Integrable Liouville Cosmological Models

The self-adjointness of Hamiltonian and the Semi-classical Wave Functions

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Outline

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Introduction

The Friedmann–Lemaître model with quintessence and phantom Liouville fields

- Dynamical Dark Energy has been modelled by quintessence[1] and phantom[2] matter, which can be realised by minimally-coupled real scalar fields with $\ell = \pm 1$[3]

$$\mathcal{S}_{\mathsf{L}} = \int \mathbb{d}^4 x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \qquad (1)$$

- The Liouville field[4] $\mathcal{V}(\phi) = V \mathbb{e}^{\lambda\phi}$ is of interest, where $\lambda, V \in \mathbb{R}$.
- Assume a flat Robertson–Walker metric $g_{\mu\nu} \, \mathbb{d}x^\mu \, \mathbb{d}x^\nu = -N^2(t) \, \mathbb{d}t^2 + \varkappa \mathbb{e}^{2\alpha(t)} \, \mathbb{d}\Omega^2_{3\mathsf{F}}$, where $\varkappa := 8\pi G$, $\mathbb{d}\Omega^2_{3\mathsf{F}} := \mathbb{d}\chi^2 + \chi^2 (\mathbb{d}\theta^2 + \sin^2\theta \, \mathbb{d}\varphi^2)$, and N lapse function.
- The total action reads $\mathcal{S} := \mathcal{S}_{\mathsf{EH}} + \mathcal{S}_{\mathsf{GHY}} + \mathcal{S}_{\mathsf{L}} = \int \mathbb{d}\Omega^2_{3\mathsf{F}} \int \mathbb{d}t \, L$, where

$$L := \varkappa^{3/2} N \mathbb{e}^{3\alpha} \left(-\frac{3}{\varkappa} \frac{\dot{\alpha}^2}{N^2} + \ell \frac{\dot{\phi}^2}{2N^2} - V \mathbb{e}^{\lambda\phi} \right). \qquad (2)$$

[1] R. R. Caldwell, R. Dave, and P. J. Steinhardt. In: *Phys. Rev. Lett.* 8 (1998), pp. 1582–1585.

[2] R. R. Caldwell. In: *Phys. Lett. B* 1-2 (2002), pp. 23–29.

[3] The signature of metric is mostly positive.

[4] Y. Nakayama. In: *International Journal of Modern Physics A* 17n18 (2004), pp. 2771–2930.

Introduction

Highlights

Integrability

Implicit trajectories can be obtained explicitly; the minisuperspace Wheeler–DeWitt equation can be integrated exactly.

Self-adjointness

The Hamiltonian is not even naturally symmetric; imposing self-adjointness leads to non-trivial physical results.

Semi-classical wave functions

Semi-classical wave functions can be constructed by the principle of constructive interference, the JWKB approximation and numerical methods.

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Decoupling the variables

Via orthogonal transformation

- By rescaling $\overline{N} := N\mathrm{e}^{-3\alpha}$ and the orthogonal transformation

$$\begin{pmatrix}\alpha(t)\\ \phi(t)\end{pmatrix} = \frac{\mathcal{s}}{g}\begin{pmatrix}\lambda & -\ell\varkappa\\ -6 & \lambda\end{pmatrix}\begin{pmatrix}\mathcal{s}_\beta\cdot\beta(t)\\ \mathcal{s}_\chi\cdot\chi(t)\end{pmatrix} \qquad \text{where } \mathcal{s}_\beta, \mathcal{s}_\chi = \pm 1, \tag{3}$$

  the effective Lagrangian eq. (2) can be decoupled ($\mathcal{s}_\beta = \mathcal{s}_\chi = +1$ from now on)

$$L = \varkappa^{3/2}\overline{N}\left(-\mathcal{s}\frac{3}{\varkappa}\frac{\dot\beta^2}{\overline{N}^2} + \ell\mathcal{s}\frac{\dot\chi^2}{2\overline{N}^2} - V\mathrm{e}^{g\chi}\right), \tag{4}$$

  where $\Delta := \lambda^2 - 6\ell\varkappa$, $\mathcal{s} := \operatorname{sgn}\Delta$ and $g := \mathcal{s}\sqrt{|\Delta|} \equiv \mathcal{s}\sqrt{\mathcal{s}\Delta}$.
- The Euler–Lagrange equations with respect to $\overline{N}$, β and χ will be called the modified first, second Friedmann equations and the Klein–Gordon equation, respectively.
- Since $\beta(t)$ is cyclic, its conjugate momentum p_β is conserved[5], and the modified second Friedmann equation can be readily integrated.

[5] The same first integral has been found in C. Lan. PhD thesis. Saint Petersburg State University, 2016, A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theor. Math. Phys.* 3 (2015), pp. 1224–1233, in canonical formalism.

Integration and the implicit trajectories

- For $p_\beta \neq 0$, fixing the *implicit gauge* $\overline{N} = -6\jmath\sqrt{\varkappa}\dot{\beta}/p_\beta$, the modified first Friedmann equation can be integrated, yielding the implicit trajectory

$$\mathrm{e}^{g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|} S^2\left(\sqrt{\frac{3}{2\varkappa}}\, g(\beta - \beta_0)\right), \tag{5}$$

in which $\upsilon := \operatorname{sgn} V$, $(\operatorname{sgn}, \operatorname{sgn})$ means $(\ell, \jmath\upsilon)$, and

$$\begin{aligned} {}_{(+,+)}S(\gamma) &:= \operatorname{sech}(\gamma), & {}_{(+,-)}S(\gamma) &:= \operatorname{csch}(\gamma), \\ {}_{(-,+)}S(\gamma) &:= \sec(\gamma), & {}_{(-,-)}S(\gamma) &:= \mathbb{i}\csc(\gamma). \end{aligned} \tag{6}$$

  - The integral for $(-,-)$ is not real.
  - The trajectories can be parametrised by β, inspiring recognising β as a 'time variable'.

- For $p_\beta = 0$, integrating the modified second Friedmann equation yields $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell\lambda\alpha/\varkappa$[6], which is the well-known power-law special solution[7].

[6] It only works for $(+,-)$ or $(-,+)$ if one checks consistency with the modified first Friedmann equation.

[7] For instance, A. R. Liddle and D. H. Lyth. *Cosmological Inflation and Large-Scale Structure*. Cambridge University, 2000, ch. 3.

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Dirac quantisation

- The primary Hamiltonian and the Hamiltonian constraint[8] reads

$$H^{\mathrm{p}} = \overline{N} H_{\perp} + v^{\overline{N}} p_{\overline{N}}, \tag{7}$$

$$H_{\perp} := -\jmath \frac{p_{\beta}^2}{12 \varkappa^{1/2}} + \ell \jmath \frac{p_{\chi}^2}{2 \varkappa^{3/2}} + \varkappa^{3/2} V \mathrm{e}^{g\chi}. \tag{8}$$

- Applying the Dirac quantisation rules with the Laplace–Beltrami operator ordering[9], one gets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(\jmath \frac{\hbar^2}{12 \varkappa^{1/2}} \partial_{\beta}^2 - \ell \jmath \frac{\hbar^2}{2 \varkappa^{3/2}} \partial_{\chi}^2 + \varkappa^{3/2} V \mathrm{e}^{g\chi} \right) \Psi. \tag{9}$$

- Equation (9) is Klein–Gordon-like, hyperbolic for $\ell = +1$ and *elliptic* for $\ell = -1$.

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[8] D. M. Gitman and I. V. Tyutin. *Quantization of Fields with Constraints*. Springer, 1990, H. J. Rothe and K. D. Rothe. *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*. World Scientific, 2010.

[9] C. Kiefer. *Quantum Gravity*. 3rd ed. Oxford University, 2012, ch. 8.

Integration of the Wheeler–DeWitt equation

The quantum-mechanical analogy

- Fourier-transforming β to $k_\beta \in \mathbb{R}$ yields the time-independent Schrödinger equation for $\widehat{H}_{\text{eff}}$

$$\ell \frac{\hbar^2 \tilde{g}^2 \nu^2}{8 M_{\text{P}}} \psi(x) = \widehat{H}_{\text{eff}} \psi(x) := \left(-\frac{\hbar^2}{2 M_{\text{P}}} \partial_x^2 + \ell \jmath \upsilon \widetilde{V} \mathrm{e}^{\tilde{g} x} \right) \psi(x), \tag{10}$$

in which $M_{\text{P}} := \sqrt{\frac{\hbar}{\varkappa}}$, $x := \sqrt{\hbar \varkappa} \cdot \chi$, $\widetilde{V} := M_{\text{P}}^{-3} |V| \geq 0$, and $\nu = \sqrt{\frac{2\varkappa}{3}} \frac{|k_\beta|}{g} \geq 0$.

- For the quintessence model $(+,+)$: $\widehat{H}_{\text{eff}}$ is bounded below; generalised eigenfunctions are the Besselian $\mathrm{K}_{\mathring{\imath}\nu}$'s.
- For the quintessence model $(+,-)$: $\widehat{H}_{\text{eff}}$ is bounded above; generalised eigenfunctions for positive spectrum / 'scattering states' are the Besselian $\mathrm{F}_{\mathring{\imath}\nu}$'s and $\mathrm{G}_{\mathring{\imath}\nu}$'s[10].
- For the phantom model $(-,+)$: $\widehat{H}_{\text{eff}}$ is bounded above; generalised eigenfunctions for negative spectrum / 'bound states' are the Besselian J_ν's.

[10] These are linear combinations of $\mathrm{J}_{\pm\mathring{\imath}\nu}$'s which are orthogonal; see T. M. Dunster. In: *SIAM Journal on Mathematical Analysis* 4 (1990), pp. 995–1018.

The self-adjointness of unbounded operators

The general theory

- Even in quantum geometrodynamics, the self-adjointness of the matter Hamiltonian is desirable, which is typically an unbounded operator in the Hilbert space $\mathscr{F}$ endowed with the Schrödinger inner product $(\cdot,\cdot)_{\mathrm{S}}$.
- Mathematically[11], an *unbounded* operator H is characterised not only by its action on a vector, but also by its domain $\mathrm{Dom}(H) \subsetneq \mathscr{F}$.
- In addition to symmetricity $(H^\dagger\phi_1, \phi_2) \equiv (H\phi_1, \phi_2)$, the self-adjointness of an unbounded operator also requires $\mathrm{Dom}(H^\dagger) = \mathrm{Dom}(H)$.
- For quantum systems not bounded below, not only the self-adjointness, but also the symmetricity of the Hamiltonian is not guaranteed automatically.
- Even when one could find a $\mathrm{Dom}_0(H)$ such that H is symmetric, one would still be left with $\mathrm{Dom}_0(H^\dagger) \supset \mathrm{Dom}_0(H)$ in general.
- Sloppily speaking, the process of extending $\mathrm{Dom}(H)$ such that $\mathrm{Dom}(H^\dagger) = \mathrm{Dom}(H)$ is called self-adjoint extension[12]; if the extension is unique, the operator is called *essentially self-adjoint*.

11 B. C. Hall. *Quantum Theory for Mathematicians*. Springer, 2013, p. 193.

12 G. Bonneau, J. Faraut, and G. Valent. In: *Am. J. Phys* 3 (2001), pp. 322–331, V. S. Araujo, F. A. B. Coutinho, and J. F. Perez. In: *Am. J. Phys* 2 (2004), pp. 203–213, A. M. Essin and D. J. Griffiths. In: *Am. J. Phys* 2 (2006), pp. 109–117, D. M. Gitman, I. V. Tyutin, and B. L. Voronov. *Self-adjoint Extensions in Quantum Mechanics*. Birkhäuser, 2012.

Separation of the variables and mode functions

- Writing $\Psi(\beta,\chi) = \varphi(\beta)\psi(\chi)$, eq. (9) can be separated into

$$\partial_\beta^2 \varphi(\beta) = k_\beta^2 \varphi(\beta); \tag{11}$$

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \qquad \mathbb{D} := -\ell \frac{6}{\varkappa}\partial_\chi^2 + \jmath\upsilon \frac{12\varkappa^2 |V|}{\hbar^2}. \tag{12}$$

- Equation (12) turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta,\chi) := \sum_{i=1}^{2} c_i \varphi_\nu^{(i)}(\gamma) \sum_{j=1}^{2} a_j \mathrm{B}_\nu^{(j)}(\sigma), \quad \nu \geq 0; \tag{13}$$

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{1}{g} k_\beta, \quad \gamma := \sqrt{\frac{3}{2\varkappa}} \frac{g}{1} \beta, \qquad \sigma^2 := \frac{8\varkappa^3 |V| \mathrm{e}^{g\chi}}{\hbar^2 g^2}, \tag{14}$$

$$_{(+,+)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{K} \text{ and } \mathrm{I}_{\mathrm{i}\nu}(\sigma), \qquad _{(+,-)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{F} \text{ and } \mathrm{G}_{\mathrm{i}\nu}(\sigma),$$

$$_{(-,+)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{J} \text{ and } \mathrm{Y}_\nu(\sigma), \qquad _{(-,-)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{K} \text{ and } \mathrm{I}_\nu(\sigma).$$

- Adapted to imaginary order, $\mathrm{F}_\nu(\sigma)$ and $\mathrm{G}_\nu(\sigma)$ are defined in[13].

[13] T. M. Dunster. In: *SIAM Journal on Mathematical Analysis* 4 (1990), pp. 995–1018.

Physical mode functions

- Physical mode functions are expected to be regular on the boundary.
- $(+,+)$: $|\mathrm{I}_{\mathbb{i}\nu}(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- $(-,+)$:
  - $\forall n \in \mathbb{N}$, $|\mathrm{Y}_n(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
  - $\forall \nu \in \mathbb{R}^+ \backslash \mathbb{N}$, choose $\mathrm{J}_{-\nu}$ instead of Y_ν, since $\mathrm{J}_{\pm\nu}$ are also linearly independent.
  - $\forall \nu \in \mathbb{R}^+ \backslash \mathbb{N}$, $|\mathrm{J}_{-\nu}(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
- $(-,-)$: $|\mathrm{K}_\nu(\sigma)| \to +\infty$ as $\alpha \to -\infty$; $|\mathrm{I}_\nu(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- These are not to be included in the space of physical wave functions. $\forall \nu \geq 0$,
  - $(+,+)$: $\mathrm{K}_{\mathbb{i}\nu}(\sigma)$ survives
  - $(+,-)$: $\mathrm{F}_{\mathbb{i}\nu}(\sigma)$ and $\mathrm{G}_{\mathbb{i}\nu}(\sigma)$ survives
  - $(-,+)$: $\mathrm{J}_\nu(\sigma)$ survives
  - $(-,-)$: drops out

Outline

Inner product for wave functions

Schrödinger product

- To make sense of amplitude and wave packet, an inner product is necessary
- In terms of a Klein–Gordon equation, call β the "temporal" variable, and χ the "spacial" variable.
- A common starting point is the *Schrödinger product*[14]

$$(\Psi_1, \Psi_2)_{\mathrm{S}} := \int \mathrm{d}\chi \, \Psi_1^*(\beta, \chi) \Psi_2(\beta, \chi); \tag{15}$$

- $(\Psi, \Psi)_{\mathrm{S}}$ is positive-definite, and the integrand $\rho_{\mathrm{S}}(\beta, \chi)$ is non-negative
- The corresponding Schrödinger current is real but does not satisfy continuity equation $\dot{\rho}_{\mathrm{S}} + \nabla \cdot \vec{j}_{\mathrm{S}} = 0$, because eq. (9) is KG-like.
- $\mathrm{K}_{\mathrm{i}\nu}$[15] for $(+,+)$, $\mathrm{F}_{\mathrm{i}\nu}$ and $\mathrm{G}_{\mathrm{i}\nu}$ for $(+,-)$ can be proved to be orthogonal and complete among themselves, as well as can be normalised.
  - $\mathrm{J}_{\mathrm{i}\nu}$'s for $(+,-)$ are not orthogonal

14 C. Kiefer. *Quantum Gravity*. 3rd ed. Oxford University, 2012, ch. 5.

15 S. B. Yakubovich. In: *Opuscula Math.* 1 (2006), pp. 161–172, A. Passian et al. In: *Journal of Mathematical Analysis and Applications* 2 (2009), pp. 380–390, R. Szmytkowski and S. Bielski. In: *Journal of Mathematical Analysis and Applications* 1 (2010), pp. 195–197.

Peculiarity of the phantom model $(-,+)$

Orthogonality for mode functions; Hermiticity for operators

- $\mathrm{J}_\nu(\sigma)$'s are not orthogonal under the Schrödinger product

$$(\mathrm{J}_\nu, \mathrm{J}_{\tilde{\nu}})_{\mathsf{S}} \propto \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{J}^*_\nu(\mathrm{e}^x)\mathrm{J}_{\tilde{\nu}}(\mathrm{e}^x) = \frac{2\sin(\pi(\nu-\tilde{\nu})/2)}{\pi(\nu^2-\tilde{\nu}^2)}, \tag{16}$$

  therefore $\mathbb{D}$ in eq. (12) is not Hermitian (though we do not need it so far)
- $\hat{p}^2_\chi$ is not Hermitian for $\{\mathrm{J}_\nu(\sigma)\}$ under the Schrödinger product

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{J}^*_\nu(-\partial^2_x \mathrm{J}_{\tilde{\nu}}) - \int_{-\infty}^{+\infty} \mathrm{d}x\, (-\partial^2_x \mathrm{J}_\nu)^* \mathrm{J}_{\tilde{\nu}} = \frac{2}{\pi} \sin\frac{\pi(\nu-\tilde{\nu})}{2}. \tag{17}$$

- In order to save Hermiticity for p^2_χ and $\mathbb{D}$ and orthogonality of the modes under Schrödinger product, one can restrict

$$\nu = 2n + \nu_0, \qquad n \in \mathbb{N}, \quad \nu_0 \in [0,2). \tag{18}$$

- Using classical singularities as boundary condition, one can fix $\nu_0 = 1$.

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Discretisation of the phantom model $(-,+)$

Levels of the phantom model are discretised if one imposes Hermiticity of squared momenta under the Schrödinger product.

- This kind of subtlety on Hermiticity is named as *self-adjoint extension*, which arises already for infinite square well[16].
- It also applies to x^{-2} potentials[17], which is of cosmological relevance[18].

[16] G. Bonneau, J. Faraut, and G. Valent. In: *Am. J. Phys* 3 (2001), pp. 322–331.

[17] A. M. Essin and D. J. Griffiths. In: *Am. J. Phys* 2 (2006), pp. 109–117, V. S. Araujo, F. A. B. Coutinho, and J. F. Perez. In: *Am. J. Phys* 2 (2004), pp. 203–213.

[18] M. Bouhmadi-López et al. In: *Physical Review D* 12 (2009).

Further inner products for wave functions

Klein–Gordon and Mostafazadeh product

- Since eq. (9) is KG-like, another popular choice is the KG product

$$(\Psi_1, \Psi_2)^g_{\mathsf{KG}} := \mathbb{i} g \Big\{ \big(\Psi_1, \partial_\beta \Psi_2\big)_{\mathsf{S}} - \big(\partial_\beta \Psi_1, \Psi_2\big)_{\mathsf{S}} \Big\}, \qquad g > 0. \tag{19}$$

  - $(\Psi, \Psi)^g_{\mathsf{KG}}$ is real but not positive-definite, so does the integrand ρ_{KG};
  - The corresponding $\vec{J}_{\mathsf{KG}}$ is conserved $\dot{\rho}_{\mathsf{KG}} + \nabla \cdot \vec{J}_{\mathsf{KG}} = 0$ and real.

- Mostafazadeh[19] found a product *for Hermitian* $\mathbb{D}$ *with positive spectrum*:

$$(\Psi_1, \Psi_2)^\kappa_{\mathsf{M}} := \kappa \Big\{ \big(\Psi_1, \mathbb{D}^{+1/2} \Psi_2\big)_{\mathsf{S}} + \big(\partial_\beta \Psi_1, \mathbb{D}^{-1/2} \partial_\beta \Psi_2\big)_{\mathsf{S}} \Big\}, \qquad \kappa > 0. \tag{20}$$

  - $(\Psi, \Psi)^\kappa_{\mathsf{M}}$ is positive-definite, but the integrand ρ^κ_{M} is complex
  - The corresponding $\vec{J}^\kappa_{\mathsf{M}}$ is conserved $\dot{\rho}^\kappa_{\mathsf{M}} + \nabla \cdot \vec{J}^\kappa_{\mathsf{M}} = 0$ but also complex.

19 A. Mostafazadeh. In: *Classical Quantum Gravity* 1 (2002), pp. 155–171.

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Mostafazadeh inner product and the corresponding density

- Real power of $\mathbb{D}$ is defined by spectral decomposition $\mathbb{D}^{\gamma} := \sum_{\nu} \nu^{2\gamma} \mathbf{P}_{\nu}$, $\mathbf{P}_{\nu} \Psi := \Psi_{\nu} (\Psi_{\nu}, \Psi)_{\mathsf{S}}$.
- It can be shown[20] that *the density*

$$\varrho^{\kappa}_{\mathsf{M}} := \kappa \Big\{ \big| \mathbb{D}^{+1/4} \Psi \big|^2 + \big| \mathbb{D}^{-1/4} \partial_{\beta} \Psi \big|^2 \Big\} \tag{21}$$

  - is equivalent to $\rho^{\kappa}_{\mathsf{M}}$ up to a boundary term

$$\int \mathrm{d}\chi \, \varrho^{\kappa}_{\mathsf{M}} = \int \mathrm{d}\chi \, \rho^{\kappa}_{\mathsf{M}} \equiv (\Psi_1, \Psi_2)^{\kappa}_{\mathsf{M}}; \tag{22}$$

  - is non-negative.
  - The corresponding current $\vec{\mathcal{J}}^{\kappa}_{\mathsf{M}}$ is real but not conserved[21].

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20 A. Mostafazadeh and F. Zamani. In: *Ann. Phys.* 9 (2006), pp. 2183–2209.

21 B. Rosenstein and L. P. Horwitz. In: *J. Phys. A: Math. Gen.* 11 (1985), pp. 2115–2121.

Wave packets of Gaussian amplitude for continuous spectrum

Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu;\overline{\nu},\sigma) := \left(\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(\nu-\overline{\nu})^2}{2\sigma^2}\right)\right)^{1/2} \tag{23}$$

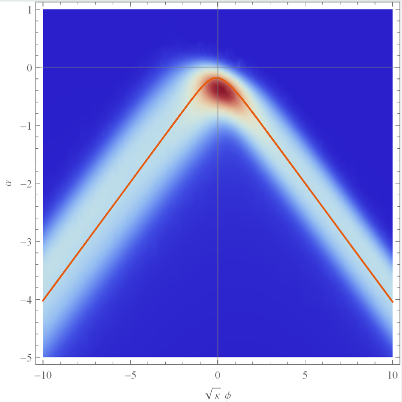
- In[22], $A\left(\nu;\overline{\nu},\sigma/\sqrt{2}\right)$ was chosen.

[22] M. P. Dąbrowski, C. Kiefer, and B. Sandhöfer. In: *Physical Review D* 4 (2006).

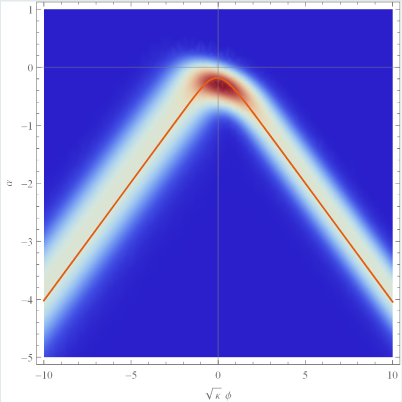
Wave packets of Gaussian amplitude for quintessence model $(+,+)$

$\mathrm{K}_{i\nu}$, with $\lambda = \varkappa^{1/2}/2$, $V = -\varkappa^{-2}$, $\bar{k}_\beta = -2$ and $\sigma_\beta = 5/4$

Schrödinger



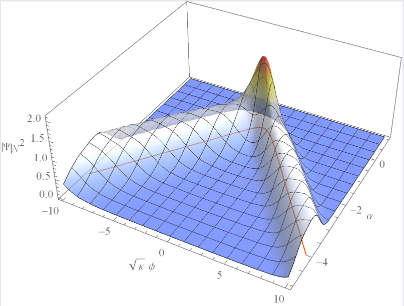
Mostafazadeh



Wave packets of Gaussian amplitude for quintessence model $(+,+)$

$\mathrm{K}_{i\nu}$, **with** $\lambda = \varkappa^{1/2}/2$, $V = -\varkappa^{-2}$, $\overline{k}_{\beta} = -2$ **and** $\sigma_{\beta} = 5/4$

Schrödinger



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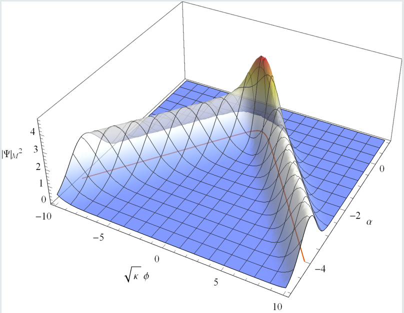

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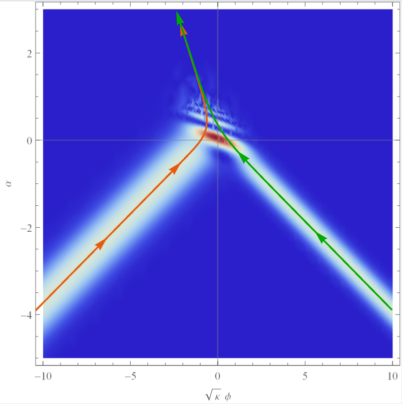
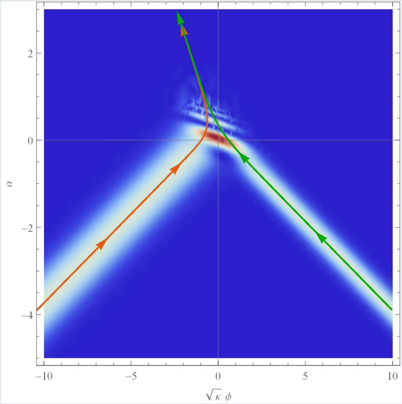
Wave packets of Gaussian amplitude for quintessence model $(+,-)$

$\mathrm{F}_{\mathring{\imath}\nu}$, **with** $\lambda = 4\varkappa^{1/2}/5$, $V = +\varkappa^{-2}$, $\overline{k}_\beta = -7/2$ **and** $\sigma_\beta = 7/5$

Schrödinger



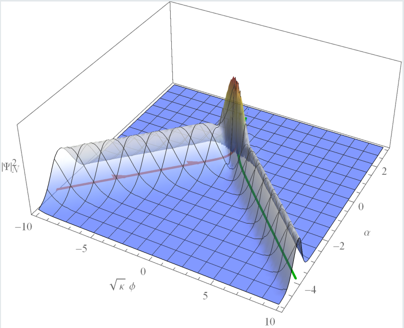
Mostafazadeh



Wave packets of Gaussian amplitude for quintessence model $(+,-)$

$\mathrm{F}_{\mathring{\imath}\nu}$, **with** $\lambda = 4\varkappa^{1/2}/5$, $V = +\varkappa^{-2}$, $\bar{k}_\beta = -7/2$ **and** $\sigma_\beta = 7/5$

Schrödinger



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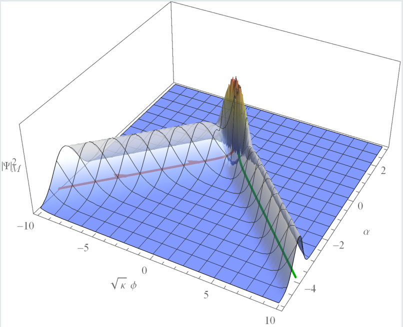
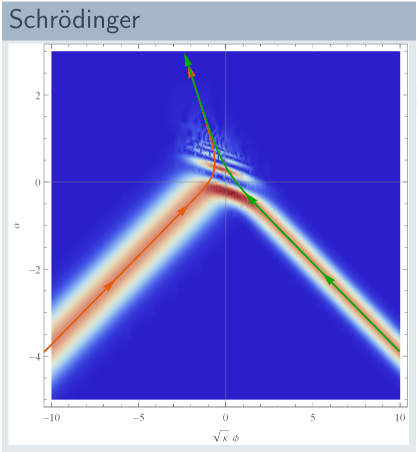
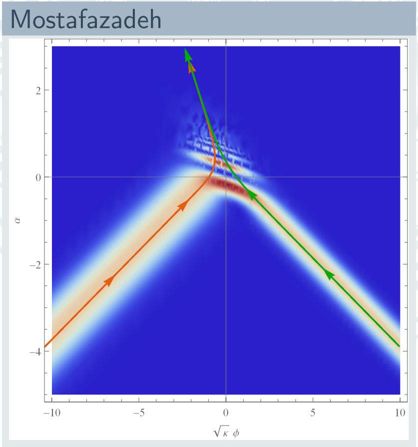


Wave packets of Gaussian amplitude for quintessence model $(+,-)$

$G_{\mathbb{i}\nu}$, with $\lambda = 4\varkappa^{1/2}/5$, $V = +\varkappa^{-2}$, $\overline{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

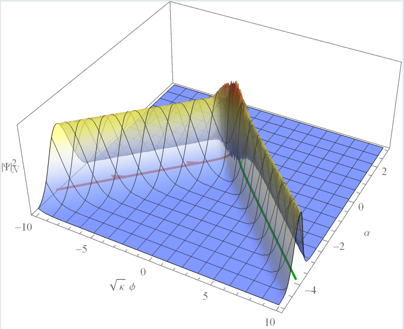




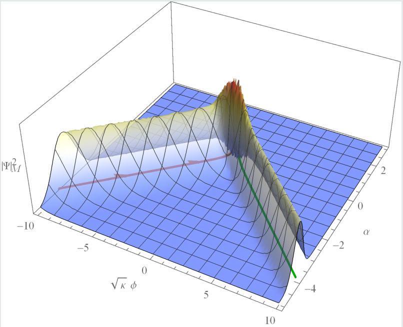
Wave packets of Gaussian amplitude for quintessence model $(+,-)$

$\mathrm{G}_{\mathfrak{i}\nu}$, **with** $\lambda = 4\varkappa^{1/2}/5$, $V = +\varkappa^{-2}$, $\bar{k}_\beta = -7/2$ **and** $\sigma_\beta = 7/5$

Schrödinger



Mostafazadeh



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Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model $(-,+)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\overline{n}) := \left(\mathrm{e}^{-\overline{n}}\frac{\overline{n}^n}{n!}\right)^{1/2} \tag{24}$$

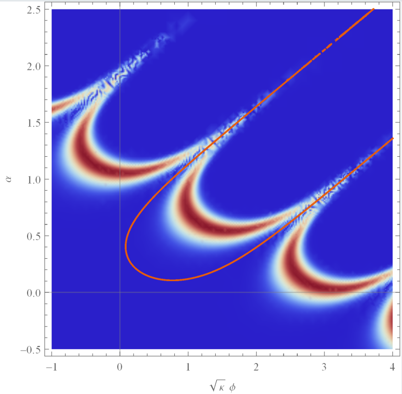
- In[23], $A_n\left(\overline{n}/\sqrt{2}\right)$ was chosen.

[23] C. Kiefer. In: *Nucl. Phys. B* 1 (1990), pp. 273–293.

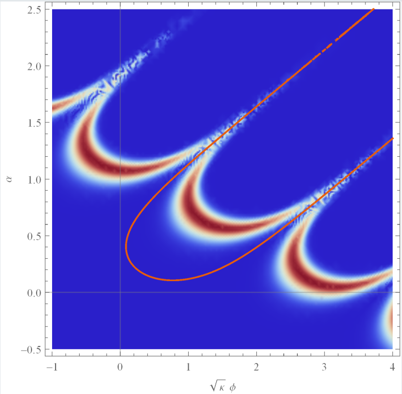
Wave packets of Poissonian amplitude for phantom model

J_{2n+1}, with $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$ and $\overline{k}_\beta = 8$

Schrödinger



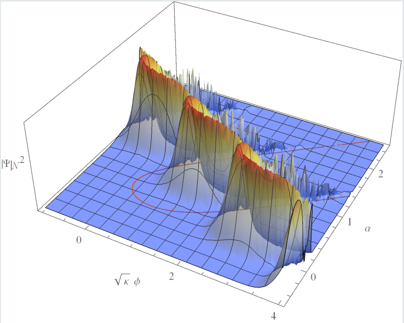
Mostafazadeh



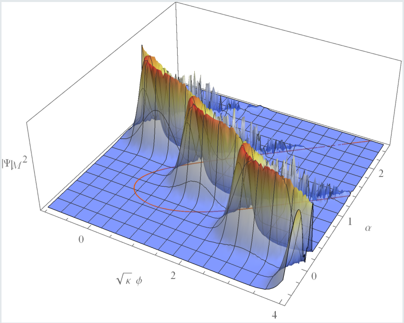
Wave packets of Poissonian amplitude for phantom model

J_{2n+1}, **with** $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$ **and** $\bar{k}_\beta = 8$

Schrödinger

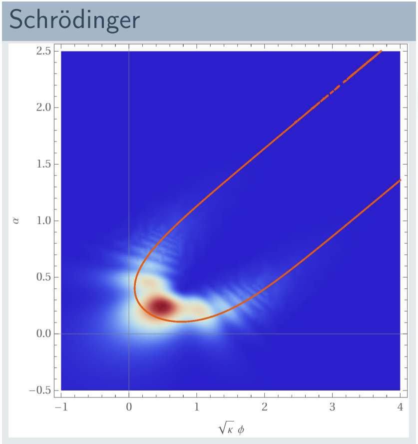


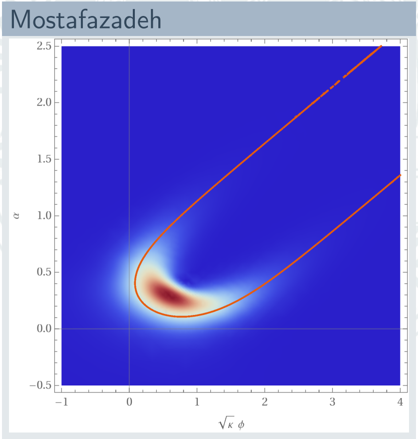
Mostafazadeh



Wave packets of Gaussian amplitude for phantom model

J_ν, with $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$, $\overline{k}_\beta = 8$ and $\sigma_\beta = 11/2$

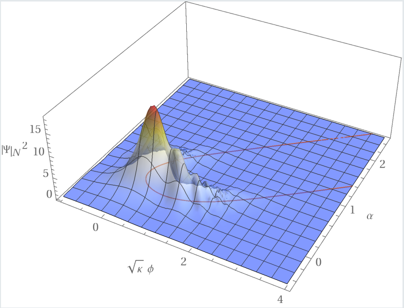




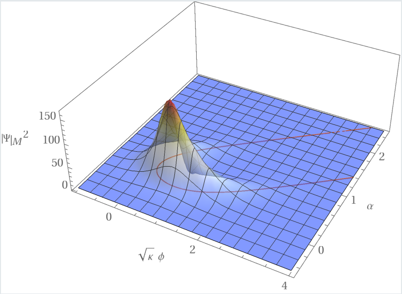
Wave packets of Gaussian amplitude for phantom model

J_ν, **with** $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$, $\overline{k}_\beta = 8$ **and** $\sigma_\beta = 11/2$

Schrödinger



Mostafazadeh



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arXiv:1802.06720

Outline

Highlights

- An integral of motion was found for Liouville cosmological models.
- Implicit trajectories in minisuperspace were obtained.
- The levels of phantom Liouville model were found to be discrete due to the Hermiticity requirement of observables.
- The semi-classical quantum mode functions and wave packets was compared directly with the classical trajectory.

Issues

- In $(+,-)$ and $(-,+)$, wave packets contain multiple branches; however, the classical universe runs only on one trajectory.
- Quantum-corrected $\overline{k}_\beta$ is to be understood.
- Wave packets with Gaussian profile are to be constructed, instead of inserting Gaussian / Poissonian amplitude artificially.
- A normalising κ for $(\cdot,\cdot)^\kappa_{\mathsf{M}}$ is to be evaluated, otherwise a quantitative comparison of $(\cdot,\cdot)_{\mathsf{S}}$ and $(\cdot,\cdot)^\kappa_{\mathsf{M}}$ is not possible.

Outlook

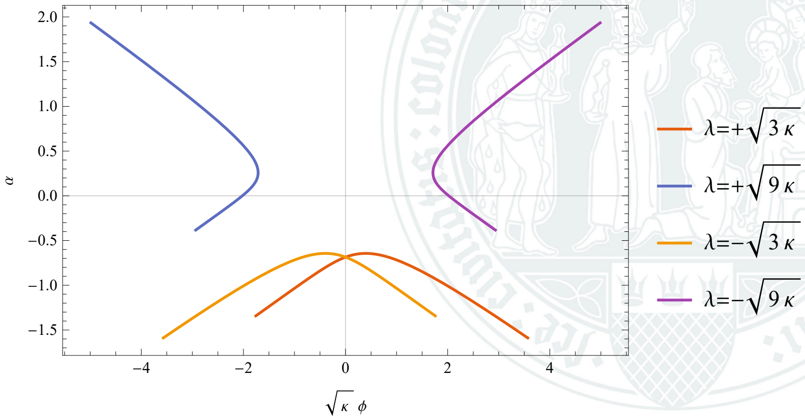
- Beyond isotropy: generalise to Bianchi models
  - Bianchi Type-I: a natural extension, under investigation
- Beyond homogeneity: cosmological perturbation, under investigation
- Beyond single field
  - Two exponential potentials: $|V_1| = |V_2|$ and special λ_i
  - Multiple Liouville fields: mixing kinetic terms needed[24]
- Beyond classic matter
  - PT-symmetric Liouville field[25]: may cross the phantom divide $w = -1$.

[24] A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theor. Math. Phys.* 3 (2015), pp. 1224–1233.

[25] A. A. Andrianov et al. In: *International Journal of Modern Physics D* 01 (2010), pp. 97–111, A. A. Andrianov, C. Lan, and O. O. Novikov. In: *Springer Proceedings in Physics*. Springer International Publishing, 2016, pp. 29–44.

Trajectories for quintessence model $(+,+)$

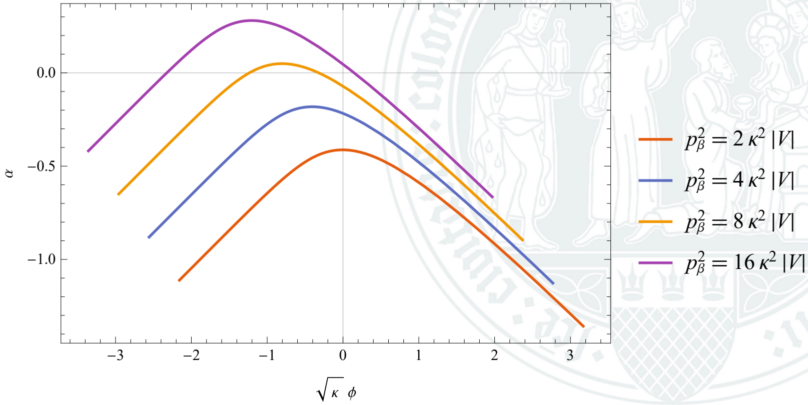
sech, with $\beta_0 = 0$, $|V| = \varkappa^{-2}$ and $p_\beta^2 = \varkappa^2\sqrt{|V|}$; varying λ



- has two asymptotes $\chi \propto \pm\beta$

Trajectories for quintessence model $(+,+)$

sech, with $\beta_0 = 0$, $|V| = \varkappa^{-2}$ and $\lambda^2 = 3\varkappa$; varying p_β



- has two asymptotes $\chi \propto \pm\beta$

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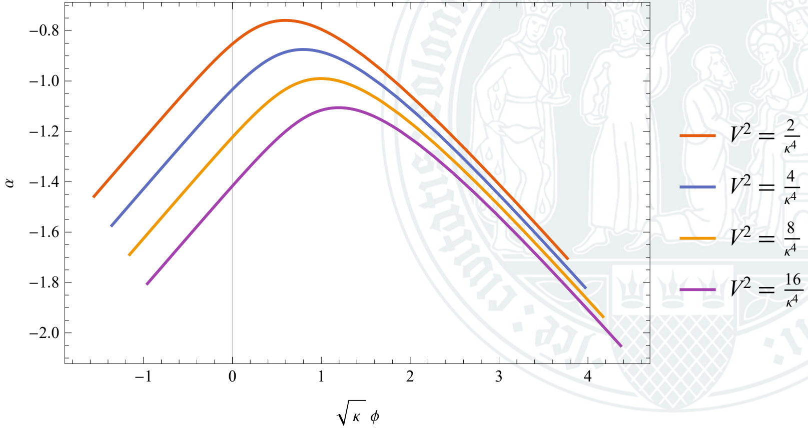
Trajectories for quintessence model $(+,+)$

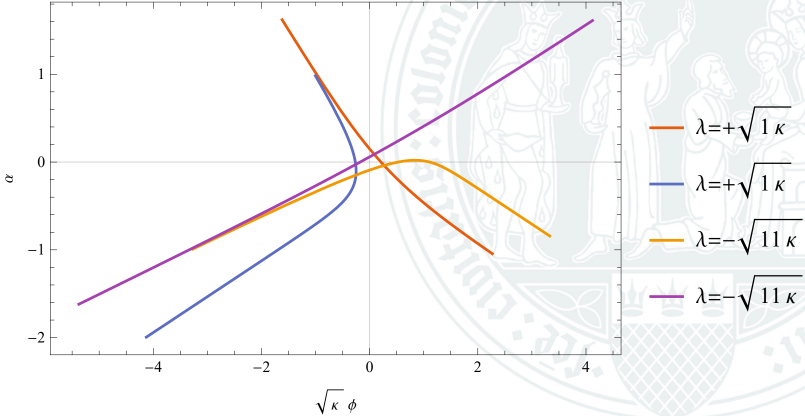
sech, **with** $\beta_0 = 0$, $\lambda^2 = 3\varkappa$ **and** $p_\beta^2 = \varkappa^2\sqrt{|V|}$; **varying** V



- has two asymptotes $\chi \propto \pm\beta$

Trajectories for quintessence model $(+,-)$

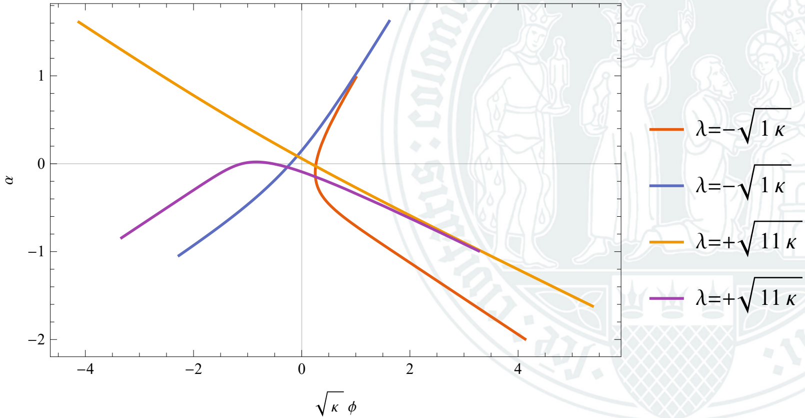
csch, **with** $\beta_0 = 0$, $|V| = \varkappa^{-2}$ **and** $p_\beta^2 = \varkappa^2\sqrt{|V|}$; **varying** λ



- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$

Trajectories for quintessence model $(+,-)$: csch

csch, with $\beta_0 = 0$, $|V| = \varkappa^{-2}$ and $p_\beta^2 = \varkappa^2\sqrt{|V|}$; varying λ



- contains two distinct solutions, separated by $\beta = 0$
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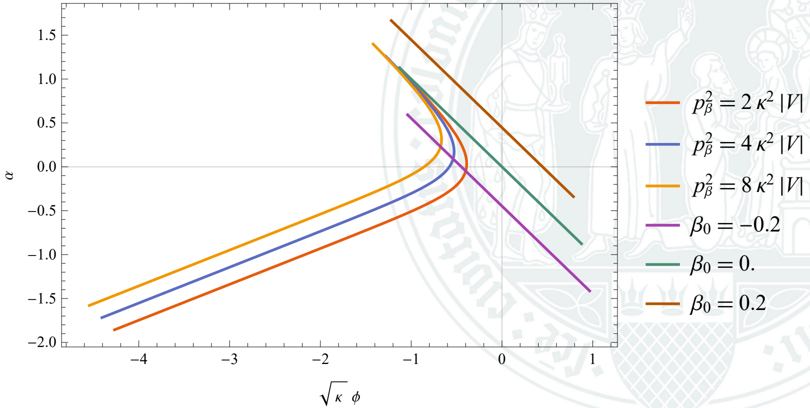
Trajectories for quintessence model $(+,-)$: csch

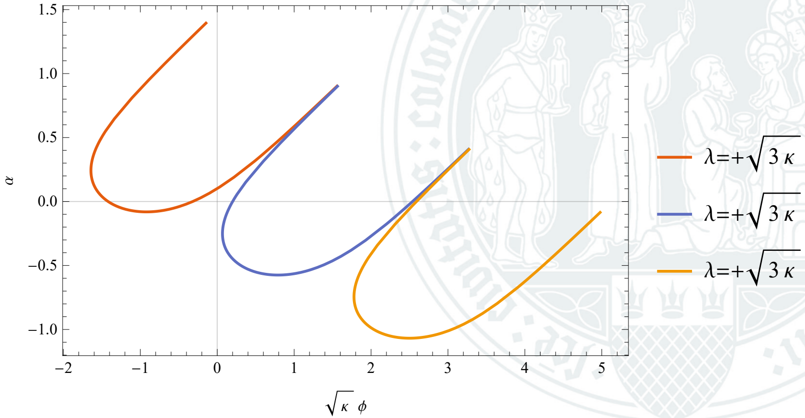
csch, **with** $V = \varkappa^{-2}$ **and** $\lambda^2 = \varkappa$; **varying** p_β



- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$

Trajectories for phantom model $(-,+)$

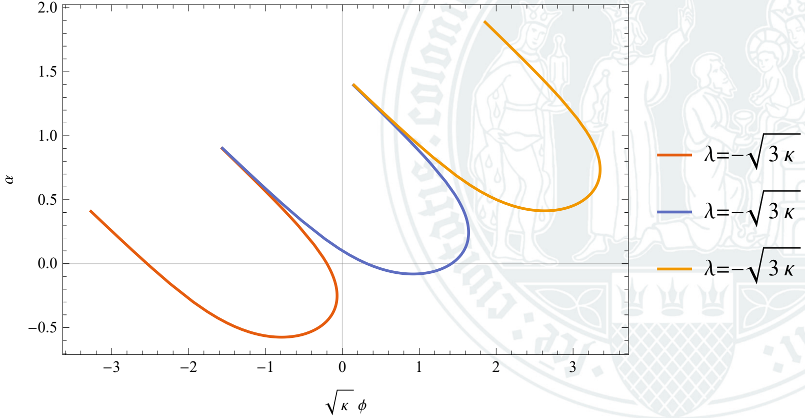
csc, **with** $V = \varkappa^{-2}$ **and** $p_\beta^2 = 3\varkappa^2\sqrt{|V|}$



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$

Trajectories for phantom model $(-,+)$

csc, **with** $V = \varkappa^{-2}$ **and** $p_\beta^2 = 3\varkappa^2\sqrt{|V|}$



- contains infinite distinct solutions
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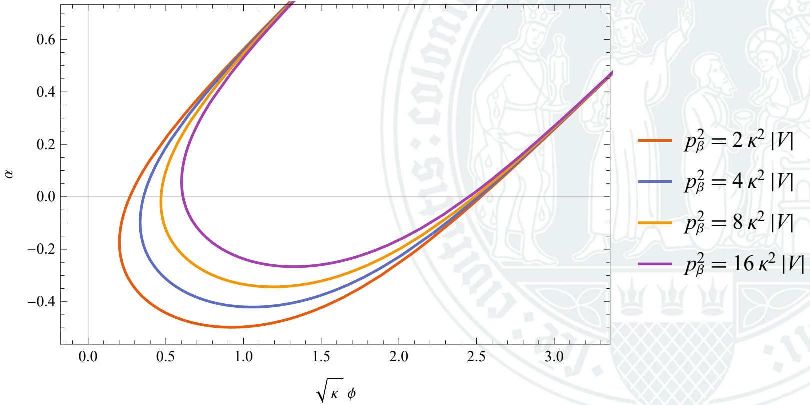
Trajectories for phantom model $(-,+)$

csc, with $V = \varkappa^{-2}$ and $\lambda^2 = 3\varkappa$; varying p_β



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$

Integration of the transformed first Friedmann equation

General integral for $p_\beta \neq 0$

In order to integrate the equation under the implicit gauge

$$\jmath \frac{p_\beta^2}{12}\left(-\ell \frac{\varkappa^{1/2}}{6}\left(\frac{\mathrm{d}\chi}{\mathrm{d}\beta}\right)^2 + \varkappa^{-1/2}\right) - \varkappa^{3/2} V \mathrm{e}^{g\chi} = 0, \tag{25}$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa}} g\beta, \qquad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} \mathrm{e}^{-g\chi}, \tag{26}$$

to get

$$\left(\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\gamma}\right)^2 + \ell(\jmath \upsilon - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-\jmath\upsilon + \tilde{\sigma}^2)}}, \tag{27}$$

which is of the standard inverse hyperbolic / trigonometric form except for $(-,-)$.

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Integration of the separated minisuperspace WDW equation

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2\psi(\chi), \qquad \mathbb{D} := -\ell\,\frac{6}{\varkappa}\partial_\chi^2 + \mathscr{s}\upsilon\,\frac{12\varkappa^2|V|}{\hbar^2}, \tag{12 rev.}$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}}\frac{k_\beta}{g}, \qquad \sigma^2 := \frac{8\varkappa^3|V|\mathrm{e}^{g\chi}}{\hbar^2 g^2}, \tag{14 rev.}$$

to get

$$\sigma^2\psi''(\sigma) + \sigma\psi'(\sigma) + \ell\,(\nu^2 - \mathscr{s}\upsilon\sigma^2)\psi(\sigma) = 0, \tag{28}$$

which is of the standard Besselian form.

offizielle