

Integrable Cosmological Models with Liouville Fields

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Outline

1. Introduction

2. Classical model and the implicit trajectories

Lagrangian formalism

3. Quantised model and the wave packets

Canonical formalism and Dirac quantisation
Semi-classical approximation
Inner product and wave packet

4. Conclusions



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Introduction

Quintessence and phantom Liouville field

- Observed accelerated expansion can be explained by a cosmological constant¹ as stationary Dark Energy, but its origin has yet to be understood.
- Dynamical Dark Energy has been modeled by quintessence² and phantom³ matter, with barotropic index⁴ $w > -1$ and $w < -1$, respectively.
- They can be realised by minimally-coupled real scalar fields with $\ell = \pm 1^5$

$$\mathcal{S} = \int d^4x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \quad (1)$$

- $\mathcal{V}(\phi) = V e^{\lambda\phi}$, $\lambda, V \in \mathbb{R}$ is of interest: Liouville field⁶.

¹ E. J. Copeland, M. Sami, and S. Tsujikawa. In: *International Journal of Modern Physics D* 15.11 (2006), pp. 1753–1935, K. Bamba et al. In: *Astrophysics and Space Science* 342.1 (2012), pp. 155–228.

² R. R. Caldwell, R. Dave, and P. J. Steinhardt. In: *Physical Review Letters* 80.8 (1998), pp. 1582–1585.

³ R. R. Caldwell. In: *Physics Letters B* 545.1-2 (2002), pp. 23–29.

⁴ Barotropic index is the w in equation of state $\rho = wp$.

⁵ The signature of metric is mostly positive.

⁶ Y. Nakayama. In: *International Journal of Modern Physics A* 19.17n18 (2004), pp. 2771–2930.



Introduction

Friedmann–Lemaître model

- Assume flat Robertson–Walker metric for a homog. and isotr. model

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + \kappa e^{2\alpha(t)} d\Omega_{3F}^2 \quad (2)$$

w/ $\kappa := 8\pi G$, $d\Omega_{3F}^2 := d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\varphi^2)$, N lapse function.

- Combined with the Liouville field, the total action reads
 $\mathcal{S} := \mathcal{S}_{EH} + \mathcal{S}_{GHY} + \mathcal{S}_L = \int d\Omega_{3F}^2 \int dt L$, where

$$L := \kappa^{3/2} N e^{3\alpha} \left(-\frac{3}{\kappa} \frac{\dot{\alpha}^2}{N^2} + \ell \frac{\dot{\phi}^2}{2N^2} - V e^{\lambda\phi} \right), \quad (3)$$

in which dot means d/dt .

- The model turns out to be integrable, both classically and quantum-mechanically, enabling one to study its full physical properties, e.g. the relation between its classical and quantum theory.



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Decoupling the variables

Via *rescaled special orthogonal transformation*

- Setting $\bar{N} := N e^{-3\alpha}$, eq. (3) transforms to

$$L = \varkappa^{3/2} \bar{N} \left(-\frac{3}{\varkappa} \frac{\dot{\alpha}^2}{\bar{N}^2} + \ell \frac{\dot{\phi}^2}{2\bar{N}^2} - V e^{\lambda\phi+6\alpha} \right) \quad (4)$$

- Defining $\Delta := \lambda^2 - 6\ell\varkappa$, $s := \operatorname{sgn} \Delta$ and $g := s\sqrt{|\Delta|} \equiv s\sqrt{s\Delta}$, the *rescaled special orthogonal transformation*

$$\begin{pmatrix} \alpha \\ \phi \end{pmatrix} = \frac{s}{g} \begin{pmatrix} \lambda & -\ell\varkappa \\ -6 & \lambda \end{pmatrix} \begin{pmatrix} s_\beta \beta \\ s_\chi \chi \end{pmatrix} \quad \text{where } s_\beta, s_\chi = \pm 1 \quad (5)$$

gives the decoupled Lagrangian

$$L = \varkappa^{3/2} \bar{N} \left(-s \frac{3}{\varkappa} \frac{\dot{\beta}^2}{\bar{N}^2} + \ell s \frac{\dot{\chi}^2}{2\bar{N}^2} - V e^{s_\chi g \chi} \right). \quad (6)$$

- The Euler–Lagrange equations w.r.t. \bar{N} , β and χ will be called the trsfed. 1st, 2nd Friedmann eqs. and the Klein–Gordon eq., respectively.



Implicit integration

General integral for $p_\beta \neq 0$

- Since β is cyclic in eq. (6), the trsfed. 2nd Friedmann eq. can be integrated⁷

$$\text{const.} \equiv p_\beta := \frac{\partial L}{\partial \dot{\beta}} = -6\mathfrak{s}\nu^{1/2} \frac{\dot{\beta}}{\bar{N}} \equiv -6\mathfrak{s}\mathfrak{s}_\beta \frac{\nu^{1/2}}{g} \frac{\lambda\dot{\alpha} + \ell\nu\dot{\phi}}{\bar{N}}. \quad (7)$$

- For $p_\beta \neq 0$, fixing the *implicit gauge* $\bar{N} = -6s\sqrt{\nu}\dot{\beta}/p_\beta$, the trsfed. 1st Friedmann equation **can be integrated to get the trajectory**

$$e^{\beta_X g\chi} = \frac{p_\beta^2}{12\nu^2|V|} S^2 \left(\mathfrak{s}_\beta \sqrt{\frac{3}{2\nu}} g\beta + C \right), \quad (8)$$

in which $\nu := \operatorname{sgn} V$, $(\operatorname{sgn}, \operatorname{sgn})$ means $(\ell, \mathfrak{s}\nu)$, and

$$\begin{array}{ll} (+,+)\mathcal{S}(\gamma) := \operatorname{sech}(\gamma), & (+,-)\mathcal{S}(\gamma) := \operatorname{csch}(\gamma), \\ (-,+)\mathcal{S}(\gamma) := \operatorname{sec}(\gamma), & (-,-)\mathcal{S}(\gamma) := i \operatorname{csc}(\gamma). \end{array} \quad (9)$$

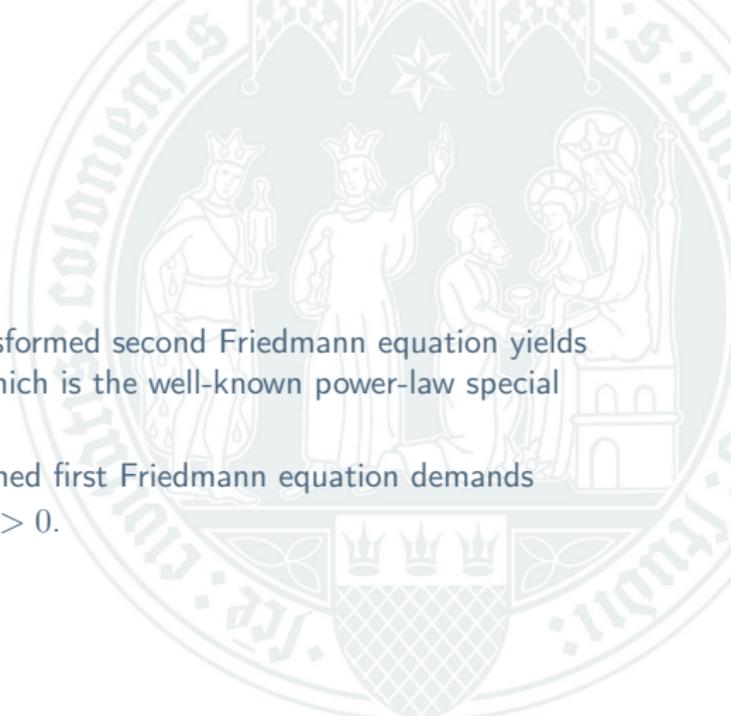
⁷The same first integral has been found in C. Lan. PhD thesis. Saint Petersburg State University, 2016, A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theoretical and Mathematical Physics* 184.3 (2015), pp. 1224–1233, in canonical formalism.



Implicit integration

Specific integral for $p_\beta = 0$

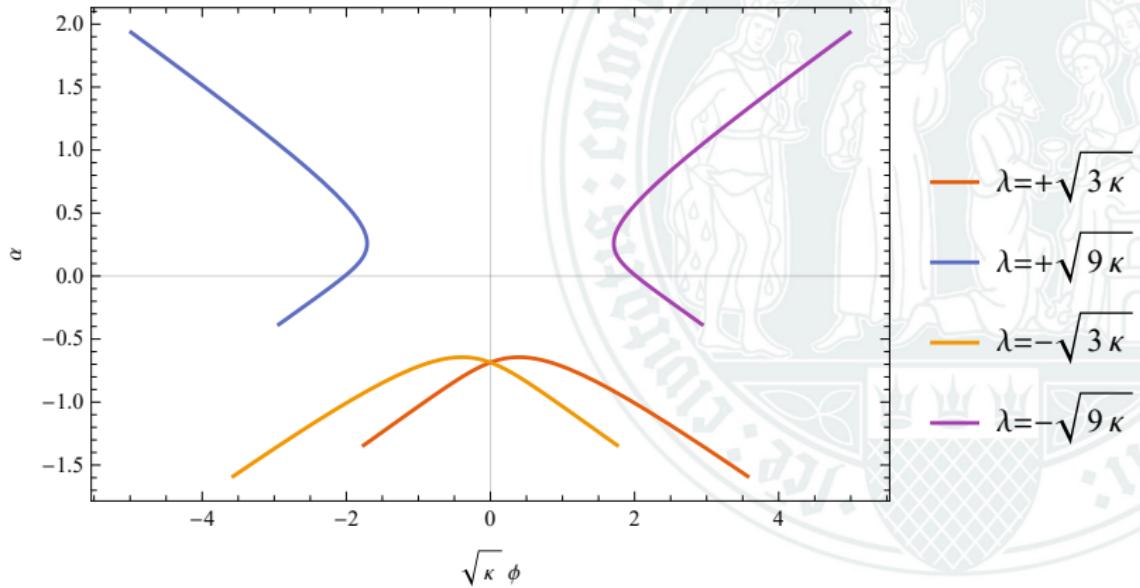
- For $p_\beta = 0$, integrating the transformed second Friedmann equation yields $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell \lambda \alpha / \nu$, which is the well-known power-law special solution⁸.
- Further integrating the transformed first Friedmann equation demands $(+, -)$ or $(-, +)$ to guarantee $\bar{N} > 0$.



⁸For instance, A. R. Liddle and D. H. Lyth. Cambridge University Press, 2000, ch. 3.

Trajectories for quintessence model $(+, +)$

sech , with $C = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

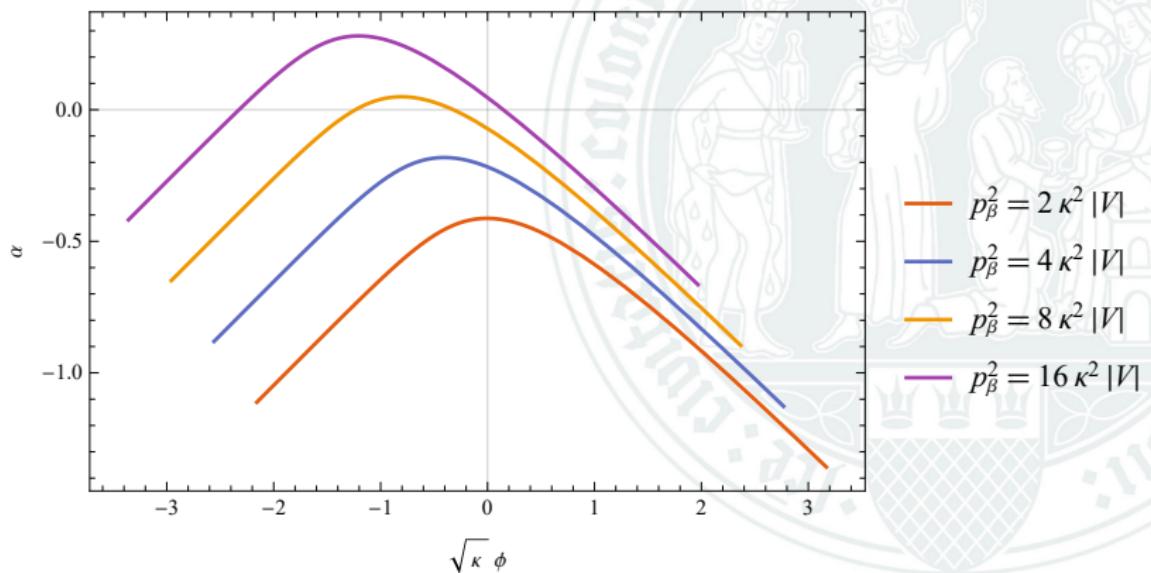


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model $(+, +)$

sech , with $C = 0$, $|V| = \nu^{-2}$ and $\lambda^2 = 3\nu$; varying p_β

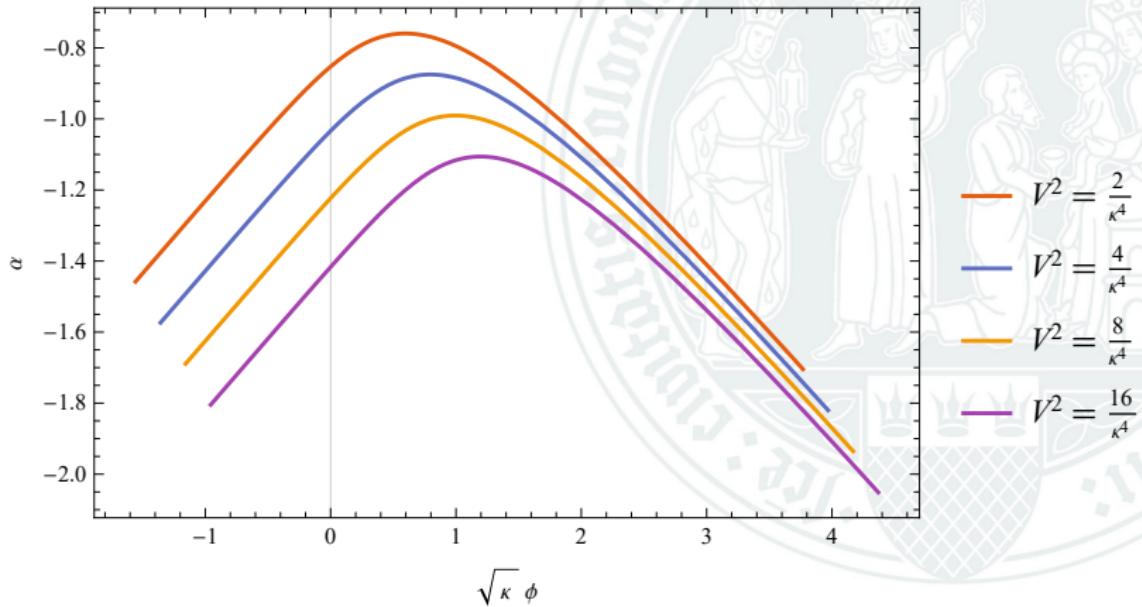


- has two asymptotes $\chi \propto \pm\beta$



Trajectories for quintessence model $(+, +)$

sech, with $C = 0$, $\lambda^2 = 3\nu$ and $p_\beta^2 = \nu^2 \sqrt{|V|}$; varying V

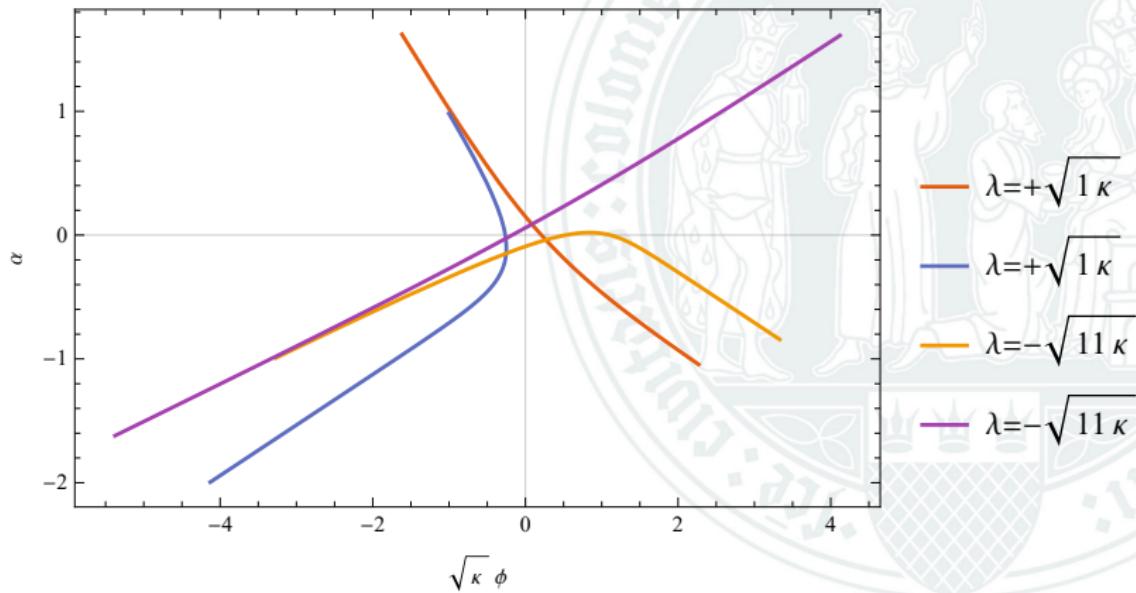


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model $(+, -)$

csch , with $C = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

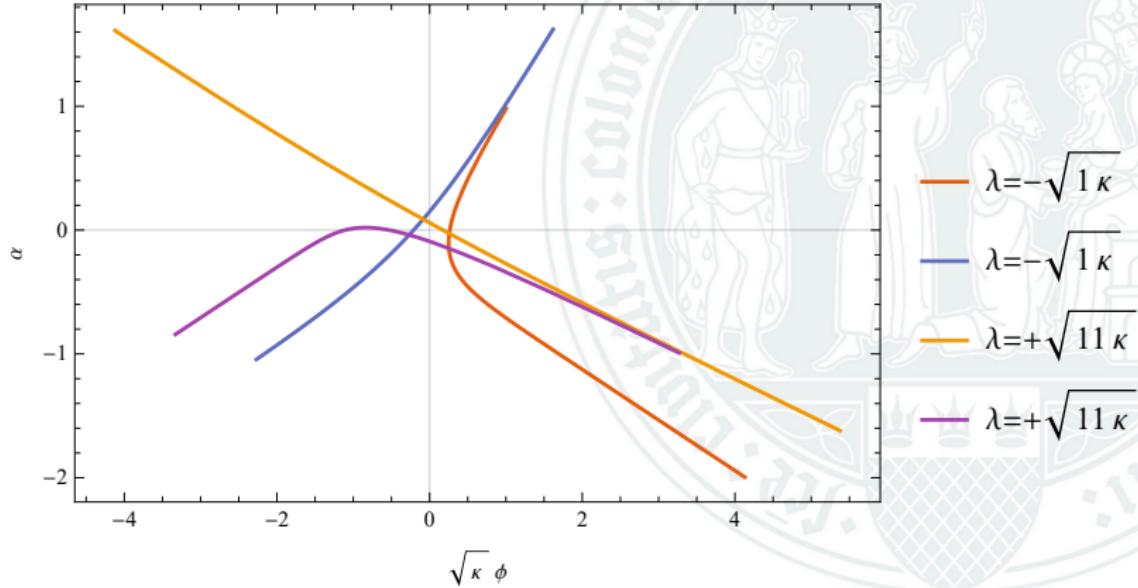


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$



Trajectories for quintessence model $(+, -)$: csch

csch, with $C = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

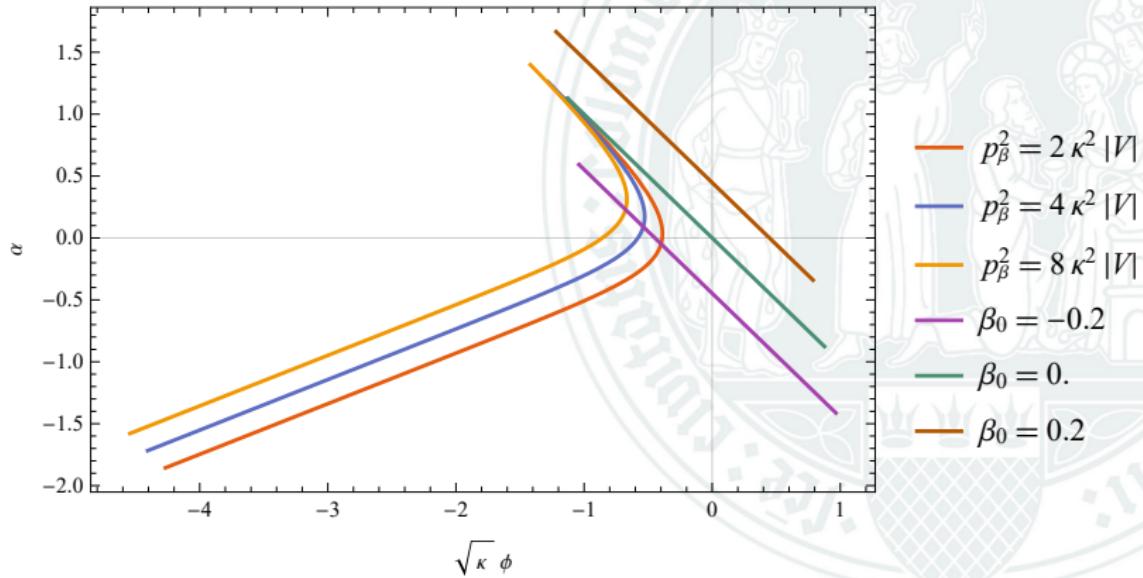


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$



Trajectories for quintessence model (+, -): csch

csch, with $V = \kappa^{-2}$ and $\lambda^2 = \kappa$; varying p_β

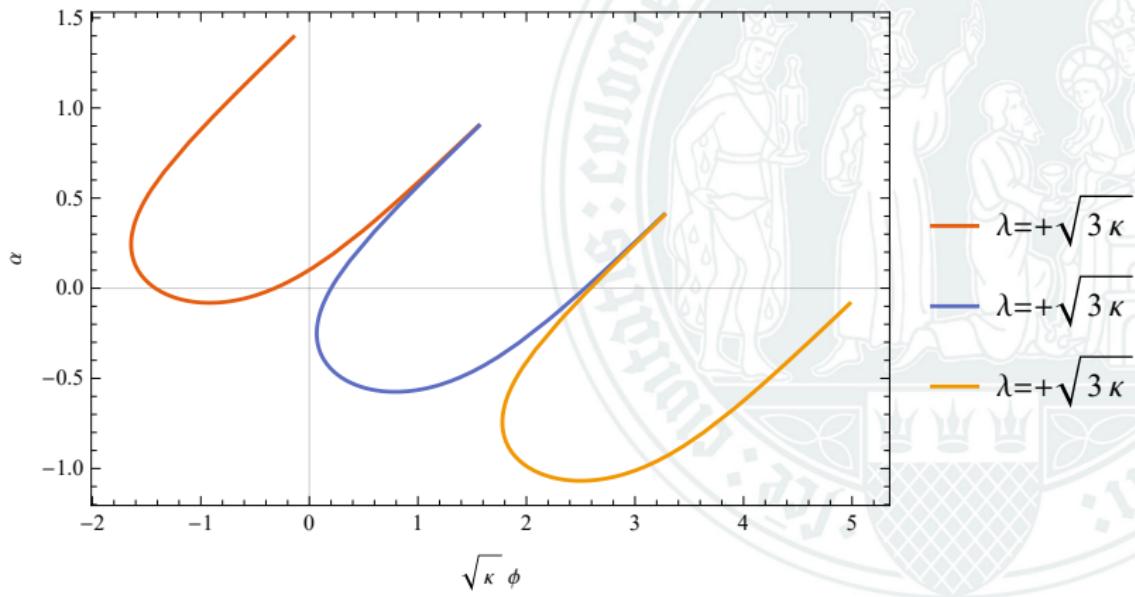


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$



Trajectories for phantom model $(-, +)$

csc, with $V = \kappa^{-2}$ and $p_\beta^2 = 3\kappa^2 \sqrt{|V|}$; varying λ

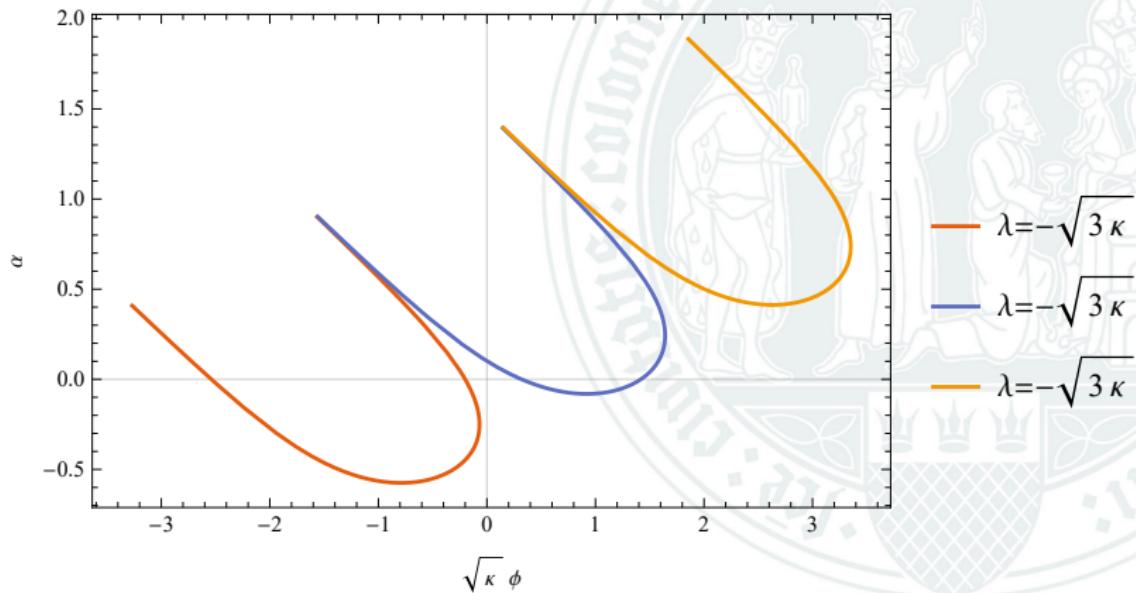


- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Trajectories for phantom model $(-, +)$

csc, with $V = \kappa^{-2}$ and $p_\beta^2 = 3\kappa^2 \sqrt{|V|}$; varying λ

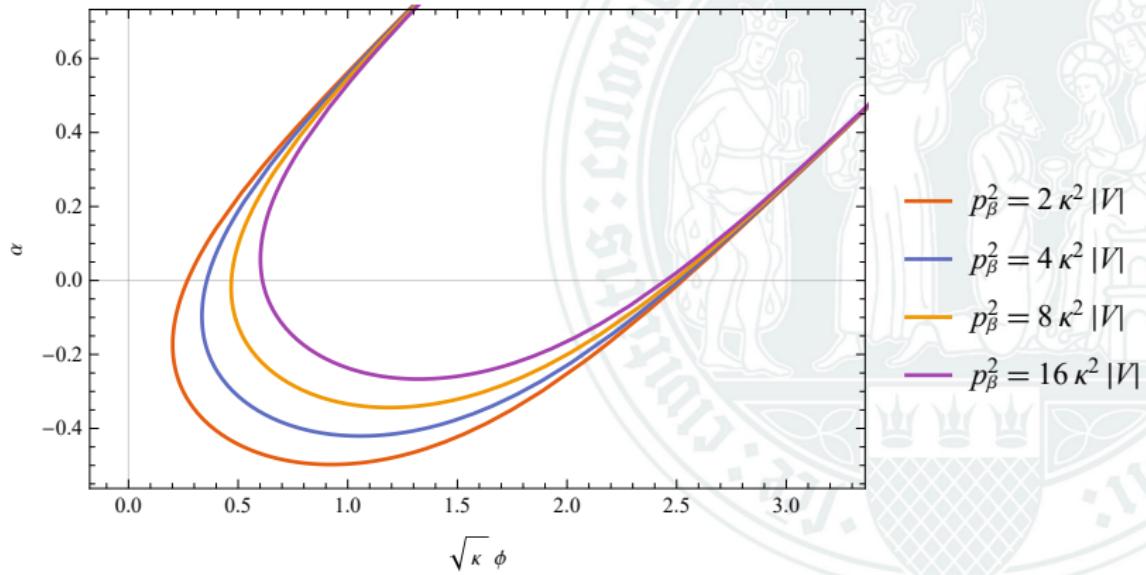


- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Trajectories for phantom model $(-, +)$

csc, with $V = \kappa^{-2}$ and $\lambda^2 = 3\kappa$; varying p_β



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Integration

Further discussions

- The integral for $(-, -)$ is not real.
- The trajectories can be parametrised by β , inspiring recognising β as a 'time variable'.
- The implicit integration enables one to compare trajectories with wave functions, see below.



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Dirac quantisation and the mode functions

- The primary Hamiltonian and the Hamiltonian constraint⁹ reads

$$H^p = \bar{N} H_{\perp} + v^{\bar{N}} p_{\bar{N}}, \quad (10)$$

$$H_{\perp} := -\mathcal{S} \frac{p_{\beta}^2}{12\kappa^{1/2}} + \ell \mathcal{S} \frac{p_{\chi}^2}{2\kappa^{3/2}} + \kappa^{3/2} V e^{g_{\beta}\chi}. \quad (11)$$

- Applying the Dirac quantisation rules with Laplace–Beltrami operator¹⁰, one gets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(\mathcal{S} \frac{\hbar^2}{12\kappa^{1/2}} \partial_{\beta}^2 - \ell \mathcal{S} \frac{\hbar^2}{2\kappa^{3/2}} \partial_{\chi}^2 + \kappa^{3/2} V e^{g_{\beta}\chi} \right) \Psi. \quad (12)$$

- Equation (12) is KG-like, hyperbolic for $\ell = +1$ and *elliptic* for $\ell = -1$.

⁹ D. M. Gitman and I. V. Tyutin. Springer Berlin Heidelberg, 1990, H. J. Rothe and K. D. Rothe. World Scientific, 2010.

¹⁰ C. Kiefer. 3rd ed. Oxford University Press, 2012, ch. 8.



Separation of the variables and mode functions

- Writing $\Psi(\beta, \chi) = \varphi(\beta)\psi(\chi)$, eq. (12) can be separated into

$$\partial_\beta^2 \varphi(\beta) = k_\beta^2 \varphi(\beta); \quad (13)$$

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \mathfrak{s} v \frac{12\varkappa^2 |V|}{\hbar^2}. \quad (14)$$

- Equation (14) turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta, \chi) := \sum_{i=1}^2 c_i \varphi_\nu^{(i)}(\gamma) \sum_{j=1}^2 a_j B_\nu^{(j)}(\sigma), \quad \nu \geq 0; \quad (15)$$

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{1}{g} k_\beta, \quad \gamma := \sqrt{\frac{3}{2\varkappa}} \frac{g}{1} \beta, \quad \sigma^2 := \frac{8\varkappa^3 |V| e^{g\beta_X \chi}}{\hbar^2 g^2}, \quad (16)$$

$${}_{(+,+)} B_\nu^{(i)}(\sigma) := K \text{ and } I_{\mathbb{B}\nu}(\sigma), \quad {}_{(+,-)} B_\nu^{(i)}(\sigma) := F \text{ and } G_{\mathbb{B}\nu}(\sigma),$$

$${}_{(-,+)} B_\nu^{(i)}(\sigma) := J \text{ and } Y_\nu(\sigma), \quad {}_{(-,-)} B_\nu^{(i)}(\sigma) := K \text{ and } I_\nu(\sigma).$$

- Adapted to imaginary order, $F_\nu(\sigma)$ and $G_\nu(\sigma)$ are defined in¹¹.

¹¹ T. M. Dunster. In: *SIAM Journal on Mathematical Analysis* 21.4 (1990), pp. 995–1018.



Physical mode functions

- Physical mode functions are expected to be regular on the boundary.
- $(+, +)$: $|I_{i\nu}(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow +\infty$
- $(-, +)$:
 - $\forall n \in \mathbb{N}$, $|Y_n(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow -\infty$.
 - $\forall \nu \in \mathbb{R}^+ \setminus \mathbb{N}$, choose $J_{-\nu}$ instead of Y_ν , since $J_{\pm\nu}$ are also linearly independent.
 - $\forall \nu \in \mathbb{R}^+ \setminus \mathbb{N}$, $|J_{-\nu}(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow -\infty$.
- $(-, -)$: $|K_\nu(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow -\infty$; $|I_\nu(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow +\infty$
- These are not to be included in the space of physical wave functions.
 $\forall \nu \geq 0$,
 - $(+, +)$: $K_{i\nu}(\sigma)$ survives
 - $(+, -)$: $F_{i\nu}(\sigma)$ and $G_{i\nu}(\sigma)$ survives
 - $(-, +)$: $J_\nu(\sigma)$ survives
 - $(-, -)$: drops out



Matching quantum number with classical first integral

Principle of constructive interference

- Write the mode function in the WKB form

$$\Psi_{k_\beta}(\beta, \chi) \sim \sqrt{\rho(\beta, \chi)} \exp\left\{ \frac{i}{\hbar} S(\beta, \chi) \right\}. \quad (17)$$

- For $S/\hbar \gg 1$ and $k_\beta \gg 1^{12}$, $S(\beta, \chi)$ becomes the Hamilton principle function in the leading-order approximation.
- A Hamilton principle function is stationary with respect to variation of integral constants¹³

$$\frac{\partial S}{\partial k_\beta} = 0. \quad (18)$$

- Demanding eq. (18) matching the classical trajectory, k_β can be related to p_β .

¹²The common form is $\hbar \rightarrow 0^+$.

¹³ U. H. Gerlach. In: *Physical Review* 177.5 (1969), pp. 1929–1941, L. D. Landau and E. M. Lifshitz. 3rd ed. Elsevier, 1976.



Matching quantum number with classical first integral

(+, +) as exemplar

- Fixing $\nu/\sigma > 1$, the asymptotic expansion reads

$$K_{\pm\nu}(\sigma) \sim \frac{\sqrt{2\pi} \cos\left(\sqrt{\nu^2 - \sigma^2} - \nu \arccos \frac{\nu}{\sigma} - \frac{\pi}{4}\right)}{(\nu^2 - \sigma^2)^{1/4} e^{\mp\nu/2}} + O\left(\frac{1}{\sigma}\right). \quad (19)$$

- There are two phases with opposite signs. Assuming c_i, a_j 's are real and applying *the principle* to $\Psi_\nu(\sigma)$, one has $\sigma/\nu = \operatorname{sech}(\beta_\beta \gamma)$, which matches the trajectory with $C = 0$ if

$$\hbar k_\beta \equiv \hbar \sqrt{\frac{3}{2\nu}} g \hbar \nu = p_\beta, \quad (20)$$

- Non-vanishing C can be compensated by the phase of c_i and a_j 's.
- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; it is not within the WKB regime.
- The conclusions also hold for $F_{\pm\nu}(\sigma), G_{\pm\nu}(\sigma)$ for $(+,-)$, and $J_\nu(\sigma)$ for $(-,+)$.



Inner product for wave functions

Schrödinger product

- To make sense of amplitude and wave packet, an inner product is necessary
- In terms of a Klein–Gordon equation, call β the “temporal” variable, and χ the “spacial” variable.
- A common starting point is the *Schrödinger product*¹⁴

$$(\Psi_1, \Psi_2)_S := \int d\chi \Psi_1^*(\beta, \chi) \Psi_2(\beta, \chi); \quad (21)$$

- $(\Psi, \Psi)_S$ is positive-definite, and the integrand $\rho_S(\beta, \chi)$ is non-negative
- The corresponding Schrödinger current is real but does not satisfy continuity equation $\dot{\rho}_S + \nabla \cdot \vec{j}_S = 0$, because eq. (12) is KG-like.
- $K_{i\nu}^{15}$ for $(+, +)$, $F_{i\nu}$ and $G_{i\nu}$ for $(+, -)$ can be proved to be orthogonal and complete among themselves, as well as can be normalised.
 - $J_{i\nu}$'s for $(+, -)$ are not orthogonal

¹⁴ C. Kiefer. 3rd ed. Oxford University Press, 2012, ch. 5.

¹⁵ S. B. Yakubovich. In: *Opuscula Math.* 26.1 (2006), pp. 161–172, A. Passian et al. In: *Journal of Mathematical Analysis and Applications* 360.2 (2009), pp. 380–390, R. Szmytkowski and S. Bielski. In: *Journal of Mathematical Analysis and Applications* 365.1 (2010), pp. 195–197.



Peculiarity of the phantom model $(-, +)$

Orthogonality for mode functions; Hermiticity for operators

- $J_\nu(\sigma)$'s are not orthogonal under the Schrödinger product

$$(J_\nu, J_{\tilde{\nu}})_S \propto \int_{-\infty}^{+\infty} dx J_\nu^*(e^x) J_{\tilde{\nu}}(e^x) = \frac{2\sin(\pi(\nu - \tilde{\nu})/2)}{\pi(\nu^2 - \tilde{\nu}^2)}, \quad (22)$$

therefore \mathbb{D} in eq. (14) is not Hermitian (though we do not need it so far)

- \hat{p}_χ^2 is not Hermitian for $\{J_\nu(\sigma)\}$ under the Schrödinger product

$$\int_{-\infty}^{+\infty} dx J_\nu^*(-\partial_x^2 J_{\tilde{\nu}}) - \int_{-\infty}^{+\infty} dx (-\partial_x^2 J_\nu)^* J_{\tilde{\nu}} = \frac{2}{\pi} \sin \frac{\pi(\nu - \tilde{\nu})}{2}. \quad (23)$$

- In order to save Hermiticity for p_χ^2 and \mathbb{D} and orthogonality of the modes under Schrödinger product, one can restrict

$$\nu = 2n + \nu_0, \quad n \in \mathbb{N}, \quad \nu_0 \in [0, 2]. \quad (24)$$

- Using classical singularities as boundary condition, one can fix $\nu_0 = 1$.



Discretisation of the phantom model $(-, +)$

Levels of the phantom model are **discretised** if one imposes Hermiticity of squared momenta under the Schrödinger product.

- This kind of subtlety on Hermiticity is named as *self-adjoint extension*, which arises already for infinite square well¹⁶.
- It also applies to x^{-2} potentials¹⁷, which is of cosmological relevance¹⁸.

¹⁶ G. Bonneau, J. Faraut, and G. Valent. In: *American Journal of Physics* 69.3 (2001), pp. 322–331.

¹⁷ A. M. Essin and D. J. Griffiths. In: *American Journal of Physics* 74.2 (2006), pp. 109–117,
V. S. Araujo, F. A. B. Coutinho, and J. F. Perez. In: *American Journal of Physics* 72.2 (2004),
pp. 203–213.

¹⁸ M. Bouhmadi-López et al. In: *Physical Review D* 79.12 (2009).



Further inner products for wave functions

Klein–Gordon and Mostafazadeh product

- Since eq. (12) is KG-like, another popular choice is the KG product

$$(\Psi_1, \Psi_2)_{\text{KG}}^g := \mathbb{i}g \left\{ (\Psi_1, \partial_\beta \Psi_2)_S - (\partial_\beta \Psi_1, \Psi_2)_S \right\}, \quad g > 0. \quad (25)$$

- $(\Psi, \Psi)_{\text{KG}}^g$ is **real** but **not positive-definite**, so does the integrand ρ_{KG} ;
- The corresponding \vec{J}_{KG} is **conserved** $\dot{\rho}_{\text{KG}} + \nabla \cdot \vec{J}_{\text{KG}} = 0$ and **real**.
- Mostafazadeh¹⁹ found a product *for Hermitian \mathbb{D} with positive spectrum*:

$$(\Psi_1, \Psi_2)_{\text{M}}^\kappa := \kappa \left\{ (\Psi_1, \mathbb{D}^{+1/2} \Psi_2)_S + (\partial_\beta \Psi_1, \mathbb{D}^{-1/2} \partial_\beta \Psi_2)_S \right\}, \quad \kappa > 0. \quad (26)$$

- $(\Psi, \Psi)_{\text{M}}^\kappa$ is **positive-definite**, but the integrand ρ_{M}^κ is **complex**
- The corresponding $\vec{J}_{\text{M}}^\kappa$ is **conserved** $\dot{\rho}_{\text{M}}^\kappa + \nabla \cdot \vec{J}_{\text{M}}^\kappa = 0$ but also **complex**.

¹⁹ A. Mostafazadeh. In: *Classical and Quantum Gravity* 20.1 (2002), pp. 155–171.



Mostafazadeh inner product and the corresponding density

- Real power of \mathbb{D} is defined by spectral decomposition $\mathbb{D}^\gamma := \sum_\nu \nu^{2\gamma} \mathbf{P}_\nu$,
 $\mathbf{P}_\nu \Psi := \Psi_\nu (\Psi_\nu, \Psi)_S$.
- It can be shown²⁰ that *the density*

$$\varrho_M^\kappa := \kappa \left\{ |\mathbb{D}^{+1/4} \Psi|^2 + |\mathbb{D}^{-1/4} \partial_\beta \Psi|^2 \right\} \quad (27)$$

- is equivalent to ρ_M^κ up to a boundary term

$$\int d\chi \varrho_M^\kappa = \int d\chi \rho_M^\kappa \equiv (\Psi_1, \Psi_2)_M^\kappa; \quad (28)$$

- is non-negative.
- The corresponding current $\vec{\mathcal{J}}_M^\kappa$ is real but not conserved²¹.

²⁰ A. Mostafazadeh and F. Zamani. In: *Annals of Physics* 321.9 (2006), pp. 2183–2209.

²¹ B. Rosenstein and L. P. Horwitz. In: *Journal of Physics A: Mathematical and General* 18.11 (1985), pp. 2115–2121.



Wave packets of Gaussian amplitude for continuous spectrum

Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu; \bar{\nu}, \sigma) := \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\nu - \bar{\nu})^2}{2\sigma^2}\right) \right)^{1/2} \quad (29)$$

- In²², $A(\nu; \bar{\nu}, \sigma/\sqrt{2})$ was chosen.

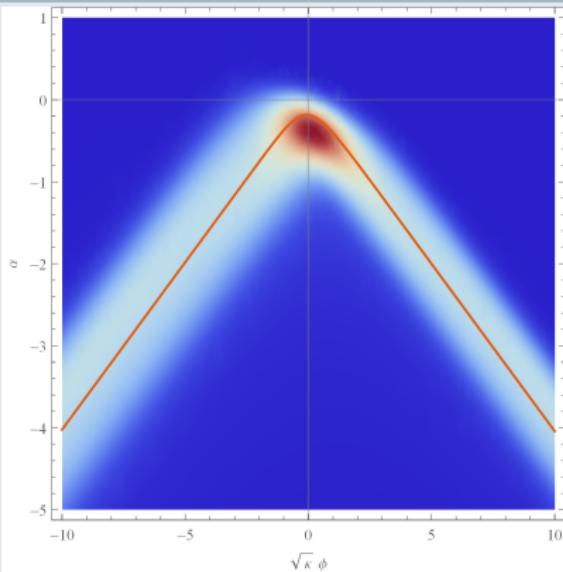
²² M. P. Dąbrowski, C. Kiefer, and B. Sandhöfer. In: *Physical Review D* 74.4 (2006).



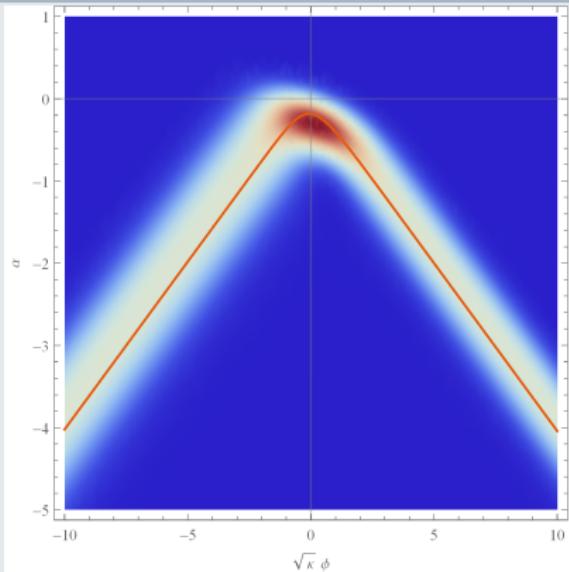
Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\parallel\nu}$, with $\lambda = \kappa^{1/2}/2$, $V = -\kappa^{-2}$, $\bar{k}_\beta = -2$ and $\sigma_\beta = 5/4$

Schrödinger



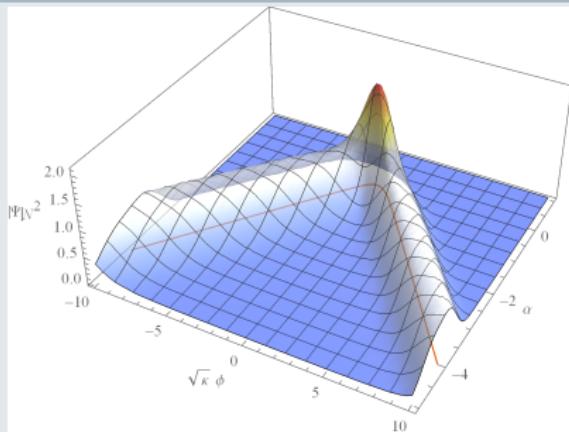
Mostafazadeh



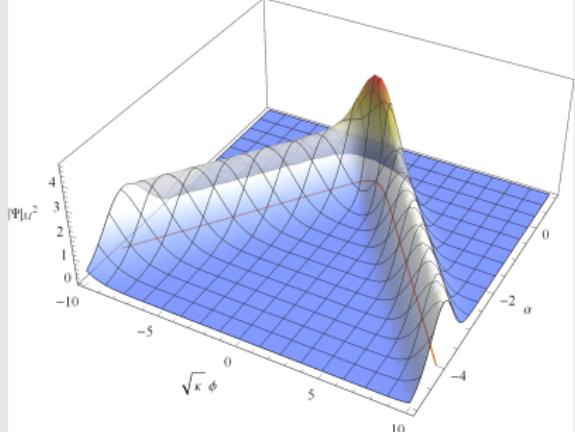
Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\parallel\nu}$, with $\lambda = \kappa^{1/2}/2$, $V = -\kappa^{-2}$, $\bar{k}_\beta = -2$ and $\sigma_\beta = 5/4$

Schrödinger



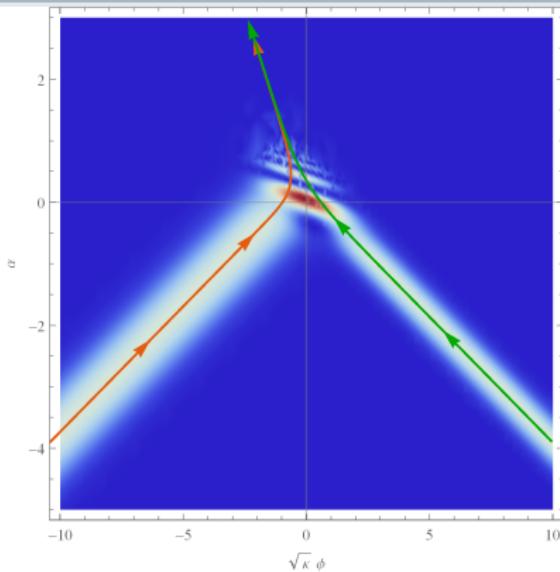
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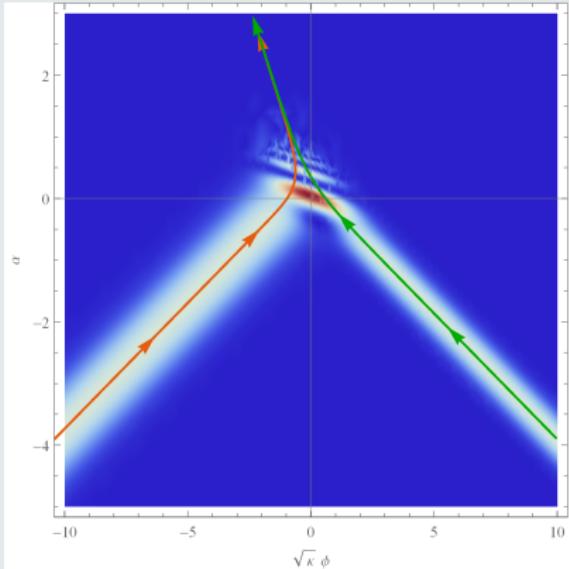
Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\pm\nu}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



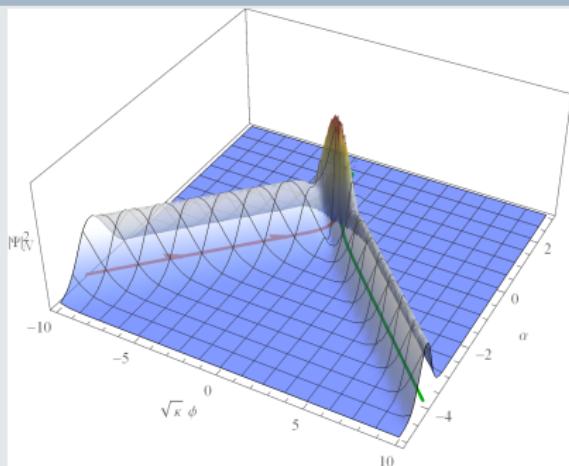
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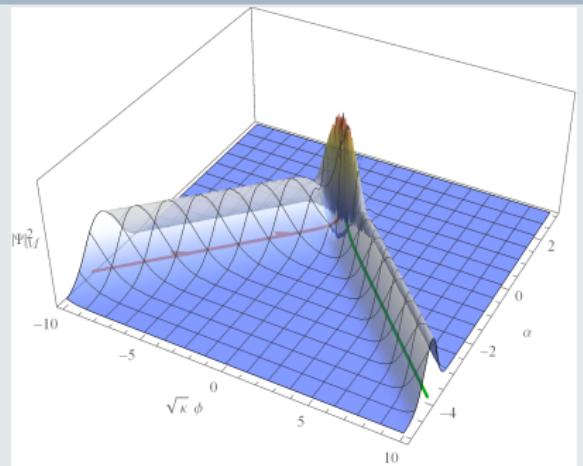
Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\tilde{\nu}}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



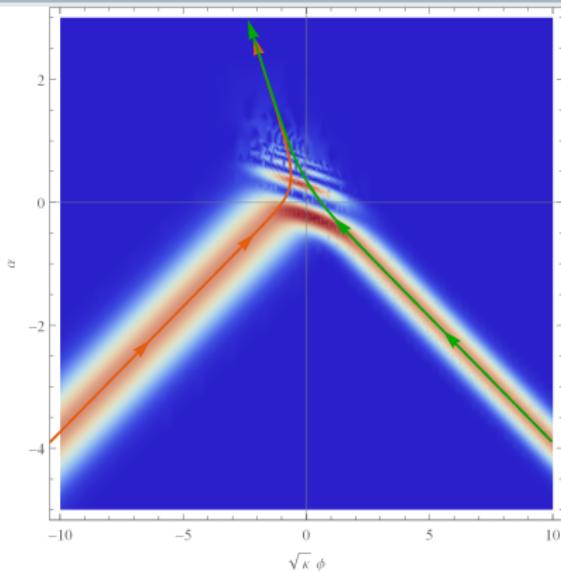
Mostafazadeh



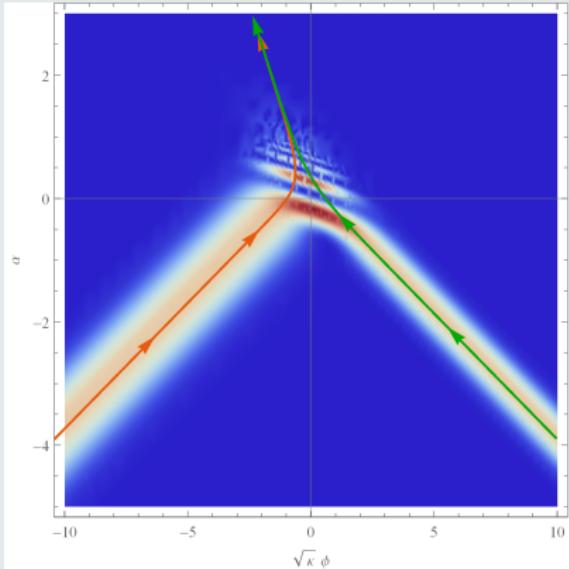
Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\text{I}\nu}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



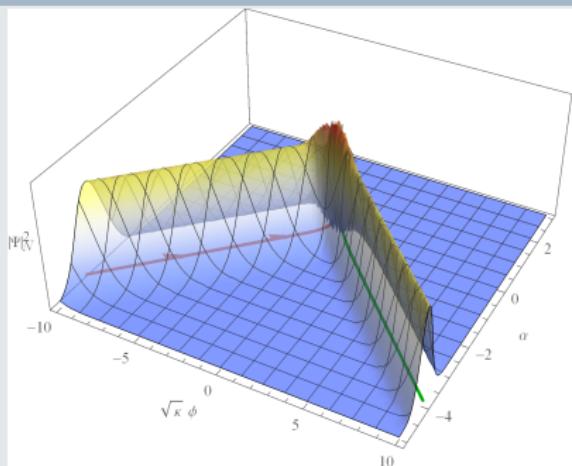
Mostafazadeh



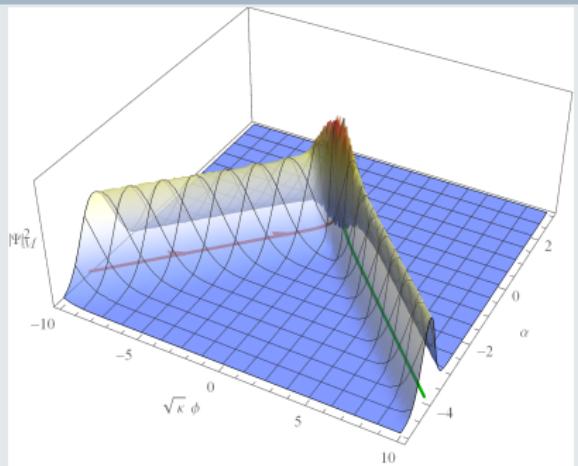
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Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model $(-, +)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\bar{n}) := \left(e^{-\bar{n}} \frac{\bar{n}^n}{n!} \right)^{1/2} \quad (30)$$

- In²³, $A_n(\bar{n}/\sqrt{2})$ was chosen.

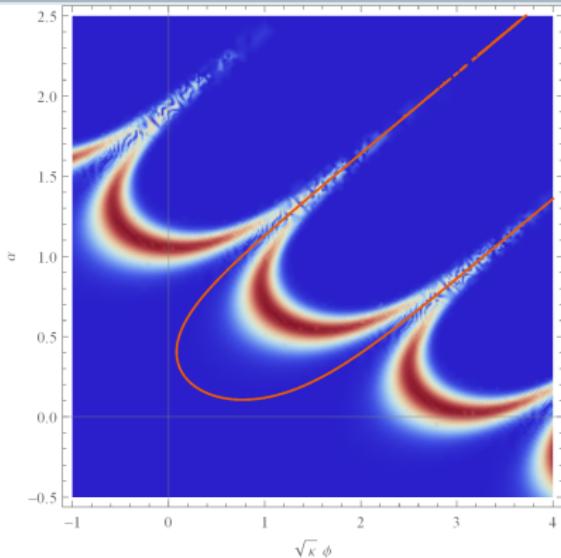
²³ C. Kiefer. In: *Nuclear Physics B* 341.1 (1990), pp. 273–293.



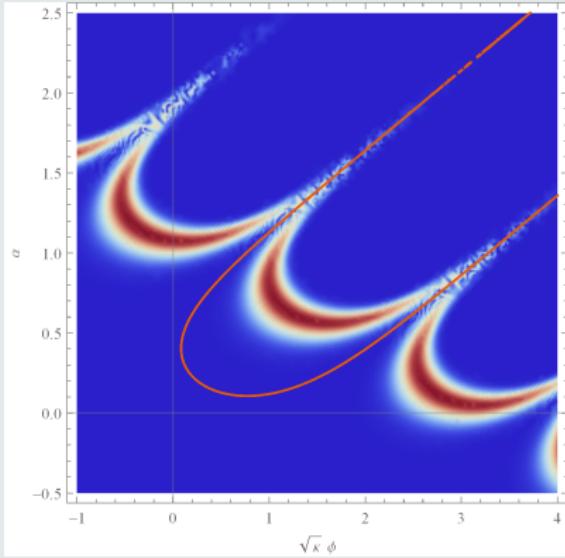
Wave packets of Poissonian amplitude for phantom model

J_{2n+1} , with $\lambda = 2\kappa^{1/2}$, $V = +\kappa^{-2}$ and $\bar{k}_\beta = 8$

Schrödinger



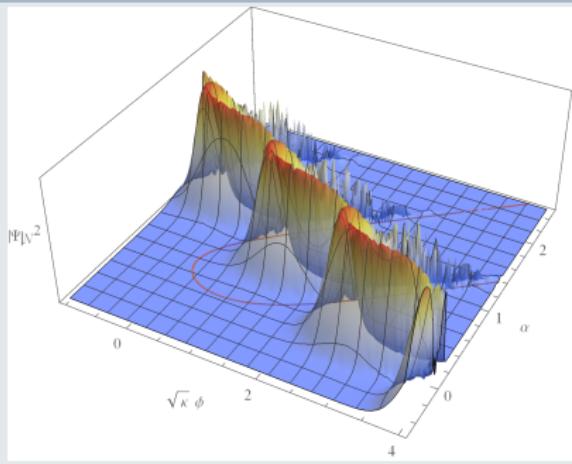
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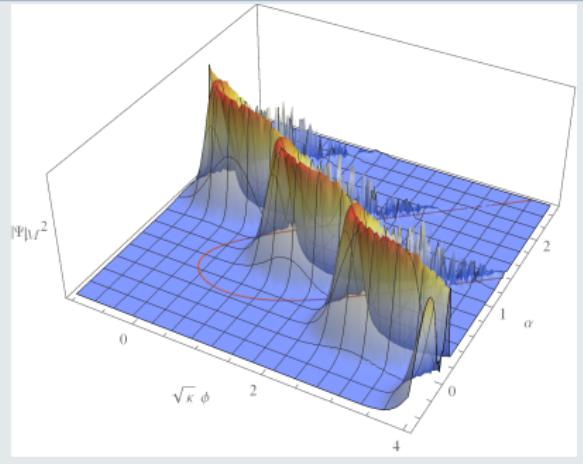
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Schrödinger



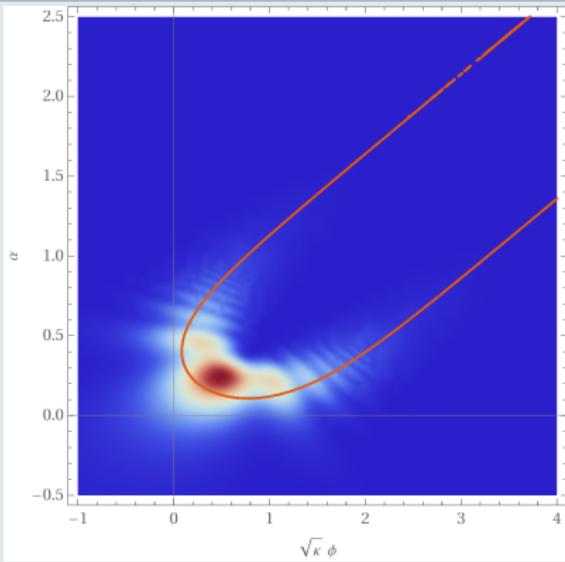
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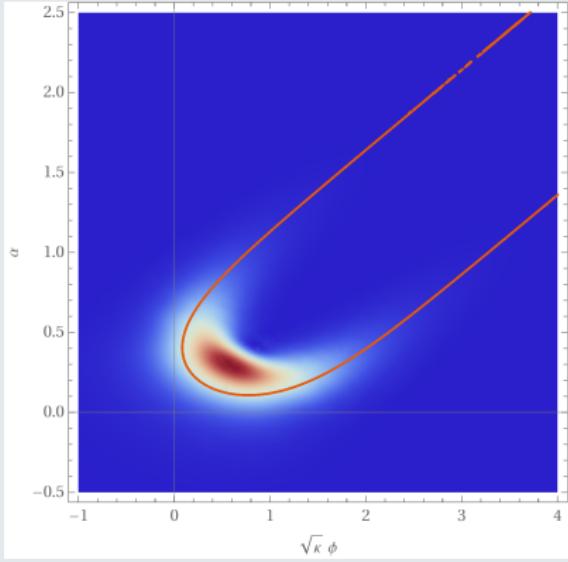
Wave packets of Gaussian amplitude for phantom model

J_ν , with $\lambda = 2\nu^{1/2}$, $V = +\nu^{-2}$, $\bar{k}_\beta = 8$ and $\sigma_\beta = 11/2$

Schrödinger



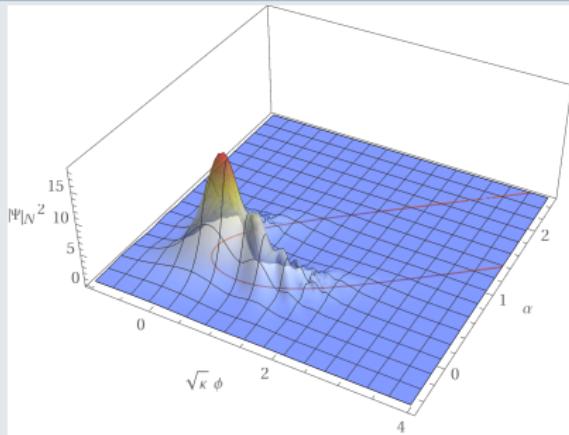
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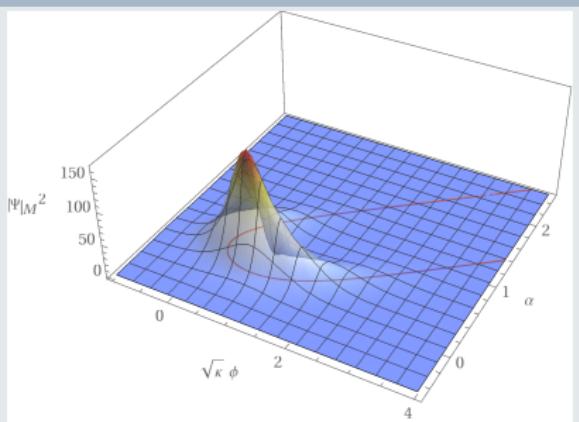
Wave packets of Gaussian amplitude for phantom model

J_ν , with $\lambda = 2\nu^{1/2}$, $V = +\nu^{-2}$, $\bar{k}_\beta = 8$ and $\sigma_\beta = 11/2$

Schrödinger



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Outline

1. Introduction

2. Classical model and the implicit trajectories

Lagrangian formalism

3. Quantised model and the wave packets

Canonical formalism and Dirac quantisation
Semi-classical approximation
Inner product and wave packet

4. Conclusions



Highlights

- An **integral of motion** was found for Liouville cosmological models.
- **Implicit trajectories** in minisuperspace were obtained.
- The levels of phantom Liouville model were found to be **discrete** due to the Hermiticity requirement of observables.
- The semi-classical quantum mode functions and wave packets was compared directly with the classical trajectory.



Issues

- In $(+, -)$ and $(-, +)$, wave packets contain multiple branches; however, the classical universe runs only on one trajectory.
- Quantum-corrected \bar{k}_β is to be understood.
- Wave packets with Gaussian profile are to be constructed, instead of inserting Gaussian / Poissonian amplitude artificially.
- A normalising κ for $(\cdot, \cdot)_M^\kappa$ is to be evaluated, otherwise a quantitative comparison of $(\cdot, \cdot)_S$ and $(\cdot, \cdot)_M^\kappa$ is not possible.



Outlook

- Beyond isotropy: generalise to Bianchi models
 - Bianchi Type-I: a natural extension, **under investigation**
- Beyond homogeneity: cosmological perturbation, **under investigation**
- Beyond single field
 - Two exponential potentials: $|V_1| = |V_2|$ and special λ_i
 - Multiple Liouville fields: mixing kinetic terms needed²⁴
- Beyond classic matter
 - PT -symmetric Liouville field²⁵: may cross the phantom divide $w = -1$.

²⁴ A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theoretical and Mathematical Physics* 184.3 (2015), pp. 1224–1233.

²⁵ A. A. Andrianov et al. In: *International Journal of Modern Physics D* 19.01 (2010), pp. 97–111, A. A. Andrianov, C. Lan, and O. O. Novikov. In: *Springer Proceedings in Physics*. Springer International Publishing, 2016, pp. 29–44.



Integration of the transformed first Friedmann equation

General integral for $p_\beta \neq 0$

In order to integrate the equation under the implicit gauge

$$\beta \frac{p_\beta^2}{12} \left(-\ell \frac{\nu^{1/2}}{6} \left(\frac{d\chi}{d\beta} \right)^2 + \nu^{-1/2} \right) - \nu^{3/2} V e^{g_\beta \chi} = 0, \quad (31)$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\nu}} g\beta, \quad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\nu^2 |V|} e^{-g_\beta \chi}, \quad (32)$$

to get

$$\left(\frac{d\tilde{\sigma}}{d\gamma} \right)^2 + \ell(\beta v - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{d\gamma}{d\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-\beta v + \tilde{\sigma}^2)}}, \quad (33)$$

which is of the standard inverse hyperbolic / trigonometric form **except for $(-, -)$.**



Integration of the separated minisuperspace WDW equation

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\nu} \partial_\chi^2 + \varsigma \nu \frac{12\nu^2 |V|}{\hbar^2}, \quad (14 \text{ rev.})$$

one can transform

$$\nu := \sqrt{\frac{2\nu}{3}} \frac{k_\beta}{g}, \quad \sigma^2 := \frac{8\nu^3 |V| e^{g_4 \chi}}{\hbar^2 g^2}, \quad (16 \text{ rev.})$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma \psi'(\sigma) + \ell(\nu^2 - \varsigma \nu \sigma^2) \psi(\sigma) = 0, \quad (34)$$

which is of the standard Besselian form.

