Integrable Liouville Cosmological Models

The self-adjointness of Hamiltonian and the Semi-classical Wave Functions

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Outline

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Introduction

The Friedmann–Lemaître model with quintessence and phantom Liouville fields

- Dynamical Dark Energy has been modelled by quintessence[1] and phantom[2] matter, which can be realised by minimally-coupled real scalar fields with $\ell = \pm 1$[3]

$$\mathcal{S}_{\mathsf{L}} = \int \mathbb{d}^4 x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \tag{1}$$

- The Liouville field[4] $\mathcal{V}(\phi) = V \mathbb{e}^{\lambda\phi}$ is of interest, where $\lambda, V \in \mathbb{R}$.
- Assume a flat Robertson–Walker metric $g_{\mu\nu} \, \mathbb{d}x^\mu \, \mathbb{d}x^\nu = -N^2(t) \, \mathbb{d}t^2 + \varkappa \mathbb{e}^{2\alpha(t)} \, \mathbb{d}\Omega^2_{3\mathsf{F}}$, where $\varkappa := 8\pi G$, $\mathbb{d}\Omega^2_{3\mathsf{F}} := \mathbb{d}\chi^2 + \chi^2 \left(\mathbb{d}\theta^2 + \sin^2\theta \, \mathbb{d}\varphi^2\right)$, and N lapse function.
- The total action reads $\mathcal{S} := \mathcal{S}_{\mathsf{EH}} + \mathcal{S}_{\mathsf{GHY}} + \mathcal{S}_{\mathsf{L}} = \int \mathbb{d}\Omega^2_{3\mathsf{F}} \int \mathbb{d}t \, L$, where

$$L := \varkappa^{3/2} N \mathbb{e}^{3\alpha} \left(-\frac{3}{\varkappa} \frac{\dot{\alpha}^2}{N^2} + \ell \frac{\dot{\phi}^2}{2N^2} - V \mathbb{e}^{\lambda\phi} \right). \tag{2}$$

[1] R. R. Caldwell *et al.*, *Phys. Rev. Lett.* **80**, 1582–1585 (Feb. 1998).
[2] R. R. Caldwell, *Phys. Lett. B* **545**, 23–29 (Oct. 2002).
[3] The signature of metric is mostly positive.
[4] Y. Nakayama, *Int. J. Mod. Phys. A* **19**, 2771–2930 (July 2004).

Introduction
Highlights

Integrability

Implicit trajectories can be obtained explicitly; the minisuperspace Wheeler–DeWitt equation can be integrated exactly.

Self-adjointness

The Hamiltonian is not even naturally symmetric; imposing self-adjointness leads to non-trivial physical results.

Semi-classical wave functions

Semi-classical wave functions can be constructed by the principle of constructive interference, the JWKB approximation and numerical methods.

Outline

Decoupling the variables

Via orthogonal transformation

- By rescaling $\overline{N} := N\mathrm{e}^{-3\alpha}$ and the orthogonal transformation

$$\begin{pmatrix} \alpha(t) \\ \phi(t) \end{pmatrix} = \frac{\jmath}{g} \begin{pmatrix} \lambda & -\ell\varkappa \\ -6 & \lambda \end{pmatrix} \begin{pmatrix} \jmath_\beta \cdot \beta(t) \\ \jmath_\chi \cdot \chi(t) \end{pmatrix} \qquad \text{where } \jmath_\beta, \jmath_\chi = \pm 1, \tag{3}$$

the effective Lagrangian eq. (2) can be decoupled ($\jmath_\beta = \jmath_\chi = +1$ from now on)

$$L = \varkappa^{3/2}\overline{N}\left(-\jmath\frac{3}{\varkappa}\frac{\dot{\beta}^2}{\overline{N}^2} + \ell\jmath\frac{\dot{\chi}^2}{2\overline{N}^2} - V\mathrm{e}^{g\chi}\right), \tag{4}$$

where $\Delta := \lambda^2 - 6\ell\varkappa$, $\jmath := \operatorname{sgn}\Delta$ and $g := \jmath\sqrt{|\Delta|} \equiv \jmath\sqrt{\jmath\Delta}$.

- The Euler–Lagrange equations with respect to $\overline{N}$, β and χ will be called the modified first, second Friedmann equations and the Klein–Gordon equation, respectively.
- Since $\beta(t)$ is cyclic, its conjugate momentum p_β is conserved[5], and the modified second Friedmann equation can be readily integrated.

[5] The same first integral has been found in C. Lan, PhD thesis, Saint Petersburg State University, 2016, https://search.rsl.ru/ru/record/01006663434, A. A. Andrianov *et al.*, *Theor. Math. Phys.* **184**, 1224–1233 (Sept. 2015), in canonical formalism.

Integration and the implicit trajectories

- For $p_\beta \neq 0$, fixing the *implicit gauge* $\overline{N} = -6\jmath\sqrt{\varkappa}\dot{\beta}/p_\beta$, the modified first Friedmann equation can be integrated, yielding the implicit trajectory

$$\mathrm{e}^{g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|} s^2\left(\sqrt{\frac{3}{2\varkappa}}\, g(\beta - \beta_0)\right), \tag{5}$$

in which $\upsilon := \operatorname{sgn} V$, $(\operatorname{sgn}, \operatorname{sgn})$ means $(\ell, \jmath\upsilon)$, and

$$\begin{aligned} {}_{(+,+)}s(\gamma) &:= \operatorname{sech}(\gamma), & {}_{(+,-)}s(\gamma) &:= \operatorname{csch}(\gamma), \\ {}_{(-,+)}s(\gamma) &:= \sec(\gamma), & {}_{(-,-)}s(\gamma) &:= \mathrm{i}\csc(\gamma). \end{aligned} \tag{6}$$

  - The integral for the phantom model with $(-,+)$ is periodic for β with $T_\beta = \sqrt{\frac{2\varkappa}{3}}\frac{2\pi}{g}$.
  - The integral for the phantom model with $(-,-)$ is not real.
  - The trajectories can be parametrised by β, inspiring recognising β as a 'time variable'.

- For $p_\beta = 0$, integrating the modified second Friedmann equation yields $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell\lambda\alpha/\varkappa$[6], which is the well-known power-law special solution[7].

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[6] It only works for $(+,-)$ or $(-,+)$ if one checks consistency with the modified first Friedmann equation.

[7] For instance, A. R. Liddle, D. H. Lyth, *Cosmological Inflation and Large-Scale Structure*, (Cambridge University, 2000) , ch. 3.

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Dirac quantisation

- The primary Hamiltonian and the Hamiltonian constraint[8] reads

$$H^{\mathsf{p}} = \overline{N} H_{\perp} + v^{\overline{N}} p_{\overline{N}}, \tag{7}$$

$$H_{\perp} := -\jmath \frac{p_\beta^2}{12\varkappa^{1/2}} + \ell \jmath \frac{p_\chi^2}{2\varkappa^{3/2}} + \varkappa^{3/2} V \mathrm{e}^{g\chi}. \tag{8}$$

- Applying the Dirac quantisation rules with the Laplace–Beltrami operator ordering[9], one gets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(\jmath \frac{\hbar^2}{12\varkappa^{1/2}} \partial_\beta^2 - \ell \jmath \frac{\hbar^2}{2\varkappa^{3/2}} \partial_\chi^2 + \varkappa^{3/2} V \mathrm{e}^{g\chi} \right) \Psi. \tag{9}$$

- Equation (9) is Klein–Gordon-like, hyperbolic for $\ell = +1$ and *elliptic* for $\ell = -1$.

[8] D. M. Gitman, I. V. Tyutin, *Quantization of Fields with Constraints*, (Springer, 1990) , H. J. Rothe, K. D. Rothe, *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*, (World Scientific, Apr. 2010) .

[9] C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012) , ch. 8.

Integration of the Wheeler–DeWitt equation

The quantum-mechanical analogy

- Fourier transforming β to $k_\beta \in \mathbb{R}$ yields the time-independent Schrödinger equation

$$\ell \frac{\hbar^2 \tilde{g}^2 \nu^2}{8M_{\mathsf{P}}} \psi(x) = \widehat{H}_{\mathsf{eff}} \psi(x) := \left(-\frac{\hbar^2}{2M_{\mathsf{P}}} \partial_x^2 + \widehat{V}_{\mathsf{eff}}(x) \right) \psi(x), \quad V_{\mathsf{eff}} = \ell s v \widetilde{V} \mathrm{e}^{\tilde{g}x}, \quad (10)$$

in which $M_{\mathsf{P}} := \sqrt{\frac{\hbar}{\varkappa}}$, $x := \sqrt{\hbar\varkappa} \cdot \chi$, $\widetilde{V} := M_{\mathsf{P}}^{-3} |V| \geq 0$, and $\nu = \sqrt{\frac{2\varkappa}{3}} \frac{|k_\beta|}{g} \geq 0$.

  - V_{eff} is a special case of the Morse potential[10].

- For the quintessence model $(+,+)$: V_{eff} is bounded below; generalised eigenfunctions are the Besselian $\mathrm{K}_{\mathrm{i}\nu}(\mathrm{e}^{\widetilde{x}})$'s, $\mathrm{e}^{2\widetilde{x}} = 8M_{\mathsf{P}} \widetilde{V} / \hbar^2 \tilde{g}^2 \cdot \mathrm{e}^{\tilde{g}x}$.
- For the quintessence model $(+,-)$: V_{eff} is bounded above; generalised eigenfunctions for positive spectrum (scattering states) are the Besselian $\mathrm{F}_{\mathrm{i}\nu}(\mathrm{e}^{\widetilde{x}})$'s and $\mathrm{G}_{\mathrm{i}\nu}(\mathrm{e}^{\widetilde{x}})$'s[11].
- For the phantom model $(-,+)$: V_{eff} is bounded above; eigenfunctions for negative eigenvalues (bound states) are the Besselian $\mathrm{J}_\nu(\mathrm{e}^{\widetilde{x}})$'s.

[10] P. M. Morse, *Phys. Rev.* **34**, 57–64 (July 1929), D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) .

[11] These are linear combinations of $\mathrm{J}_{\pm\mathrm{i}\nu}(\mathrm{e}^{\widetilde{x}})$; see T. M. Dunster, *SIAM J. Math. Anal.* **21**, 995–1018 (July 1990).

The self-adjointness of unbounded operators

The general theory

- Even in quantum geometrodynamics, the self-adjointness of the matter Hamiltonian is desirable, which is typically an unbounded operator in the Hilbert space $\mathbf{F}$ endowed with the Schrödinger inner product $(\cdot,\cdot)_S$.
- Mathematically, an *unbounded* operator H is characterised not only by its action on a vector, but also by its domain $\mathrm{Dom}(H) \subsetneq \mathbf{F}$[12].
- In addition to the *symmetricity* $(\phi_1, H^\dagger\phi_2) \equiv (H\phi_1, \phi_2)$, the self-adjointness of an unbounded operator also requires $\mathrm{Dom}(H^\dagger) = \mathrm{Dom}(H)$.
- For quantum systems not bounded below, not only the self-adjointness, but also the symmetricity of the Hamiltonian is not guaranteed automatically.
- Even when one could find a $\mathrm{Dom}_0(H)$ such that H is symmetric, one would still be left with $\mathrm{Dom}_0(H^\dagger) \supseteq \mathrm{Dom}_0(H)$ in general.
- Sloppily speaking, the process of extending $\mathrm{Dom}(H)$ such that $\mathrm{Dom}(H^\dagger) = \mathrm{Dom}(H)$ is called self-adjoint extension[13]; if the extension is unique, the operator is called *essentially self-adjoint*.

[12] B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267 , ch. 9.

[13] D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) , G. Bonneau *et al.*, *Am. J. Phys* **69**, 322–331 (Mar. 28, 2001), V. S. Araujo *et al.*, *Am. J. Phys* **72**, 203–213 (Feb. 2004), A. M. Essin, D. J. Griffiths, *Am. J. Phys* **74**, 109–117 (Feb. 2006).

The self-adjointness of the Hamiltonian

Results for the special Morse potential[15]

- For the quintessence model $(+,+)$: $\widehat{H}_{\text{eff}}$ is essentially self-adjoint on $C_c^\infty(\mathbb{R})$[14].
- For $(+,-)$ and $(-,+)$: $\widehat{H}_{\text{eff}}$ is not self-adjoint but admits a family of self-adjoint extensions parametrised by $a \in [0,2)$.
- For the quintessence model $(+,-)$, the generalised eigenfunctions are

$$ {}_{(+,-)}\psi_\nu \propto \mathrm{F}_{\mathrm{i}\nu} \cos\frac{\pi a_\geq}{2} + \mathrm{G}_{\mathrm{i}\nu} \sin\frac{\pi a_\geq}{2} \tag{11} $$

  with asymptotic behaviour ${}_{(+,-)}\psi_\nu(\mathrm{e}^{\widetilde{x}}) \sim \mathrm{e}^{-\widetilde{x}/2} \cos\left(\mathrm{e}^{\widetilde{x}} - \frac{\pi a}{2} - \frac{\pi}{4}\right)$ as $x \to +\infty$.
- For the phantom model $(-,+)$, the eigenfunctions are

$$ {}_{(-,+)}\psi_\nu \propto \mathrm{J}_\nu, \qquad \nu = 2n + a_<,\ n \in \mathbb{Z}_\geq, \tag{12} $$

  and the corresponding full wave-functions are Bloch-periodic ${}_{(-,+)}\Psi_n(\beta,\chi) = \mathrm{e}^{\mathrm{i}\pi a_<}\, {}_{(-,+)}\Psi_n(\beta + T_\beta, \chi)$.
  - Equation (12) can also be shown by noting $\left(\mathrm{J}_\mu, \widehat{H}_{\text{eff}}\, \mathrm{J}_\nu\right)_{\mathsf{S}} - \left(\widehat{H}_{\text{eff}}\, \mathrm{J}_\mu, \mathrm{J}_\nu\right)_{\mathsf{S}} \propto \sin\frac{\pi(\mu-\nu)}{2}$ and then imposing symmetricity.
  - Imposing (anti-)periodic boundary condition would fix $a_< = 0\ (1)$.

[14] See also B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267 , sec. 9.9.
[15] D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) , sec. 8.5.

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The self-adjointness of the Hamiltonian

Discussions

- The self-adjointness of $\widehat{H}_{\text{eff}}$ is non-trivial and not unique for $(+,-)$ and $(-,+)$.
- For the phantom model $(-,+)$, it discretises the spectrum.
- The self-adjointness has been discussed in quantum cosmology in e.g.[16] (in the operator ordering scenario) and more recently in[17].
- The self-adjoint extension is also non-trivial for the x^{-2} potential[18], which is of cosmological relevance as well[19].

[16] C. R. Almeida *et al.*, *Gravitation Cosmol.* **21**, 191–199 (July 2015).
[17] S. Gryb, K. P. Y. Thébault, arXiv: 1801.05789 (gr-qc) (Jan. 17, 2018).
[18] A. M. Essin, D. J. Griffiths, *Am. J. Phys* **74**, 109–117 (Feb. 2006).
[19] M. Bouhmadi-López *et al.*, *Phys. Rev. D* **79**, arXiv: 0905.2421 (gr-qc) (June 2009).

Outline

Inner product for wave functions

Schrödinger product

- To make sense of amplitude and wave packet, an inner product is necessary
- In terms of a Klein–Gordon equation, call β the "temporal" variable, and χ the "spacial" variable.
- A common starting point is the *Schrödinger product*[20]

$$(\Psi_1, \Psi_2)_{\mathrm{S}} := \int \mathrm{d}\chi\, \Psi_1^*(\beta, \chi)\Psi_2(\beta, \chi); \tag{13}$$

- $(\Psi, \Psi)_{\mathrm{S}}$ is positive-definite, and the integrand $\rho_{\mathrm{S}}(\beta, \chi)$ is non-negative
- The corresponding Schrödinger current is real but does not satisfy continuity equation $\dot{\rho}_{\mathrm{S}} + \nabla \cdot \vec{j}_{\mathrm{S}} = 0$, because eq. (9) is KG-like.
- $\mathrm{K}_{\mathbb{i}\nu}$[21] for $(+,+)$, $\mathrm{F}_{\mathbb{i}\nu}$ and $\mathrm{G}_{\mathbb{i}\nu}$ for $(+,-)$ can be proved to be orthogonal and complete among themselves, as well as can be normalised.
    - $\mathrm{J}_{\mathbb{i}\nu}$'s for $(+,-)$ are not orthogonal

[20] C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012) , ch. 5.

[21] S. B. Yakubovich, *Opuscula Math.* **26**, 161–172 (2006), A. Passian *et al.*, *J. Math. Anal. Appl.* **360**, 380–390 (Dec. 2009), R. Szmytkowski, S. Bielski, *J. Math. Anal. Appl.* **365**, 195–197 (May 2010).

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Further inner products for wave functions

Klein–Gordon and Mostafazadeh product

- Since eq. (9) is KG-like, another popular choice is the KG product

$$(\Psi_1, \Psi_2)^g_{\mathsf{KG}} := \mathbb{i} g\Big\{\big(\Psi_1, \partial_\beta \Psi_2\big)_{\mathsf{S}} - \big(\partial_\beta \Psi_1, \Psi_2\big)_{\mathsf{S}}\Big\}, \qquad g > 0. \tag{14}$$

  - $(\Psi, \Psi)^g_{\mathsf{KG}}$ is real but not positive-definite, so does the integrand ρ_{KG};
  - The corresponding $\vec{J}_{\mathsf{KG}}$ is conserved $\dot{\rho}_{\mathsf{KG}} + \nabla \cdot \vec{J}_{\mathsf{KG}} = 0$ and real.

- Mostafazadeh[22] found a product *for Hermitian $\mathbb{D}$ with positive spectrum*:

$$(\Psi_1, \Psi_2)^\kappa_{\mathsf{M}} := \kappa\Big\{\big(\Psi_1, \mathbb{D}^{+1/2}\Psi_2\big)_{\mathsf{S}} + \big(\partial_\beta \Psi_1, \mathbb{D}^{-1/2}\partial_\beta \Psi_2\big)_{\mathsf{S}}\Big\}, \qquad \kappa > 0. \tag{15}$$

  - $(\Psi, \Psi)^\kappa_{\mathsf{M}}$ is positive-definite, but the integrand ρ^κ_{M} is complex
  - The corresponding $\vec{J}^\kappa_{\mathsf{M}}$ is conserved $\dot{\rho}^\kappa_{\mathsf{M}} + \nabla \cdot \vec{J}^\kappa_{\mathsf{M}} = 0$ but also complex.

[22] A. Mostafazadeh, *Class. Quantum Grav.* **20**, 155–171 (Dec. 2002).

Mostafazadeh inner product and the corresponding density

- Real power of $\mathbb{D}$ is defined by spectral decomposition $\mathbb{D}^\gamma := \sum_\nu \nu^{2\gamma} \mathbf{P}_\nu$, $\mathbf{P}_\nu \Psi := \Psi_\nu (\Psi_\nu, \Psi)_{\mathsf{S}}$.
- It can be shown[23] that *the density*

$$\varrho_{\mathsf{M}}^\kappa := \kappa \left\{ \left| \mathbb{D}^{+1/4} \Psi \right|^2 + \left| \mathbb{D}^{-1/4} \partial_\beta \Psi \right|^2 \right\} \tag{16}$$

  - is equivalent to ρ_{M}^κ up to a boundary term

$$\int \mathrm{d}\chi \, \varrho_{\mathsf{M}}^\kappa = \int \mathrm{d}\chi \, \rho_{\mathsf{M}}^\kappa \equiv (\Psi_1, \Psi_2)_{\mathsf{M}}^\kappa ; \tag{17}$$

  - is non-negative.
  - The corresponding current $\vec{\mathcal{J}}_{\mathsf{M}}^\kappa$ is real but not conserved[24].

[23] A. Mostafazadeh, F. Zamani, *Ann. Phys.* **321**, 2183–2209 (Sept. 2006).
[24] B. Rosenstein, L. P. Horwitz, *J. Phys. A: Math. Gen.* **18**, 2115–2121 (Aug. 1985).

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Wave packets of Gaussian amplitude for continuous spectrum

Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu; \overline{\nu}, \sigma) := \left(\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(\nu - \overline{\nu})^2}{2\sigma^2} \right) \right)^{1/2} \tag{18}$$

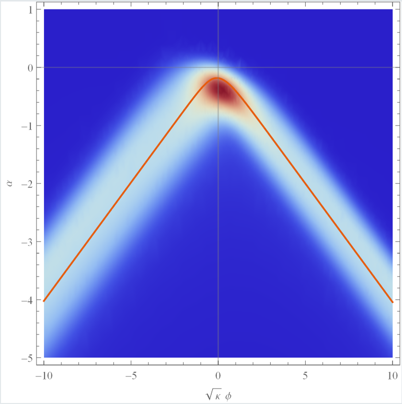
- In[25], $A\left(\nu; \overline{\nu}, \sigma/\sqrt{2}\right)$ was chosen.

[25] M. P. Dąbrowski *et al.*, *Phys. Rev. D* **74**, arXiv: `hep-th/0605229 (hep-th)` (Aug. 2006).

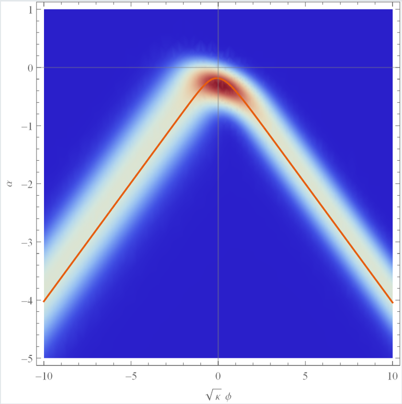
Wave packets of Gaussian amplitude for quintessence model $(+,+)$

$K_{i\nu}$, **with** $\lambda = \varkappa^{1/2}/2$, $V = -\varkappa^{-2}$, $\bar{k}_\beta = -2$ **and** $\sigma_\beta = 5/4$

Schrödinger



Mostafazadeh



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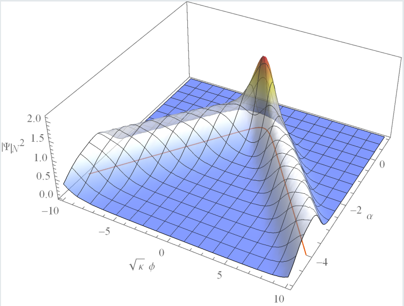

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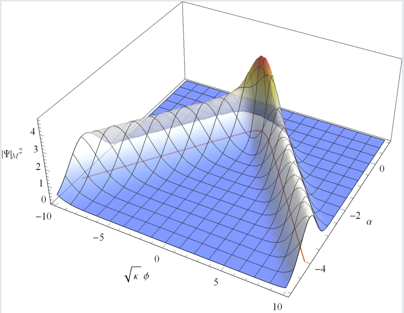
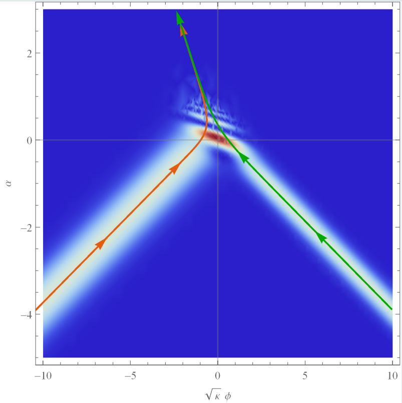


Wave packets of Gaussian amplitude for quintessence model $(+,-)$

$\mathrm{F}_{\mathring{\imath}\nu}$, **with** $\lambda = 4\varkappa^{1/2}/5$, $V = +\varkappa^{-2}$, $\overline{k}_\beta = -7/2$ **and** $\sigma_\beta = 7/5$

Schrödinger



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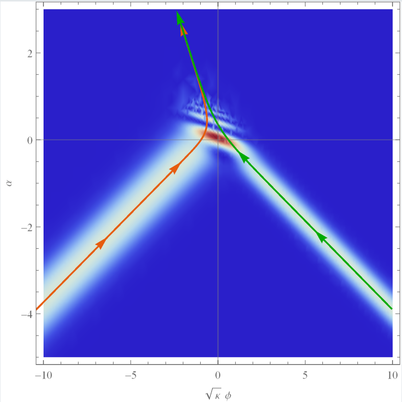

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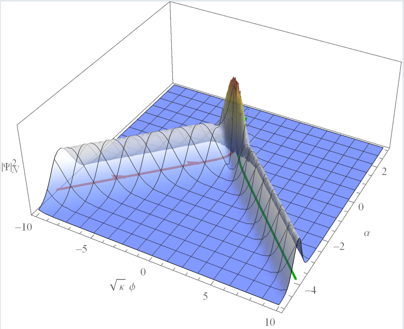
Schrödinger



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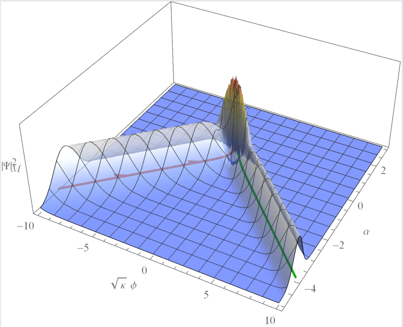
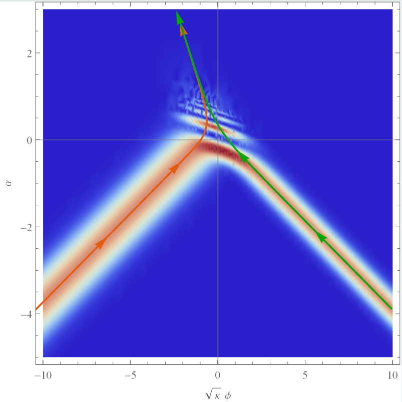


Wave packets of Gaussian amplitude for quintessence model $(+,-)$

$\mathrm{G}_{\mathbb{i}\nu}$, **with** $\lambda = 4\varkappa^{1/2}/5$, $V = +\varkappa^{-2}$, $\bar{k}_\beta = -7/2$ **and** $\sigma_\beta = 7/5$

Schrödinger



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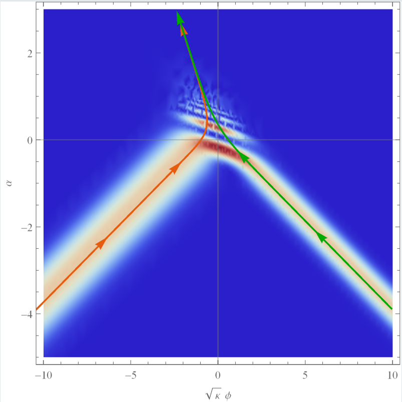

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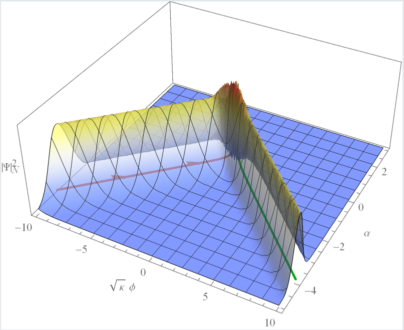
$G_{\mathbb{i}\nu}$, **with** $\lambda = 4\varkappa^{1/2}/5$, $V = +\varkappa^{-2}$, $\overline{k}_\beta = -7/2$ **and** $\sigma_\beta = 7/5$

Schrödinger



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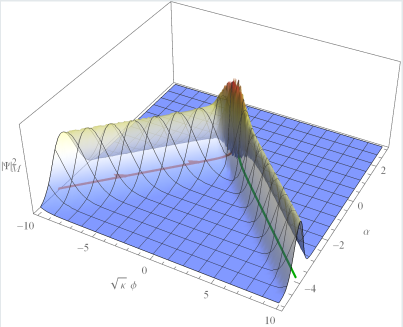

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Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model $(-,+)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\overline{n}) := \left(\mathrm{e}^{-\overline{n}}\frac{\overline{n}^n}{n!}\right)^{1/2} \tag{19}$$

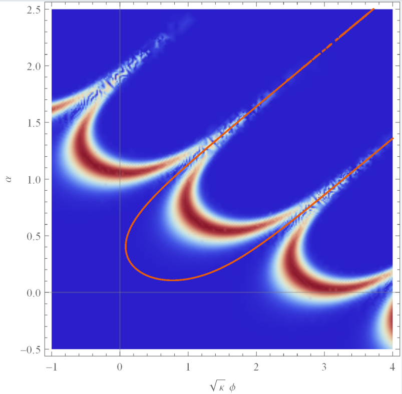
- In[26], $A_n\left(\overline{n}/\sqrt{2}\right)$ was chosen.

[26] C. Kiefer, *Nucl. Phys. B* **341**, 273–293 (Sept. 1990).

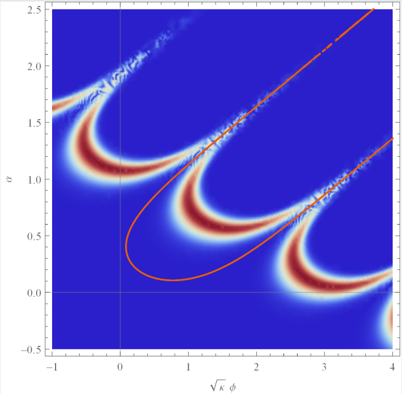
Wave packets of Poissonian amplitude for phantom model

J_{2n+1}, with $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$ and $\overline{k}_\beta = 8$

Schrödinger



Mostafazadeh



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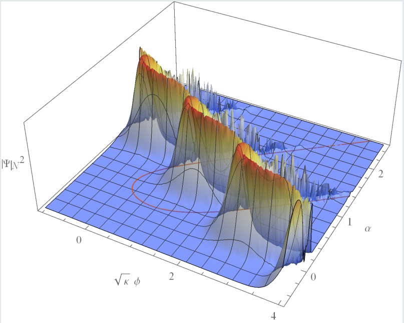
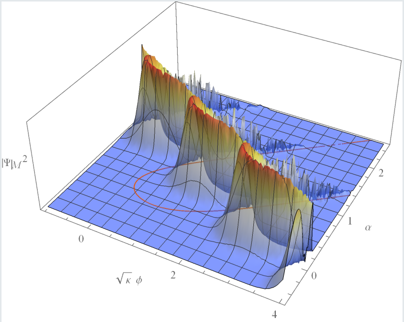
Wave packets of Poissonian amplitude for phantom model

J_{2n+1}, with $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$ and $\bar{k}_\beta = 8$

Schrödinger



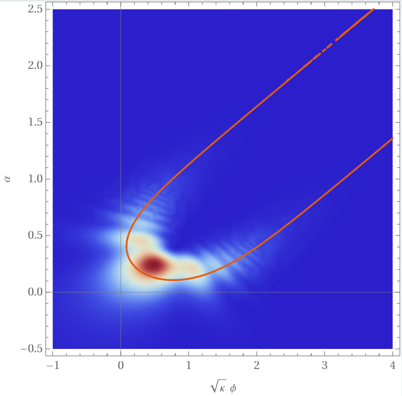
Mostafazadeh



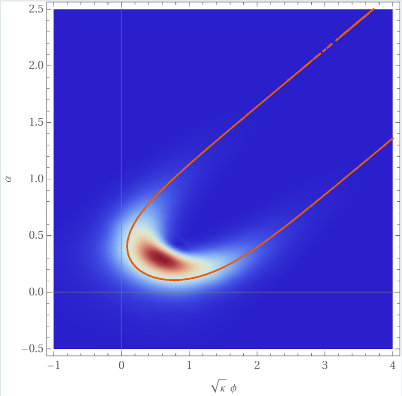
Wave packets of Gaussian amplitude for phantom model

J_ν, **with** $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$, $\bar{k}_\beta = 8$ **and** $\sigma_\beta = 11/2$

Schrödinger



Mostafazadeh



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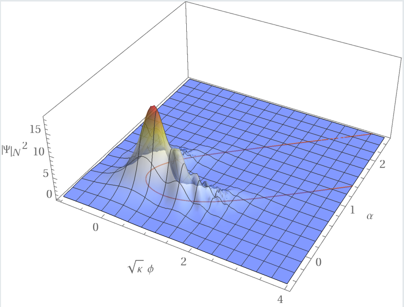
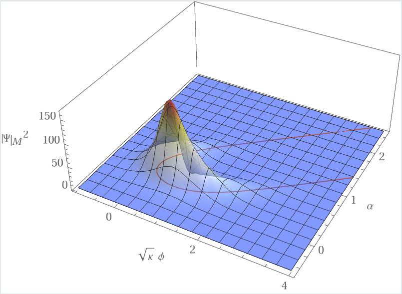
Wave packets of Gaussian amplitude for phantom model

J_ν, **with** $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$, $\overline{k}_\beta = 8$ **and** $\sigma_\beta = 11/2$

Schrödinger



Mostafazadeh



Outline

Highlights

- An integral of motion was found for Liouville cosmological models.
- Implicit trajectories in minisuperspace were obtained.
- The levels of phantom Liouville model were found to be discrete due to the Hermiticity requirement of observables.
- The semi-classical quantum mode functions and wave packets was compared directly with the classical trajectory.

Issues

- In $(+,-)$ and $(-,+)$, wave packets contain multiple branches; however, the classical universe runs only on one trajectory.
- Quantum-corrected $\overline{k}_\beta$ is to be understood.
- Wave packets with Gaussian profile are to be constructed, instead of inserting Gaussian / Poissonian amplitude artificially.
- A normalising κ for $(\cdot,\cdot)_{\mathsf{M}}^{\kappa}$ is to be evaluated, otherwise a quantitative comparison of $(\cdot,\cdot)_{\mathsf{S}}$ and $(\cdot,\cdot)_{\mathsf{M}}^{\kappa}$ is not possible.

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Outlook

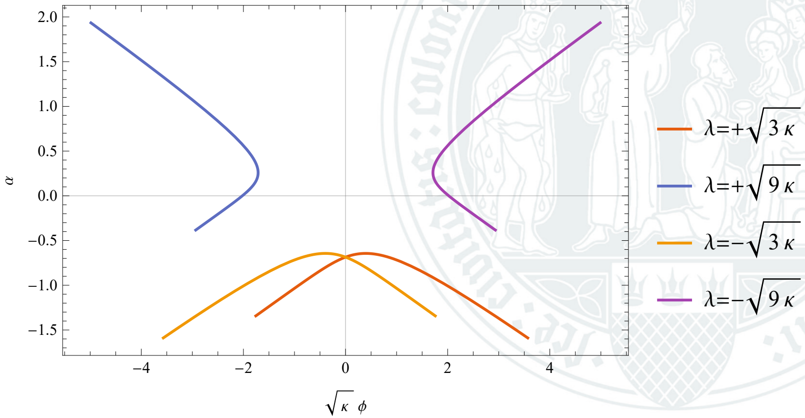
- Beyond isotropy: generalise to Bianchi models
  - Bianchi Type-I: a natural extension, under investigation
- Beyond homogeneity: cosmological perturbation, under investigation
- Beyond single field
  - Two exponential potentials: $|V_1| = |V_2|$ and special λ_i
  - Multiple Liouville fields: mixing kinetic terms needed[27]
- Beyond classic matter
  - PT-symmetric Liouville field[28]: may cross the phantom divide $w = -1$.

[27] A. A. Andrianov *et al.*, *Theor. Math. Phys.* **184**, 1224–1233 (Sept. 2015).

[28] A. A. Andrianov *et al.*, *Int. J. Mod. Phys. D* **19**, 97–111 (Jan. 2010), A. A. Andrianov *et al.*, in *Springer Proceedings in Physics* (Springer International Publishing, 2016), pp. 29–44.

Trajectories for quintessence model $(+,+)$

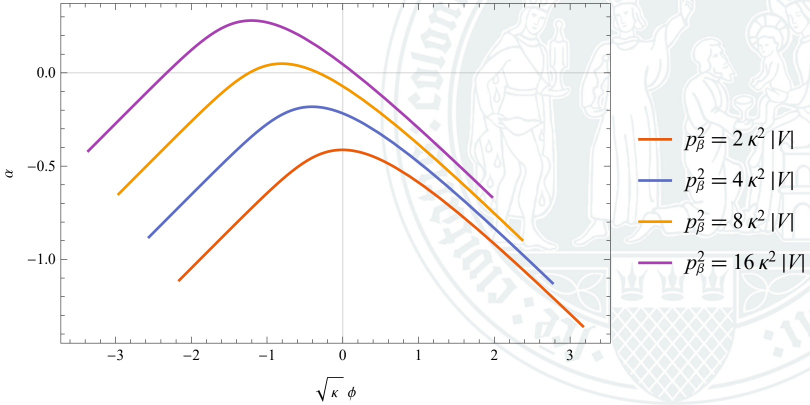
sech, with $\beta_0 = 0$, $|V| = \varkappa^{-2}$ and $p_\beta^2 = \varkappa^2\sqrt{|V|}$; varying λ



- has two asymptotes $\chi \propto \pm\beta$

Trajectories for quintessence model $(+,+)$

sech, **with** $\beta_0 = 0$, $|V| = \varkappa^{-2}$ **and** $\lambda^2 = 3\varkappa$; **varying** p_β



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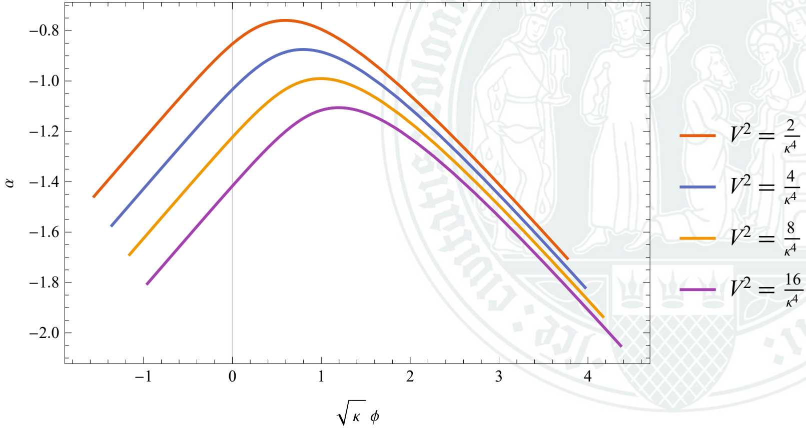
Trajectories for quintessence model $(+,+)$

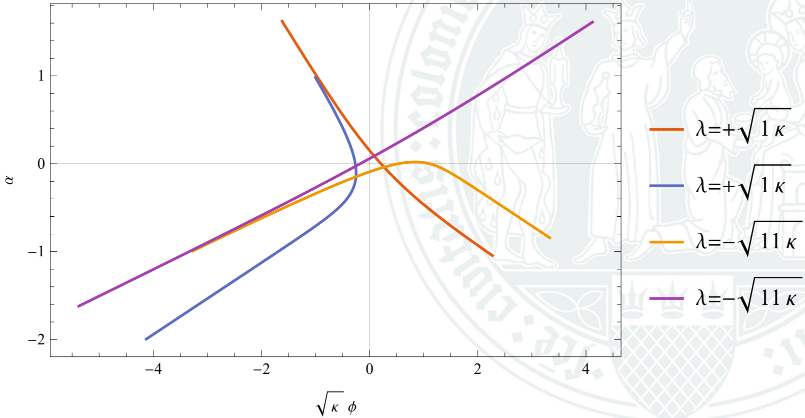
sech, **with** $\beta_0 = 0$, $\lambda^2 = 3\varkappa$ **and** $p_\beta^2 = \varkappa^2\sqrt{|V|}$; **varying** V



- has two asymptotes $\chi \propto \pm\beta$

Trajectories for quintessence model $(+,-)$

csch, with $\beta_0 = 0$, $|V| = \varkappa^{-2}$ and $p_\beta^2 = \varkappa^2\sqrt{|V|}$; varying λ



- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$

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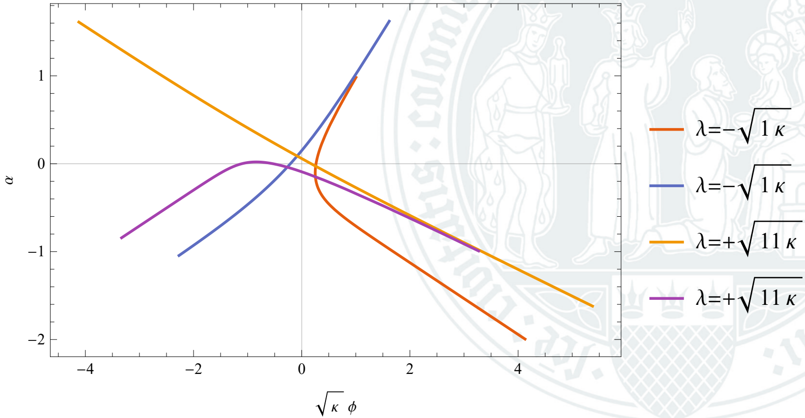
Trajectories for quintessence model $(+,-)$: csch

csch, **with** $\beta_0 = 0$, $|V| = \varkappa^{-2}$ **and** $p_\beta^2 = \varkappa^2\sqrt{|V|}$; **varying** λ



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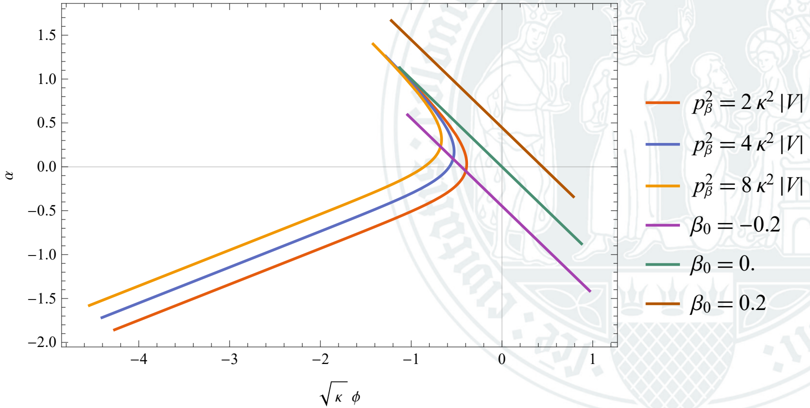

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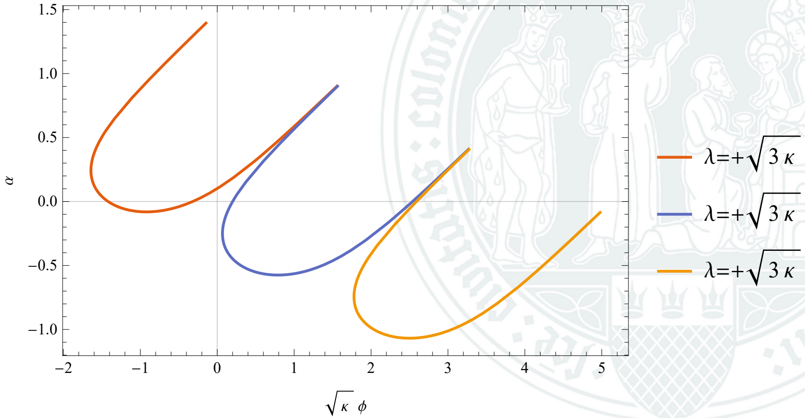
Trajectories for phantom model $(-,+)$

csc, **with** $V = \varkappa^{-2}$ **and** $p_\beta^2 = 3\varkappa^2\sqrt{|V|}$



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$

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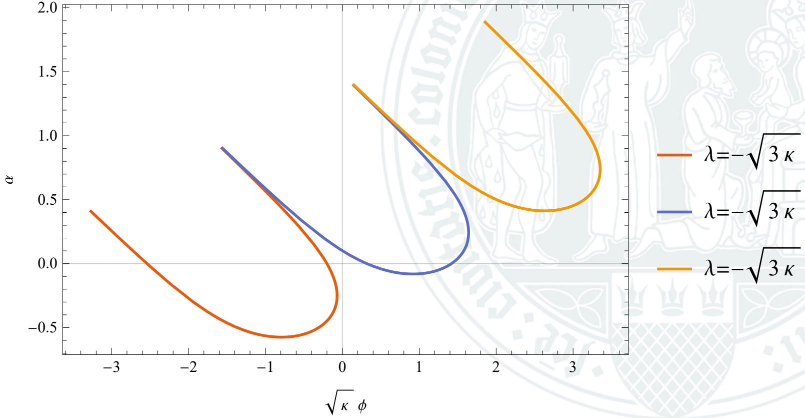
Trajectories for phantom model $(-,+)$

csc, **with** $V = \varkappa^{-2}$ **and** $p_\beta^2 = 3\varkappa^2\sqrt{|V|}$



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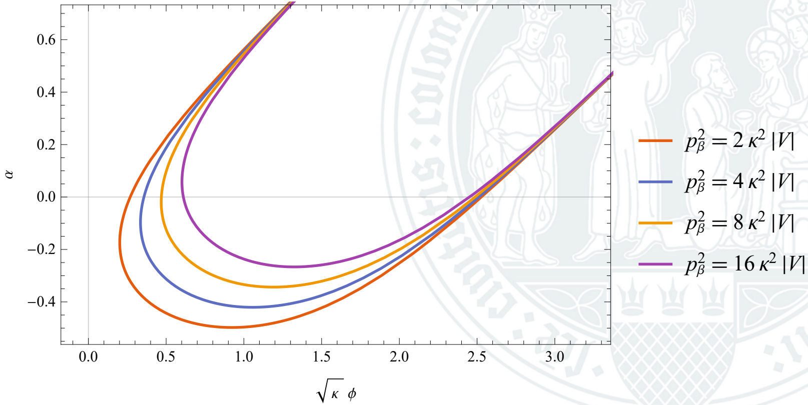

- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$

In order to integrate the equation under the implicit gauge

$$\jmath \frac{p_\beta^2}{12}\left(-\ell \frac{\varkappa^{1/2}}{6}\left(\frac{\mathrm{d}\chi}{\mathrm{d}\beta}\right)^2 + \varkappa^{-1/2}\right) - \varkappa^{3/2} V \mathrm{e}^{g\chi} = 0, \tag{20}$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa}} g\beta, \qquad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} \mathrm{e}^{-g\chi}, \tag{21}$$

to get

$$\left(\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\gamma}\right)^2 + \ell(\jmath v - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-\jmath v + \tilde{\sigma}^2)}}, \tag{22}$$

which is of the standard inverse hyperbolic / trigonometric form except for $(-,-)$.

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Integration of the separated minisuperspace WDW equation

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \qquad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \jmath v \frac{12\varkappa^2 |V|}{\hbar^2}, \tag{?? rev.}$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{k_\beta}{g}, \qquad \sigma^2 := \frac{8\varkappa^3 |V| \mathrm{e}^{g\chi}}{\hbar^2 g^2}, \tag{?? rev.}$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma \psi'(\sigma) + \ell (\nu^2 - \jmath v \sigma^2) \psi(\sigma) = 0, \tag{23}$$

which is of the standard Besselian form.

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