

Integrable Liouville Cosmological Models

Self-adjointness of the Hamiltonian and Semi-classical Wave Functions

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Outline

1. Introduction

2. The classical model and the implicit trajectories

Lagrangian formalism and a first integral

3. The Dirac quantisation and the self-adjointness

Canonical formalism and Dirac quantisation

4. The semi-classical wave packets

Semi-classical approximation
Inner product and wave packet

5. Conclusions



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Introduction

Friedmann–Lemaître model with quintessence and phantom Liouville fields

- Dynamical Dark Energy has been modelled by quintessence¹ and phantom² matter, which can be realised by minimally-coupled real scalar fields with $\ell = \pm 1$ ³

$$\mathcal{S}_L = \int d^4x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \quad (1)$$

- The Liouville field⁴ $\mathcal{V}(\phi) = V e^{\lambda\phi}$ is of interest, where $\lambda, V \in \mathbb{R}$.
- Assume a flat Robertson–Walker metric $g_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + \varkappa e^{2\alpha(t)} d\Omega_{3F}^2$, where $\varkappa := 8\pi G$, $d\Omega_{3F}^2 := d\chi^2 + \chi^2(d\theta^2 + \sin^2 \theta d\varphi^2)$, and N lapse function
- The total action reads $\mathcal{S} := \mathcal{S}_{EH} + \mathcal{S}_{GHY} + \mathcal{S}_L = \int d\Omega_{3F}^2 \int dt L$, where

$$L := \varkappa^{3/2} N e^{3\alpha} \left(-\frac{3}{\varkappa} \frac{\dot{\alpha}^2}{N^2} + \ell \frac{\dot{\phi}^2}{2N^2} - V e^{\lambda\phi} \right). \quad (2)$$

¹ R. R. Caldwell, R. Dave, and P. J. Steinhardt. In: *Phys. Rev. Lett.* 80.8 (1998), pp. 1582–1585.

² R. R. Caldwell. In: *Phys. Lett. B* 545.1-2 (2002), pp. 23–29.

³The signature of metric is mostly positive.

⁴ Y. Nakayama. In: *International Journal of Modern Physics A* 19.17n18 (2004), pp. 2771–2930.



Introduction

Highlights

Integrability

Implicit trajectories can be obtained explicitly; the minisuperspace Wheeler–DeWitt equation can be integrated exactly.

Self-adjointness

The Hamiltonian is not even naturally symmetric; imposing self-adjointness leads to non-trivial physical results.

Semi-classical wave functions

Semi-classical wave functions can be constructed by the principle of constructive interference, the JWKB approximation and numerical methods.



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Decoupling the variables

Via orthogonal transformation

- By rescaling $\bar{N} := Ne^{-3\alpha}$ and the orthogonal transformation

$$\begin{pmatrix} \alpha(t) \\ \phi(t) \end{pmatrix} = \frac{\varsigma}{g} \begin{pmatrix} \lambda & -\ell\kappa \\ -6 & \lambda \end{pmatrix} \begin{pmatrix} \varsigma_\beta \cdot \beta(t) \\ \varsigma_\chi \cdot \chi(t) \end{pmatrix} \quad \text{where } \varsigma_\beta, \varsigma_\chi = \pm 1, \quad (3)$$

the effective Lagrangian eq. (2) can be decoupled ($\varsigma_\beta = \varsigma_\chi = +1$ from now on)

$$L = \kappa^{3/2} \bar{N} \left(-\varsigma \frac{3}{\kappa} \frac{\dot{\beta}^2}{\bar{N}^2} + \ell \varsigma \frac{\dot{\chi}^2}{2\bar{N}^2} - V e^{g\chi} \right), \quad (4)$$

where $\Delta := \lambda^2 - 6\ell\kappa$, $\varsigma := \operatorname{sgn} \Delta$ and $g := \varsigma \sqrt{|\Delta|} \equiv \varsigma \sqrt{3\Delta}$.

- The Euler–Lagrange equations with respect to \bar{N} , β and χ will be called the modified first, second Friedmann equations and the Klein–Gordon equation, respectively.
- Since $\beta(t)$ is cyclic, its conjugate momentum p_β is conserved⁵, and the modified second Friedmann equation can be readily integrated.

⁵The same first integral has been found in C. Lan. PhD thesis. Saint Petersburg State University, 2016, A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theor. Math. Phys.* 184.3 (2015), pp. 1224–1233, in canonical formalism.



Integration and the implicit trajectories

- For $p_\beta \neq 0$, fixing the *implicit gauge* $\bar{N} = -6\mathfrak{s}\sqrt{\varkappa}\dot{\beta}/p_\beta$, the modified first Friedmann equation can be integrated, yielding the **implicit** trajectory

$$\mathbb{e}^{g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|} S^2 \left(\sqrt{\frac{3}{2\varkappa}} g(\beta - \beta_0) \right), \quad (5)$$

in which $v := \operatorname{sgn} V$, $(\operatorname{sgn}, \operatorname{sgn})$ means $(\ell, \mathfrak{s}v)$, and

$$\begin{aligned} (+,+)\,S(\gamma) &:= \operatorname{sech}(\gamma), & (+,-)\,S(\gamma) &:= \operatorname{csch}(\gamma), \\ (-,+)\,S(\gamma) &:= \sec(\gamma), & (-,-)\,S(\gamma) &:= \mathfrak{i} \csc(\gamma). \end{aligned} \quad (6)$$

- The integral for $(-, -)$ is not real.
- The trajectories can be parametrised by β , inspiring recognising β as a ‘time variable’.
- For $p_\beta = 0$, integrating the modified second Friedmann equation yields $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell \lambda \alpha / \varkappa^6$, which is the well-known power-law special solution⁷.

⁶It only works for $(+, -)$ or $(-, +)$ if one checks consistency with the modified first Friedmann equation.

⁷For instance, A. R. Liddle and D. H. Lyth. *Cosmological Inflation and Large-Scale Structure*. Cambridge University, 2000, ch. 3.



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Dirac quantisation

- The primary Hamiltonian and the Hamiltonian constraint⁸ reads

$$H^p = \bar{N} H_{\perp} + v^{\bar{N}} p_{\bar{N}}, \quad (7)$$

$$H_{\perp} := -\beta \frac{p_{\beta}^2}{12\nu^{1/2}} + \ell \beta \frac{p_{\chi}^2}{2\nu^{3/2}} + \nu^{3/2} V e^{g\chi}. \quad (8)$$

- Applying the Dirac quantisation rules with the Laplace–Beltrami operator ordering⁹, one gets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(\beta \frac{\hbar^2}{12\nu^{1/2}} \partial_{\beta}^2 - \ell \beta \frac{\hbar^2}{2\nu^{3/2}} \partial_{\chi}^2 + \nu^{3/2} V e^{g\chi} \right) \Psi. \quad (9)$$

- Equation (9) is Klein–Gordon-like, hyperbolic for $\ell = +1$ and *elliptic* for $\ell = -1$.

⁸ D. M. Gitman and I. V. Tyutin. *Quantization of Fields with Constraints*. Springer, 1990, H. J. Rothe and K. D. Rothe. *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*. World Scientific, 2010.

⁹ C. Kiefer. *Quantum Gravity*. 3rd ed. Oxford University, 2012, ch. 8.



Integration of the Wheeler–DeWitt equation

Quantum-mechanical analogy

- Fourier-transforming β to k_β yields

$$\ell E\psi(x) = \widehat{H}_{\text{eff}}\psi(x) := \left(-\frac{\hbar^2}{2M_P} \partial_x^2 + \ell s v \widetilde{V} e^{\tilde{g}x} \right) \psi(x), \quad (10)$$

in which $E := \hbar^2 k_\beta^2 / 12\nu^{1/2} \geq 0$, $M_P := \nu^{-1/2}$, $x := \nu\chi$, and $\widetilde{V} := \nu^{3/2}|V|$.

- For the quintessence model $(+, +)$: \widehat{H}_{eff} bounded below; generalised eigenfunctions are the Besselian $K_{\pm E}$'s.
- For the quintessence model $(+, -)$: \widehat{H}_{eff} bounded **above**; generalised eigenfunctions for positive spectrum / 'scattering states' are the Besselian $F_{\pm E}$'s and $G_{\pm E}$ 's¹⁰.
- For the phantom model $(-, +)$: \widehat{H}_{eff} bounded **above**; generalised eigenfunctions for negative spectrum / 'bound states' are the Besselian J_E 's.

¹⁰

T. M. Dunster. In: *SIAM Journal on Mathematical Analysis* 21.4 (1990), pp. 995–1018.



Separation of the variables and mode functions

- Writing $\Psi(\beta, \chi) = \varphi(\beta)\psi(\chi)$, eq. (9) can be separated into

$$\partial_\beta^2 \varphi(\beta) = k_\beta^2 \varphi(\beta); \quad (11)$$

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \imath v \frac{12\varkappa^2 |V|}{\hbar^2}. \quad (12)$$

- Equation (12) turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta, \chi) := \sum_{i=1}^2 c_i \varphi_\nu^{(i)}(\gamma) \sum_{j=1}^2 a_j B_\nu^{(j)}(\sigma), \quad \nu \geq 0; \quad (13)$$

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{1}{g} k_\beta, \quad \gamma := \sqrt{\frac{3}{2\varkappa}} \frac{g}{1} \beta, \quad \sigma^2 := \frac{8\varkappa^3 |V| e^{g\chi}}{\hbar^2 g^2}, \quad (14)$$

$${}_{(+,+)} B_\nu^{(i)}(\sigma) := K \text{ and } I_{\mathbb{B}\nu}(\sigma), \quad {}_{(+,-)} B_\nu^{(i)}(\sigma) := F \text{ and } G_{\mathbb{B}\nu}(\sigma),$$

$${}_{(-,+)} B_\nu^{(i)}(\sigma) := J \text{ and } Y_\nu(\sigma), \quad {}_{(--)} B_\nu^{(i)}(\sigma) := K \text{ and } I_\nu(\sigma).$$

- Adapted to imaginary order, $F_\nu(\sigma)$ and $G_\nu(\sigma)$ are defined in¹¹.

¹¹

T. M. Dunster. In: *SIAM Journal on Mathematical Analysis* 21.4 (1990), pp. 995–1018.



Physical mode functions

- Physical mode functions are expected to be regular on the boundary.
- $(+, +)$: $|I_{\pm\nu}(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow +\infty$
- $(-, +)$:
 - $\forall n \in \mathbb{N}$, $|Y_n(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow -\infty$.
 - $\forall \nu \in \mathbb{R}^+ \setminus \mathbb{N}$, choose $J_{-\nu}$ instead of Y_ν , since $J_{\pm\nu}$ are also linearly independent.
 - $\forall \nu \in \mathbb{R}^+ \setminus \mathbb{N}$, $|J_{-\nu}(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow -\infty$.
- $(-, -)$: $|K_\nu(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow -\infty$; $|I_\nu(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow +\infty$
- These are not to be included in the space of physical wave functions. $\forall \nu \geq 0$,
 - $(+, +)$: $K_{\pm\nu}(\sigma)$ survives
 - $(+, -)$: $F_{\pm\nu}(\sigma)$ and $G_{\pm\nu}(\sigma)$ survives
 - $(-, +)$: $J_\nu(\sigma)$ survives
 - $(-, -)$: drops out



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Matching quantum number with classical first integral

Principle of constructive interference

- Write the mode function in the WKB form

$$\Psi_{k_\beta}(\beta, \chi) \sim \sqrt{\rho(\beta, \chi)} \exp\left\{ \frac{i}{\hbar} S(\beta, \chi) \right\}. \quad (15)$$

- For $S/\hbar \gg 1$ and $k_\beta \gg 1^{12}$, $S(\beta, \chi)$ becomes the Hamilton principle function in the leading-order approximation.
- A Hamilton principle function is stationary with respect to variation of integral constants¹³

$$\frac{\partial S}{\partial k_\beta} = 0. \quad (16)$$

- Demanding eq. (16) matching the classical trajectory, k_β can be related to p_β .

¹²The common form is $\hbar \rightarrow 0^+$.

¹³ U. H. Gerlach. In: *Phys. Rev.* 177.5 (1969), pp. 1929–1941, L. D. Landau and E. M. Lifshitz. *Mechanics*. 3rd ed. Elsevier, 1976.



Matching quantum number with classical first integral

(+, +) as exemplar

- Fixing $\nu/\sigma > 1$, the asymptotic expansion reads

$$K_{\pm\nu}(\sigma) \sim \frac{\sqrt{2\pi} \cos\left(\sqrt{\nu^2 - \sigma^2} - \nu \arccos \frac{\nu}{\sigma} - \frac{\pi}{4}\right)}{(\nu^2 - \sigma^2)^{1/4} e^{\mp\nu/2}} + O\left(\frac{1}{\sigma}\right). \quad (17)$$

- There are two phases with opposite signs. Assuming c_i, a_j 's are real and applying the principle to $\Psi_\nu(\sigma)$, one has $\sigma/\nu = \operatorname{sech}(\gamma)$, which matches the trajectory with $\beta_0 = 0$ if

$$\hbar k_\beta \equiv \hbar \sqrt{\frac{3}{2\pi}} g \hbar \nu = p_\beta, \quad (18)$$

- Non-vanishing C can be compensated by the phase of c_i and a_j 's.
- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; it is not within the WKB regime.
- The conclusions also hold for $F_{\pm\nu}(\sigma), G_{\pm\nu}(\sigma)$ for $(+,-)$, and $J_\nu(\sigma)$ for $(-,+)$.



Inner product for wave functions

Schrödinger product

- To make sense of amplitude and wave packet, an inner product is necessary
- In terms of a Klein–Gordon equation, call β the “temporal” variable, and χ the “spacial” variable.
- A common starting point is the *Schrödinger product*¹⁴

$$(\Psi_1, \Psi_2)_S := \int d\chi \Psi_1^*(\beta, \chi) \Psi_2(\beta, \chi); \quad (19)$$

- $(\Psi, \Psi)_S$ is **positive-definite**, and the integrand $\rho_S(\beta, \chi)$ is **non-negative**
- The corresponding Schrödinger current is **real** but **does not satisfy continuity equation** $\dot{\rho}_S + \nabla \cdot \vec{j}_S = 0$, because eq. (9) is KG-like.
- $K_{\parallel\nu}$ ¹⁵ for $(+, +)$, $F_{\parallel\nu}$ and $G_{\parallel\nu}$ for $(+, -)$ can be proved to be **orthogonal** and **complete** among themselves, as well as **can be normalised**.
 - $J_{\parallel\nu}$'s for $(+, -)$ are **not orthogonal**

¹⁴ C. Kiefer. *Quantum Gravity*. 3rd ed. Oxford University, 2012, ch. 5.

¹⁵ S. B. Yakubovich. In: *Opuscula Math.* 26.1 (2006), pp. 161–172, A. Passian et al. In: *Journal of Mathematical Analysis and Applications* 360.2 (2009), pp. 380–390, R. Szmytkowski and S. Bielski. In: *Journal of Mathematical Analysis and Applications* 365.1 (2010), pp. 195–197.



Peculiarity of the phantom model $(-, +)$

Orthogonality for mode functions; Hermiticity for operators

- $J_\nu(\sigma)$'s are not orthogonal under the Schrödinger product

$$(J_\nu, J_{\tilde{\nu}})_S \propto \int_{-\infty}^{+\infty} dx J_\nu^*(e^x) J_{\tilde{\nu}}(e^x) = \frac{2\sin(\pi(\nu - \tilde{\nu})/2)}{\pi(\nu^2 - \tilde{\nu}^2)}, \quad (20)$$

therefore \mathbb{D} in eq. (12) is not Hermitian (though we do not need it so far)

- \hat{p}_χ^2 is not Hermitian for $\{J_\nu(\sigma)\}$ under the Schrödinger product

$$\int_{-\infty}^{+\infty} dx J_\nu^*(-\partial_x^2 J_{\tilde{\nu}}) - \int_{-\infty}^{+\infty} dx (-\partial_x^2 J_\nu)^* J_{\tilde{\nu}} = \frac{2}{\pi} \sin \frac{\pi(\nu - \tilde{\nu})}{2}. \quad (21)$$

- In order to save Hermiticity for p_χ^2 and \mathbb{D} and orthogonality of the modes under Schrödinger product, one can restrict

$$\nu = 2n + \nu_0, \quad n \in \mathbb{N}, \quad \nu_0 \in [0, 2). \quad (22)$$

- Using classical singularities as boundary condition, one can fix $\nu_0 = 1$.



Discretisation of the phantom model $(-, +)$

Levels of the phantom model are **discretised** if one imposes Hermiticity of squared momenta under the Schrödinger product.

- This kind of subtlety on Hermiticity is named as *self-adjoint extension*, which arises already for infinite square well¹⁶.
- It also applies to x^{-2} potentials¹⁷, which is of cosmological relevance¹⁸.

¹⁶ G. Bonneau, J. Faraut, and G. Valent. In: *Am. J. Phys* 69.3 (2001), pp. 322–331.

¹⁷ A. M. Essin and D. J. Griffiths. In: *Am. J. Phys* 74.2 (2006), pp. 109–117, V. S. Araujo, F. A. B. Coutinho, and J. F. Perez. In: *Am. J. Phys* 72.2 (2004), pp. 203–213.

¹⁸ M. Bouhmadi-López et al. In: *Physical Review D* 79.12 (2009).



Further inner products for wave functions

Klein–Gordon and Mostafazadeh product

- Since eq. (9) is KG-like, another popular choice is the KG product

$$(\Psi_1, \Psi_2)_{\text{KG}}^g := \mathbb{i}g \left\{ (\Psi_1, \partial_\beta \Psi_2)_S - (\partial_\beta \Psi_1, \Psi_2)_S \right\}, \quad g > 0. \quad (23)$$

- $(\Psi, \Psi)_{\text{KG}}^g$ is **real** but **not positive-definite**, so does the integrand ρ_{KG} ;
- The corresponding \vec{J}_{KG} is **conserved** $\dot{\rho}_{\text{KG}} + \nabla \cdot \vec{J}_{\text{KG}} = 0$ and **real**.
- Mostafazadeh¹⁹ found a product *for Hermitian \mathbb{D} with positive spectrum*:

$$(\Psi_1, \Psi_2)_{\text{M}}^\kappa := \kappa \left\{ (\Psi_1, \mathbb{D}^{+1/2} \Psi_2)_S + (\partial_\beta \Psi_1, \mathbb{D}^{-1/2} \partial_\beta \Psi_2)_S \right\}, \quad \kappa > 0. \quad (24)$$

- $(\Psi, \Psi)_{\text{M}}^\kappa$ is **positive-definite**, but the integrand ρ_{M}^κ is **complex**
- The corresponding $\vec{J}_{\text{M}}^\kappa$ is **conserved** $\dot{\rho}_{\text{M}}^\kappa + \nabla \cdot \vec{J}_{\text{M}}^\kappa = 0$ but also **complex**.

¹⁹

A. Mostafazadeh. In: *Classical Quantum Gravity* 20.1 (2002), pp. 155–171.



Mostafazadeh inner product and the corresponding density

- Real power of \mathbb{D} is defined by spectral decomposition $\mathbb{D}^\gamma := \sum_\nu \nu^{2\gamma} \mathbf{P}_\nu$,
 $\mathbf{P}_\nu \Psi := \Psi_\nu (\Psi_\nu, \Psi)_S$.
- It can be shown²⁰ that *the density*

$$\varrho_M^\kappa := \kappa \left\{ |\mathbb{D}^{+1/4} \Psi|^2 + |\mathbb{D}^{-1/4} \partial_\beta \Psi|^2 \right\} \quad (25)$$

- is equivalent to ρ_M^κ up to a boundary term

$$\int d\chi \varrho_M^\kappa = \int d\chi \rho_M^\kappa \equiv (\Psi_1, \Psi_2)_M^\kappa; \quad (26)$$

- is non-negative.
- The corresponding current $\vec{\mathcal{J}}_M^\kappa$ is real but not conserved²¹.

²⁰ A. Mostafazadeh and F. Zamani. In: *Ann. Phys.* 321.9 (2006), pp. 2183–2209.

²¹ B. Rosenstein and L. P. Horwitz. In: *J. Phys. A: Math. Gen.* 18.11 (1985), pp. 2115–2121.



Wave packets of Gaussian amplitude for continuous spectrum

Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu; \bar{\nu}, \sigma) := \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\nu - \bar{\nu})^2}{2\sigma^2}\right) \right)^{1/2} \quad (27)$$

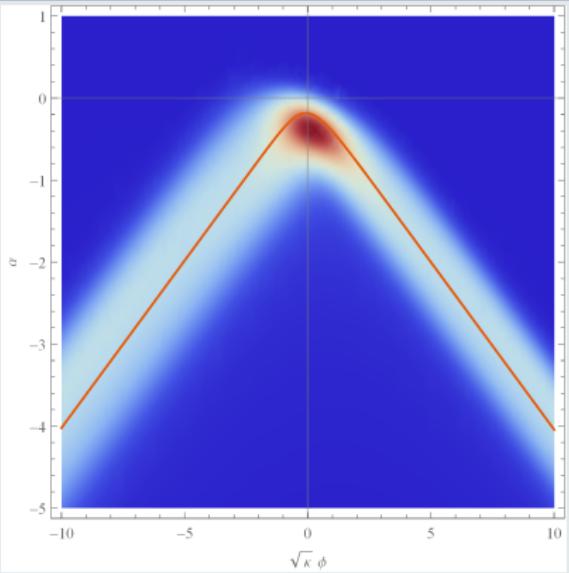
- In²², $A(\nu; \bar{\nu}, \sigma/\sqrt{2})$ was chosen.



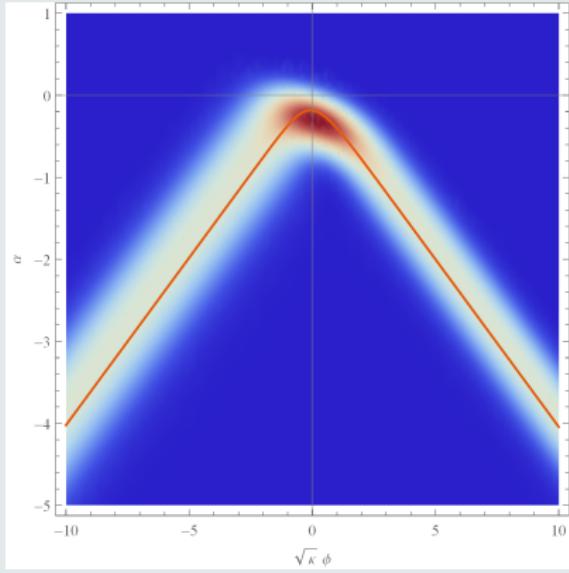
Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\ell\nu}$, with $\lambda = \varkappa^{1/2}/2$, $V = -\varkappa^{-2}$, $\bar{k}_\beta = -2$ and $\sigma_\beta = 5/4$

Schrödinger



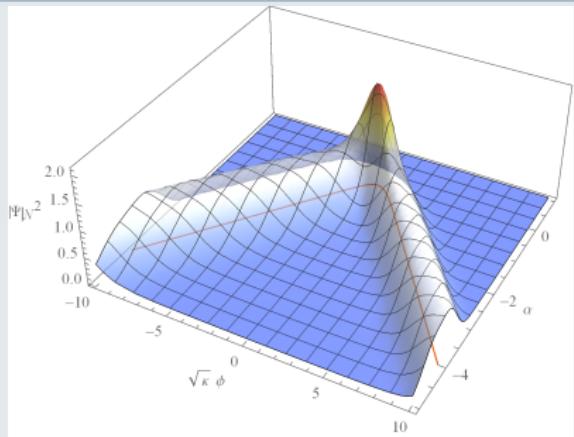
Mostafazadeh



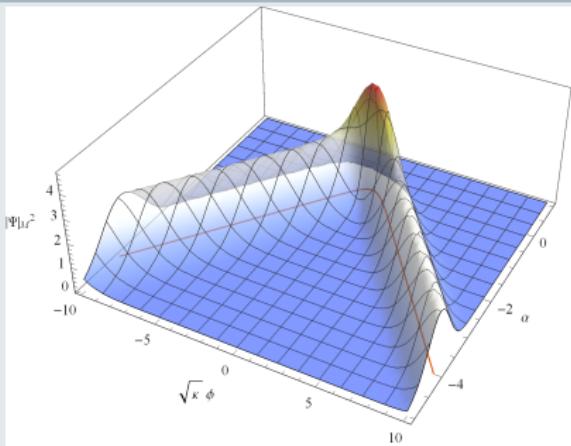
Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\ell\nu}$, with $\lambda = \varkappa^{1/2}/2$, $V = -\varkappa^{-2}$, $\bar{k}_\beta = -2$ and $\sigma_\beta = 5/4$

Schrödinger



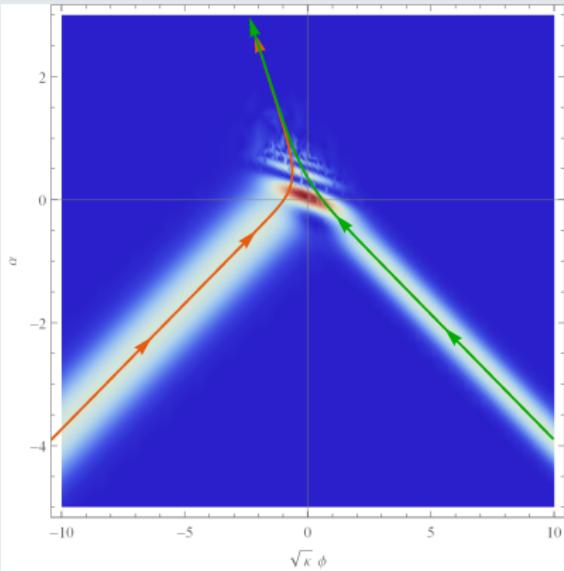
Mostafazadeh



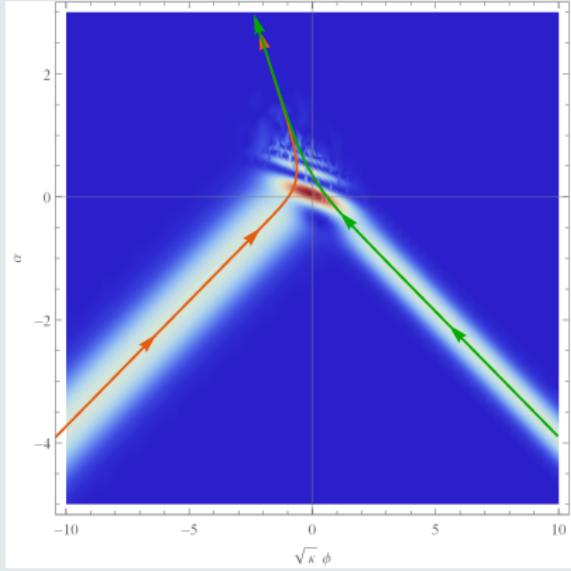
Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\parallel\nu}$, with $\lambda = 4\nu^{1/2}/5$, $V = +\nu^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



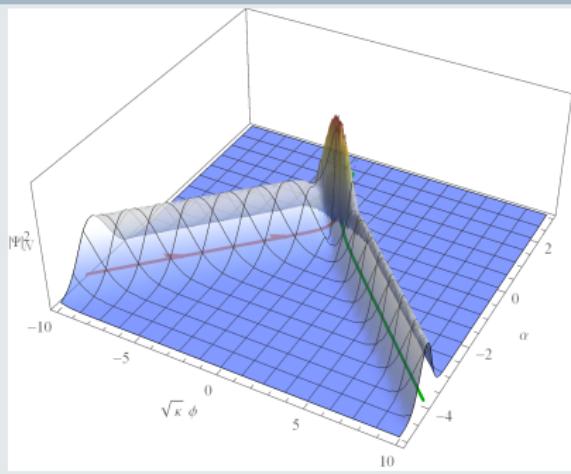
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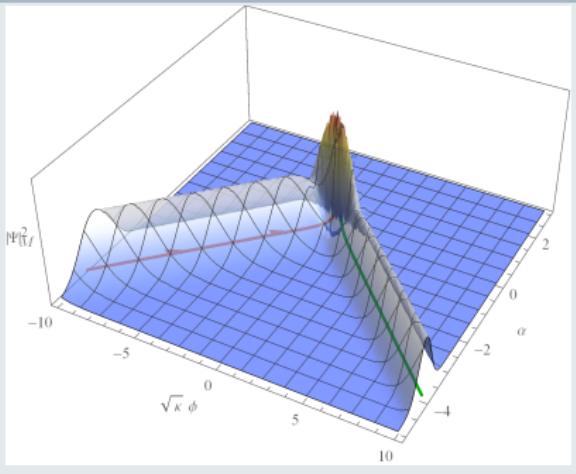
Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\parallel\nu}$, with $\lambda = 4\nu^{1/2}/5$, $V = +\nu^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



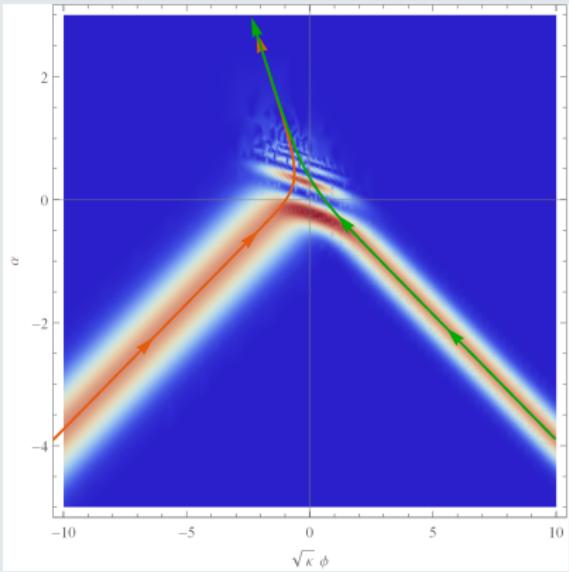
Mostafazadeh



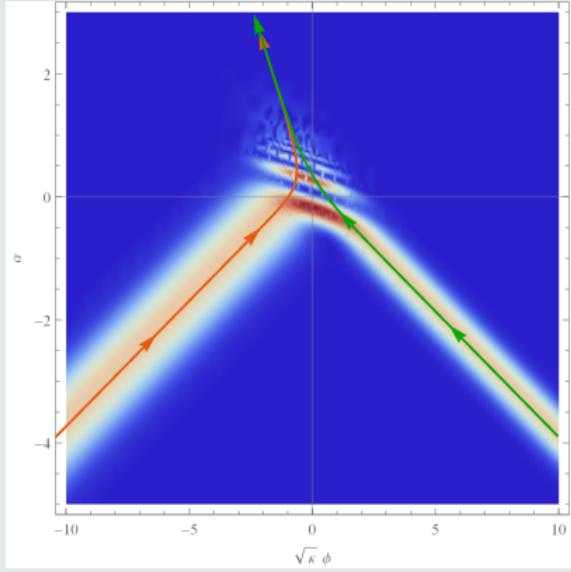
Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\parallel\nu}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



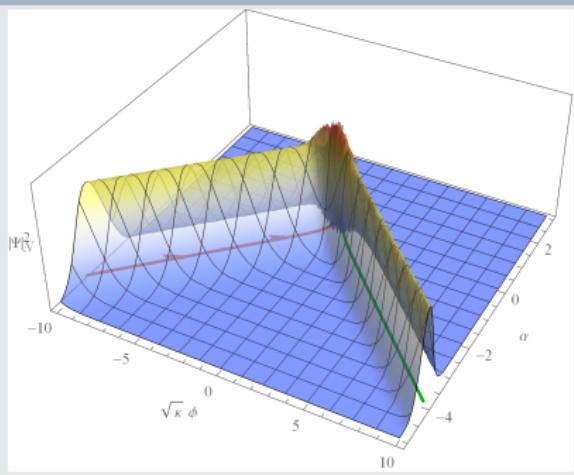
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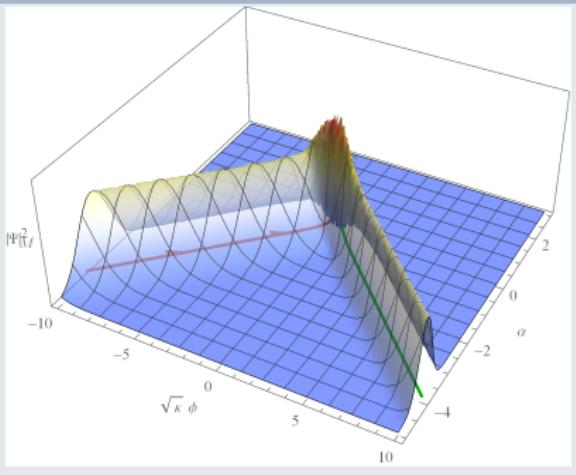
Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\parallel\nu}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



Mostafazadeh



Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model $(-, +)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\bar{n}) := \left(e^{-\bar{n}} \frac{\bar{n}^n}{n!} \right)^{1/2} \quad (28)$$

- In²³, $A_n(\bar{n}/\sqrt{2})$ was chosen.

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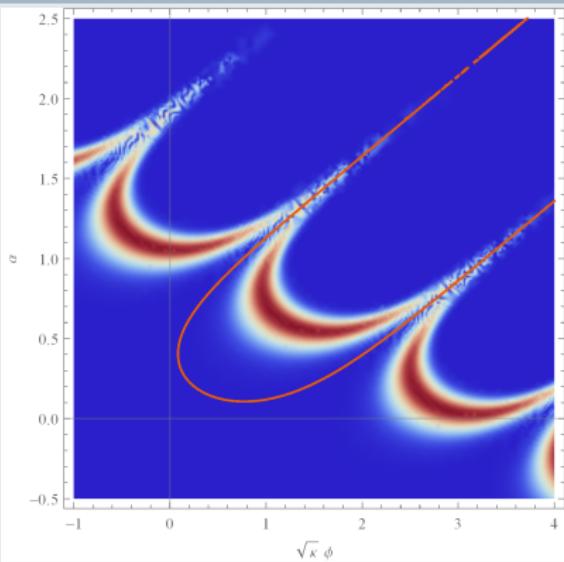
C. Kiefer. In: *Nucl. Phys. B* 341.1 (1990), pp. 273–293.



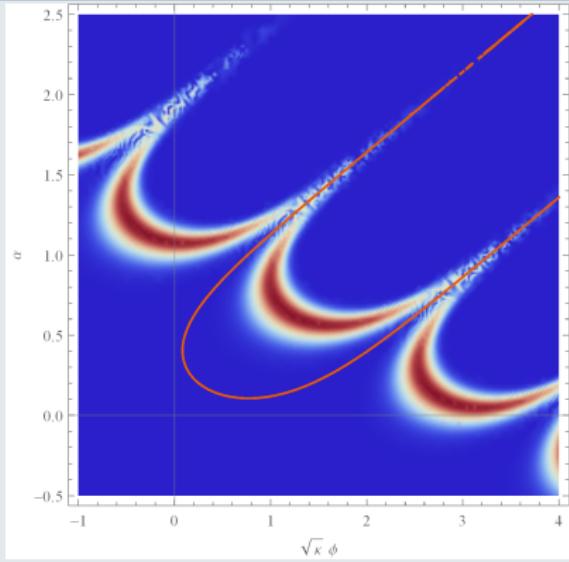
Wave packets of Poissonian amplitude for phantom model

J_{2n+1} , with $\lambda = 2\kappa^{1/2}$, $V = +\kappa^{-2}$ and $\bar{k}_\beta = 8$

Schrödinger



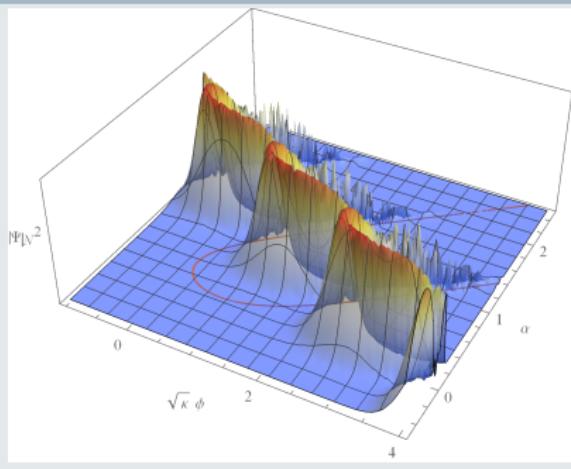
Mostafazadeh



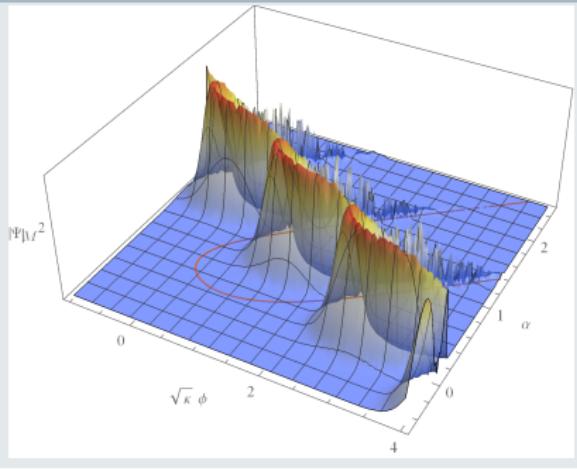
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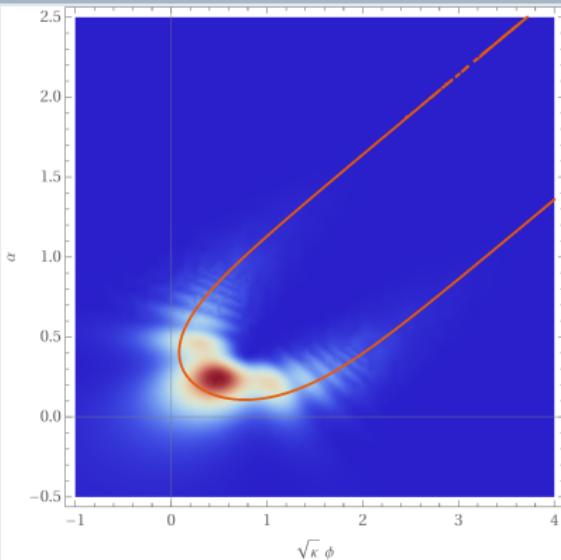
Mostafazadeh



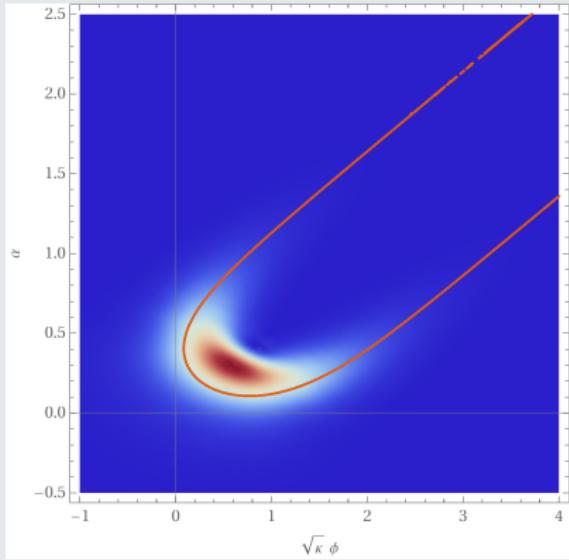
Wave packets of Gaussian amplitude for phantom model

J_ν , with $\lambda = 2\kappa^{1/2}$, $V = +\kappa^{-2}$, $\bar{k}_\beta = 8$ and $\sigma_\beta = 11/2$

Schrödinger



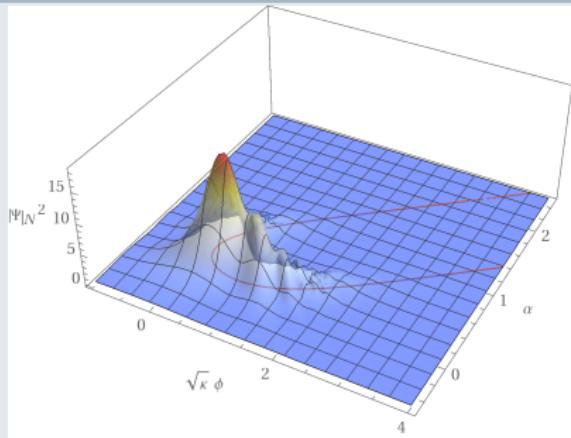
Mostafazadeh



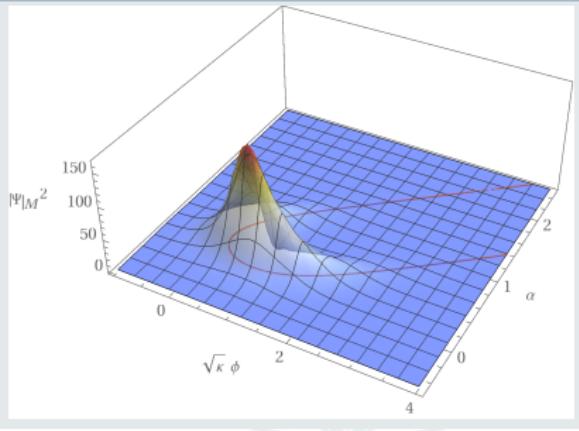
Wave packets of Gaussian amplitude for phantom model

J_ν , with $\lambda = 2\nu^{1/2}$, $V = +\nu^{-2}$, $\bar{k}_\beta = 8$ and $\sigma_\beta = 11/2$

Schrödinger



Mostafazadeh



Outline

1. Introduction
2. The classical model and the implicit trajectories
Lagrangian formalism and a first integral
3. The Dirac quantisation and the self-adjointness
Canonical formalism and Dirac quantisation
4. The semi-classical wave packets
Semi-classical approximation
Inner product and wave packet
5. Conclusions



Highlights

- An **integral of motion** was found for Liouville cosmological models.
- **Implicit trajectories** in minisuperspace were obtained.
- The levels of phantom Liouville model were found to be **discrete** due to the Hermiticity requirement of observables.
- The semi-classical quantum mode functions and wave packets was compared directly with the classical trajectory.



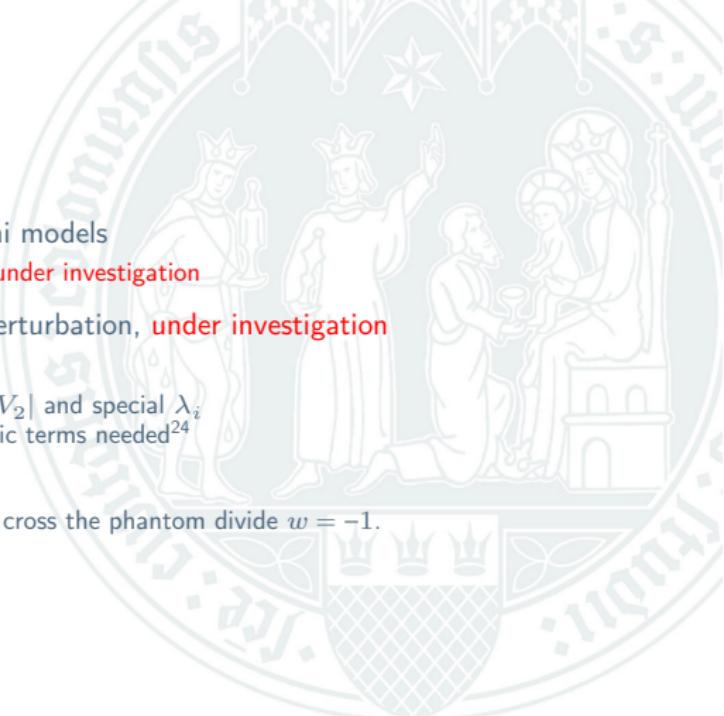
Issues

- In $(+, -)$ and $(-, +)$, wave packets contain multiple branches; however, the classical universe runs only on one trajectory.
- Quantum-corrected \bar{k}_β is to be understood.
- Wave packets with Gaussian profile are to be constructed, instead of inserting Gaussian / Poissonian amplitude artificially.
- A normalising κ for $(\cdot, \cdot)_M^\kappa$ is to be evaluated, otherwise a quantitative comparison of $(\cdot, \cdot)_S$ and $(\cdot, \cdot)_M^\kappa$ is not possible.



Outlook

- Beyond isotropy: generalise to Bianchi models
 - Bianchi Type-I: a natural extension, **under investigation**
- Beyond homogeneity: cosmological perturbation, **under investigation**
- Beyond single field
 - Two exponential potentials: $|V_1| = |V_2|$ and special λ_i
 - Multiple Liouville fields: mixing kinetic terms needed²⁴
- Beyond classic matter
 - PT -symmetric Liouville field²⁵: may cross the phantom divide $w = -1$.



²⁴ A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theor. Math. Phys.* 184.3 (2015), pp. 1224–1233.

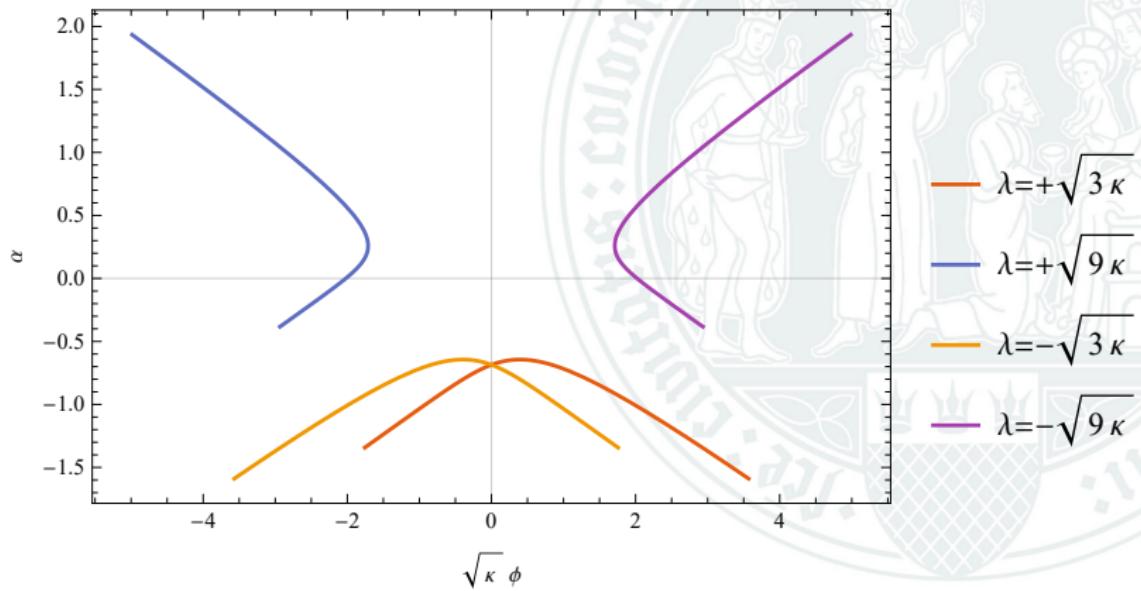
²⁵ A. A. Andrianov et al. In: *International Journal of Modern Physics D* 19.01 (2010), pp. 97–111,

A. A. Andrianov, C. Lan, and O. O. Novikov. In: *Springer Proceedings in Physics*. Springer International Publishing, 2016, pp. 29–44.



Trajectories for quintessence model $(+, +)$

sech, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

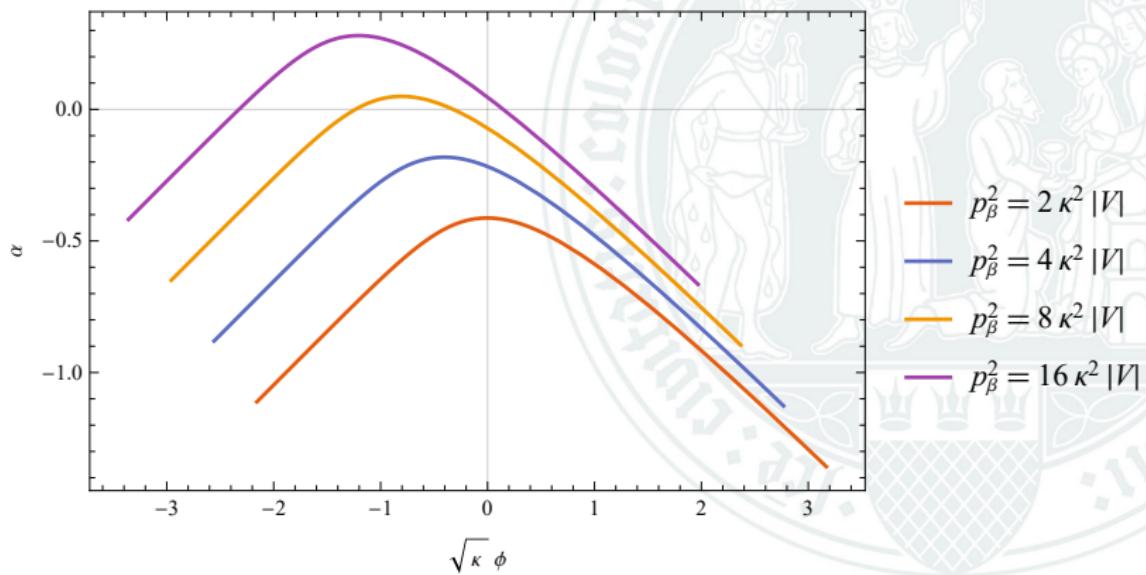


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model (+, +)

sech, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $\lambda^2 = 3\kappa$; varying p_β

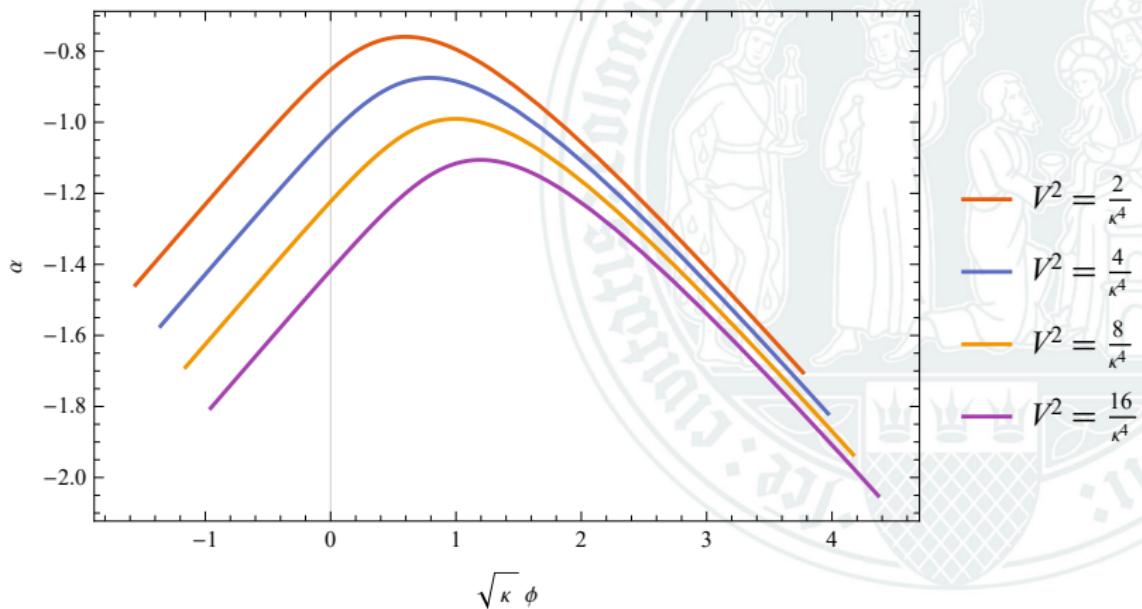


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model (+, +)

sech, with $\beta_0 = 0$, $\lambda^2 = 3\kappa$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying V

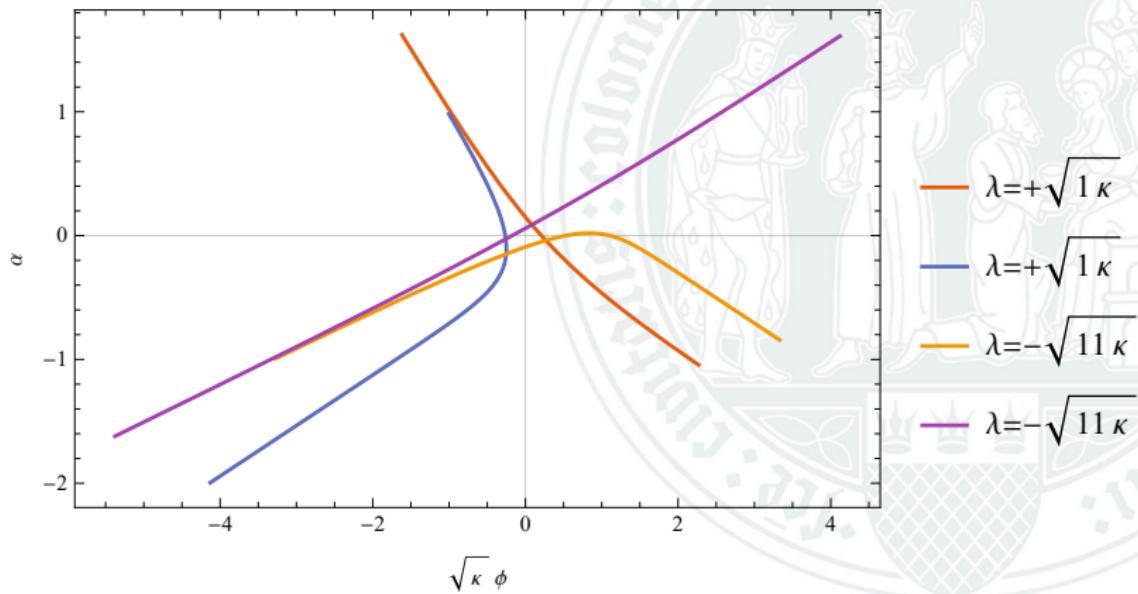


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model $(+, -)$

csch, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

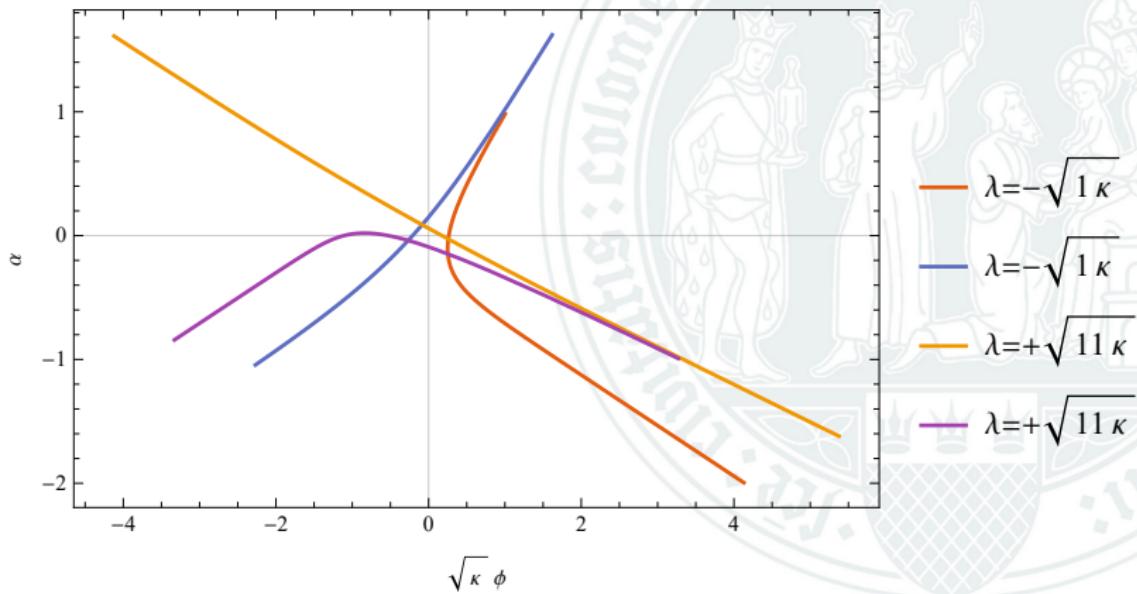


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm \beta$ and $\beta = 0$



Trajectories for quintessence model $(+, -)$: csch

csch, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

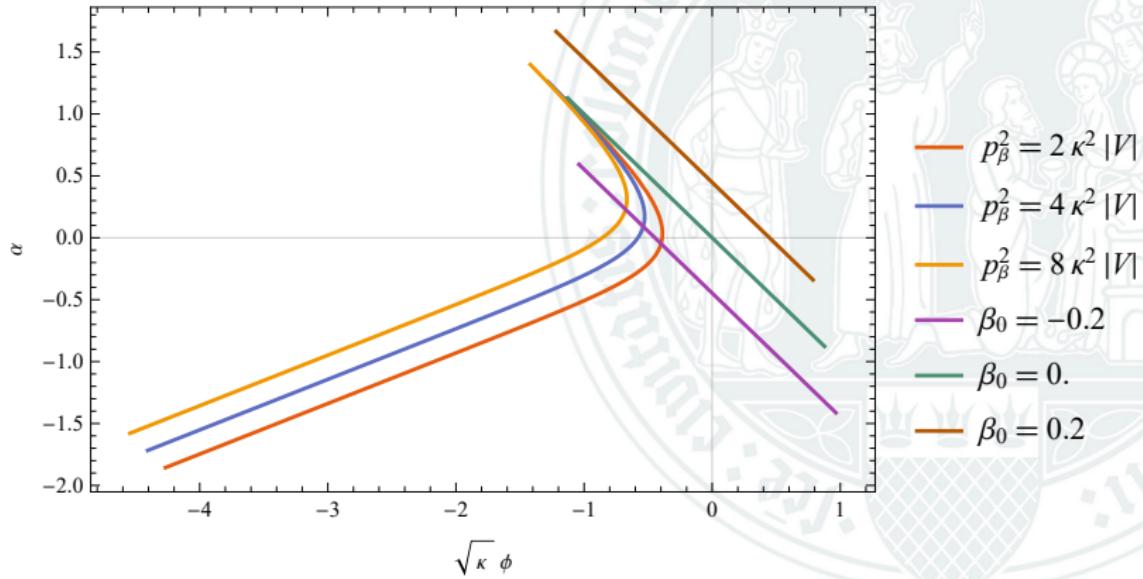


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$



Trajectories for quintessence model (+, -): csch

csch, with $V = \kappa^{-2}$ and $\lambda^2 = \kappa$; varying p_β

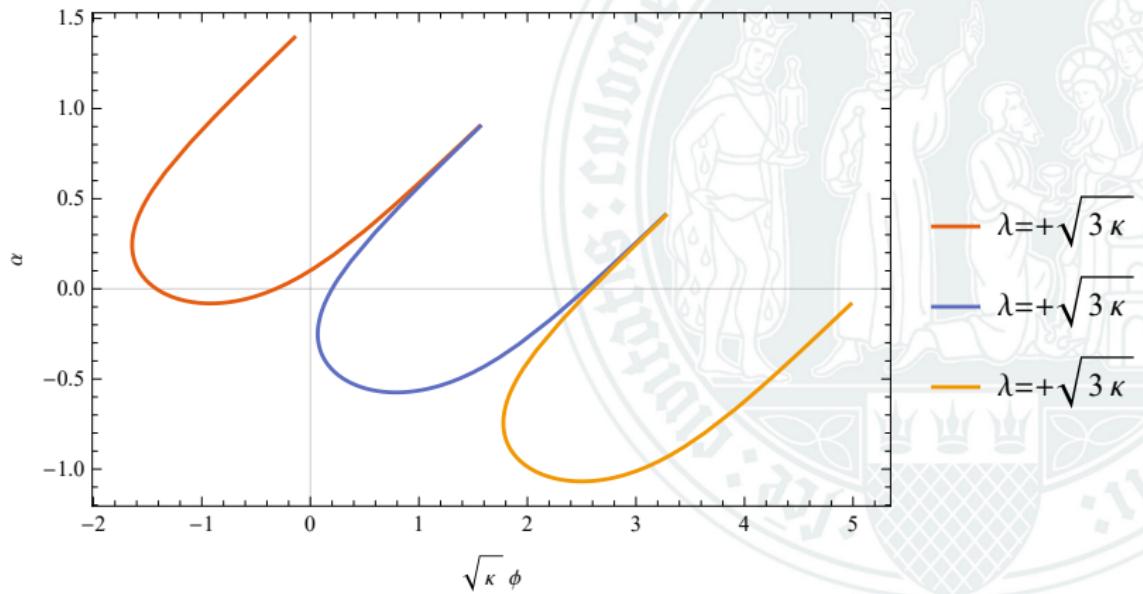


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$



Trajectories for phantom model $(-, +)$

csc, with $V = \varkappa^{-2}$ and $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

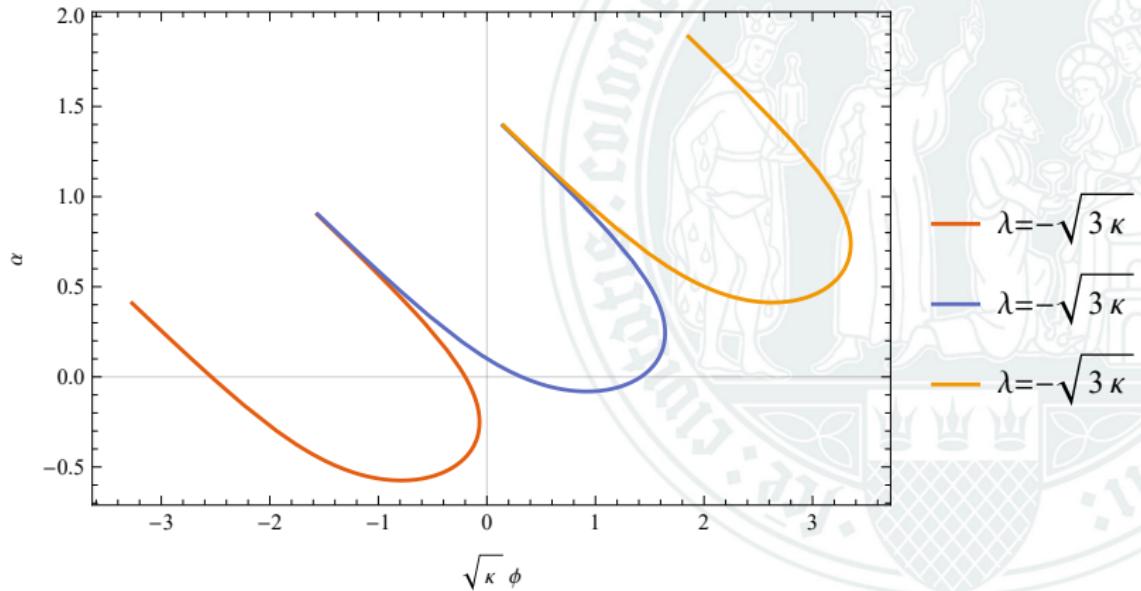


- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Trajectories for phantom model $(-, +)$

csc, with $V = \varkappa^{-2}$ and $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

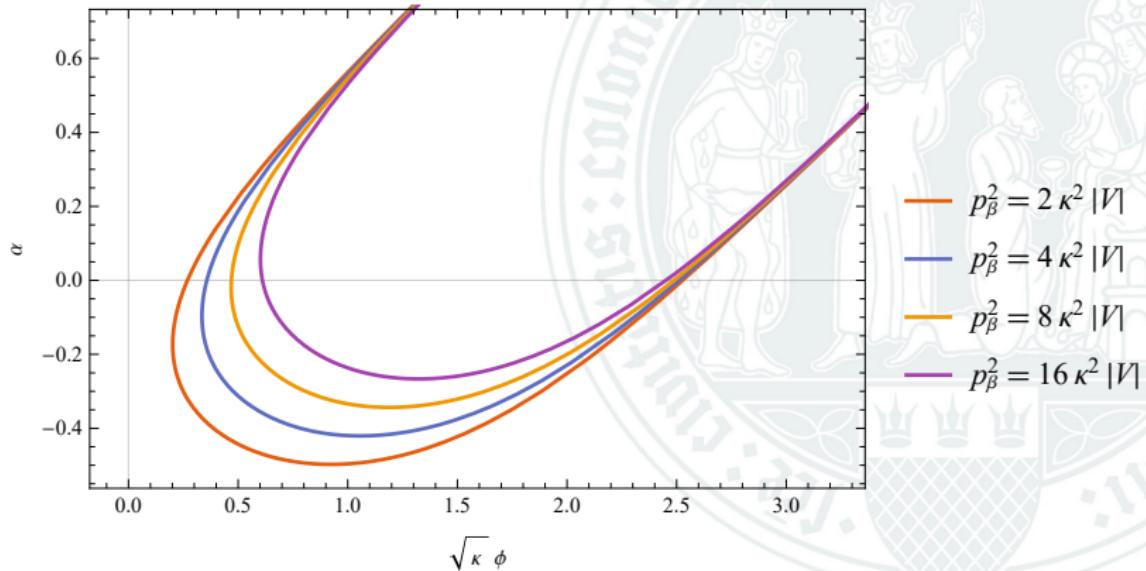


- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Trajectories for phantom model $(-, +)$

csc, with $V = \kappa^{-2}$ and $\lambda^2 = 3\kappa$; varying p_β



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Integration of the transformed first Friedmann equation

General integral for $p_\beta \neq 0$

In order to integrate the equation under the implicit gauge

$$s \frac{p_\beta^2}{12} \left(-\ell \frac{\varkappa^{1/2}}{6} \left(\frac{d\chi}{d\beta} \right)^2 + \varkappa^{-1/2} \right) - \varkappa^{3/2} V e^{g\chi} = 0, \quad (29)$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa} g\beta}, \quad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} e^{-g\chi}, \quad (30)$$

to get

$$\left(\frac{d\tilde{\sigma}}{d\gamma} \right)^2 + \ell(s\upsilon - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{d\gamma}{d\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-s\upsilon + \tilde{\sigma}^2)}}, \quad (31)$$

which is of the standard inverse hyperbolic / trigonometric form **except for $(-, -)$** .



Integration of the separated minisuperspace WDW equation

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \mathfrak{s} v \frac{12\varkappa^2 |V|}{\hbar^2}, \quad (12 \text{ rev.})$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{k_\beta}{g}, \quad \sigma^2 := \frac{8\varkappa^3 |V| e^{g\chi}}{\hbar^2 g^2}, \quad (14 \text{ rev.})$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma \psi'(\sigma) + \ell(\nu^2 - \mathfrak{s} v \sigma^2) \psi(\sigma) = 0, \quad (32)$$

which is of the standard Besselian form.



