

Integrable Liouville Minisuperspace Model

可積的劉維爾小超空間模型

Self-adjointness of the Energy Density and Semi-classical Wave Functions
能量密度的自伴性與準經典波函數

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Outline

1. Introduction

2. The classical model and the implicit trajectories

Lagrangian formalism and the first integral

3. Dirac quantisation and self-adjointness of the energy density

Dirac quantisation and the quantum-mechanical analogy

Self-adjointness of the energy density

4. The semi-classical wave functions

The Wentzel–Kramers–Brillouin approximation

Inner products and wave packets

5. Conclusions



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Introduction

The Friedmann–Lemaître model with quintessence and phantom Liouville fields

- Dynamical Dark Energy has been modelled by quintessence¹ and phantom² matter, which can be realised by minimally-coupled real scalar fields with $\ell = \pm 1$ ³

$$\mathcal{S}_L = \int d^4x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \quad (1)$$

- The Liouville field⁴ $\mathcal{V}(\phi) = V e^{\lambda\phi}$ is of interest, where λ, V are real parameters.
- Assume a flat Robertson–Walker metric $g_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + \kappa e^{2\alpha(t)} d\Omega_{3F}^2$, where $\kappa := 8\pi G$, $d\Omega_{3F}^2 := d\chi^2 + \chi^2(d\theta^2 + \sin^2 \theta d\varphi^2)$, and N lapse function.
- The total minisuperspace action reads $\mathcal{S} := \mathcal{S}_{EH} + \mathcal{S}_{GHY} + \mathcal{S}_L = \int d\Omega_{3F}^2 \int dt L$, where

$$L := \kappa^{3/2} N e^{3\alpha} \left(-\frac{3}{\kappa} \frac{\dot{\alpha}^2}{N^2} + \ell \frac{\dot{\phi}^2}{2N^2} - V e^{\lambda\phi} \right). \quad (2)$$

¹R. R. Caldwell *et al.*, *Phys. Rev. Lett.* **80**, 1582–1585 (Aug. 7, 1997).

²R. R. Caldwell, *Phys. Lett. B* **545**, 23–29 (Aug. 16, 1999).

³The signature of metric is $(-, +, \dots, +)$.

⁴Y. Nakayama, *Int. J. Mod. Phys. A* **19**, 2771–2930 (Feb. 2, 2004).



Integrability

With the help of an integral of motion, implicit trajectories can be obtained explicitly, and the minisuperspace Wheeler–DeWitt equation can be integrated exactly.

Self-adjointness

Imposing self-adjointness of the energy density leads to removal of the degeneracy and discretisation of the spectrum.

Semi-classical wave functions

Semi-classical wave functions can be constructed by the principle of constructive interference, the Wentzel–Kramers–Brillouin approximation and numerical methods.



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Decoupling the variables

Via orthogonal transformation

- By rescaling $\bar{N} := N e^{-3\alpha}$ and an orthogonal transformation from (α, ϕ) to (β, χ) , the system can be **decoupled**

$$L = \varkappa^{3/2} \bar{N} \left(-\beta \frac{3}{\varkappa} \frac{\dot{\beta}^2}{\bar{N}^2} + \ell \beta \frac{\dot{\chi}^2}{2\bar{N}^2} - V e^{g\chi} \right), \quad (3)$$

where β is the sign of $(\lambda^2 - 6\ell\varkappa)$, and $g = \beta \sqrt{|\lambda^2 - 6\ell\varkappa|}$ takes the place of λ .

- Using the **conservation of p_β** ⁵, $\bar{N}(t)$ can be eliminated, leading to the exact integrations of the equations of motion as **implicit trajectories** (without t).
 - $p_\beta \neq 0$: four cases according to $(\ell, \beta v)$, v is the sign of V ; see below.
 - $p_\beta = 0$: well-known power-law special solution for $(+, -)$ and $(-, +)$; read literature⁶.

⁵The same first integral has been found in C. Lan, PhD thesis, Saint Petersburg State University, 2016, <https://search.rsl.ru/ru/record/01006663434>, A. A. Andrianov et al., *Theor. Math. Phys.* **184**, 1224–1233 (Mar. 18, 2015), in canonical formalism.

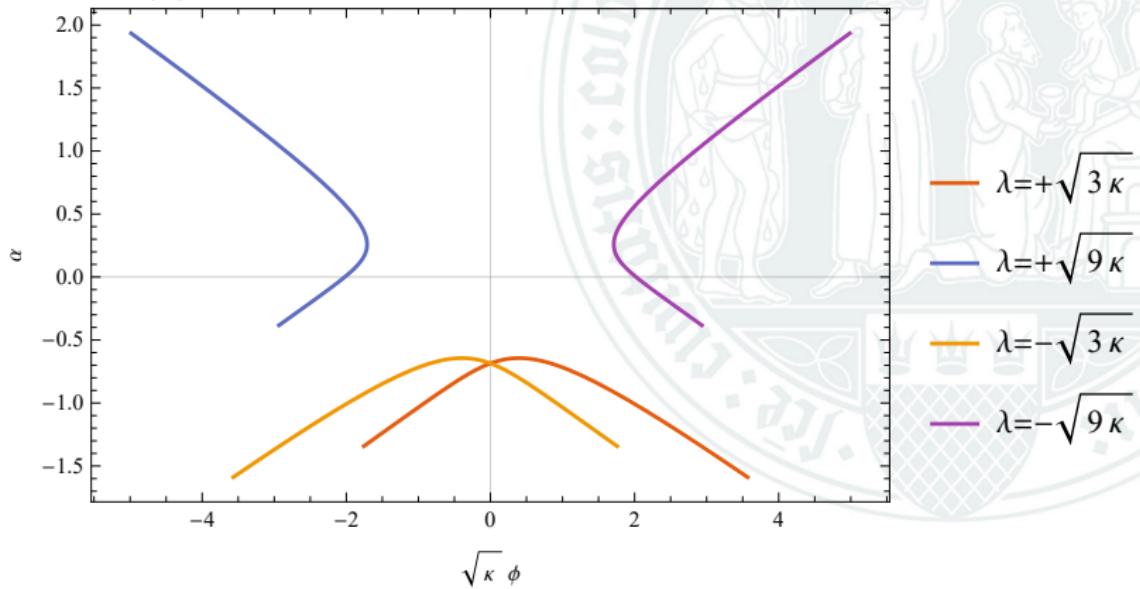
⁶For instance, A. R. Liddle, D. H. Lyth, *Cosmological Inflation and Large-Scale Structure*, (Cambridge University, 2000) , ch. 3.



Minisuperspace trajectories for quintessence model (+, +)

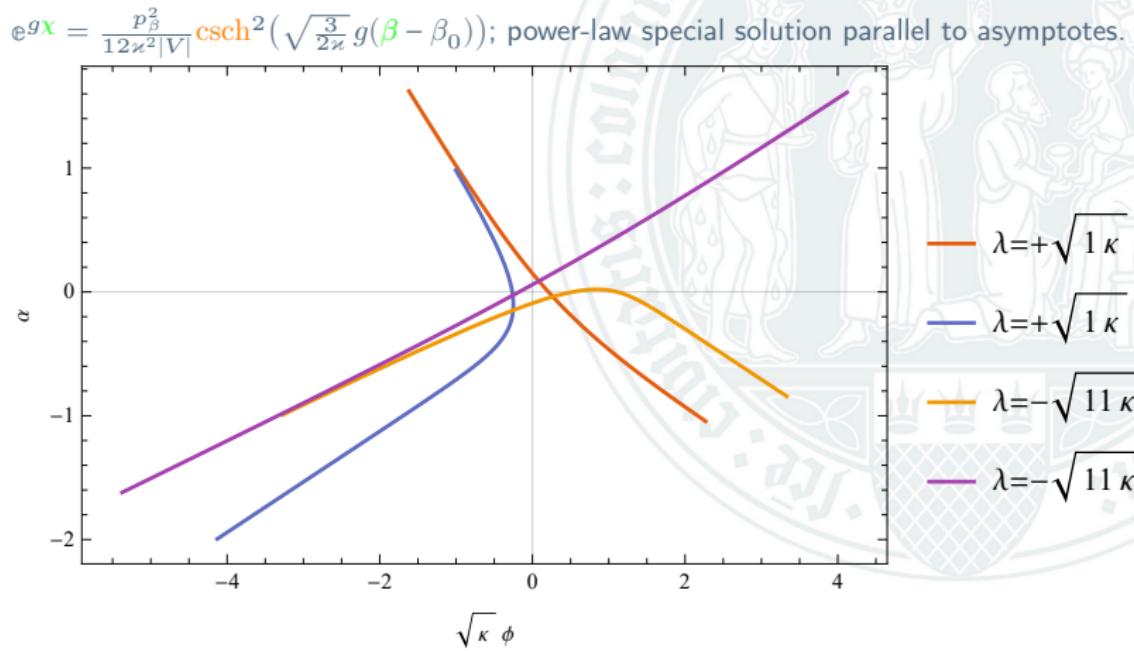
sech, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

$$e^{g\chi} = \frac{p_\beta^2}{12\kappa^2|V|} \operatorname{sech}^2\left(\sqrt{\frac{3}{2\kappa}} g(\beta - \beta_0)\right); \text{ no power-law special solution.}$$



Trajectories for quintessence model (+, -)

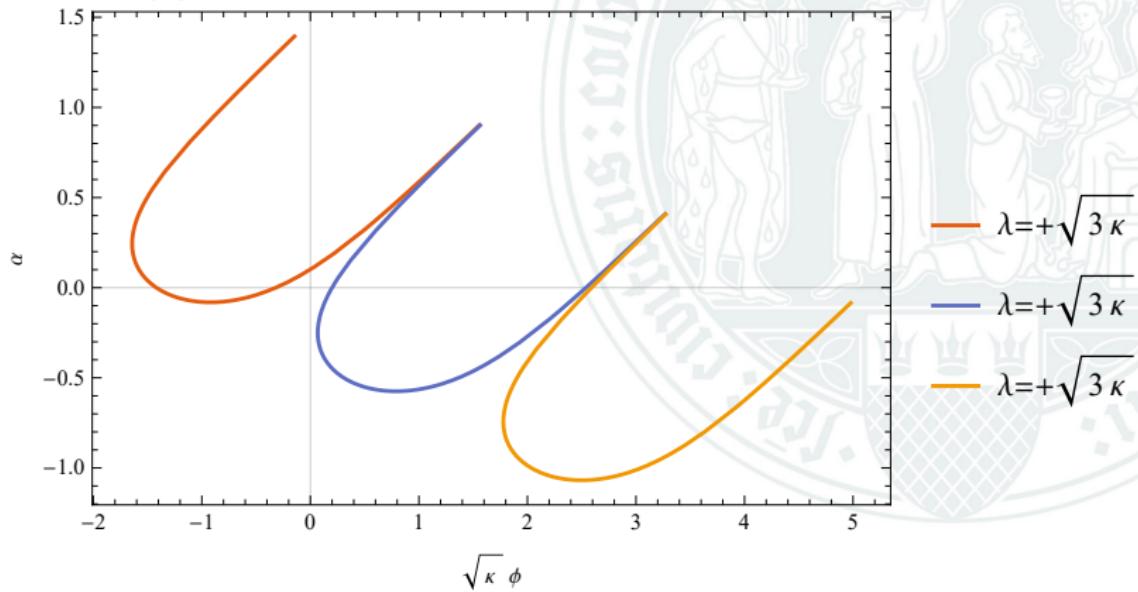
csh, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ



Minisuperspace trajectories for phantom model $(-, +)$

sec, with $V = \varkappa^{-2}$ and $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

$e^{g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|} \sec^2\left(\sqrt{\frac{3}{2\varkappa}} g(\beta - \beta_0)\right)$, power-law special solution parallel to asymptotes.



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Dirac quantisation

- The primary Hamiltonian of this constrained, singular system⁷ reads

$$H^p = \bar{N} H_{\perp} + v^{\bar{N}} p_{\bar{N}}, \text{ in which the Hamiltonian constraint is}$$

$$H_{\perp} := -\beta \frac{p_{\beta}^2}{12\nu^{1/2}} + \ell \beta \frac{p_{\chi}^2}{2\nu^{3/2}} + \nu^{3/2} V e^{g\chi}. \quad (4)$$

- Applying the Dirac quantisation rules with the Laplace–Beltrami operator ordering⁸, one gets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(\beta \frac{\hbar^2}{12\nu^{1/2}} \partial_{\beta}^2 - \ell \beta \frac{\hbar^2}{2\nu^{3/2}} \partial_{\chi}^2 + \nu^{3/2} V e^{g\chi} \right) \Psi. \quad (5)$$

- Equation (5) is Klein–Gordon-like, hyperbolic for $\ell = +1$ and **elliptic** for $\ell = -1$.
 - The Klein–Gordon nature of the Wheeler–DeWitt equation often results in the non-existence of a probability interpretation. But see below!

⁷D. M. Gitman, I. V. Tyutin, *Quantization of Fields with Constraints*, (Springer, 1990), H. J. Rothe, K. D. Rothe, *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*, (World Scientific, Apr. 2010).

⁸C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012), ch. 8.



Integration of the Wheeler–DeWitt equation

The quantum-mechanical analogy

- Fourier transforming β to $k_\beta \in \mathbb{R}$ yields the time-independent Schrödinger equation

$$E\psi(x) = \widehat{H}_{\text{eff}}\psi := \left(-\frac{\hbar^2}{2M_p} \partial_x^2 + V_{\text{eff}} \right) \psi, \quad V_{\text{eff}} := \ell s v \widetilde{V} e^{\tilde{g}x}, \quad (6)$$

in which $E \propto \ell\nu^2$, $M_p := \sqrt{\frac{\hbar}{\pi}}$, $x \propto \chi$, $\widetilde{V} \propto |V| \geq 0$, and $\nu \propto |k_\beta| \geq 0$.

- \widehat{H}_{eff} plays the role of energy density in cosmology.
- The Liouville potential V_{eff} is a special case of the Morse potential⁹.
- The Starobinsky model in the Einstein frame is related to the Morse potential.

- Let $e^{2y} := 8M_p \widetilde{V} / \hbar^2 \tilde{g}^2 \cdot e^{\tilde{g}x}$; there are four cases according to (ℓ, sv) , and the mode functions are modified and unmodified Bessel functions of imaginary and real orders

	$E \geq 0$	$E < 0$
$V_{\text{eff}} > 0$	$(+, +); K_{il\nu}(e^y)$	$(-, -); \text{none}$
$V_{\text{eff}} < 0$	$(+, -); F_{il\nu}(e^y), G_{il\nu}(e^y)$ ¹⁰	$(-, +); J_\nu(e^y)$

⁹P. M. Morse, *Phys. Rev.* **34**, 57–64 (July 1929), D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012).

¹⁰These are linear combinations of $J_{\pm il\nu}(e^y)$; see T. M. Dunster, *SIAM J. Math. Anal.* **21**, 995–1018 (July 1990).



Self-adjointness of the energy density

Results for the Liouville potential¹⁴

- k_β being real enforces the reality of E and the self-adjointness of \widehat{H}_{eff} .
- For $V_{\text{eff}} > 0$: \widehat{H}_{eff} is **essentially self-adjoint**¹¹. Nothing special for quintessence (+,+).
- For $V_{\text{eff}} < 0$: \widehat{H}_{eff} admits a **family of self-adjoint extensions**.
- Quintessence (+,-): “eigenfunctions” are linear combinations of F and G; **the degeneracy is removed**
- Phantom (-,+): the spectrum becomes **discrete**

$$\psi_{-, \nu}(e^y) \propto J_\nu(e^y), \quad \nu = 2n + a_<, \quad n \in \mathbb{Z}_\geq, \quad (7)$$

and the corresponding full wave-functions are Bloch-periodic. Imposing (anti-)periodic boundary condition fixes $a_< = 0$ (1).

- The self-adjointness has been discussed in quantum cosmology in e.g.¹² (operator ordering) and more recently in¹³ (Λ as a quantum number).

¹¹ See also B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267 , sec. 9.9.

¹² C. R. Almeida et al., *Gravitation Cosmol.* **21**, 191–199 (Jan. 17, 2015).

¹³ S. Gryb, K. P. Y. Thébault, arXiv: 1801.05789 (gr-qc) (Jan. 17, 2018).

¹⁴ See also D. M. Gitman et al., *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) , sec. 8.5.



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Matching quantum number with classical first integral

Apply the principle of constructive interference; take $(+, +)$ as an example

- Write the mode function in the Wentzel–Kramers–Brillouin form

$\Psi_\nu(\beta, \chi) \sim \sqrt{\rho(\beta, \chi)} \exp\left\{\frac{1}{\hbar} S(\beta, \chi)\right\}$. For $S/\hbar \gg 1$ and $\nu \gg 1$, the phase $S(\beta, \chi)$ becomes the Hamilton principal function in the leading-order approximation, which is stationary with respect to the variation of integral constants $\partial S / \partial \nu = 0$.

- Fixing $\nu/\sigma > 1$, where $\sigma^2 \propto e^{g\chi}$, $\nu \propto k_\beta$, $\gamma \propto \beta$,

$$K_{\mathbb{I}\nu}(\sigma) \sim \frac{\sqrt{2\pi} \cos\left(\sqrt{\nu^2 - \sigma^2} - \nu \arccos \frac{\nu}{\sigma} - \frac{\pi}{4}\right)}{(\nu^2 - \sigma^2)^{1/4} e^{\pm\nu/2}} + O\left(\frac{1}{\sigma}\right). \quad (8)$$

- Applying the principle, one has $\sigma/\nu = \operatorname{sech}(\beta_\beta \gamma)$, which matches the trajectory if $\hbar k_\beta \equiv \hbar \sqrt{\frac{3}{2\omega}} g \nu = p_\beta$, so that the quantum k_β is matched to the classical p_β .
- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; it is not within the Wentzel–Kramers–Brillouin regime.
- The conclusions also hold for $F_{\mathbb{I}\nu}(\sigma)$, $G_{\mathbb{I}\nu}(\sigma)$ for $(+, -)$, and $J_\nu(\sigma)$ for $(-, +)$.



Inner products for wave functions

Schrödinger and Mostafazadeh inner products

- Take (β, χ) as ‘temporal’ and ‘spatial’. Begin with the Schrödinger product¹⁵

$$(\Psi_1, \Psi_2)_S := \int d\chi \Psi_1^*(\beta, \chi) \Psi_2(\beta, \chi); \quad (9)$$

- $(\Psi, \Psi)_S$ positive-definite; the integrand $\rho_S(\beta, \chi)$ non-negative
- The Schrödinger current real but no continuity since the WdW eq. is Klein–Gordon-like
- Mostafazadeh¹⁶: for self-adjoint \widehat{H}_{eff} with positive spectrum,

$$(\Psi_1, \Psi_2)_M \propto (\Psi_1, \widehat{H}_{\text{eff}}^{+1/2} \Psi_2)_S + (\dot{\Psi}_1, \widehat{H}_{\text{eff}}^{-1/2} \dot{\Psi}_2)_S \quad (10)$$

- Real power of \widehat{H}_{eff} is defined by spectral decomposition
- For the density¹⁷

$$\rho_M \propto |\widehat{H}_{\text{eff}}^{+1/4} \Psi|^2 + |\widehat{H}_{\text{eff}}^{-1/4} \dot{\Psi}|^2, \quad (11)$$

continuity can be partially recovered. A candidate for probability density.

¹⁵C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012), ch. 5.

¹⁶A. Mostafazadeh, *Class. Quantum Grav.* **20**, 155–171 (Sept. 6, 2002).

¹⁷A. Mostafazadeh, F. Zamani, *Ann. Phys.* **321**, 2183–2209 (Feb. 17, 2006).



Wave packets of Gaussian amplitude for continuous spectrum

Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu; \bar{\nu}, \sigma) := \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\nu - \bar{\nu})^2}{2\sigma^2}\right) \right)^{1/2} \quad (12)$$

- In¹⁸, $A(\nu; \bar{\nu}, \sigma/\sqrt{2})$ was chosen.

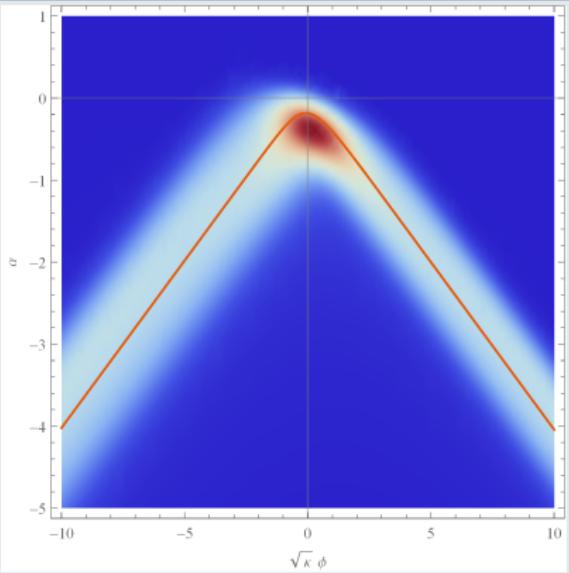
¹⁸M. P. Dąbrowski et al., *Phys. Rev. D* **74**, arXiv: hep-th/0605229 (hep-th) (May 23, 2006).



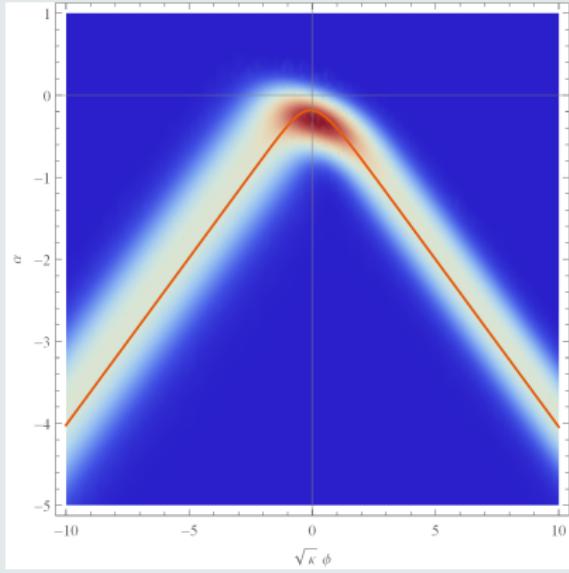
Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\ell\nu}$, with $\lambda = \varkappa^{1/2}/2$, $V = -\varkappa^{-2}$, $\bar{k}_\beta = -2$ and $\sigma_\beta = 5/4$

Schrödinger



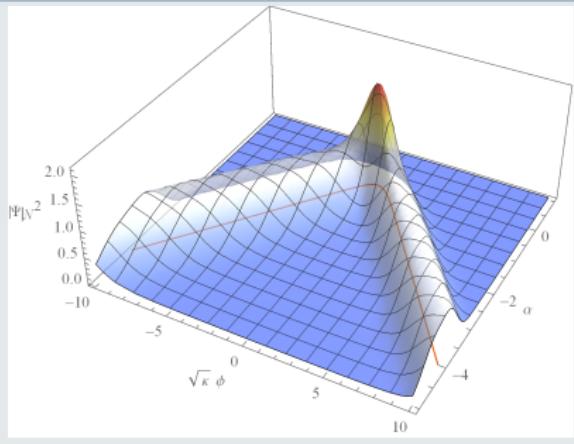
Mostafazadeh



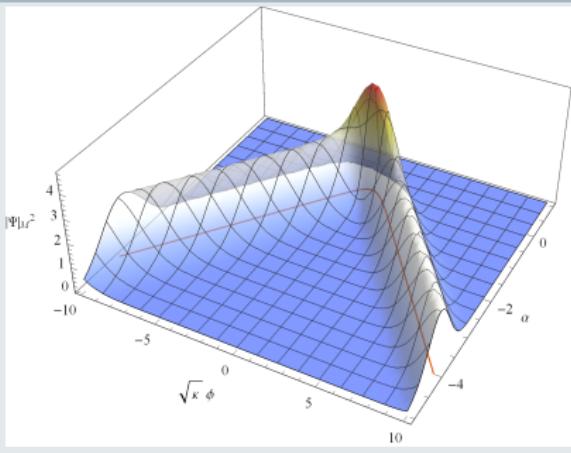
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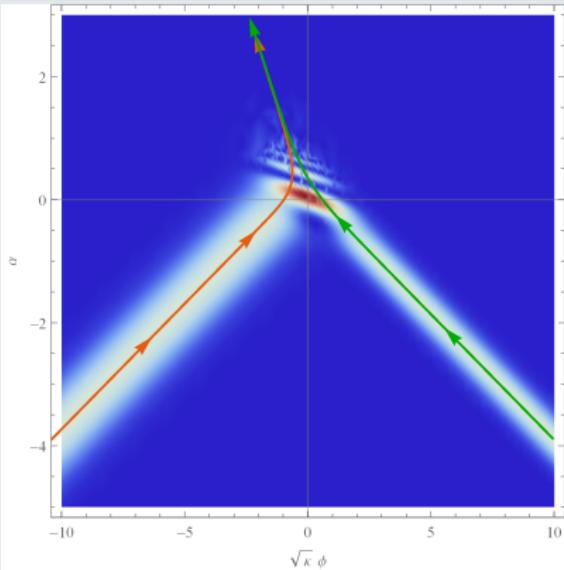
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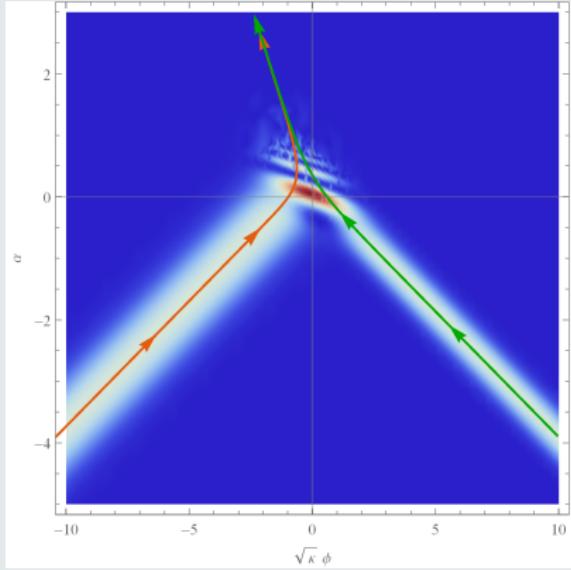
Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\parallel\nu}$, with $\lambda = 4\nu^{1/2}/5$, $V = +\nu^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



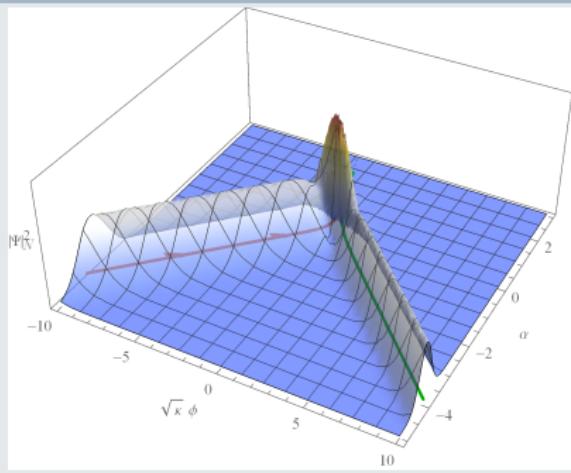
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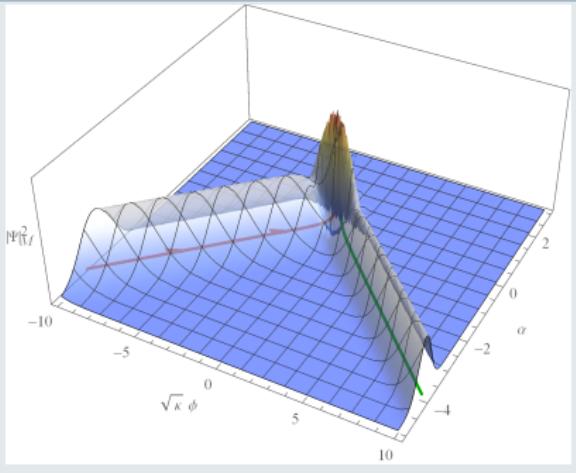
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Schrödinger



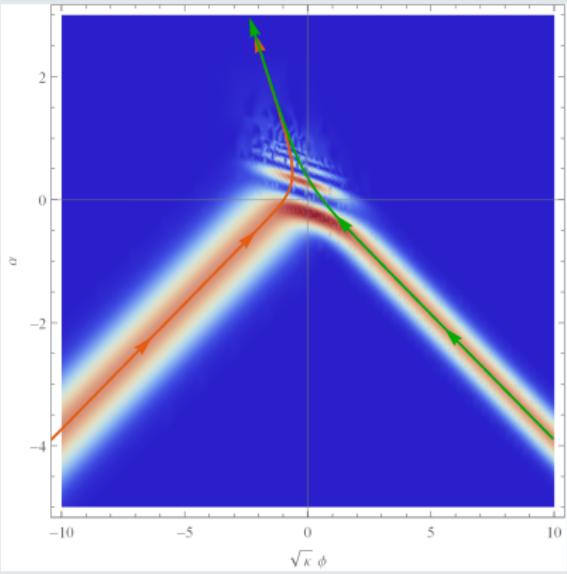
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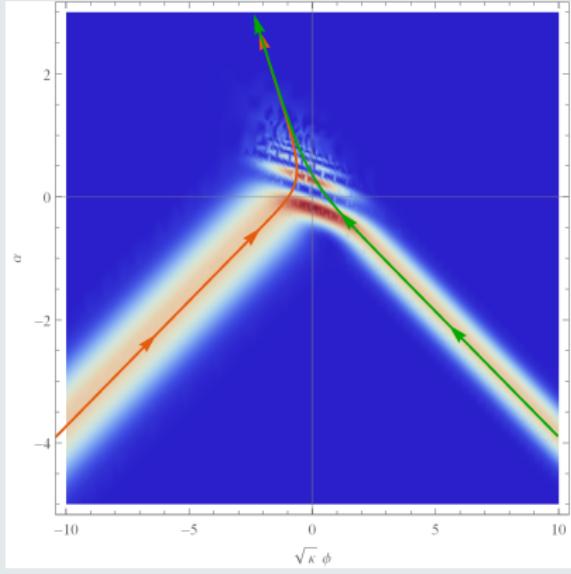
Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\parallel\nu}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



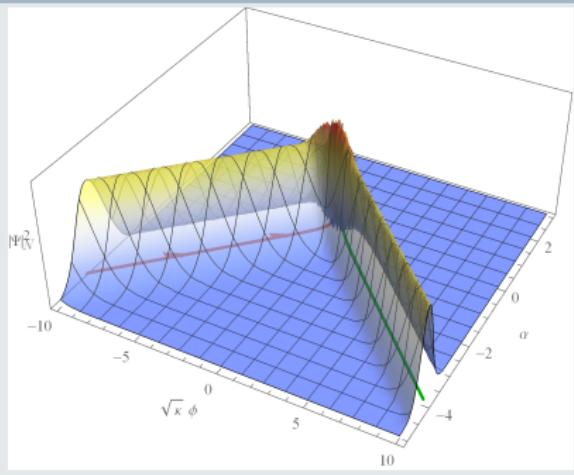
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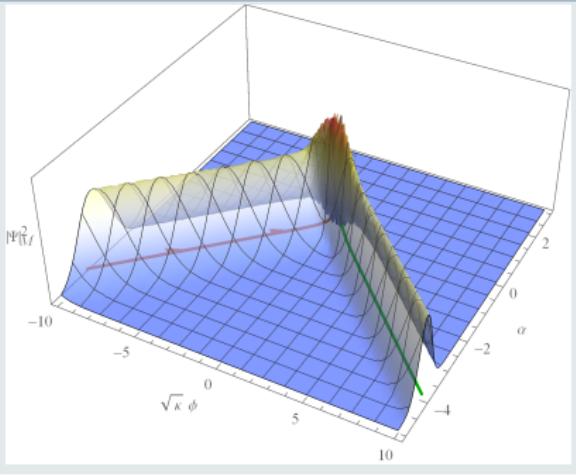
Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\parallel\nu}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



Mostafazadeh



Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model $(-, +)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\bar{n}) := \left(e^{-\bar{n}} \frac{\bar{n}^n}{n!} \right)^{1/2} \quad (13)$$

- In¹⁹, $A_n(\bar{n}/\sqrt{2})$ was chosen.

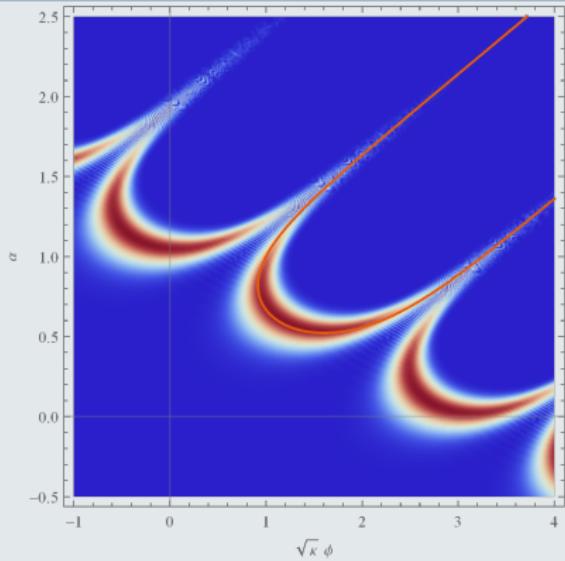
¹⁹C. Kiefer, *Nucl. Phys. B* **341**, 273–293 (Sept. 1990).



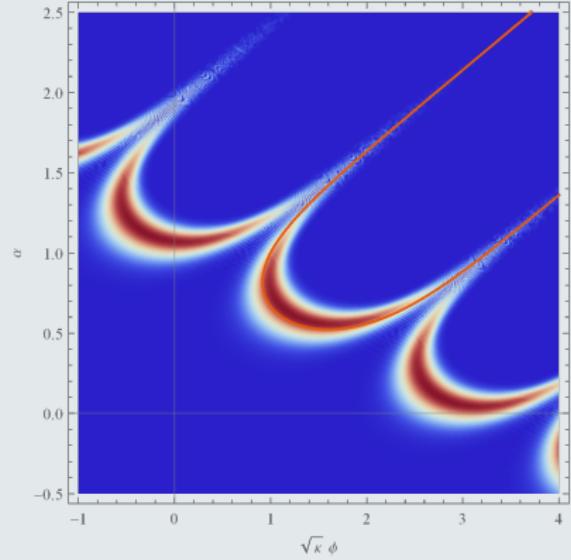
Wave packets of Poissonian amplitude for phantom model

J_{2n+1} , with $\lambda = 2\kappa^{1/2}$, $V = +\kappa^{-2}$ and $\bar{k}_\beta = 8$

Schrödinger



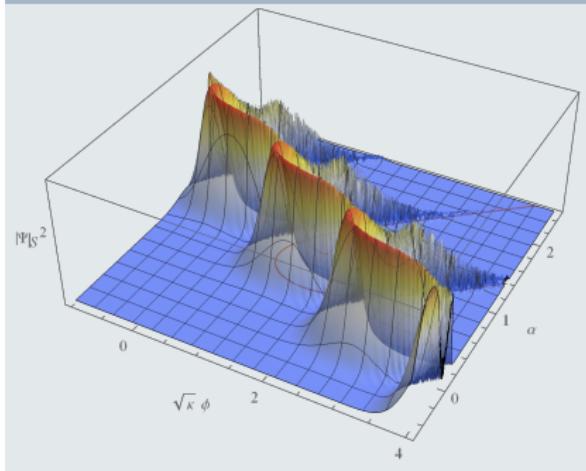
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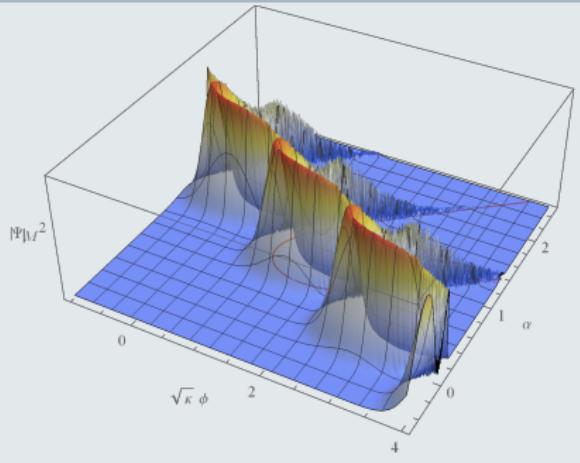
Wave packets of Poissonian amplitude for phantom model

J_{2n+1} , with $\lambda = 2\kappa^{1/2}$, $V = +\kappa^{-2}$ and $\bar{k}_\beta = 8$

Schrödinger



Mostafazadeh



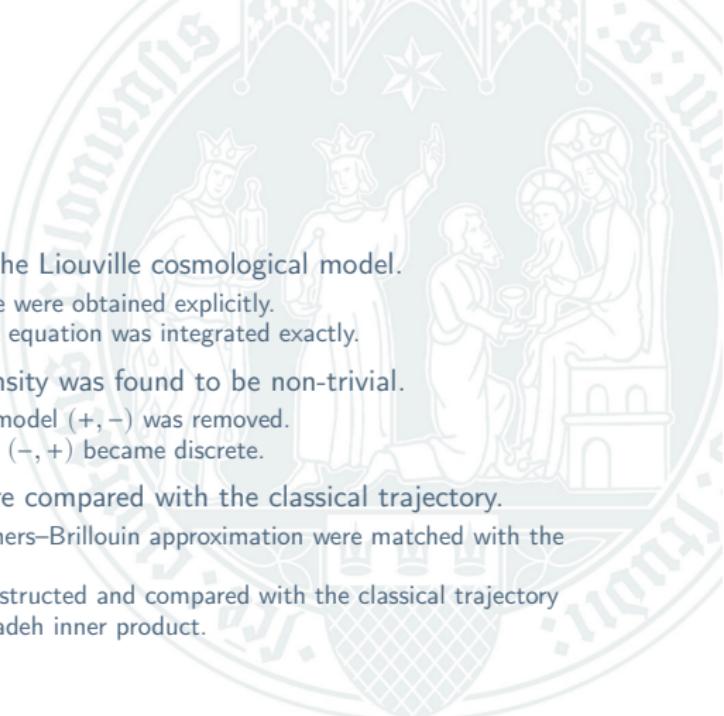
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Highlights

Revisited



- An **integral of motion** was found for the Liouville cosmological model.
 - Implicit trajectories in minisuperspace were obtained explicitly.
 - The minisuperspace Wheeler–DeWitt equation was integrated exactly.
- The **self-adjointness** of the energy density was found to be non-trivial.
 - The degeneracy of the quintessence model $(+, -)$ was removed.
 - The spectrum of the phantom model $(-, +)$ became discrete.
- The **semi-classical wave functions** were compared with the classical trajectory.
 - Mode functions in the Wentzel–Kramers–Brillouin approximation were matched with the classical Hamilton principal function.
 - Semi-classical wave packets were constructed and compared with the classical trajectory under the Schrödinger and Mostafazadeh inner product.

Outlook

- Beyond isotropy: Bianchi models²⁰
- Beyond homogeneity: cosmological perturbations²¹ (current work)
- Beyond single Liouville field: multiple potentials / multiple fields²²
- Beyond classical matter and Schrödinger inner product: PT -symmetric fields²³

²⁰A. Y. Kamenshchik *et al.*, *Phys. Rev. D* **95**, arXiv: 1702.02314 (gr-qc) (Feb. 8, 2017).

²¹D. Brizuela *et al.*, *Phys. Rev. D* **93**, arXiv: 1511.05545 (gr-qc) (Nov. 17, 2015), D. Brizuela *et al.*, *Phys. Rev. D* **94**, arXiv: 1611.02932 (gr-qc) (Nov. 9, 2016), O. O. Novikov, arXiv: 1712.03939 (hep-th) (Dec. 11, 2017), A. Y. Kamenshchik *et al.*, *Class. Quantum Grav.* **35**, 015012 (Sept. 29, 2017), C. Kiefer, D. Wichmann, *Gen. Relativ. Gravit.* (2018) **50**:66–50, arXiv: 1802.01422 (gr-qc) (Feb. 5, 2018).

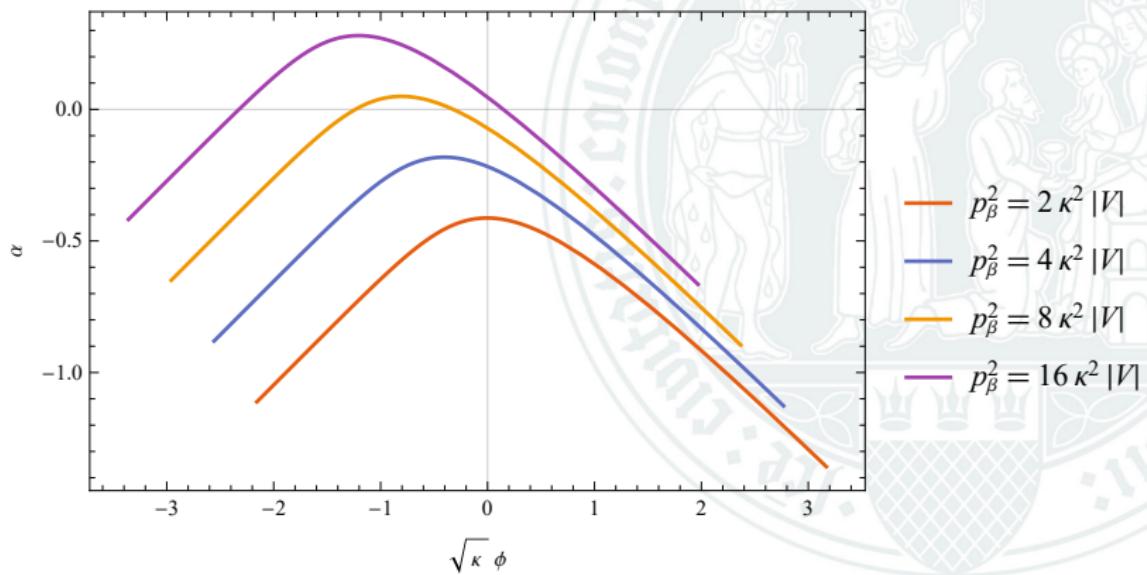
²²A. A. Andrianov *et al.*, *Theor. Math. Phys.* **184**, 1224–1233 (Mar. 18, 2015).

²³A. A. Andrianov *et al.*, *Int. J. Mod. Phys. D* **19**, 97–111 (Jan. 2010), A. A. Andrianov *et al.*, in *Springer Proceedings in Physics* (Springer International Publishing, 2016), pp. 29–44.



Trajectories for quintessence model (+, +)

sech, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $\lambda^2 = 3\kappa$; varying p_β

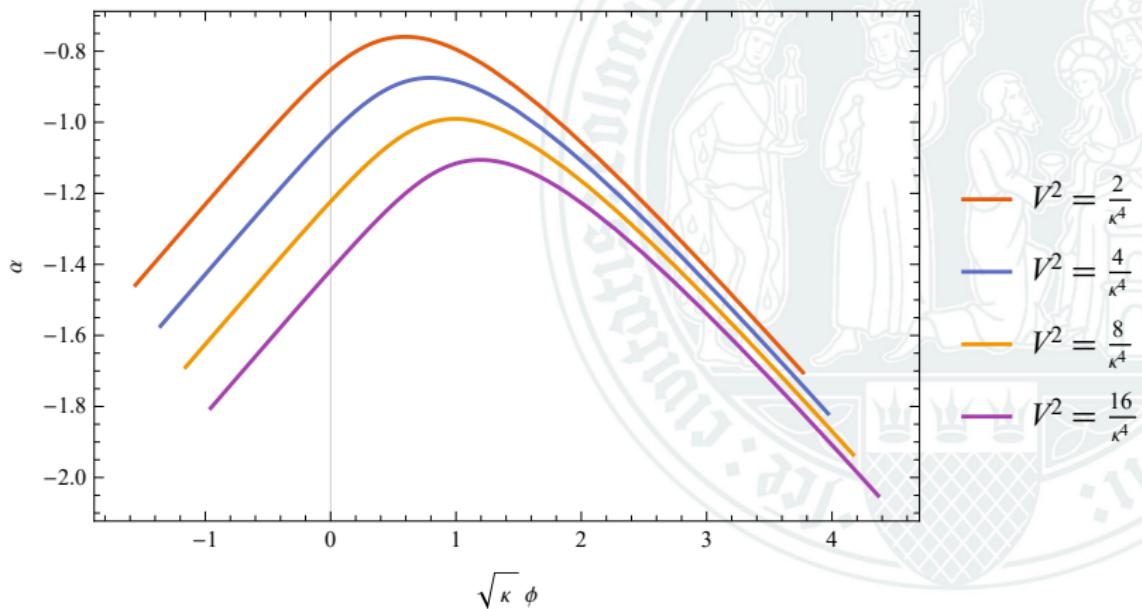


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model (+, +)

sech, with $\beta_0 = 0$, $\lambda^2 = 3\kappa$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying V

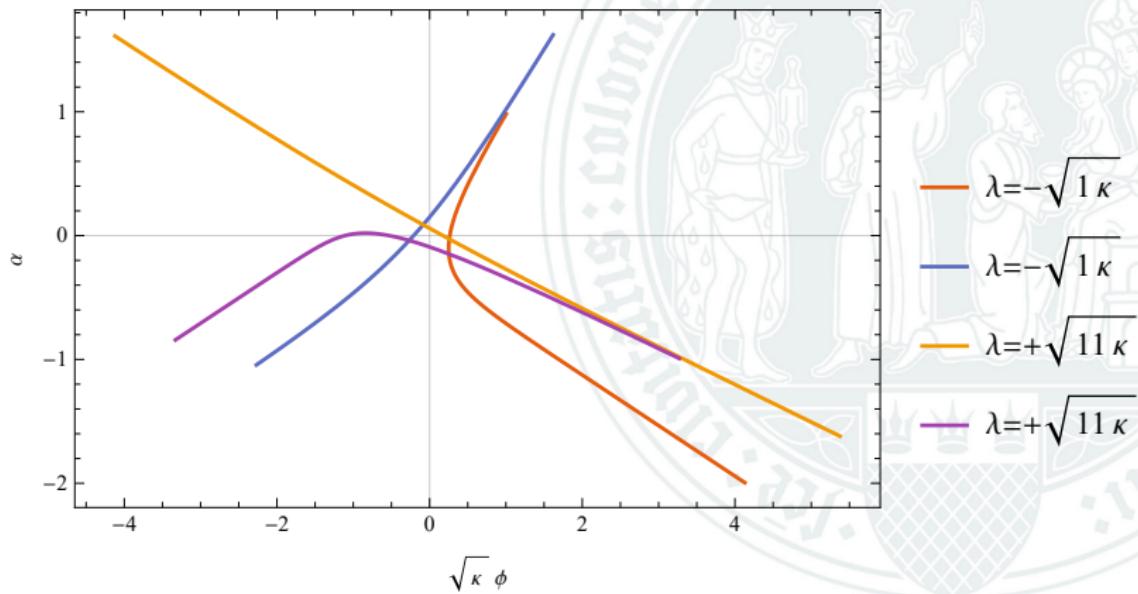


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model $(+, -)$: csch

csch, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

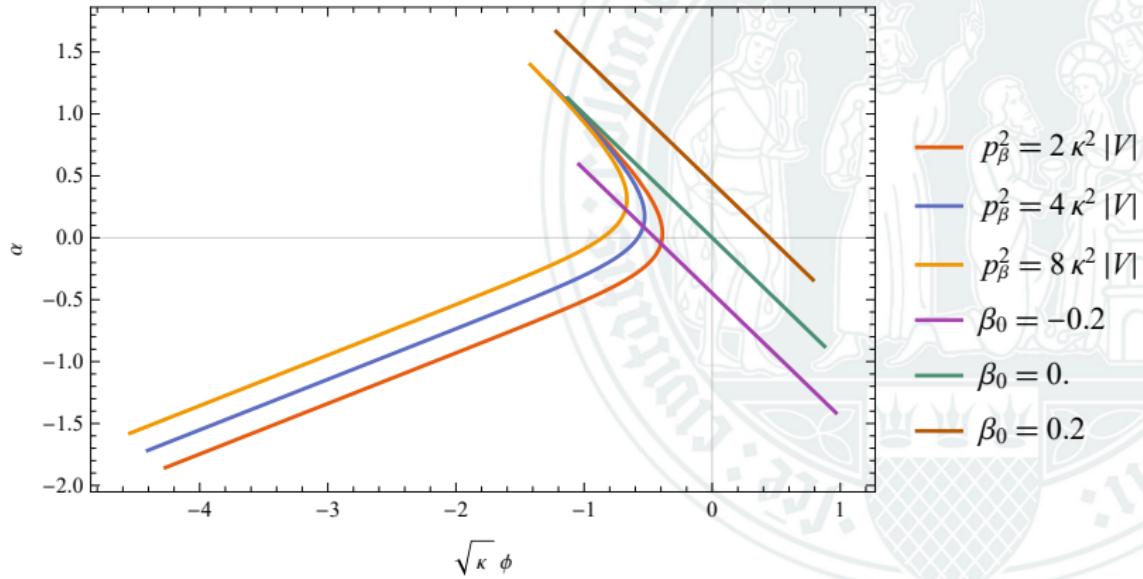


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$



Trajectories for quintessence model (+, -): csch

csch, with $V = \kappa^{-2}$ and $\lambda^2 = \kappa$; varying p_β

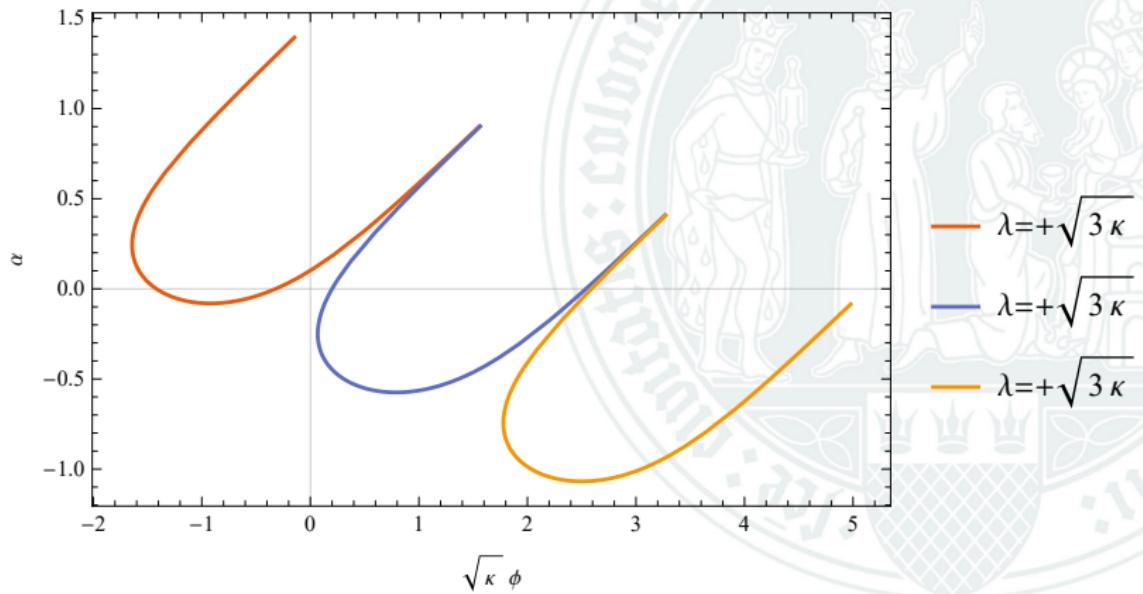


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$



Trajectories for phantom model $(-, +)$

sec, with $V = \varkappa^{-2}$ and $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

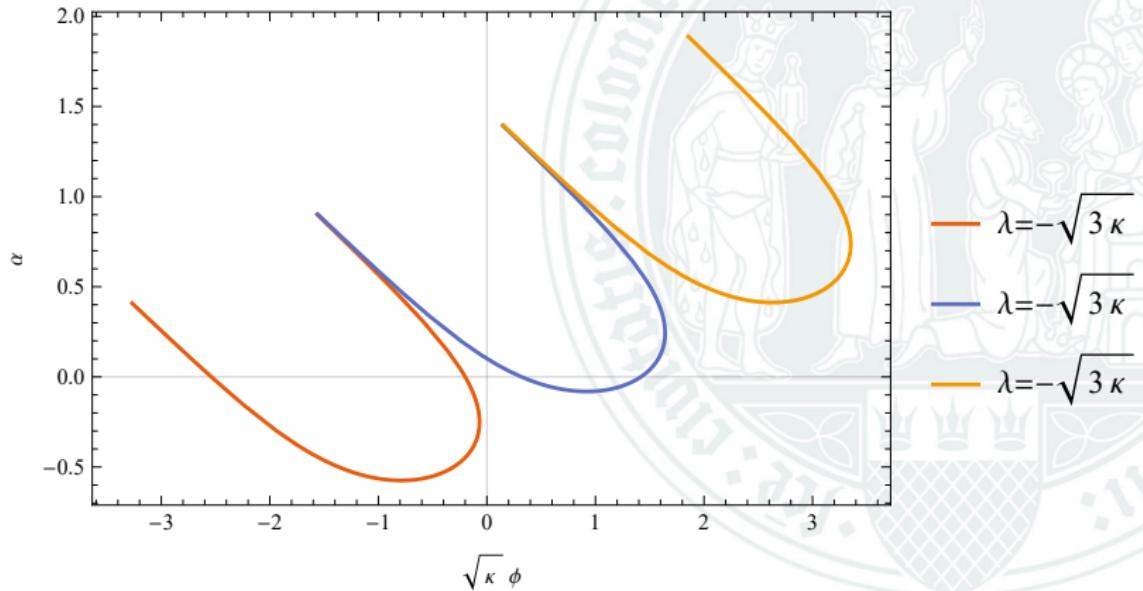


- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Trajectories for phantom model $(-, +)$

csc, with $V = \varkappa^{-2}$ and $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

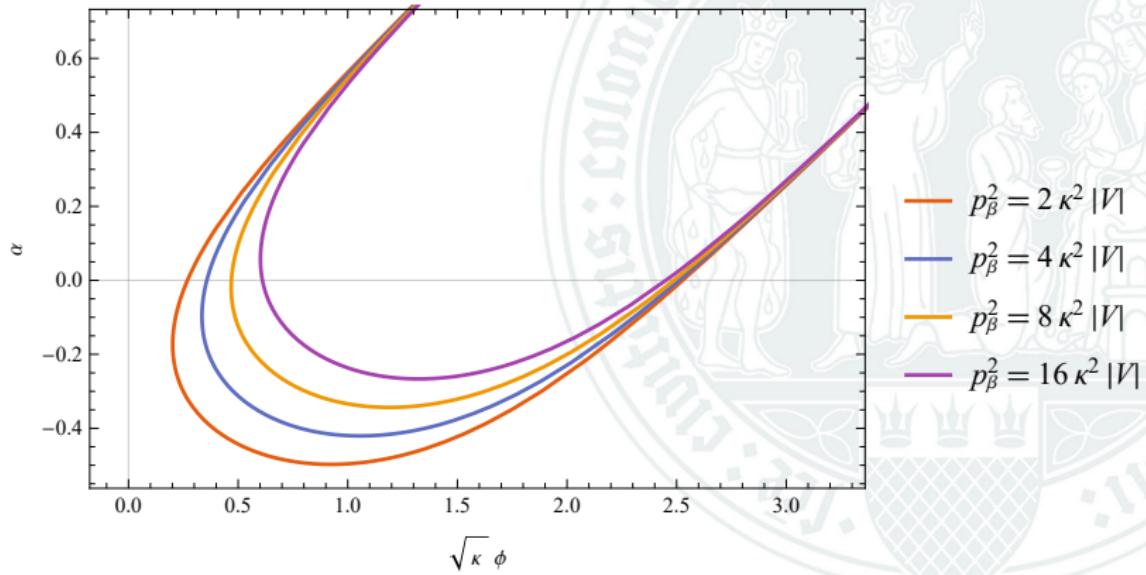


- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Trajectories for phantom model $(-, +)$

csc, with $V = \kappa^{-2}$ and $\lambda^2 = 3\kappa$; varying p_β



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Integration of the transformed first Friedmann equation

General integral for $p_\beta \neq 0$

In order to integrate the equation under the implicit gauge

$$s \frac{p_\beta^2}{12} \left(-\ell \frac{\varkappa^{1/2}}{6} \left(\frac{d\chi}{d\beta} \right)^2 + \varkappa^{-1/2} \right) - \varkappa^{3/2} V e^{g\chi} = 0, \quad (14)$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa} g\beta}, \quad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} e^{-g\chi}, \quad (15)$$

to get

$$\left(\frac{d\tilde{\sigma}}{d\gamma} \right)^2 + \ell(s\nu - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{d\gamma}{d\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-s\nu + \tilde{\sigma}^2)}}, \quad (16)$$

which is of the standard inverse hyperbolic / trigonometric form **except for $(-, -)$** .



Integration of the separated minisuperspace WDW equation

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \mathfrak{s} v \frac{12\varkappa^2 |V|}{\hbar^2}, \quad (17)$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{k_\beta}{g}, \quad \sigma^2 := \frac{8\varkappa^3 |V| e^{g\chi}}{\hbar^2 g^2}, \quad (18)$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma \psi'(\sigma) + \ell(\nu^2 - \mathfrak{s} v \sigma^2) \psi(\sigma) = 0, \quad (19)$$

which is of the standard Besselian form.



Self-adjointness of unbounded operators

General theory

- Even in quantum geometrodynamics, the self-adjointness of the energy density is desirable, which is typically an unbounded operator in the Hilbert space \mathbf{F} endowed with the Schrödinger inner product $(\cdot, \cdot)_S$.
- Mathematically, an *unbounded* operator H is characterised not only by its action on a vector, but also by its domain $\text{Dom}(H) \subsetneq \mathbf{F}^{24}$.
- In addition to the *symmetry* $(H^\dagger \phi_1, \phi_2) \equiv (\phi_1, H\phi_2)$, the self-adjointness of an unbounded operator also requires $\text{Dom}(H^\dagger) = \text{Dom}(H)$.
- For quantum systems not bounded below, not only the self-adjointness, but also the symmetry of the energy density is not guaranteed automatically.
- Even when one could find a $\text{Dom}_0(H)$ such that H is symmetric, one would still be left with $\text{Dom}_0(H^\dagger) \supsetneq \text{Dom}_0(H)$ in general.
- Sloppily speaking, the process of extending $\text{Dom}(H)$ such that $\text{Dom}(H^\dagger) = \text{Dom}(H)$ is called **self-adjoint extension**²⁵; if the extension is unique, the operator is called *essentially self-adjoint*.

²⁴B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267 , ch. 9.

²⁵D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) , G. Bonneau *et al.*, *Am. J. Phys* **69**, 322–331 (Mar. 28, 2001), V. S. Araujo *et al.*, *Am. J. Phys* **72**, 203–213 (Feb. 2004), A. M. Essin, D. J. Griffiths, *Am. J. Phys* **74**, 109–117 (Feb. 2006).

