Integrable Cosmological Models with Liouville Fields

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Introduction

Introduction

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- Flat Robertson–Walker metric $\mathrm{d}s^2 = -N^2(t)\,\mathrm{d}t^2 + \varkappa^{-1}\mathrm{e}^{2\alpha(t)}\,\mathrm{d}\Omega_3^2$
    - $\varkappa := 8\pi G$; $\mathrm{d}\Omega_3^2$ dimensionless spacial metric
- Real Klein–Gordon field with potential $V\mathrm{e}^{\lambda\phi}$ (Liouville), where $\lambda, V \in \mathbb{R}$, and kinetic term with sign $\ell = \pm 1$ (quintessence / phantom model).
- Total action $\mathcal{S} := S_{\mathsf{EH}} + S_{\mathsf{GHY}} + S_{\mathsf{L}} = \int \mathrm{d}\Omega_3^2 \int \mathrm{d}t\, L$,

$$L := \varkappa^{3/2} N \mathrm{e}^{3\alpha}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{N^2} + \ell\,\frac{\dot{\phi}^2}{2N^2} - V\mathrm{e}^{\lambda\phi}\right), \tag{1}$$

in which dot means $\mathrm{d}/\mathrm{d}t$ and $\ell = \pm 1$.

Decoupling the variables

Via *rescaled* special orthogonal transformation

- Setting $\overline{N} := N \mathbb{e}^{-3\alpha}$, eq. (1) transforms to

$$L = \varkappa^{3/2} \overline{N} \left(-\frac{3}{\varkappa} \frac{\dot{\alpha}^2}{\overline{N}^2} + \ell \frac{\dot{\phi}^2}{2\overline{N}^2} - V \mathbb{e}^{\lambda\phi + 6\alpha} \right) \tag{2}$$

- Defining $\Delta := \lambda^2 - 6\ell\varkappa$, $\mathscr{s} := \operatorname{sgn} \Delta$ and $g := \mathscr{s}\sqrt{|\Delta|} \equiv \mathscr{s}\sqrt{\mathscr{s}\Delta}$, the *rescaled* special orthogonal transformation

$$\begin{pmatrix} \alpha \\ \phi \end{pmatrix} = \frac{\mathscr{s}}{g} \begin{pmatrix} \lambda & -\ell\varkappa \\ -6 & \lambda \end{pmatrix} \begin{pmatrix} \mathscr{s}_\beta \beta \\ \mathscr{s}_\chi \chi \end{pmatrix} \qquad \text{where } \mathscr{s}_\beta, \mathscr{s}_\chi = \pm 1 \tag{3}$$

gives the decoupled Lagrangian

$$L = \varkappa^{3/2} \overline{N} \left(-\mathscr{s} \frac{3}{\varkappa} \frac{\dot{\beta}^2}{\overline{N}^2} + \ell \mathscr{s} \frac{\dot{\chi}^2}{2\overline{N}^2} - V \mathbb{e}^{\mathscr{s}_\chi g \chi} \right). \tag{4}$$

- The Euler–Lagrange equations w.r.t. $\overline{N}$, β and χ will be called the trsfed. 1st, 2nd Friedmann eqs. and the Klein–Gordon eq., respectively.

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Possible generalisation of the decoupling technique

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- Beyond isotropy
  - Bianchi Type-I: under investigation
- Beyond one Liouville field
  - Two exponential potentials: $V_1 = V_2$ and special λ_i
  - Multiple Liouville fields: mixing kinetic terms[1]
- Beyond classic matter
  - Non-Hermitian, PT-symmetric Liouville fields[2]: may avoid big-rip etc.

[1] A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theoretical and Mathematical Physics* 184.3 (2015), pp. 1224–1233.

[2] A. A. Andrianov et al. In: *International Journal of Modern Physics D* 19.01 (2010), pp. 97–111, A. A. Andrianov, C. Lan, and O. O. Novikov. In: *Springer Proceedings in Physics*. Springer International Publishing, 2016, pp. 29–44.

Implicitised integration

General integral for $p_\beta \neq 0$

- Since β is cyclic in eq. (4), the trsfed. 2nd Friedmann eq. can be integrated[3]

$$\text{const.} \equiv p_\beta := \frac{\partial L}{\partial \dot\beta} = -6\jmath\varkappa^{1/2}\frac{\dot\beta}{\overline{N}} \equiv -6\jmath\jmath_\beta \frac{\varkappa^{1/2}}{g}\frac{\lambda\dot\alpha + \ell\varkappa\dot\phi}{\overline{N}}. \tag{5}$$

- For $p_\beta \neq 0$, fixing the *implicitising gauge* $\overline{N} = -6\jmath\sqrt{\varkappa}\dot\beta/p_\beta$, the trsfed. 1st Friedmann equation can be integrated

$$\mathrm{e}^{\jmath_\chi g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|} S^2\left(\jmath_\beta\sqrt{\frac{3}{2\varkappa}}\,g\beta + C\right), \tag{6}$$

in which $v := \operatorname{sgn} V$, $(\operatorname{sgn}, \operatorname{sgn})$ means $(\ell, \jmath v)$, and

$$\begin{aligned} {}_{(+,+)}S(\gamma) &:= \operatorname{sech}(\gamma), & {}_{(+,-)}S(\gamma) &:= \operatorname{csch}(\gamma), \\ {}_{(-,+)}S(\gamma) &:= \sec(\gamma), & {}_{(-,-)}S(\gamma) &:= \mathbb{i}\csc(\gamma). \end{aligned} \tag{7}$$

[3] The same first integral in canonical formalism has been found in C. Lan. PhD thesis. Saint Petersburg State University, 2016, A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theoretical and Mathematical Physics* 184.3 (2015), pp. 1224–1233.

Implicitised integration

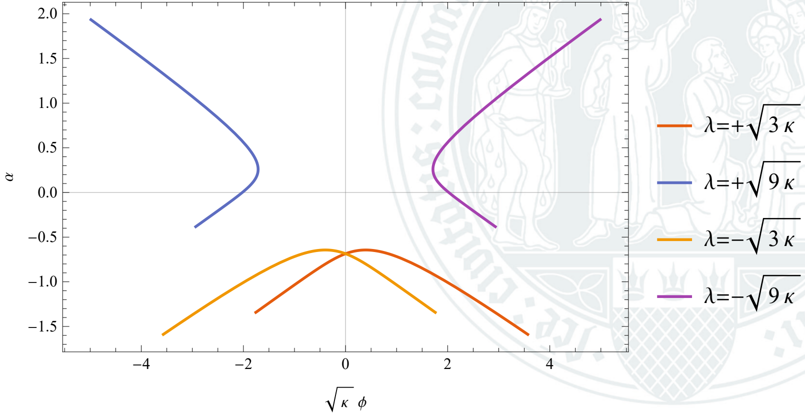
Specific integral for $p_\beta = 0$

- For $p_\beta = 0$, one has $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell\,\lambda\alpha/\varkappa$, which is the familiar power-law special solution[4].
- Further integrating the first Friedmann equation demands $(+,-)$ or $(-,+)$ to guarantee $\overline{N} > 0$, and the result is automatically consistent with the trsfed. Klein–Gordon equation.

[4] M. P. Dąbrowski, C. Kiefer, and B. Sandhöfer. In: *Physical Review D* 74.4 (2006).

Trajectories for quintessence model $(+,+)$

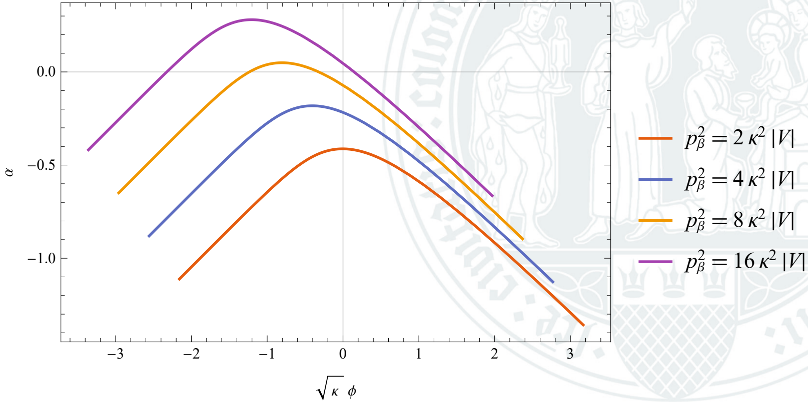
sech, **with** $C=0$, $|V|=\varkappa^{-2}$ **and** $p_\beta^2=\varkappa^2\sqrt{|V|}$, **varying** λ



- has two asymptotes

Trajectories for quintessence model $(+,+)$

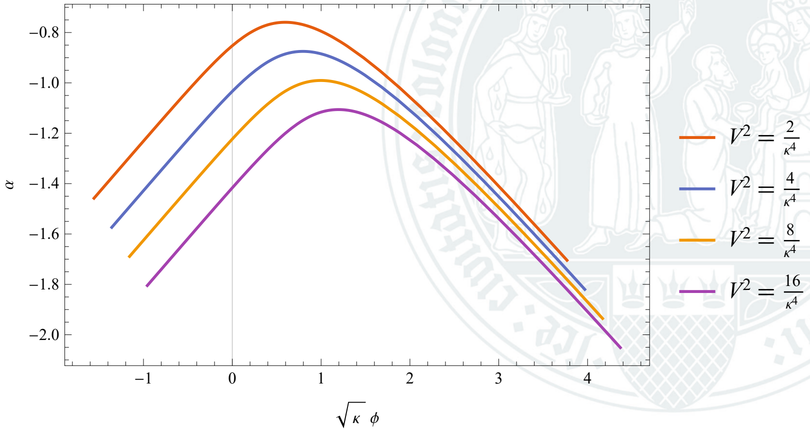
sech, with $C = 0$, $|V| = \varkappa^{-2}$ and $\lambda^2 = 3\varkappa$, varying p_β



- has two asymptotes $\chi \propto \pm\beta$

Trajectories for quintessence model $(+,+)$

sech, **with** $C = 0$, $\lambda^2 = 3\varkappa$ **and** $p_\beta^2 = \varkappa^2\sqrt{|V|}$, **varying** V



- has two asymptotes $\chi \propto \pm\beta$

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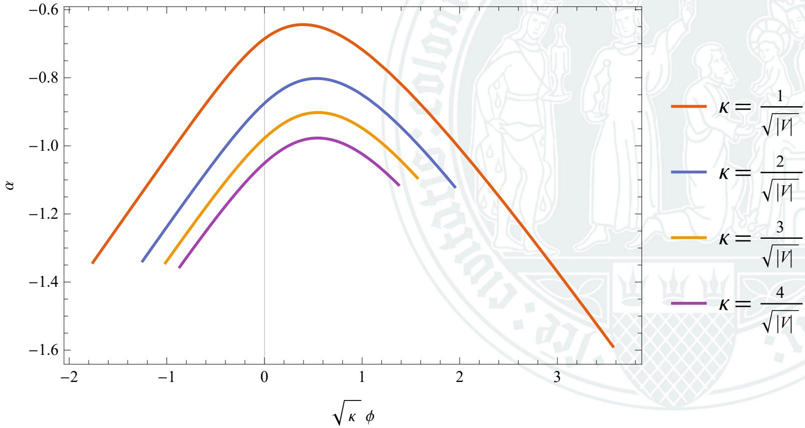
Trajectories for quintessence model $(+,+)$

sech, with $C = 0$, $\lambda^2 = 3\varkappa$ and $p_\beta^2 = \varkappa^2\sqrt{|V|}$, varying $\varkappa$



- has two asymptotes $\chi \propto \pm\beta$

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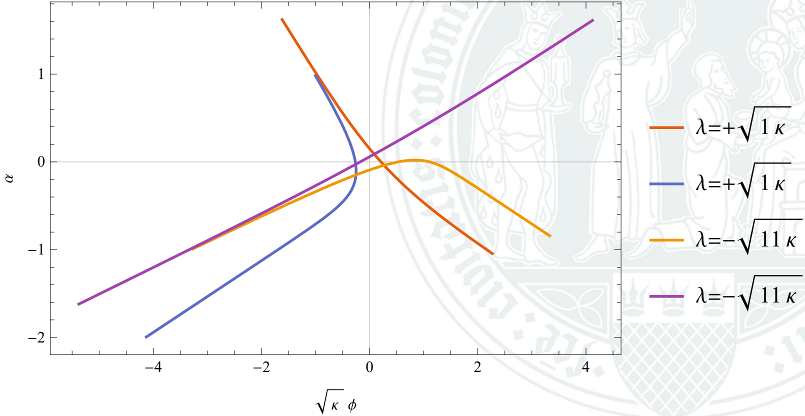
Trajectories for quintessence model $(+,-)$

csch, with $C = 0$, $|V| = \varkappa^{-2}$ and $p_\beta^2 = \varkappa^2\sqrt{|V|}$, varying λ



- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$

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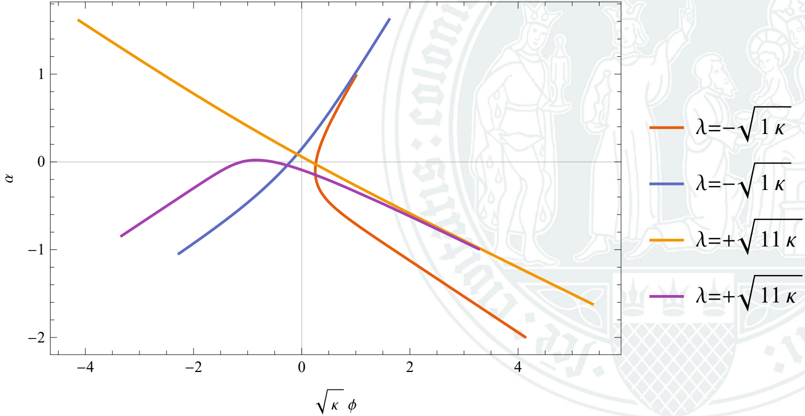
Trajectories for quintessence model $(+,-)$: csch

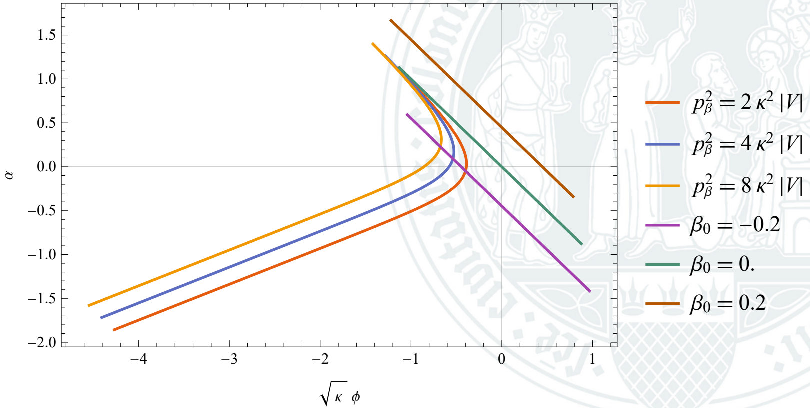
csch, with $C = 0$, $|V| = \varkappa^{-2}$ and $p_\beta^2 = \varkappa^2 \sqrt{|V|}$, varying λ



- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$

Trajectories for quintessence model $(+,-)$: csch

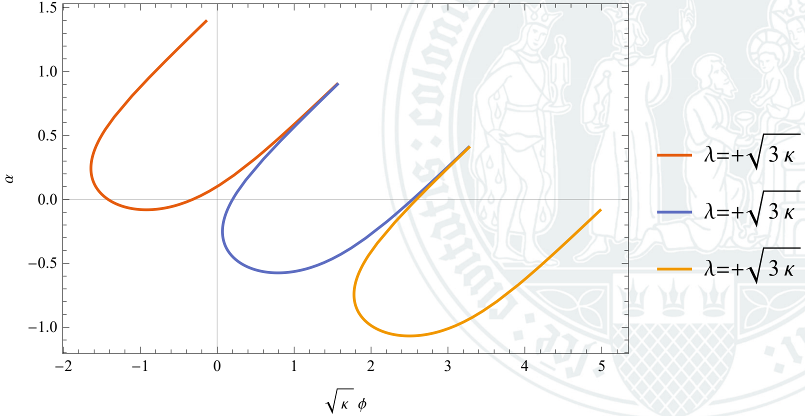
csch, with $V = \varkappa^{-2}$ and $\lambda^2 = \varkappa$, varying p_β



- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$

Trajectories for phantom model $(-,+)$

csc, **with** $C=0$, $V=\varkappa^{-2}$ **and** $p_\beta^2 = 3\varkappa^2\sqrt{|V|}$, **varying** λ



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto n\pi$
- is β-even for $C = 0$

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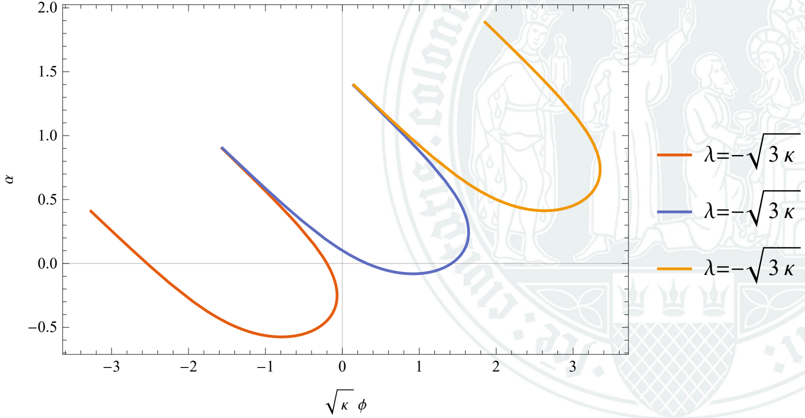
Trajectories for phantom model $(-,+)$

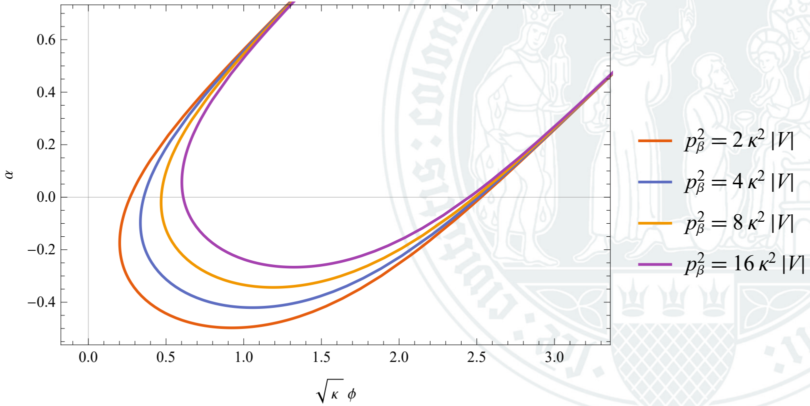
csc, **with** $C = 0$, $V = \varkappa^{-2}$ **and** $p_\beta^2 = 3\varkappa^2\sqrt{|V|}$, **varying** λ



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto n\mathbb{\pi}$
- is β-even for $C = 0$

Trajectories for phantom model $(-,+)$

csc, **with** $C=0$, $|V| = \varkappa^{-2}$ **and** $\lambda^2 = 3\varkappa$, **varying** p_β



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto n\pi$
- is β-even for $C = 0$

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- The integrals are consistent with the trsfed. Klein–Gordon equation.
- The integral for $(-,-)$ is not real.
- The implicitised integral enables one to compare trajectories with wave functions, see below.

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Dirac quantisation and the mode functions
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- The primary Hamiltonian and the Hamiltonian constraint reads

$$H^{\mathsf{p}} = \overline{N} H_{\perp} + v^{\overline{N}} p_{\overline{N}}, \tag{8}$$

$$H_{\perp} := -\mathfrak{s}\,\frac{p_\beta^2}{12\varkappa^{1/2}} + \ell\,\mathfrak{s}\,\frac{p_\chi^2}{2\varkappa^{3/2}} + \varkappa^{3/2} V \mathrm{e}^{g\mathfrak{s}_\chi \chi}. \tag{9}$$

- Applying the Dirac quantisation rules with Laplace–Beltrami opeator, one gets the mss. Wheeler–DeWitt eq. with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(\mathfrak{s}\,\frac{\hbar^2}{12\varkappa^{1/2}} \partial_\beta^2 - \ell\,\mathfrak{s}\,\frac{\hbar^2}{2\varkappa^{3/2}} \partial_\chi^2 + \varkappa^{3/2} V \mathrm{e}^{g\mathfrak{s}_\chi \chi} \right) \Psi. \tag{10}$$

- Equation (10) is KG-like, hyperbolic for $\ell = +1$ and *elliptic* for $\ell = -1$.

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Separation of the variables and mode functions

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- Writing $\Psi(\beta,\chi) = \varphi(\beta)\psi(\chi)$, eq. (10) can be separated into

$$\partial_\beta^2 \varphi(\beta) = k_\beta^2 \varphi(\beta); \tag{11}$$

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \qquad \mathbb{D} := -\ell \frac{6}{\varkappa}\partial_\chi^2 + s\upsilon \frac{12\varkappa^2 |V|}{\hbar^2}. \tag{12}$$

- Equation (12) turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta,\chi) := \sum_{i=1}^{2} c_i \varphi_\nu^{(i)}(\gamma) \sum_{j=1}^{2} a_j \mathrm{B}_\nu^{(j)}(\sigma), \quad \nu \geq 0; \tag{13}$$

$$\nu := \sqrt{\frac{2\varkappa}{3}\frac{1}{g}}\, k_\beta, \quad \gamma := \sqrt{\frac{3}{2\varkappa}\frac{g}{1}}\, \beta, \qquad \sigma^2 := \frac{8\varkappa^3 |V| \mathrm{e}^{g s_\chi \chi}}{\hbar^2 g^2}, \tag{14}$$

$${}_{(+,+)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{K} \text{ and } \mathrm{I}_{\mathbb{i}\nu}(\sigma), \qquad {}_{(+,-)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{F} \text{ and } \mathrm{G}_{\mathbb{i}\nu}(\sigma),$$

$${}_{(-,+)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{J} \text{ and } \mathrm{Y}_\nu(\sigma), \qquad {}_{(-,-)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{K} \text{ and } \mathrm{I}_\nu(\sigma).$$

- Adapted to imaginary order, $\mathrm{F}_\nu(\sigma)$ and $\mathrm{G}_\nu(\sigma)$ are defined in[5]; also see below.

[5] T. M. Dunster. In: *SIAM Journal on Mathematical Analysis* 21.4 (1990), pp. 995–1018.

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Physical mode functions

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- Physical mode functions are expected to be regular on the boundary.
- $(+,+)$: $|\mathrm{I}_{\mathbb{i}\nu}(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- $(-,+)$:
  - $\forall n \in \mathbb{N}$, $|\mathrm{Y}_n(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
  - $\forall \nu \in \mathbb{R}^+ \backslash \mathbb{N}$, choose $\mathrm{J}_{-\nu}$ instead of Y_ν, since $\mathrm{J}_{\pm\nu}$ are also linearly independent.
  - $\forall \nu \in \mathbb{R}^+ \backslash \mathbb{N}$, $|\mathrm{J}_{-\nu}(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
- $(-,-)$: $|\mathrm{K}_\nu(\sigma)| \to +\infty$ as $\alpha \to -\infty$; $|\mathrm{I}_\nu(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- These are not to be included in the space of physical wave functions. $\forall \nu \geq 0$,
  - $(+,+)$: $\mathrm{K}_{\mathbb{i}\nu}(\sigma)$ survives
  - $(+,-)$: F and $\mathrm{G}_{\mathbb{i}\nu}(\sigma)$ survives
  - $(-,+)$: $\mathrm{J}_\nu(\sigma)$ survives
  - $(-,-)$: drops out

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Matching quantum number with classical first integral

Principle of constructive interference

- Baustelle
- In order to match the quantum number k_β (or *linearly*, ν) with the classical first integral p_β, one may apply the *principle of constructive interference*[6].
- Writing the mode function in the WKB form

$$\Psi_{k_\beta}(\beta,\chi) \sim \sqrt{\rho}\,\mathbb{e}^{\mathbb{i}S/\hbar}, \qquad S/\hbar \gg 1 \text{ and } k_\beta \gg 1, \tag{15}$$

the principle demands that $\partial S/\partial k_\beta = 0$ be equivalent to the classical trajectory.

[6] C. Kiefer. In: *Nuclear Physics B* 341.1 (1990), pp. 273–293.

Matching quantum number with classical first integral

$(+,+)$ as exemplar

- Fixing $\nu/\sigma > 1$, the asymptotic expansion reads

$$\mathrm{K}_{\mathrm{i}\nu}(\sigma) \sim \frac{\sqrt{2\pi}\cos\left(\sqrt{\nu^2-\sigma^2} - \nu \arccos\frac{\nu}{\sigma} - \frac{\pi}{4}\right)}{\left(\nu^2-\sigma^2\right)^{1/4} \mathrm{e}^{\pi\nu/2}} + O\left(\frac{1}{\sigma}\right). \tag{16}$$

  - There are two phases with opposite signs. Assuming c_i, a_j's are real and applying *the principle* to $\Psi_\nu(\sigma)$, one has$\sigma/\nu = \mathrm{sech}(\jmath_\beta \gamma)$, which matches the trajectory with $C = 0$ if

$$\hbar k_\beta \equiv \hbar\sqrt{\frac{3}{2\varkappa} g\hbar\nu} = p_\beta, \tag{17}$$

  - Non-vanishing C can be compensated by the phase of c_i and a_j's.

- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; it is not within the WKB regime.

- The conclusions also hold for $\mathrm{F}_{\mathrm{i}\nu}(\sigma)$, $\mathrm{G}_{\mathrm{i}\nu}(\sigma)$ for $(+,-)$, and $\mathrm{J}_\nu(\sigma)$ for $(-,+)$.

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Inner product for wave functions

Schrödinger product

- To make sense of amplitude and wave packet, an inner product is necessary
- A common starting point is the *Schrödinger product*

$$(\Psi_1, \Psi_2)_{\mathrm{S}} := \int \mathrm{d}\chi\, \Psi_1^*(\beta, \chi)\Psi_2(\beta, \chi); \tag{18}$$

- $(\Psi, \Psi)_{\mathrm{S}}$ is positive-definite, and the integrand $\rho_{\mathrm{S}}(\beta, \chi)$ is non-negative
- The corresponding Schrödinger current is real but does not satisfy continuity equation $\dot{\rho}_{\mathrm{S}} + \nabla \cdot \vec{j}_{\mathrm{S}} = 0$, because eq. (10) is KG-like.
- $\mathrm{K}_{\mathbb{i}\nu}$[7] for $(+,+)$, $\mathrm{F}_{\mathbb{i}\nu}$ and $\mathrm{G}_{\mathbb{i}\nu}$ for $(+,-)$ can be proved to be orthogonal and complete among themselves, as well as can be normalised.
    - $\mathrm{J}_{\mathbb{i}\nu}$'s for $(+,-)$ are not orthogonal

[7] S. B. Yakubovich. In: *Opuscula Math.* 26.1 (2006), pp. 161–172, A. Passian et al. In: *Journal of Mathematical Analysis and Applications* 360.2 (2009), pp. 380–390, R. Szmytkowski and S. Bielski. In: *Journal of Mathematical Analysis and Applications* 365.1 (2010), pp. 195–197.

Peculiarity of the phantom model $(-,+)$

Orthogonality for mode functions; Hermiticity for operators

- $\mathrm{J}_\nu(\sigma)$'s are not orthogonal under the Schrödinger product

$$(\mathrm{J}_\nu, \mathrm{J}_{\tilde{\nu}})_{\mathsf{S}} \propto \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{J}_\nu^*(\mathrm{e}^x) \mathrm{J}_{\tilde{\nu}}(\mathrm{e}^x) = \frac{2\sin(\pi(\nu-\tilde{\nu})/2)}{\pi(\nu^2-\tilde{\nu}^2)}, \tag{19}$$

  therefore $\mathbb{D}$ in eq. (12) is not Hermitian (though we do not need it so far)

- $\hat{p}_\chi^2$ is not Hermitian for $\{\mathrm{J}_\nu(\sigma)\}$ under the Schrödinger product

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{J}_\nu^*(-\partial_x^2 \mathrm{J}_{\tilde{\nu}}) - \int_{-\infty}^{+\infty} \mathrm{d}x\, (-\partial_x^2 \mathrm{J}_\nu)^* \mathrm{J}_{\tilde{\nu}} = \frac{2}{\pi} \sin\frac{\pi(\nu-\tilde{\nu})}{2}. \tag{20}$$

- In order to save Hermiticity for p_χ^2 and $\mathbb{D}$ and orthogonality of the modes under Schrödinger product, one can restrict

$$\nu = 2n + \nu_0, \qquad n \in \mathbb{N}, \quad \nu_0 \in [0,2)\,. \tag{21}$$

- The classical trajectory is β-even; imposing the same condition fixes $\nu_0 = 1$.

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Discritisation of the phantom model $(-,+)$ 233

Levels of the phantom model are discretised

...if one imposes Hermiticity of squared momenta under the Schrödinger product

- This kind of subtlety on Hermiticity is named as *self-adjoint extension*, which arises already for infinite square well[8].
- It also applies to x^{-2} potentials[9], which is of cosmological relevance[10].

8 G. Bonneau, J. Faraut, and G. Valent. In: *American Journal of Physics* 69.3 (2001), pp. 322–331.
9 A. M. Essin and D. J. Griffiths. In: *American Journal of Physics* 74.2 (2006), pp. 109–117, V. S. Araujo, F. A. B. Coutinho, and J. F. Perez. In: *American Journal of Physics* 72.2 (2004), pp. 203–213.
10 M. Bouhmadi-López et al. In: *Physical Review D* 79.12 (2009).

Further inner products for wave functions

Klein–Gordon and Mostafazadeh product

- Since eq. (10) is KG-like, another popular choice is the KG product

$$(\Psi_1, \Psi_2)^g_{\mathsf{KG}} := \mathbb{i} g \Big\{ \big(\Psi_1, \dot{\Psi}_2\big)_{\mathsf{S}} - \big(\dot{\Psi}_1, \Psi_2\big)_{\mathsf{S}} \Big\}, \qquad g > 0. \tag{22}$$

  - $(\Psi, \Psi)^g_{\mathsf{KG}}$ is real but not positive-definite, so does the integrand ρ_{KG};
  - The corresponding $\vec{J}_{\mathsf{KG}}$ is conserved $\dot{\rho}_{\mathsf{KG}} + \nabla \cdot \vec{J}_{\mathsf{KG}} = 0$ and real.

- Mostafazadeh[11] found a product *for Hermitian* $\mathbb{D}$ *with positive spectrum*:

$$(\Psi_1, \Psi_2)^\kappa_{\mathsf{M}} := \kappa \Big\{ \big(\Psi_1, \mathbb{D}^{+1/2} \Psi_2\big)_{\mathsf{S}} + \big(\dot{\Psi}_1, \mathbb{D}^{-1/2} \dot{\Psi}_2\big)_{\mathsf{S}} \Big\}, \qquad \kappa > 0. \tag{23}$$

  - $(\Psi, \Psi)^\kappa_{\mathsf{M}}$ is positive-definite, but the integrand ρ^κ_{M} is complex
  - The corresponding $\vec{J}^\kappa_{\mathsf{M}}$ is conserved $\dot{\rho}^\kappa_{\mathsf{M}} + \nabla \cdot \vec{J}^\kappa_{\mathsf{M}} = 0$ but also complex.

[11] A. Mostafazadeh. In: *Classical and Quantum Gravity* 20.1 (2002), pp. 155–171.

Mostafazadeh inner product and the corresponding density

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- The real power of $\mathbb{D}$ is defined by spectral decomposition $\mathbb{D}^{\gamma} := \sum_{\nu} \nu^{2\gamma} \mathbf{P}_{\nu}$, $\mathbf{P}_{\nu} \Psi := \Psi_{\nu} (\Psi_{\nu}, \Psi)_{\mathsf{S}}$.

- It can be shown[12] that *the density*

$$\varrho_{\mathsf{M}}^{\kappa} := \kappa \left\{ \left| \mathbb{D}^{+1/4} \Psi \right|^2 + \left| \mathbb{D}^{-1/4} \dot{\Psi} \right|^2 \right\} \tag{24}$$

  - is equivalent to $\rho_{\mathsf{M}}^{\kappa}$ up to a boundary term $\int \mathrm{d}\chi \, \varrho_{\mathsf{M}}^{\kappa} = \int \mathrm{d}\chi \, \rho_{\mathsf{M}}^{\kappa} \equiv (\Psi_1, \Psi_2)_{\mathsf{M}}^{\kappa}$;
  - is non-negative.
  - The corresponding current $\vec{\mathcal{J}}_{\mathsf{M}}^{\kappa}$ is real but not conserved[13].

---

[12] A. Mostafazadeh and F. Zamani. In: *Annals of Physics* 321.9 (2006), pp. 2183–2209.

[13] B. Rosenstein and L. P. Horwitz. In: *Journal of Physics A: Mathematical and General* 18.11 (1985), pp. 2115–2121.

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu;\overline{\nu},\sigma) := \left(\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(\nu-\overline{\nu})^2}{2\sigma^2}\right)\right)^{1/2} \tag{25}$$

- In[14], $A\left(\nu;\overline{\nu},\sigma/\sqrt{2}\right)$ was chosen.

[14] M. P. Dąbrowski, C. Kiefer, and B. Sandhöfer. In: *Physical Review D* 74.4 (2006).

Wave packets with Gaussian amplitude and the corresponding trajectory

Quintessence model $(+,+)$ with $\mathrm{K}_{i\nu}$

Wave packets with Gaussian amplitude and the corresponding trajectory

Quintessence model $(+,-)$ with $\mathrm{F}_{\mathbb{i}\nu}$

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Wave packets with Gaussian amplitude and the corresponding trajectory

Quintessence model $(+,-)$ with $\mathrm{G}_{\mathbb{i}\nu}$

Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model $(-,+)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\overline{n}) := \left(\mathbb{e}^{-\overline{n}} \frac{\overline{n}^n}{n!} \right)^{1/2} \tag{26}$$

- In[15], $A_n\left(\overline{n}/\sqrt{2}\right)$ was chosen.

[15] C. Kiefer. In: *Nuclear Physics B* 341.1 (1990), pp. 273–293.

Wave packets with Poissonian amplitude and the corresponding trajectory

Phantom model $(-,+)$ with discrete J_{2n+1}

Wave packets with Gaussian amplitude and the corresponding trajectory

Phantom model $(-,+)$ with continuous J_ν

Issues

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- A normalising κ for $(\cdot,\cdot)^{\kappa}_{\mathsf{M}}$ has not yet been able to be evaluated, hence a quantitative comparison of $(\cdot,\cdot)_{\mathsf{S}}$ and $(\cdot,\cdot)^{\kappa}_{\mathsf{M}}$ is not yet possible.
- Wave packets with Gaussian profile are to be constructed, instead of inserting Gaussian / Possonian amplitude
- Quantum-corrected $\overline{\nu}$ is to be understood

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Outlook
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- Beyond classic matter: PT-symmetric instead of phantom field
- Beyond homogeneity: cosmological perturbation

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Integration of the transformed first Friedmann equation
General integral for $p_\beta \neq 0$

In order to integrate the equation under the implicitising gauge

$$\jmath \frac{p_\beta^2}{12}\left(-\ell \frac{\varkappa^{1/2}}{6}\left(\frac{\mathrm{d}\chi}{\mathrm{d}\beta}\right)^2 + \varkappa^{-1/2}\right) - \varkappa^{3/2} V \mathrm{e}^{g\jmath_\chi \chi} = 0, \tag{27}$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa}} g\beta, \qquad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} \mathrm{e}^{-g\jmath_\chi \chi}, \tag{28}$$

to get

$$\left(\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\gamma}\right)^2 + \ell(\jmath v - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-\jmath v + \tilde{\sigma}^2)}}, \tag{29}$$

which is of the standard inverse hyperbolic / trigonometric form except for $(-,-)$.

Integration of the separated minisuperspace WDW equation 233

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \qquad \mathbb{D} := -\ell \frac{6}{\varkappa}\partial_\chi^2 + \mathscr{s}\upsilon \frac{12\varkappa^2 |V|}{\hbar^2}, \tag{12 rev.}$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}}\frac{k_\beta}{g}, \qquad \sigma^2 := \frac{8\varkappa^3 |V| \mathrm{e}^{g\mathscr{s}_\chi \chi}}{\hbar^2 g^2}, \tag{14 rev.}$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma\psi'(\sigma) + \ell(\nu^2 - \mathscr{s}\upsilon\sigma^2)\psi(\sigma) = 0, \tag{30}$$

which is of the standard Besselian form.

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Laden des Themes

Das Theme kann mit den folgenden Optionen geladen werden

```
\usetheme[%
% uk,      %% Farben aller Fakultaeten
wiso,     %% Wiso-Fakultaet
% jura,    %% Rechtswissenschaftliche Fakultaet
% medizin, %% Medizinische Fakultaet
% philo,   %% Philosophische Fakultaet
% matnat,  %% Mathematisch-Naturwissenschaftliche Fakultaet
% human,   %% Humanwissenschaftliche Fakultaet
% verw,    %% Universitaetsverwaltung
]{UzK}
```

block-Umgebungen

Standard (block)

Verwendet die Farbe "'Blaugrau Mittel"' als Blocktitel-Hintergrund

exampleblock

Bei Verwendung der Fußzeile mit allen Fakultätsfarben Titelhintergrund in Wiso-Grün, sonst in der jeweiligen Fakultätsfarbe

alertblock

Verwendet das Rot der Folientitel