

Integrable Liouville Cosmological Models

The self-adjointness of Hamiltonian and the Semi-classical Wave Functions

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March 23, 2018



Outline

1. Introduction

2. The classical model and the implicit trajectories

Lagrangian formalism and the first integral

3. Dirac quantisation and the self-adjointness of Hamiltonian

Dirac quantisation and the quantum-mechanical analogy
The self-adjointness of Hamiltonian

4. The semi-classical wave packets

Inner product and wave packet

5. Conclusions



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Introduction

The Friedmann–Lemaître model with quintessence and phantom Liouville fields

- Dynamical Dark Energy has been modelled by quintessence¹ and phantom² matter, which can be realised by minimally-coupled real scalar fields with $\ell = \pm 1$ ³

$$\mathcal{S}_L = \int d^4x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \quad (1)$$

- The Liouville field⁴ $\mathcal{V}(\phi) = V e^{\lambda\phi}$ is of interest, where $\lambda, V \in \mathbb{R}$.
- Assume a flat Robertson–Walker metric $g_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + \varkappa e^{2\alpha(t)} d\Omega_{3F}^2$, where $\varkappa := 8\pi G$, $d\Omega_{3F}^2 := d\chi^2 + \chi^2(d\theta^2 + \sin^2 \theta d\varphi^2)$, and N lapse function.
- The total action reads $\mathcal{S} := \mathcal{S}_{EH} + \mathcal{S}_{GHY} + \mathcal{S}_L = \int d\Omega_{3F}^2 \int dt L$, where

$$L := \varkappa^{3/2} N e^{3\alpha} \left(-\frac{3}{\varkappa} \frac{\dot{\alpha}^2}{N^2} + \ell \frac{\dot{\phi}^2}{2N^2} - V e^{\lambda\phi} \right). \quad (2)$$

¹R. R. Caldwell *et al.*, *Phys. Rev. Lett.* **80**, 1582–1585 (Feb. 1998).

²R. R. Caldwell, *Phys. Lett. B* **545**, 23–29 (Oct. 2002).

³The signature of metric is mostly positive.

⁴Y. Nakayama, *Int. J. Mod. Phys. A* **19**, 2771–2930 (July 2004).



Introduction

Highlights

Integrability

Implicit trajectories can be obtained explicitly; the minisuperspace Wheeler–DeWitt equation can be integrated exactly.

Self-adjointness

The Hamiltonian is not even naturally symmetric; imposing self-adjointness leads to removal of degeneracy and discretisation of levels.

Semi-classical wave functions

Semi-classical wave functions can be constructed by the principle of constructive interference, the JWKB approximation and numerical methods.



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Decoupling the variables

Via orthogonal transformation

- By rescaling $\bar{N} := N e^{-3\alpha}$ and the orthogonal transformation

$$\begin{pmatrix} \alpha(t) \\ \phi(t) \end{pmatrix} = \frac{\varsigma}{g} \begin{pmatrix} \lambda & -\ell \varkappa \\ -6 & \lambda \end{pmatrix} \begin{pmatrix} \varsigma_\beta \cdot \beta(t) \\ \varsigma_\chi \cdot \chi(t) \end{pmatrix} \quad \text{where } \varsigma_\beta, \varsigma_\chi = \pm 1, \quad (3)$$

the effective Lagrangian eq. (2) can be decoupled ($\varsigma_\beta = \varsigma_\chi = +1$ from now on)

$$L = \varkappa^{3/2} \bar{N} \left(-\varsigma \frac{3}{\varkappa} \frac{\dot{\beta}^2}{\bar{N}^2} + \ell \varsigma \frac{\dot{\chi}^2}{2\bar{N}^2} - V e^{g\chi} \right), \quad (4)$$

where $\Delta := \lambda^2 - 6\ell\varkappa$, $\varsigma := \operatorname{sgn} \Delta$ and $g := \varsigma \sqrt{|\Delta|} \equiv \varsigma \sqrt{3\Delta}$.

- The Euler–Lagrange equations with respect to \bar{N} , β and χ will be called the modified first, second Friedmann equations and the Klein–Gordon equation, respectively.
- Since $\beta(t)$ is cyclic, its conjugate momentum p_β is conserved⁵, and the modified second Friedmann equation can be readily integrated.

⁵The same first integral has been found in C. Lan, PhD thesis, Saint Petersburg State University, 2016, <https://search.rsl.ru/ru/record/01006663434>, A. A. Andrianov et al., *Theor. Math. Phys.* **184**, 1224–1233 (Sept. 2015), in canonical formalism.



Integration and the implicit trajectories

- For $p_\beta \neq 0$, fixing the *implicit gauge* $\bar{N} = -6\sqrt{\nu}\dot{\beta}/p_\beta$, the modified first Friedmann equation can be integrated, yielding the **implicit** trajectory

$$e^{g\chi} = \frac{p_\beta^2}{12\nu^2|V|} s^2 \left(\sqrt{\frac{3}{2\nu}} g(\beta - \beta_0) \right), \quad (5)$$

in which $v := \operatorname{sgn} V$, $(\operatorname{sgn}, \operatorname{sgn})$ means (ℓ, sv) , and

$$\begin{aligned} s_{++}(\gamma) &:= \operatorname{sech}(\gamma), & s_{+-}(\gamma) &:= \operatorname{csch}(\gamma), \\ s_{-+}(\gamma) &:= \sec(\gamma), & s_{--}(\gamma) &:= i \csc(\gamma). \end{aligned} \quad (6)$$

- The integral for the phantom model with $(-, +)$ is periodic for β with $T_\beta = \sqrt{\frac{2\nu}{3}} \frac{2\pi}{g}$.
 - The integral for the phantom model with $(-, -)$ is not real.
 - The trajectories can be parametrised by β , inspiring recognising β as a ‘time variable’.
- For $p_\beta = 0$, integrating the modified second Friedmann equation yields $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell \lambda \alpha / \nu^6$, which is the well-known power-law special solution⁷.

⁶It only works for $(+, -)$ or $(-, +)$ if one checks consistency with the modified first Friedmann equation.

⁷For instance, A. R. Liddle, D. H. Lyth, *Cosmological Inflation and Large-Scale Structure*, (Cambridge University, 2000) , ch. 3.



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Dirac quantisation

- The primary Hamiltonian and the Hamiltonian constraint⁸ reads

$$H^p = \bar{N} H_{\perp} + v^{\bar{N}} p_{\bar{N}}, \quad (7)$$

$$H_{\perp} := -\beta \frac{p_{\beta}^2}{12\nu^{1/2}} + \ell \beta \frac{p_{\chi}^2}{2\nu^{3/2}} + \nu^{3/2} V e^{g\chi}. \quad (8)$$

- Applying the Dirac quantisation rules with the Laplace–Beltrami operator ordering⁹, one gets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(\beta \frac{\hbar^2}{12\nu^{1/2}} \partial_{\beta}^2 - \ell \beta \frac{\hbar^2}{2\nu^{3/2}} \partial_{\chi}^2 + \nu^{3/2} V e^{g\chi} \right) \Psi. \quad (9)$$

- Equation (9) is Klein–Gordon-like, hyperbolic for $\ell = +1$ and *elliptic* for $\ell = -1$.

⁸D. M. Gitman, I. V. Tyutin, *Quantization of Fields with Constraints*, (Springer, 1990), H. J. Rothe, K. D. Rothe, *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*, (World Scientific, Apr. 2010).

⁹C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012), ch. 8.



Integration of the Wheeler–DeWitt equation

The quantum-mechanical analogy

- Fourier transforming β to $k_\beta \in \mathbb{R}$ yields the time-independent Schrödinger equation

$$E\psi(x) := \ell \frac{\hbar^2 \tilde{g}^2 \nu^2}{8M_P} \psi = \widehat{H}_{\text{eff}} \psi := \left(-\frac{\hbar^2}{2M_P} \partial_x^2 + \widehat{V}_{\text{eff}}(x) \right) \psi, \quad V_{\text{eff}} := \ell \wp \widetilde{V} e^{\tilde{g}x}, \quad (10)$$

in which $M_P := \sqrt{\frac{\hbar}{\nu}}$, $x := \sqrt{\hbar}\wp \cdot \chi$, $\widetilde{V} := M_P^{-3}|V| \geq 0$, and $\nu = \sqrt{\frac{2\nu}{3}} \frac{|k_\beta|}{g} \geq 0$.

- V_{eff} is a special case of the Morse potential¹⁰.

- Let $e^{2\tilde{x}} = 8M_P \widetilde{V}/\hbar^2 \tilde{g}^2 \cdot e^{\tilde{g}x}$

	$E \geq 0$	$E < 0$
$V_{\text{eff}} > 0$	$(+, +); K_{\frac{1}{2}\nu}(e^{\tilde{x}})$	$(-, -); \text{none}$
$V_{\text{eff}} < 0$	$(+, -); F_{\frac{1}{2}\nu}(e^{\tilde{x}}), G_{\frac{1}{2}\nu}(e^{\tilde{x}})$	$(-, +); J_\nu(e^{\tilde{x}})$

¹⁰P. M. Morse, *Phys. Rev.* **34**, 57–64 (July 1929), D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012).



The self-adjointness of unbounded operators

The general theory

- Even in quantum geometrodynamics, the self-adjointness of the matter Hamiltonian is desirable, which is typically an unbounded operator in the Hilbert space \mathbf{F} endowed with the Schrödinger inner product $(\cdot, \cdot)_S$.
- Mathematically, an *unbounded* operator H is characterised not only by its action on a vector, but also by its domain $\text{Dom}(H) \subsetneq \mathbf{F}^{11}$.
- In addition to the *symmetry* $(H^\dagger \phi_1, \phi_2) \equiv (\phi_1, H\phi_2)$, the self-adjointness of an unbounded operator also requires $\text{Dom}(H^\dagger) = \text{Dom}(H)$.
- For quantum systems not bounded below, not only the self-adjointness, but also the symmetry of the Hamiltonian is not guaranteed automatically.
- Even when one could find a $\text{Dom}_0(H)$ such that H is symmetric, one would still be left with $\text{Dom}_0(H^\dagger) \supsetneq \text{Dom}_0(H)$ in general.
- Sloppily speaking, the process of extending $\text{Dom}(H)$ such that $\text{Dom}(H^\dagger) = \text{Dom}(H)$ is called **self-adjoint extension**¹²; if the extension is unique, the operator is called *essentially self-adjoint*.

¹¹B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267 , ch. 9.

¹²D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) , G. Bonneau *et al.*, *Am. J. Phys* **69**, 322–331 (Mar. 28, 2001), V. S. Araujo *et al.*, *Am. J. Phys* **72**, 203–213 (Feb. 2004), A. M. Essin, D. J. Griffiths, *Am. J. Phys* **74**, 109–117 (Feb. 2006).



The self-adjointness of the Hamiltonian

Results for the Liouville potential¹⁶

- For $V_{\text{eff}} > 0$: \widehat{H}_{eff} is essentially self-adjoint¹³.
 - For the quintessence model $(+, +)$, nothing special.
- For $V_{\text{eff}} < 0$: \widehat{H}_{eff} admits a family of self-adjoint extensions.
 - For the quintessence model $(+, -)$, the generalised eigenfunction is a linear combination of F and G, and the degeneracy is removed.
 - For the phantom model $(-, +)$, the eigenfunctions are discretised

$$\psi_{+, \nu} \propto J_\nu, \quad \nu = 2n + a_<, \quad n \in \mathbb{Z}_\geq, \quad (11)$$

and the corresponding full wave-functions are Bloch-periodic

$\Psi_{+, n}(\beta, \chi) = e^{i\pi a_<} \Psi_{-, n}(\beta + T_\beta, \chi)$. Imposing (anti-)periodic boundary condition would fix $a_< = 0$ (1).

- The self-adjointness has been discussed in quantum cosmology in e.g.¹⁴ (in the operator ordering scenario) and more recently in¹⁵.

¹³See also B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267, sec. 9.9.

¹⁴C. R. Almeida et al., *Gravitation Cosmol.* **21**, 191–199 (July 2015).

¹⁵S. Gryb, K. P. Y. Thébault, arXiv: 1801.05789 (gr-qc) (Jan. 17, 2018).

¹⁶D. M. Gitman et al., *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012), sec. 8.5.



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Inner products for wave functions

Schrödinger and Mostafazadeh inner product

- A common starting point is the Schrödinger product¹⁷

$$(\Psi_1, \Psi_2)_S := \int d\chi \Psi_1^*(\beta, \chi) \Psi_2(\beta, \chi); \quad (12)$$

- $(\Psi, \Psi)_S$ is positive-definite, and the integrand $\rho_S(\beta, \chi)$ is non-negative
- The corresponding Schrödinger current is real but does not satisfy continuity equation
- Mostafazadeh¹⁸ found an inner product for self-adjoint \mathbb{D} with positive spectrum:

$$(\Psi_1, \Psi_2)_M^\kappa := \kappa \left\{ (\Psi_1, \mathbb{D}^{+1/2} \Psi_2)_S + (\partial_\beta \Psi_1, \mathbb{D}^{-1/2} \partial_\beta \Psi_2)_S \right\}, \quad \kappa > 0. \quad (13)$$

- Real power of \mathbb{D} is defined by spectral decomposition
- It can be shown¹⁹ that the density

$$\varrho_M^\kappa := \kappa \left\{ |\mathbb{D}^{+1/4} \Psi|^2 + |\mathbb{D}^{-1/4} \partial_\beta \Psi|^2 \right\} \quad (14)$$

- is equivalent to the integrand ρ_M^κ up to a boundary term $\int d\chi \varrho_M^\kappa = \int d\chi \rho_M^\kappa \equiv (\Psi_1, \Psi_2)_M^\kappa$
- is non-negative
- The corresponding current \vec{J}_M^κ is real but not conserved²⁰.

¹⁷C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012), ch. 5.

¹⁸A. Mostafazadeh, *Class. Quantum Grav.* **20**, 155–171 (Dec. 2002).

¹⁹A. Mostafazadeh, F. Zamani, *Ann. Phys.* **321**, 2183–2209 (Sept. 2006).

²⁰B. Rosenstein, L. P. Horwitz, *J. Phys. A: Math. Gen.* **18**, 2115–2121 (Aug. 1985).



Wave packets of Gaussian amplitude for continuous spectrum

Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu; \bar{\nu}, \sigma) := \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\nu - \bar{\nu})^2}{2\sigma^2}\right) \right)^{1/2} \quad (15)$$

- In²¹, $A(\nu; \bar{\nu}, \sigma/\sqrt{2})$ was chosen.

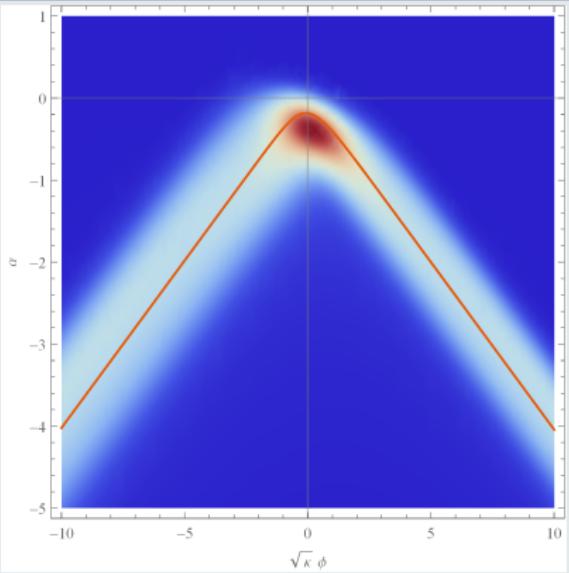
²¹M. P. Dąbrowski et al., *Phys. Rev. D* **74**, arXiv: hep-th/0605229 (hep-th) (Aug. 2006).



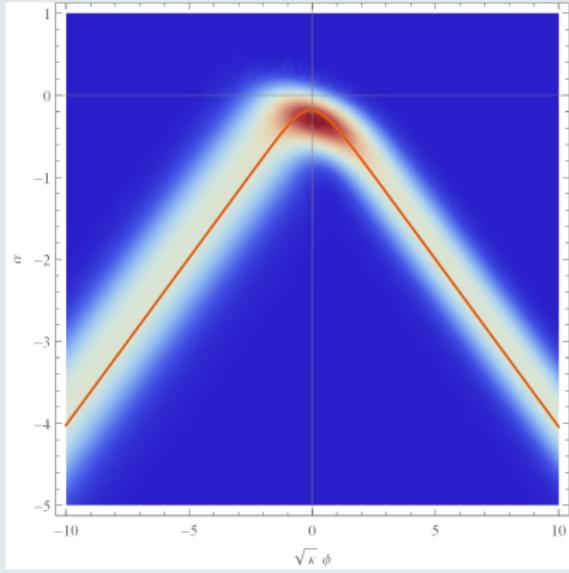
Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\ell\nu}$, with $\lambda = \varkappa^{1/2}/2$, $V = -\varkappa^{-2}$, $\bar{k}_\beta = -2$ and $\sigma_\beta = 5/4$

Schrödinger



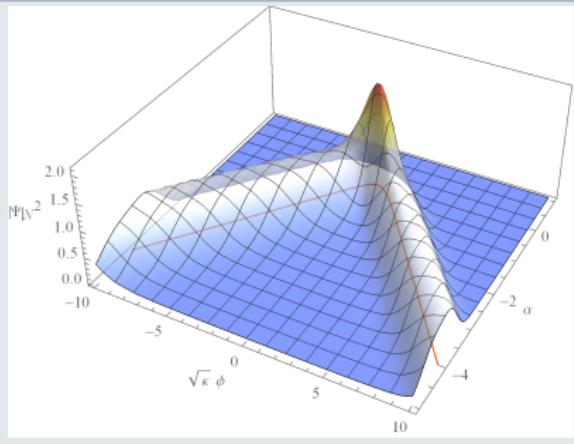
Mostafazadeh



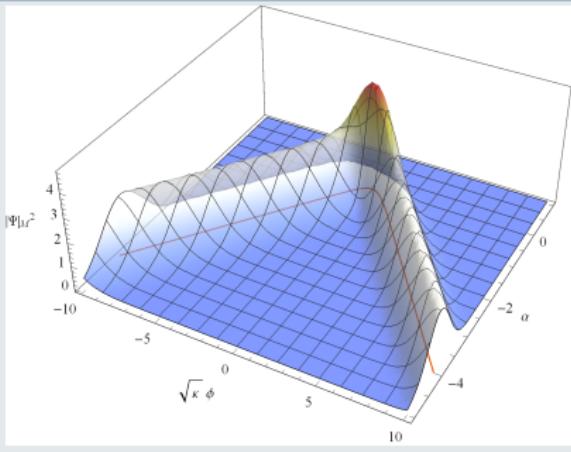
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Schrödinger



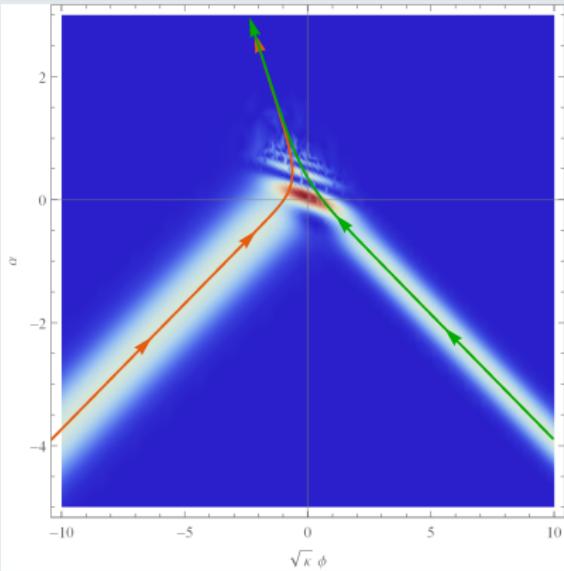
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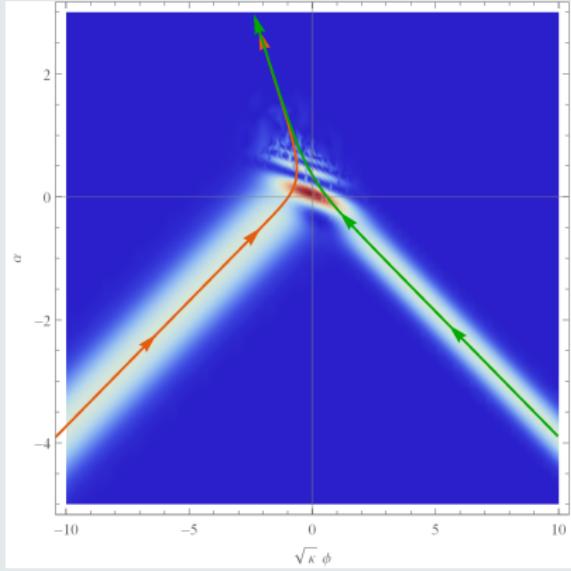
Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\parallel\nu}$, with $\lambda = 4\nu^{1/2}/5$, $V = +\nu^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



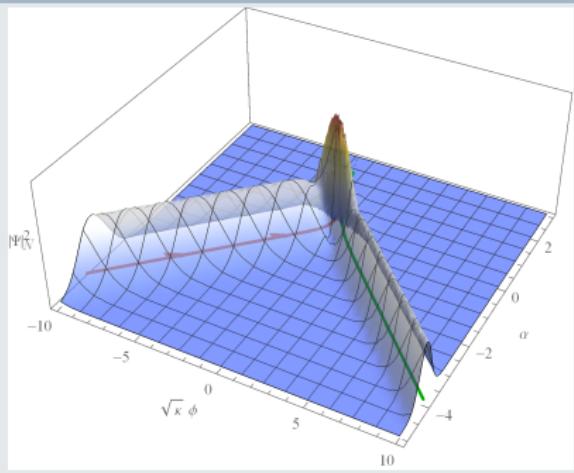
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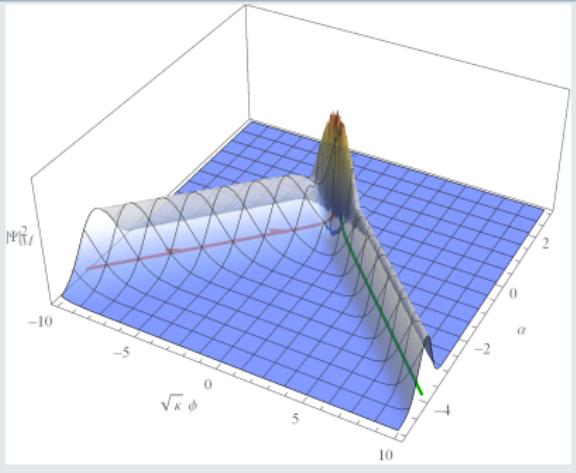
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Schrödinger



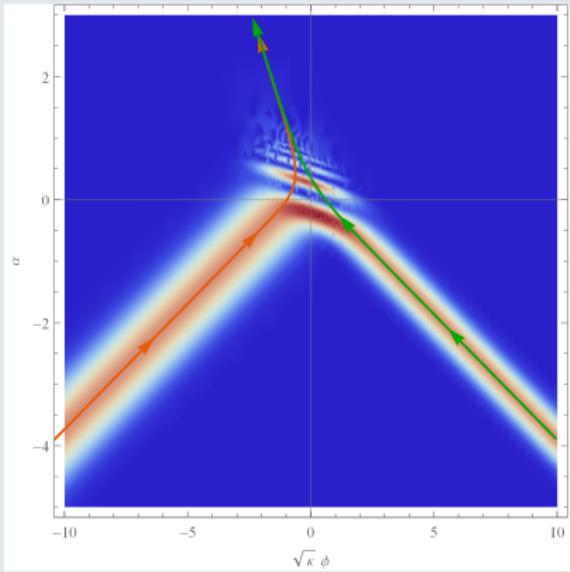
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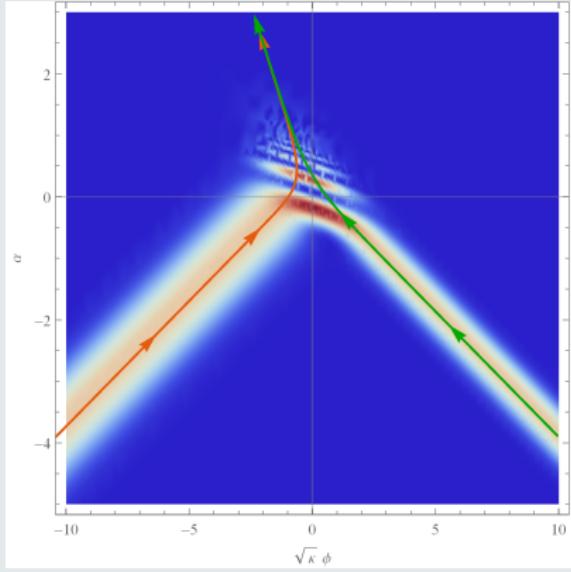
Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\parallel\nu}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



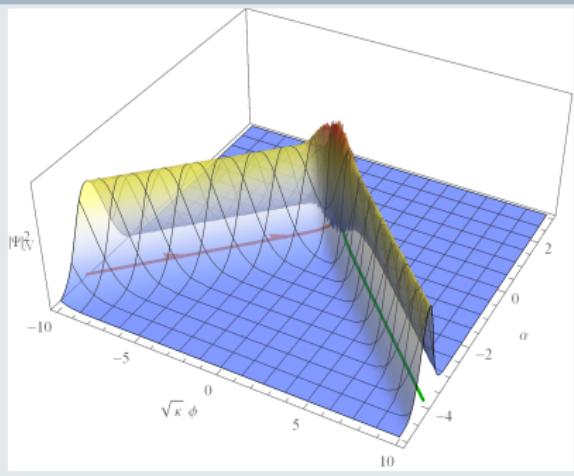
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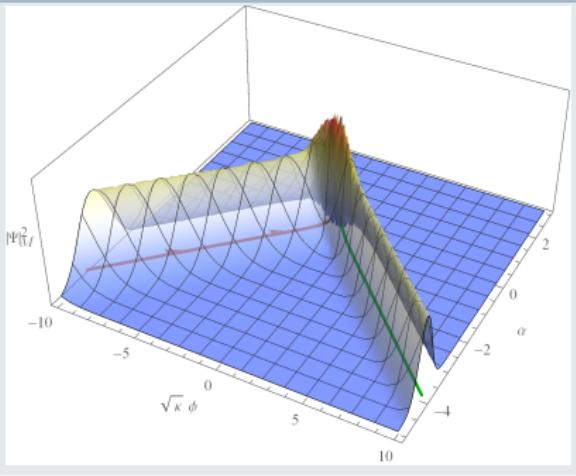
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Schrödinger



Mostafazadeh



Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model $(-, +)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\bar{n}) := \left(e^{-\bar{n}} \frac{\bar{n}^n}{n!} \right)^{1/2} \quad (16)$$

- In²², $A_n(\bar{n}/\sqrt{2})$ was chosen.

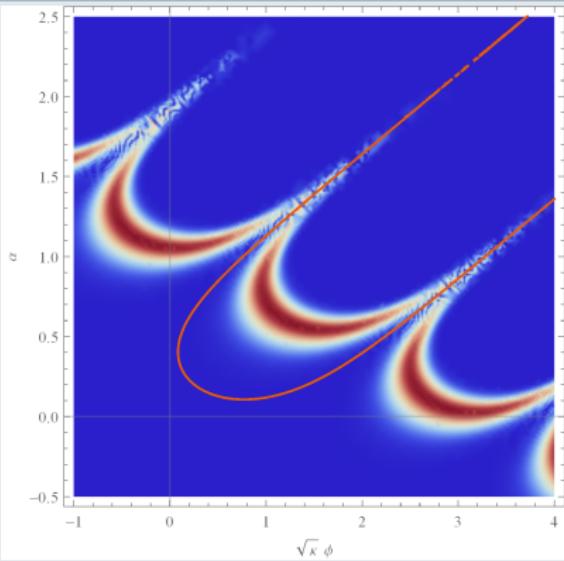
²²C. Kiefer, *Nucl. Phys. B* **341**, 273–293 (Sept. 1990).



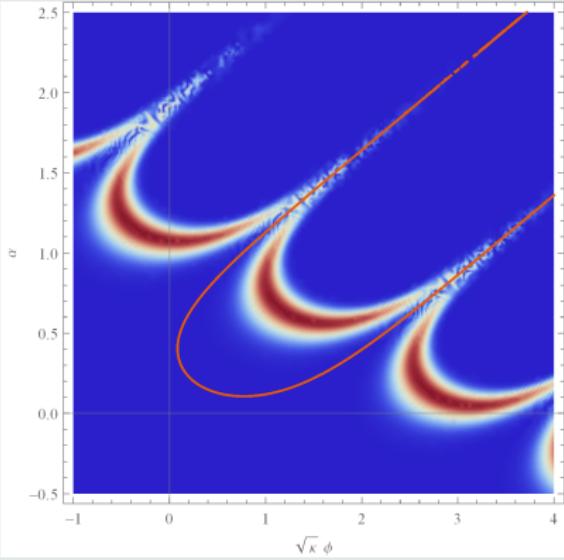
Wave packets of Poissonian amplitude for phantom model

J_{2n+1} , with $\lambda = 2\kappa^{1/2}$, $V = +\kappa^{-2}$ and $\bar{k}_\beta = 8$

Schrödinger



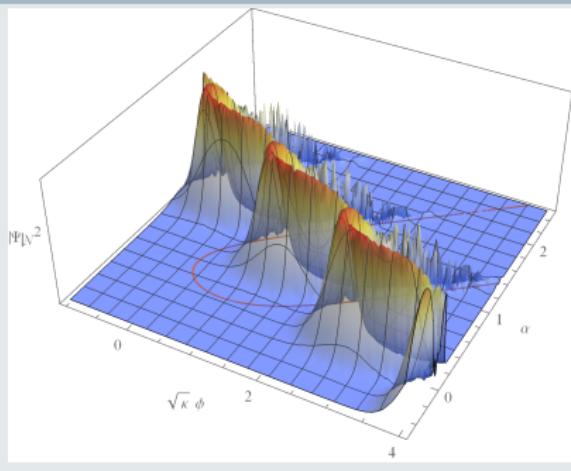
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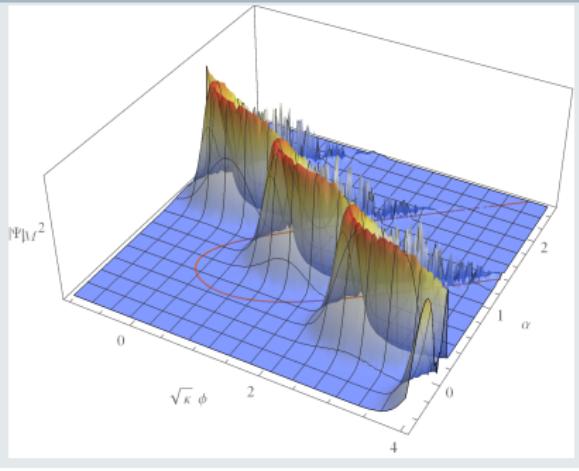
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J_{2n+1} , with $\lambda = 2\kappa^{1/2}$, $V = +\kappa^{-2}$ and $\bar{k}_\beta = 8$

Schrödinger



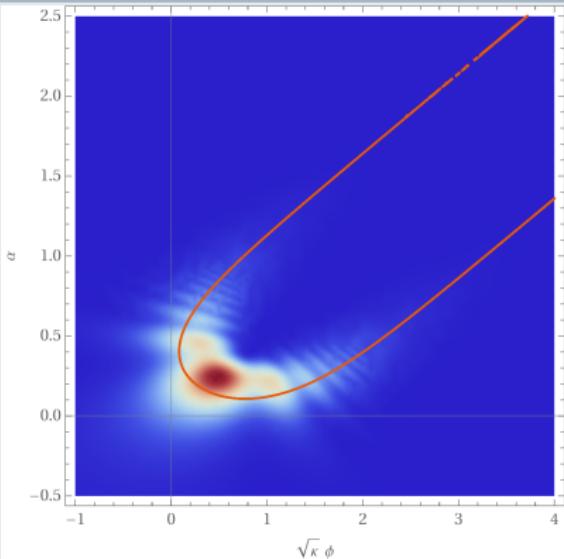
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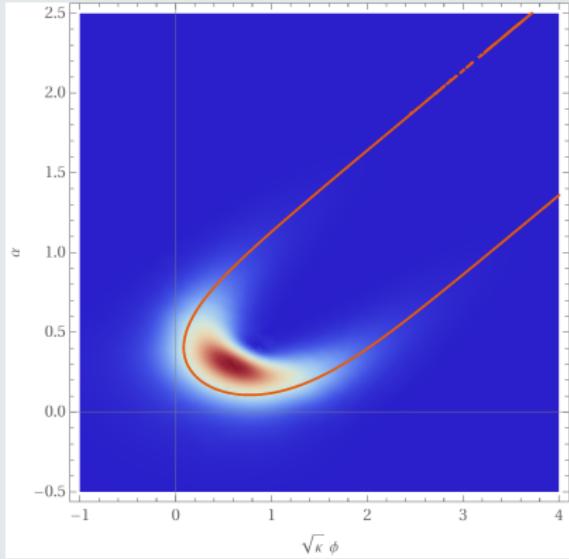
Wave packets of Gaussian amplitude for phantom model

J_ν , with $\lambda = 2\kappa^{1/2}$, $V = +\kappa^{-2}$, $\bar{k}_\beta = 8$ and $\sigma_\beta = 11/2$

Schrödinger



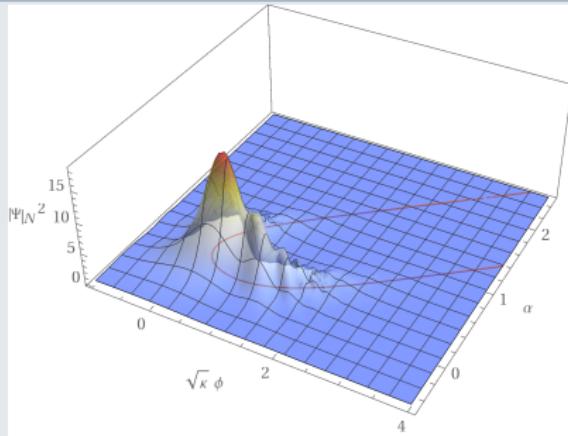
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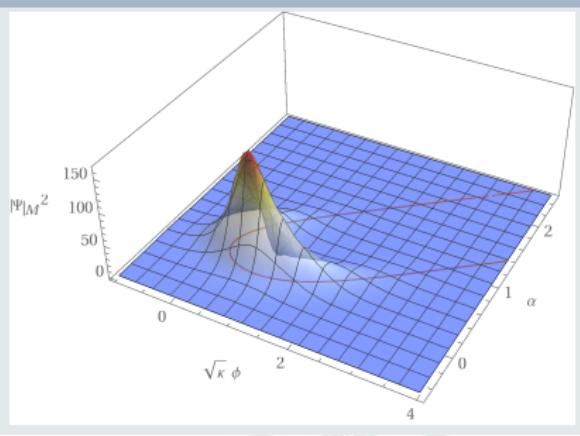
Wave packets of Gaussian amplitude for phantom model

J_ν , with $\lambda = 2\nu^{1/2}$, $V = +\nu^{-2}$, $\bar{k}_\beta = 8$ and $\sigma_\beta = 11/2$

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Highlights

- An **integral of motion** was found for Liouville cosmological models.
- **Implicit trajectories** in minisuperspace were obtained.
- The levels of phantom Liouville model were found to be **discrete** due to the Hermiticity requirement of observables.
- The semi-classical quantum mode functions and wave packets was compared directly with the classical trajectory.



Issues

- In $(+, -)$ and $(-, +)$, wave packets contain multiple branches; however, the classical universe runs only on one trajectory.
- Quantum-corrected \bar{k}_β is to be understood.
- Wave packets with Gaussian profile are to be constructed, instead of inserting Gaussian / Poissonian amplitude artificially.
- A normalising κ for $(\cdot, \cdot)_M^\kappa$ is to be evaluated, otherwise a quantitative comparison of $(\cdot, \cdot)_S$ and $(\cdot, \cdot)_M^\kappa$ is not possible.



Outlook

- Beyond isotropy: generalise to Bianchi models
 - Bianchi Type-I: a natural extension, **under investigation**
- Beyond homogeneity: cosmological perturbation, **under investigation**
- Beyond single field
 - Two exponential potentials: $|V_1| = |V_2|$ and special λ_i
 - Multiple Liouville fields: mixing kinetic terms needed²³
- Beyond classic matter
 - PT -symmetric Liouville field²⁴: may cross the phantom divide $w = -1$.

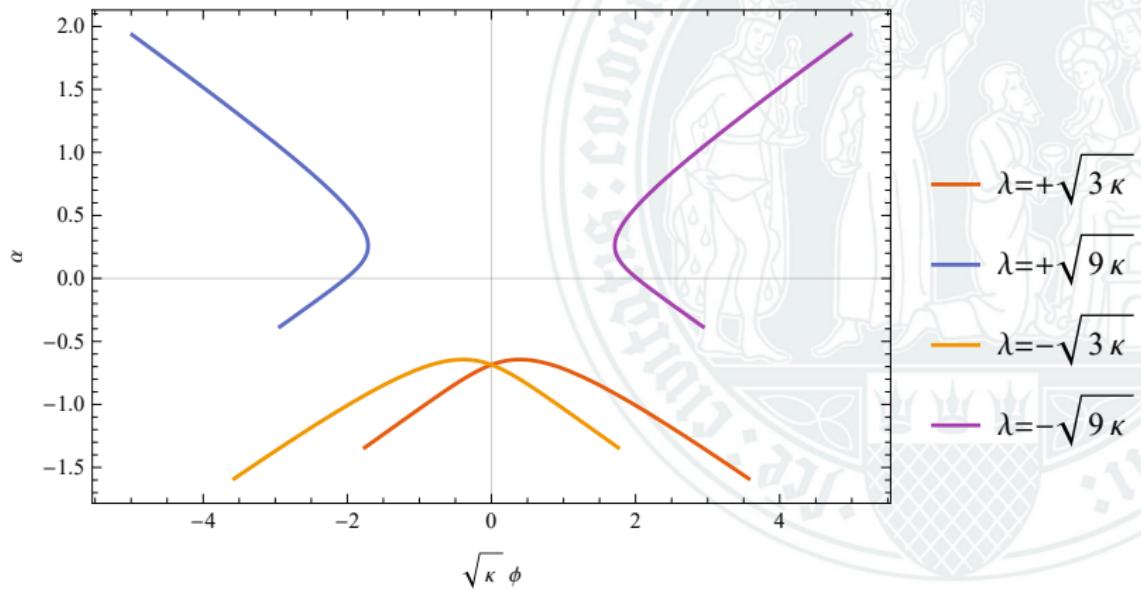
²³ A. A. Andrianov *et al.*, *Theor. Math. Phys.* **184**, 1224–1233 (Sept. 2015).

²⁴ A. A. Andrianov *et al.*, *Int. J. Mod. Phys. D* **19**, 97–111 (Jan. 2010), A. A. Andrianov *et al.*, in *Springer Proceedings in Physics* (Springer International Publishing, 2016), pp. 29–44.



Trajectories for quintessence model $(+, +)$

sech, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

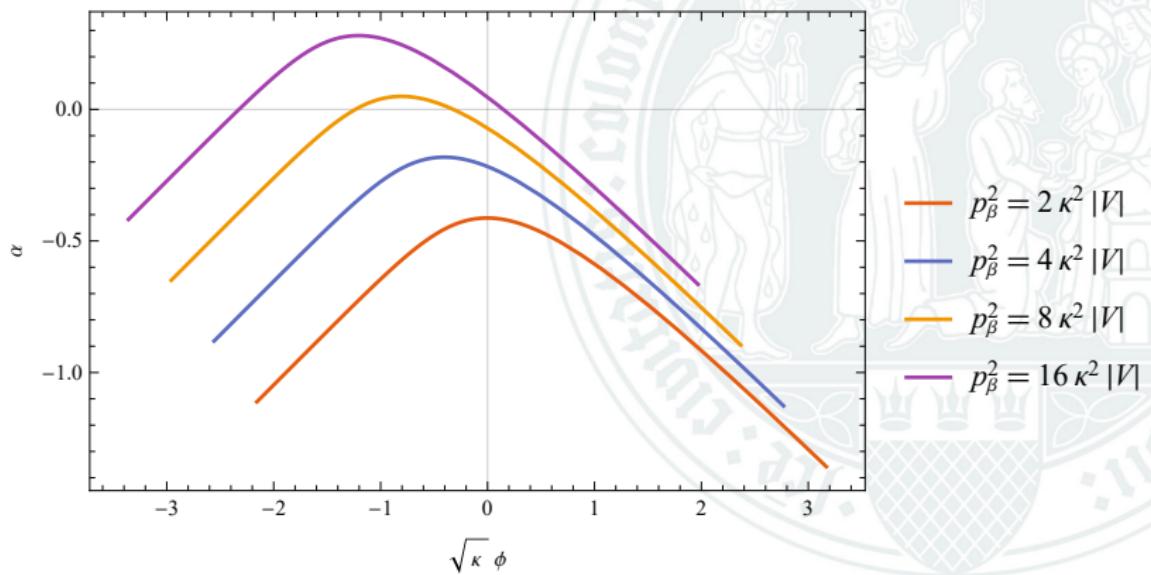


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model (+, +)

sech, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $\lambda^2 = 3\kappa$; varying p_β

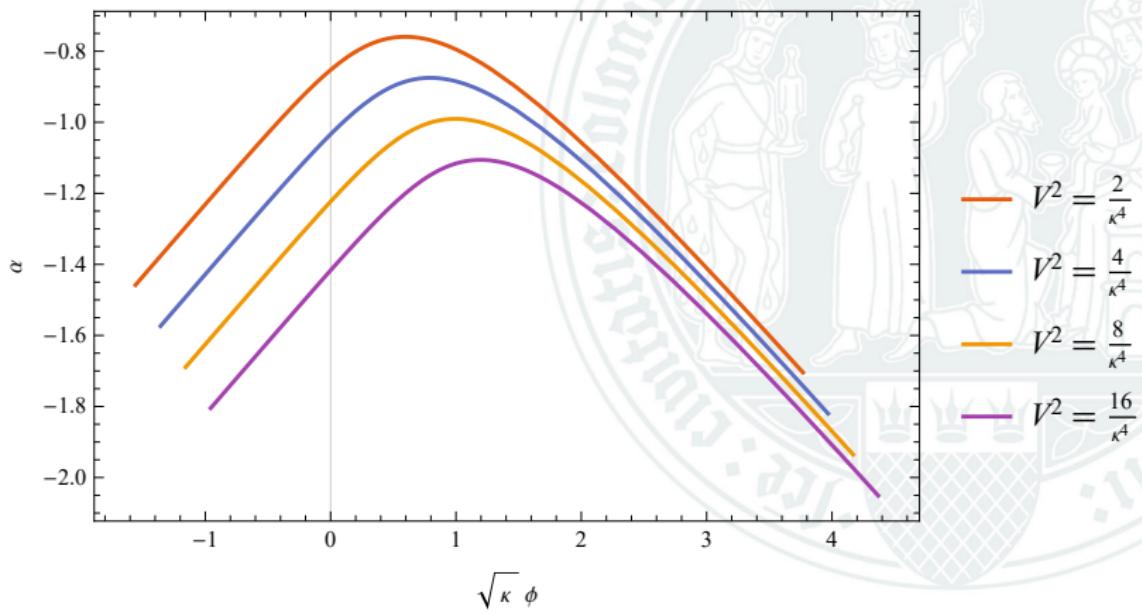


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model (+, +)

sech, with $\beta_0 = 0$, $\lambda^2 = 3\kappa$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying V

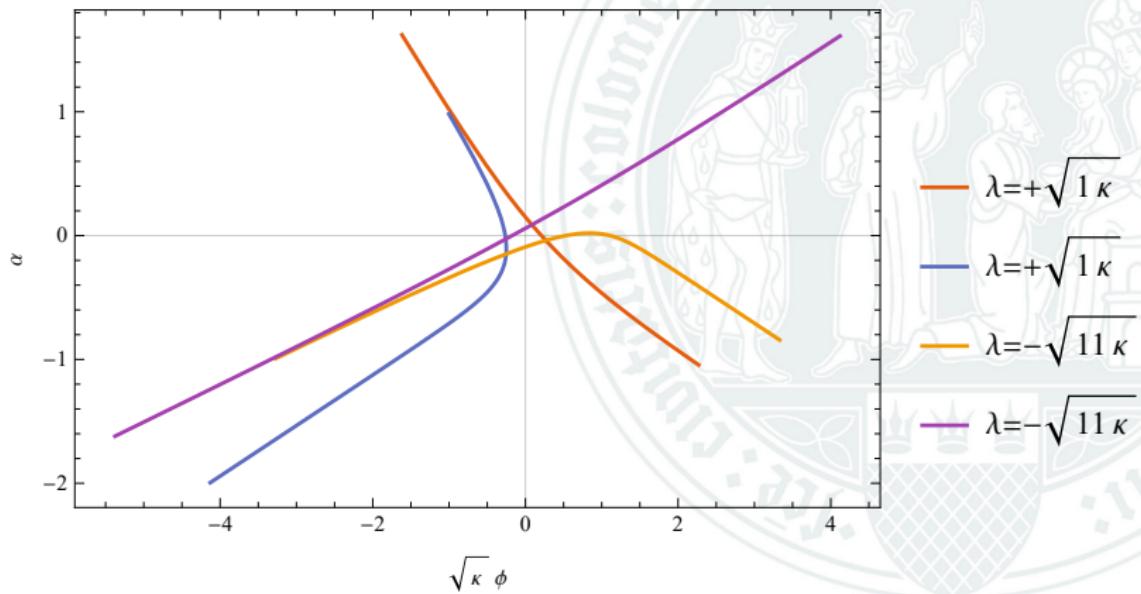


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model (+, -)

csch, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

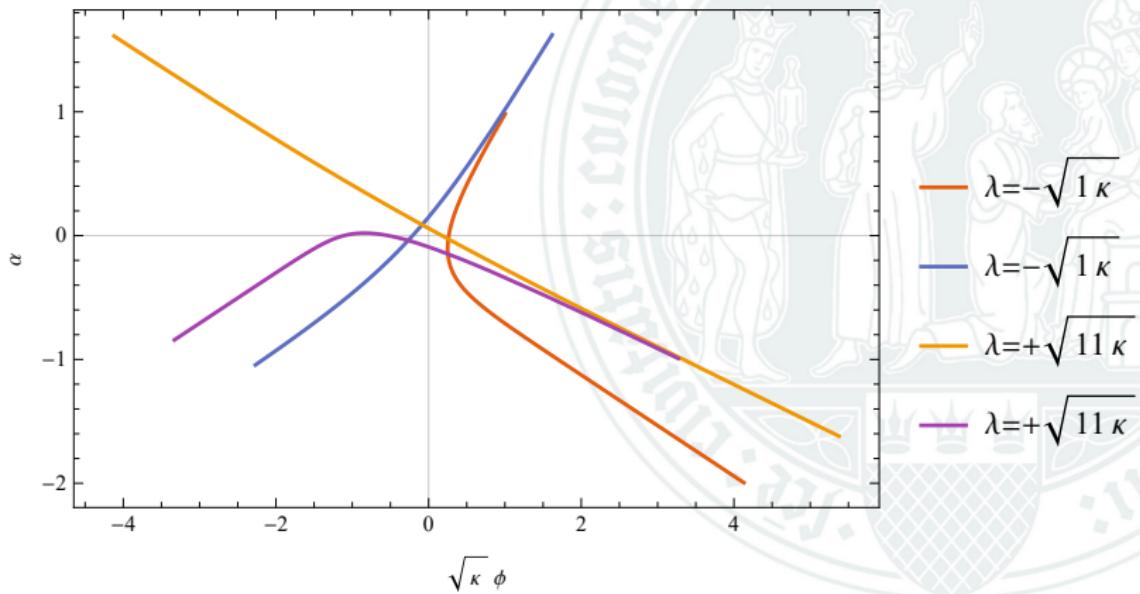


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm \beta$ and $\beta = 0$



Trajectories for quintessence model $(+, -)$: csch

csch, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

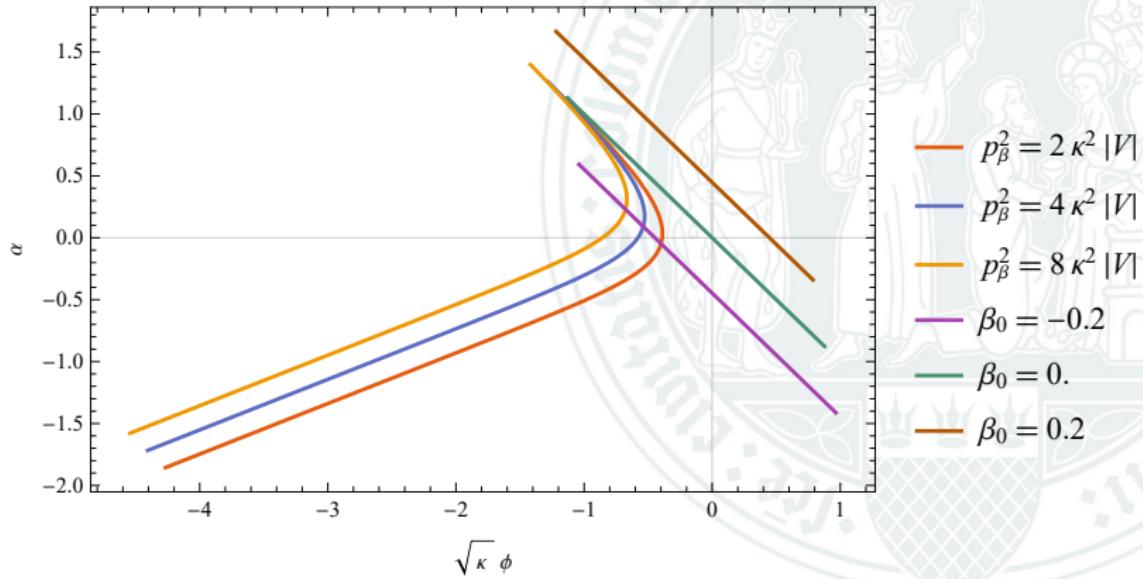


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm \beta$ and $\beta = 0$



Trajectories for quintessence model (+, -): csch

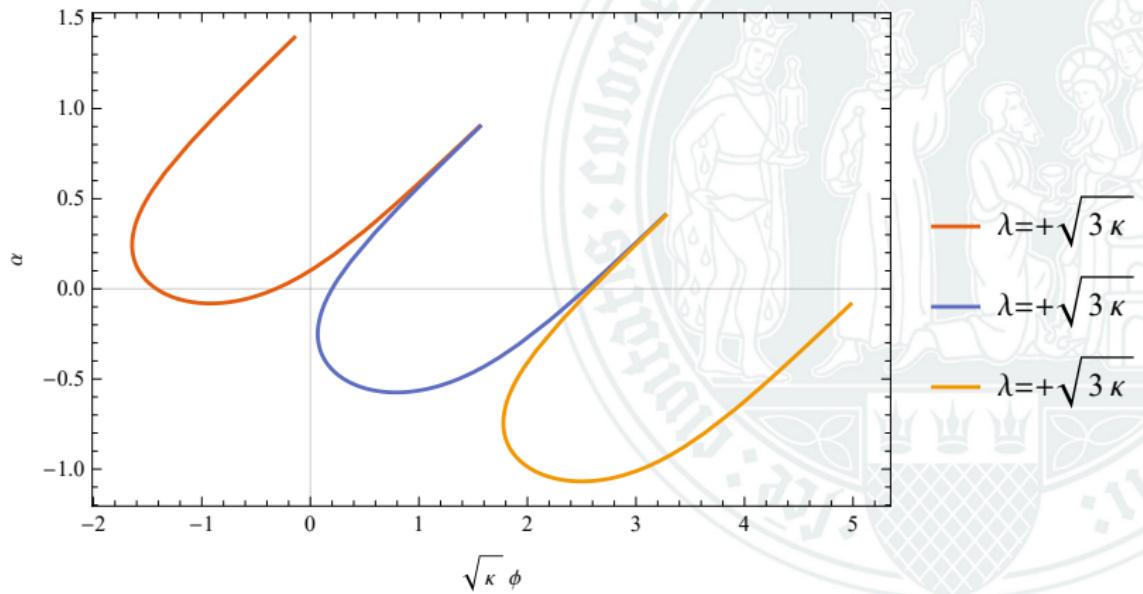
csch, with $V = \kappa^{-2}$ and $\lambda^2 = \kappa$; varying p_β



- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$

Trajectories for phantom model $(-, +)$

csc, with $V = \varkappa^{-2}$ and $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

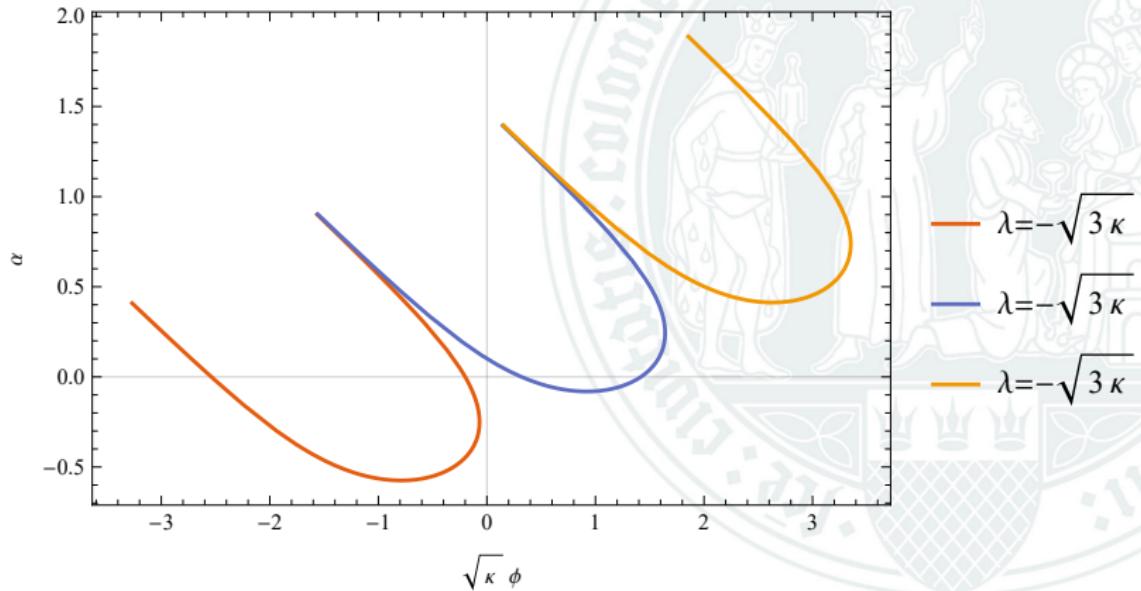


- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Trajectories for phantom model $(-, +)$

csc, with $V = \varkappa^{-2}$ and $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

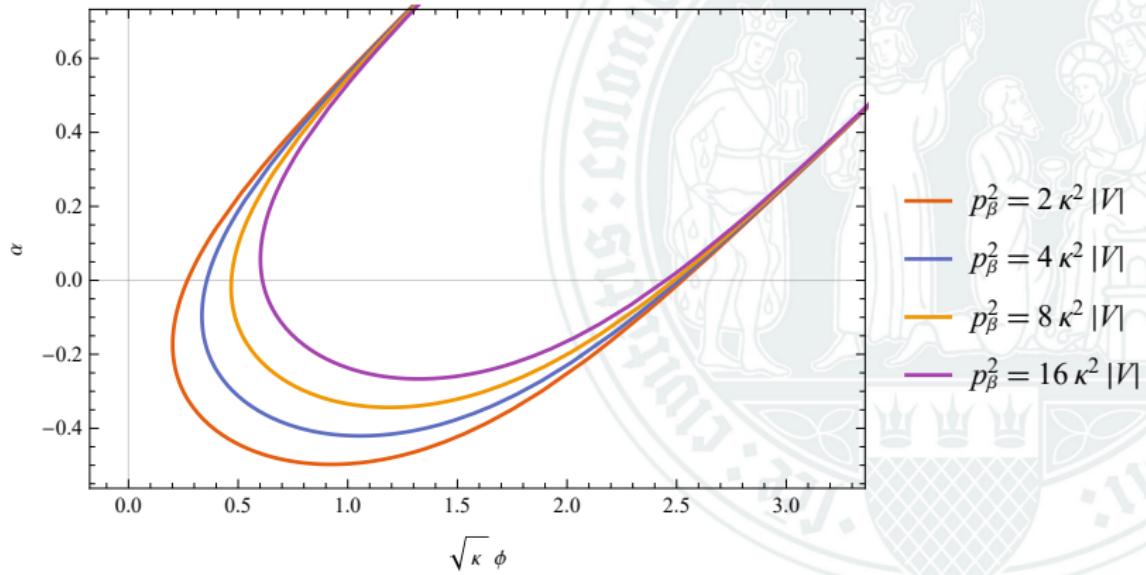


- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Trajectories for phantom model $(-, +)$

csc, with $V = \kappa^{-2}$ and $\lambda^2 = 3\kappa$; varying p_β



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Integration of the transformed first Friedmann equation

General integral for $p_\beta \neq 0$

In order to integrate the equation under the implicit gauge

$$s \frac{p_\beta^2}{12} \left(-\ell \frac{\varkappa^{1/2}}{6} \left(\frac{d\chi}{d\beta} \right)^2 + \varkappa^{-1/2} \right) - \varkappa^{3/2} V e^{g\chi} = 0, \quad (17)$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa} g\beta}, \quad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} e^{-g\chi}, \quad (18)$$

to get

$$\left(\frac{d\tilde{\sigma}}{d\gamma} \right)^2 + \ell(s\upsilon - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{d\gamma}{d\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-s\upsilon + \tilde{\sigma}^2)}}, \quad (19)$$

which is of the standard inverse hyperbolic / trigonometric form **except for $(-, -)$.**



Integration of the separated minisuperspace WDW equation

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \mathfrak{s} v \frac{12\varkappa^2 |V|}{\hbar^2}, \quad (\text{?? rev.})$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{k_\beta}{g}, \quad \sigma^2 := \frac{8\varkappa^3 |V| e^{g\chi}}{\hbar^2 g^2}, \quad (\text{?? rev.})$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma \psi'(\sigma) + \ell(\nu^2 - \mathfrak{s} v \sigma^2) \psi(\sigma) = 0, \quad (20)$$

which is of the standard Besselian form.



