

# Integrable Liouville Cosmological Models

## Self-adjointness of the Hamiltonian and Semi-classical Wave Functions

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# Outline

## 1. Introduction

## 2. The classical model and the implicit trajectories

Lagrangian formalism

## 3. The Dirac quantisation and the self-adjointness

Canonical formalism and Dirac quantisation

## 4. The semi-classical wave packets

Semi-classical approximation  
Inner product and wave packet

## 5. Conclusions



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# Introduction

## Friedmann–Lemaître model with quintessence and phantom Liouville fields

- Dynamical Dark Energy has been modelled by quintessence<sup>1</sup> and phantom<sup>2</sup> matter, which can be realised by minimally-coupled real scalar fields with  $\ell = \pm 1^3$

$$\mathcal{S}_L = \int d^4x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \quad (1)$$

- The Liouville field<sup>4</sup>  $\mathcal{V}(\phi) = V e^{\lambda\phi}$  is of interest, where  $\lambda, V \in \mathbb{R}$ .
- Assume a flat Robertson–Walker metric  $g_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + \varkappa e^{2\alpha(t)} d\Omega_{3F}^2$ , where  $\varkappa := 8\pi G$ ,  $d\Omega_{3F}^2 := d\chi^2 + \chi^2(d\theta^2 + \sin^2 \theta d\varphi^2)$ , and  $N$  lapse function
- The total action reads  $\mathcal{S} := \mathcal{S}_{EH} + \mathcal{S}_{GHY} + \mathcal{S}_L = \int d\Omega_{3F}^2 \int dt L$ , where

$$L := \varkappa^{3/2} N e^{3\alpha} \left( -\frac{3}{\varkappa} \frac{\dot{\alpha}^2}{N^2} + \ell \frac{\dot{\phi}^2}{2N^2} - V e^{\lambda\phi} \right). \quad (2)$$

<sup>1</sup>R. R. Caldwell, R. Dave, and P. J. Steinhardt. "Cosmological Imprint of an Energy Component with General Equation of State". In: *Phys. Rev. Lett.* 80.8 (Feb. 1998), pp. 1582–1585.

<sup>2</sup>R. R. Caldwell. "A phantom menace? Cosmological consequences of a dark energy component with super-negative equation of state". In: *Phys. Lett. B* 545.1-2 (Oct. 2002), pp. 23–29.

<sup>3</sup>The signature of metric is mostly positive.

<sup>4</sup>Y. Nakayama. "Liouville field theory: A Decade after the revolution". In: *International Journal of Modern Physics A* 19.17n18 (July 2004), pp. 2771–2930.



# Introduction

## Highlights

### Integrability

Implicit trajectories can be obtained explicitly; the minisuperspace Wheeler–DeWitt equation can be integrated exactly.

### Self-adjointness

The Hamiltonian is not even naturally symmetric; imposing self-adjointness leads to non-trivial physical results.

### Semi-classical wave functions

Semi-classical wave functions can be constructed by the principle of constructive interference, the JWKB approximation and numerical methods.



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# Decoupling the variables

Via orthogonal transformation

- By rescaling  $\bar{N} := N e^{-3\alpha}$  and the orthogonal transformation

$$\begin{pmatrix} \alpha(t) \\ \phi(t) \end{pmatrix} = \frac{s}{g} \begin{pmatrix} \lambda & -\ell \nu \\ -6 & \lambda \end{pmatrix} \begin{pmatrix} s_\beta \cdot \beta(t) \\ s_\chi \cdot \chi(t) \end{pmatrix} \quad \text{where } s_\beta, s_\chi = \pm 1, \quad (3)$$

the effective Lagrangian eq. (2) can be decoupled ( $s_\beta = s_\chi = +1$  from now on)

$$L = \nu^{3/2} \bar{N} \left( -s \frac{3}{\nu} \frac{\dot{\beta}^2}{\bar{N}^2} + \ell s \frac{\dot{\chi}^2}{2\bar{N}^2} - V e^{g\chi} \right), \quad (4)$$

where  $\Delta := \lambda^2 - 6\ell\nu$ ,  $s := \operatorname{sgn} \Delta$  and  $g := s\sqrt{|\Delta|} \equiv s\sqrt{3\Delta}$ .

- The Euler–Lagrange equations with respect to  $\bar{N}$ ,  $\beta$  and  $\chi$  will be called the modified first, second Friedmann equations and the Klein–Gordon equation, respectively.
- Since  $\beta(t)$  is cyclic, its conjugate momentum  $p_\beta$  is conserved<sup>5</sup>, and the modified second Friedmann equation can be readily integrated.

<sup>5</sup>The same first integral has been found in C. Lan. "The integrable cosmological model with scalar field and its extension to PT-symmetric theory". PhD thesis. Saint Petersburg State University, 2016. URL: <https://search.rsl.ru/ru/record/01006663434>, A. A. Andrianov, O. O. Novikov, and C. Lan. "Quantum cosmology of multifield scalar matter: Some exact solutions". In: *Theor. Math. Phys.* 184.3 (Sept. 2015), pp. 1224–1233, in canonical formalism.



# Integration and the implicit trajectories

- For  $p_\beta \neq 0$ , fixing the *implicit gauge*  $\bar{N} = -6\mathfrak{s}\sqrt{\varkappa}\dot{\beta}/p_\beta$ , the modified first Friedmann equation can be integrated, yielding the **implicit** trajectory

$$\mathbb{e}^{g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|} S^2 \left( \sqrt{\frac{3}{2\varkappa}} g(\beta - \beta_0) \right), \quad (5)$$

in which  $v := \operatorname{sgn} V$ ,  $(\operatorname{sgn}, \operatorname{sgn})$  means  $(\ell, \mathfrak{s}v)$ , and

$$\begin{aligned} (+,+)\,S(\gamma) &:= \operatorname{sech}(\gamma), & (+,-)\,S(\gamma) &:= \operatorname{csch}(\gamma), \\ (-,+)\,S(\gamma) &:= \sec(\gamma), & (-,-)\,S(\gamma) &:= \mathfrak{i} \csc(\gamma). \end{aligned} \quad (6)$$

- The integral for  $(-, -)$  is not real.
- The trajectories can be parametrised by  $\beta$ , inspiring recognising  $\beta$  as a ‘time variable’.
- For  $p_\beta = 0$ , integrating the modified second Friedmann equation yields  $\beta \equiv \beta_0$  or  $\phi - \phi_0 = -\ell \lambda \alpha / \varkappa^6$ , which is the well-known power-law special solution<sup>7</sup>.

<sup>6</sup>It only works for  $(+, -)$  or  $(-, +)$  if one checks consistency with the modified first Friedmann equation.

<sup>7</sup>For instance, A. R. Liddle and D. H. Lyth. *Cosmological Inflation and Large-Scale Structure*. Cambridge University, 2000, ch. 3.



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# Dirac quantisation

- The primary Hamiltonian and the Hamiltonian constraint<sup>8</sup> reads

$$H^p = \bar{N} H_{\perp} + v^{\bar{N}} p_{\bar{N}}, \quad (7)$$

$$H_{\perp} := -\beta \frac{p_{\beta}^2}{12\kappa^{1/2}} + \ell \beta \frac{p_{\chi}^2}{2\kappa^{3/2}} + \kappa^{3/2} V e^{g\chi}. \quad (8)$$

- Applying the Dirac quantisation rules with the Laplace–Beltrami operator ordering<sup>9</sup>, one gets the minisuperspace Wheeler–DeWitt equation with  $(\beta, \chi)$

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left( \beta \frac{\hbar^2}{12\kappa^{1/2}} \partial_{\beta}^2 - \ell \beta \frac{\hbar^2}{2\kappa^{3/2}} \partial_{\chi}^2 + \kappa^{3/2} V e^{g\chi} \right) \Psi. \quad (9)$$

- Equation (9) is Klein–Gordon-like, hyperbolic for  $\ell = +1$  and *elliptic* for  $\ell = -1$ .

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<sup>8</sup>D. M. Gitman and I. V. Tyutin. *Quantization of Fields with Constraints*. Springer Berlin Heidelberg, 1990.  
H. J. Rothe and K. D. Rothe. *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*. World Scientific, Apr. 2010.

<sup>9</sup>C. Kiefer. *Quantum Gravity*. 3rd ed. Oxford University, Apr. 2012, ch. 8.



# Integration of the Wheeler–DeWitt equation

Quantum-mechanical analogy

- Fourier-transforming  $\beta$  to  $k_\beta$  yields

$$\ell E \psi(x) = \widehat{H}_{\text{eff}} \psi(x) := \left( -\frac{\hbar^2}{2M_p} \partial_x^2 + \ell s v \widetilde{V} e^{\tilde{g}x} \right) \psi(x), \quad (10)$$

in which  $E := \hbar^2 k_\beta^2 / 12\nu^{1/2} \geq 0$ ,  $M_p := \nu^{-1/2}$ ,  $x := \nu\chi$ , and  $\widetilde{V} := \nu^{3/2}|V|$ .

- For the quintessence model  $(+, +)$ :  $\widehat{H}_{\text{eff}}$  bounded below; generalised eigenfunctions are the Besselian  $K_{\pm E}$ 's.
- For the quintessence model  $(+, -)$ :  $\widehat{H}_{\text{eff}}$  bounded **above**; generalised eigenfunctions for positive spectrum / 'scattering states' are the Besselian  $F_{\pm E}$ 's and  $G_{\pm E}$ 's<sup>10</sup>.
- For the phantom model  $(-, +)$ :  $\widehat{H}_{\text{eff}}$  bounded **above**; generalised eigenfunctions for negative spectrum / 'bound states' are the Besselian  $J_E$ 's.

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<sup>10</sup>T. M. Dunster. "Bessel Functions of Purely Imaginary Order, with an Application to Second-Order Linear Differential Equations Having a Large Parameter". In: *SIAM Journal on Mathematical Analysis* 21.4 (July 1990), pp. 995–1018.



# Separation of the variables and mode functions

- Writing  $\Psi(\beta, \chi) = \varphi(\beta)\psi(\chi)$ , eq. (9) can be separated into

$$\partial_\beta^2 \varphi(\beta) = k_\beta^2 \varphi(\beta); \quad (11)$$

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \imath v \frac{12\varkappa^2 |V|}{\hbar^2}. \quad (12)$$

- Equation (12) turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta, \chi) := \sum_{i=1}^2 c_i \varphi_\nu^{(i)}(\gamma) \sum_{j=1}^2 a_j B_\nu^{(j)}(\sigma), \quad \nu \geq 0; \quad (13)$$

$$\nu := \sqrt{\frac{2\varkappa}{3} \frac{1}{g}} k_\beta, \quad \gamma := \sqrt{\frac{3}{2\varkappa} \frac{g}{1}} \beta, \quad \sigma^2 := \frac{8\varkappa^3 |V| e^{g\chi}}{\hbar^2 g^2}, \quad (14)$$

$${}_{(+,+)} B_\nu^{(i)}(\sigma) := K \text{ and } I_{\mathbb{B}\nu}(\sigma), \quad {}_{(+,-)} B_\nu^{(i)}(\sigma) := F \text{ and } G_{\mathbb{B}\nu}(\sigma),$$

$${}_{(-,+)} B_\nu^{(i)}(\sigma) := J \text{ and } Y_\nu(\sigma), \quad {}_{(-,-)} B_\nu^{(i)}(\sigma) := K \text{ and } I_\nu(\sigma).$$

- Adapted to imaginary order,  $F_\nu(\sigma)$  and  $G_\nu(\sigma)$  are defined in<sup>11</sup>.

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<sup>11</sup>T. M. Dunster. "Bessel Functions of Purely Imaginary Order, with an Application to Second-Order Linear Differential Equations Having a Large Parameter". In: *SIAM Journal on Mathematical Analysis* 21.4 (July 1990), pp. 995–1018.



# Physical mode functions

- Physical mode functions are expected to be regular on the boundary.
- $(+, +)$ :  $|I_{\pm\nu}(\sigma)| \rightarrow +\infty$  as  $\alpha \rightarrow +\infty$
- $(-, +)$ :
  - $\forall n \in \mathbb{N}$ ,  $|Y_n(\sigma)| \rightarrow +\infty$  as  $\alpha \rightarrow -\infty$ .
  - $\forall \nu \in \mathbb{R}^+ \setminus \mathbb{N}$ , choose  $J_{-\nu}$  instead of  $Y_\nu$ , since  $J_{\pm\nu}$  are also linearly independent.
  - $\forall \nu \in \mathbb{R}^+ \setminus \mathbb{N}$ ,  $|J_{-\nu}(\sigma)| \rightarrow +\infty$  as  $\alpha \rightarrow -\infty$ .
- $(-, -)$ :  $|K_\nu(\sigma)| \rightarrow +\infty$  as  $\alpha \rightarrow -\infty$ ;  $|I_\nu(\sigma)| \rightarrow +\infty$  as  $\alpha \rightarrow +\infty$
- These are not to be included in the space of physical wave functions.  $\forall \nu \geq 0$ ,
  - $(+, +)$ :  $K_{\pm\nu}(\sigma)$  survives
  - $(+, -)$ :  $F_{\pm\nu}(\sigma)$  and  $G_{\pm\nu}(\sigma)$  survives
  - $(-, +)$ :  $J_\nu(\sigma)$  survives
  - $(-, -)$ : drops out



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# Matching quantum number with classical first integral

## Principle of constructive interference

- Write the mode function in the WKB form

$$\Psi_{k_\beta}(\beta, \chi) \sim \sqrt{\rho(\beta, \chi)} \exp\left\{ \frac{i}{\hbar} S(\beta, \chi) \right\}. \quad (15)$$

- For  $S/\hbar \gg 1$  and  $k_\beta \gg 1^{12}$ ,  $S(\beta, \chi)$  becomes the Hamilton principle function in the leading-order approximation.
- A Hamilton principle function is stationary with respect to variation of integral constants<sup>13</sup>

$$\frac{\partial S}{\partial k_\beta} = 0. \quad (16)$$

- Demanding eq. (16) matching the classical trajectory,  $k_\beta$  can be related to  $p_\beta$ .

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<sup>12</sup>The common form is  $\hbar \rightarrow 0^+$ .

<sup>13</sup>U. H. Gerlach. "Derivation of the Ten Einstein Field Equations from the Semiclassical Approximation to Quantum Geometrodynamics". In: *Phys. Rev.* 177.5 (Jan. 1969), pp. 1929–1941, L. D. Landau and E. M. Lifshitz. *Mechanics*. 3rd ed. Elsevier, 1976.



# Matching quantum number with classical first integral

(+, +) as exemplar

- Fixing  $\nu/\sigma > 1$ , the asymptotic expansion reads

$$K_{\pm\nu}(\sigma) \sim \frac{\sqrt{2\pi} \cos\left(\sqrt{\nu^2 - \sigma^2} - \nu \arccos \frac{\nu}{\sigma} - \frac{\pi}{4}\right)}{(\nu^2 - \sigma^2)^{1/4} e^{\mp\nu/2}} + O\left(\frac{1}{\sigma}\right). \quad (17)$$

- There are two phases with opposite signs. Assuming  $c_i, a_j$ 's are real and applying the principle to  $\Psi_\nu(\sigma)$ , one has  $\sigma/\nu = \operatorname{sech}(\gamma)$ , which matches the trajectory with  $\beta_0 = 0$  if

$$\hbar k_\beta \equiv \hbar \sqrt{\frac{3}{2\pi}} g \hbar \nu = p_\beta, \quad (18)$$

- Non-vanishing  $C$  can be compensated by the phase of  $c_i$  and  $a_j$ 's.
- Fixing  $\nu/\sigma < 1$ , the asymptotic expansion is not oscillatory, but exponential; it is not within the WKB regime.
- The conclusions also hold for  $F_{\pm\nu}(\sigma), G_{\pm\nu}(\sigma)$  for  $(+,-)$ , and  $J_\nu(\sigma)$  for  $(-,+)$ .



# Inner product for wave functions

## Schrödinger product

- To make sense of amplitude and wave packet, an inner product is necessary
- In terms of a Klein–Gordon equation, call  $\beta$  the “temporal” variable, and  $\chi$  the “spacial” variable.
- A common starting point is the *Schrödinger product*<sup>14</sup>

$$(\Psi_1, \Psi_2)_S := \int d\chi \Psi_1^*(\beta, \chi) \Psi_2(\beta, \chi); \quad (19)$$

- $(\Psi, \Psi)_S$  is **positive-definite**, and the integrand  $\rho_S(\beta, \chi)$  is **non-negative**
- The corresponding Schrödinger current is **real** but **does not satisfy continuity equation**  $\dot{\rho}_S + \nabla \cdot \vec{j}_S = 0$ , because eq. (9) is KG-like.
- $K_{\parallel\nu}^{15}$  for  $(+, +)$ ,  $F_{\parallel\nu}$  and  $G_{\parallel\nu}$  for  $(+, -)$  can be proved to be **orthogonal** and **complete** among themselves, as well as **can be normalised**.
  - $J_{\parallel\nu}$ 's for  $(+, -)$  are **not orthogonal**

<sup>14</sup> C. Kiefer. *Quantum Gravity*. 3rd ed. Oxford University, Apr. 2012, ch. 5.

<sup>15</sup> S. B. Yakubovich. “A distribution associated with the Kontorovich-Lebedev transform”. In: *Opuscula Math.* 26.1 (2006), pp. 161–172. URL: [http://www.opuscula.agh.edu.pl/vol26/1/art/opuscula\\_math\\_2612.pdf](http://www.opuscula.agh.edu.pl/vol26/1/art/opuscula_math_2612.pdf), A. Passian et al. “On the orthogonality of the MacDonald’s functions”. In: *Journal of Mathematical Analysis and Applications* 360.2 (Dec. 2009), pp. 380–390, R. Szmytkowski and S. Bielski. “Comment on the orthogonality of the Macdonald functions of imaginary order”. In: *Journal of Mathematical Analysis and Applications* 365.1 (May 2010), pp. 195–197.



# Peculiarity of the phantom model $(-, +)$

Orthogonality for mode functions; Hermiticity for operators

- $J_\nu(\sigma)$ 's are not orthogonal under the Schrödinger product

$$(J_\nu, J_{\tilde{\nu}})_S \propto \int_{-\infty}^{+\infty} dx J_\nu^*(e^x) J_{\tilde{\nu}}(e^x) = \frac{2\sin(\pi(\nu - \tilde{\nu})/2)}{\pi(\nu^2 - \tilde{\nu}^2)}, \quad (20)$$

therefore  $\mathbb{D}$  in eq. (12) is not Hermitian (though we do not need it so far)

- $\hat{p}_\chi^2$  is not Hermitian for  $\{J_\nu(\sigma)\}$  under the Schrödinger product

$$\int_{-\infty}^{+\infty} dx J_\nu^*(-\partial_x^2 J_{\tilde{\nu}}) - \int_{-\infty}^{+\infty} dx (-\partial_x^2 J_\nu)^* J_{\tilde{\nu}} = \frac{2}{\pi} \sin \frac{\pi(\nu - \tilde{\nu})}{2}. \quad (21)$$

- In order to save Hermiticity for  $p_\chi^2$  and  $\mathbb{D}$  and orthogonality of the modes under Schrödinger product, one can restrict

$$\nu = 2n + \nu_0, \quad n \in \mathbb{N}, \quad \nu_0 \in [0, 2). \quad (22)$$

- Using classical singularities as boundary condition, one can fix  $\nu_0 = 1$ .



# Discretisation of the phantom model $(-, +)$

Levels of the phantom model are **discretised** if one imposes Hermiticity of squared momenta under the Schrödinger product.

- This kind of subtlety on Hermiticity is named as *self-adjoint extension*, which arises already for infinite square well<sup>16</sup>.
- It also applies to  $x^{-2}$  potentials<sup>17</sup>, which is of cosmological relevance<sup>18</sup>.

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<sup>16</sup>G. Bonneau, J. Faraut, and G. Valent. "Self-adjoint extensions of operators and the teaching of quantum mechanics". In: *Am. J. Phys* 69.3 (Mar. 2001), pp. 322–331.

<sup>17</sup>A. M. Essin and D. J. Griffiths. "Quantum mechanics of the  $1 / x^2$  potential". In: *Am. J. Phys* 74.2 (Feb. 2006), pp. 109–117, V. S. Araujo, F. A. B. Coutinho, and J. F. Perez. "Operator domains and self-adjoint operators". In: *Am. J. Phys* 72.2 (Feb. 2004), pp. 203–213.

<sup>18</sup>M. Bouhmadi-López et al. "Quantum fate of singularities in a dark-energy dominated universe". In: *Physical Review D* 79.12 (June 2009).



# Further inner products for wave functions

Klein–Gordon and Mostafazadeh product

- Since eq. (9) is KG-like, another popular choice is the KG product

$$(\Psi_1, \Psi_2)_{\text{KG}}^g := \mathbb{i}g \left\{ (\Psi_1, \partial_\beta \Psi_2)_S - (\partial_\beta \Psi_1, \Psi_2)_S \right\}, \quad g > 0. \quad (23)$$

- $(\Psi, \Psi)_{\text{KG}}^g$  is **real** but **not positive-definite**, so does the integrand  $\rho_{\text{KG}}$ :
- The corresponding  $\vec{J}_{\text{KG}}$  is **conserved**  $\dot{\rho}_{\text{KG}} + \nabla \cdot \vec{J}_{\text{KG}} = 0$  and **real**.
- Mostafazadeh<sup>19</sup> found a product *for Hermitian  $\mathbb{D}$  with positive spectrum*:

$$(\Psi_1, \Psi_2)_M^\kappa := \kappa \left\{ (\Psi_1, \mathbb{D}^{+1/2} \Psi_2)_S + (\partial_\beta \Psi_1, \mathbb{D}^{-1/2} \partial_\beta \Psi_2)_S \right\}, \quad \kappa > 0. \quad (24)$$

- $(\Psi, \Psi)_M^\kappa$  is **positive-definite**, but the integrand  $\rho_M^\kappa$  is **complex**
- The corresponding  $\vec{J}_M^\kappa$  is **conserved**  $\dot{\rho}_M^\kappa + \nabla \cdot \vec{J}_M^\kappa = 0$  but also **complex**.

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<sup>19</sup> A. Mostafazadeh. "Hilbert space structures on the solution space of Klein–Gordon-type evolution equations". In: *Classical Quantum Gravity* 20.1 (Dec. 2002), pp. 155–171.



# Mostafazadeh inner product and the corresponding density

- Real power of  $\mathbb{D}$  is defined by spectral decomposition  $\mathbb{D}^\gamma := \sum_\nu \nu^{2\gamma} \mathbf{P}_\nu$ ,  
 $\mathbf{P}_\nu \Psi := \Psi_\nu (\Psi_\nu, \Psi)_S$ .
- It can be shown<sup>20</sup> that *the density*

$$\varrho_M^\kappa := \kappa \left\{ |\mathbb{D}^{+1/4} \Psi|^2 + |\mathbb{D}^{-1/4} \partial_\beta \Psi|^2 \right\} \quad (25)$$

- is equivalent to  $\rho_M^\kappa$  up to a boundary term

$$\int d\chi \varrho_M^\kappa = \int d\chi \rho_M^\kappa \equiv (\Psi_1, \Psi_2)_M^\kappa; \quad (26)$$

- is non-negative.
- The corresponding current  $\vec{\mathcal{J}}_M^\kappa$  is real but not conserved<sup>21</sup>.

<sup>20</sup> A. Mostafazadeh and F. Zamani. "Quantum mechanics of Klein–Gordon fields I: Hilbert Space, localized states, and chiral symmetry". In: *Ann. Phys.* 321.9 (Sept. 2006), pp. 2183–2209.

<sup>21</sup> B. Rosenstein and L. P. Horwitz. "Probability current versus charge current of a relativistic particle". In: *J. Phys. A: Math. Gen.* 18.11 (Aug. 1985), pp. 2115–2121.



# Wave packets of Gaussian amplitude for continuous spectrum

Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu; \bar{\nu}, \sigma) := \left( \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{(\nu - \bar{\nu})^2}{2\sigma^2}\right) \right)^{1/2} \quad (27)$$

- In<sup>22</sup>,  $A(\nu; \bar{\nu}, \sigma/\sqrt{2})$  was chosen.

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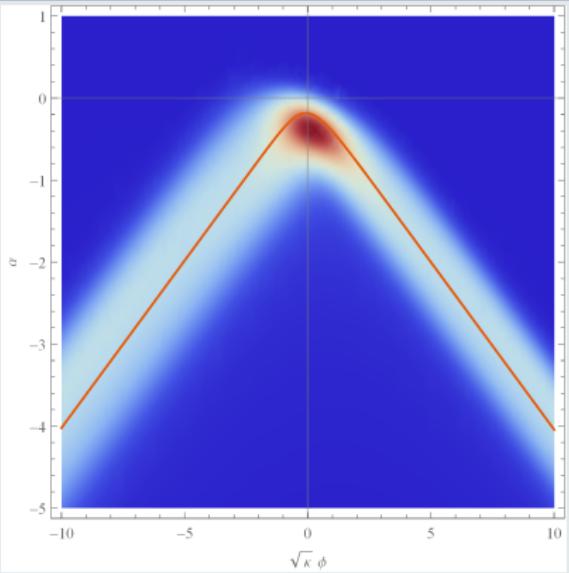
<sup>22</sup>M. P. Dąbrowski, C. Kiefer, and B. Sandhöfer. "Quantum phantom cosmology". In: *Physical Review D* 74.4 (Aug. 2006).



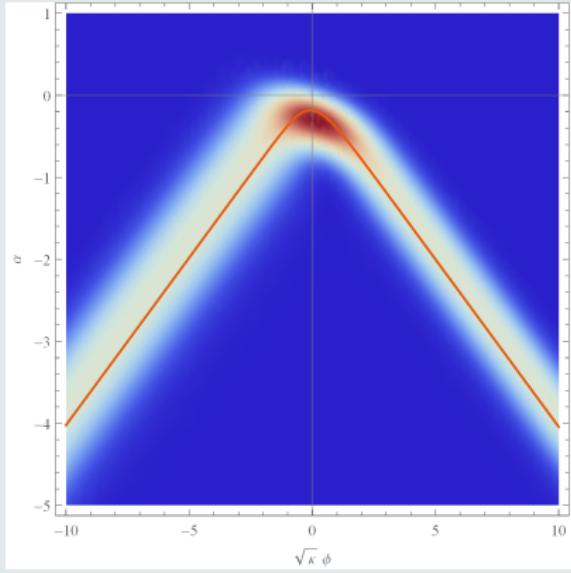
# Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\ell\nu}$ , with  $\lambda = \varkappa^{1/2}/2$ ,  $V = -\varkappa^{-2}$ ,  $\bar{k}_\beta = -2$  and  $\sigma_\beta = 5/4$

Schrödinger



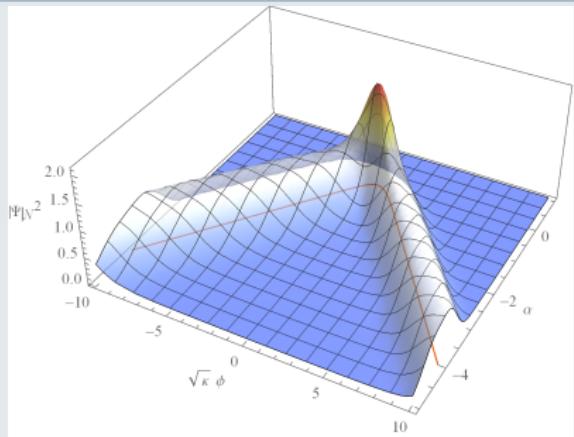
Mostafazadeh



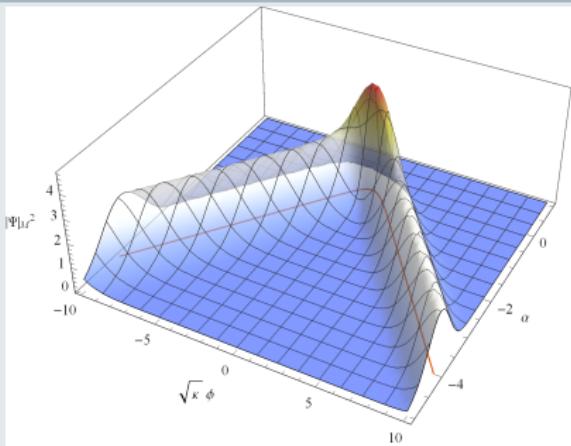
# Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\ell\nu}$ , with  $\lambda = \varkappa^{1/2}/2$ ,  $V = -\varkappa^{-2}$ ,  $\bar{k}_\beta = -2$  and  $\sigma_\beta = 5/4$

Schrödinger



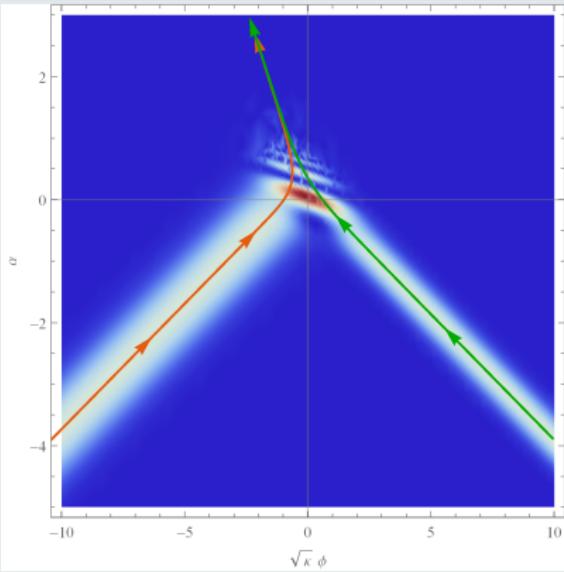
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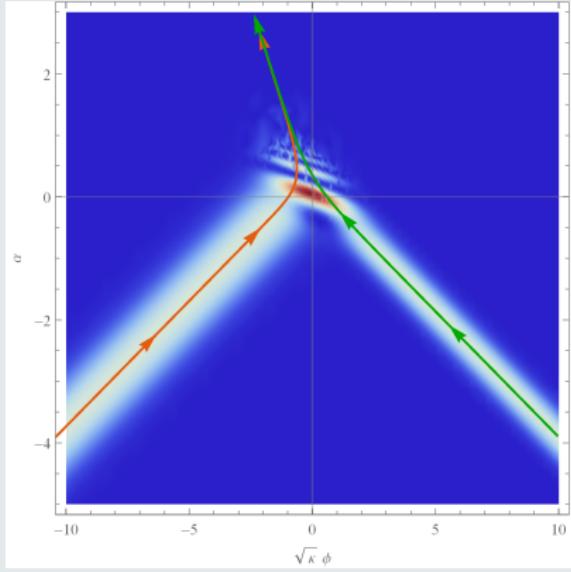
# Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\parallel\nu}$ , with  $\lambda = 4\nu^{1/2}/5$ ,  $V = +\nu^{-2}$ ,  $\bar{k}_\beta = -7/2$  and  $\sigma_\beta = 7/5$

Schrödinger



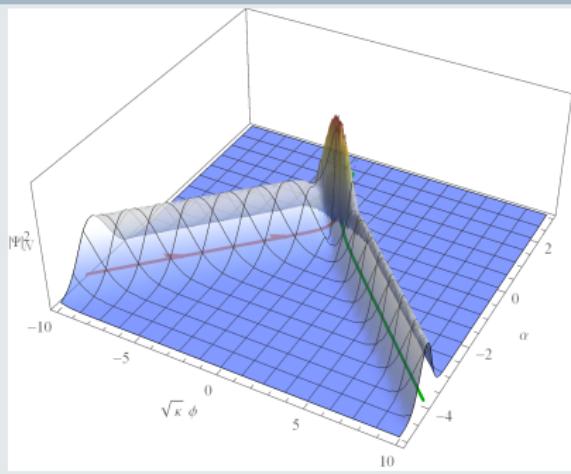
Mostafazadeh



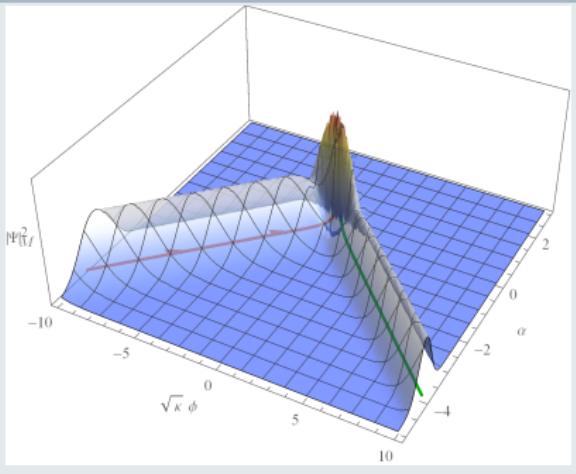
# Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\parallel\nu}$ , with  $\lambda = 4\nu^{1/2}/5$ ,  $V = +\nu^{-2}$ ,  $\bar{k}_\beta = -7/2$  and  $\sigma_\beta = 7/5$

Schrödinger



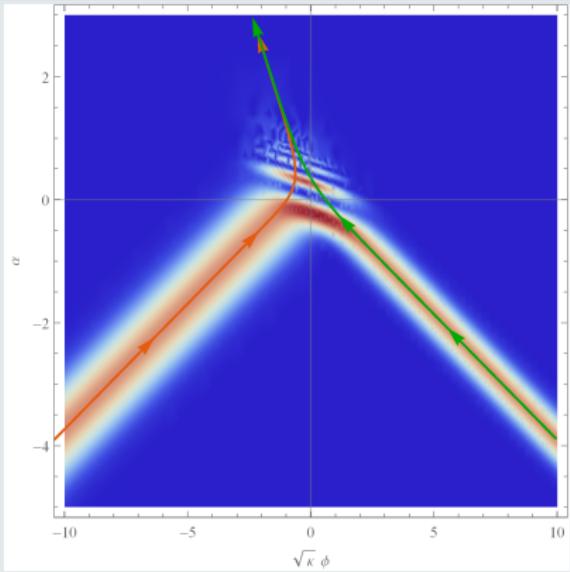
Mostafazadeh



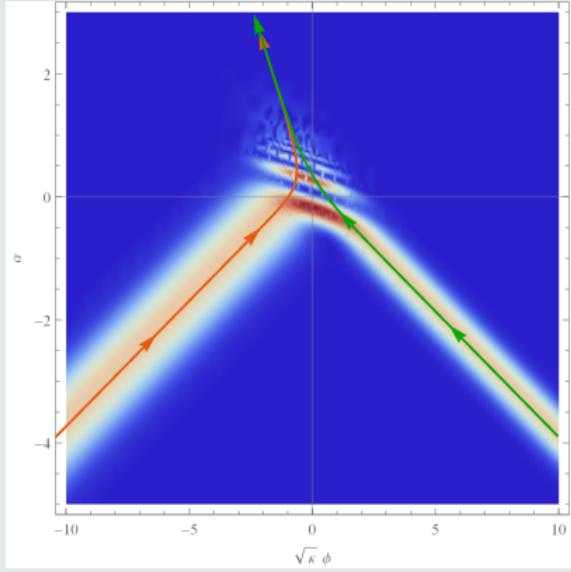
# Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\parallel\nu}$ , with  $\lambda = 4\kappa^{1/2}/5$ ,  $V = +\kappa^{-2}$ ,  $\bar{k}_\beta = -7/2$  and  $\sigma_\beta = 7/5$

Schrödinger



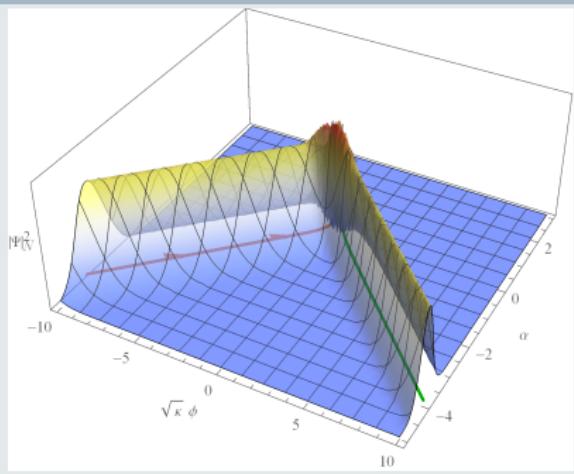
Mostafazadeh



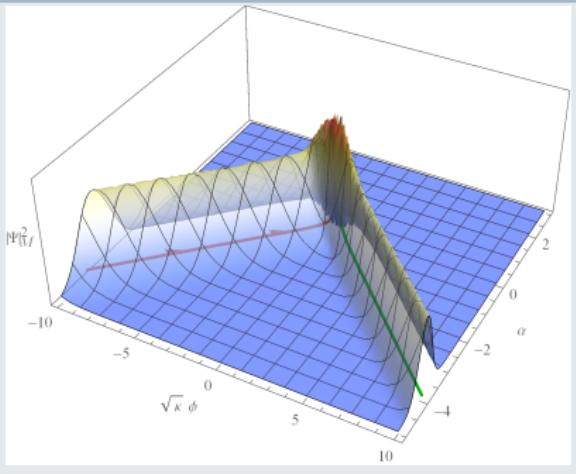
# Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\parallel\nu}$ , with  $\lambda = 4\kappa^{1/2}/5$ ,  $V = +\kappa^{-2}$ ,  $\bar{k}_\beta = -7/2$  and  $\sigma_\beta = 7/5$

Schrödinger



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# Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model  $(-, +)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\bar{n}) := \left( e^{-\bar{n}} \frac{\bar{n}^n}{n!} \right)^{1/2} \quad (28)$$

- In<sup>23</sup>,  $A_n(\bar{n}/\sqrt{2})$  was chosen.

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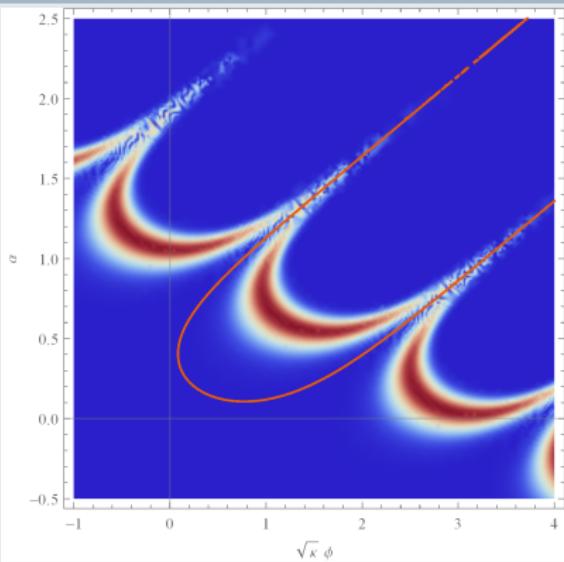
<sup>23</sup>C. Kiefer. "Wave packets in quantum cosmology and the cosmological constant". In: *Nucl. Phys. B* 341.1 (Sept. 1990), pp. 273–293.



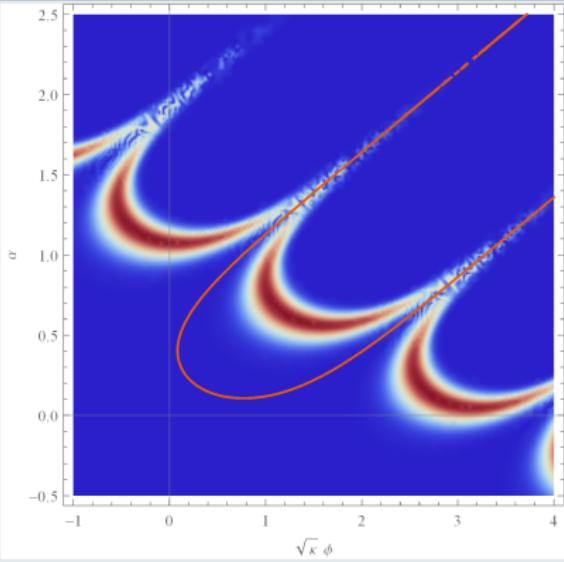
# Wave packets of Poissonian amplitude for phantom model

$J_{2n+1}$ , with  $\lambda = 2\kappa^{1/2}$ ,  $V = +\kappa^{-2}$  and  $\bar{k}_\beta = 8$

Schrödinger



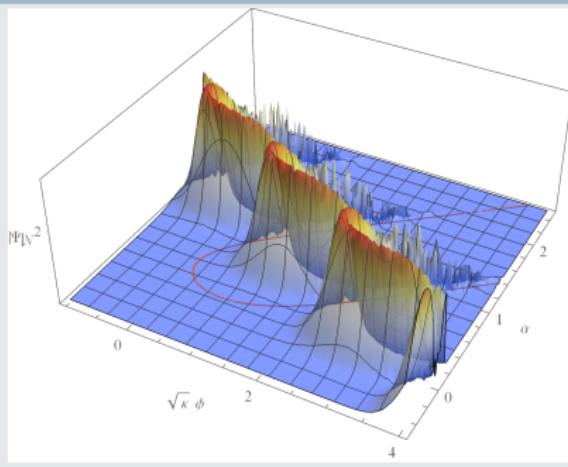
Mostafazadeh



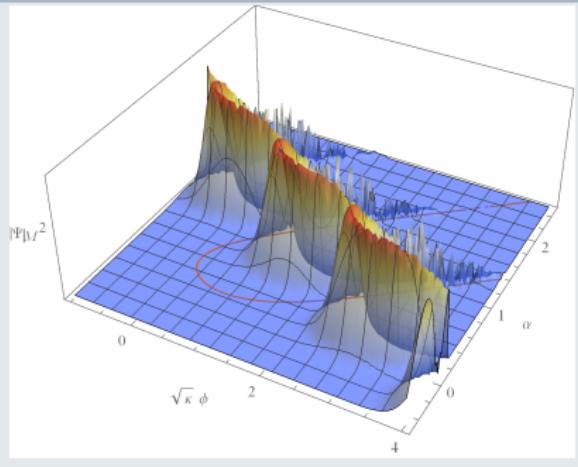
# Wave packets of Poissonian amplitude for phantom model

$J_{2n+1}$ , with  $\lambda = 2\kappa^{1/2}$ ,  $V = +\kappa^{-2}$  and  $\bar{k}_\beta = 8$

Schrödinger



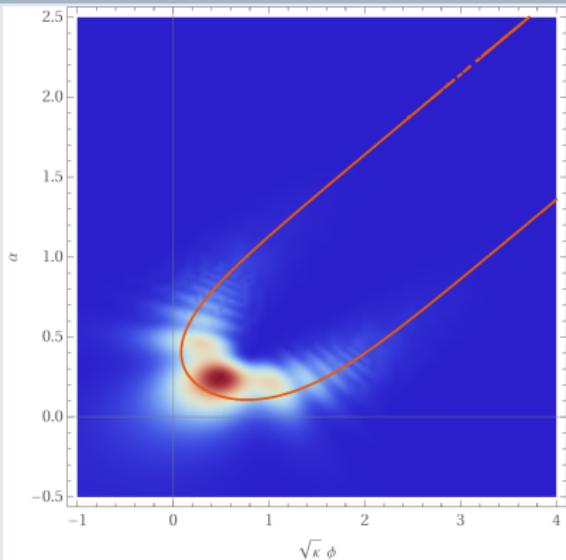
Mostafazadeh



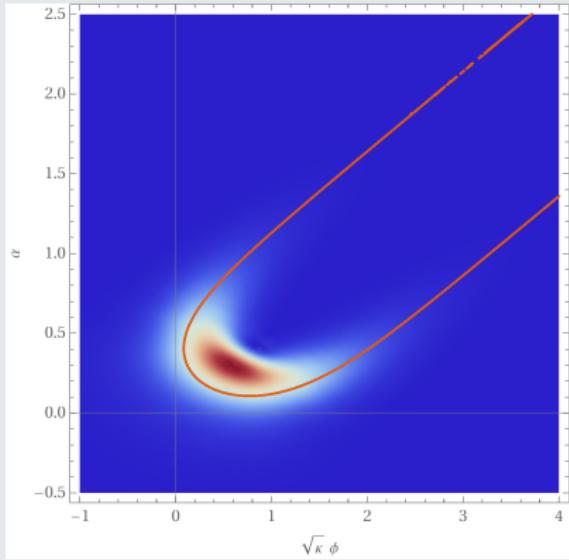
# Wave packets of Gaussian amplitude for phantom model

$J_\nu$ , with  $\lambda = 2\kappa^{1/2}$ ,  $V = +\kappa^{-2}$ ,  $\bar{k}_\beta = 8$  and  $\sigma_\beta = 11/2$

Schrödinger



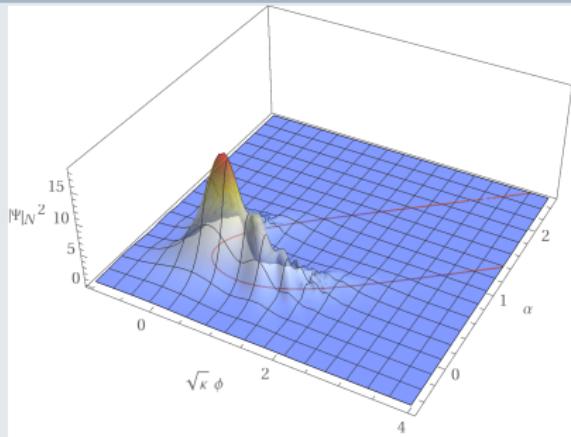
Mostafazadeh



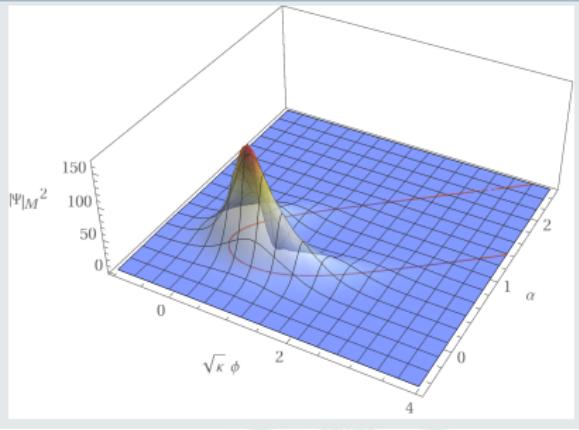
# Wave packets of Gaussian amplitude for phantom model

$J_\nu$ , with  $\lambda = 2\nu^{1/2}$ ,  $V = +\nu^{-2}$ ,  $\bar{k}_\beta = 8$  and  $\sigma_\beta = 11/2$

Schrödinger



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# Outline

1. Introduction
2. The classical model and the implicit trajectories  
Lagrangian formalism
3. The Dirac quantisation and the self-adjointness  
Canonical formalism and Dirac quantisation
4. The semi-classical wave packets  
Semi-classical approximation  
Inner product and wave packet
5. Conclusions



# Highlights

- An **integral of motion** was found for Liouville cosmological models.
- **Implicit trajectories** in minisuperspace were obtained.
- The levels of phantom Liouville model were found to be **discrete** due to the Hermiticity requirement of observables.
- The semi-classical quantum mode functions and wave packets was compared directly with the classical trajectory.



## Issues

- In  $(+, -)$  and  $(-, +)$ , wave packets contain multiple branches; however, the classical universe runs only on one trajectory.
- Quantum-corrected  $\bar{k}_\beta$  is to be understood.
- Wave packets with Gaussian profile are to be constructed, instead of inserting Gaussian / Poissonian amplitude artificially.
- A normalising  $\kappa$  for  $(\cdot, \cdot)_M^\kappa$  is to be evaluated, otherwise a quantitative comparison of  $(\cdot, \cdot)_S$  and  $(\cdot, \cdot)_M^\kappa$  is not possible.



# Outlook

- Beyond isotropy: generalise to Bianchi models
  - Bianchi Type-I: a natural extension, **under investigation**
- Beyond homogeneity: cosmological perturbation, **under investigation**
- Beyond single field
  - Two exponential potentials:  $|V_1| = |V_2|$  and special  $\lambda_i$
  - Multiple Liouville fields: mixing kinetic terms needed<sup>24</sup>
- Beyond classic matter
  - $PT$ -symmetric Liouville field<sup>25</sup>: may cross the phantom divide  $w = -1$ .

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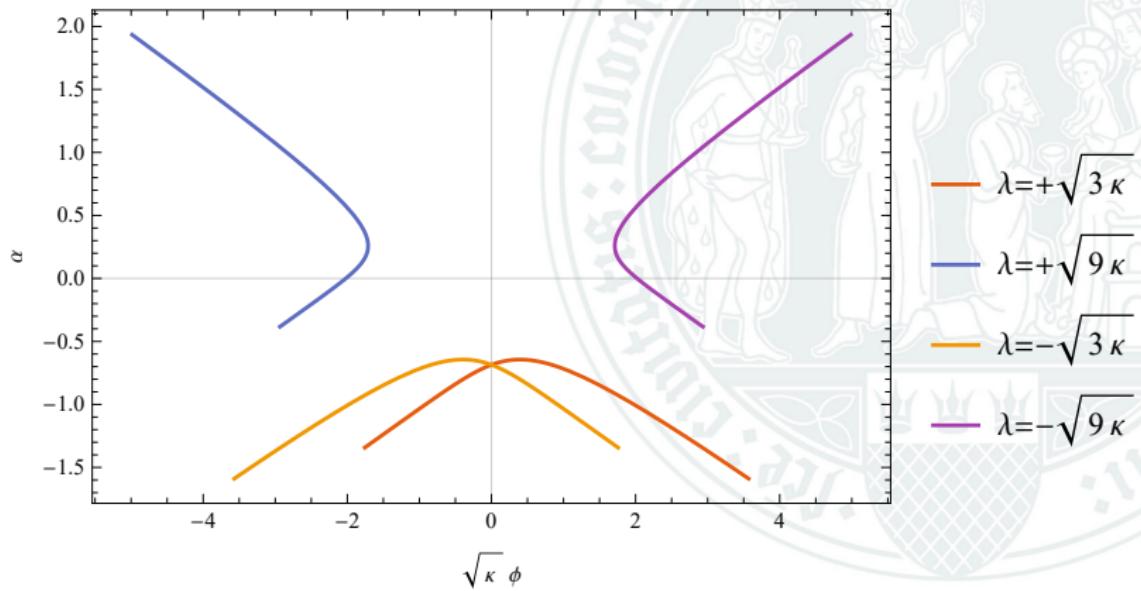
<sup>24</sup> A. A. Andrianov, O. O. Novikov, and C. Lan. "Quantum cosmology of multifield scalar matter: Some exact solutions". In: *Theor. Math. Phys.* 184.3 (Sept. 2015), pp. 1224–1233.

<sup>25</sup> A. A. Andrianov et al. "Phantom cosmology based on  $PT$  symmetry". In: *International Journal of Modern Physics D* 19.01 (Jan. 2010), pp. 97–111, A. A. Andrianov, C. Lan, and O. O. Novikov. " $PT$  Symmetric Classical and Quantum Cosmology". In: *Springer Proceedings in Physics*. Springer International Publishing, 2016, pp. 29–44.



# Trajectories for quintessence model $(+, +)$

sech, with  $\beta_0 = 0$ ,  $|V| = \kappa^{-2}$  and  $p_\beta^2 = \kappa^2 \sqrt{|V|}$ ; varying  $\lambda$

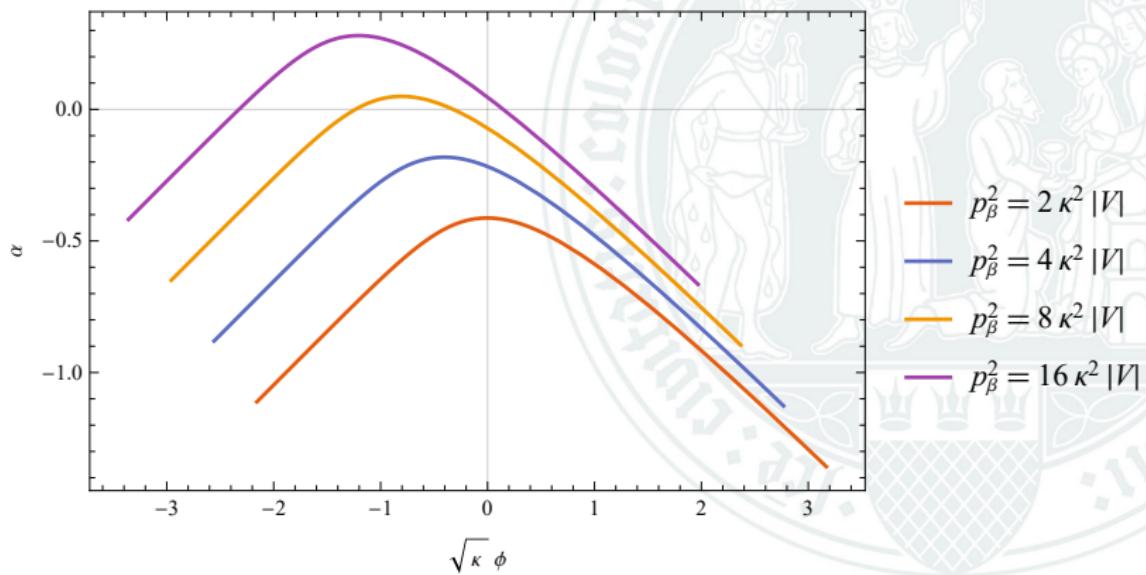


- has two asymptotes  $\chi \propto \pm \beta$



# Trajectories for quintessence model (+, +)

sech, with  $\beta_0 = 0$ ,  $|V| = \kappa^{-2}$  and  $\lambda^2 = 3\kappa$ ; varying  $p_\beta$

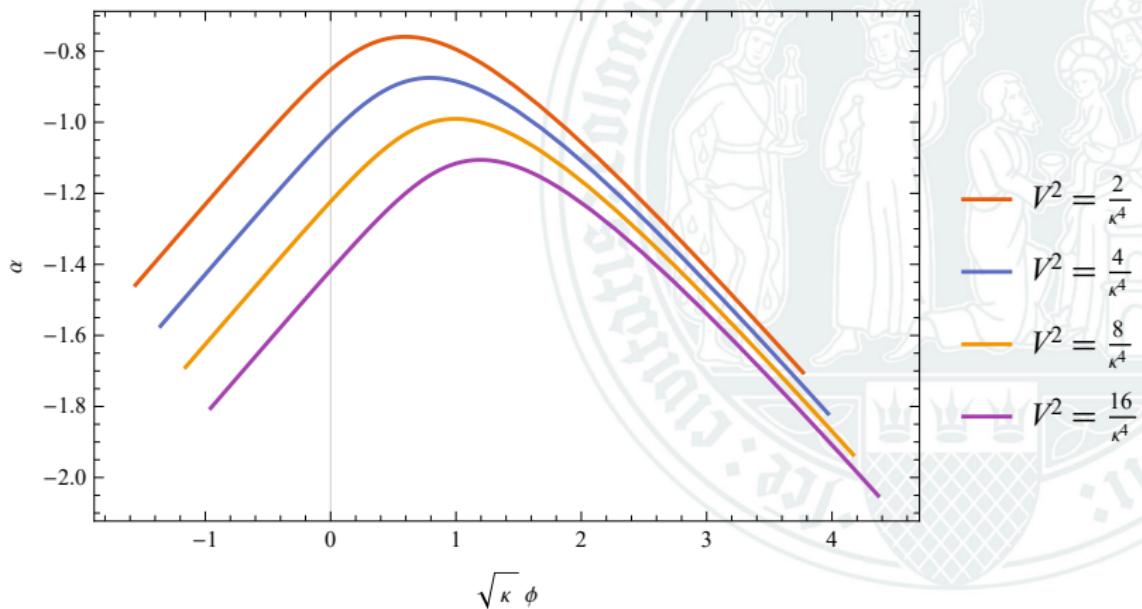


- has two asymptotes  $\chi \propto \pm \beta$



# Trajectories for quintessence model (+, +)

sech, with  $\beta_0 = 0$ ,  $\lambda^2 = 3\kappa$  and  $p_\beta^2 = \kappa^2 \sqrt{|V|}$ ; varying  $V$

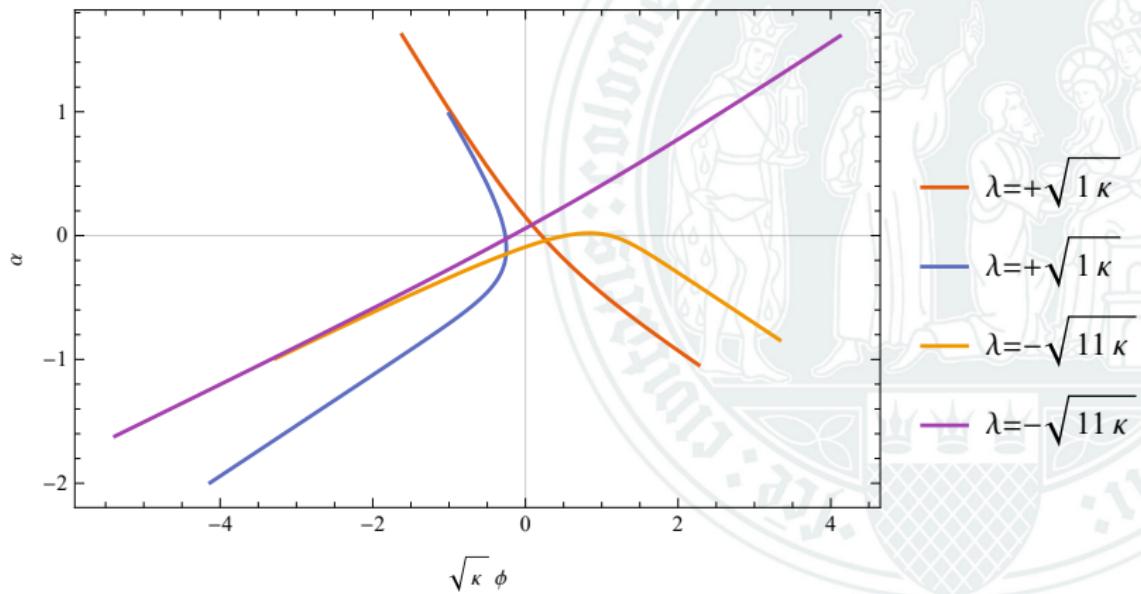


- has two asymptotes  $\chi \propto \pm \beta$



# Trajectories for quintessence model $(+, -)$

csch, with  $\beta_0 = 0$ ,  $|V| = \kappa^{-2}$  and  $p_\beta^2 = \kappa^2 \sqrt{|V|}$ ; varying  $\lambda$

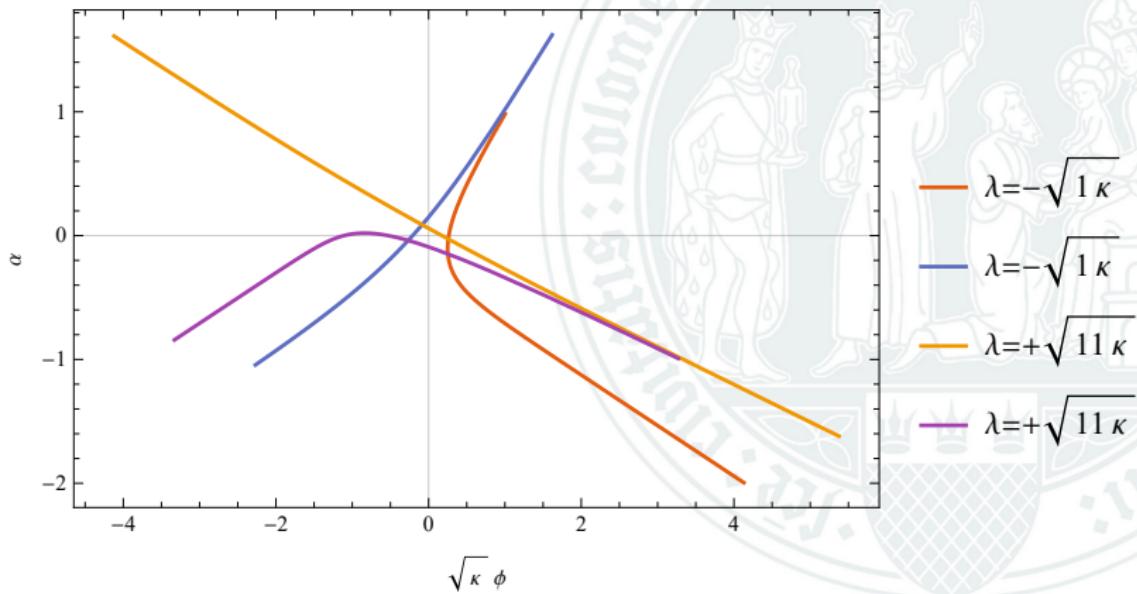


- contains two distinct solutions, separated by  $\beta = 0$
- has three asymptotes  $\chi \propto \pm\beta$  and  $\beta = 0$



# Trajectories for quintessence model $(+, -)$ : csch

csch, with  $\beta_0 = 0$ ,  $|V| = \kappa^{-2}$  and  $p_\beta^2 = \kappa^2 \sqrt{|V|}$ ; varying  $\lambda$

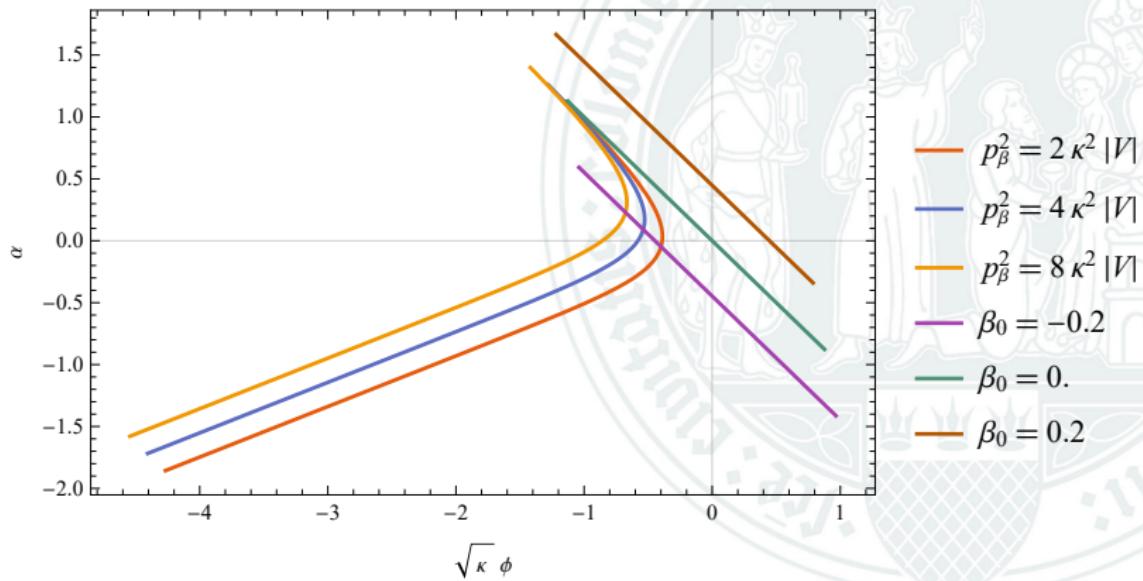


- contains two distinct solutions, separated by  $\beta = 0$
- has three asymptotes  $\chi \propto \pm\beta$  and  $\beta = 0$



# Trajectories for quintessence model (+, -): csch

csch, with  $V = \kappa^{-2}$  and  $\lambda^2 = \kappa$ ; varying  $p_\beta$

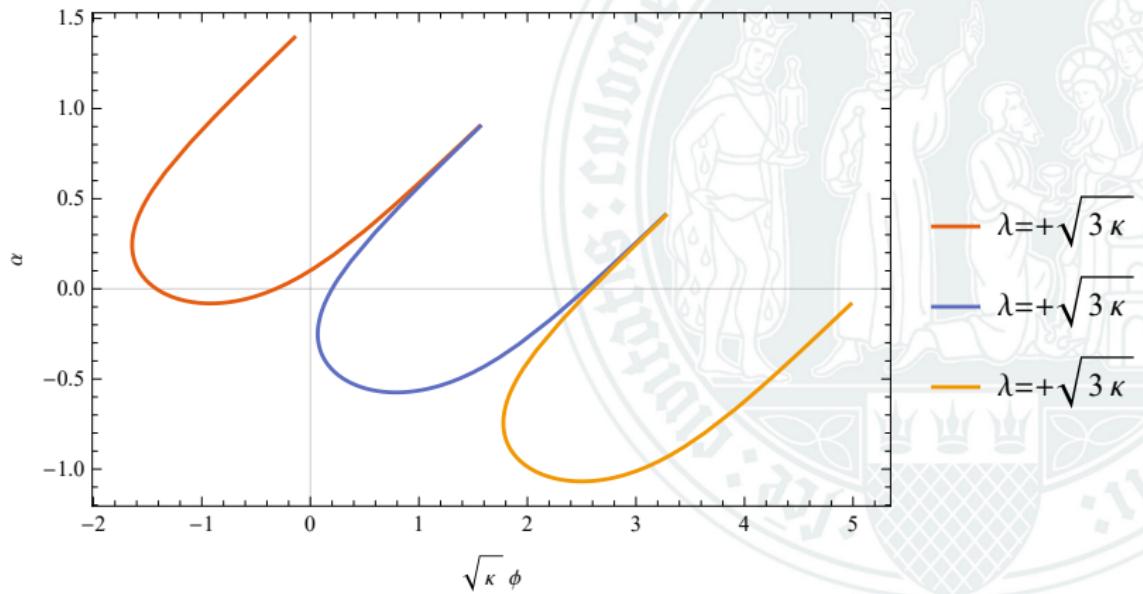


- contains two distinct solutions, separated by  $\beta = 0$
- has three asymptotes  $\chi \propto \pm\beta$  and  $\beta = 0$



# Trajectories for phantom model $(-, +)$

csc, with  $V = \varkappa^{-2}$  and  $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

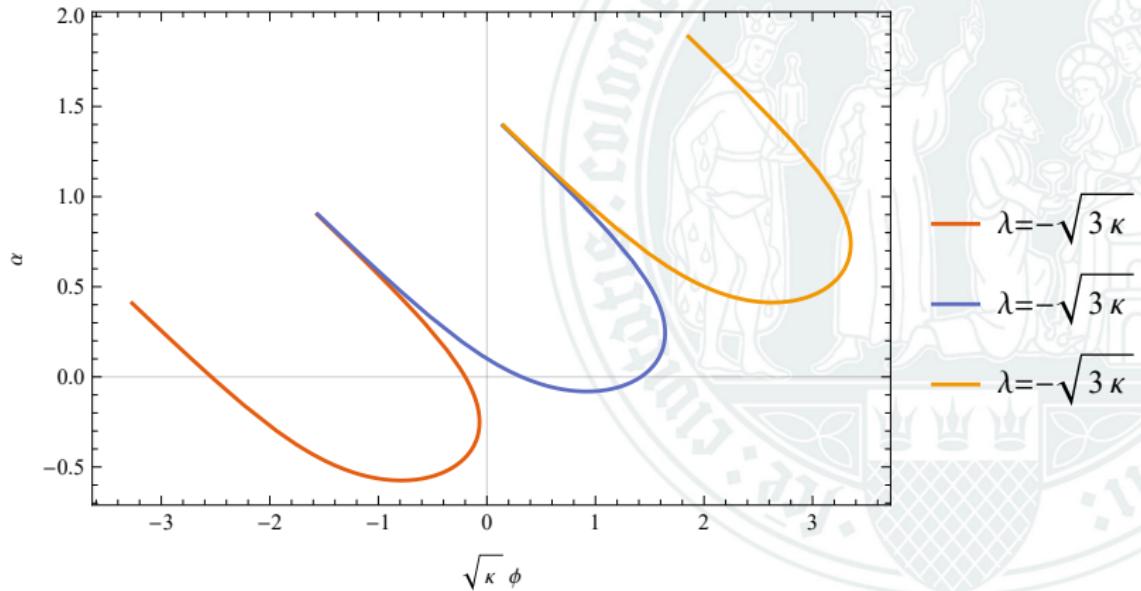


- contains infinite distinct solutions
- has infinite parallel asymptotes  $\beta \propto (n + 1/2)\pi$



# Trajectories for phantom model $(-, +)$

csc, with  $V = \varkappa^{-2}$  and  $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

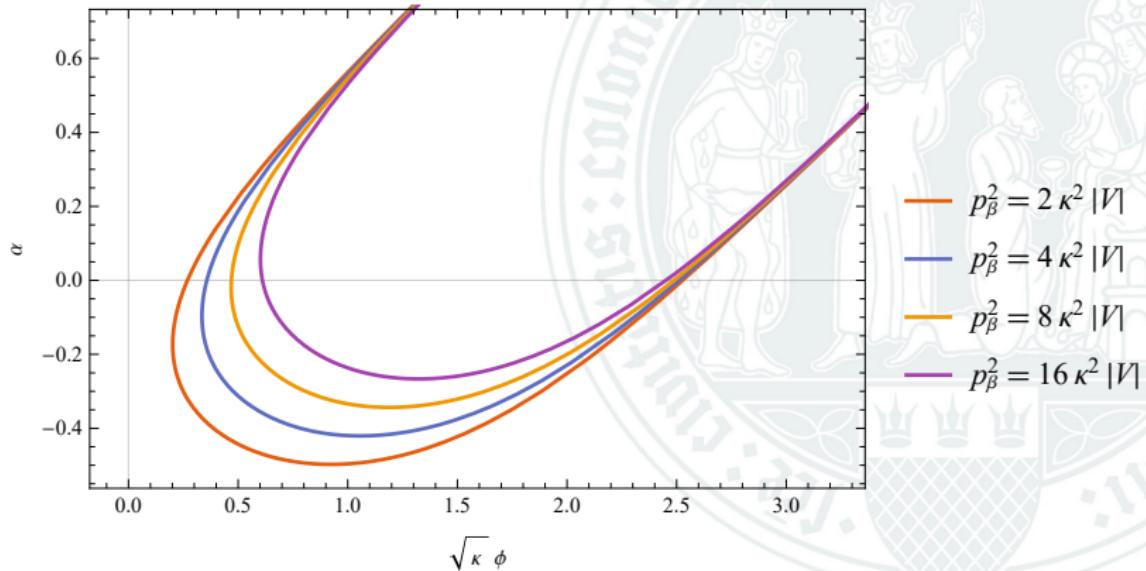


- contains infinite distinct solutions
- has infinite parallel asymptotes  $\beta \propto (n + 1/2)\pi$



# Trajectories for phantom model $(-, +)$

csc, with  $V = \kappa^{-2}$  and  $\lambda^2 = 3\kappa$ ; varying  $p_\beta$



- contains infinite distinct solutions
- has infinite parallel asymptotes  $\beta \propto (n + 1/2)\pi$



# Integration of the transformed first Friedmann equation

General integral for  $p_\beta \neq 0$

In order to integrate the equation under the implicit gauge

$$s \frac{p_\beta^2}{12} \left( -\ell \frac{\varkappa^{1/2}}{6} \left( \frac{d\chi}{d\beta} \right)^2 + \varkappa^{-1/2} \right) - \varkappa^{3/2} V e^{g\chi} = 0, \quad (29)$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa} g\beta}, \quad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} e^{-g\chi}, \quad (30)$$

to get

$$\left( \frac{d\tilde{\sigma}}{d\gamma} \right)^2 + \ell(s\upsilon - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{d\gamma}{d\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-s\upsilon + \tilde{\sigma}^2)}}, \quad (31)$$

which is of the standard inverse hyperbolic / trigonometric form **except for  $(-, -)$ .**



# Integration of the separated minisuperspace WDW equation

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \mathfrak{s} v \frac{12\varkappa^2 |V|}{\hbar^2}, \quad (12 \text{ rev.})$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{k_\beta}{g}, \quad \sigma^2 := \frac{8\varkappa^3 |V| e^{g\chi}}{\hbar^2 g^2}, \quad (14 \text{ rev.})$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma \psi'(\sigma) + \ell(\nu^2 - \mathfrak{s} v \sigma^2) \psi(\sigma) = 0, \quad (32)$$

which is of the standard Besselian form.



