

Integrable Liouville Cosmological Models

The self-adjointness of Hamiltonian and the Semi-classical Wave Functions

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Outline

1. Introduction

2. The classical model and the implicit trajectories

Lagrangian formalism and the first integral

3. Dirac quantisation and the self-adjointness of Hamiltonian

Dirac quantisation and the quantum-mechanical analogy
The self-adjointness of Hamiltonian

4. The semi-classical wave packets

Inner product and wave packet

5. Conclusions



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Introduction

The Friedmann–Lemaître model with quintessence and phantom Liouville fields

- Dynamical Dark Energy has been modelled by quintessence¹ and phantom² matter, which can be realised by minimally-coupled real scalar fields with $\ell = \pm 1$ ³

$$\mathcal{S}_L = \int d^4x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \quad (1)$$

- The Liouville field⁴ $\mathcal{V}(\phi) = V e^{\lambda\phi}$ is of interest, where $\lambda, V \in \mathbb{R}$.
- Assume a flat Robertson–Walker metric $g_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + \varkappa e^{2\alpha(t)} d\Omega_{3F}^2$, where $\varkappa := 8\pi G$, $d\Omega_{3F}^2 := d\chi^2 + \chi^2(d\theta^2 + \sin^2 \theta d\varphi^2)$, and N lapse function.
- The total action reads $\mathcal{S} := \mathcal{S}_{EH} + \mathcal{S}_{GHY} + \mathcal{S}_L = \int d\Omega_{3F}^2 \int dt L$, where

$$L := \varkappa^{3/2} N e^{3\alpha} \left(-\frac{3}{\varkappa} \frac{\dot{\alpha}^2}{N^2} + \ell \frac{\dot{\phi}^2}{2N^2} - V e^{\lambda\phi} \right). \quad (2)$$

¹R. R. Caldwell *et al.*, *Phys. Rev. Lett.* **80**, 1582–1585 (Feb. 1998).

²R. R. Caldwell, *Phys. Lett. B* **545**, 23–29 (Oct. 2002).

³The signature of metric is mostly positive.

⁴Y. Nakayama, *International Journal of Modern Physics A* **19**, 2771–2930 (July 2004).



Introduction

Highlights

Integrability

Implicit trajectories can be obtained explicitly; the minisuperspace Wheeler–DeWitt equation can be integrated exactly.

Self-adjointness

The Hamiltonian is not even naturally symmetric; imposing self-adjointness leads to non-trivial physical results.

Semi-classical wave functions

Semi-classical wave functions can be constructed by the principle of constructive interference, the JWKB approximation and numerical methods.



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Decoupling the variables

Via orthogonal transformation

- By rescaling $\bar{N} := Ne^{-3\alpha}$ and the orthogonal transformation

$$\begin{pmatrix} \alpha(t) \\ \phi(t) \end{pmatrix} = \frac{\varsigma}{g} \begin{pmatrix} \lambda & -\ell\kappa \\ -6 & \lambda \end{pmatrix} \begin{pmatrix} \varsigma_\beta \cdot \beta(t) \\ \varsigma_\chi \cdot \chi(t) \end{pmatrix} \quad \text{where } \varsigma_\beta, \varsigma_\chi = \pm 1, \quad (3)$$

the effective Lagrangian eq. (2) can be decoupled ($\varsigma_\beta = \varsigma_\chi = +1$ from now on)

$$L = \kappa^{3/2} \bar{N} \left(-\varsigma \frac{3}{\kappa} \frac{\dot{\beta}^2}{\bar{N}^2} + \ell \varsigma \frac{\dot{\chi}^2}{2\bar{N}^2} - V e^{g\chi} \right), \quad (4)$$

where $\Delta := \lambda^2 - 6\ell\kappa$, $\varsigma := \operatorname{sgn} \Delta$ and $g := \varsigma \sqrt{|\Delta|} \equiv \varsigma \sqrt{3\Delta}$.

- The Euler–Lagrange equations with respect to \bar{N} , β and χ will be called the modified first, second Friedmann equations and the Klein–Gordon equation, respectively.
- Since $\beta(t)$ is cyclic, its conjugate momentum p_β is conserved⁵, and the modified second Friedmann equation can be readily integrated.

⁵The same first integral has been found in C. Lan, PhD thesis, Saint Petersburg State University, 2016, <https://search.rsl.ru/ru/record/01006663434>, A. A. Andrianov et al., *Theor. Math. Phys.* **184**, 1224–1233 (Sept. 2015), in canonical formalism.



Integration and the implicit trajectories

- For $p_\beta \neq 0$, fixing the *implicit gauge* $\bar{N} = -6\mathfrak{s}\sqrt{\varkappa}\dot{\beta}/p_\beta$, the modified first Friedmann equation can be integrated, yielding the **implicit** trajectory

$$\mathbb{e}^{g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|} S^2 \left(\sqrt{\frac{3}{2\varkappa}} g(\beta - \beta_0) \right), \quad (5)$$

in which $v := \operatorname{sgn} V$, $(\operatorname{sgn}, \operatorname{sgn})$ means $(\ell, \mathfrak{s}v)$, and

$$\begin{aligned} (+,+)\,S(\gamma) &:= \operatorname{sech}(\gamma), & (+,-)\,S(\gamma) &:= \operatorname{csch}(\gamma), \\ (-,+)\,S(\gamma) &:= \sec(\gamma), & (-,-)\,S(\gamma) &:= \mathfrak{i} \csc(\gamma). \end{aligned} \quad (6)$$

- The integral for $(-, -)$ is not real.
- The trajectories can be parametrised by β , inspiring recognising β as a ‘time variable’.
- For $p_\beta = 0$, integrating the modified second Friedmann equation yields $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell \lambda \alpha / \varkappa^6$, which is the well-known power-law special solution⁷.

⁶It only works for $(+, -)$ or $(-, +)$ if one checks consistency with the modified first Friedmann equation.

⁷For instance, A. R. Liddle, D. H. Lyth, *Cosmological Inflation and Large-Scale Structure*, (Cambridge University, 2000) , ch. 3.



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Dirac quantisation

- The primary Hamiltonian and the Hamiltonian constraint⁸ reads

$$H^p = \bar{N} H_{\perp} + v^{\bar{N}} p_{\bar{N}}, \quad (7)$$

$$H_{\perp} := -\beta \frac{p_{\beta}^2}{12\nu^{1/2}} + \ell \beta \frac{p_{\chi}^2}{2\nu^{3/2}} + \nu^{3/2} V e^{g\chi}. \quad (8)$$

- Applying the Dirac quantisation rules with the Laplace–Beltrami operator ordering⁹, one gets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(\beta \frac{\hbar^2}{12\nu^{1/2}} \partial_{\beta}^2 - \ell \beta \frac{\hbar^2}{2\nu^{3/2}} \partial_{\chi}^2 + \nu^{3/2} V e^{g\chi} \right) \Psi. \quad (9)$$

- Equation (9) is Klein–Gordon-like, hyperbolic for $\ell = +1$ and *elliptic* for $\ell = -1$.

⁸D. M. Gitman, I. V. Tyutin, *Quantization of Fields with Constraints*, (Springer, 1990), H. J. Rothe, K. D. Rothe, *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*, (World Scientific, Apr. 2010).

⁹C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012), ch. 8.



Integration of the Wheeler–DeWitt equation

The quantum-mechanical analogy

- Fourier transforming β to $k_\beta \in \mathbb{R}$ yields the time-independent Schrödinger equation

$$\ell \frac{\hbar^2 \tilde{g}^2 \nu^2}{8M_P} \psi(x) = \widehat{H}_{\text{eff}} \psi(x) := \left(-\frac{\hbar^2}{2M_P} \partial_x^2 + \widehat{V}_{\text{eff}}(x) \right) \psi(x), \quad V_{\text{eff}} = \ell s \nu \widetilde{V} e^{\tilde{g} x}, \quad (10)$$

in which $M_P := \sqrt{\frac{\hbar}{\nu}}$, $x := \sqrt{\hbar\nu} \cdot \chi$, $\widetilde{V} := M_P^{-3} |V| \geq 0$, and $\nu = \sqrt{\frac{2\nu}{3}} \frac{|k_\beta|}{g} \geq 0$.

- V_{eff} is a special case of the Morse potential¹⁰.
- For the quintessence model $(+, +)$: V_{eff} is bounded **below**; generalised eigenfunctions are the Besselian $K_{\pm\nu}$'s.
- For the quintessence model $(+, -)$: V_{eff} is bounded **above**; generalised eigenfunctions for positive spectrum (scattering states) are the Besselian $F_{\pm\nu}$'s and $G_{\pm\nu}$ 's¹¹.
- For the phantom model $(-, +)$: V_{eff} is bounded **above**; eigenfunctions for negative eigenvalues (bound states) are the Besselian J_ν 's.

¹⁰P. M. Morse, *Physical Review* **34**, 57–64 (July 1929), D. M. Gitman et al., *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012).

¹¹These are linear combinations of $J_{\pm\nu}$'s which are orthogonal; see T. M. Dunster, *SIAM Journal on Mathematical Analysis* **21**, 995–1018 (July 1990).



The self-adjointness of unbounded operator

The general theory

- Even in quantum geometrodynamics, the self-adjointness of the matter Hamiltonian is desirable, which is typically an unbounded operator in the Hilbert space \mathcal{F} endowed with the Schrödinger inner product $(\cdot, \cdot)_S$.
- Mathematically, an *unbounded* operator H is characterised not only by its action on a vector, but also by its domain $\text{Dom}(H) \subsetneq \mathcal{F}^{12}$.
- In addition to the *symmetry* $(\phi_1, H^\dagger \phi_2) \equiv (H\phi_1, \phi_2)$, the self-adjointness of an unbounded operator also requires $\text{Dom}(H^\dagger) = \text{Dom}(H)$.
- For quantum systems not bounded below, not only the self-adjointness, but also the symmetry of the Hamiltonian is not guaranteed automatically.
- Even when one could find a $\text{Dom}_0(H)$ such that H is symmetric, one would still be left with $\text{Dom}_0(H^\dagger) \supsetneq \text{Dom}_0(H)$ in general.
- Sloppily speaking, the process of extending $\text{Dom}(H)$ such that $\text{Dom}(H^\dagger) = \text{Dom}(H)$ is called **self-adjoint extension**¹³; if the extension is unique, the operator is called *essentially self-adjoint*.

¹²B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267 , ch. 9.

¹³D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) , G. Bonneau *et al.*, *Am. J. Phys* **69**, 322–331 (Mar. 2001), V. S. Araujo *et al.*, *Am. J. Phys* **72**, 203–213 (Feb. 2004), A. M. Essin, D. J. Griffiths, *Am. J. Phys* **74**, 109–117 (Feb. 2006).



The self-adjointness of the Hamiltonian

Results for the special Morse potential¹⁵

- For the quintessence model $(+, +)$: \widehat{H}_{eff} is **essentially self-adjoint** on $C_c^\infty(\mathbb{R})$ ¹⁴.
- For $(+, -)$ and $(-, +)$: \widehat{H}_{eff} is **not self-adjoint** but admits a family of self-adjoint extensions parametrised by $a \in [0, 2)$, where the (generalised) eigenfunctions are

$$\Xi_\nu \propto F_{i\nu} \cos \frac{\pi a_1}{2} + G_{i\nu} \sin \frac{\pi a_1}{2}; \quad (11)$$

$$\Phi_\nu \propto J_{2n+a_2}, \quad \nu = 2n + a_2, n \in \mathbb{Z}_{\geq}. \quad (12)$$

- For $(-, +)$, this can be shown by noting $(J_\mu, \hat{p}^2 J_\nu)_S - (\hat{p}^2 J_\mu, J_\nu)_S \propto \sin \frac{\pi(\mu-\nu)}{2}$; Imposing symmetricity leads to the condition for eq. (12).
 - For $(-, +)$, in terms of the full wave-function,
- $$\Psi(\beta, \chi) = e^{i\pi a_2} \Psi(\beta+, \chi) \quad (13)$$
- Imposing (anti-)periodic boundary condition would fix $a_2 = 0$ (1).
 - Using classical singularities as boundary condition, one can fix $\nu_0 = 1$.

¹⁴ See also B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267, sec. 9.9.

¹⁵ D. M. Gitman et al., *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012), sec. 8.5.



Discretisation of the phantom model $(-, +)$

Levels of the phantom model are **discretised** if one imposes Hermiticity of squared momenta under the Schrödinger product.

- This kind of subtlety on Hermiticity is named as *self-adjoint extension*, which arises already for infinite square well¹⁶.
- It also applies to x^{-2} potentials¹⁷, which is of cosmological relevance¹⁸.

¹⁶G. Bonneau *et al.*, *Am. J. Phys* **69**, 322–331 (Mar. 2001).

¹⁷V. S. Araujo *et al.*, *Am. J. Phys* **72**, 203–213 (Feb. 2004), A. M. Essin, D. J. Griffiths, *Am. J. Phys* **74**, 109–117 (Feb. 2006).

¹⁸M. Bouhmadi-López *et al.*, *Physical Review D* **79** (June 2009).



Separation of the variables and mode functions

- Writing $\Psi(\beta, \chi) = \varphi(\beta)\psi(\chi)$, eq. (9) can be separated into

$$\partial_\beta^2 \varphi(\beta) = k_\beta^2 \varphi(\beta); \quad (14)$$

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \imath v \frac{12\varkappa^2 |V|}{\hbar^2}. \quad (15)$$

- Equation (15) turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta, \chi) := \sum_{i=1}^2 c_i \varphi_\nu^{(i)}(\gamma) \sum_{j=1}^2 a_j B_\nu^{(j)}(\sigma), \quad \nu \geq 0; \quad (16)$$

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{1}{g} k_\beta, \quad \gamma := \sqrt{\frac{3}{2\varkappa}} \frac{g}{1} \beta, \quad \sigma^2 := \frac{8\varkappa^3 |V| e^{g\chi}}{\hbar^2 g^2}, \quad (17)$$

$${}_{(+,+)} B_\nu^{(i)}(\sigma) := K \text{ and } I_{\mathbb{B}\nu}(\sigma), \quad {}_{(+,-)} B_\nu^{(i)}(\sigma) := F \text{ and } G_{\mathbb{B}\nu}(\sigma),$$

$${}_{(-,+)} B_\nu^{(i)}(\sigma) := J \text{ and } Y_\nu(\sigma), \quad {}_{(--)} B_\nu^{(i)}(\sigma) := K \text{ and } I_\nu(\sigma).$$

- Adapted to imaginary order, $F_\nu(\sigma)$ and $G_\nu(\sigma)$ are defined in¹⁹.

¹⁹T. M. Dunster, *SIAM Journal on Mathematical Analysis* **21**, 995–1018 (July 1990).



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Inner product for wave functions

Schrödinger product

- To make sense of amplitude and wave packet, an inner product is necessary
- In terms of a Klein–Gordon equation, call β the “temporal” variable, and χ the “spacial” variable.
- A common starting point is the *Schrödinger product*²⁰

$$(\Psi_1, \Psi_2)_S := \int d\chi \Psi_1^*(\beta, \chi) \Psi_2(\beta, \chi); \quad (18)$$

- $(\Psi, \Psi)_S$ is **positive-definite**, and the integrand $\rho_S(\beta, \chi)$ is **non-negative**
- The corresponding Schrödinger current is **real** but **does not satisfy continuity equation** $\dot{\rho}_S + \nabla \cdot \vec{j}_S = 0$, because eq. (9) is KG-like.
- $K_{\parallel\nu}$ ²¹ for $(+, +)$, $F_{\parallel\nu}$ and $G_{\parallel\nu}$ for $(+, -)$ can be proved to be **orthogonal** and **complete** among themselves, as well as **can be normalised**.
 - $J_{\parallel\nu}$'s for $(+, -)$ are **not orthogonal**

²⁰C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012), ch. 5.

²¹S. B. Yakubovich, *Opuscula Math.* **26**, 161–172 (2006), A. Passian et al., *Journal of Mathematical Analysis and Applications* **360**, 380–390 (Dec. 2009), R. Szymtkowski, S. Bielski, *Journal of Mathematical Analysis and Applications* **365**, 195–197 (May 2010).



Further inner products for wave functions

Klein–Gordon and Mostafazadeh product

- Since eq. (9) is KG-like, another popular choice is the KG product

$$(\Psi_1, \Psi_2)_{\text{KG}}^g := \mathbb{i}g \left\{ (\Psi_1, \partial_\beta \Psi_2)_S - (\partial_\beta \Psi_1, \Psi_2)_S \right\}, \quad g > 0. \quad (19)$$

- $(\Psi, \Psi)_{\text{KG}}^g$ is **real** but **not positive-definite**, so does the integrand ρ_{KG} ;
- The corresponding \vec{J}_{KG} is **conserved** $\dot{\rho}_{\text{KG}} + \nabla \cdot \vec{J}_{\text{KG}} = 0$ and **real**.
- Mostafazadeh²² found a product *for Hermitian \mathbb{D} with positive spectrum*:

$$(\Psi_1, \Psi_2)_M^\kappa := \kappa \left\{ (\Psi_1, \mathbb{D}^{+1/2} \Psi_2)_S + (\partial_\beta \Psi_1, \mathbb{D}^{-1/2} \partial_\beta \Psi_2)_S \right\}, \quad \kappa > 0. \quad (20)$$

- $(\Psi, \Psi)_M^\kappa$ is **positive-definite**, but the integrand ρ_M^κ is **complex**
- The corresponding \vec{J}_M^κ is **conserved** $\dot{\rho}_M^\kappa + \nabla \cdot \vec{J}_M^\kappa = 0$ but also **complex**.

²²A. Mostafazadeh, *Classical Quantum Gravity* **20**, 155–171 (Dec. 2002).



Mostafazadeh inner product and the corresponding density

- Real power of \mathbb{D} is defined by spectral decomposition $\mathbb{D}^\gamma := \sum_\nu \nu^{2\gamma} \mathbf{P}_\nu$,
 $\mathbf{P}_\nu \Psi := \Psi_\nu (\Psi_\nu, \Psi)_S$.
- It can be shown²³ that *the density*

$$\varrho_M^\kappa := \kappa \left\{ |\mathbb{D}^{+1/4} \Psi|^2 + |\mathbb{D}^{-1/4} \partial_\beta \Psi|^2 \right\} \quad (21)$$

- is equivalent to ρ_M^κ up to a boundary term

$$\int d\chi \varrho_M^\kappa = \int d\chi \rho_M^\kappa \equiv (\Psi_1, \Psi_2)_M^\kappa; \quad (22)$$

- is non-negative.
- The corresponding current $\vec{\mathcal{J}}_M^\kappa$ is real but not conserved²⁴.

²³A. Mostafazadeh, F. Zamani, *Ann. Phys.* **321**, 2183–2209 (Sept. 2006).

²⁴B. Rosenstein, L. P. Horwitz, *J. Phys. A: Math. Gen.* **18**, 2115–2121 (Aug. 1985).



Wave packets of Gaussian amplitude for continuous spectrum

Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu; \bar{\nu}, \sigma) := \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\nu - \bar{\nu})^2}{2\sigma^2}\right) \right)^{1/2} \quad (23)$$

- In²⁵, $A(\nu; \bar{\nu}, \sigma/\sqrt{2})$ was chosen.

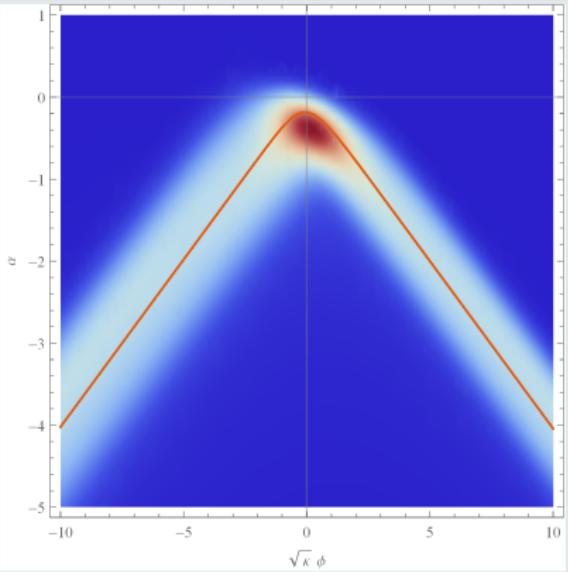
²⁵M. P. Dąbrowski et al., *Physical Review D* **74** (Aug. 2006).



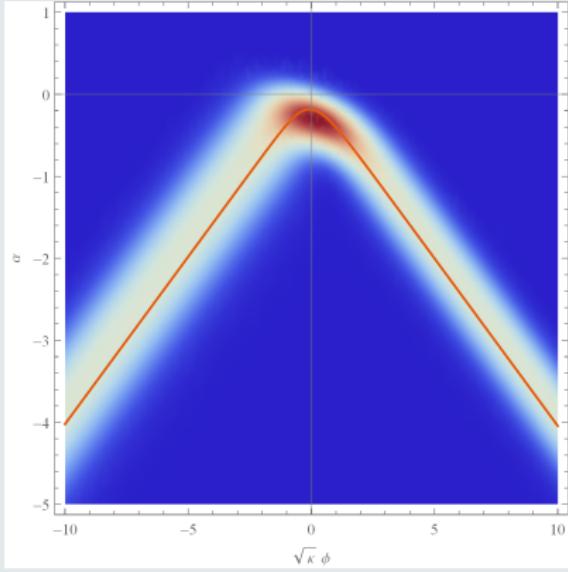
Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\ell\nu}$, with $\lambda = \varkappa^{1/2}/2$, $V = -\varkappa^{-2}$, $\bar{k}_\beta = -2$ and $\sigma_\beta = 5/4$

Schrödinger



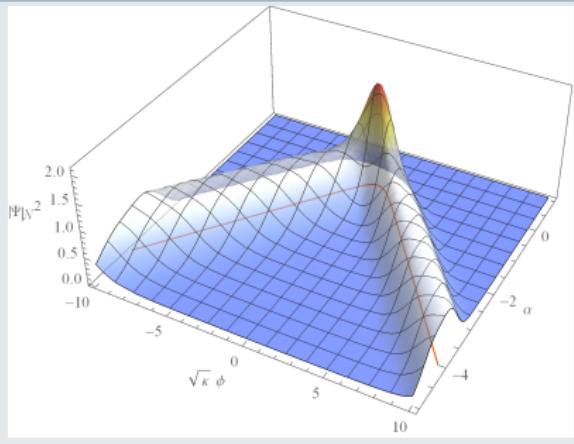
Mostafazadeh



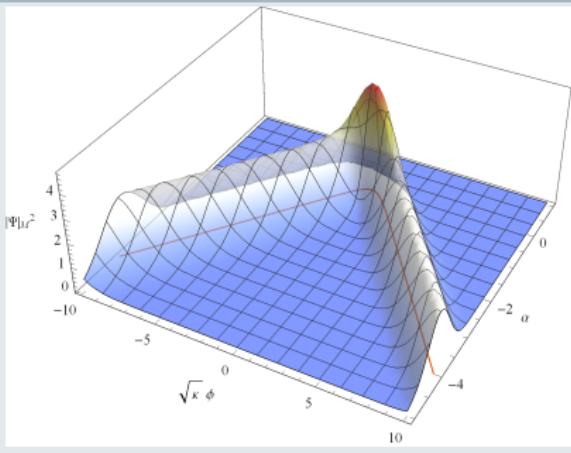
Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\ell\nu}$, with $\lambda = \varkappa^{1/2}/2$, $V = -\varkappa^{-2}$, $\bar{k}_\beta = -2$ and $\sigma_\beta = 5/4$

Schrödinger



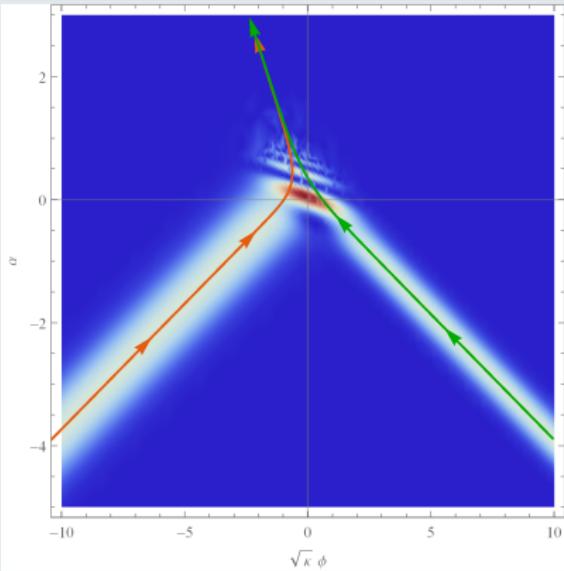
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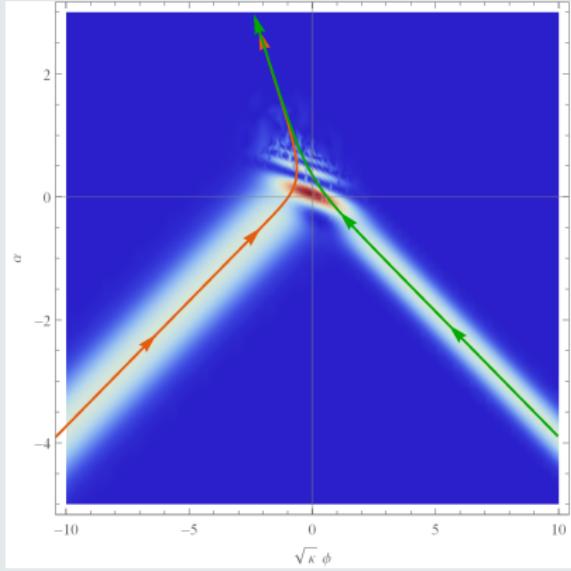
Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\parallel\nu}$, with $\lambda = 4\nu^{1/2}/5$, $V = +\nu^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



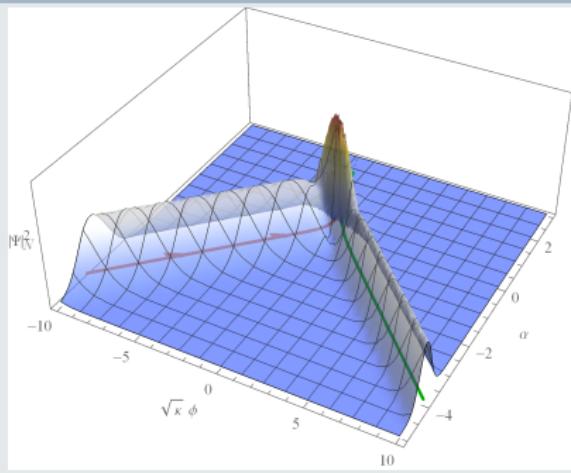
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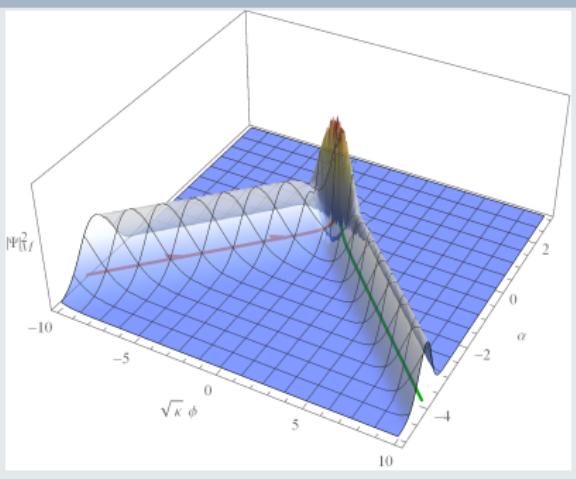
Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\parallel\nu}$, with $\lambda = 4\nu^{1/2}/5$, $V = +\nu^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



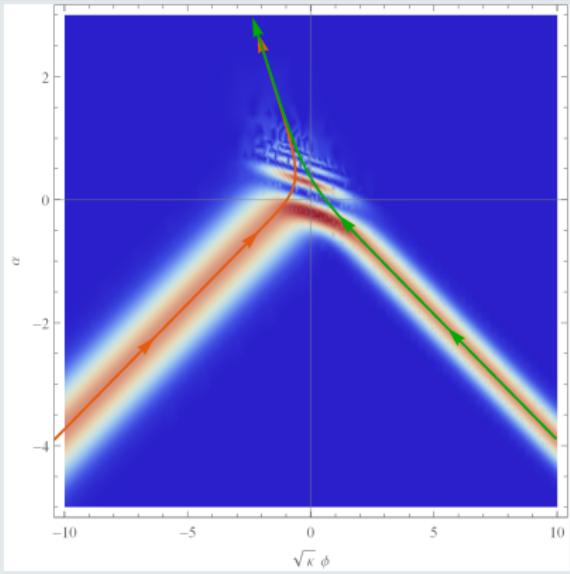
Mostafazadeh



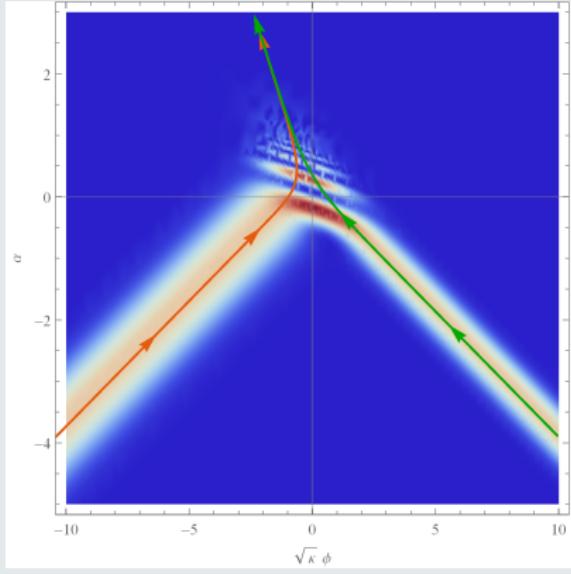
Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\parallel\nu}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



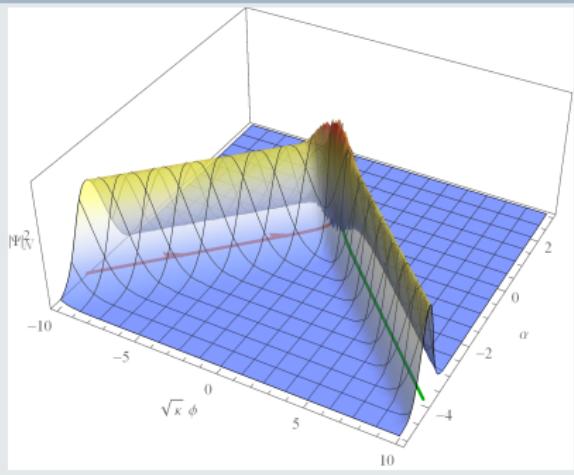
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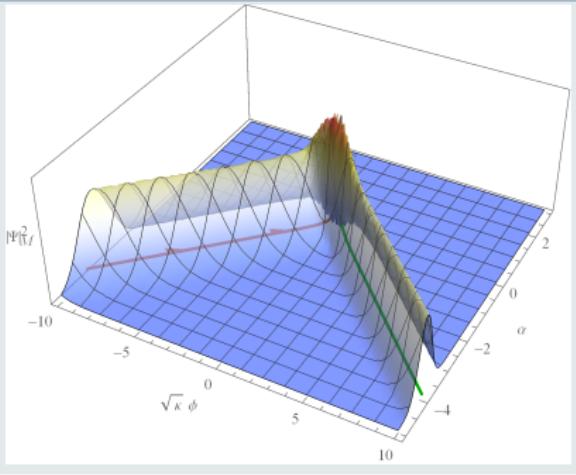
Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\parallel\nu}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



Mostafazadeh



Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model $(-, +)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\bar{n}) := \left(e^{-\bar{n}} \frac{\bar{n}^n}{n!} \right)^{1/2} \quad (24)$$

- In²⁶, $A_n(\bar{n}/\sqrt{2})$ was chosen.

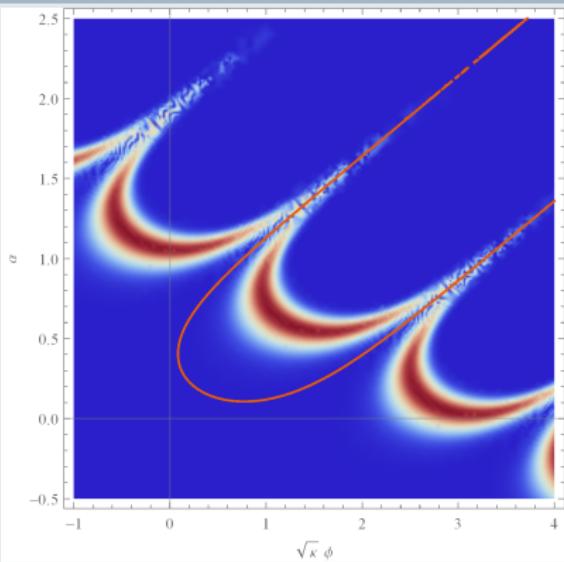
²⁶C. Kiefer, *Nucl. Phys. B* **341**, 273–293 (Sept. 1990).



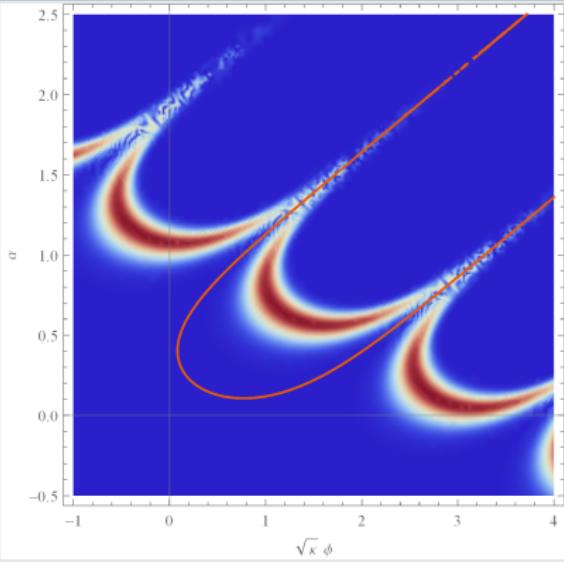
Wave packets of Poissonian amplitude for phantom model

J_{2n+1} , with $\lambda = 2\kappa^{1/2}$, $V = +\kappa^{-2}$ and $\bar{k}_\beta = 8$

Schrödinger



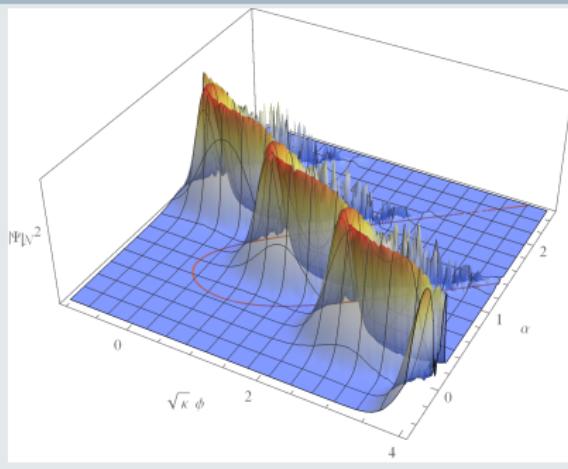
Mostafazadeh



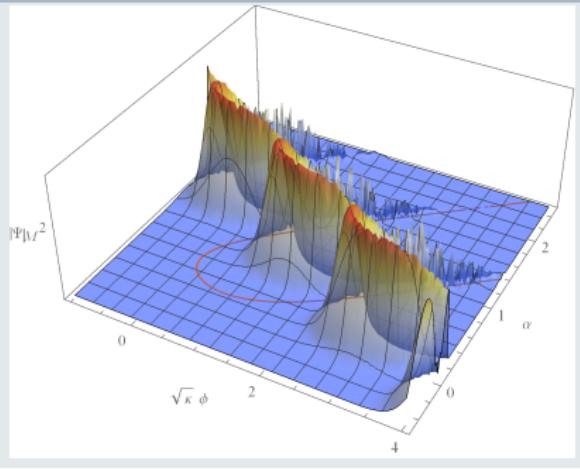
Wave packets of Poissonian amplitude for phantom model

J_{2n+1} , with $\lambda = 2\kappa^{1/2}$, $V = +\kappa^{-2}$ and $\bar{k}_\beta = 8$

Schrödinger



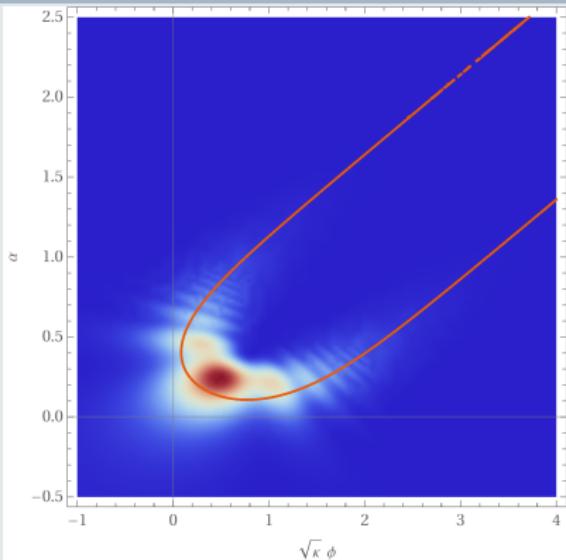
Mostafazadeh



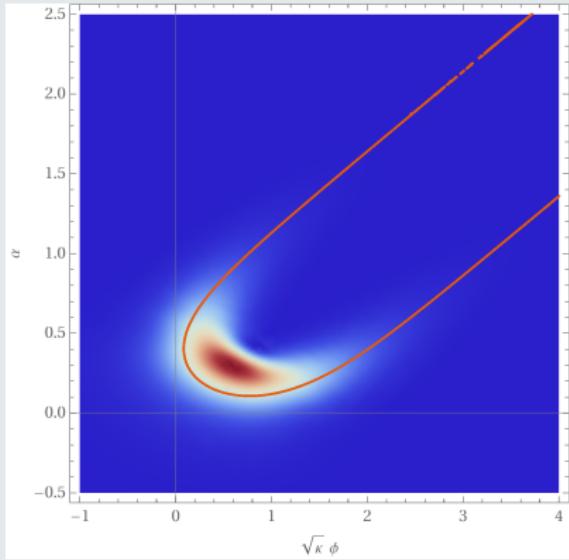
Wave packets of Gaussian amplitude for phantom model

J_ν , with $\lambda = 2\kappa^{1/2}$, $V = +\kappa^{-2}$, $\bar{k}_\beta = 8$ and $\sigma_\beta = 11/2$

Schrödinger



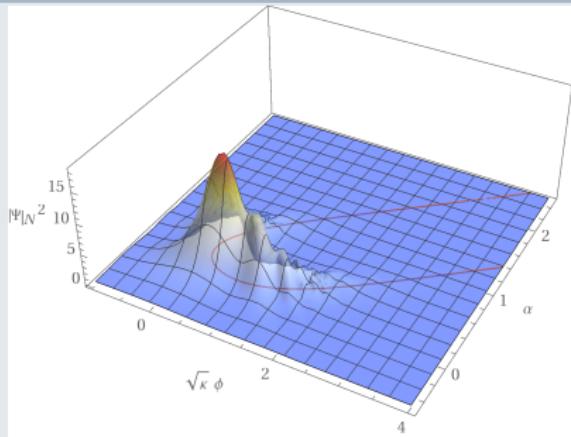
Mostafazadeh



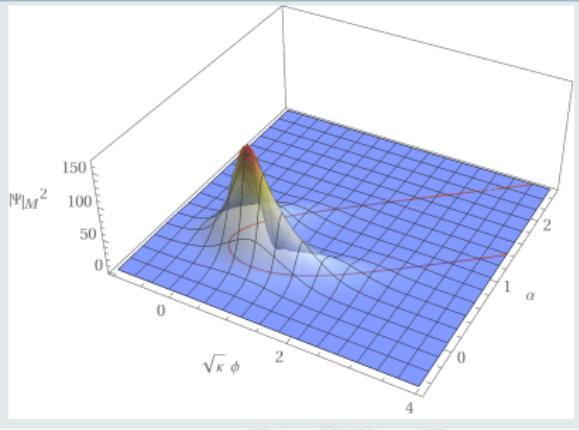
Wave packets of Gaussian amplitude for phantom model

J_ν , with $\lambda = 2\nu^{1/2}$, $V = +\nu^{-2}$, $\bar{k}_\beta = 8$ and $\sigma_\beta = 11/2$

Schrödinger



Mostafazadeh



Outline

1. Introduction
2. The classical model and the implicit trajectories
Lagrangian formalism and the first integral
3. Dirac quantisation and the self-adjointness of Hamiltonian
Dirac quantisation and the quantum-mechanical analogy
The self-adjointness of Hamiltonian
4. The semi-classical wave packets
Inner product and wave packet
5. Conclusions



Highlights

- An **integral of motion** was found for Liouville cosmological models.
- **Implicit trajectories** in minisuperspace were obtained.
- The levels of phantom Liouville model were found to be **discrete** due to the Hermiticity requirement of observables.
- The semi-classical quantum mode functions and wave packets was compared directly with the classical trajectory.



Issues

- In $(+, -)$ and $(-, +)$, wave packets contain multiple branches; however, the classical universe runs only on one trajectory.
- Quantum-corrected \bar{k}_β is to be understood.
- Wave packets with Gaussian profile are to be constructed, instead of inserting Gaussian / Poissonian amplitude artificially.
- A normalising κ for $(\cdot, \cdot)_M^\kappa$ is to be evaluated, otherwise a quantitative comparison of $(\cdot, \cdot)_S$ and $(\cdot, \cdot)_M^\kappa$ is not possible.



Outlook

- Beyond isotropy: generalise to Bianchi models
 - Bianchi Type-I: a natural extension, **under investigation**
- Beyond homogeneity: cosmological perturbation, **under investigation**
- Beyond single field
 - Two exponential potentials: $|V_1| = |V_2|$ and special λ_i
 - Multiple Liouville fields: mixing kinetic terms needed²⁷
- Beyond classic matter
 - PT -symmetric Liouville field²⁸: may cross the phantom divide $w = -1$.

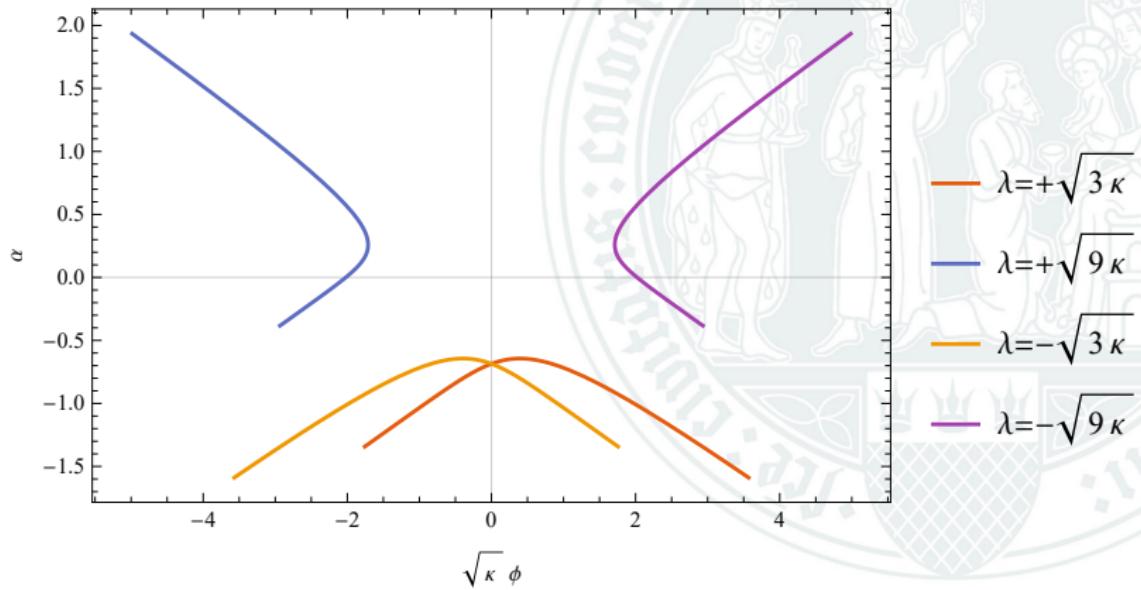
²⁷ A. A. Andrianov *et al.*, *Theor. Math. Phys.* **184**, 1224–1233 (Sept. 2015).

²⁸ A. A. Andrianov *et al.*, *International Journal of Modern Physics D* **19**, 97–111 (Jan. 2010), A. A. Andrianov *et al.*, in *Springer Proceedings in Physics* (Springer International Publishing, 2016), pp. 29–44.



Trajectories for quintessence model $(+, +)$

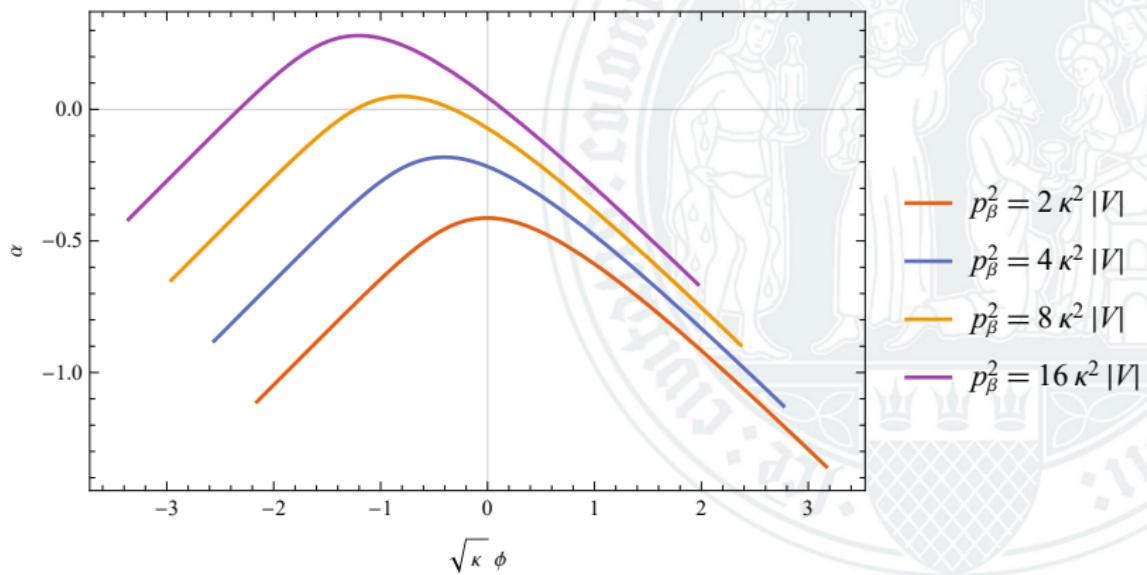
sech, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ



- has two asymptotes $\chi \propto \pm \beta$

Trajectories for quintessence model (+, +)

sech, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $\lambda^2 = 3\kappa$; varying p_β

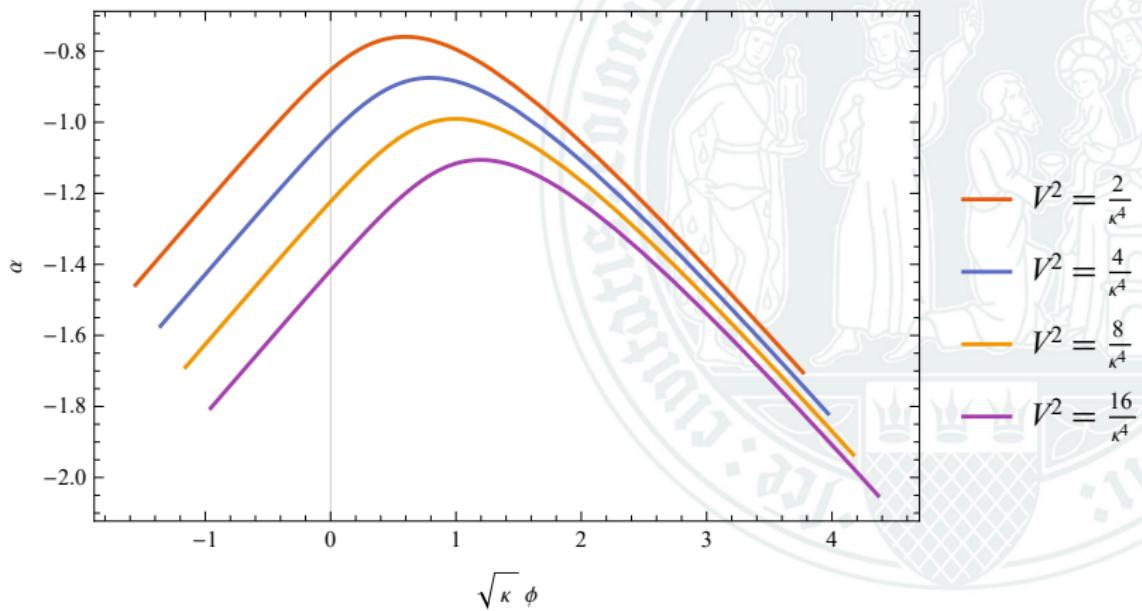


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model (+, +)

sech, with $\beta_0 = 0$, $\lambda^2 = 3\kappa$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying V

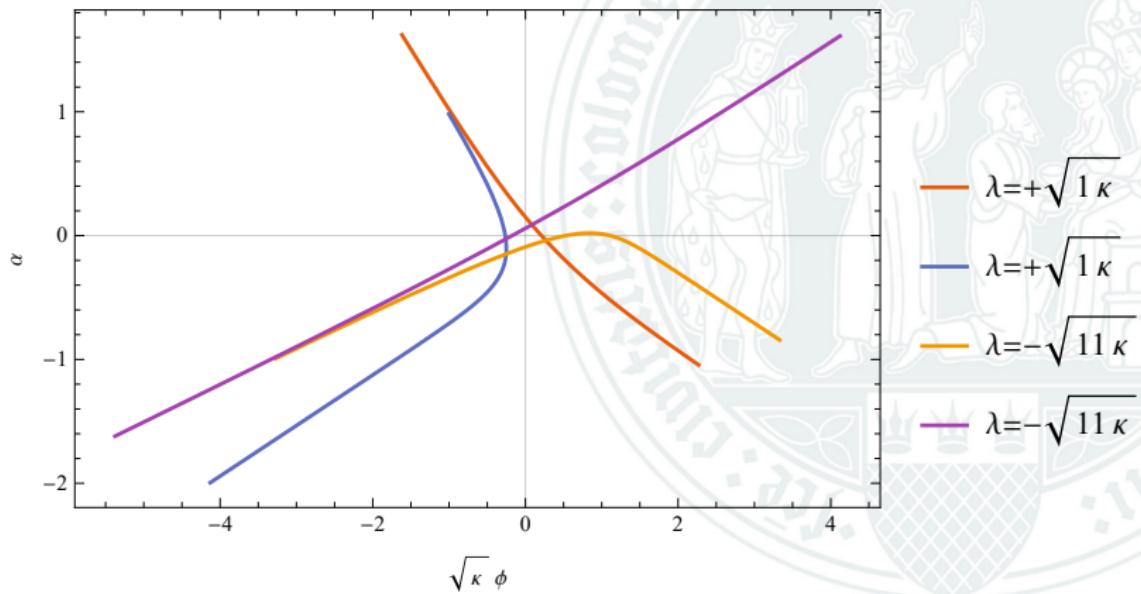


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model (+, -)

csch, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

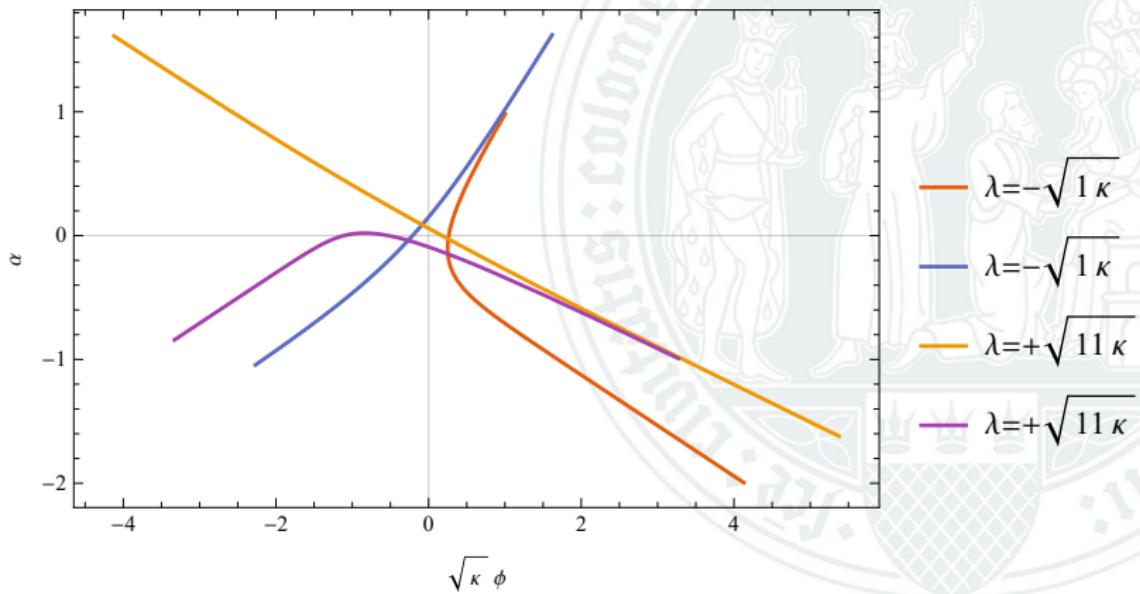


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm \beta$ and $\beta = 0$



Trajectories for quintessence model $(+, -)$: csch

csch, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

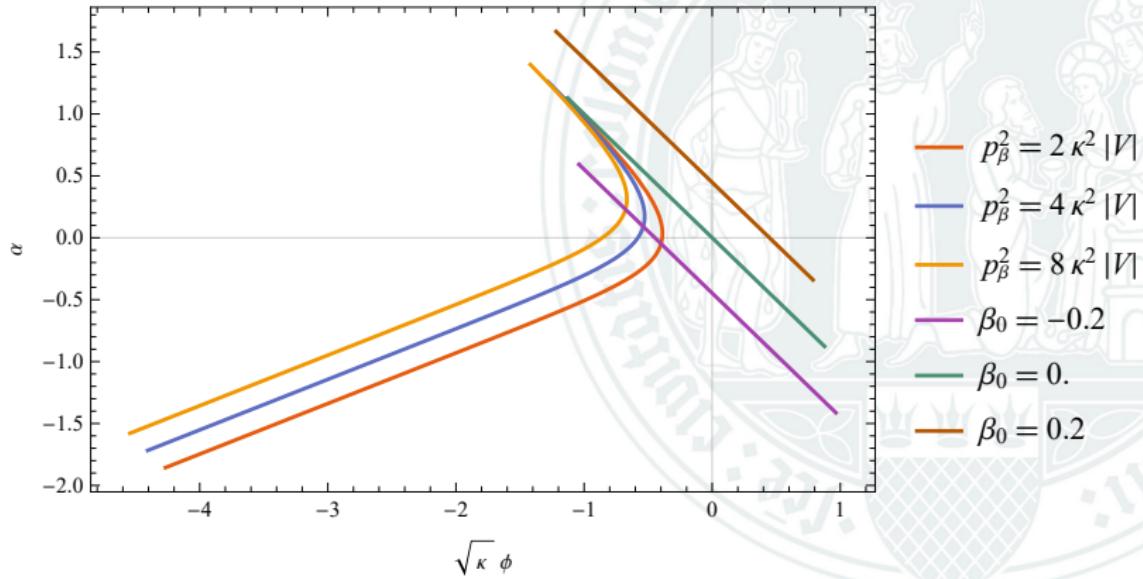


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Trajectories for quintessence model (+, -): csch

csch, with $V = \kappa^{-2}$ and $\lambda^2 = \kappa$; varying p_β

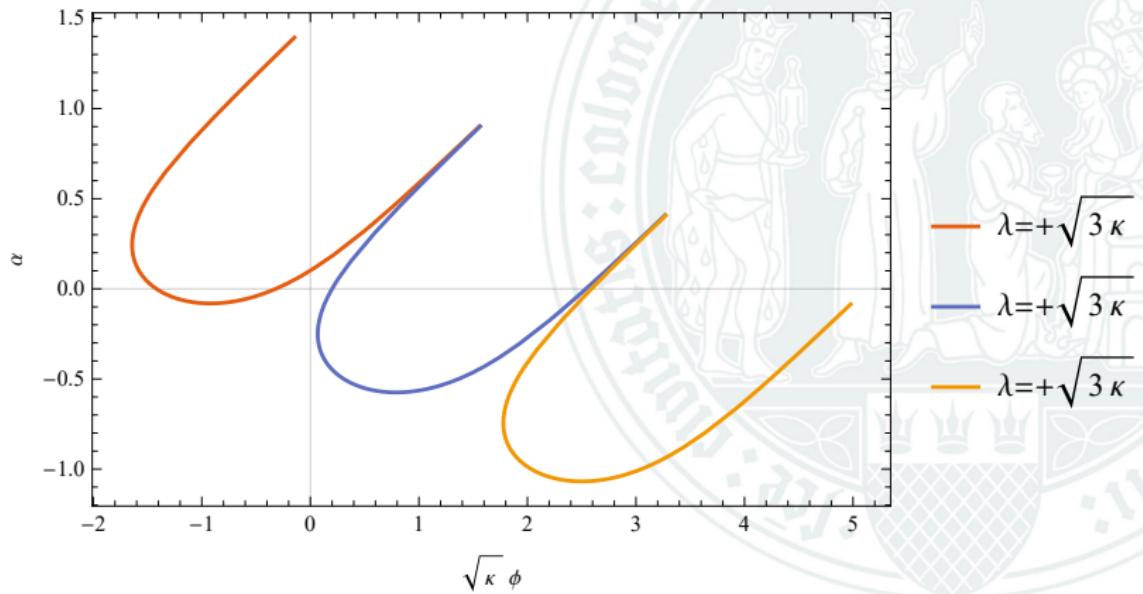


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm \beta$ and $\beta = 0$



Trajectories for phantom model $(-, +)$

csc, with $V = \varkappa^{-2}$ and $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

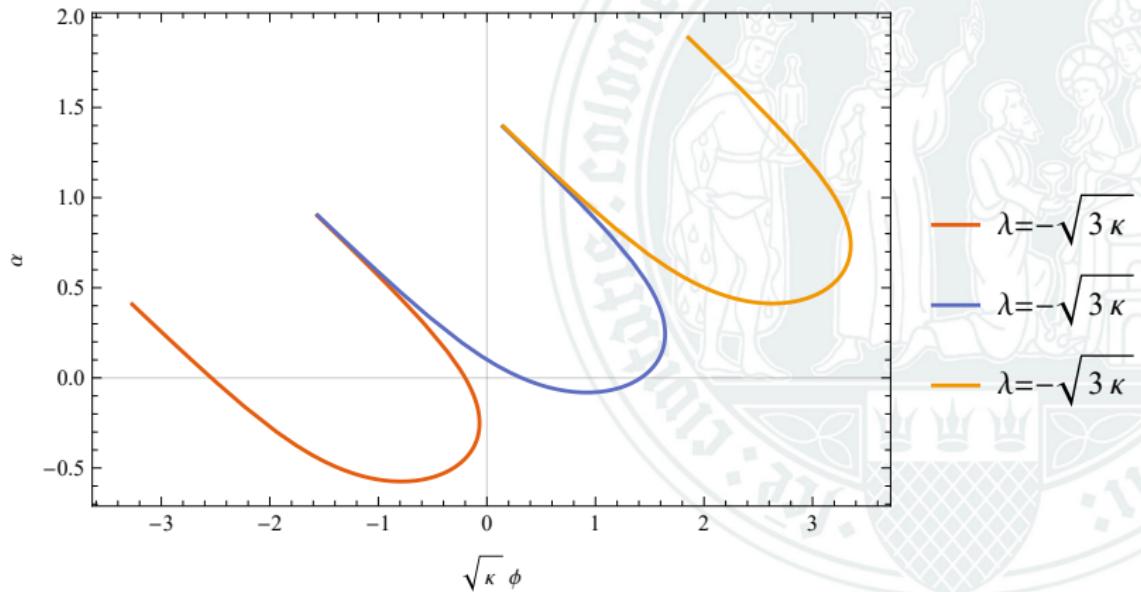


- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Trajectories for phantom model $(-, +)$

csc, with $V = \varkappa^{-2}$ and $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

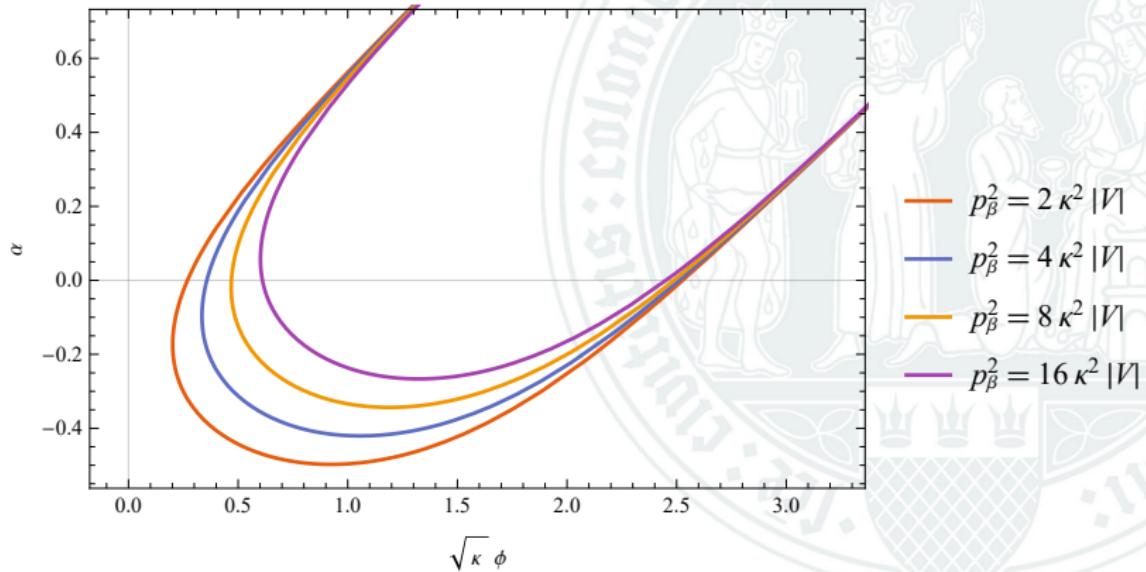


- contains infinite distinct solutions
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Trajectories for phantom model $(-, +)$

csc, with $V = \kappa^{-2}$ and $\lambda^2 = 3\kappa$; varying p_β



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Integration of the transformed first Friedmann equation

General integral for $p_\beta \neq 0$

In order to integrate the equation under the implicit gauge

$$s \frac{p_\beta^2}{12} \left(-\ell \frac{\varkappa^{1/2}}{6} \left(\frac{d\chi}{d\beta} \right)^2 + \varkappa^{-1/2} \right) - \varkappa^{3/2} V e^{g\chi} = 0, \quad (25)$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa} g\beta}, \quad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} e^{-g\chi}, \quad (26)$$

to get

$$\left(\frac{d\tilde{\sigma}}{d\gamma} \right)^2 + \ell(s\upsilon - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{d\gamma}{d\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-s\upsilon + \tilde{\sigma}^2)}}, \quad (27)$$

which is of the standard inverse hyperbolic / trigonometric form **except for $(-, -)$.**



Integration of the separated minisuperspace WDW equation

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \mathfrak{s} v \frac{12\varkappa^2 |V|}{\hbar^2}, \quad (15 \text{ rev.})$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{k_\beta}{g}, \quad \sigma^2 := \frac{8\varkappa^3 |V| e^{g\chi}}{\hbar^2 g^2}, \quad (17 \text{ rev.})$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma \psi'(\sigma) + \ell(\nu^2 - \mathfrak{s} v \sigma^2) \psi(\sigma) = 0, \quad (28)$$

which is of the standard Besselian form.



