

# Integrable Cosmological Models with Liouville Fields

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# Outline

## 1. Introduction

## 2. Classical model and the implicit trajectories

Lagrangian formalism

## 3. Quantised model and the wave packets

Canonical formalism and Dirac quantisation  
Semi-classical approximation  
Inner product and wave packet

## 4. Conclusions



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# Introduction

## Quintessence and phantom Liouville field

- Observed accelerated expansion can be explained by a cosmological constant<sup>1</sup> as stationary Dark Energy, but its origin has yet to be understood.
- Dynamical Dark Energy has been modeled by quintessence<sup>2</sup> and phantom<sup>3</sup> matter, with barotropic index<sup>4</sup>  $w > -1$  and  $w < -1$ , respectively.
- They can be realised by minimally-coupled real scalar fields with  $\ell = \pm 1^5$

$$\mathcal{S} = \int d^4x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \quad (1)$$

- $\mathcal{V}(\phi) = V e^{\lambda\phi}$ ,  $\lambda, V \in \mathbb{R}$  is of interest: Liouville field<sup>6</sup>.

<sup>1</sup> E. J. Copeland, M. Sami, and S. Tsujikawa. In: *International Journal of Modern Physics D* 15.11 (2006), pp. 1753–1935, K. Bamba et al. In: *Astrophysics and Space Science* 342.1 (2012), pp. 155–228.

<sup>2</sup> R. R. Caldwell, R. Dave, and P. J. Steinhardt. In: *Physical Review Letters* 80.8 (1998), pp. 1582–1585.

<sup>3</sup> R. R. Caldwell. In: *Physics Letters B* 545.1-2 (2002), pp. 23–29.

<sup>4</sup> Barotropic index is the  $w$  in equation of state  $\rho = wp$ .

<sup>5</sup> The signature of metric is mostly positive.

<sup>6</sup> Y. Nakayama. In: *International Journal of Modern Physics A* 19.17n18 (2004), pp. 2771–2930.



# Introduction

## Friedmann–Lemaître model

- Assume flat Robertson–Walker metric for a homog. and isotr. model

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + \kappa e^{2\alpha(t)} d\Omega_{3F}^2 \quad (2)$$

w/  $\kappa := 8\pi G$ ,  $d\Omega_{3F}^2 := d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\varphi^2)$ ,  $N$  lapse function.

- Combined with the Liouville field, the total action reads  
 $\mathcal{S} := \mathcal{S}_{EH} + \mathcal{S}_{GHY} + \mathcal{S}_L = \int d\Omega_{3F}^2 \int dt L$ , where

$$L := \kappa^{3/2} N e^{3\alpha} \left( -\frac{3}{\kappa} \frac{\dot{\alpha}^2}{N^2} + \ell \frac{\dot{\phi}^2}{2N^2} - V e^{\lambda\phi} \right), \quad (3)$$

in which dot means  $d/dt$ .

- The model turns out to be integrable, both classically and quantum-mechanically, enabling one to study its full physical properties, e.g. the relation between its classical and quantum theory.



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# Decoupling the variables

Via *rescaled special orthogonal transformation*

- Setting  $\bar{N} := N e^{-3\alpha}$ , eq. (3) transforms to

$$L = \varkappa^{3/2} \bar{N} \left( -\frac{3}{\varkappa} \frac{\dot{\alpha}^2}{\bar{N}^2} + \ell \frac{\dot{\phi}^2}{2\bar{N}^2} - V e^{\lambda\phi+6\alpha} \right) \quad (4)$$

- Defining  $\Delta := \lambda^2 - 6\ell\varkappa$ ,  $s := \operatorname{sgn} \Delta$  and  $g := s\sqrt{|\Delta|} \equiv s\sqrt{s\Delta}$ , the *rescaled special orthogonal transformation*

$$\begin{pmatrix} \alpha \\ \phi \end{pmatrix} = \frac{s}{g} \begin{pmatrix} \lambda & -\ell\varkappa \\ -6 & \lambda \end{pmatrix} \begin{pmatrix} s_\beta \beta \\ s_\chi \chi \end{pmatrix} \quad \text{where } s_\beta, s_\chi = \pm 1 \quad (5)$$

gives the decoupled Lagrangian

$$L = \varkappa^{3/2} \bar{N} \left( -s \frac{3}{\varkappa} \frac{\dot{\beta}^2}{\bar{N}^2} + \ell s \frac{\dot{\chi}^2}{2\bar{N}^2} - V e^{s_\chi g \chi} \right). \quad (6)$$

- The Euler–Lagrange equations w.r.t.  $\bar{N}$ ,  $\beta$  and  $\chi$  will be called the trsfed. 1st, 2nd Friedmann eqs. and the Klein–Gordon eq., respectively.



# Implicit integration

General integral for  $p_\beta \neq 0$

- Since  $\beta$  is cyclic in eq. (6), the trsfed. 2nd Friedmann eq. can be integrated<sup>7</sup>

$$\text{const.} \equiv p_\beta := \frac{\partial L}{\partial \dot{\beta}} = -6s\nu^{1/2} \frac{\dot{\beta}}{\bar{N}} \equiv -6s s_\beta \frac{\nu^{1/2}}{g} \frac{\lambda \dot{\alpha} + \ell \nu \dot{\phi}}{\bar{N}}. \quad (7)$$

- For  $p_\beta \neq 0$ , fixing the *implicit gauge*  $\bar{N} = -6s\sqrt{\nu}\dot{\beta}/p_\beta$ , the trsfed. 1st Friedmann equation **can be integrated to get the trajectory**

$$e^{\beta_X g\chi} = \frac{p_\beta^2}{12\nu^2|V|} S^2 \left( s_\beta \sqrt{\frac{3}{2\nu}} g\beta + C \right), \quad (8)$$

in which  $\nu := \operatorname{sgn} V$ ,  $(\operatorname{sgn}, \operatorname{sgn})$  means  $(\ell, s\nu)$ , and

$$\begin{array}{ll} (+,+)\mathcal{S}(\gamma) := \operatorname{sech}(\gamma), & (+,-)\mathcal{S}(\gamma) := \operatorname{csch}(\gamma), \\ (-,+)\mathcal{S}(\gamma) := \operatorname{sec}(\gamma), & (-,-)\mathcal{S}(\gamma) := i \operatorname{csc}(\gamma). \end{array} \quad (9)$$

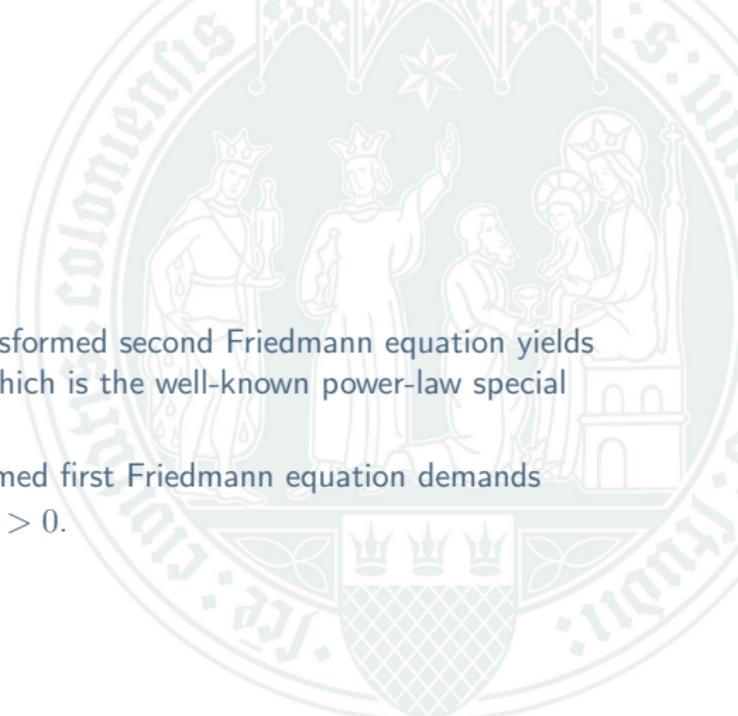
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<sup>7</sup>The same first integral has been found in C. Lan. PhD thesis. Saint Petersburg State University, 2016, A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theoretical and Mathematical Physics* 184.3 (2015), pp. 1224–1233, in canonical formalism.



# Implicit integration

Specific integral for  $p_\beta = 0$



- For  $p_\beta = 0$ , integrating the transformed second Friedmann equation yields  $\beta \equiv \beta_0$  or  $\phi - \phi_0 = -\ell \lambda \alpha / \nu$ , which is the well-known power-law special solution<sup>8</sup>.
- Further integrating the transformed first Friedmann equation demands  $(+, -)$  or  $(-, +)$  to guarantee  $\bar{N} > 0$ .

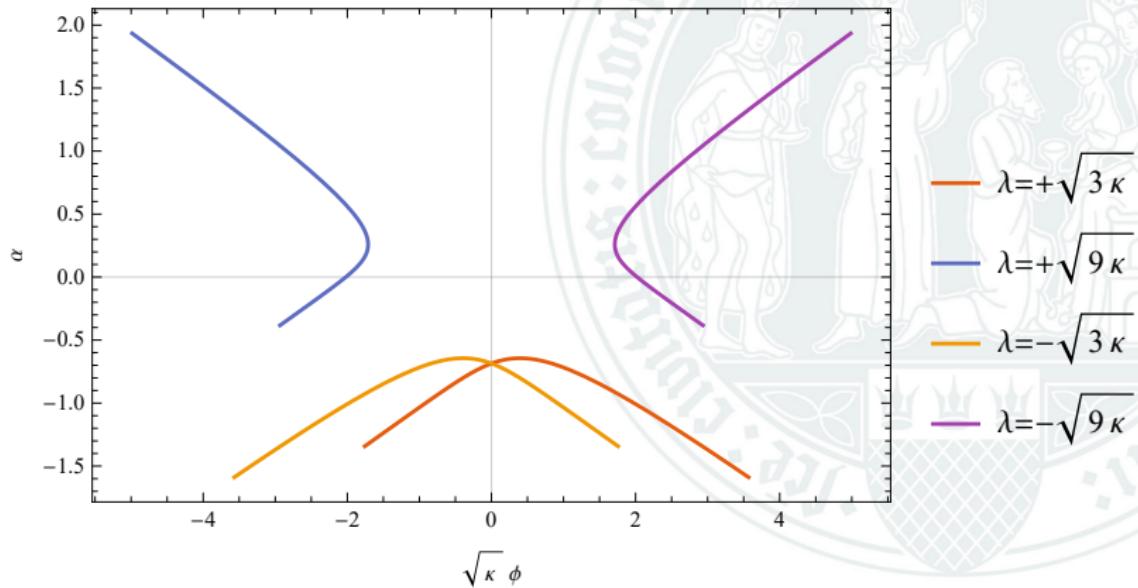
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<sup>8</sup>For instance, A. R. Liddle and D. H. Lyth. Cambridge University Press, 2000, ch. 3.



# Trajectories for quintessence model $(+, +)$

$\text{sech}$ , with  $C = 0$ ,  $|V| = \kappa^{-2}$  and  $p_\beta^2 = \kappa^2 \sqrt{|V|}$ ; varying  $\lambda$

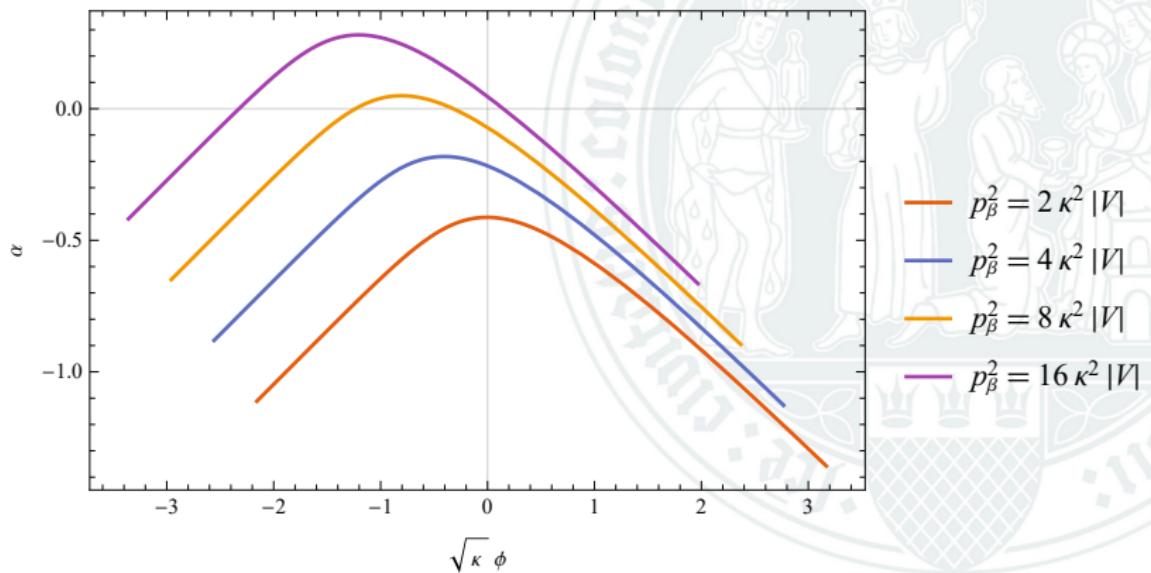


- has two asymptotes  $\chi \propto \pm \beta$



# Trajectories for quintessence model $(+, +)$

$\text{sech}$ , with  $C = 0$ ,  $|V| = \nu^{-2}$  and  $\lambda^2 = 3\nu$ ; varying  $p_\beta$

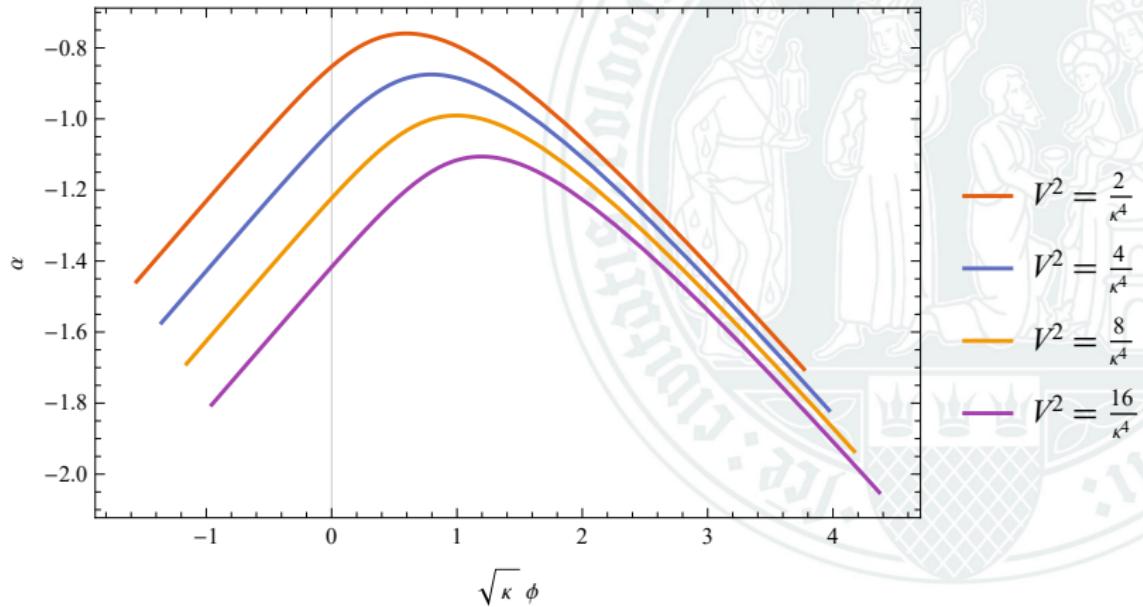


- has two asymptotes  $\chi \propto \pm\beta$



# Trajectories for quintessence model $(+, +)$

sech, with  $C = 0$ ,  $\lambda^2 = 3\nu$  and  $p_\beta^2 = \nu^2 \sqrt{|V|}$ ; varying  $V$

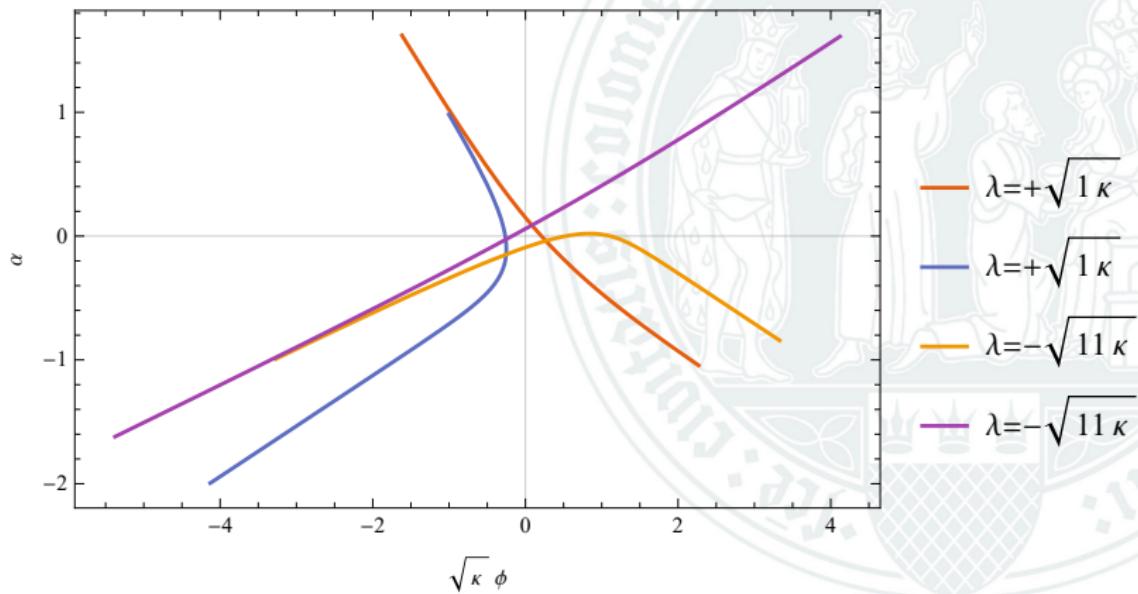


- has two asymptotes  $\chi \propto \pm \beta$



# Trajectories for quintessence model $(+, -)$

$\text{csch}$ , with  $C = 0$ ,  $|V| = \kappa^{-2}$  and  $p_\beta^2 = \kappa^2 \sqrt{|V|}$ ; varying  $\lambda$

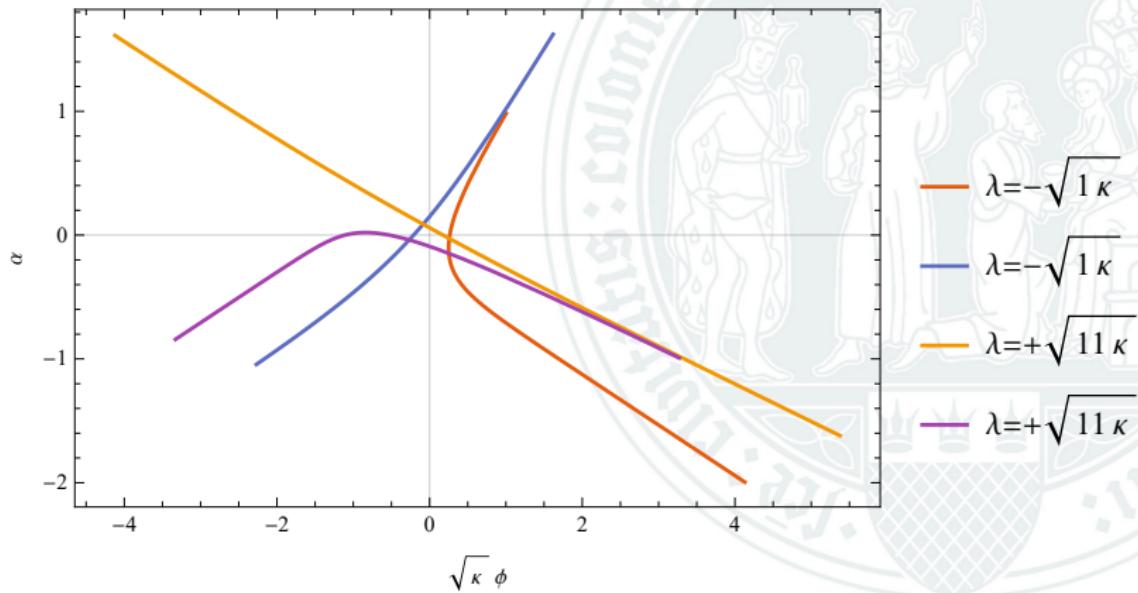


- contains two distinct solutions, separated by  $\beta = 0$
- has three asymptotes  $\chi \propto \pm\beta$  and  $\beta = 0$



# Trajectories for quintessence model $(+, -)$ : csch

csch, with  $C = 0$ ,  $|V| = \kappa^{-2}$  and  $p_\beta^2 = \kappa^2 \sqrt{|V|}$ ; varying  $\lambda$

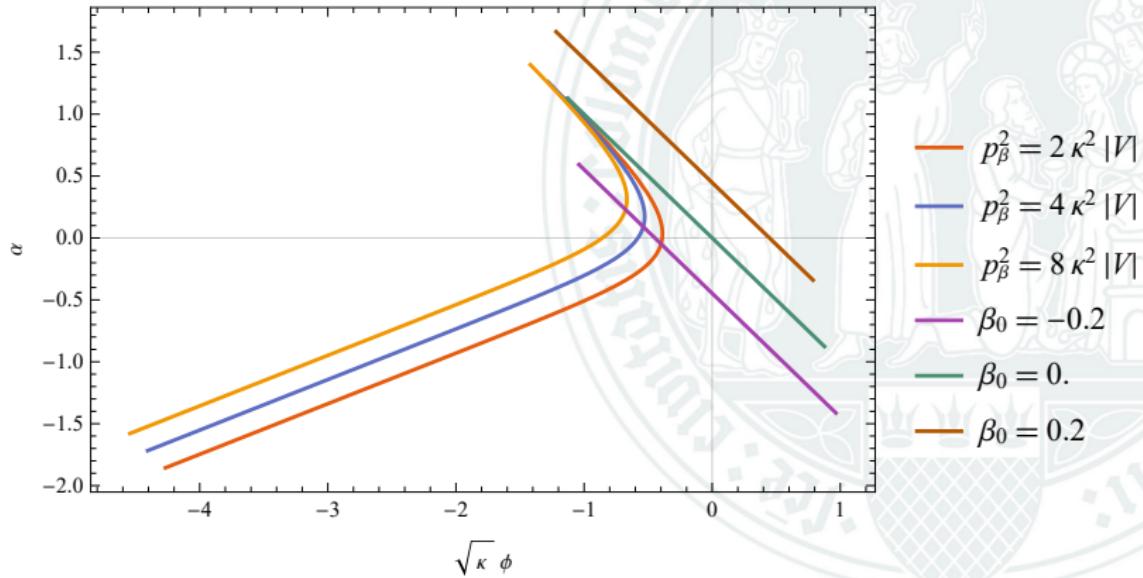


- contains two distinct solutions, separated by  $\beta = 0$
- has three asymptotes  $\chi \propto \pm\beta$  and  $\beta = 0$



# Trajectories for quintessence model (+, -): csch

csch, with  $V = \kappa^{-2}$  and  $\lambda^2 = \kappa$ ; varying  $p_\beta$

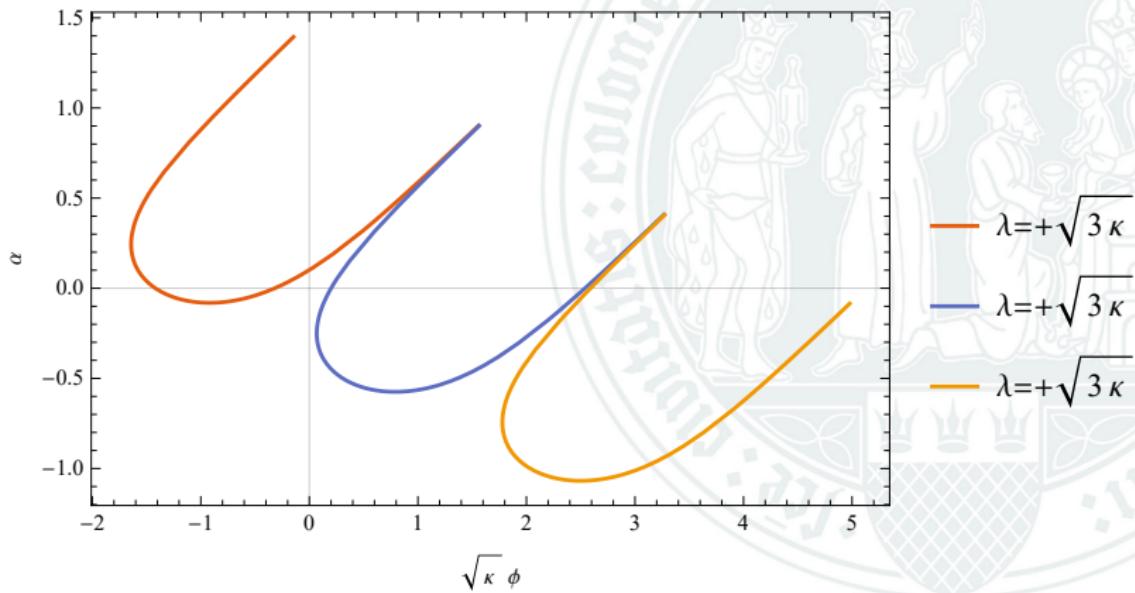


- contains two distinct solutions, separated by  $\beta = 0$
- has three asymptotes  $\chi \propto \pm\beta$  and  $\beta = 0$



# Trajectories for phantom model $(-, +)$

csc, with  $V = \kappa^{-2}$  and  $p_\beta^2 = 3\kappa^2 \sqrt{|V|}$ ; varying  $\lambda$

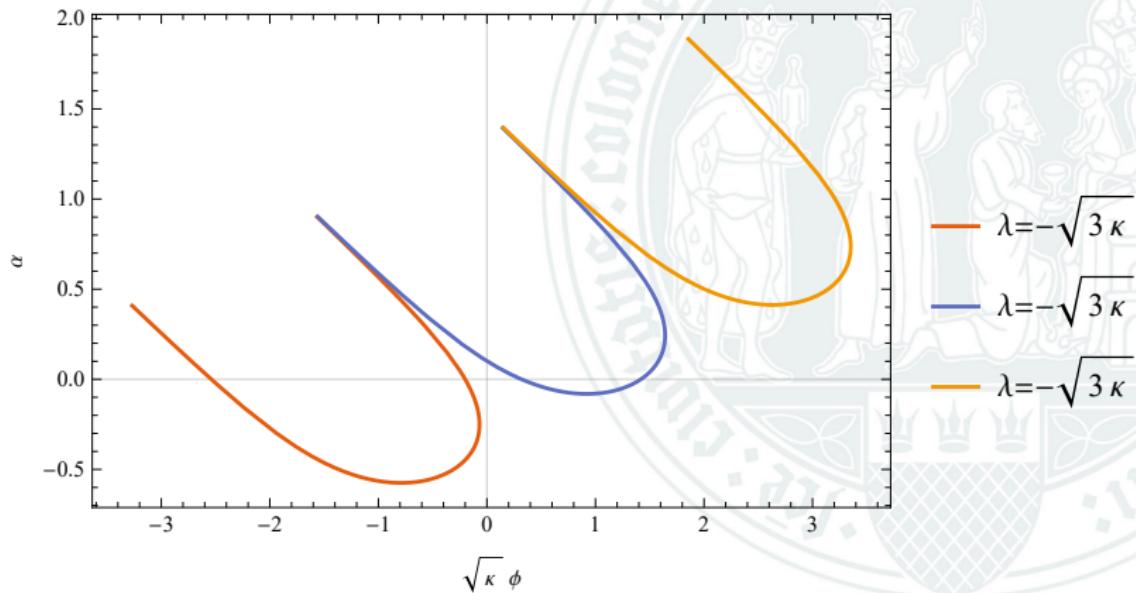


- contains infinite distinct solutions
- has infinite parallel asymptotes  $\beta \propto (n + 1/2)\pi$



# Trajectories for phantom model $(-, +)$

csc, with  $V = \kappa^{-2}$  and  $p_\beta^2 = 3\kappa^2 \sqrt{|V|}$ ; varying  $\lambda$

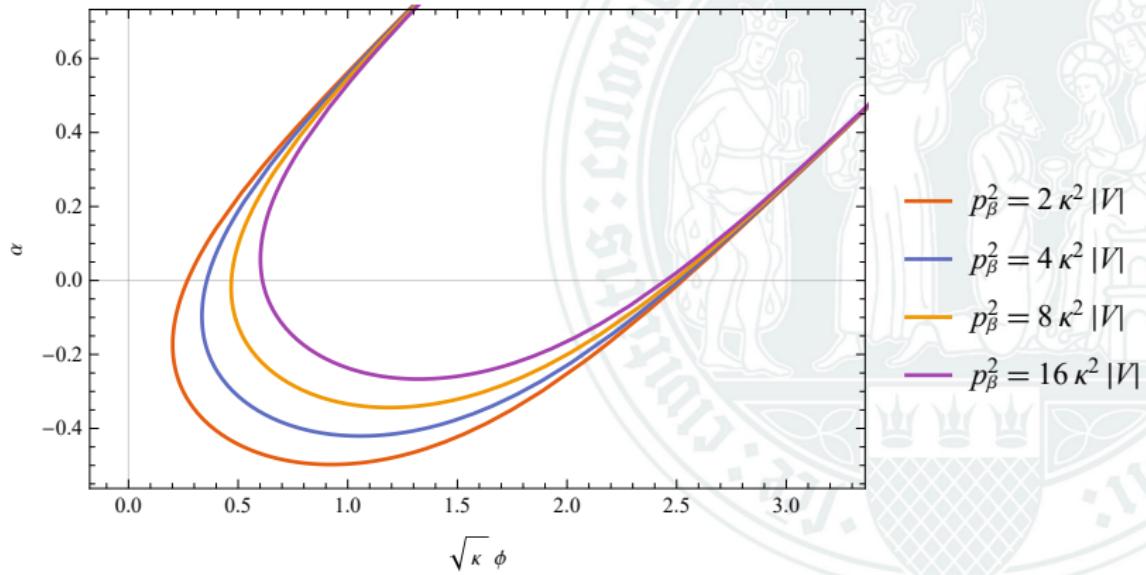


- contains infinite distinct solutions
- has infinite parallel asymptotes  $\beta \propto (n + 1/2)\pi$



# Trajectories for phantom model $(-, +)$

csc, with  $V = \kappa^{-2}$  and  $\lambda^2 = 3\kappa$ ; varying  $p_\beta$



- contains infinite distinct solutions
- has infinite parallel asymptotes  $\beta \propto (n + 1/2)\pi$



# Integration

## Further discussions

- The integral for  $(-, -)$  is not real.
- The trajectories can be parametrised by  $\beta$ , inspiring recognising  $\beta$  as a 'time variable'.
- The implicit integration enables one to compare trajectories with wave functions, see below.



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# Dirac quantisation and the mode functions

- The primary Hamiltonian and the Hamiltonian constraint<sup>9</sup> reads

$$H^p = \bar{N} H_{\perp} + v^{\bar{N}} p_{\bar{N}}, \quad (10)$$

$$H_{\perp} := -\mathcal{S} \frac{p_{\beta}^2}{12\kappa^{1/2}} + \ell \mathcal{S} \frac{p_{\chi}^2}{2\kappa^{3/2}} + \kappa^{3/2} V e^{g_{\beta}\chi}. \quad (11)$$

- Applying the Dirac quantisation rules with Laplace–Beltrami opeator<sup>10</sup>, one gets the minisuperspace Wheeler–DeWitt equation with  $(\beta, \chi)$

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left( \mathcal{S} \frac{\hbar^2}{12\kappa^{1/2}} \partial_{\beta}^2 - \ell \mathcal{S} \frac{\hbar^2}{2\kappa^{3/2}} \partial_{\chi}^2 + \kappa^{3/2} V e^{g_{\beta}\chi} \right) \Psi. \quad (12)$$

- Equation (12) is KG-like, hyperbolic for  $\ell = +1$  and *elliptic* for  $\ell = -1$ .

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<sup>9</sup> D. M. Gitman and I. V. Tyutin. Springer Berlin Heidelberg, 1990, H. J. Rothe and K. D. Rothe. WORLD SCIENTIFIC, 2010.

<sup>10</sup> C. Kiefer. 3rd ed. Oxford University Press, 2012, ch. 8.



## Separation of the variables and mode functions

- Writing  $\Psi(\beta, \chi) = \varphi(\beta)\psi(\chi)$ , eq. (12) can be separated into

$$\partial_\beta^2 \varphi(\beta) = k_\beta^2 \varphi(\beta); \quad (13)$$

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \mathfrak{s} v \frac{12\varkappa^2 |V|}{\hbar^2}. \quad (14)$$

- Equation (14) turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta, \chi) := \sum_{i=1}^2 c_i \varphi_\nu^{(i)}(\gamma) \sum_{j=1}^2 a_j B_\nu^{(j)}(\sigma), \quad \nu \geq 0; \quad (15)$$

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{1}{g} k_\beta, \quad \gamma := \sqrt{\frac{3}{2\varkappa}} \frac{g}{1} \beta, \quad \sigma^2 := \frac{8\varkappa^3 |V| e^{g\beta_X \chi}}{\hbar^2 g^2}, \quad (16)$$

$${}_{(+,+)} B_\nu^{(i)}(\sigma) := K \text{ and } I_{\mathbb{B}\nu}(\sigma), \quad {}_{(+,-)} B_\nu^{(i)}(\sigma) := F \text{ and } G_{\mathbb{B}\nu}(\sigma),$$

$${}_{(-,+)} B_\nu^{(i)}(\sigma) := J \text{ and } Y_\nu(\sigma), \quad {}_{(-,-)} B_\nu^{(i)}(\sigma) := K \text{ and } I_\nu(\sigma).$$

- Adapted to imaginary order,  $F_\nu(\sigma)$  and  $G_\nu(\sigma)$  are defined in<sup>11</sup>.

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<sup>11</sup> T. M. Dunster. In: *SIAM Journal on Mathematical Analysis* 21.4 (1990), pp. 995–1018.



# Physical mode functions

- Physical mode functions are expected to be regular on the boundary.
- $(+, +)$ :  $|I_{i\nu}(\sigma)| \rightarrow +\infty$  as  $\alpha \rightarrow +\infty$
- $(-, +)$ :
  - $\forall n \in \mathbb{N}$ ,  $|Y_n(\sigma)| \rightarrow +\infty$  as  $\alpha \rightarrow -\infty$ .
  - $\forall \nu \in \mathbb{R}^+ \setminus \mathbb{N}$ , choose  $J_{-\nu}$  instead of  $Y_\nu$ , since  $J_{\pm\nu}$  are also linearly independent.
  - $\forall \nu \in \mathbb{R}^+ \setminus \mathbb{N}$ ,  $|J_{-\nu}(\sigma)| \rightarrow +\infty$  as  $\alpha \rightarrow -\infty$ .
- $(-, -)$ :  $|K_\nu(\sigma)| \rightarrow +\infty$  as  $\alpha \rightarrow -\infty$ ;  $|I_\nu(\sigma)| \rightarrow +\infty$  as  $\alpha \rightarrow +\infty$
- These are not to be included in the space of physical wave functions.  
 $\forall \nu \geq 0$ ,
  - $(+, +)$ :  $K_{i\nu}(\sigma)$  survives
  - $(+, -)$ :  $F_{i\nu}(\sigma)$  and  $G_{i\nu}(\sigma)$  survives
  - $(-, +)$ :  $J_\nu(\sigma)$  survives
  - $(-, -)$ : drops out



# Matching quantum number with classical first integral

## Principle of constructive interference

- Write the mode function in the WKB form

$$\Psi_{k_\beta}(\beta, \chi) \sim \sqrt{\rho(\beta, \chi)} \exp\left\{ \frac{i}{\hbar} S(\beta, \chi) \right\}. \quad (17)$$

- For  $S/\hbar \gg 1$  and  $k_\beta \gg 1^{12}$ ,  $S(\beta, \chi)$  becomes the Hamilton principle function in the leading-order approximation.
- A Hamilton principle function is stationary with respect to variation of integral constants<sup>13</sup>

$$\frac{\partial S}{\partial k_\beta} = 0. \quad (18)$$

- Demanding eq. (18) matching the classical trajectory,  $k_\beta$  can be related to  $p_\beta$ .

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<sup>12</sup>The common form is  $\hbar \rightarrow 0^+$ .

<sup>13</sup> U. H. Gerlach. In: *Physical Review* 177.5 (1969), pp. 1929–1941, L. D. Landau and E. M. Lifshitz. 3rd ed. Elsevier, 1976.



# Matching quantum number with classical first integral

(+, +) as exemplar

- Fixing  $\nu/\sigma > 1$ , the asymptotic expansion reads

$$K_{\pm\nu}(\sigma) \sim \frac{\sqrt{2\pi} \cos\left(\sqrt{\nu^2 - \sigma^2} - \nu \arccos \frac{\nu}{\sigma} - \frac{\pi}{4}\right)}{(\nu^2 - \sigma^2)^{1/4} e^{\mp\nu/2}} + O\left(\frac{1}{\sigma}\right). \quad (19)$$

- There are two phases with opposite signs. Assuming  $c_i, a_j$ 's are real and applying *the principle* to  $\Psi_\nu(\sigma)$ , one has  $\sigma/\nu = \operatorname{sech}(\beta_\beta \gamma)$ , which matches the trajectory with  $C = 0$  if

$$\hbar k_\beta \equiv \hbar \sqrt{\frac{3}{2\nu}} g \hbar \nu = p_\beta, \quad (20)$$

- Non-vanishing  $C$  can be compensated by the phase of  $c_i$  and  $a_j$ 's.
- Fixing  $\nu/\sigma < 1$ , the asymptotic expansion is not oscillatory, but exponential; it is not within the WKB regime.
- The conclusions also hold for  $F_{\pm\nu}(\sigma), G_{\pm\nu}(\sigma)$  for  $(+,-)$ , and  $J_\nu(\sigma)$  for  $(-,+)$ .



# Inner product for wave functions

## Schrödinger product

- To make sense of amplitude and wave packet, an inner product is necessary
- In terms of a Klein–Gordon equation, call  $\beta$  the “temporal” variable, and  $\chi$  the “spacial” variable.
- A common starting point is the *Schrödinger product*<sup>14</sup>

$$(\Psi_1, \Psi_2)_S := \int d\chi \Psi_1^*(\beta, \chi) \Psi_2(\beta, \chi); \quad (21)$$

- $(\Psi, \Psi)_S$  is **positive-definite**, and the integrand  $\rho_S(\beta, \chi)$  is **non-negative**
- The corresponding Schrödinger current is **real** but **does not satisfy continuity equation**  $\dot{\rho}_S + \nabla \cdot \vec{j}_S = 0$ , because eq. (12) is KG-like.
- $K_{i\nu}^{15}$  for  $(+, +)$ ,  $F_{i\nu}$  and  $G_{i\nu}$  for  $(+, -)$  can be proved to be **orthogonal** and **complete** among themselves, as well as **can be normalised**.
  - $J_{i\nu}$ 's for  $(+, -)$  are **not orthogonal**

<sup>14</sup> C. Kiefer. 3rd ed. Oxford University Press, 2012, ch. 5.

<sup>15</sup> S. B. Yakubovich. In: *Opuscula Math.* 26.1 (2006), pp. 161–172, A. Passian et al. In: *Journal of Mathematical Analysis and Applications* 360.2 (2009), pp. 380–390, R. Szmytkowski and S. Bielski. In: *Journal of Mathematical Analysis and Applications* 365.1 (2010), pp. 195–197.



# Peculiarity of the phantom model $(-, +)$

Orthogonality for mode functions; Hermiticity for operators

- $J_\nu(\sigma)$ 's are not orthogonal under the Schrödinger product

$$(J_\nu, J_{\tilde{\nu}})_S \propto \int_{-\infty}^{+\infty} dx J_\nu^*(e^x) J_{\tilde{\nu}}(e^x) = \frac{2\sin(\pi(\nu - \tilde{\nu})/2)}{\pi(\nu^2 - \tilde{\nu}^2)}, \quad (22)$$

therefore  $\mathbb{D}$  in eq. (14) is not Hermitian (though we do not need it so far)

- $\hat{p}_\chi^2$  is not Hermitian for  $\{J_\nu(\sigma)\}$  under the Schrödinger product

$$\int_{-\infty}^{+\infty} dx J_\nu^*(-\partial_x^2 J_{\tilde{\nu}}) - \int_{-\infty}^{+\infty} dx (-\partial_x^2 J_\nu)^* J_{\tilde{\nu}} = \frac{2}{\pi} \sin \frac{\pi(\nu - \tilde{\nu})}{2}. \quad (23)$$

- In order to save Hermiticity for  $p_\chi^2$  and  $\mathbb{D}$  and orthogonality of the modes under Schrödinger product, one can restrict

$$\nu = 2n + \nu_0, \quad n \in \mathbb{N}, \quad \nu_0 \in [0, 2]. \quad (24)$$

- Using classical singularities as boundary condition, one can fix  $\nu_0 = 1$ .



## Discretisation of the phantom model $(-, +)$

Levels of the phantom model are **discretised** if one imposes Hermiticity of squared momenta under the Schrödinger product.

- This kind of subtlety on Hermiticity is named as *self-adjoint extension*, which arises already for infinite square well<sup>16</sup>.
- It also applies to  $x^{-2}$  potentials<sup>17</sup>, which is of cosmological relevance<sup>18</sup>.

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<sup>16</sup> G. Bonneau, J. Faraut, and G. Valent. In: *American Journal of Physics* 69.3 (2001), pp. 322–331.

<sup>17</sup> A. M. Essin and D. J. Griffiths. In: *American Journal of Physics* 74.2 (2006), pp. 109–117,  
V. S. Araujo, F. A. B. Coutinho, and J. F. Perez. In: *American Journal of Physics* 72.2 (2004),  
pp. 203–213.

<sup>18</sup> M. Bouhmadi-López et al. In: *Physical Review D* 79.12 (2009).



# Further inner products for wave functions

## Klein–Gordon and Mostafazadeh product

- Since eq. (12) is KG-like, another popular choice is the KG product

$$(\Psi_1, \Psi_2)_{\text{KG}}^g := \mathbb{i}g \left\{ (\Psi_1, \partial_\beta \Psi_2)_S - (\partial_\beta \Psi_1, \Psi_2)_S \right\}, \quad g > 0. \quad (25)$$

- $(\Psi, \Psi)_{\text{KG}}^g$  is **real** but **not positive-definite**, so does the integrand  $\rho_{\text{KG}}$ ;
- The corresponding  $\vec{J}_{\text{KG}}$  is **conserved**  $\dot{\rho}_{\text{KG}} + \nabla \cdot \vec{J}_{\text{KG}} = 0$  and **real**.
- Mostafazadeh<sup>19</sup> found a product *for Hermitian  $\mathbb{D}$  with positive spectrum*:

$$(\Psi_1, \Psi_2)_{\text{M}}^\kappa := \kappa \left\{ (\Psi_1, \mathbb{D}^{+1/2} \Psi_2)_S + (\partial_\beta \Psi_1, \mathbb{D}^{-1/2} \partial_\beta \Psi_2)_S \right\}, \quad \kappa > 0. \quad (26)$$

- $(\Psi, \Psi)_{\text{M}}^\kappa$  is **positive-definite**, but the integrand  $\rho_{\text{M}}^\kappa$  is **complex**
- The corresponding  $\vec{J}_{\text{M}}^\kappa$  is **conserved**  $\dot{\rho}_{\text{M}}^\kappa + \nabla \cdot \vec{J}_{\text{M}}^\kappa = 0$  but also **complex**.

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<sup>19</sup> A. Mostafazadeh. In: *Classical and Quantum Gravity* 20.1 (2002), pp. 155–171.



# Mostafazadeh inner product and the corresponding density

- Real power of  $\mathbb{D}$  is defined by spectral decomposition  $\mathbb{D}^\gamma := \sum_\nu \nu^{2\gamma} \mathbf{P}_\nu$ ,  
 $\mathbf{P}_\nu \Psi := \Psi_\nu (\Psi_\nu, \Psi)_S$ .
- It can be shown<sup>20</sup> that *the density*

$$\varrho_M^\kappa := \kappa \left\{ |\mathbb{D}^{+1/4} \Psi|^2 + |\mathbb{D}^{-1/4} \partial_\beta \Psi|^2 \right\} \quad (27)$$

- is equivalent to  $\rho_M^\kappa$  up to a boundary term

$$\int d\chi \varrho_M^\kappa = \int d\chi \rho_M^\kappa \equiv (\Psi_1, \Psi_2)_M^\kappa; \quad (28)$$

- is non-negative.
- The corresponding current  $\vec{\mathcal{J}}_M^\kappa$  is real but not conserved<sup>21</sup>.

<sup>20</sup> A. Mostafazadeh and F. Zamani. In: *Annals of Physics* 321.9 (2006), pp. 2183–2209.

<sup>21</sup> B. Rosenstein and L. P. Horwitz. In: *Journal of Physics A: Mathematical and General* 18.11 (1985), pp. 2115–2121.



# Wave packets of Gaussian amplitude for continuous spectrum

## Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu; \bar{\nu}, \sigma) := \left( \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\nu - \bar{\nu})^2}{2\sigma^2}\right) \right)^{1/2} \quad (29)$$

- In<sup>22</sup>,  $A(\nu; \bar{\nu}, \sigma/\sqrt{2})$  was chosen.

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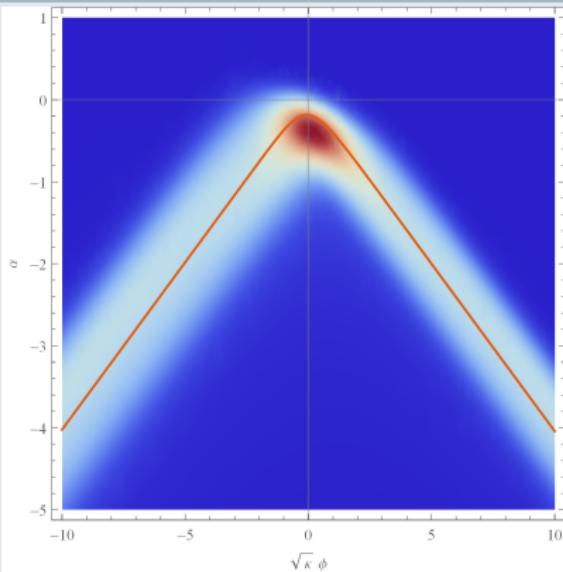
<sup>22</sup> M. P. Dąbrowski, C. Kiefer, and B. Sandhöfer. In: *Physical Review D* 74.4 (2006).



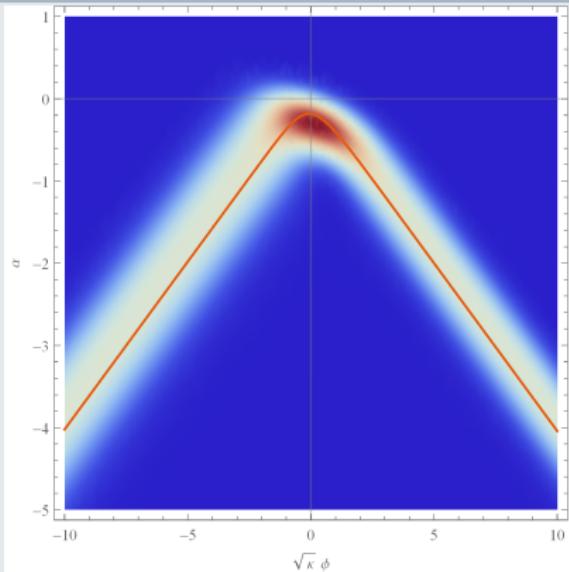
# Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\parallel\nu}$ , with  $\lambda = \kappa^{1/2}/2$ ,  $V = -\kappa^{-2}$ ,  $\bar{k}_\beta = -2$  and  $\sigma_\beta = 5/4$

Schödinger



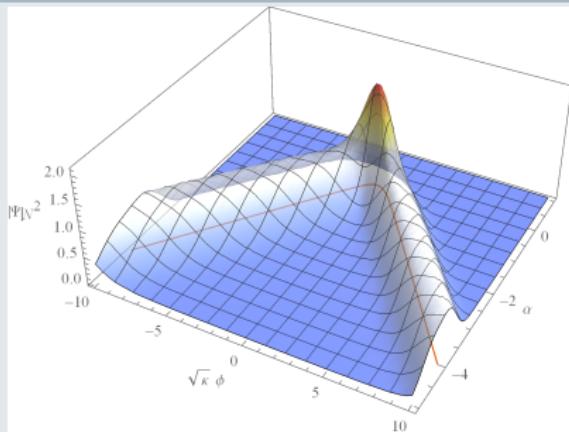
Mostafazadeh



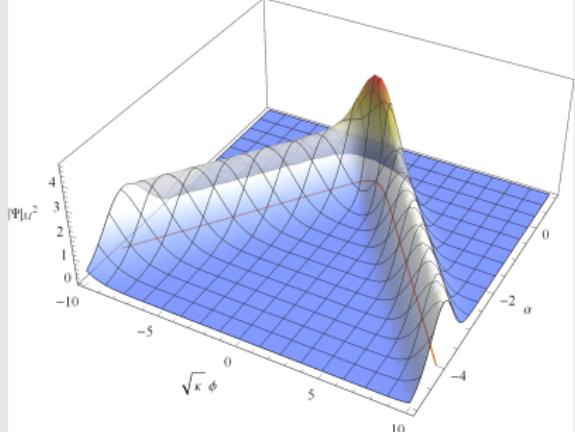
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Schödinger



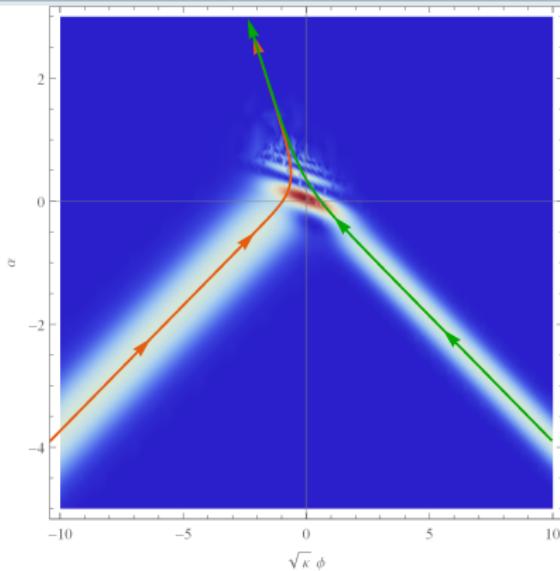
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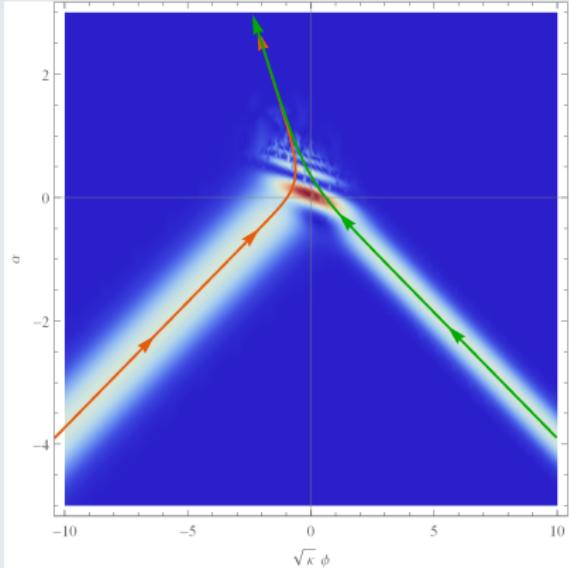
# Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\text{I}\nu}$ , with  $\lambda = 4\kappa^{1/2}/5$ ,  $V = +\kappa^{-2}$ ,  $\bar{k}_\beta = -7/2$  and  $\sigma_\beta = 7/5$

Schödinger



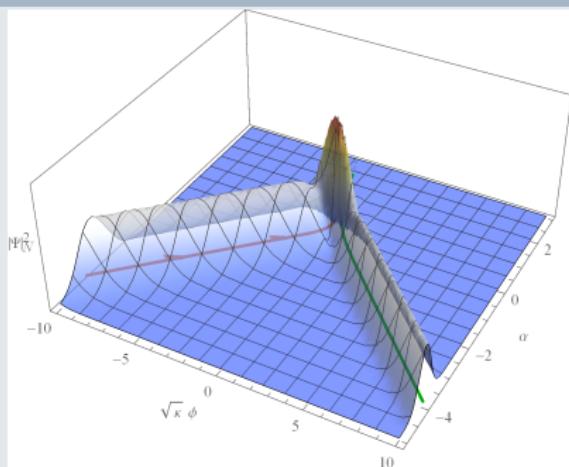
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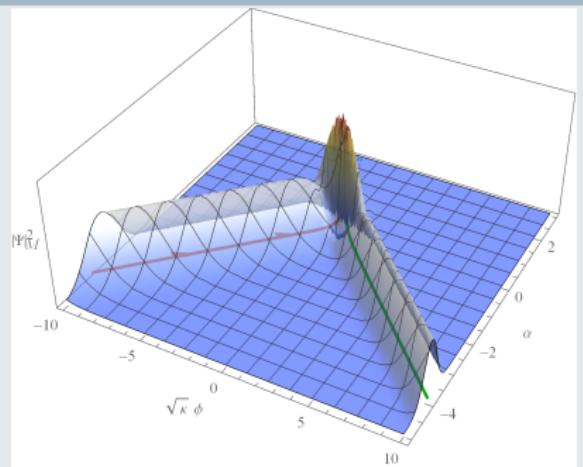
# Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\text{I}\nu}$ , with  $\lambda = 4\kappa^{1/2}/5$ ,  $V = +\kappa^{-2}$ ,  $\bar{k}_\beta = -7/2$  and  $\sigma_\beta = 7/5$

Schödinger



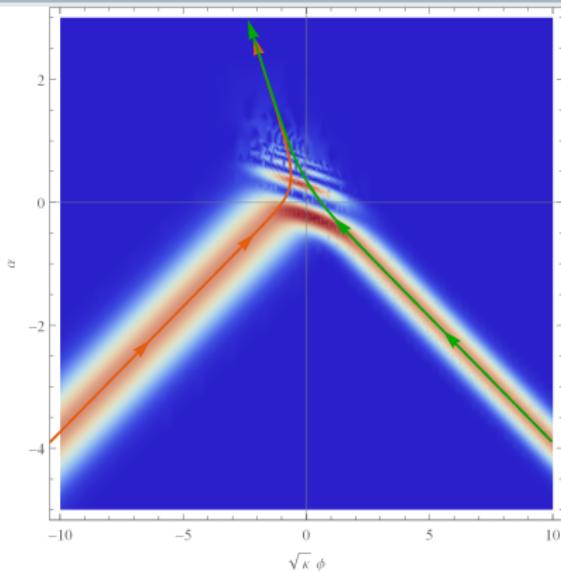
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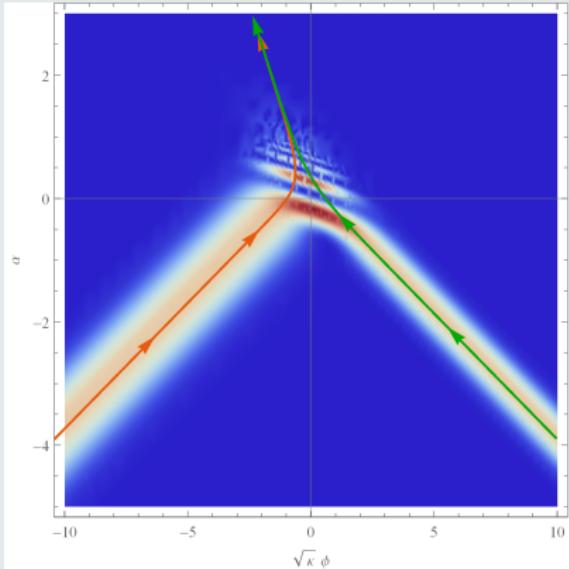
# Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\text{I}\nu}$ , with  $\lambda = 4\kappa^{1/2}/5$ ,  $V = +\kappa^{-2}$ ,  $\bar{k}_\beta = -7/2$  and  $\sigma_\beta = 7/5$

Schödinger



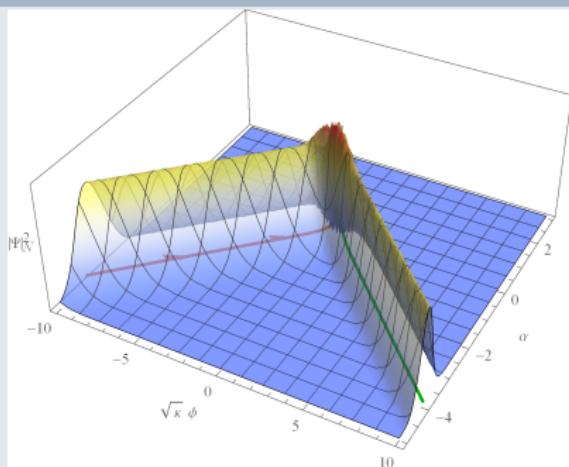
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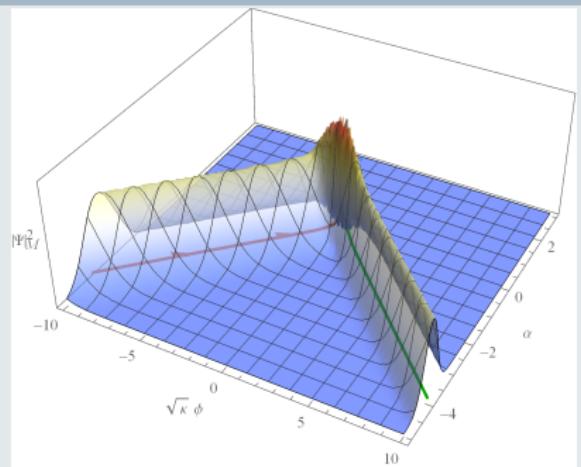
# Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\mu\nu}$ , with  $\lambda = 4\kappa^{1/2}/5$ ,  $V = +\kappa^{-2}$ ,  $\bar{k}_\beta = -7/2$  and  $\sigma_\beta = 7/5$

Schödinger



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# Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model  $(-, +)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\bar{n}) := \left( e^{-\bar{n}} \frac{\bar{n}^n}{n!} \right)^{1/2} \quad (30)$$

- In<sup>23</sup>,  $A_n(\bar{n}/\sqrt{2})$  was chosen.

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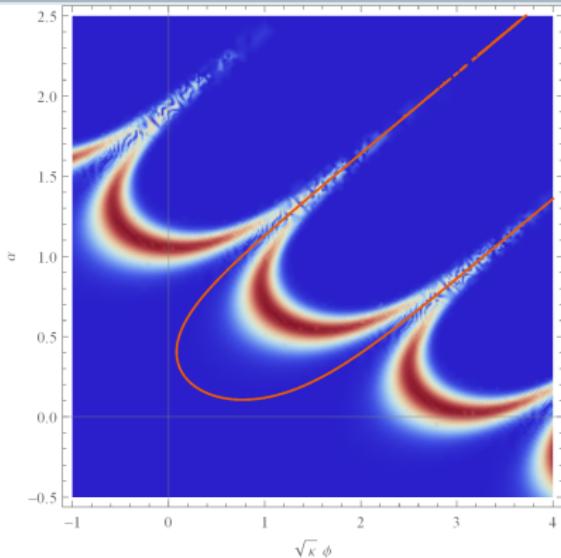
<sup>23</sup> C. Kiefer. In: *Nuclear Physics B* 341.1 (1990), pp. 273–293.



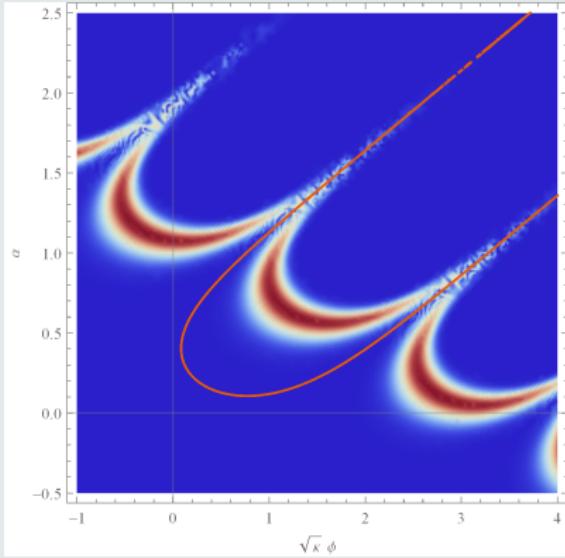
# Wave packets of Poissonian amplitude for phantom model

$J_{2n+1}$ , with  $\lambda = 2\kappa^{1/2}$ ,  $V = +\kappa^{-2}$  and  $\bar{k}_\beta = 8$

Schödinger



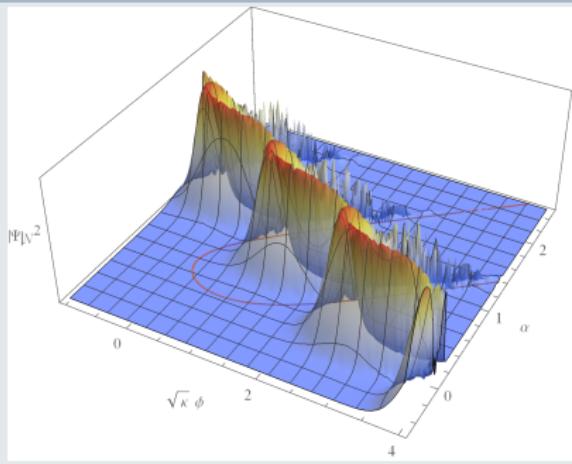
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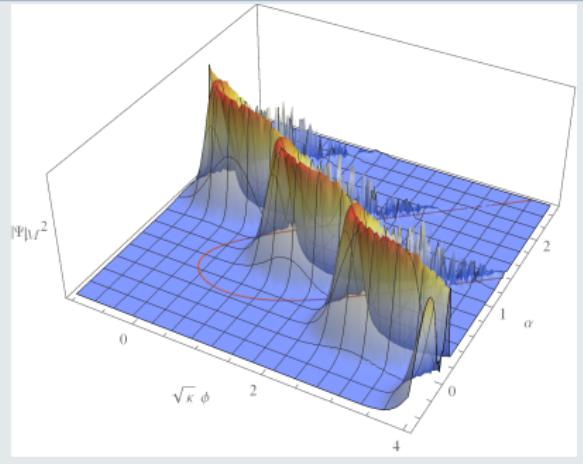
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Schödinger



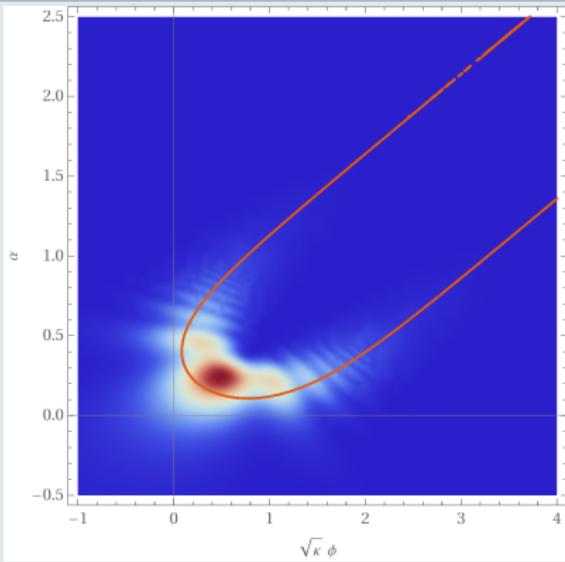
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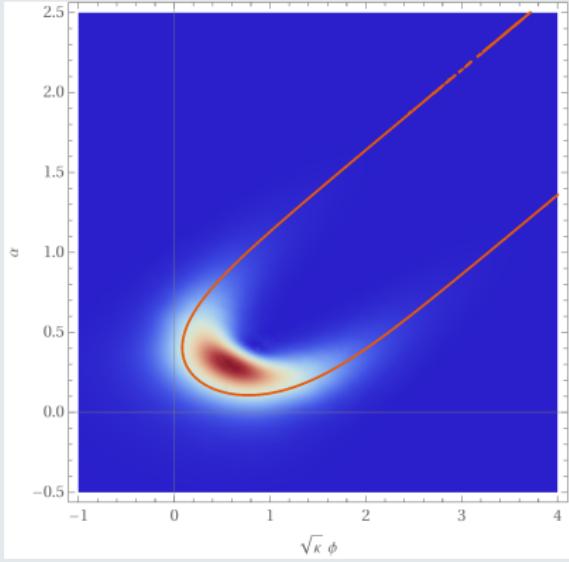
# Wave packets of Gaussian amplitude for phantom model

$J_\nu$ , with  $\lambda = 2\nu^{1/2}$ ,  $V = +\nu^{-2}$ ,  $\bar{k}_\beta = 8$  and  $\sigma_\beta = 11/2$

Schödinger



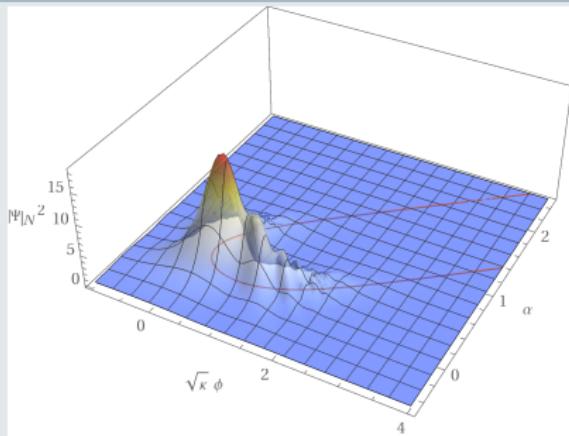
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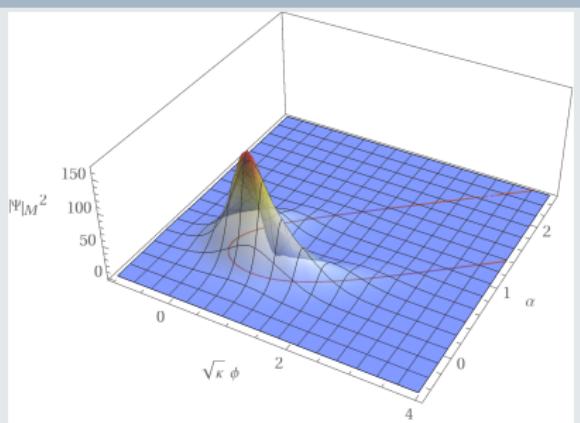
# Wave packets of Gaussian amplitude for phantom model

$J_\nu$ , with  $\lambda = 2\nu^{1/2}$ ,  $V = +\nu^{-2}$ ,  $\bar{k}_\beta = 8$  and  $\sigma_\beta = 11/2$

Schödinger



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# Outline

## 1. Introduction

## 2. Classical model and the implicit trajectories

Lagrangian formalism

## 3. Quantised model and the wave packets

Canonical formalism and Dirac quantisation  
Semi-classical approximation  
Inner product and wave packet

## 4. Conclusions



## Highlights

- An **integral of motion** was found for Liouville cosmological models.
- **Implicit trajectories** in minisuperspace were obtained.
- The levels of phantom Liouville model were found to be **discrete** due to the Hermiticity requirement of observables.
- The semi-classical quantum mode functions and wave packets was compared directly with the classical trajectory.



## Issues

- In  $(+, -)$  and  $(-, +)$ , wave packets contain multiple branches; however, the classical universe runs only on one trajectory.
- Quantum-corrected  $\bar{k}_\beta$  is to be understood.
- Wave packets with Gaussian profile are to be constructed, instead of inserting Gaussian / Poissonian amplitude artificially.
- A normalising  $\kappa$  for  $(\cdot, \cdot)_M^\kappa$  is to be evaluated, otherwise a quantitative comparison of  $(\cdot, \cdot)_S$  and  $(\cdot, \cdot)_M^\kappa$  is not possible.



# Outlook

- Beyond isotropy: generalise to Bianchi models
  - Bianchi Type-I: a natural extension, **under investigation**
- Beyond homogeneity: cosmological perturbation, **under investigation**
- Beyond single field
  - Two exponential potentials:  $|V_1| = |V_2|$  and special  $\lambda_i$
  - Multiple Liouville fields: mixing kinetic terms needed<sup>24</sup>
- Beyond classic matter
  - $PT$ -symmetric Liouville field<sup>25</sup>: may cross the phantom divide  $w = -1$ .

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<sup>24</sup> A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theoretical and Mathematical Physics* 184.3 (2015), pp. 1224–1233.

<sup>25</sup> A. A. Andrianov et al. In: *International Journal of Modern Physics D* 19.01 (2010), pp. 97–111, A. A. Andrianov, C. Lan, and O. O. Novikov. In: *Springer Proceedings in Physics*. Springer International Publishing, 2016, pp. 29–44.



# Integration of the transformed first Friedmann equation

General integral for  $p_\beta \neq 0$

In order to integrate the equation under the implicit gauge

$$\beta \frac{p_\beta^2}{12} \left( -\ell \frac{\nu^{1/2}}{6} \left( \frac{d\chi}{d\beta} \right)^2 + \nu^{-1/2} \right) - \nu^{3/2} V e^{g_\beta \chi} = 0, \quad (31)$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\nu}} g\beta, \quad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\nu^2 |V|} e^{-g_\beta \chi}, \quad (32)$$

to get

$$\left( \frac{d\tilde{\sigma}}{d\gamma} \right)^2 + \ell(\beta v - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{d\gamma}{d\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-\beta v + \tilde{\sigma}^2)}}, \quad (33)$$

which is of the standard inverse hyperbolic / trigonometric form **except for  $(-, -)$ .**



# Integration of the separated minisuperspace WDW equation

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\nu} \partial_\chi^2 + \varsigma \nu \frac{12\nu^2 |V|}{\hbar^2}, \quad (14 \text{ rev.})$$

one can transform

$$\nu := \sqrt{\frac{2\nu}{3}} \frac{k_\beta}{g}, \quad \sigma^2 := \frac{8\nu^3 |V| e^{g_4 \chi}}{\hbar^2 g^2}, \quad (16 \text{ rev.})$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma \psi'(\sigma) + \ell(\nu^2 - \varsigma \nu \sigma^2) \psi(\sigma) = 0, \quad (34)$$

which is of the standard Besselian form.

