[11] found a product *for Hermitian* $\mathbb{D}$ *with positive spectrum*:

$$(\Psi_1, \Psi_2)^\kappa_{\mathsf{M}} := \kappa\Big\{\big(\Psi_1, \mathbb{D}^{+1/2}\Psi_2\big)_{\mathsf{S}} + \big(\dot{\Psi}_1, \mathbb{D}^{-1/2}\dot{\Psi}_2\big)_{\mathsf{S}}\Big\}, \qquad \kappa > 0. \tag{24}$$

  - $(\Psi, \Psi)^\kappa_{\mathsf{M}}$ is positive-definite, but the integrand ρ^κ_{M} is complex
  - The corresponding $\vec{J}^\kappa_{\mathsf{M}}$ is conserved $\dot{\rho}^\kappa_{\mathsf{M}} + \nabla \cdot \vec{J}^\kappa_{\mathsf{M}} = 0$ but also complex.

[11] A. Mostafazadeh. In: *Classical and Quantum Gravity* 20.1 (2002), pp. 155–171.

Mostafazadeh inner product and the corresponding density 233

- The real power of $\mathbb{D}$ is defined by spectral decomposition $\mathbb{D}^{\gamma} := \sum_{\nu} \nu^{2\gamma} \mathbf{P}_{\nu}$, $\mathbf{P}_{\nu} \Psi := \Psi_{\nu} (\Psi_{\nu}, \Psi)_{\mathsf{S}}$.
- It can be shown[12] that *the density*

$$\varrho^{\kappa}_{\mathsf{M}} := \kappa \left\{ \left| \mathbb{D}^{+1/4} \Psi \right|^2 + \left| \mathbb{D}^{-1/4} \dot{\Psi} \right|^2 \right\} \tag{25}$$

  - is equivalent to $\rho^{\kappa}_{\mathsf{M}}$ up to a boundary term $\int \mathrm{d}\chi \, \varrho^{\kappa}_{\mathsf{M}} = \int \mathrm{d}\chi \, \rho^{\kappa}_{\mathsf{M}} \equiv (\Psi_1, \Psi_2)^{\kappa}_{\mathsf{M}}$;
  - is non-negative.
  - The corresponding current $\vec{\mathcal{J}}^{\kappa}_{\mathsf{M}}$ is real but not conserved[13].

[12] A. Mostafazadeh and F. Zamani. In: *Annals of Physics* 321.9 (2006), pp. 2183–2209.
[13] B. Rosenstein and L. P. Horwitz. In: *Journal of Physics A: Mathematical and General* 18.11 (1985), pp. 2115–2121.

Wave packets with Gaussian amplitude for continuous spectrum

Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu; \overline{\nu}, \sigma) := \left(\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(\nu - \overline{\nu})^2}{2\sigma^2} \right) \right)^{1/2} \tag{26}$$

- In[14], $A\left(\nu; \overline{\nu}, \sigma/\sqrt{2}\right)$ was chosen.

14 M. P. Dąbrowski, C. Kiefer, and B. Sandhöfer. In: *Physical Review D* 74.4 (2006).

Wave packets with Gaussian amplitude and the corresponding trajectory

Quintessence model $(+,+)$ with $\mathrm{K}_{i\nu}$

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Wave packets with Gaussian amplitude and the corresponding trajectory

Quintessence model $(+,-)$ with $\mathrm{F}_{\mathbb{i}\nu}$

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Wave packets with Gaussian amplitude and the corresponding trajectory

Quintessence model $(+,-)$ with $\mathrm{G}_{\mathbb{i}\nu}$

Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model $(-,+)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\overline{n}) := \left(\mathbb{e}^{-\overline{n}} \frac{\overline{n}^n}{n!} \right)^{1/2} \tag{27}$$

- In[15], $A_n\left(\overline{n}/\sqrt{2}\right)$ was chosen.

[15] C. Kiefer. In: *Nuclear Physics B* 341.1 (1990), pp. 273–293.

Wave packets with Poissonian amplitude and the corresponding trajectory

Phantom model $(-,+)$ with discrete J_{2n+1}

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Wave packets with Gaussian amplitude and the corresponding trajectory

Phantom model $(-,+)$ with continuous J_ν

Issues
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- A normalising κ for $(\cdot,\cdot)^{\kappa}_{\mathsf{M}}$ has not yet been able to be evaluated, hence a quantitative comparison of $(\cdot,\cdot)_{\mathsf{S}}$ and $(\cdot,\cdot)^{\kappa}_{\mathsf{M}}$ is not yet possible.
- Wave packets with Gaussian profile are to be constructed, instead of inserting Gaussian / Possonian amplitude
- Quantum-corrected $\overline{\nu}$ is to be understood

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Outlook
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- Beyond classic matter: PT-symmetric instead of phantom field
- Beyond homogeneity: cosmological perturbation

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Integration of the transformed first Friedmann equation

$p_\beta \neq 0$

In order to integrate the equation under the implicitising gauge

$$\mathscr{s}\frac{p_\beta^2}{12}\left(-\ell\frac{\varkappa^{1/2}}{6}\left(\frac{\mathrm{d}\chi}{\mathrm{d}\beta}\right)^2+\varkappa^{-1/2}\right)-\varkappa^{3/2}V\mathrm{e}^{g\mathscr{s}_\chi\chi}=0, \tag{28}$$

one can substitute

$$\gamma:=\sqrt{\frac{3}{2\varkappa}}g\beta, \qquad \tilde{\sigma}^2:=\frac{p_\beta^2}{12\varkappa^2|V|}\mathrm{e}^{-g\mathscr{s}_\chi\chi}, \tag{29}$$

to get

$$\left(\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\gamma}\right)^2+\ell\left(\mathscr{s}\upsilon-\tilde{\sigma}^2\right)=0 \quad\Rightarrow\quad \frac{\mathrm{d}\gamma}{\mathrm{d}\tilde{\sigma}}=\pm\frac{1}{\sqrt{\ell\left(-\mathscr{s}\upsilon+\tilde{\sigma}^2\right)}}, \tag{30}$$

which is of the standard inverse hyperbolic / trigonometric form except for $(-,-)$.

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Integration of the separated minisuperspace WDW equation
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In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \qquad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \mathscr{s} v \frac{12\varkappa^2 |V|}{\hbar^2}, \tag{13 rev.}$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{k_\beta}{g}, \qquad \sigma^2 := \frac{8\varkappa^3 |V| \mathrm{e}^{g \mathscr{s}_\chi \chi}}{\hbar^2 g^2}, \tag{15 rev.}$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma \psi'(\sigma) + \ell (\nu^2 - \mathscr{s} v \sigma^2) \psi(\sigma) = 0, \tag{31}$$

which is of the standard Besselian form.

Laden des Themes

Das Theme kann mit den folgenden Optionen geladen werden

```
\usetheme[%
% uk,      %% Farben aller Fakultaeten
wiso,     %% Wiso-Fakultaet
% jura,    %% Rechtswissenschaftliche Fakultaet
% medizin, %% Medizinische Fakultaet
% philo,   %% Philosophische Fakultaet
% matnat,  %% Mathematisch-Naturwissenschaftliche Fakultaet
% human,   %% Humanwissenschaftliche Fakultaet
% verw,    %% Universitaetsverwaltung
]{UzK}
```

block-Umgebungen

Standard (block)

Verwendet die Farbe "'Blaugrau Mittel"' als Blocktitel-Hintergrund

exampleblock

Bei Verwendung der Fußzeile mit allen Fakultätsfarben Titelhintergrund in Wiso-Grün, sonst in der jeweiligen Fakultätsfarbe

alertblock

Verwendet das Rot der Folientitel