Integrable Liouville Cosmological Models

Self-adjointness of the Hamiltonian and Semi-classical Wave Functions

Alexander A. Andrianov[1,4] Chen Lan[2] Oleg O. Novikov[1] Yi-Fan Wang[3]

[1] Saint-Petersburg State University, Ulyanovskaya str. 1, Petrodvorets, Sankt-Petersburg 198504, Russland

[2] ELI-ALPS Research Institute, Budapesti út 5, H-67228 Szeged, Ungarn

[3] Institut für Theoretische Physik, Universität zu Köln, Zülpicher Straße 77, D-50937 Köln, Deutschland

[4] Institut de Ciències del Cosmos, Universitat de Barcelona, Martí i Franquès 1, E-08028 Barcelona, Spanien

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Outline

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Introduction

The Friedmann–Lemaître model with quintessence and phantom Liouville fields

- Dynamical Dark Energy has been modelled by quintessence[1] and phantom[2] matter, which can be realised by minimally-coupled real scalar fields with $\ell = \pm 1$[3]

$$\mathcal{S}_{\mathsf{L}} = \int \mathrm{d}^4 x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \tag{1}$$

- The Liouville field[4] $\mathcal{V}(\phi) = V \mathrm{e}^{\lambda\phi}$ is of interest, where $\lambda, V \in \mathbb{R}$.
- Assume a flat Robertson–Walker metric $g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = -N^2(t)\, \mathrm{d}t^2 + \varkappa \mathrm{e}^{2\alpha(t)}\, \mathrm{d}\Omega_{3\mathsf{F}}^2$, where $\varkappa := 8\pi G$, $\mathrm{d}\Omega_{3\mathsf{F}}^2 := \mathrm{d}\chi^2 + \chi^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right)$, and N lapse function.
- The total action reads $\mathcal{S} := \mathcal{S}_{\mathsf{EH}} + \mathcal{S}_{\mathsf{GHY}} + \mathcal{S}_{\mathsf{L}} = \int \mathrm{d}\Omega_{3\mathsf{F}}^2 \int \mathrm{d}t\, L$, where

$$L := \varkappa^{3/2} N \mathrm{e}^{3\alpha} \left(-\frac{3}{\varkappa} \frac{\dot{\alpha}^2}{N^2} + \ell \frac{\dot{\phi}^2}{2N^2} - V \mathrm{e}^{\lambda\phi} \right). \tag{2}$$

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[1] R. R. Caldwell *et al.*, *Phys. Rev. Lett.* **80**, 1582–1585 (Aug. 7, 1997).

[2] R. R. Caldwell, *Phys. Lett. B* **545**, 23–29 (Aug. 16, 1999).

[3] The signature of metric is $(-, +, \dots, +)$.

[4] Y. Nakayama, *Int. J. Mod. Phys. A* **19**, 2771–2930 (Feb. 2, 2004).

Integrability

With the help of an integral of motion, implicit trajectories can be obtained explicitly, and the minisuperspace Wheeler–DeWitt equation can be integrated exactly.

Self-adjointness

Imposing self-adjointness of the Hamiltonian leads to removal of the degeneracy and discretisation of the spectrum.

Semi-classical wave functions

Semi-classical wave functions can be constructed by the principle of constructive interference, the Wentzel–Kramers–Brillouin approximation and numerical methods.

Outline

Decoupling the variables

Via orthogonal transformation

- By rescaling $\overline{N} := N\mathrm{e}^{-3\alpha}$ and the orthogonal transformation

$$\begin{pmatrix}\alpha(t)\\ \phi(t)\end{pmatrix} = \frac{s}{g}\begin{pmatrix}\lambda & -\ell\varkappa\\ -6 & \lambda\end{pmatrix}\begin{pmatrix}\beta(t)\\ \chi(t)\end{pmatrix}, \tag{3}$$

the system can be decoupled ($\Delta := \lambda^2 - 6\ell\varkappa$, $s := \operatorname{sgn}\Delta$ and $g := s\sqrt{|\Delta|} \equiv s\sqrt{s\Delta}$)

$$L = \varkappa^{3/2}\overline{N}\left(-s\frac{3}{\varkappa}\frac{\dot\beta^2}{\overline{N}^2} + \ell s\frac{\dot\chi^2}{2\overline{N}^2} - V\mathrm{e}^{g\chi}\right). \tag{4}$$

- Using the conservation of p_β[5], $\overline{N}(t)$ can be eliminated, leading to the exact integrations of the equations of motion as implicit trajectories.
  - $p_\beta \neq 0$: four cases according to (ℓ, sv), $v := \operatorname{sgn}V$.
  - $p_\beta = 0$: well-known power-law special solution for $(+,-)$ and $(-,+)$[6].

[5] The same first integral has been found in C. Lan, PhD thesis, Saint Petersburg State University, 2016, `https://search.rsl.ru/ru/record/01006663434`, A. A. Andrianov *et al.*, *Theor. Math. Phys.* **184**, 1224–1233 (Mar. 18, 2015), in canonical formalism.

[6] For instance, A. R. Liddle, D. H. Lyth, *Cosmological Inflation and Large-Scale Structure*, (Cambridge University, 2000) , ch. 3.

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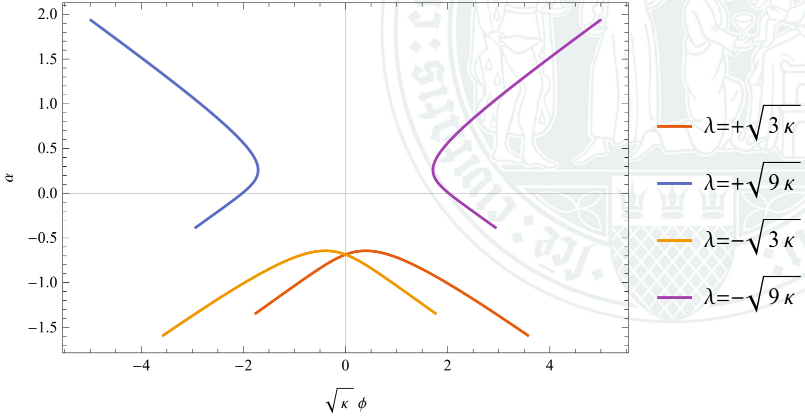
Trajectories for quintessence model $(+,+)$

sech, with $\beta_0 = 0$, $|V| = \varkappa^{-2}$ and $p_\beta^2 = \varkappa^2\sqrt{|V|}$; varying λ

$$\mathrm{e}^{g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|}\operatorname{sech}^2\left(\sqrt{\frac{3}{2\varkappa}}\, g(\beta - \beta_0)\right) \tag{5}$$



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Trajectories for quintessence model $(+,-)$

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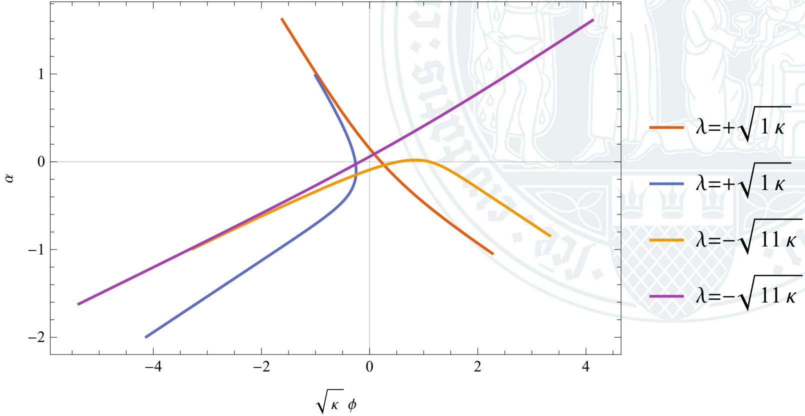
$$\mathrm{e}^{g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|}\operatorname{csch}^2\left(\sqrt{\frac{3}{2\varkappa}}\,g(\beta-\beta_0)\right) \tag{6}$$



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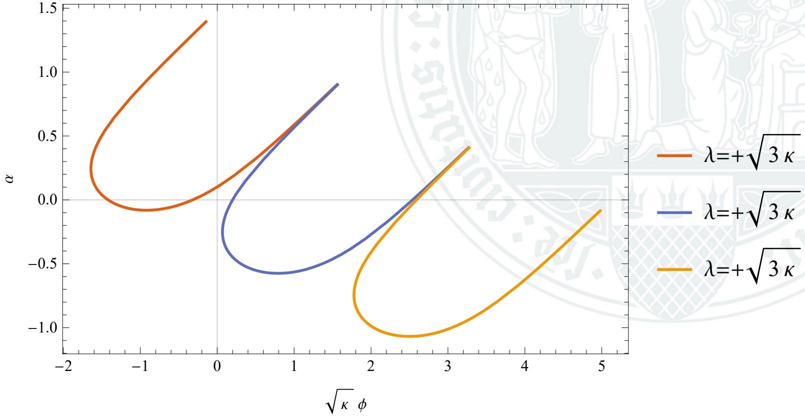
Trajectories for phantom model $(-,+)$

sec, **with** $V=\varkappa^{-2}$ **and** $p_\beta^2 = 3\varkappa^2\sqrt{|V|}$

$$e^{g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|}\sec^2\left(\sqrt{\frac{3}{2\varkappa}}\,g(\beta-\beta_0)\right) \tag{7}$$



Outline

Dirac quantisation

- The primary Hamiltonian and the Hamiltonian constraint[7] reads

$$H^{\mathsf{p}} = \overline{N} H_\perp + v^{\overline{N}} p_{\overline{N}}, \tag{8}$$

$$H_\perp := -\jmath \frac{p_\beta^2}{12\varkappa^{1/2}} + \ell \jmath \frac{p_\chi^2}{2\varkappa^{3/2}} + \varkappa^{3/2} V \mathbb{e}^{g\chi}. \tag{9}$$

- Applying the Dirac quantisation rules with the Laplace–Beltrami operator ordering[8], one gets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_\perp \Psi(\beta, \chi) := \left(\jmath \frac{\hbar^2}{12\varkappa^{1/2}} \partial_\beta^2 - \ell \jmath \frac{\hbar^2}{2\varkappa^{3/2}} \partial_\chi^2 + \varkappa^{3/2} V \mathbb{e}^{g\chi} \right) \Psi. \tag{10}$$

- Equation (10) is Klein–Gordon-like, hyperbolic for $\ell = +1$ and *elliptic* for $\ell = -1$.

[7] D. M. Gitman, I. V. Tyutin, *Quantization of Fields with Constraints*, (Springer, 1990) , H. J. Rothe, K. D. Rothe, *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*, (World Scientific, Apr. 2010) .

[8] C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012) , ch. 8.

Integration of the Wheeler–DeWitt equation

The quantum-mechanical analogy

- Fourier transforming β to $k_\beta \in \mathbb{R}$ yields the time-independent Schrödinger equation

$$E\psi(x) := \ell\frac{\hbar^2\tilde{g}^2\nu^2}{8M_\mathsf{P}}\psi = \widehat{H}_\mathsf{eff}\psi := \left(-\frac{\hbar^2}{2M_\mathsf{P}}\partial_x^2 + V_\mathsf{eff}\right)\psi, \quad V_\mathsf{eff} := \ell\, \jmath v \widetilde{V}\mathbb{e}^{\tilde{g}x}, \tag{11}$$

in which $M_\mathsf{P} := \sqrt{\frac{\hbar}{\varkappa}}$, $x := \sqrt{\hbar\varkappa}\cdot\chi$, $\widetilde{V} := M_\mathsf{P}^{-3}|V| \geq 0$, and $\nu = \sqrt{\frac{2\varkappa}{3}}\frac{|k_\beta|}{g} \geq 0$.

  - The Liouville potential V_eff is a special case of the Morse potential[9].
  - The Starobinsky model in the Einstein frame is related to the Morse potential.

- Let $\mathbb{e}^{2y} := 8M_\mathsf{P}\widetilde{V}/\hbar^2\tilde{g}^2 \cdot \mathbb{e}^{\tilde{g}x}$; there are four cases according to $(\ell, \jmath v)$, and the mode functions are

	$E \geq 0$	$E < 0$
$V_\mathsf{eff} > 0$	$(+,+)$; $\mathrm{K}_{\mathbb{i}\nu}(\mathbb{e}^y)$	$(-,-)$; none
$V_\mathsf{eff} < 0$	$(+,-)$; $\mathrm{F}_{\mathbb{i}\nu}(\mathbb{e}^y)$, $\mathrm{G}_{\mathbb{i}\nu}(\mathbb{e}^y)$[10]	$(-,+)$; $\mathrm{J}_\nu(\mathbb{e}^y)$

[9] P. M. Morse, *Phys. Rev.* **34**, 57–64 (July 1929), D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) .

[10] These are linear combinations of $\mathrm{J}_{\pm\mathbb{i}\nu}(\mathbb{e}^{\tilde{x}})$; see T. M. Dunster, *SIAM J. Math. Anal.* **21**, 995–1018 (July 1990).

Self-adjointness of the Hamiltonian

Results for the Liouville potential[14]

- $k_\beta \in \mathbb{R}$, enforcing the self-adjointness of $\widehat{H}_{\text{eff}}$.
- For $V_{\text{eff}} > 0$: $\widehat{H}_{\text{eff}}$ is essentially self-adjoint[11].
  - Quintessence $(+,+)$: nothing special.
- For $V_{\text{eff}} < 0$: $\widehat{H}_{\text{eff}}$ admits a family of self-adjoint extensions.
  - Quintessence $(+,-)$: "eigenfunctions" are linear combinations of F and G; the degeneracy is removed
  - Phantom $(-,+)$: the spectrum becomes discrete

$$\psi_{-+,\nu}(\mathbb{e}^y) \propto \mathrm{J}_\nu(\mathbb{e}^y), \qquad \nu = 2n + a_<, \; n \in \mathbb{Z}_\geq, \tag{12}$$

  and the corresponding full wave-functions are Bloch-periodic. Imposing (anti-)periodic boundary condition fixes $a_< = 0\ (1)$.

- The self-adjointness has been discussed in quantum cosmology in e.g.[12] (operator ordering) and more recently in[13] (Λ as a quantum number).

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[11] See also B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267 , sec. 9.9.
[12] C. R. Almeida *et al.*, *Gravitation Cosmol.* **21**, 191–199 (Jan. 17, 2015).
[13] S. Gryb, K. P. Y. Thébault, arXiv: 1801.05789 (gr-qc) (Jan. 17, 2018).
[14] See also D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) , sec. 8.5.

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Matching quantum number with classical first integral

Principle of constructive interference; $(+,+)$ as the example

- Write the mode function in the Wentzel–Kramers–Brillouin form $\Psi_\nu(\beta,\chi) \sim \sqrt{\rho(\beta,\chi)}\exp\{\frac{i}{\hbar}S(\beta,\chi)\}$. For $S/\hbar \gg 1$ and $\nu \gg 1$, $S(\beta,\chi)$ becomes the Hamilton principal function in the leading-order approximation, which is stationary with respect to the variation of integral constants $\partial S/\partial\nu = 0$.

- Fixing $\nu/\sigma > 1$,

$$\mathrm{K}_{i\nu}(\sigma) \sim \frac{\sqrt{2\pi}\cos\left(\sqrt{\nu^2-\sigma^2} - \nu\arccos\frac{\nu}{\sigma} - \frac{\pi}{4}\right)}{\left(\nu^2-\sigma^2\right)^{1/4} e^{\pi\nu/2}} + O\left(\frac{1}{\sigma}\right). \quad (13)$$

- Applying *the principle* to the phase of $\Psi_\nu(\sigma)$, one has $\sigma/\nu = \operatorname{sech}(s_\beta\gamma)$, which matches the trajectory if $\hbar k_\beta \equiv \hbar\sqrt{\frac{3}{2\varkappa}}g\nu = p_\beta$.
- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; it is not within the Wentzel–Kramers–Brillouin regime.

- The conclusions also hold for $\mathrm{F}_{i\nu}(\sigma)$, $\mathrm{G}_{i\nu}(\sigma)$ for $(+,-)$, and $\mathrm{J}_\nu(\sigma)$ for $(-,+)$.

Inner products for wave functions

Schrödinger and Mostafazadeh inner product

- A common starting point is the *Schrödinger product*[15]

$$(\Psi_1, \Psi_2)_{\mathsf{S}} := \int \mathrm{d}\chi\, \Psi_1^*(\beta, \chi) \Psi_2(\beta, \chi); \tag{14}$$

  - $(\Psi, \Psi)_{\mathsf{S}}$ is positive-definite, and the integrand $\rho_{\mathsf{S}}(\beta, \chi)$ is non-negative
  - The corresponding Schrödinger current is real but does not satisfy continuity equation

- Mostafazadeh[16]: *for self-adjoint* $\mathbb{D}$ *with positive spectrum* in $(-\partial_t^2 + \mathbb{D})\Psi = 0$:

$$(\Psi_1, \Psi_2)_{\mathsf{M}}^{\kappa} := \kappa \Big\{ \big(\Psi_1, \mathbb{D}^{+1/2} \Psi_2\big)_{\mathsf{S}} + \big(\partial_\beta \Psi_1, \mathbb{D}^{-1/2} \partial_\beta \Psi_2\big)_{\mathsf{S}} \Big\}, \qquad \kappa > 0. \tag{15}$$

  - Real power of $\mathbb{D}$ is defined by spectral decomposition
  - It can be shown[17] that *the density*

$$\varrho_{\mathsf{M}}^{\kappa} := \kappa \Big\{ \big|\mathbb{D}^{+1/4} \Psi\big|^2 + \big|\mathbb{D}^{-1/4} \partial_\beta \Psi\big|^2 \Big\} \tag{16}$$

    - is equivalent to the integrand $\rho_{\mathsf{M}}^{\kappa}$ up to a boundary term $\int \mathrm{d}\chi\, \varrho_{\mathsf{M}}^{\kappa} = \int \mathrm{d}\chi\, \rho_{\mathsf{M}}^{\kappa} \equiv (\Psi_1, \Psi_2)_{\mathsf{M}}^{\kappa}$
    - is non-negative
    - The corresponding current $\vec{\mathcal{J}}_{\mathsf{M}}^{\kappa}$ is real but not conserved[18].

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[15] C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012) , ch. 5.
[16] A. Mostafazadeh, *Class. Quantum Grav.* **20**, 155–171 (Sept. 6, 2002).
[17] A. Mostafazadeh, F. Zamani, *Ann. Phys.* **321**, 2183–2209 (Feb. 17, 2006).
[18] B. Rosenstein, L. P. Horwitz, *J. Phys. A: Math. Gen.* **18**, 2115–2121 (Aug. 1985).

Wave packets of Gaussian amplitude for continuous spectrum

Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu; \overline{\nu}, \sigma) := \left(\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(\nu - \overline{\nu})^2}{2\sigma^2} \right) \right)^{1/2} \tag{17}$$

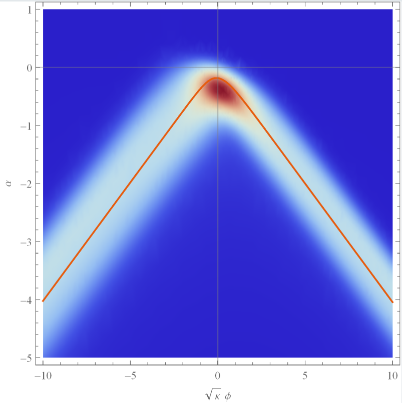
- In[19], $A\left(\nu; \overline{\nu}, \sigma/\sqrt{2}\right)$ was chosen.

[19] M. P. Dąbrowski *et al.*, *Phys. Rev. D* **74**, arXiv: hep-th/0605229 (hep-th) (May 23, 2006).

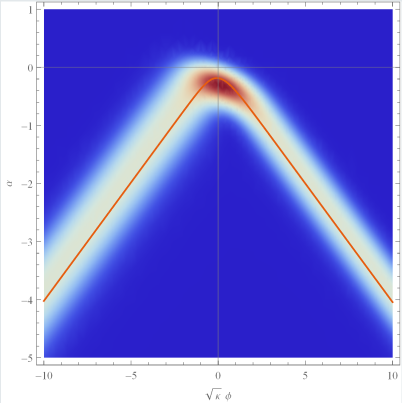
Wave packets of Gaussian amplitude for quintessence model $(+,+)$

$\mathrm{K}_{i\nu}$, **with** $\lambda = \varkappa^{1/2}/2$, $V = -\varkappa^{-2}$, $\overline{k}_\beta = -2$ **and** $\sigma_\beta = 5/4$

Schrödinger



Mostafazadeh



Wave packets of Gaussian amplitude for quintessence model $(+,+)$

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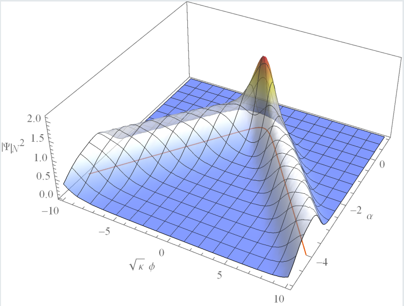
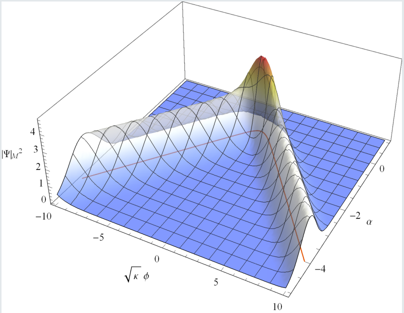


Mostafazadeh



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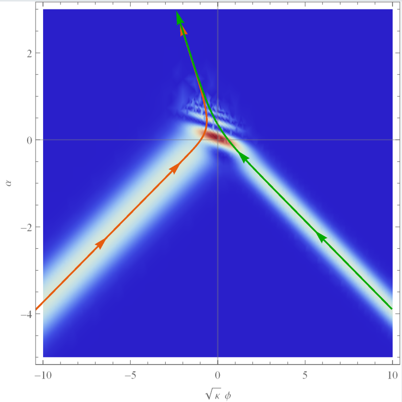
Wave packets of Gaussian amplitude for quintessence model $(+,-)$

$\mathrm{F}_{\mathring{\imath}\nu}$, **with** $\lambda = 4\varkappa^{1/2}/5$, $V = +\varkappa^{-2}$, $\overline{k}_\beta = -7/2$ **and** $\sigma_\beta = 7/5$

Schrödinger



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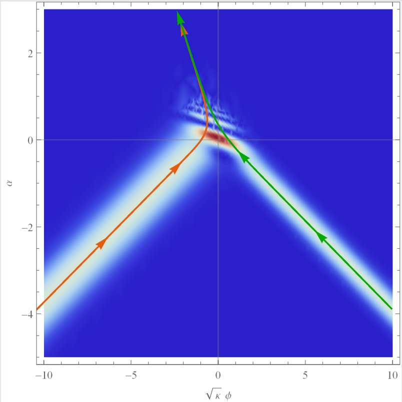
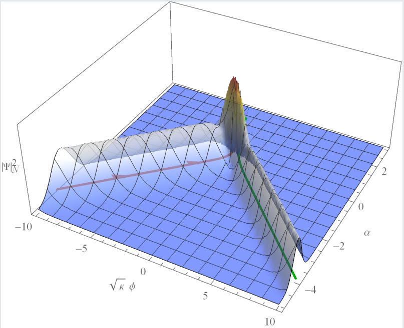


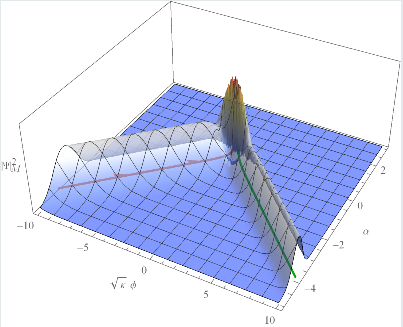
Wave packets of Gaussian amplitude for quintessence model $(+,-)$

$\mathrm{F}_{\mathring{\imath}\nu}$, **with** $\lambda = 4\varkappa^{1/2}/5$, $V = +\varkappa^{-2}$, $\bar{k}_\beta = -7/2$ **and** $\sigma_\beta = 7/5$

Schrödinger



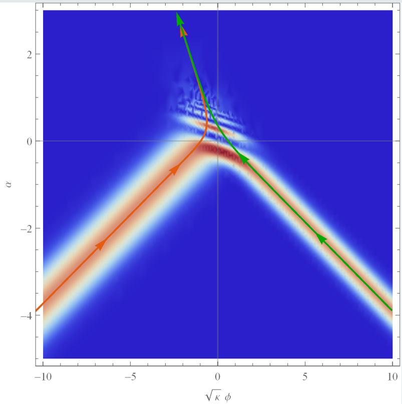
Mostafazadeh



Wave packets of Gaussian amplitude for quintessence model $(+,-)$

$\mathrm{G}_{\mathfrak{i}\nu}$, **with** $\lambda = 4\varkappa^{1/2}/5$, $V = +\varkappa^{-2}$, $\bar{k}_\beta = -7/2$ **and** $\sigma_\beta = 7/5$

Schrödinger



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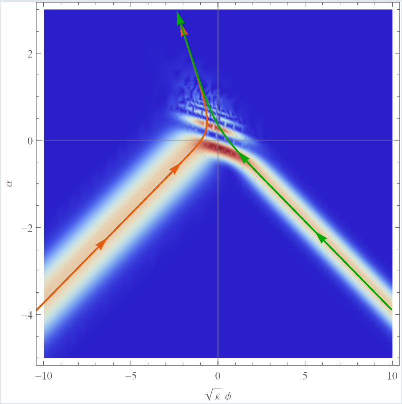

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Wave packets of Gaussian amplitude for quintessence model $(+,-)$

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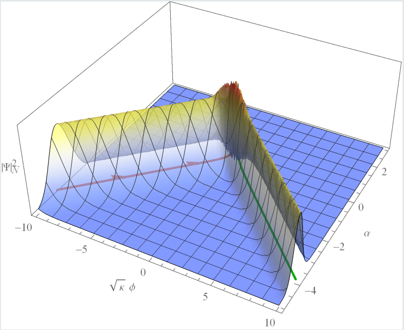
Schrödinger



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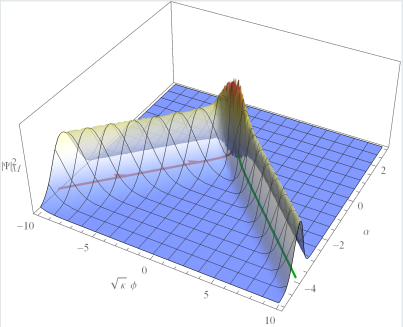


Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model $(-,+)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

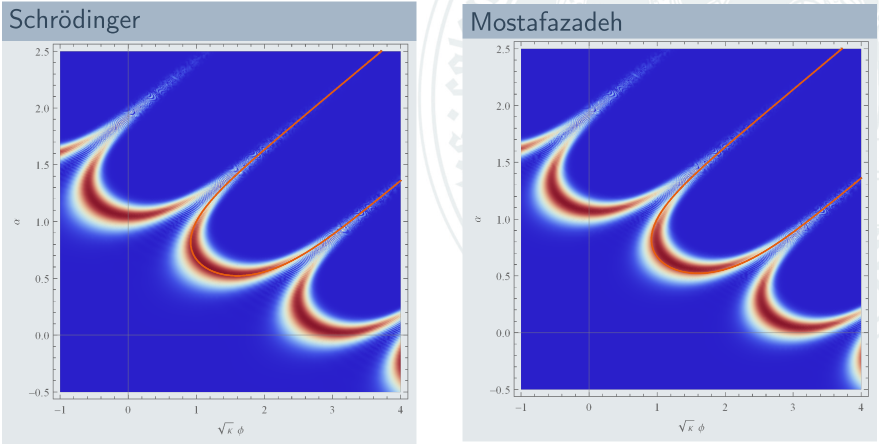
$$A_n(\overline{n}) := \left(\mathbb{e}^{-\overline{n}} \frac{\overline{n}^n}{n!} \right)^{1/2} \tag{18}$$

- In[20], $A_n\left(\overline{n}/\sqrt{2}\right)$ was chosen.

[20] C. Kiefer, *Nucl. Phys. B* **341**, 273–293 (Sept. 1990).

Wave packets of Poissonian amplitude for phantom model

J_{2n+1}, with $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$ and $\bar{k}_\beta = 8$



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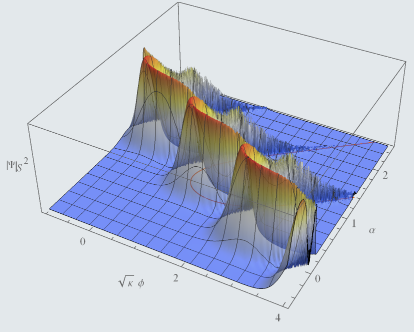
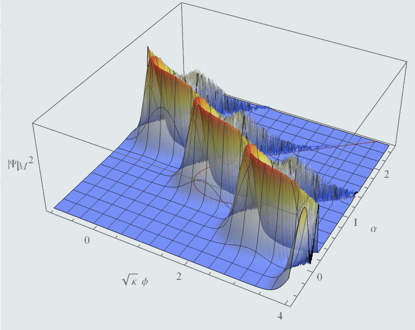
Wave packets of Poissonian amplitude for phantom model

J_{2n+1}, with $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$ and $\overline{k}_\beta = 8$

Schrödinger



Mostafazadeh



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Highlights
Revisited

- An integral of motion was found for the Liouville cosmological model.
  - Implicit trajectories in minisuperspace were obtained explicitly.
  - The minisuperspace Wheeler–DeWitt equation was integrated exactly.
- The self-adjointness of the matter Hamiltonian was found to be non-trivial.
  - The degeneracy of the quintessence model $(+,-)$ was removed.
  - The spectrum of the phantom model $(-,+)$ became discrete.
- The semi-classical wave functions were compared with the classical trajectory.
  - Mode functions in the Wentzel–Kramers–Brillouin approximation were matched with the classical Hamilton principal function.
  - Semi-classical wave packets were constructed and compared with the classical trajectory under the Schrödinger and Mostafazadeh inner product.

Outlook

- Beyond isotropy: Bianchi models[21]
- Beyond homogeneity: cosmological perturbations[22]
- Beyond single Liouville field: multiple potentials / multiple fields[23]
- Beyond classical matter and Schrödinger inner product: PT-symmetric field[24]

[21] A. Y. Kamenshchik *et al.*, *Phys. Rev. D* **95**, arXiv: 1702.02314 (gr-qc) (Feb. 8, 2017).

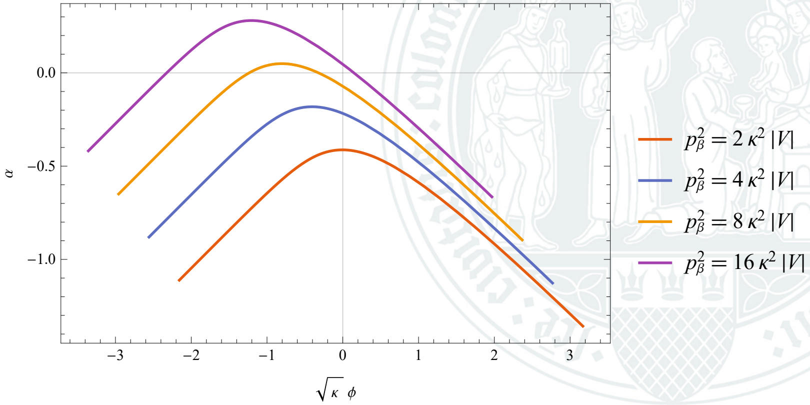
[22] D. Brizuela *et al.*, *Phys. Rev. D* **93**, arXiv: 1511.05545 (gr-qc) (Nov. 17, 2015), D. Brizuela *et al.*, *Phys. Rev. D* **94**, arXiv: 1611.02932 (gr-qc) (Nov. 9, 2016), O. O. Novikov, arXiv: 1712.03939 (hep-th) (Dec. 11, 2017), A. Y. Kamenshchik *et al.*, *Class. Quantum Grav.* **35**, 015012 (Sept. 29, 2017), C. Kiefer, D. Wichmann, arXiv: 1802.01422 (gr-qc) (Feb. 5, 2018).

[23] A. A. Andrianov *et al.*, *Theor. Math. Phys.* **184**, 1224–1233 (Mar. 18, 2015).

[24] A. A. Andrianov *et al.*, *Int. J. Mod. Phys. D* **19**, 97–111 (Jan. 2010), A. A. Andrianov *et al.*, in *Springer Proceedings in Physics* (Springer International Publishing, 2016), pp. 29–44.

Trajectories for quintessence model $(+,+)$

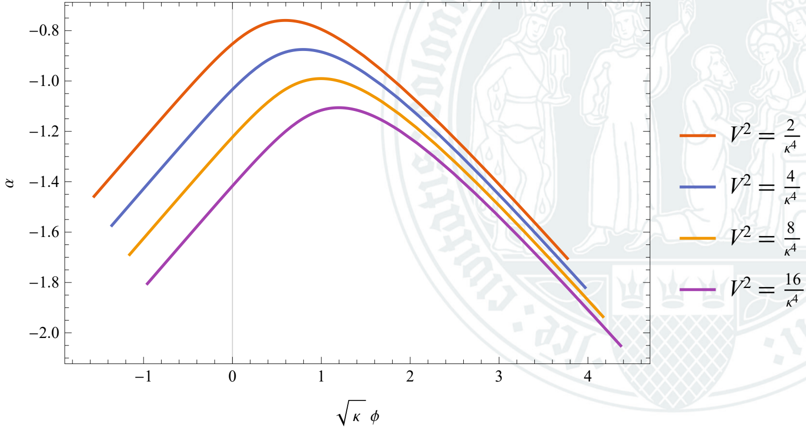
sech, with $\beta_0 = 0$, $|V| = \varkappa^{-2}$ and $\lambda^2 = 3\varkappa$; varying p_β



- has two asymptotes $\chi \propto \pm\beta$

Trajectories for quintessence model $(+,+)$

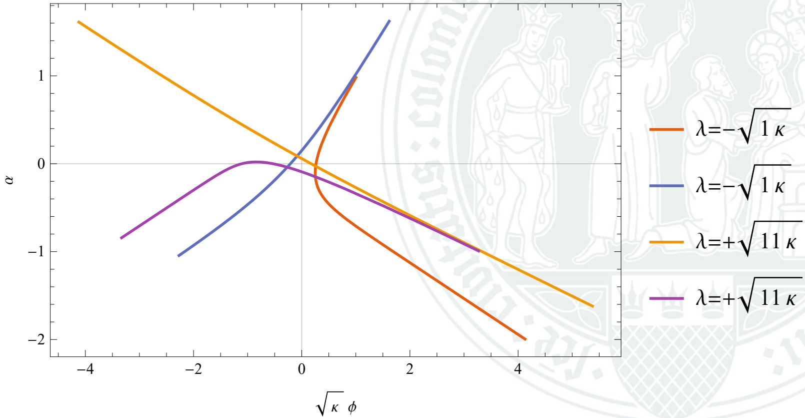
sech, with $\beta_0 = 0$, $\lambda^2 = 3\varkappa$ and $p_\beta^2 = \varkappa^2\sqrt{|V|}$; varying V



- has two asymptotes $\chi \propto \pm\beta$

Trajectories for quintessence model $(+,-)$: csch

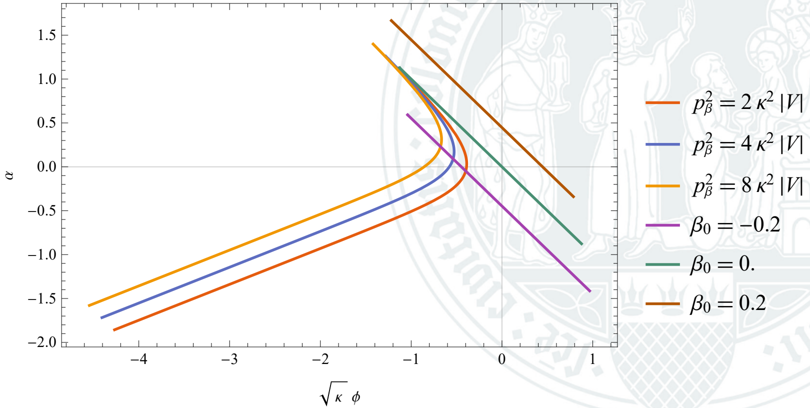
csch, with $\beta_0 = 0$, $|V| = \varkappa^{-2}$ and $p_\beta^2 = \varkappa^2 \sqrt{|V|}$; varying λ



- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$

Trajectories for quintessence model $(+,-)$: csch

csch, **with** $V = \varkappa^{-2}$ **and** $\lambda^2 = \varkappa$; **varying** p_β

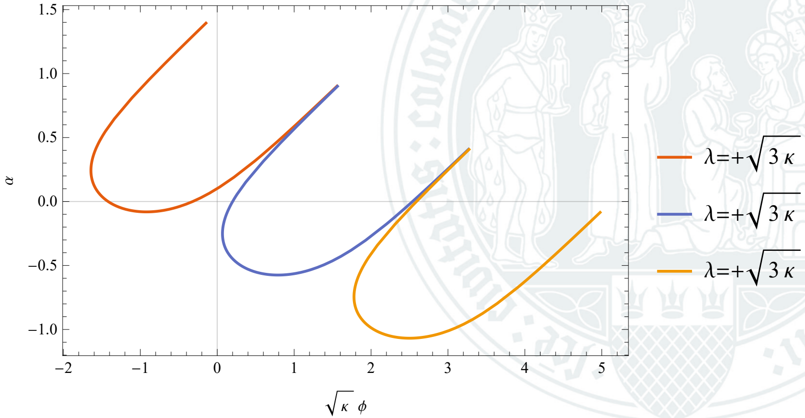


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$

Andrianov, Lan, Novikov, Wang
SPBU, ELI-ALPS, UzK, UB
arXiv:1802.06720

Trajectories for phantom model $(-,+)$

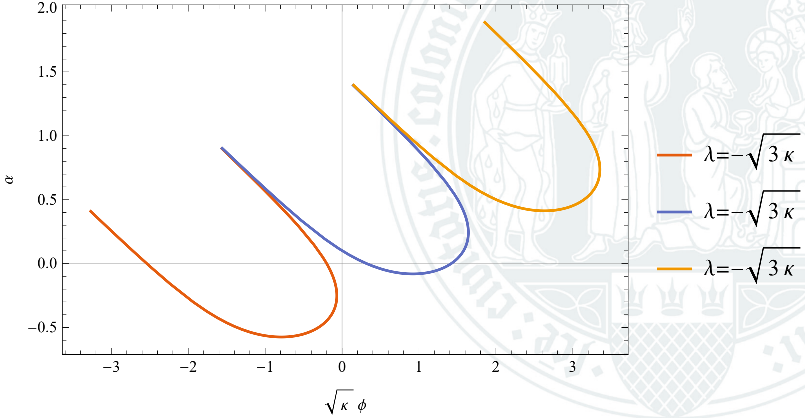
sec, **with** $V = \varkappa^{-2}$ **and** $p_\beta^2 = 3\varkappa^2\sqrt{|V|}$



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$

Trajectories for phantom model $(-,+)$

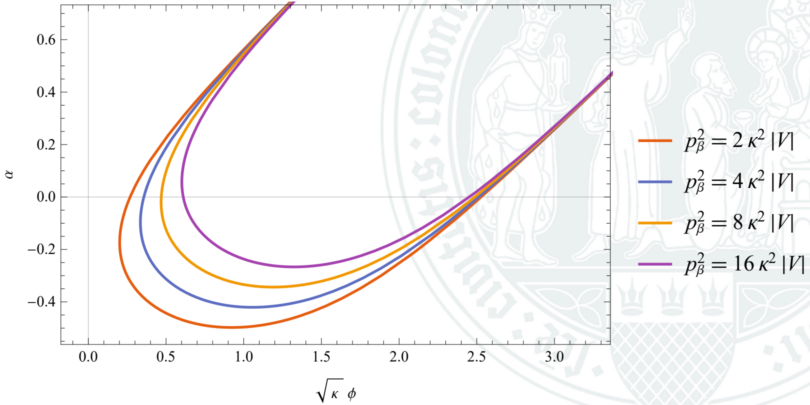
csc, **with** $V = \varkappa^{-2}$ **and** $p_\beta^2 = 3\varkappa^2\sqrt{|V|}$



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$

Trajectories for phantom model $(-,+)$

csc, with $V = \varkappa^{-2}$ and $\lambda^2 = 3\varkappa$; varying p_β



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$

In order to integrate the equation under the implicit gauge

$$\jmath \frac{p_\beta^2}{12}\left(-\ell \frac{\varkappa^{1/2}}{6}\left(\frac{\mathrm{d}\chi}{\mathrm{d}\beta}\right)^2 + \varkappa^{-1/2}\right) - \varkappa^{3/2} V \mathrm{e}^{g\chi} = 0, \tag{19}$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa}} g\beta, \qquad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} \mathrm{e}^{-g\chi}, \tag{20}$$

to get

$$\left(\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\gamma}\right)^2 + \ell(\jmath\upsilon - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-\jmath\upsilon + \tilde{\sigma}^2)}}, \tag{21}$$

which is of the standard inverse hyperbolic / trigonometric form except for $(-,-)$.

Integration of the separated minisuperspace WDW equation

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \qquad \mathbb{D} := -\ell \frac{6}{\varkappa}\partial_\chi^2 + \jmath\upsilon \frac{12\varkappa^2 |V|}{\hbar^2}, \tag{22}$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}}\frac{k_\beta}{g}, \qquad \sigma^2 := \frac{8\varkappa^3 |V| \mathrm{e}^{g\chi}}{\hbar^2 g^2}, \tag{23}$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma\psi'(\sigma) + \ell(\nu^2 - \jmath\upsilon\sigma^2)\psi(\sigma) = 0, \tag{24}$$

which is of the standard Besselian form.

Self-adjointness of unbounded operators

General theory

- Even in quantum geometrodynamics, the self-adjointness of the matter Hamiltonian is desirable, which is typically an unbounded operator in the Hilbert space $\mathbf{F}$ endowed with the Schrödinger inner product $(\cdot, \cdot)_S$.
- Mathematically, an *unbounded* operator H is characterised not only by its action on a vector, but also by its domain $\mathrm{Dom}(H) \subsetneq \mathbf{F}$[25].
- In addition to the *symmetricity* $(H^\dagger \phi_1, \phi_2) \equiv (\phi_1, H\phi_2)$, the self-adjointness of an unbounded operator also requires $\mathrm{Dom}(H^\dagger) = \mathrm{Dom}(H)$.
- For quantum systems not bounded below, not only the self-adjointness, but also the symmetricity of the Hamiltonian is not guaranteed automatically.
- Even when one could find a $\mathrm{Dom}_0(H)$ such that H is symmetric, one would still be left with $\mathrm{Dom}_0(H^\dagger) \supseteq \mathrm{Dom}_0(H)$ in general.
- Sloppily speaking, the process of extending $\mathrm{Dom}(H)$ such that $\mathrm{Dom}(H^\dagger) = \mathrm{Dom}(H)$ is called self-adjoint extension[26]; if the extension is unique, the operator is called *essentially self-adjoint*.

[25] B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267 , ch. 9.

[26] D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) , G. Bonneau *et al.*, *Am. J. Phys* **69**, 322–331 (Mar. 28, 2001), V. S. Araujo *et al.*, *Am. J. Phys* **72**, 203–213 (Feb. 2004), A. M. Essin, D. J. Griffiths, *Am. J. Phys* **74**, 109–117 (Feb. 2006).