

# Integrable Liouville Cosmological Models

## The self-adjointness of Hamiltonian and the Semi-classical Wave Functions

Alexander A. Andrianov<sup>1,4</sup> Chen Lan<sup>2</sup> Oleg O. Novikov<sup>1</sup> Yi-Fan Wang<sup>3</sup>

<sup>1</sup>Saint-Petersburg State University, Ulyanovskaya str. 1, Petrodvorets, Sankt-Petersburg 198504, Russland

<sup>2</sup>ELI-ALPS Research Institute, Budapesti út 5, H-67228 Szeged, Ungarn

<sup>3</sup>Institut für Theoretische Physik, Universität zu Köln, Zülpicher Straße 77, D-50937 Köln, Deutschland

<sup>4</sup>Institut de Ciències del Cosmos, Universitat de Barcelona, Martí i Franquès 1, E-08028 Barcelona, Spanien

March 23, 2018



# Outline

## 1. Introduction

## 2. The classical model and the implicit trajectories

Lagrangian formalism and the first integral

## 3. Dirac quantisation and the self-adjointness of Hamiltonian

Dirac quantisation and the quantum-mechanical analogy

The self-adjointness of Hamiltonian

## 4. The semi-classical wave packets

The Wentzel–Kramers–Brillouin approximation

The inner product and wave packet

## 5. Conclusions



# Outline

1. Introduction
2. The classical model and the implicit trajectories  
Lagrangian formalism and the first integral
3. Dirac quantisation and the self-adjointness of Hamiltonian  
Dirac quantisation and the quantum-mechanical analogy  
The self-adjointness of Hamiltonian
4. The semi-classical wave packets  
The Wentzel–Kramers–Brillouin approximation  
The inner product and wave packet
5. Conclusions



# Introduction

## The Friedmann–Lemaître model with quintessence and phantom Liouville fields

- Dynamical Dark Energy has been modelled by quintessence<sup>1</sup> and phantom<sup>2</sup> matter, which can be realised by minimally-coupled real scalar fields with  $\ell = \pm 1$ <sup>3</sup>

$$\mathcal{S}_L = \int d^4x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \quad (1)$$

- The Liouville field<sup>4</sup>  $\mathcal{V}(\phi) = V e^{\lambda\phi}$  is of interest, where  $\lambda, V \in \mathbb{R}$ .
- Assume a flat Robertson–Walker metric  $g_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + \kappa e^{2\alpha(t)} d\Omega_{3F}^2$ , where  $\kappa := 8\pi G$ ,  $d\Omega_{3F}^2 := d\chi^2 + \chi^2(d\theta^2 + \sin^2 \theta d\varphi^2)$ , and  $N$  lapse function.
- The total action reads  $\mathcal{S} := \mathcal{S}_{EH} + \mathcal{S}_{GHY} + \mathcal{S}_L = \int d\Omega_{3F}^2 \int dt L$ , where

$$L := \kappa^{3/2} N e^{3\alpha} \left( -\frac{3}{\kappa} \frac{\dot{\alpha}^2}{N^2} + \ell \frac{\dot{\phi}^2}{2N^2} - V e^{\lambda\phi} \right). \quad (2)$$

<sup>1</sup>R. R. Caldwell *et al.*, *Phys. Rev. Lett.* **80**, 1582–1585 (Feb. 1998).

<sup>2</sup>R. R. Caldwell, *Phys. Lett. B* **545**, 23–29 (Oct. 2002).

<sup>3</sup>The signature of metric is  $(-, +, \dots, +)$ .

<sup>4</sup>Y. Nakayama, *Int. J. Mod. Phys. A* **19**, 2771–2930 (July 2004).



# Introduction

## Highlights

### Integrability

Implicit trajectories can be obtained explicitly; the minisuperspace Wheeler–DeWitt equation can be integrated exactly.

### Self-adjointness

The Hamiltonian can be asymmetric; imposing self-adjointness leads to removal of degeneracy and discretisation of levels.

### Semi-classical wave functions

Semi-classical wave functions can be constructed by the principle of constructive interference, the Wentzel–Kramers–Brillouin approximation and numerical methods.



# Outline

1. Introduction

## 2. The classical model and the implicit trajectories

Lagrangian formalism and the first integral

## 3. Dirac quantisation and the self-adjointness of Hamiltonian

Dirac quantisation and the quantum-mechanical analogy

The self-adjointness of Hamiltonian

## 4. The semi-classical wave packets

The Wentzel–Kramers–Brillouin approximation

The inner product and wave packet

5. Conclusions



# Decoupling the variables

Via orthogonal transformation

- By rescaling  $\bar{N} := N e^{-3\alpha}$  and the orthogonal transformation

$$\begin{pmatrix} \alpha(t) \\ \phi(t) \end{pmatrix} = \frac{s}{g} \begin{pmatrix} \lambda & -\ell \varkappa \\ -6 & \lambda \end{pmatrix} \begin{pmatrix} \beta(t) \\ \chi(t) \end{pmatrix}, \quad (3)$$

the system can be decoupled ( $\Delta := \lambda^2 - 6\ell\varkappa$ ,  $s := \operatorname{sgn} \Delta$  and  $g := s\sqrt{|\Delta|} \equiv s\sqrt{s\Delta}$ )

$$L = \varkappa^{3/2} \bar{N} \left( -s \frac{3}{\pi} \frac{\dot{\beta}^2}{\bar{N}^2} + \ell s \frac{\dot{\chi}^2}{2\bar{N}^2} - V e^{g\chi} \right). \quad (4)$$

- Using the conservation of  $p_\beta$ <sup>5</sup>,  $\bar{N}(t)$  can be eliminated, leading to the exact integrations of the equations of motion as **implicit trajectories**.
  - $p_\beta \neq 0$ : four cases according to  $(\ell, sv)$ ,  $v := \operatorname{sgn} V$ .
  - $p_\beta = 0$ : well-known power-law special solution for  $(+, -)$  and  $(-, +)$ .

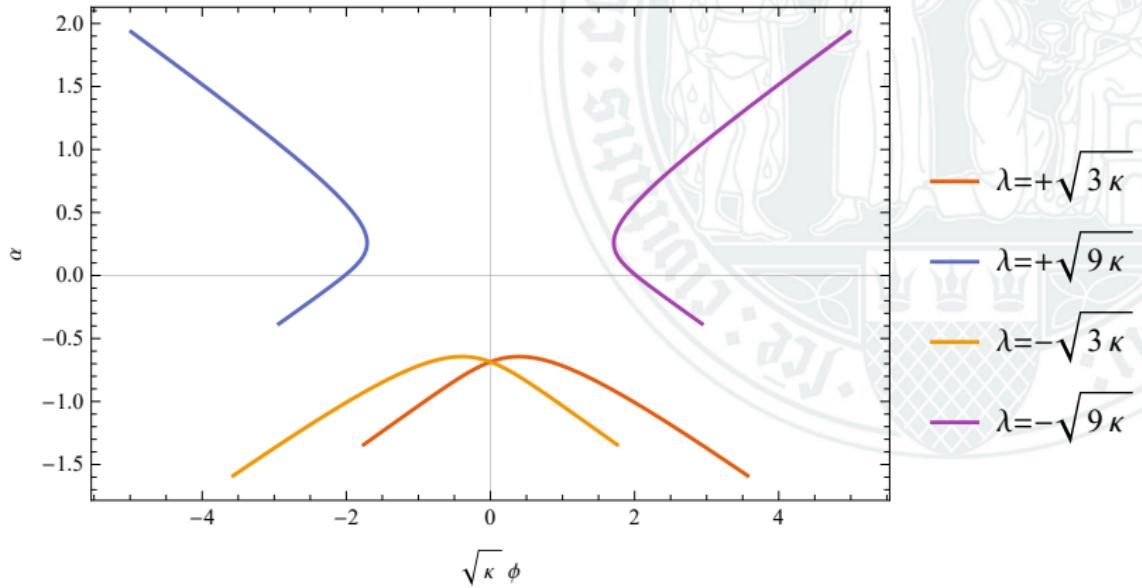
<sup>5</sup>The same first integral has been found in C. Lan, PhD thesis, Saint Petersburg State University, 2016, <https://search.rsl.ru/ru/record/01006663434>, A. A. Andrianov et al., *Theor. Math. Phys.* **184**, 1224–1233 (Sept. 2015), in canonical formalism.



# Trajectories for quintessence model (+, +)

sech, with  $\beta_0 = 0$ ,  $|V| = \kappa^{-2}$  and  $p_\beta^2 = \kappa^2 \sqrt{|V|}$ ; varying  $\lambda$

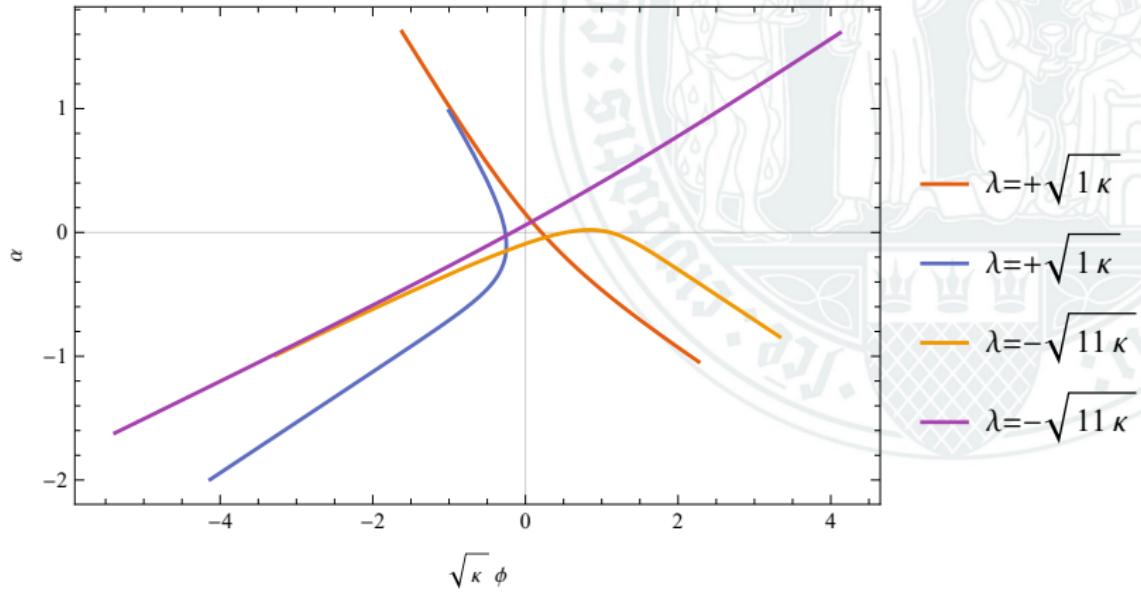
$$e^{g\chi} = \frac{p_\beta^2}{12\kappa^2|V|} \operatorname{sech}^2\left(\sqrt{\frac{3}{2\kappa}} g(\beta - \beta_0)\right) \quad (5)$$



# Trajectories for quintessence model $(+, -)$

$\text{csch}$ , with  $\beta_0 = 0$ ,  $|V| = \kappa^{-2}$  and  $p_\beta^2 = \kappa^2 \sqrt{|V|}$ ; varying  $\lambda$

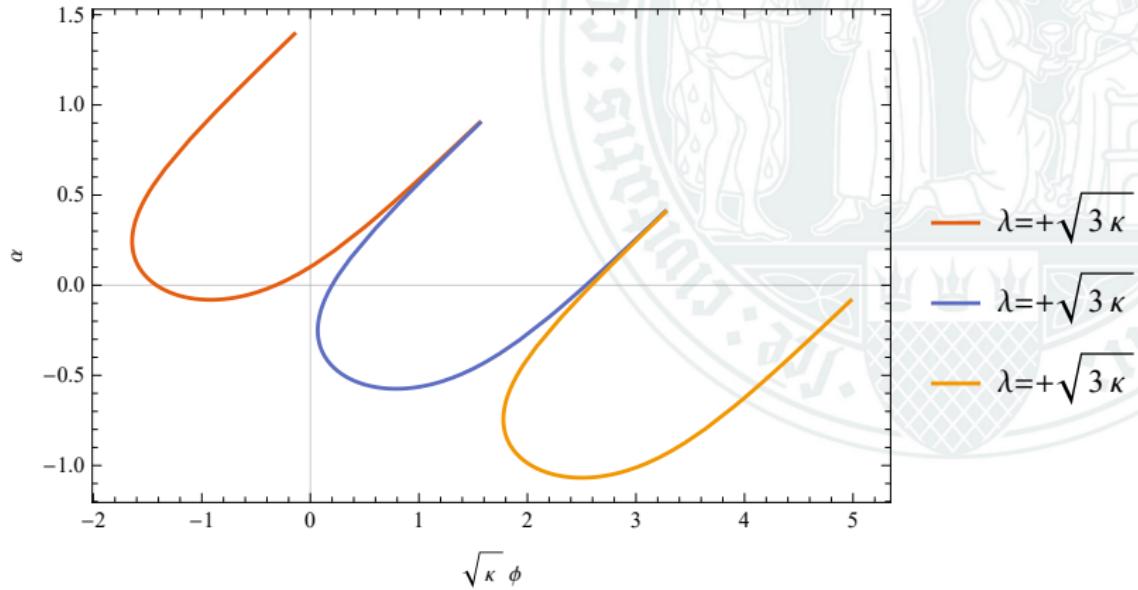
$$e^{g\chi} = \frac{p_\beta^2}{12\kappa^2|V|} \text{csch}^2\left(\sqrt{\frac{3}{2\kappa}} g(\beta - \beta_0)\right) \quad (6)$$



# Trajectories for phantom model $(-, +)$

sec, with  $V = \varkappa^{-2}$  and  $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

$$\mathbb{E}^{g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|} \sec^2\left(\sqrt{\frac{3}{2\varkappa}} g(\beta - \beta_0)\right) \quad (7)$$



# Outline

1. Introduction
2. The classical model and the implicit trajectories  
Lagrangian formalism and the first integral
3. Dirac quantisation and the self-adjointness of Hamiltonian  
Dirac quantisation and the quantum-mechanical analogy  
The self-adjointness of Hamiltonian
4. The semi-classical wave packets  
The Wentzel–Kramers–Brillouin approximation  
The inner product and wave packet
5. Conclusions



# Dirac quantisation

- The primary Hamiltonian and the Hamiltonian constraint<sup>6</sup> reads

$$H^p = \bar{N} H_{\perp} + v^{\bar{N}} p_{\bar{N}}, \quad (8)$$

$$H_{\perp} := -\beta \frac{p_{\beta}^2}{12\nu^{1/2}} + \ell \beta \frac{p_{\chi}^2}{2\nu^{3/2}} + \nu^{3/2} V e^{g\chi}. \quad (9)$$

- Applying the Dirac quantisation rules with the Laplace–Beltrami operator ordering<sup>7</sup>, one gets the minisuperspace Wheeler–DeWitt equation with  $(\beta, \chi)$

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left( \beta \frac{\hbar^2}{12\nu^{1/2}} \partial_{\beta}^2 - \ell \beta \frac{\hbar^2}{2\nu^{3/2}} \partial_{\chi}^2 + \nu^{3/2} V e^{g\chi} \right) \Psi. \quad (10)$$

- Equation (10) is Klein–Gordon-like, hyperbolic for  $\ell = +1$  and *elliptic* for  $\ell = -1$ .

---

<sup>6</sup>D. M. Gitman, I. V. Tyutin, *Quantization of Fields with Constraints*, (Springer, 1990), H. J. Rothe, K. D. Rothe, *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*, (World Scientific, Apr. 2010).

<sup>7</sup>C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012), ch. 8.



# Integration of the Wheeler–DeWitt equation

The quantum-mechanical analogy

- Fourier transforming  $\beta$  to  $k_\beta \in \mathbb{R}$  yields the time-independent Schrödinger equation

$$E\psi(x) := \ell \frac{\hbar^2 \tilde{g}^2 \nu^2}{8M_P} \psi = \widehat{H}_{\text{eff}} \psi := \left( -\frac{\hbar^2}{2M_P} \partial_x^2 + V_{\text{eff}} \right) \psi, \quad V_{\text{eff}} := \ell s v \widetilde{V} e^{\tilde{g}x}, \quad (11)$$

in which  $M_P := \sqrt{\frac{\hbar}{\nu}}$ ,  $x := \sqrt{\hbar\nu} \cdot \chi$ ,  $\widetilde{V} := M_P^{-3} |V| \geq 0$ , and  $\nu = \sqrt{\frac{2\nu}{3}} \frac{|k_\beta|}{\tilde{g}} \geq 0$ .

- The Liouville potential  $V_{\text{eff}}$  is a special case of the Morse potential<sup>8</sup>.
- Let  $e^{2y} := 8M_P \widetilde{V} / \hbar^2 \tilde{g}^2 \cdot e^{\tilde{g}x}$

|                      | $E \geq 0$                             | $E < 0$              |
|----------------------|--|----------------------|
| $V_{\text{eff}} > 0$ | (+, +); $K_{i\nu}(e^y)$                | (-, -); none         |
| $V_{\text{eff}} < 0$ | (+, -); $F_{i\nu}(e^y), G_{i\nu}(e^y)$ | (-, +); $J_\nu(e^y)$ |

<sup>8</sup>P. M. Morse, *Phys. Rev.* **34**, 57–64 (July 1929), D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012).



# Self-adjointness of the Hamiltonian

Results for the Liouville potential<sup>12</sup>

- $k_\beta \in \mathbb{R}$ , enforcing the self-adjointness of  $\widehat{H}_{\text{eff}}$ .
- For  $V_{\text{eff}} > 0$ :  $\widehat{H}_{\text{eff}}$  is **essentially self-adjoint**<sup>9</sup>.
  - Quintessence (+, +): nothing special.
- For  $V_{\text{eff}} < 0$ :  $\widehat{H}_{\text{eff}}$  admits a **family of self-adjoint extensions**.
  - Quintessence (+, -): "eigenfunctions" are lin. combi. of F and G; **the degeneracy is removed**
  - Phantom (-, +): the eigenfunctions are **discretised**

$$\psi_{-, \nu}(e^y) \propto J_\nu(e^y), \quad \nu = 2n + a_<, \quad n \in \mathbb{Z}_\geq, \quad (12)$$

and the corresponding full wave-functions are Bloch-periodic Imposing (anti-)periodic boundary condition fixes  $a_< = 0$  (1).

- The self-adjointness has been discussed in quantum cosmology in e.g.<sup>10</sup> (operator ordering) and more recently in<sup>11</sup> ( $\Lambda$  as a quantum number).

<sup>9</sup>See also B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267 , sec. 9.9.

<sup>10</sup>C. R. Almeida et al., *Gravitation Cosmol.* **21**, 191–199 (July 2015).

<sup>11</sup>S. Gryb, K. P. Y. Thébault, arXiv: 1801.05789 (gr-qc) (Jan. 17, 2018).

<sup>12</sup>See also D. M. Gitman et al., *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) , sec. 8.5.



# Outline

1. Introduction
2. The classical model and the implicit trajectories  
Lagrangian formalism and the first integral

3. Dirac quantisation and the self-adjointness of Hamiltonian  
Dirac quantisation and the quantum-mechanical analogy  
The self-adjointness of Hamiltonian

4. The semi-classical wave packets  
The Wentzel–Kramers–Brillouin approximation  
The inner product and wave packet

5. Conclusions



# Matching quantum number with classical first integral

Principle of constructive interference;  $(+, +)$  as the example

- Write the mode function in the Wentzel–Kramers–Brillouin form  
 $\Psi_\nu(\beta, \chi) \sim \sqrt{\rho(\beta, \chi)} \exp\left\{\frac{i}{\hbar} S(\beta, \chi)\right\}$ . For  $S/\hbar \gg 1$  and  $\nu \gg 1$ ,  $S(\beta, \chi)$  becomes the Hamilton principal function in the leading-order approximation, which is stationary with respect to the variation of integral constants  $\partial S / \partial \nu = 0$ .
- Fixing  $\nu/\sigma > 1$ ,

$$K_{\pm\nu}(\sigma) \sim \frac{\sqrt{2\pi} \cos\left(\sqrt{\nu^2 - \sigma^2} - \nu \arccos \frac{\nu}{\sigma} - \frac{\pi}{4}\right)}{(\nu^2 - \sigma^2)^{1/4} e^{\mp\nu/2}} + O\left(\frac{1}{\sigma}\right). \quad (13)$$

- Applying *the principle* to the phase of  $\Psi_\nu(\sigma)$ , one has  $\sigma/\nu = \operatorname{sech}(\beta_\beta \gamma)$ , which matches the trajectory if  $\hbar k_\beta \equiv \hbar \sqrt{\frac{3}{2\nu}} g\nu = p_\beta$ .
- Fixing  $\nu/\sigma < 1$ , the asymptotic expansion is not oscillatory, but exponential; it is not within the Wentzel–Kramers–Brillouin regime.
- The conclusions also hold for  $F_{\pm\nu}(\sigma)$ ,  $G_{\pm\nu}(\sigma)$  for  $(+, -)$ , and  $J_\nu(\sigma)$  for  $(-, +)$ .



# Inner products for wave functions

Schrödinger and Mostafazadeh inner product

- A common starting point is the Schrödinger product<sup>13</sup>

$$(\Psi_1, \Psi_2)_S := \int d\chi \Psi_1^*(\beta, \chi) \Psi_2(\beta, \chi); \quad (14)$$

- $(\Psi, \Psi)_S$  is positive-definite, and the integrand  $\rho_S(\beta, \chi)$  is non-negative
- The corresponding Schrödinger current is real but does not satisfy continuity equation
- Mostafazadeh<sup>14</sup> found an inner product for self-adjoint  $\mathbb{D}$  with positive spectrum:

$$(\Psi_1, \Psi_2)_M^\kappa := \kappa \left\{ (\Psi_1, \mathbb{D}^{+1/2} \Psi_2)_S + (\partial_\beta \Psi_1, \mathbb{D}^{-1/2} \partial_\beta \Psi_2)_S \right\}, \quad \kappa > 0. \quad (15)$$

- Real power of  $\mathbb{D}$  is defined by spectral decomposition
- It can be shown<sup>15</sup> that the density

$$\varrho_M^\kappa := \kappa \left\{ |\mathbb{D}^{+1/4} \Psi|^2 + |\mathbb{D}^{-1/4} \partial_\beta \Psi|^2 \right\} \quad (16)$$

- is equivalent to the integrand  $\rho_M^\kappa$  up to a boundary term  $\int d\chi \varrho_M^\kappa = \int d\chi \rho_M^\kappa \equiv (\Psi_1, \Psi_2)_M^\kappa$
- is non-negative
- The corresponding current  $\vec{\mathcal{J}}_M^\kappa$  is real but not conserved<sup>16</sup>.

<sup>13</sup>C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012), ch. 5.

<sup>14</sup>A. Mostafazadeh, *Class. Quantum Grav.* **20**, 155–171 (Dec. 2002).

<sup>15</sup>A. Mostafazadeh, F. Zamani, *Ann. Phys.* **321**, 2183–2209 (Sept. 2006).

<sup>16</sup>B. Rosenstein, L. P. Horwitz, *J. Phys. A: Math. Gen.* **18**, 2115–2121 (Aug. 1985).



# Wave packets of Gaussian amplitude for continuous spectrum

Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu; \bar{\nu}, \sigma) := \left( \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\nu - \bar{\nu})^2}{2\sigma^2}\right) \right)^{1/2} \quad (17)$$

- In<sup>17</sup>,  $A(\nu; \bar{\nu}, \sigma/\sqrt{2})$  was chosen.

---

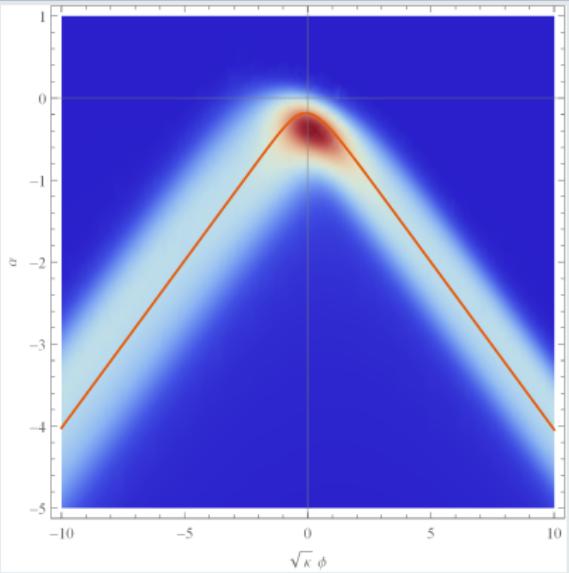
<sup>17</sup> M. P. Dąbrowski et al., *Phys. Rev. D* **74**, arXiv: hep-th/0605229 (hep-th) (Aug. 2006).



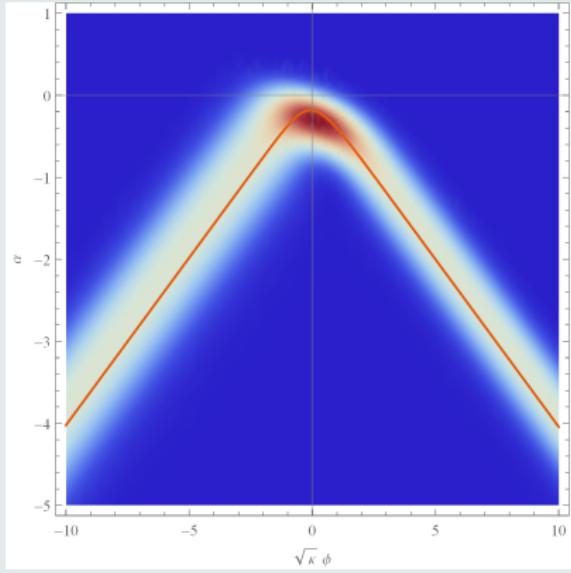
# Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\ell\nu}$ , with  $\lambda = \varkappa^{1/2}/2$ ,  $V = -\varkappa^{-2}$ ,  $\bar{k}_\beta = -2$  and  $\sigma_\beta = 5/4$

Schrödinger



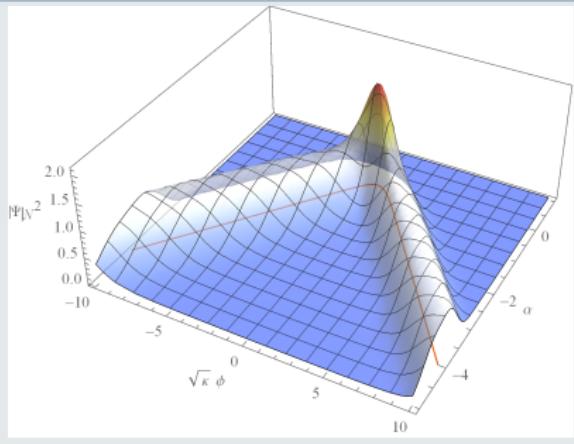
Mostafazadeh



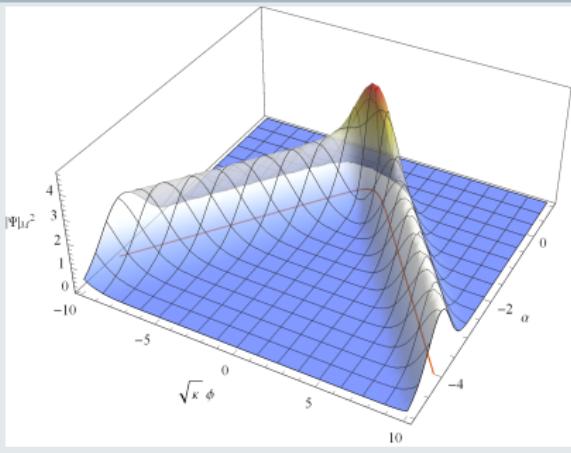
# Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\ell\nu}$ , with  $\lambda = \varkappa^{1/2}/2$ ,  $V = -\varkappa^{-2}$ ,  $\bar{k}_\beta = -2$  and  $\sigma_\beta = 5/4$

Schrödinger



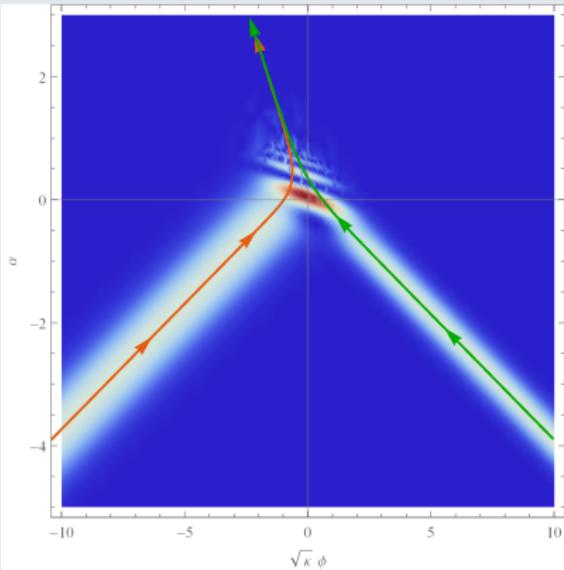
Mostafazadeh



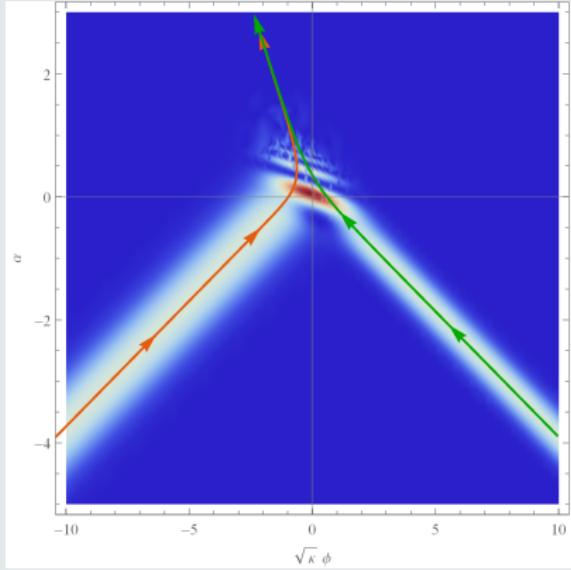
# Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\parallel\nu}$ , with  $\lambda = 4\nu^{1/2}/5$ ,  $V = +\nu^{-2}$ ,  $\bar{k}_\beta = -7/2$  and  $\sigma_\beta = 7/5$

Schrödinger



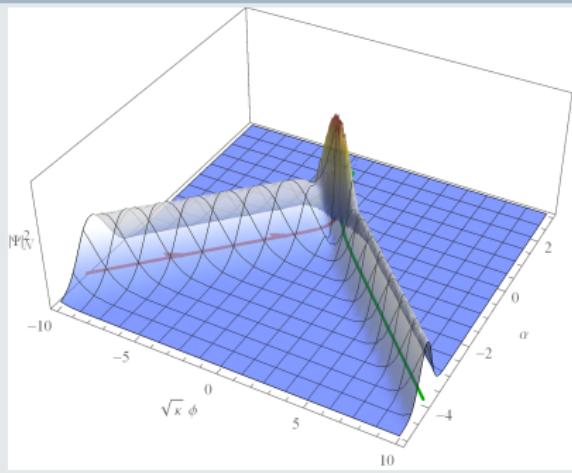
Mostafazadeh



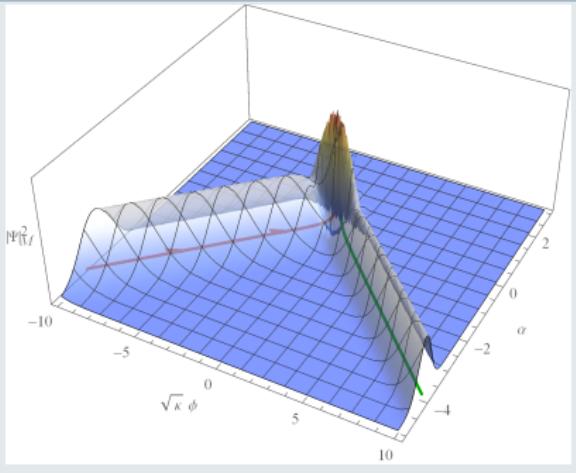
# Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\parallel\nu}$ , with  $\lambda = 4\nu^{1/2}/5$ ,  $V = +\nu^{-2}$ ,  $\bar{k}_\beta = -7/2$  and  $\sigma_\beta = 7/5$

Schrödinger



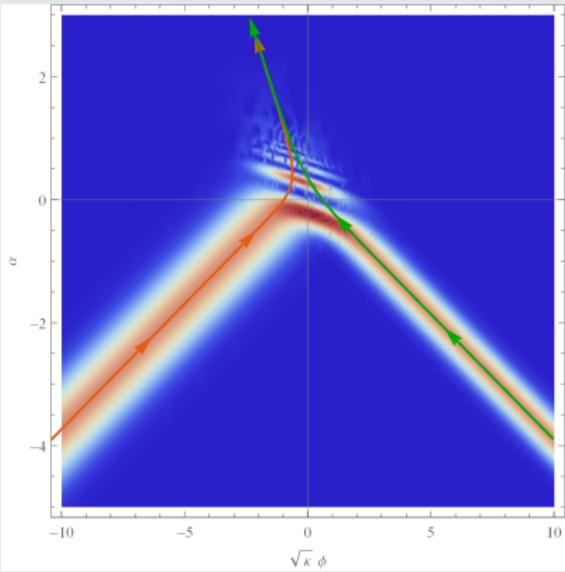
Mostafazadeh



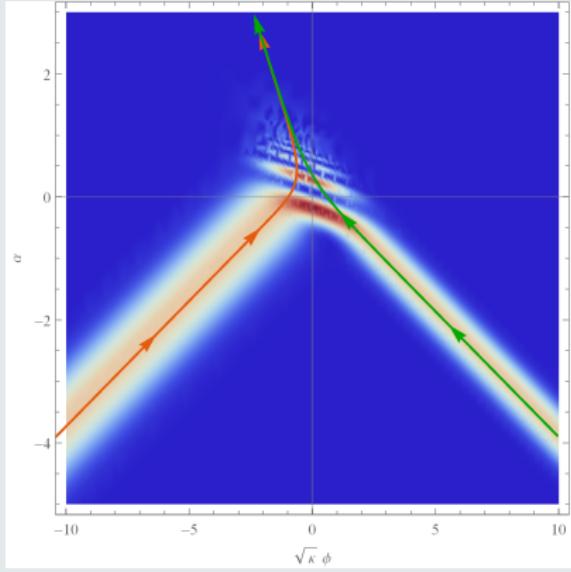
# Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\parallel\nu}$ , with  $\lambda = 4\kappa^{1/2}/5$ ,  $V = +\kappa^{-2}$ ,  $\bar{k}_\beta = -7/2$  and  $\sigma_\beta = 7/5$

Schrödinger



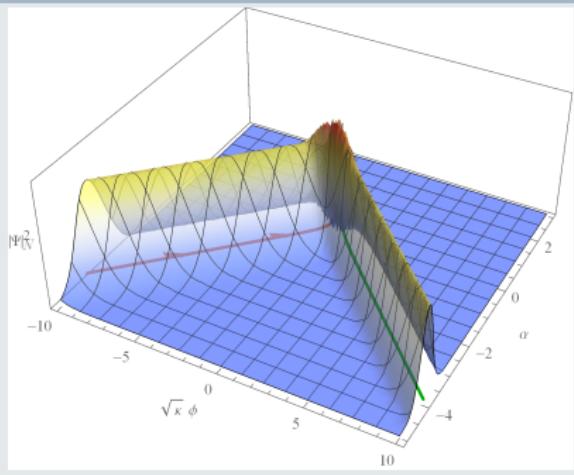
Mostafazadeh



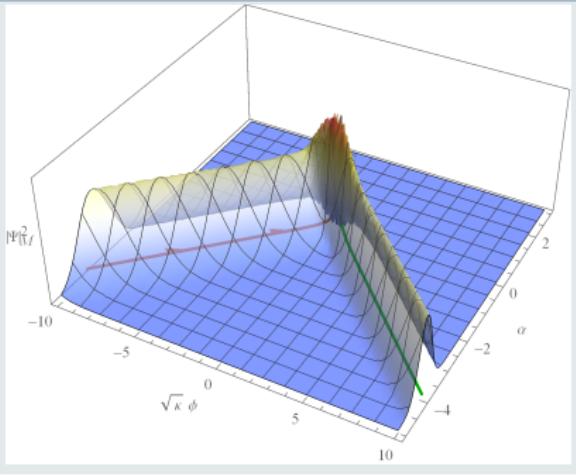
# Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\parallel\nu}$ , with  $\lambda = 4\kappa^{1/2}/5$ ,  $V = +\kappa^{-2}$ ,  $\bar{k}_\beta = -7/2$  and  $\sigma_\beta = 7/5$

Schrödinger



Mostafazadeh



# Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model  $(-, +)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\bar{n}) := \left( e^{-\bar{n}} \frac{\bar{n}^n}{n!} \right)^{1/2} \quad (18)$$

- In<sup>18</sup>C,  $A_n(\bar{n}/\sqrt{2})$  was chosen.

---

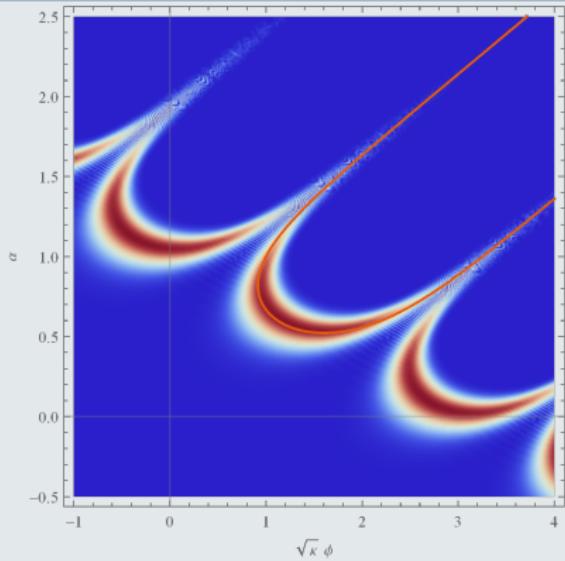
<sup>18</sup>C. Kiefer, *Nucl. Phys. B* **341**, 273–293 (Sept. 1990).



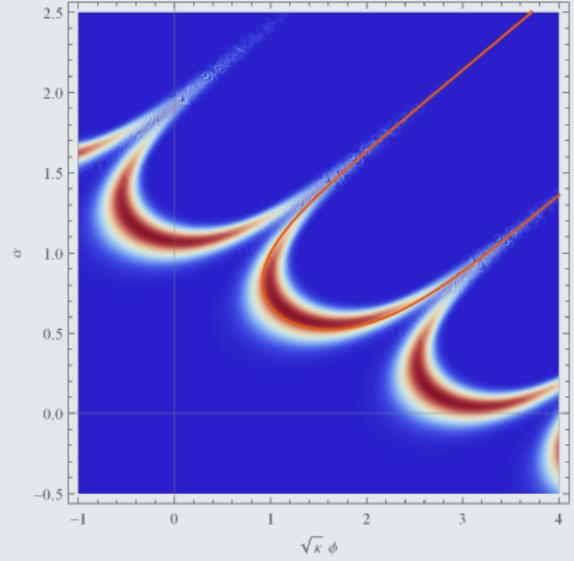
# Wave packets of Poissonian amplitude for phantom model

$J_{2n+1}$ , with  $\lambda = 2\kappa^{1/2}$ ,  $V = +\kappa^{-2}$  and  $\bar{k}_\beta = 8$

Schrödinger



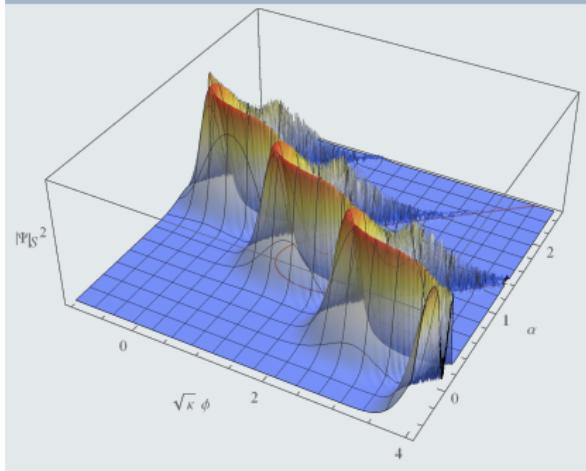
Mostafazadeh



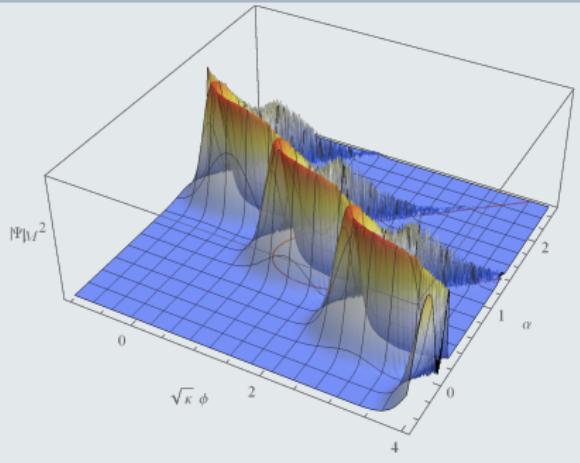
# Wave packets of Poissonian amplitude for phantom model

$J_{2n+1}$ , with  $\lambda = 2\kappa^{1/2}$ ,  $V = +\kappa^{-2}$  and  $\bar{k}_\beta = 8$

Schrödinger



Mostafazadeh



# Outline

1. Introduction
2. The classical model and the implicit trajectories  
Lagrangian formalism and the first integral
3. Dirac quantisation and the self-adjointness of Hamiltonian  
Dirac quantisation and the quantum-mechanical analogy  
The self-adjointness of Hamiltonian
4. The semi-classical wave packets  
The Wentzel–Kramers–Brillouin approximation  
The inner product and wave packet
5. Conclusions



# Highlights

Revisited

- An **integral of motion** was found for the Liouville cosmological model.
  - Implicit trajectories in minisuperspace were obtained explicitly.
  - The minisuperspace Wheeler–Dewitt equation was integrated exactly.
- The **self-adjointness** of matter Hamiltonian was found to be non-trivial.
  - The degeneracy of the quintessence model  $(+, -)$  was removed.
  - The levels of the phantom model  $(-, +)$  were discretised.
- The **semi-classical wave functions** were compared with the classical trajectory.
  - Mode functions in the Wentzel–Kramers–Brillouin approximation were matched with the classical Hamilton principal function.
  - Semi-classical wave packets were constructed and compared with the classical trajectory under the Schrödinger and Mostafazadeh inner product.



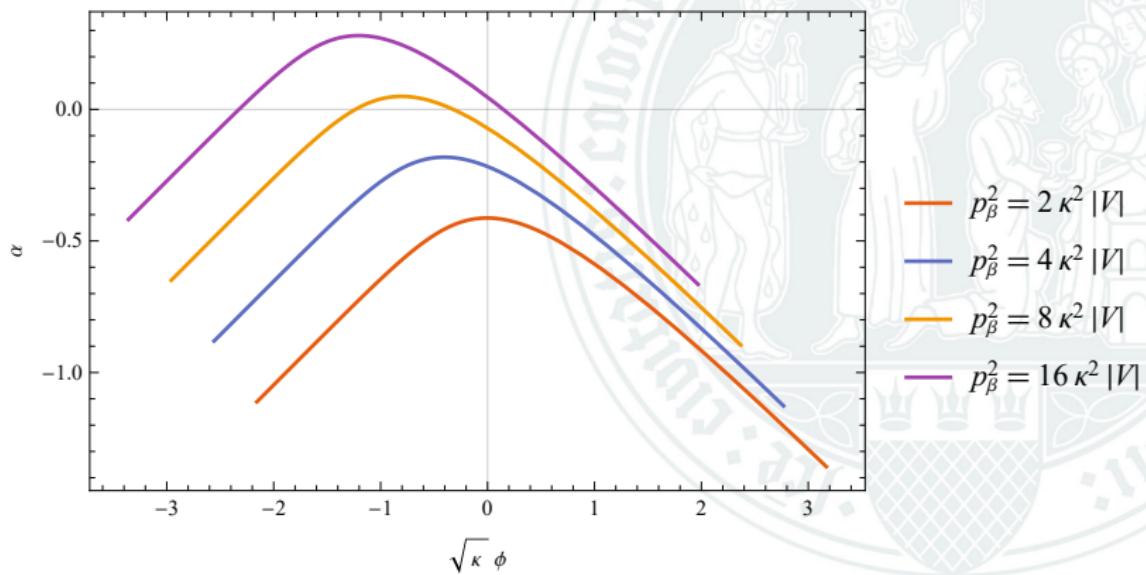
# Outlook

- Beyond isotropy: Bianchi models
- Beyond homogeneity: cosmological perturbations
- Beyond single Liouville field: multiple potentials / multiple fields
- Beyond classical matter and Schrödinger inner product:  $PT$ -symmetric field



# Trajectories for quintessence model (+, +)

sech, with  $\beta_0 = 0$ ,  $|V| = \kappa^{-2}$  and  $\lambda^2 = 3\kappa$ ; varying  $p_\beta$

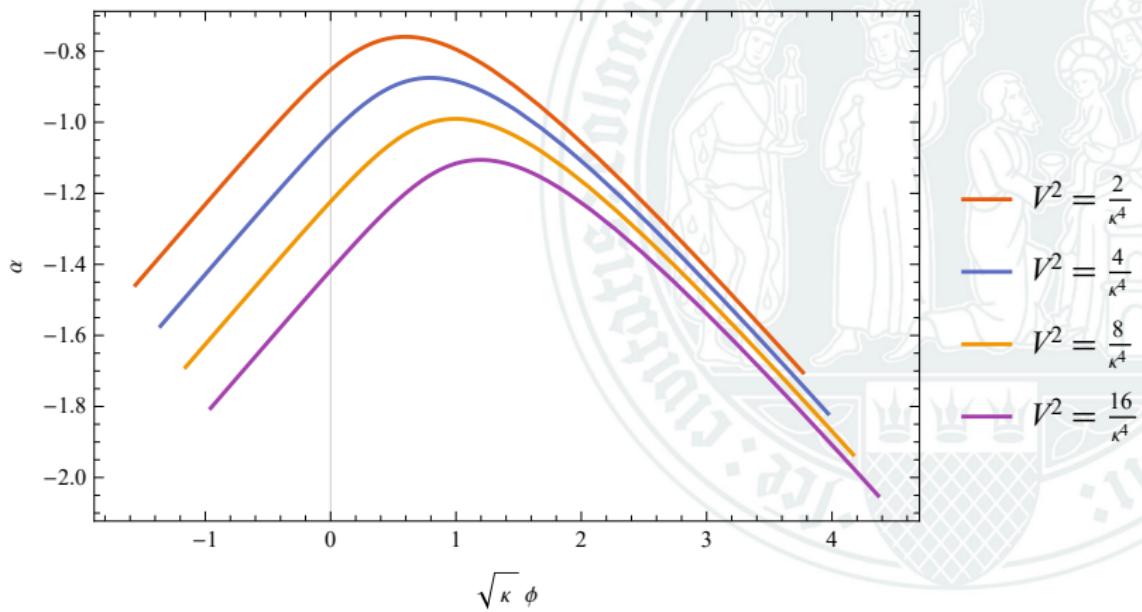


- has two asymptotes  $\chi \propto \pm \beta$



# Trajectories for quintessence model (+, +)

sech, with  $\beta_0 = 0$ ,  $\lambda^2 = 3\kappa$  and  $p_\beta^2 = \kappa^2 \sqrt{|V|}$ ; varying  $V$

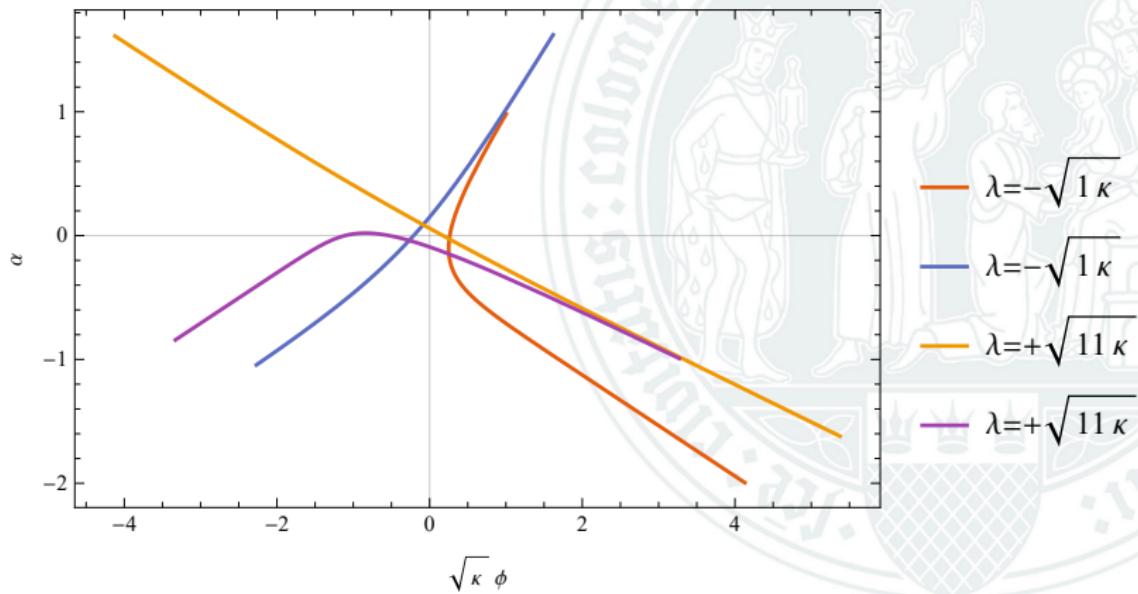


- has two asymptotes  $\chi \propto \pm \beta$



# Trajectories for quintessence model $(+, -)$ : csch

csch, with  $\beta_0 = 0$ ,  $|V| = \kappa^{-2}$  and  $p_\beta^2 = \kappa^2 \sqrt{|V|}$ ; varying  $\lambda$

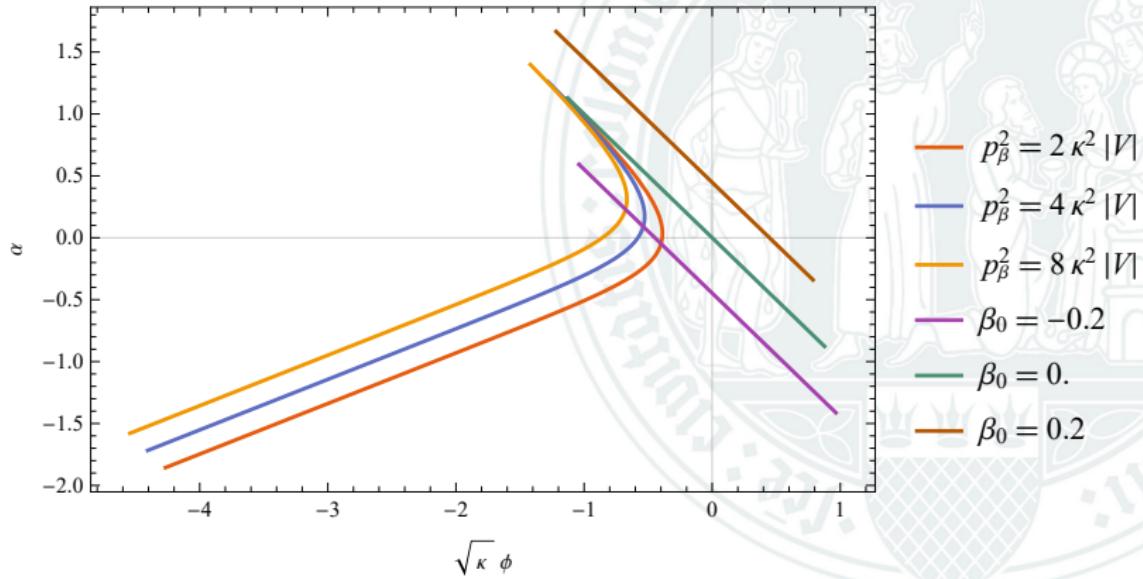


- contains two distinct solutions, separated by  $\beta = 0$
- has three asymptotes  $\chi \propto \pm \beta$  and  $\beta = 0$



# Trajectories for quintessence model (+, -): csch

csch, with  $V = \kappa^{-2}$  and  $\lambda^2 = \kappa$ ; varying  $p_\beta$

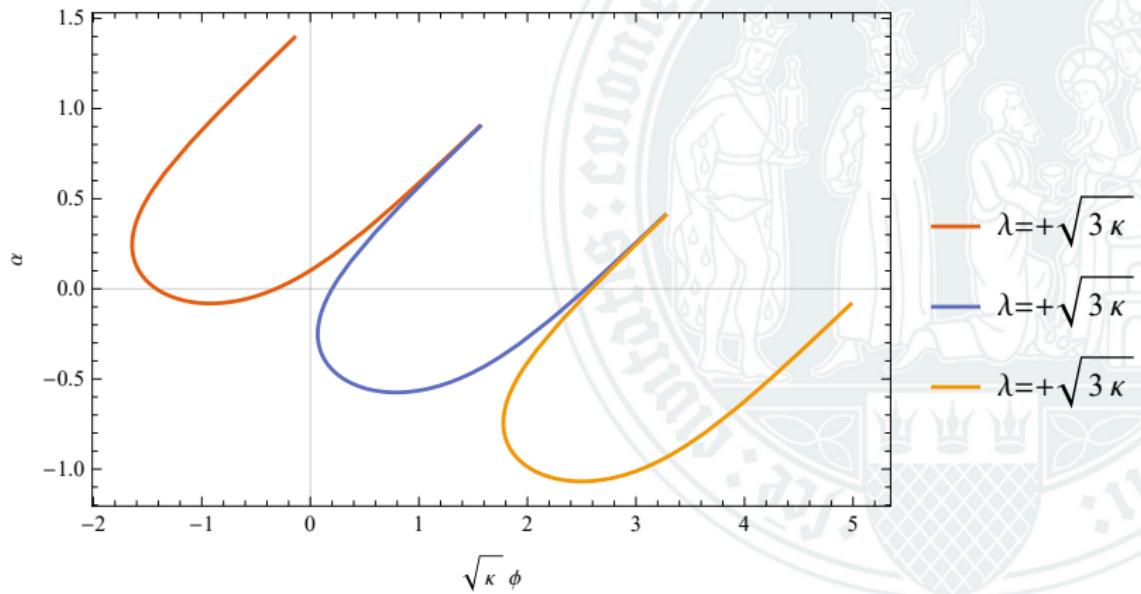


- contains two distinct solutions, separated by  $\beta = 0$
- has three asymptotes  $\chi \propto \pm \beta$  and  $\beta = 0$



# Trajectories for phantom model $(-, +)$

sec, with  $V = \varkappa^{-2}$  and  $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

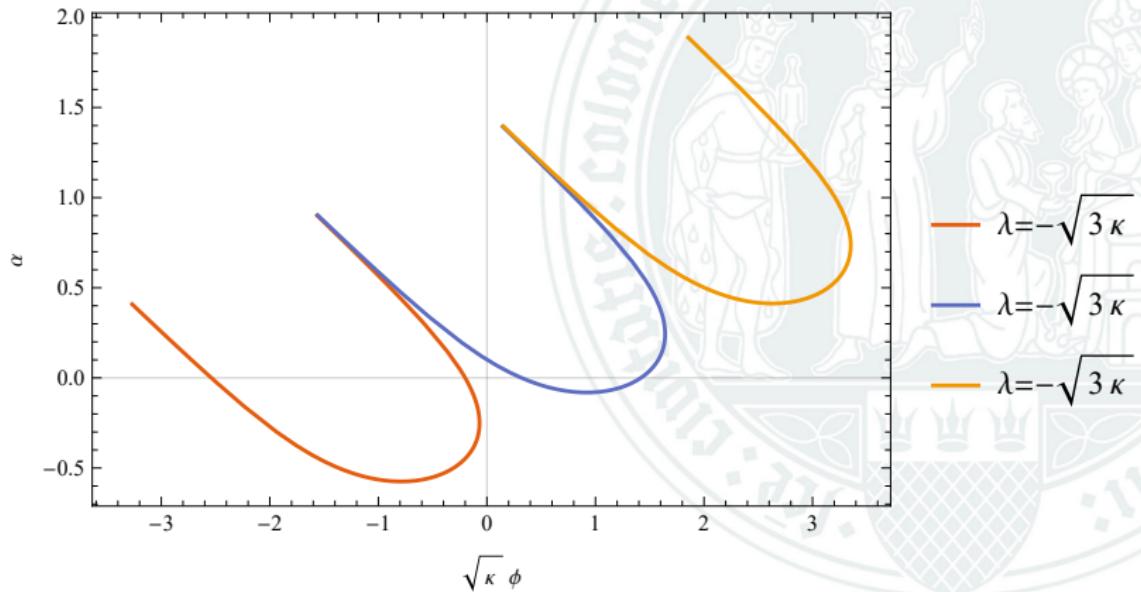


- contains infinite distinct solutions
- has infinite parallel asymptotes  $\beta \propto (n + 1/2)\pi$



# Trajectories for phantom model $(-, +)$

csc, with  $V = \varkappa^{-2}$  and  $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

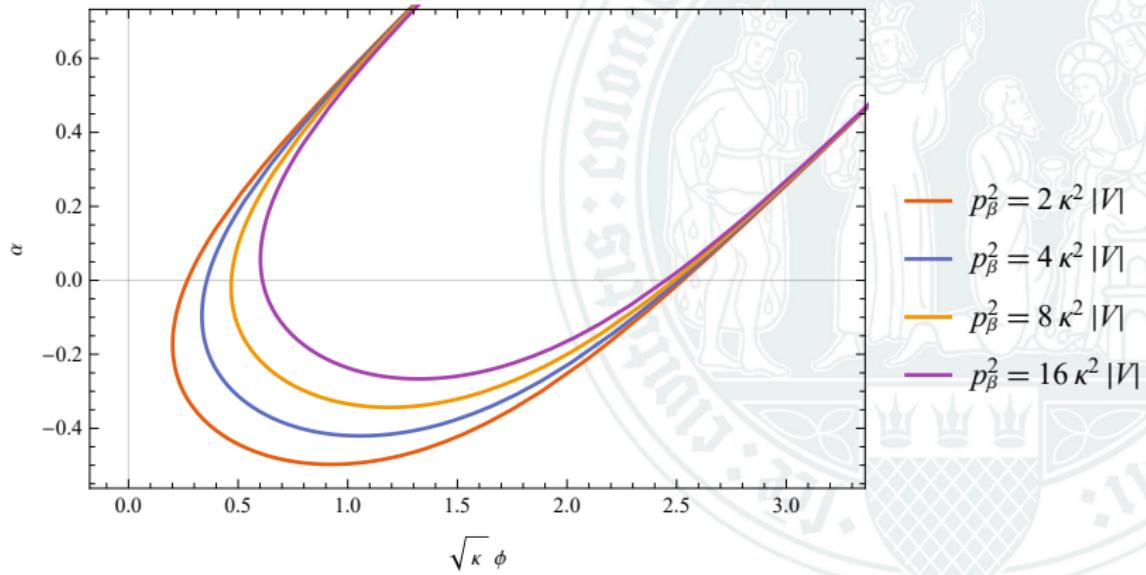


- contains infinite distinct solutions
- has infinite parallel asymptotes  $\beta \propto (n + 1/2)\pi$



# Trajectories for phantom model $(-, +)$

csc, with  $V = \kappa^{-2}$  and  $\lambda^2 = 3\kappa$ ; varying  $p_\beta$



- contains infinite distinct solutions
- has infinite parallel asymptotes  $\beta \propto (n + 1/2)\pi$



# Integration of the transformed first Friedmann equation

General integral for  $p_\beta \neq 0$

In order to integrate the equation under the implicit gauge

$$s \frac{p_\beta^2}{12} \left( -\ell \frac{\varkappa^{1/2}}{6} \left( \frac{d\chi}{d\beta} \right)^2 + \varkappa^{-1/2} \right) - \varkappa^{3/2} V e^{g\chi} = 0, \quad (19)$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa} g\beta}, \quad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} e^{-g\chi}, \quad (20)$$

to get

$$\left( \frac{d\tilde{\sigma}}{d\gamma} \right)^2 + \ell(s\upsilon - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{d\gamma}{d\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-s\upsilon + \tilde{\sigma}^2)}}, \quad (21)$$

which is of the standard inverse hyperbolic / trigonometric form **except for  $(-, -)$** .



# Integration of the separated minisuperspace WDW equation

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \mathfrak{s} v \frac{12\varkappa^2 |V|}{\hbar^2}, \quad (22)$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{k_\beta}{g}, \quad \sigma^2 := \frac{8\varkappa^3 |V| e^{g\chi}}{\hbar^2 g^2}, \quad (23)$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma \psi'(\sigma) + \ell(\nu^2 - \mathfrak{s} v \sigma^2) \psi(\sigma) = 0, \quad (24)$$

which is of the standard Besselian form.



# Self-adjointness of unbounded operators

## General theory

- Even in quantum geometrodynamics, the self-adjointness of the matter Hamiltonian is desirable, which is typically an unbounded operator in the Hilbert space  $\mathbf{F}$  endowed with the Schrödinger inner product  $(\cdot, \cdot)_S$ .
- Mathematically, an *unbounded* operator  $H$  is characterised not only by its action on a vector, but also by its domain  $\text{Dom}(H) \subsetneq \mathbf{F}^{19}$ .
- In addition to the *symmetry*  $(H^\dagger \phi_1, \phi_2) \equiv (\phi_1, H\phi_2)$ , the self-adjointness of an unbounded operator also requires  $\text{Dom}(H^\dagger) = \text{Dom}(H)$ .
- For quantum systems not bounded below, not only the self-adjointness, but also the symmetry of the Hamiltonian is not guaranteed automatically.
- Even when one could find a  $\text{Dom}_0(H)$  such that  $H$  is symmetric, one would still be left with  $\text{Dom}_0(H^\dagger) \supsetneq \text{Dom}_0(H)$  in general.
- Sloppily speaking, the process of extending  $\text{Dom}(H)$  such that  $\text{Dom}(H^\dagger) = \text{Dom}(H)$  is called **self-adjoint extension**<sup>20</sup>; if the extension is unique, the operator is called *essentially self-adjoint*.

---

<sup>19</sup>B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267 , ch. 9.

<sup>20</sup>D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) , G. Bonneau *et al.*, *Am. J. Phys* **69**, 322–331 (Mar. 28, 2001), V. S. Araujo *et al.*, *Am. J. Phys* **72**, 203–213 (Feb. 2004), A. M. Essin, D. J. Griffiths, *Am. J. Phys* **74**, 109–117 (Feb. 2006).

