

Integrable Cosmological Models with Liouville Fields

Alexander A. Andrianov^{1,4} Chen Lan² Oleg O. Novikov¹
Yi-Fan Wang³

¹Saint-Petersburg State University, Ulyanovskaya str. 1, Petrodvorets, Sankt-Petersburg 198504, Russland

²ELI-ALPS Research Institute, Budapesti út 5, H-67228 Szeged, Ungarn

³Institut für Theoretische Physik, Universität zu Köln, Zülpicher Straße 77, D-50937 Köln, Deutschland

⁴Institut de Ciències del Cosmos, Universitat de Barcelona, Martí i Franquès 1, E-08028 Barcelona, Spanien

December 12, 2017



Outline

1. Introduction

2. Classical model and the implicit trajectories

Lagrangian formalism

3. Quantised model and the wave packets

Canonical formalism and Dirac quantisation
Semi-classical approximation
Inner product and wave packet

4. Conclusions



Outline

1. Introduction

2. Classical model and the implicit trajectories

Lagrangian formalism

3. Quantised model and the wave packets

Canonical formalism and Dirac quantisation
Semi-classical approximation
Inner product and wave packet

4. Conclusions



Introduction

Quintessence and phantom Liouville field

- Observed accelerated expansion can be explained by a cosmological constant¹ as stationary Dark Energy, but its origin has yet to be understood.
- Dynamical Dark Energy has been modeled by quintessence² and phantom³ matter, with barotropic index⁴ $w > -1$ and $w < -1$, respectively.
- They can be realised by minimally-coupled real scalar fields with $\ell = \pm 1^5$

$$\mathcal{S} = \int d^4x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \quad (1)$$

- $\mathcal{V}(\phi) = V e^{\lambda\phi}$, $\lambda, V \in \mathbb{R}$ is of interest: Liouville field⁶.

¹ E. J. Copeland, M. Sami, and S. Tsujikawa. In: *International Journal of Modern Physics D* 15.11 (2006), pp. 1753–1935, K. Bamba et al. In: *Astrophys. Space Sci.* 342.1 (2012), pp. 155–228.

² R. R. Caldwell, R. Dave, and P. J. Steinhardt. In: *Phys. Rev. Lett.* 80.8 (1998), pp. 1582–1585.

³ R. R. Caldwell. In: *Phys. Lett. B* 545.1-2 (2002), pp. 23–29.

⁴ Barotropic index is the w in equation of state $\rho = wp$.

⁵ The signature of metric is mostly positive.

⁶ Y. Nakayama. In: *International Journal of Modern Physics A* 19.17n18 (2004), pp. 2771–2930.



Introduction

Friedmann–Lemaître model

- Assume flat Robertson–Walker metric for a homog. and isotr. model

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + \kappa e^{2\alpha(t)} d\Omega_{3F}^2 \quad (2)$$

w/ $\kappa := 8\pi G$, $d\Omega_{3F}^2 := d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\varphi^2)$, N lapse function.

- Combined with the Liouville field, the total action reads
 $\mathcal{S} := \mathcal{S}_{EH} + \mathcal{S}_{GHY} + \mathcal{S}_L = \int d\Omega_{3F}^2 \int dt L$, where

$$L := \kappa^{3/2} N e^{3\alpha} \left(-\frac{3}{\kappa} \frac{\dot{\alpha}^2}{N^2} + \ell \frac{\dot{\phi}^2}{2N^2} - V e^{\lambda\phi} \right), \quad (3)$$

in which dot means d/dt .

- The model turns out to be integrable, both classically and quantum-mechanically, enabling one to study its full physical properties, e.g. the relation between its classical and quantum theory.



Outline

1. Introduction

2. Classical model and the implicit trajectories

Lagrangian formalism

3. Quantised model and the wave packets

Canonical formalism and Dirac quantisation
Semi-classical approximation
Inner product and wave packet

4. Conclusions



Decoupling the variables

Via *rescaled special orthogonal transformation*

- Setting $\bar{N} := N e^{-3\alpha}$, eq. (3) transforms to

$$L = \varkappa^{3/2} \bar{N} \left(-\frac{3}{\varkappa} \frac{\dot{\alpha}^2}{\bar{N}^2} + \ell \frac{\dot{\phi}^2}{2\bar{N}^2} - V e^{\lambda\phi+6\alpha} \right) \quad (4)$$

- Defining $\Delta := \lambda^2 - 6\ell\varkappa$, $s := \operatorname{sgn} \Delta$ and $g := s\sqrt{|\Delta|} \equiv s\sqrt{s\Delta}$, the *rescaled special orthogonal transformation*

$$\begin{pmatrix} \alpha \\ \phi \end{pmatrix} = \frac{s}{g} \begin{pmatrix} \lambda & -\ell\varkappa \\ -6 & \lambda \end{pmatrix} \begin{pmatrix} s_\beta \beta \\ s_\chi \chi \end{pmatrix} \quad \text{where } s_\beta, s_\chi = \pm 1 \quad (5)$$

gives the decoupled Lagrangian

$$L = \varkappa^{3/2} \bar{N} \left(-s \frac{3}{\varkappa} \frac{\dot{\beta}^2}{\bar{N}^2} + \ell s \frac{\dot{\chi}^2}{2\bar{N}^2} - V e^{s_\chi g \chi} \right). \quad (6)$$

- The Euler–Lagrange equations w.r.t. \bar{N} , β and χ will be called the trsfed. 1st, 2nd Friedmann eqs. and the Klein–Gordon eq., respectively.



Implicit integration

General integral for $p_\beta \neq 0$

- Since β is cyclic in eq. (6), the trsfed. 2nd Friedmann eq. can be integrated⁷

$$\text{const.} \equiv p_\beta := \frac{\partial L}{\partial \dot{\beta}} = -6\mathfrak{s}\varkappa^{1/2} \frac{\dot{\beta}}{\bar{N}} \equiv -6\mathfrak{s}\mathfrak{s}_\beta \frac{\varkappa^{1/2}}{g} \frac{\lambda\dot{\alpha} + \ell\varkappa\dot{\phi}}{\bar{N}}. \quad (7)$$

- For $p_\beta \neq 0$, fixing the *implicit gauge* $\bar{N} = -6\mathfrak{s}\sqrt{\varkappa}\dot{\beta}/p_\beta$, the trsfed. 1st Friedmann equation **can be integrated to get the trajectory**

$$e^{\mathfrak{s}_X g X} = \frac{p_\beta^2}{12\varkappa^2 |V|} S^2 \left(\mathfrak{s}_\beta \sqrt{\frac{3}{2\varkappa}} g(\beta - \beta_0) \right), \quad (8)$$

in which $v := \operatorname{sgn} V$, $(\operatorname{sgn}, \operatorname{sgn})$ means $(\ell, \mathfrak{s}v)$, and

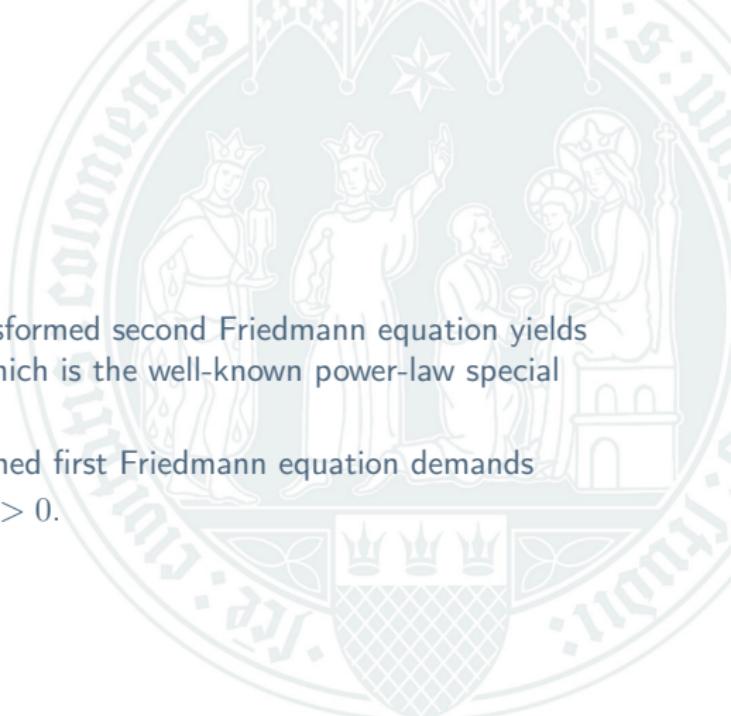
$$\begin{array}{ll} (+,+)\mathcal{S}(\gamma) := \operatorname{sech}(\gamma), & (+,-)\mathcal{S}(\gamma) := \operatorname{csch}(\gamma), \\ (-,+)\mathcal{S}(\gamma) := \operatorname{sec}(\gamma), & (-,-)\mathcal{S}(\gamma) := i \operatorname{csc}(\gamma). \end{array} \quad (9)$$

⁷The same first integral has been found in C. Lan. PhD thesis. Saint Petersburg State University, 2016, A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theor. Math. Phys.* 184.3 (2015), pp. 1224–1233, in canonical formalism.



Implicit integration

Specific integral for $p_\beta = 0$



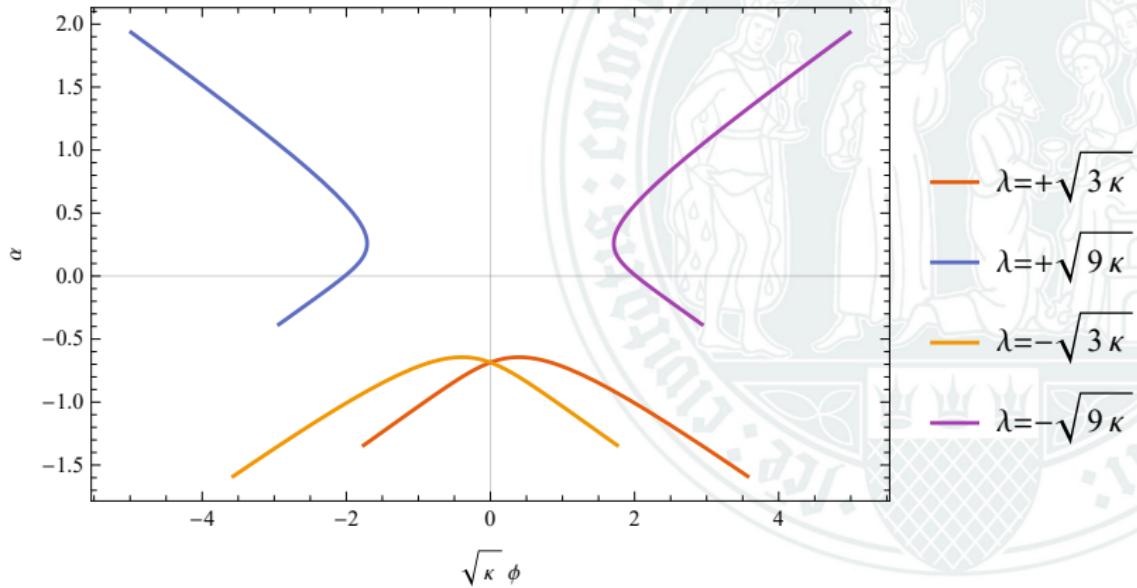
- For $p_\beta = 0$, integrating the transformed second Friedmann equation yields $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell \lambda \alpha / \nu$, which is the well-known power-law special solution⁸.
- Further integrating the transformed first Friedmann equation demands $(+, -)$ or $(-, +)$ to guarantee $\bar{N} > 0$.

⁸For instance, A. R. Liddle and D. H. Lyth. *Cosmological Inflation and Large-Scale Structure*. Cambridge University Press, 2000, ch. 3.



Trajectories for quintessence model $(+, +)$

sech , with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

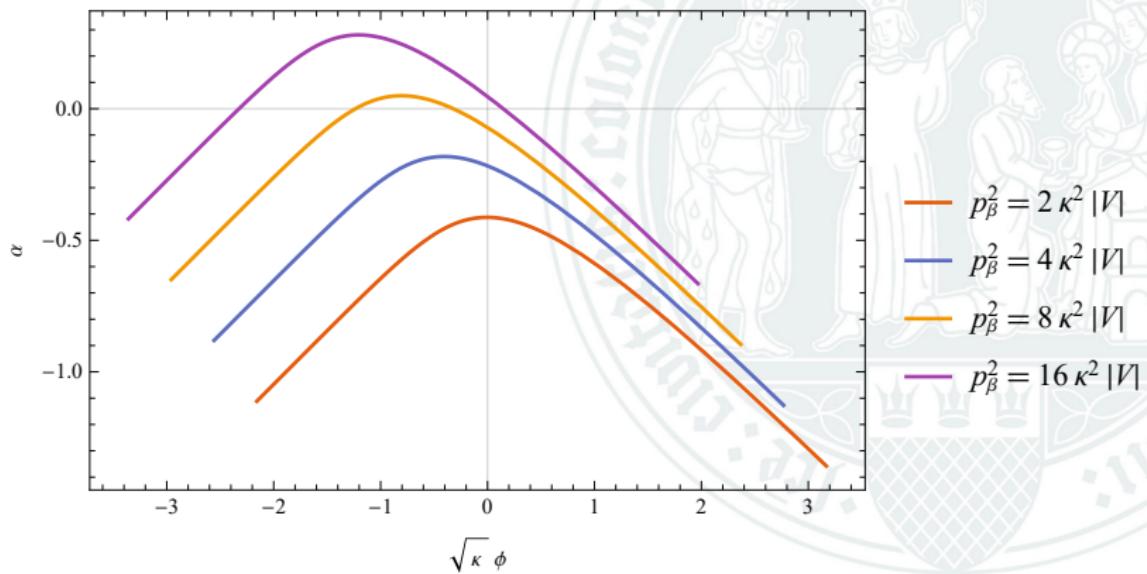


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model (+, +)

sech , with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $\lambda^2 = 3\kappa$; varying p_β

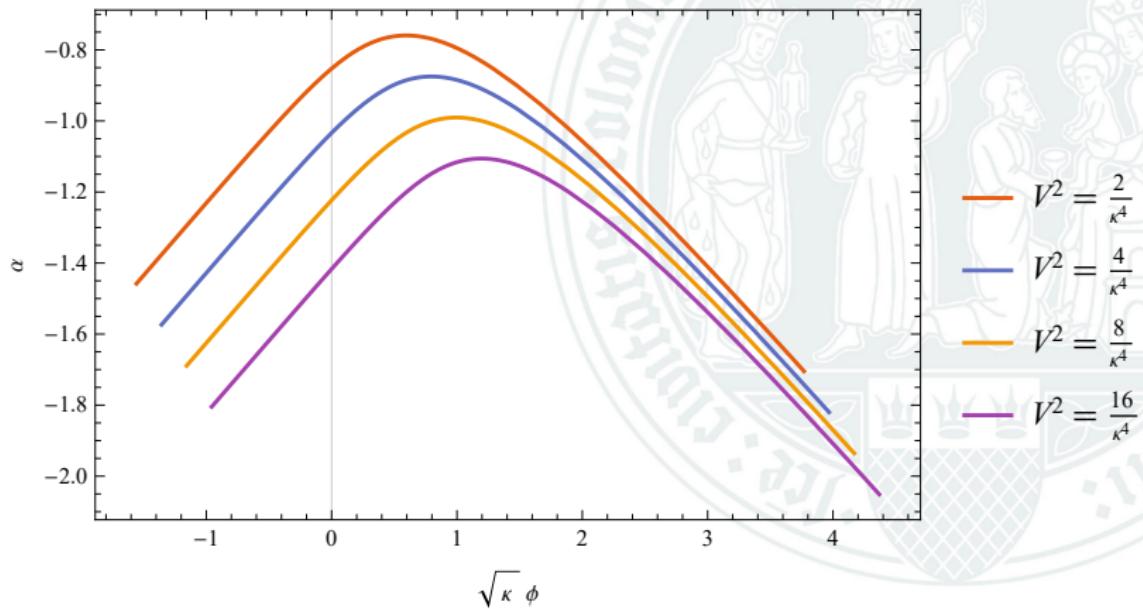


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model (+, +)

sech, with $\beta_0 = 0$, $\lambda^2 = 3\nu$ and $p_\beta^2 = \nu^2 \sqrt{|V|}$; varying V

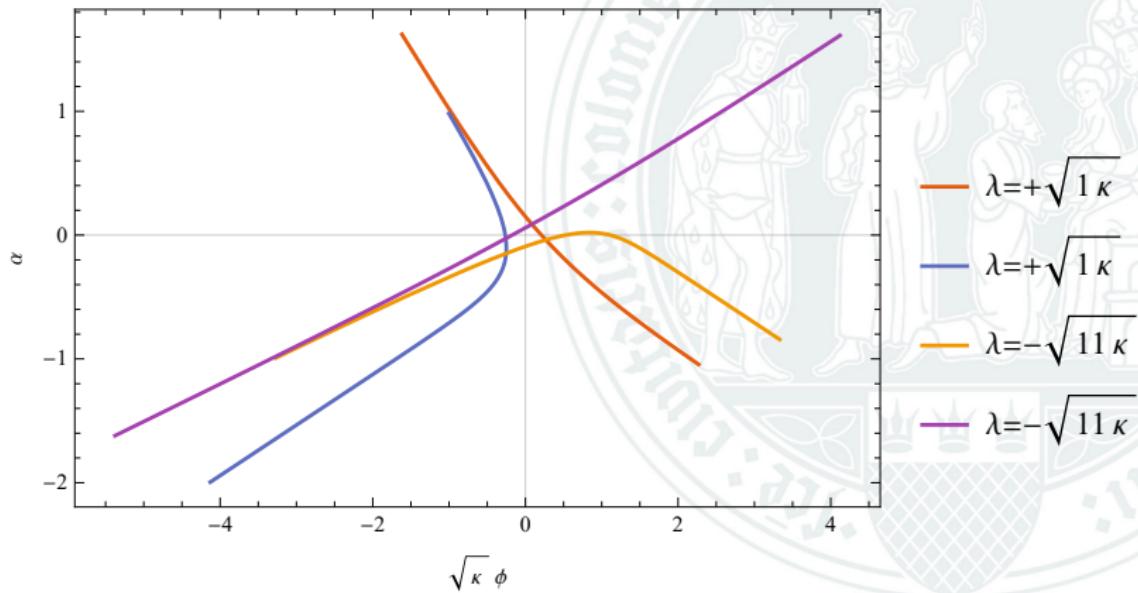


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model $(+, -)$

csch , with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

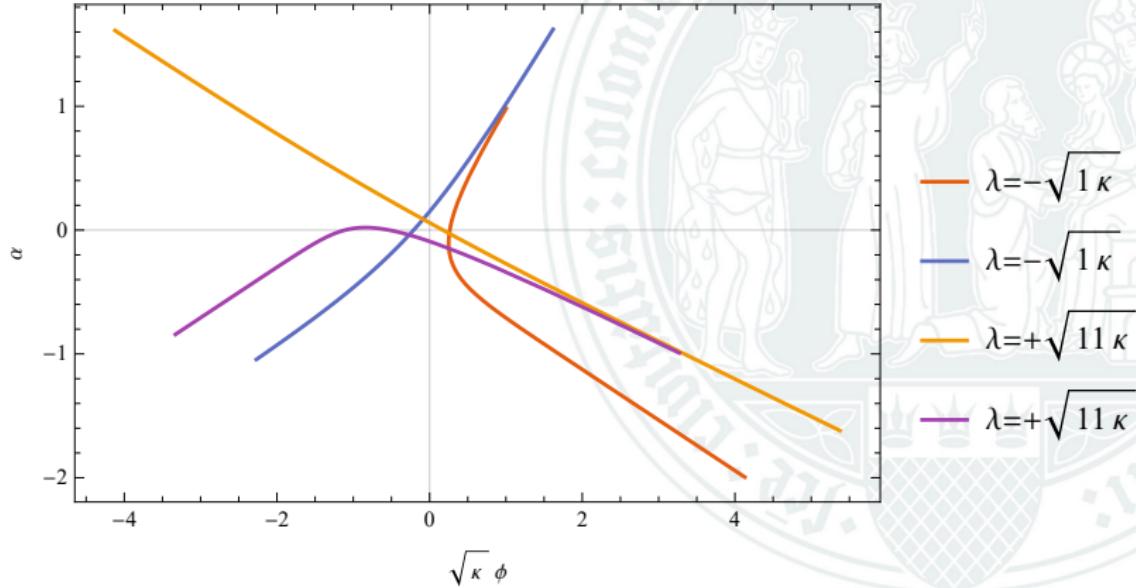


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$



Trajectories for quintessence model $(+, -)$: csch

csch, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

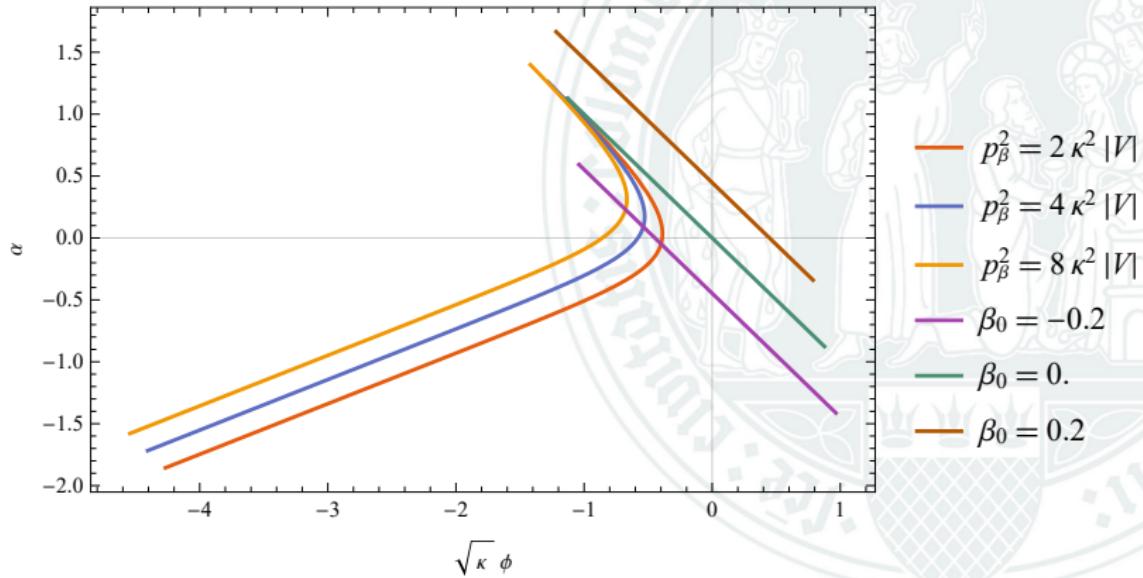


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$



Trajectories for quintessence model (+, -): csch

csch, with $V = \kappa^{-2}$ and $\lambda^2 = \kappa$; varying p_β

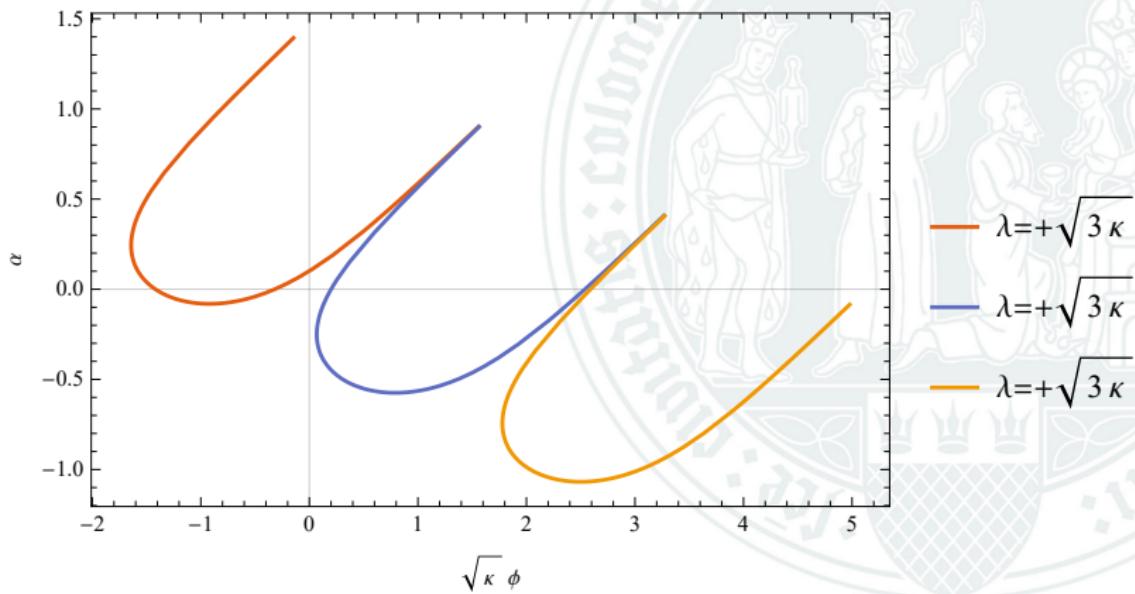


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$



Trajectories for phantom model $(-, +)$

csc, with $V = \kappa^{-2}$ and $p_\beta^2 = 3\kappa^2 \sqrt{|V|}$

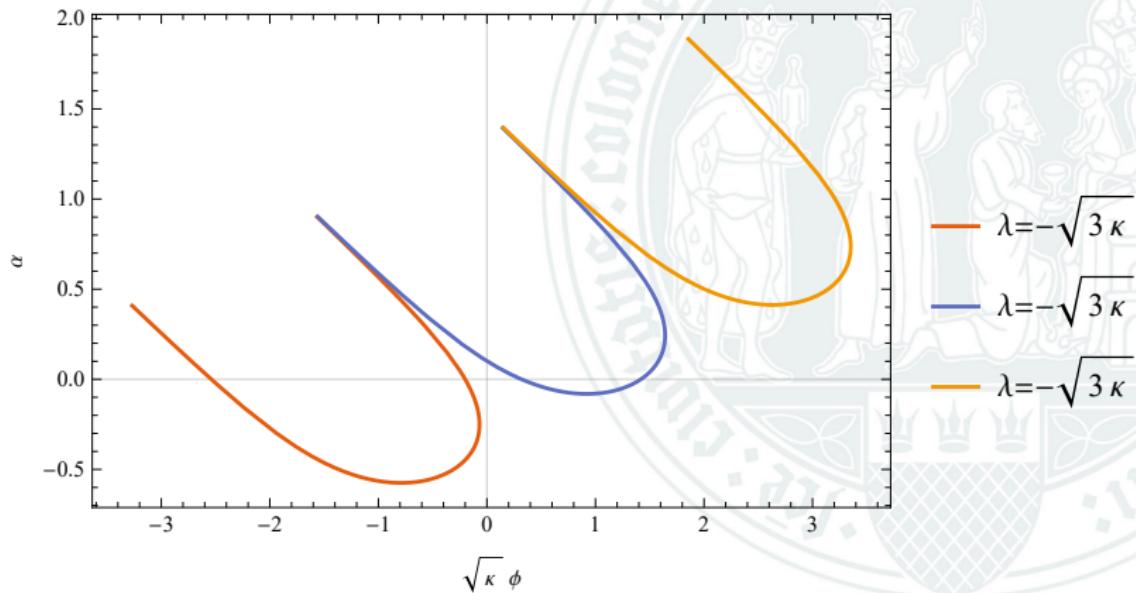


- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Trajectories for phantom model $(-, +)$

csc, with $V = \kappa^{-2}$ and $p_\beta^2 = 3\kappa^2 \sqrt{|V|}$

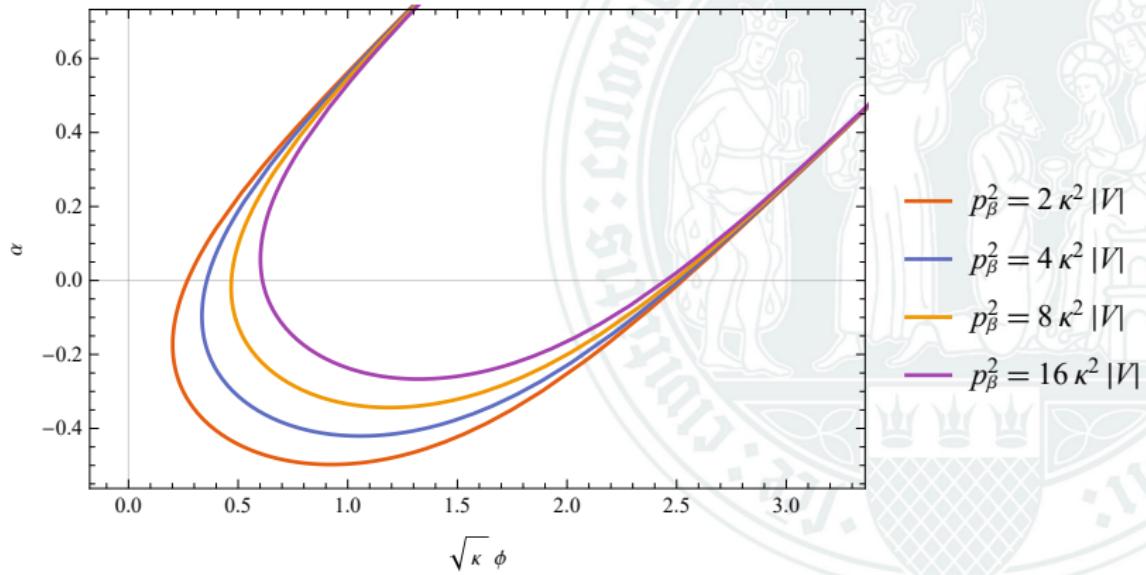


- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Trajectories for phantom model $(-, +)$

csc, with $V = \kappa^{-2}$ and $\lambda^2 = 3\kappa$; varying p_β



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Integration

Further discussions

- The integral for $(-, -)$ is not real.
- The trajectories can be parametrised by β , inspiring recognising β as a 'time variable'.
- The implicit integration enables one to compare trajectories with wave functions, see below.



Outline

1. Introduction
2. Classical model and the implicit trajectories
Lagrangian formalism
3. Quantised model and the wave packets
Canonical formalism and Dirac quantisation
Semi-classical approximation
Inner product and wave packet
4. Conclusions



Dirac quantisation and the mode functions

- The primary Hamiltonian and the Hamiltonian constraint⁹ reads

$$H^P = \overline{N} H_{\perp} + v^{\overline{N}} p_{\overline{N}}, \quad (10)$$

$$H_{\perp} := -s \frac{p_{\beta}^2}{12\kappa^{1/2}} + \ell s \frac{p_{\chi}^2}{2\kappa^{3/2}} + \kappa^{3/2} V e^{g_s \chi \chi}. \quad (11)$$

- Applying the Dirac quantisation rules with Laplace–Beltrami operator¹⁰, one gets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(s \frac{\hbar^2}{12\kappa^{1/2}} \partial_{\beta}^2 - \ell s \frac{\hbar^2}{2\kappa^{3/2}} \partial_{\chi}^2 + \kappa^{3/2} V e^{g_s \chi \chi} \right) \Psi. \quad (12)$$

- Equation (12) is KG-like, hyperbolic for $\ell = +1$ and elliptic for $\ell = -1$.

⁹ D. M. Gitman and I. V. Tyutin. *Quantization of Fields with Constraints*. Springer Berlin Heidelberg, 1990, H. J. Rothe and K. D. Rothe. *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*. World Scientific, 2010.

¹⁰ C. Kiefer. *Quantum Gravity*. 3rd ed. Oxford University Press, 2012, ch. 8.



Separation of the variables and mode functions

- Writing $\Psi(\beta, \chi) = \varphi(\beta)\psi(\chi)$, eq. (12) can be separated into

$$\partial_\beta^2 \varphi(\beta) = k_\beta^2 \varphi(\beta); \quad (13)$$

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \mathfrak{s} v \frac{12\varkappa^2 |V|}{\hbar^2}. \quad (14)$$

- Equation (14) turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta, \chi) := \sum_{i=1}^2 c_i \varphi_\nu^{(i)}(\gamma) \sum_{j=1}^2 a_j B_\nu^{(j)}(\sigma), \quad \nu \geq 0; \quad (15)$$

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{1}{g} k_\beta, \quad \gamma := \sqrt{\frac{3}{2\varkappa}} \frac{g}{1} \beta, \quad \sigma^2 := \frac{8\varkappa^3 |V| e^{g\beta_X \chi}}{\hbar^2 g^2}, \quad (16)$$

$${}_{(+,+)} B_\nu^{(i)}(\sigma) := K \text{ and } I_{\mathbb{B}\nu}(\sigma), \quad {}_{(+,-)} B_\nu^{(i)}(\sigma) := F \text{ and } G_{\mathbb{B}\nu}(\sigma),$$

$${}_{(-,+)} B_\nu^{(i)}(\sigma) := J \text{ and } Y_\nu(\sigma), \quad {}_{(-,-)} B_\nu^{(i)}(\sigma) := K \text{ and } I_\nu(\sigma).$$

- Adapted to imaginary order, $F_\nu(\sigma)$ and $G_\nu(\sigma)$ are defined in¹¹.

¹¹ T. M. Dunster. In: *SIAM Journal on Mathematical Analysis* 21.4 (1990), pp. 995–1018.



Physical mode functions

- Physical mode functions are expected to be regular on the boundary.
- $(+, +)$: $|I_{i\nu}(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow +\infty$
- $(-, +)$:
 - $\forall n \in \mathbb{N}$, $|Y_n(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow -\infty$.
 - $\forall \nu \in \mathbb{R}^+ \setminus \mathbb{N}$, choose $J_{-\nu}$ instead of Y_ν , since $J_{\pm\nu}$ are also linearly independent.
 - $\forall \nu \in \mathbb{R}^+ \setminus \mathbb{N}$, $|J_{-\nu}(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow -\infty$.
- $(-, -)$: $|K_\nu(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow -\infty$; $|I_\nu(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow +\infty$
- These are not to be included in the space of physical wave functions.
 $\forall \nu \geq 0$,
 - $(+, +)$: $K_{i\nu}(\sigma)$ survives
 - $(+, -)$: $F_{i\nu}(\sigma)$ and $G_{i\nu}(\sigma)$ survives
 - $(-, +)$: $J_\nu(\sigma)$ survives
 - $(-, -)$: drops out



Matching quantum number with classical first integral

Principle of constructive interference

- Write the mode function in the WKB form

$$\Psi_{k_\beta}(\beta, \chi) \sim \sqrt{\rho(\beta, \chi)} \exp\left\{ \frac{i}{\hbar} S(\beta, \chi) \right\}. \quad (17)$$

- For $S/\hbar \gg 1$ and $k_\beta \gg 1^{12}$, $S(\beta, \chi)$ becomes the Hamilton principle function in the leading-order approximation.
- A Hamilton principle function is stationary with respect to variation of integral constants¹³

$$\frac{\partial S}{\partial k_\beta} = 0. \quad (18)$$

- Demanding eq. (18) matching the classical trajectory, k_β can be related to p_β .

¹²The common form is $\hbar \rightarrow 0^+$.

¹³ U. H. Gerlach. In: *Phys. Rev.* 177.5 (1969), pp. 1929–1941, L. D. Landau and E. M. Lifshitz. *Mechanics.* 3rd ed. Elsevier, 1976.



Matching quantum number with classical first integral

(+, +) as exemplar

- Fixing $\nu/\sigma > 1$, the asymptotic expansion reads

$$K_{\pm\nu}(\sigma) \sim \frac{\sqrt{2\pi} \cos\left(\sqrt{\nu^2 - \sigma^2} - \nu \arccos \frac{\nu}{\sigma} - \frac{\pi}{4}\right)}{(\nu^2 - \sigma^2)^{1/4} e^{\mp\nu/2}} + O\left(\frac{1}{\sigma}\right). \quad (19)$$

- There are two phases with opposite signs. Assuming c_i, a_j 's are real and applying *the principle* to $\Psi_\nu(\sigma)$, one has $\sigma/\nu = \operatorname{sech}(\beta_\beta \gamma)$, which matches the trajectory with $\beta_0 = 0$ if

$$\hbar k_\beta \equiv \hbar \sqrt{\frac{3}{2\nu}} g \hbar \nu = p_\beta, \quad (20)$$

- Non-vanishing C can be compensated by the phase of c_i and a_j 's.
- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; it is not within the WKB regime.
- The conclusions also hold for $F_{\pm\nu}(\sigma), G_{\pm\nu}(\sigma)$ for $(+,-)$, and $J_\nu(\sigma)$ for $(-,+)$.



Inner product for wave functions

Schrödinger product

- To make sense of amplitude and wave packet, an inner product is necessary
- In terms of a Klein–Gordon equation, call β the “temporal” variable, and χ the “spacial” variable.
- A common starting point is the *Schrödinger product*¹⁴

$$(\Psi_1, \Psi_2)_S := \int d\chi \Psi_1^*(\beta, \chi) \Psi_2(\beta, \chi); \quad (21)$$

- $(\Psi, \Psi)_S$ is positive-definite, and the integrand $\rho_S(\beta, \chi)$ is non-negative
- The corresponding Schrödinger current is real but does not satisfy continuity equation $\dot{\rho}_S + \nabla \cdot \vec{j}_S = 0$, because eq. (12) is KG-like.
- $K_{i\nu}$ ¹⁵ for $(+, +)$, $F_{i\nu}$ and $G_{i\nu}$ for $(+, -)$ can be proved to be orthogonal and complete among themselves, as well as can be normalised.
 - $J_{i\nu}$'s for $(+, -)$ are not orthogonal

¹⁴ C. Kiefer. *Quantum Gravity*. 3rd ed. Oxford University Press, 2012, ch. 5.

¹⁵ S. B. Yakubovich. In: *Opuscula Math.* 26.1 (2006), pp. 161–172, A. Passian et al. In: *Journal of Mathematical Analysis and Applications* 360.2 (2009), pp. 380–390, R. Szymkowski and S. Bielski. In: *Journal of Mathematical Analysis and Applications* 365.1 (2010), pp. 195–197.



Peculiarity of the phantom model $(-, +)$

Orthogonality for mode functions; Hermiticity for operators

- $J_\nu(\sigma)$'s are not orthogonal under the Schrödinger product

$$(J_\nu, J_{\tilde{\nu}})_S \propto \int_{-\infty}^{+\infty} dx J_\nu^*(e^x) J_{\tilde{\nu}}(e^x) = \frac{2\sin(\pi(\nu - \tilde{\nu})/2)}{\pi(\nu^2 - \tilde{\nu}^2)}, \quad (22)$$

therefore \mathbb{D} in eq. (14) is not Hermitian (though we do not need it so far)

- \hat{p}_χ^2 is not Hermitian for $\{J_\nu(\sigma)\}$ under the Schrödinger product

$$\int_{-\infty}^{+\infty} dx J_\nu^*(-\partial_x^2 J_{\tilde{\nu}}) - \int_{-\infty}^{+\infty} dx (-\partial_x^2 J_\nu)^* J_{\tilde{\nu}} = \frac{2}{\pi} \sin \frac{\pi(\nu - \tilde{\nu})}{2}. \quad (23)$$

- In order to save Hermiticity for p_χ^2 and \mathbb{D} and orthogonality of the modes under Schrödinger product, one can restrict

$$\nu = 2n + \nu_0, \quad n \in \mathbb{N}, \quad \nu_0 \in [0, 2]. \quad (24)$$

- Using classical singularities as boundary condition, one can fix $\nu_0 = 1$.



Discretisation of the phantom model $(-, +)$

Levels of the phantom model are **discretised** if one imposes Hermiticity of squared momenta under the Schrödinger product.

- This kind of subtlety on Hermiticity is named as *self-adjoint extension*, which arises already for infinite square well¹⁶.
- It also applies to x^{-2} potentials¹⁷, which is of cosmological relevance¹⁸.

¹⁶ G. Bonneau, J. Faraut, and G. Valent. In: *Am. J. Phys* 69.3 (2001), pp. 322–331.

¹⁷ A. M. Essin and D. J. Griffiths. In: *Am. J. Phys* 74.2 (2006), pp. 109–117, V. S. Araujo, F. A. B. Coutinho, and J. F. Perez. In: *Am. J. Phys* 72.2 (2004), pp. 203–213.

¹⁸ M. Bouhmadi-López et al. In: *Physical Review D* 79.12 (2009).



Further inner products for wave functions

Klein–Gordon and Mostafazadeh product

- Since eq. (12) is KG-like, another popular choice is the KG product

$$(\Psi_1, \Psi_2)_{\text{KG}}^g := \mathbb{i}g \left\{ (\Psi_1, \partial_\beta \Psi_2)_S - (\partial_\beta \Psi_1, \Psi_2)_S \right\}, \quad g > 0. \quad (25)$$

- $(\Psi, \Psi)_{\text{KG}}^g$ is **real** but **not positive-definite**, so does the integrand ρ_{KG} ;
- The corresponding \vec{J}_{KG} is **conserved** $\dot{\rho}_{\text{KG}} + \nabla \cdot \vec{J}_{\text{KG}} = 0$ and **real**.
- Mostafazadeh¹⁹ found a product *for Hermitian \mathbb{D} with positive spectrum*:

$$(\Psi_1, \Psi_2)_{\text{M}}^\kappa := \kappa \left\{ (\Psi_1, \mathbb{D}^{+1/2} \Psi_2)_S + (\partial_\beta \Psi_1, \mathbb{D}^{-1/2} \partial_\beta \Psi_2)_S \right\}, \quad \kappa > 0. \quad (26)$$

- $(\Psi, \Psi)_{\text{M}}^\kappa$ is **positive-definite**, but the integrand ρ_{M}^κ is **complex**
- The corresponding $\vec{J}_{\text{M}}^\kappa$ is **conserved** $\dot{\rho}_{\text{M}}^\kappa + \nabla \cdot \vec{J}_{\text{M}}^\kappa = 0$ but also **complex**.

¹⁹

A. Mostafazadeh. In: *Classical Quantum Gravity* 20.1 (2002), pp. 155–171.



Mostafazadeh inner product and the corresponding density

- Real power of \mathbb{D} is defined by spectral decomposition $\mathbb{D}^\gamma := \sum_\nu \nu^{2\gamma} \mathbf{P}_\nu$,
 $\mathbf{P}_\nu \Psi := \Psi_\nu (\Psi_\nu, \Psi)_S$.
- It can be shown²⁰ that *the density*

$$\varrho_M^\kappa := \kappa \left\{ |\mathbb{D}^{+1/4} \Psi|^2 + |\mathbb{D}^{-1/4} \partial_\beta \Psi|^2 \right\} \quad (27)$$

- is equivalent to ρ_M^κ up to a boundary term

$$\int d\chi \varrho_M^\kappa = \int d\chi \rho_M^\kappa \equiv (\Psi_1, \Psi_2)_M^\kappa; \quad (28)$$

- is non-negative.
- The corresponding current $\vec{\mathcal{J}}_M^\kappa$ is real but not conserved²¹.

²⁰ A. Mostafazadeh and F. Zamani. In: *Ann. Phys.* 321.9 (2006), pp. 2183–2209.

²¹ B. Rosenstein and L. P. Horwitz. In: *J. Phys. A: Math. Gen.* 18.11 (1985), pp. 2115–2121.



Wave packets of Gaussian amplitude for continuous spectrum

Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu; \bar{\nu}, \sigma) := \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\nu - \bar{\nu})^2}{2\sigma^2}\right) \right)^{1/2} \quad (29)$$

- In²², $A(\nu; \bar{\nu}, \sigma/\sqrt{2})$ was chosen.

22

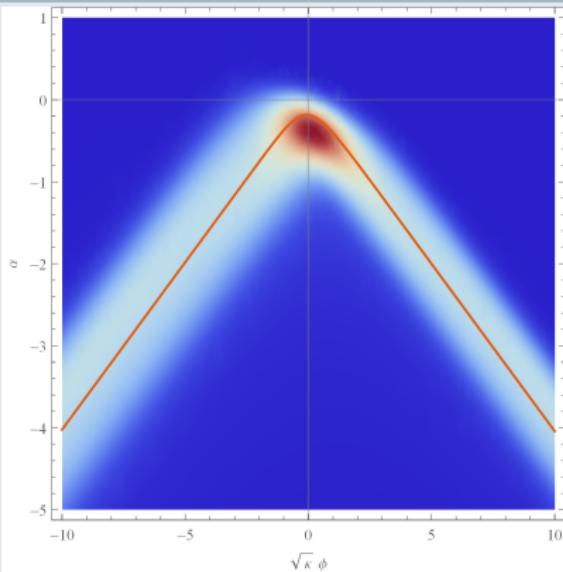
M. P. Dąbrowski, C. Kiefer, and B. Sandhöfer. In: *Physical Review D* 74.4 (2006).



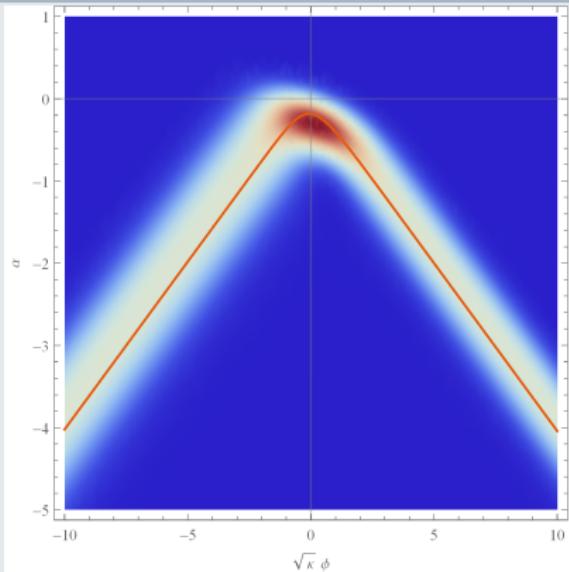
Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\parallel\nu}$, with $\lambda = \kappa^{1/2}/2$, $V = -\kappa^{-2}$, $\bar{k}_\beta = -2$ and $\sigma_\beta = 5/4$

Schrödinger



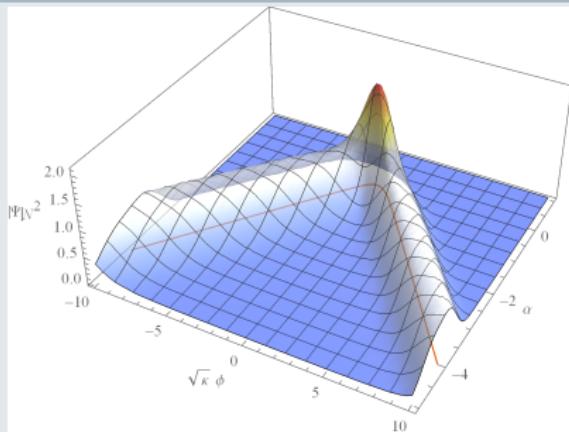
Mostafazadeh



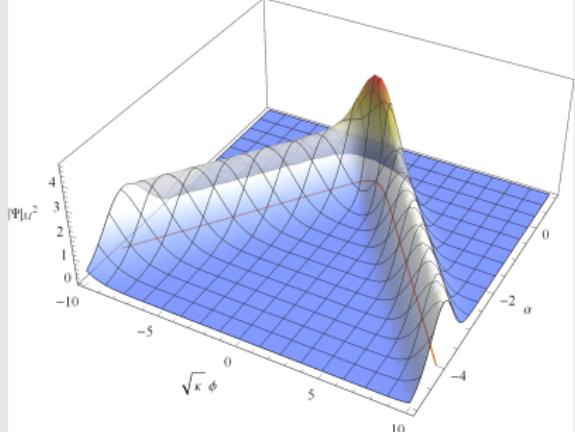
Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\parallel\nu}$, with $\lambda = \kappa^{1/2}/2$, $V = -\kappa^{-2}$, $\bar{k}_\beta = -2$ and $\sigma_\beta = 5/4$

Schrödinger



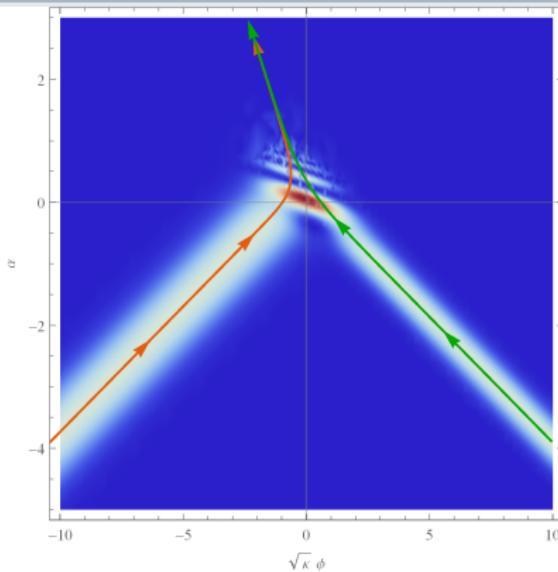
Mostafazadeh



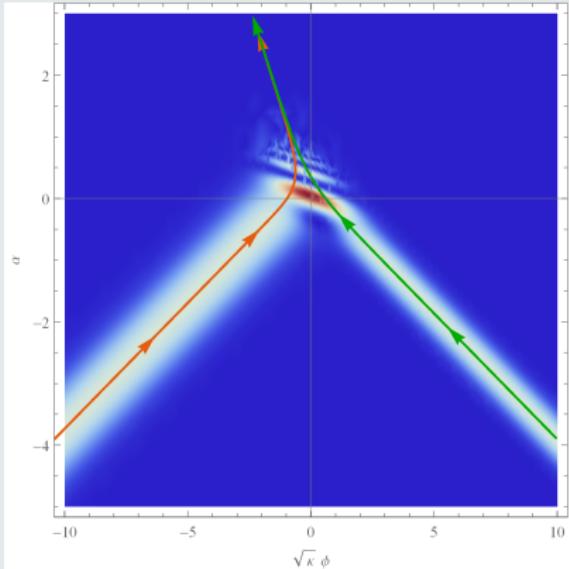
Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\pm\nu}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



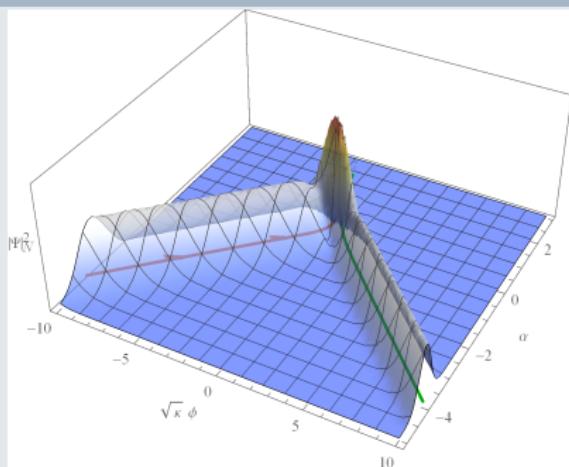
Mostafazadeh



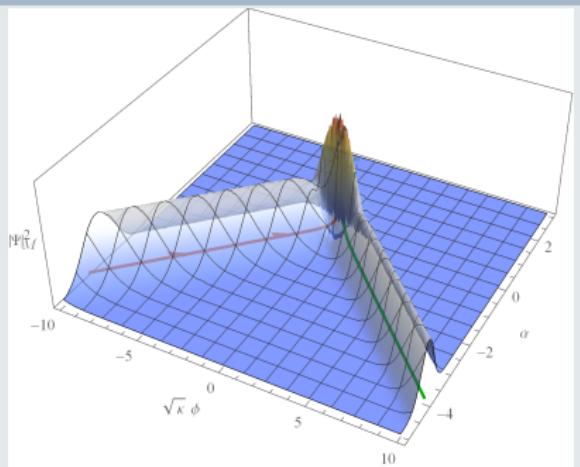
Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\tilde{\nu}}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



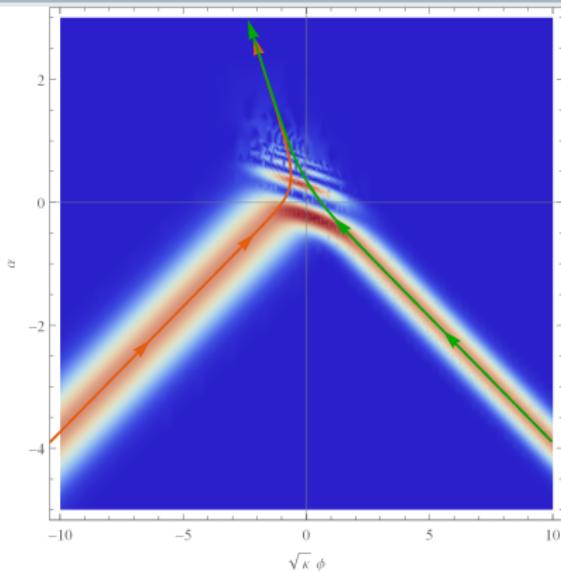
Mostafazadeh



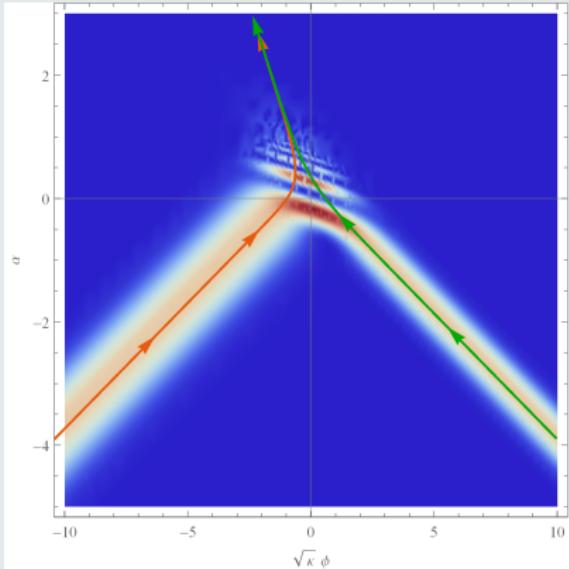
Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\text{I}\nu}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



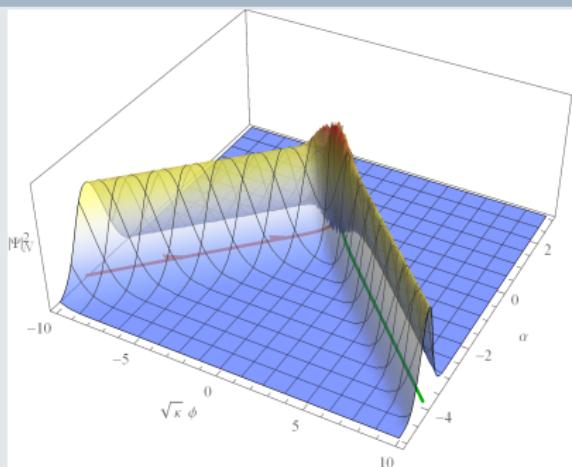
Mostafazadeh



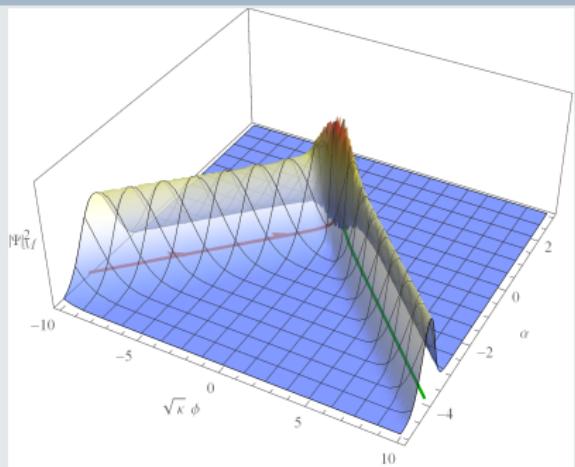
Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\mu\nu}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



Mostafazadeh



Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model $(-, +)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\bar{n}) := \left(e^{-\bar{n}} \frac{\bar{n}^n}{n!} \right)^{1/2} \quad (30)$$

- In²³, $A_n(\bar{n}/\sqrt{2})$ was chosen.

²³

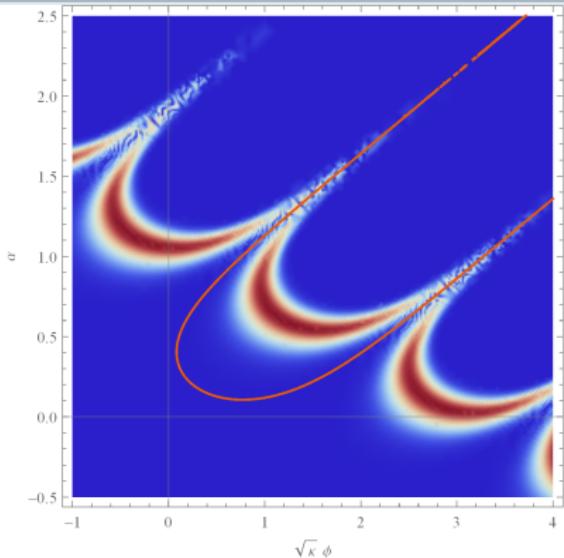
C. Kiefer. In: *Nucl. Phys. B* 341.1 (1990), pp. 273–293.



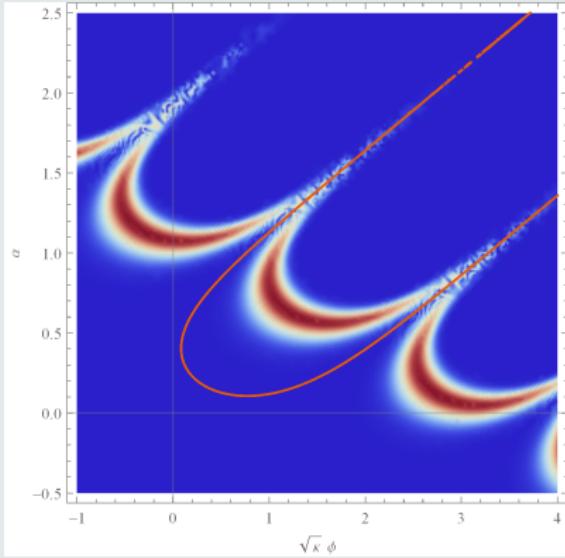
Wave packets of Poissonian amplitude for phantom model

J_{2n+1} , with $\lambda = 2\kappa^{1/2}$, $V = +\kappa^{-2}$ and $\bar{k}_\beta = 8$

Schrödinger



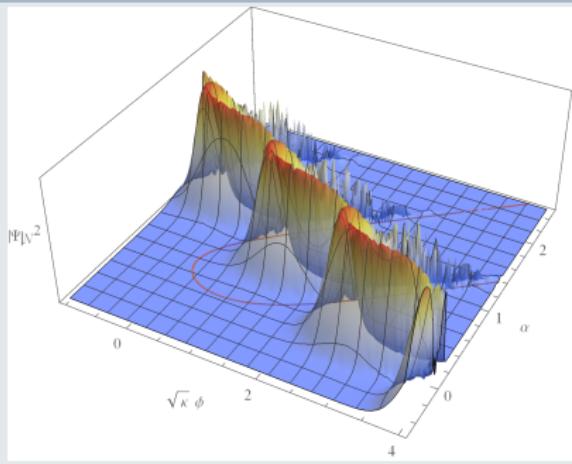
Mostafazadeh



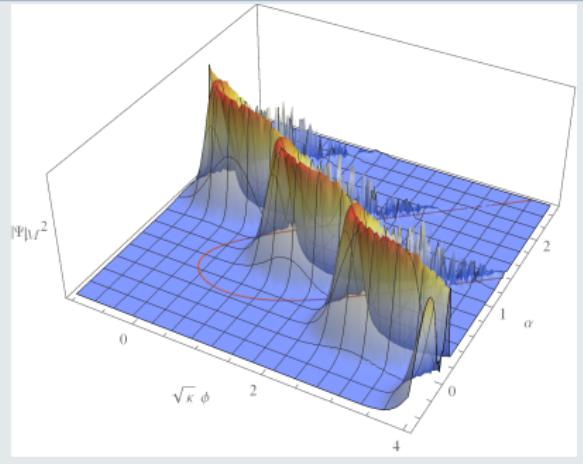
Wave packets of Poissonian amplitude for phantom model

J_{2n+1} , with $\lambda = 2\kappa^{1/2}$, $V = +\kappa^{-2}$ and $\bar{k}_\beta = 8$

Schrödinger



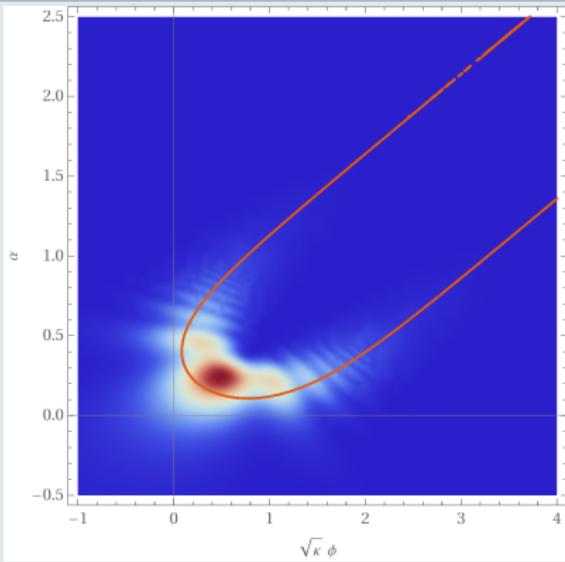
Mostafazadeh



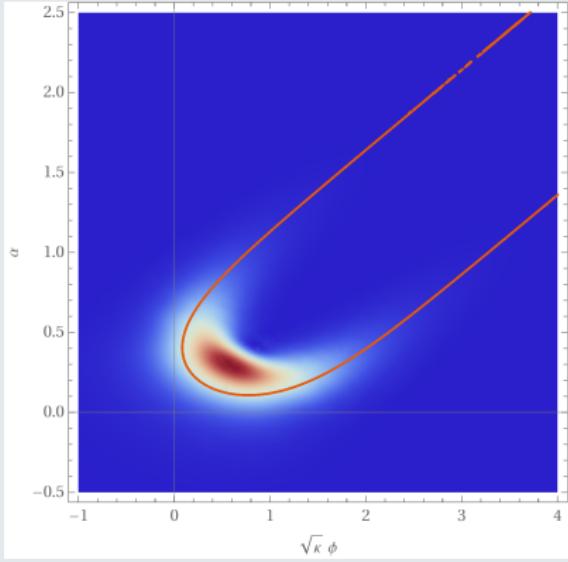
Wave packets of Gaussian amplitude for phantom model

J_ν , with $\lambda = 2\nu^{1/2}$, $V = +\nu^{-2}$, $\bar{k}_\beta = 8$ and $\sigma_\beta = 11/2$

Schrödinger



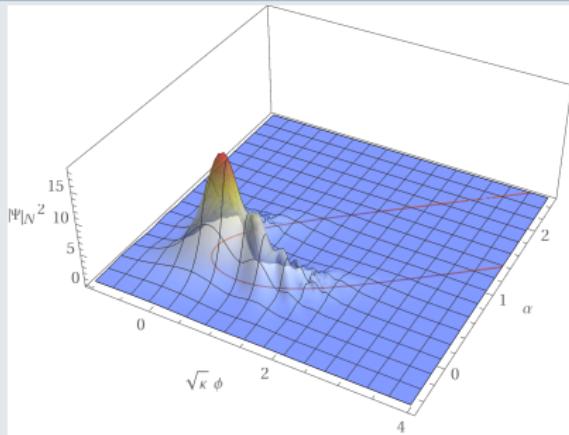
Mostafazadeh



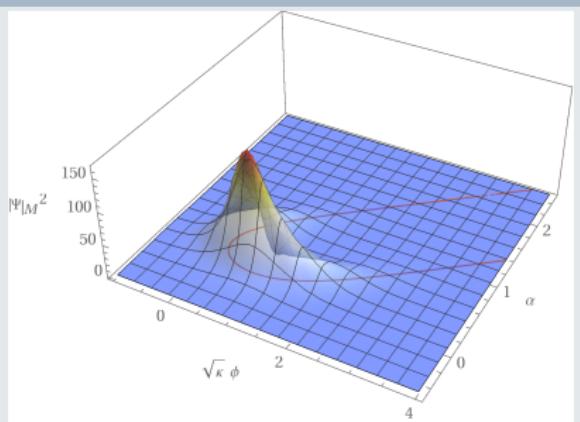
Wave packets of Gaussian amplitude for phantom model

J_ν , with $\lambda = 2\nu^{1/2}$, $V = +\nu^{-2}$, $\bar{k}_\beta = 8$ and $\sigma_\beta = 11/2$

Schrödinger



Mostafazadeh



Outline

1. Introduction

2. Classical model and the implicit trajectories

Lagrangian formalism

3. Quantised model and the wave packets

Canonical formalism and Dirac quantisation
Semi-classical approximation
Inner product and wave packet

4. Conclusions



Highlights

- An **integral of motion** was found for Liouville cosmological models.
- **Implicit trajectories** in minisuperspace were obtained.
- The levels of phantom Liouville model were found to be **discrete** due to the Hermiticity requirement of observables.
- The semi-classical quantum mode functions and wave packets was compared directly with the classical trajectory.



Issues

- In $(+, -)$ and $(-, +)$, wave packets contain multiple branches; however, the classical universe runs only on one trajectory.
- Quantum-corrected \bar{k}_β is to be understood.
- Wave packets with Gaussian profile are to be constructed, instead of inserting Gaussian / Poissonian amplitude artificially.
- A normalising κ for $(\cdot, \cdot)_M^\kappa$ is to be evaluated, otherwise a quantitative comparison of $(\cdot, \cdot)_S$ and $(\cdot, \cdot)_M^\kappa$ is not possible.



Outlook

- Beyond isotropy: generalise to Bianchi models
 - Bianchi Type-I: a natural extension, **under investigation**
- Beyond homogeneity: cosmological perturbation, **under investigation**
- Beyond single field
 - Two exponential potentials: $|V_1| = |V_2|$ and special λ_i
 - Multiple Liouville fields: mixing kinetic terms needed²⁴
- Beyond classic matter
 - PT -symmetric Liouville field²⁵: may cross the phantom divide $w = -1$.

²⁴ A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theor. Math. Phys.* 184.3 (2015), pp. 1224–1233.

²⁵ A. A. Andrianov et al. In: *International Journal of Modern Physics D* 19.01 (2010), pp. 97–111, A. A. Andrianov, C. Lan, and O. O. Novikov. In: *Springer Proceedings in Physics*. Springer International Publishing, 2016, pp. 29–44.



Integration of the transformed first Friedmann equation

General integral for $p_\beta \neq 0$

In order to integrate the equation under the implicit gauge

$$\beta \frac{p_\beta^2}{12} \left(-\ell \frac{\nu^{1/2}}{6} \left(\frac{d\chi}{d\beta} \right)^2 + \nu^{-1/2} \right) - \nu^{3/2} V e^{g_\beta \chi} = 0, \quad (31)$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\nu}} g\beta, \quad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\nu^2 |V|} e^{-g_\beta \chi}, \quad (32)$$

to get

$$\left(\frac{d\tilde{\sigma}}{d\gamma} \right)^2 + \ell(\beta v - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{d\gamma}{d\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-\beta v + \tilde{\sigma}^2)}}, \quad (33)$$

which is of the standard inverse hyperbolic / trigonometric form **except for $(-, -)$.**



Integration of the separated minisuperspace WDW equation

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\nu} \partial_\chi^2 + \varsigma \nu \frac{12\nu^2 |V|}{\hbar^2}, \quad (14 \text{ rev.})$$

one can transform

$$\nu := \sqrt{\frac{2\nu}{3}} \frac{k_\beta}{g}, \quad \sigma^2 := \frac{8\nu^3 |V| e^{g_4 \chi}}{\hbar^2 g^2}, \quad (16 \text{ rev.})$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma \psi'(\sigma) + \ell(\nu^2 - \varsigma \nu \sigma^2) \psi(\sigma) = 0, \quad (34)$$

which is of the standard Besselian form.

