

Integrable Liouville Cosmological Models

Self-adjointness of the Hamiltonian and Semi-classical Wave Functions

Alexander A. Andrianov^{1,4} Chen Lan² Oleg O. Novikov¹ Yi-Fan Wang³

¹Saint-Petersburg State University, Ulyanovskaya str. 1, Petrodvorets, Sankt-Petersburg 198504, Russland

²ELI-ALPS Research Institute, Budapesti út 5, H-67228 Szeged, Ungarn

³Institut für Theoretische Physik, Universität zu Köln, Zülpicher Straße 77, D-50937 Köln, Deutschland

⁴Institut de Ciències del Cosmos, Universitat de Barcelona, Martí i Franquès 1, E-08028 Barcelona, Spanien

March 23, 2018



Outline

1. Introduction

2. The classical model and the implicit trajectories

Lagrangian formalism and the first integral

3. Dirac quantisation and self-adjointness of the Hamiltonian

Dirac quantisation and the quantum-mechanical analogy

Self-adjointness of the Hamiltonian

4. The semi-classical wave functions

The Wentzel–Kramers–Brillouin approximation

Inner products and wave packets

5. Conclusions



Outline

1. Introduction
2. The classical model and the implicit trajectories
Lagrangian formalism and the first integral
3. Dirac quantisation and self-adjointness of the Hamiltonian
Dirac quantisation and the quantum-mechanical analogy
Self-adjointness of the Hamiltonian
4. The semi-classical wave functions
The Wentzel–Kramers–Brillouin approximation
Inner products and wave packets
5. Conclusions



Introduction

The Friedmann–Lemaître model with quintessence and phantom Liouville fields

- Dynamical Dark Energy has been modelled by quintessence¹ and phantom² matter, which can be realised by minimally-coupled real scalar fields with $\ell = \pm 1$ ³

$$\mathcal{S}_L = \int d^4x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \quad (1)$$

- The Liouville field⁴ $\mathcal{V}(\phi) = V e^{\lambda\phi}$ is of interest, where $\lambda, V \in \mathbb{R}$.
- Assume a flat Robertson–Walker metric $g_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + \varkappa e^{2\alpha(t)} d\Omega_{3F}^2$, where $\varkappa := 8\pi G$, $d\Omega_{3F}^2 := d\chi^2 + \chi^2(d\theta^2 + \sin^2 \theta d\varphi^2)$, and N lapse function.
- The total action reads $\mathcal{S} := \mathcal{S}_{EH} + \mathcal{S}_{GHY} + \mathcal{S}_L = \int d\Omega_{3F}^2 \int dt L$, where

$$L := \varkappa^{3/2} N e^{3\alpha} \left(-\frac{3}{\varkappa} \frac{\dot{\alpha}^2}{N^2} + \ell \frac{\dot{\phi}^2}{2N^2} - V e^{\lambda\phi} \right). \quad (2)$$

¹R. R. Caldwell *et al.*, *Phys. Rev. Lett.* **80**, 1582–1585 (Aug. 7, 1997).

²R. R. Caldwell, *Phys. Lett. B* **545**, 23–29 (Aug. 16, 1999).

³The signature of metric is $(-, +, \dots, +)$.

⁴Y. Nakayama, *Int. J. Mod. Phys. A* **19**, 2771–2930 (Feb. 2, 2004).



Introduction

Highlights

Integrability

With the help of an integral of motion, implicit trajectories can be obtained explicitly, and the minisuperspace Wheeler–DeWitt equation can be integrated exactly.

Self-adjointness

Imposing self-adjointness of the Hamiltonian leads to removal of the degeneracy and discretisation of the spectrum.

Semi-classical wave functions

Semi-classical wave functions can be constructed by the principle of constructive interference, the Wentzel–Kramers–Brillouin approximation and numerical methods.



Outline

1. Introduction
2. The classical model and the implicit trajectories
Lagrangian formalism and the first integral
3. Dirac quantisation and self-adjointness of the Hamiltonian
Dirac quantisation and the quantum-mechanical analogy
Self-adjointness of the Hamiltonian
4. The semi-classical wave functions
The Wentzel–Kramers–Brillouin approximation
Inner products and wave packets
5. Conclusions



Decoupling the variables

Via orthogonal transformation

- By rescaling $\bar{N} := N e^{-3\alpha}$ and the orthogonal transformation

$$\begin{pmatrix} \alpha(t) \\ \phi(t) \end{pmatrix} = \frac{s}{g} \begin{pmatrix} \lambda & -\ell \nu \\ -6 & \lambda \end{pmatrix} \begin{pmatrix} \beta(t) \\ \chi(t) \end{pmatrix}, \quad (3)$$

the system can be decoupled ($\Delta := \lambda^2 - 6\ell\nu$, $s := \operatorname{sgn} \Delta$ and $g := s\sqrt{|\Delta|} \equiv s\sqrt{s\Delta}$)

$$L = \nu^{3/2} \bar{N} \left(-s \frac{3}{\nu} \frac{\dot{\beta}^2}{\bar{N}^2} + \ell s \frac{\dot{\chi}^2}{2\bar{N}^2} - V e^{g\chi} \right). \quad (4)$$

- Using the **conservation of p_β** ⁵, $\bar{N}(t)$ can be eliminated, leading to the exact integrations of the equations of motion as **implicit trajectories**.
 - $p_\beta \neq 0$: four cases according to $(\ell, s\nu)$, $v := \operatorname{sgn} V$.
 - $p_\beta = 0$: well-known power-law special solution for $(+, -)$ and $(-, +)$.

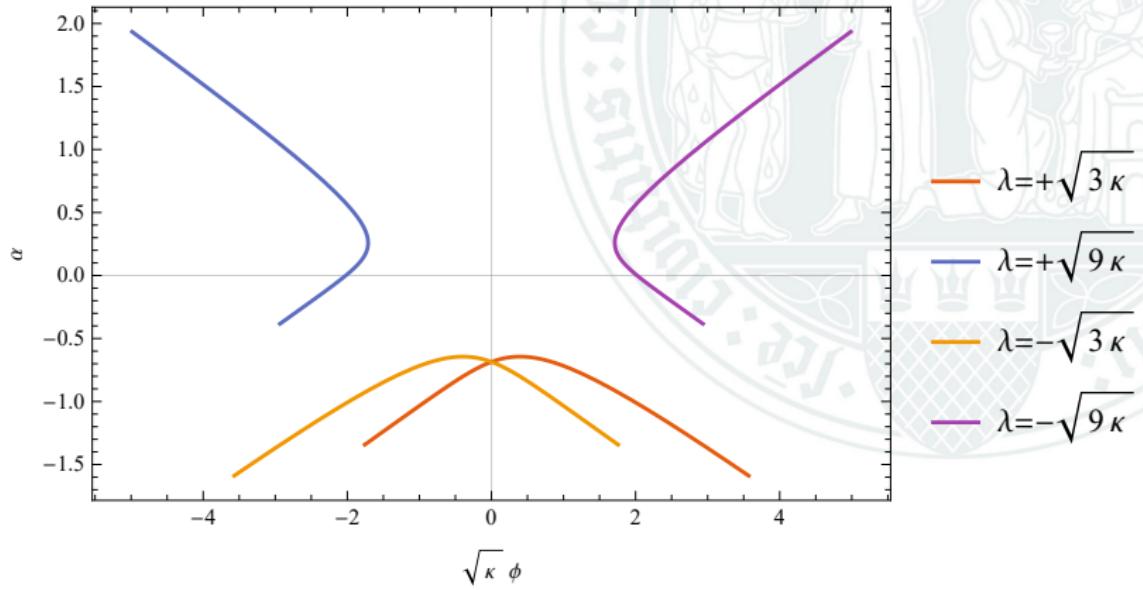
⁵The same first integral has been found in C. Lan, PhD thesis, Saint Petersburg State University, 2016, <https://search.rsl.ru/ru/record/01006663434>, A. A. Andrianov et al., *Theor. Math. Phys.* **184**, 1224–1233 (Mar. 18, 2015), in canonical formalism.



Trajectories for quintessence model (+, +)

sech, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

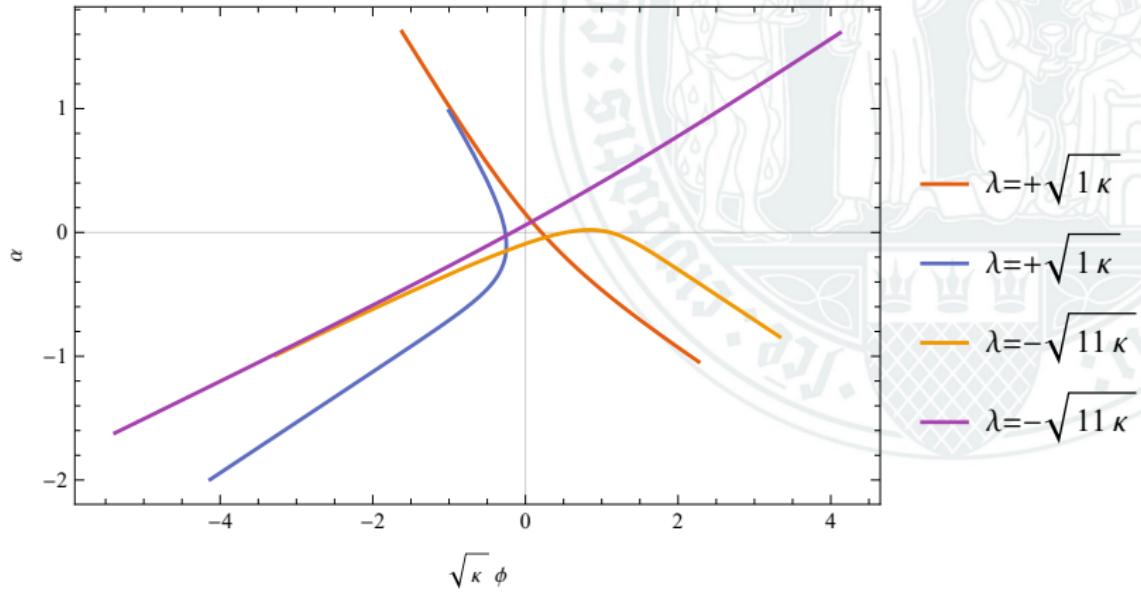
$$e^{g\chi} = \frac{p_\beta^2}{12\kappa^2|V|} \operatorname{sech}^2\left(\sqrt{\frac{3}{2\kappa}} g(\beta - \beta_0)\right) \quad (5)$$



Trajectories for quintessence model $(+, -)$

csch , with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

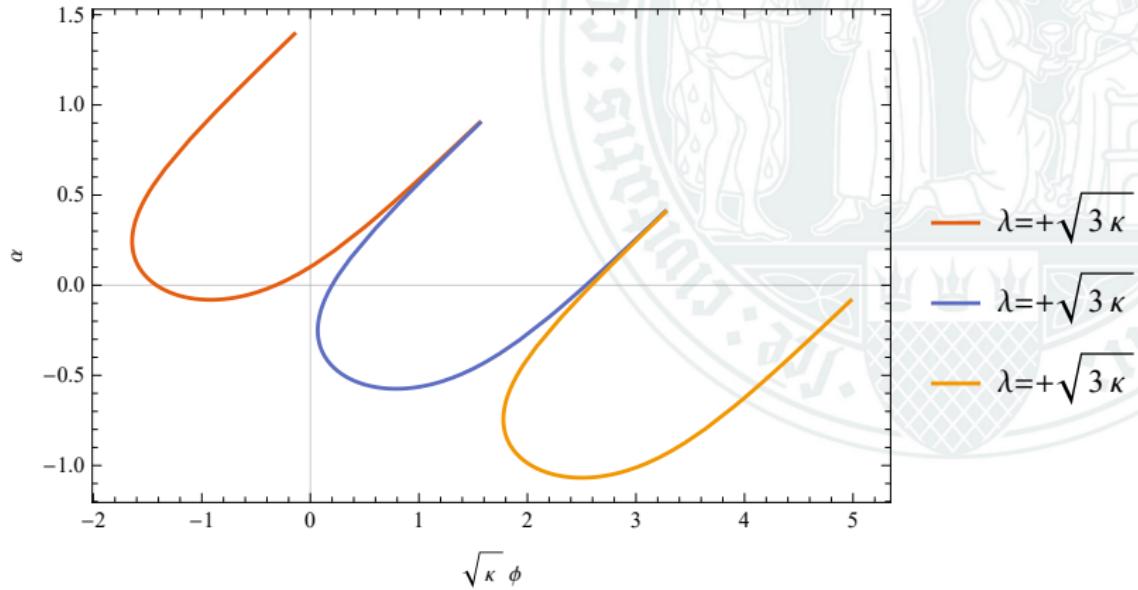
$$e^{g\chi} = \frac{p_\beta^2}{12\kappa^2|V|} \text{csch}^2\left(\sqrt{\frac{3}{2\kappa}} g(\beta - \beta_0)\right) \quad (6)$$



Trajectories for phantom model $(-, +)$

sec, with $V = \varkappa^{-2}$ and $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

$$\mathbb{E}^{g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|} \sec^2\left(\sqrt{\frac{3}{2\varkappa}} g(\beta - \beta_0)\right) \quad (7)$$



Outline

1. Introduction
2. The classical model and the implicit trajectories
Lagrangian formalism and the first integral
3. Dirac quantisation and self-adjointness of the Hamiltonian
Dirac quantisation and the quantum-mechanical analogy
Self-adjointness of the Hamiltonian
4. The semi-classical wave functions
The Wentzel–Kramers–Brillouin approximation
Inner products and wave packets
5. Conclusions



Dirac quantisation

- The primary Hamiltonian and the Hamiltonian constraint⁶ reads

$$H^p = \bar{N} H_{\perp} + v^{\bar{N}} p_{\bar{N}}, \quad (8)$$

$$H_{\perp} := -\beta \frac{p_{\beta}^2}{12\nu^{1/2}} + \ell \beta \frac{p_{\chi}^2}{2\nu^{3/2}} + \nu^{3/2} V e^{g\chi}. \quad (9)$$

- Applying the Dirac quantisation rules with the Laplace–Beltrami operator ordering⁷, one gets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(\beta \frac{\hbar^2}{12\nu^{1/2}} \partial_{\beta}^2 - \ell \beta \frac{\hbar^2}{2\nu^{3/2}} \partial_{\chi}^2 + \nu^{3/2} V e^{g\chi} \right) \Psi. \quad (10)$$

- Equation (10) is Klein–Gordon-like, hyperbolic for $\ell = +1$ and *elliptic* for $\ell = -1$.

⁶D. M. Gitman, I. V. Tyutin, *Quantization of Fields with Constraints*, (Springer, 1990), H. J. Rothe, K. D. Rothe, *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*, (World Scientific, Apr. 2010).

⁷C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012), ch. 8.



Integration of the Wheeler–DeWitt equation

The quantum-mechanical analogy

- Fourier transforming β to $k_\beta \in \mathbb{R}$ yields the time-independent Schrödinger equation

$$E\psi(x) := \ell \frac{\hbar^2 \tilde{g}^2 \nu^2}{8M_P} \psi = \widehat{H}_{\text{eff}} \psi := \left(-\frac{\hbar^2}{2M_P} \partial_x^2 + V_{\text{eff}} \right) \psi, \quad V_{\text{eff}} := \ell s v \widetilde{V} e^{\tilde{g}x}, \quad (11)$$

in which $M_P := \sqrt{\frac{\hbar}{\nu}}$, $x := \sqrt{\hbar\nu} \cdot \chi$, $\widetilde{V} := M_P^{-3} |V| \geq 0$, and $\nu = \sqrt{\frac{2\nu}{3} \frac{|k_\beta|}{g}} \geq 0$.

- The Liouville potential V_{eff} is a special case of the Morse potential⁸.
- The Starobinsky model in the Einstein frame is related to the Morse potential.
- Let $e^{2y} := 8M_P \widetilde{V}/\hbar^2 \tilde{g}^2 \cdot e^{\tilde{g}x}$; there are four cases according to (ℓ, sv) , and the mode functions are

	$E \geq 0$	$E < 0$
$V_{\text{eff}} > 0$	$(+, +); K_{i\nu}(e^y)$	$(-, -); \text{none}$
$V_{\text{eff}} < 0$	$(+, -); F_{i\nu}(e^y), G_{i\nu}(e^y)$ ⁹	$(-, +); J_\nu(e^y)$

⁸P. M. Morse, *Phys. Rev.* **34**, 57–64 (July 1929), D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012).

⁹These are linear combinations of $J_{\pm i\nu}(e^{\tilde{x}})$; see T. M. Dunster, *SIAM J. Math. Anal.* **21**, 995–1018 (July 1990).



Self-adjointness of the Hamiltonian

Results for the Liouville potential¹³

- $k_\beta \in \mathbb{R}$, enforcing the self-adjointness of \widehat{H}_{eff} .
- For $V_{\text{eff}} > 0$: \widehat{H}_{eff} is **essentially self-adjoint**¹⁰.
 - Quintessence (+, +): nothing special.
- For $V_{\text{eff}} < 0$: \widehat{H}_{eff} admits **a family of self-adjoint extensions**.
 - Quintessence (+, -): "eigenfunctions" are linear combinations of F and G; **the degeneracy is removed**
 - Phantom (-, +): the spectrum becomes **discrete**

$$\psi_{+, \nu}(\mathfrak{e}^y) \propto J_\nu(\mathfrak{e}^y), \quad \nu = 2n + a_<, \quad n \in \mathbb{Z}_\geq, \quad (12)$$

and the corresponding full wave-functions are Bloch-periodic. Imposing (anti-)periodic boundary condition fixes $a_< = 0$ (1).

- The self-adjointness has been discussed in quantum cosmology in e.g.¹¹ (operator ordering) and more recently in¹² (Λ as a quantum number).

¹⁰See also B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267, sec. 9.9.

¹¹C. R. Almeida et al., *Gravitation Cosmol.* **21**, 191–199 (Jan. 17, 2015).

¹²S. Gryb, K. P. Y. Thébault, arXiv: 1801.05789 (gr-qc) (Jan. 17, 2018).

¹³See also D. M. Gitman et al., *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012), sec. 8.5.



Outline

1. Introduction
2. The classical model and the implicit trajectories
Lagrangian formalism and the first integral
3. Dirac quantisation and self-adjointness of the Hamiltonian
Dirac quantisation and the quantum-mechanical analogy
Self-adjointness of the Hamiltonian
4. The semi-classical wave functions
The Wentzel–Kramers–Brillouin approximation
Inner products and wave packets
5. Conclusions



Matching quantum number with classical first integral

Principle of constructive interference; $(+, +)$ as the example

- Write the mode function in the Wentzel–Kramers–Brillouin form
 $\Psi_\nu(\beta, \chi) \sim \sqrt{\rho(\beta, \chi)} \exp\left\{\frac{i}{\hbar} S(\beta, \chi)\right\}$. For $S/\hbar \gg 1$ and $\nu \gg 1$, $S(\beta, \chi)$ becomes the Hamilton principal function in the leading-order approximation, which is stationary with respect to the variation of integral constants $\partial S / \partial \nu = 0$.
- Fixing $\nu/\sigma > 1$,

$$K_{\pm\nu}(\sigma) \sim \frac{\sqrt{2\pi} \cos\left(\sqrt{\nu^2 - \sigma^2} - \nu \arccos \frac{\nu}{\sigma} - \frac{\pi}{4}\right)}{(\nu^2 - \sigma^2)^{1/4} e^{\mp\nu/2}} + O\left(\frac{1}{\sigma}\right). \quad (13)$$

- Applying *the principle* to the phase of $\Psi_\nu(\sigma)$, one has $\sigma/\nu = \operatorname{sech}(\beta_\beta \gamma)$, which matches the trajectory if $\hbar k_\beta \equiv \hbar \sqrt{\frac{3}{2\nu}} g\nu = p_\beta$.
- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; it is not within the Wentzel–Kramers–Brillouin regime.
- The conclusions also hold for $F_{\pm\nu}(\sigma)$, $G_{\pm\nu}(\sigma)$ for $(+, -)$, and $J_\nu(\sigma)$ for $(-, +)$.



Inner products for wave functions

Schrödinger and Mostafazadeh inner product

- A common starting point is the Schrödinger product¹⁴

$$(\Psi_1, \Psi_2)_S := \int d\chi \Psi_1^*(\beta, \chi) \Psi_2(\beta, \chi); \quad (14)$$

- $(\Psi, \Psi)_S$ is **positive-definite**, and the integrand $\rho_S(\beta, \chi)$ is **non-negative**
- The corresponding Schrödinger current is **real** but **does not satisfy continuity equation**
- Mostafazadeh¹⁵: *for self-adjoint \mathbb{D} with positive spectrum in $(-\partial_t^2 + \mathbb{D})\Psi = 0$:*

$$(\Psi_1, \Psi_2)_M^\kappa := \kappa \left\{ (\Psi_1, \mathbb{D}^{+1/2} \Psi_2)_S + (\partial_\beta \Psi_1, \mathbb{D}^{-1/2} \partial_\beta \Psi_2)_S \right\}, \quad \kappa > 0. \quad (15)$$

- Real power of \mathbb{D} is defined by spectral decomposition
- It can be shown¹⁶ that *the density*

$$\varrho_M^\kappa := \kappa \left\{ |\mathbb{D}^{+1/4} \Psi|^2 + |\mathbb{D}^{-1/4} \partial_\beta \Psi|^2 \right\} \quad (16)$$

- is **equivalent to the integrand** ρ_M^κ up to a boundary term $\int d\chi \varrho_M^\kappa = \int d\chi \rho_M^\kappa \equiv (\Psi_1, \Psi_2)_M^\kappa$
- is **non-negative**
- The corresponding current $\vec{\mathcal{J}}_M^\kappa$ is **real** but **not conserved**¹⁷.

¹⁴ C. Kiefer, *Quantum Gravity*, (Oxford University, ed. 3, Apr. 2012), ch. 5.

¹⁵ A. Mostafazadeh, *Class. Quantum Grav.* **20**, 155–171 (Sept. 6, 2002).

¹⁶ A. Mostafazadeh, F. Zamani, *Ann. Phys.* **321**, 2183–2209 (Feb. 17, 2006).

¹⁷ B. Rosenstein, L. P. Horwitz, *J. Phys. A: Math. Gen.* **18**, 2115–2121 (Aug. 1985).



Wave packets of Gaussian amplitude for continuous spectrum

Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu; \bar{\nu}, \sigma) := \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\nu - \bar{\nu})^2}{2\sigma^2}\right) \right)^{1/2} \quad (17)$$

- In¹⁸, $A(\nu; \bar{\nu}, \sigma/\sqrt{2})$ was chosen.

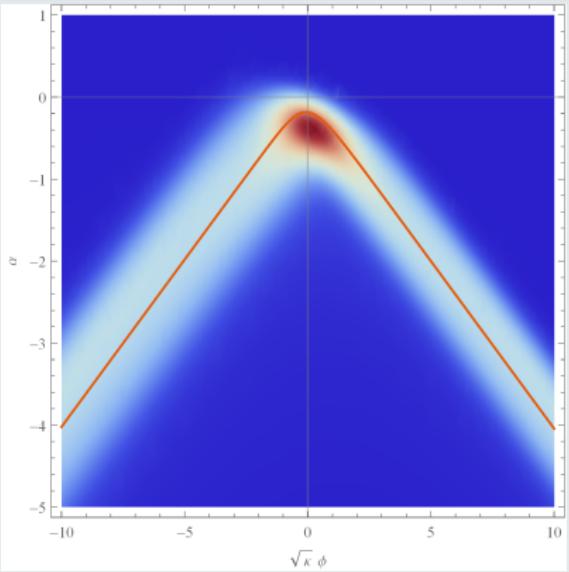
¹⁸M. P. Dąbrowski et al., *Phys. Rev. D* **74**, arXiv: hep-th/0605229 (hep-th) (May 23, 2006).



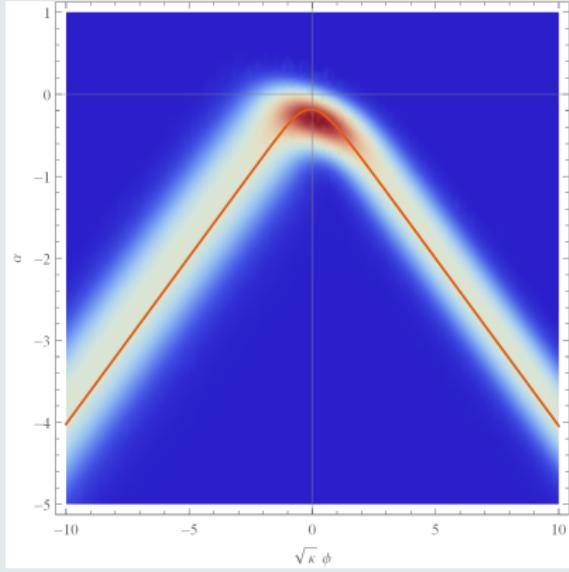
Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\ell\nu}$, with $\lambda = \varkappa^{1/2}/2$, $V = -\varkappa^{-2}$, $\bar{k}_\beta = -2$ and $\sigma_\beta = 5/4$

Schrödinger



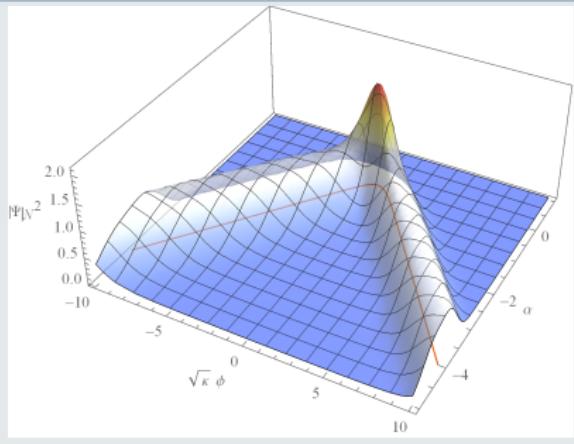
Mostafazadeh



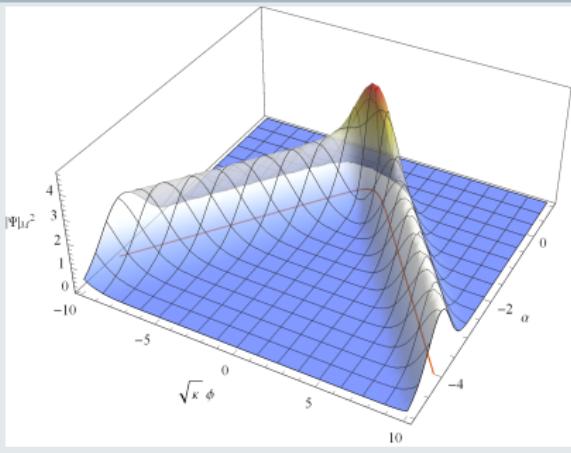
Wave packets of Gaussian amplitude for quintessence model (+, +)

$K_{\ell\nu}$, with $\lambda = \varkappa^{1/2}/2$, $V = -\varkappa^{-2}$, $\bar{k}_\beta = -2$ and $\sigma_\beta = 5/4$

Schrödinger



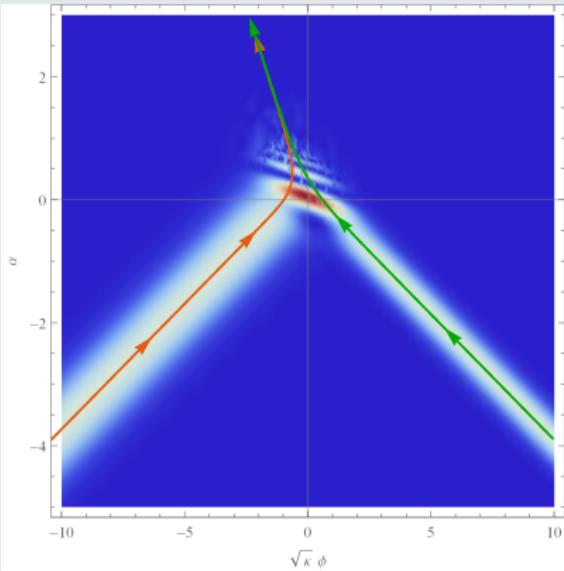
Mostafazadeh



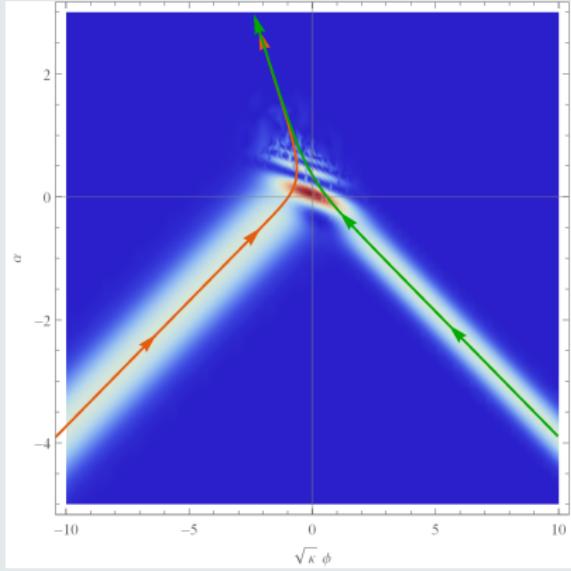
Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\parallel\nu}$, with $\lambda = 4\nu^{1/2}/5$, $V = +\nu^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



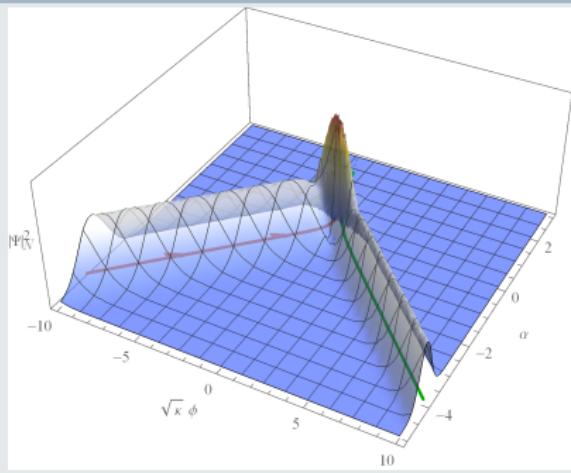
Mostafazadeh



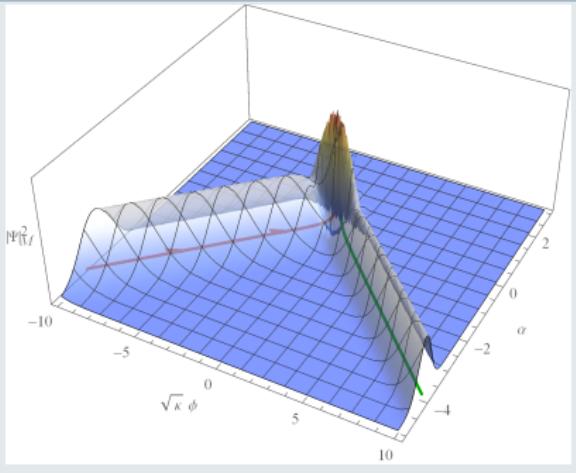
Wave packets of Gaussian amplitude for quintessence model (+, -)

$F_{\parallel\nu}$, with $\lambda = 4\nu^{1/2}/5$, $V = +\nu^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



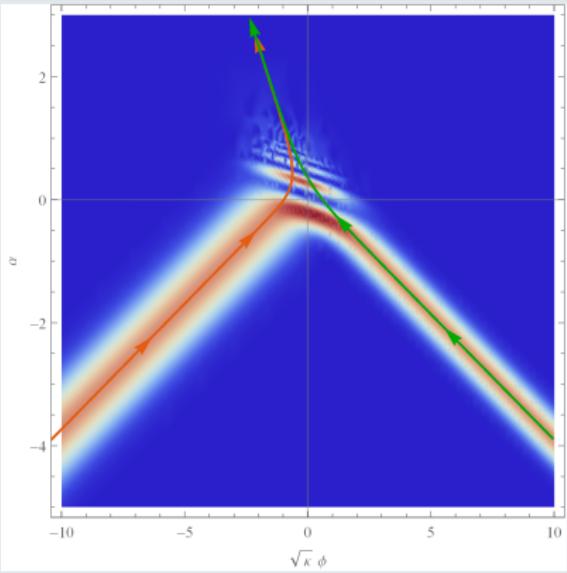
Mostafazadeh



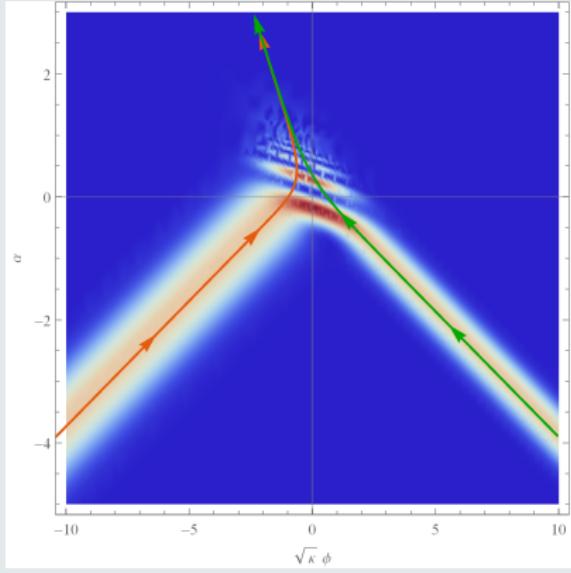
Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\parallel\nu}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



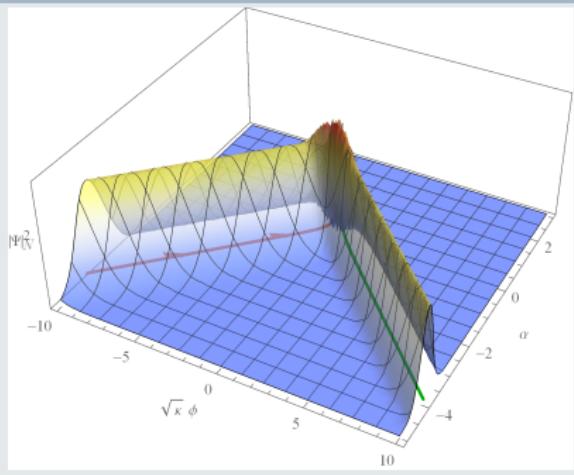
Mostafazadeh



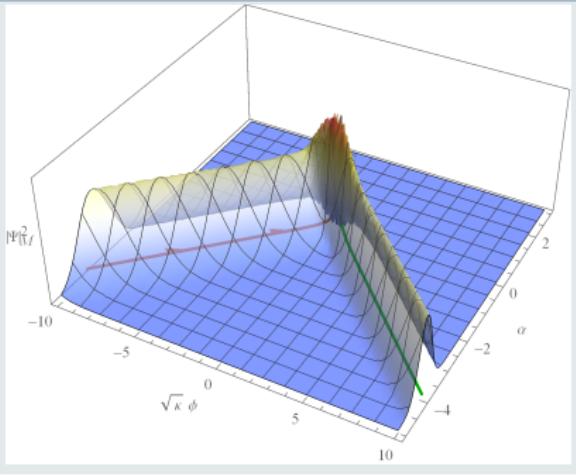
Wave packets of Gaussian amplitude for quintessence model (+, -)

$G_{\parallel\nu}$, with $\lambda = 4\kappa^{1/2}/5$, $V = +\kappa^{-2}$, $\bar{k}_\beta = -7/2$ and $\sigma_\beta = 7/5$

Schrödinger



Mostafazadeh



Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model $(-, +)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\bar{n}) := \left(e^{-\bar{n}} \frac{\bar{n}^n}{n!} \right)^{1/2} \quad (18)$$

- In¹⁹, $A_n(\bar{n}/\sqrt{2})$ was chosen.

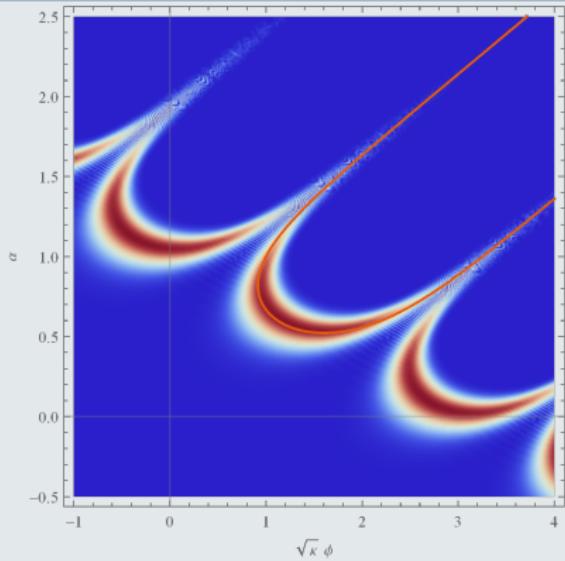
¹⁹C. Kiefer, *Nucl. Phys. B* **341**, 273–293 (Sept. 1990).



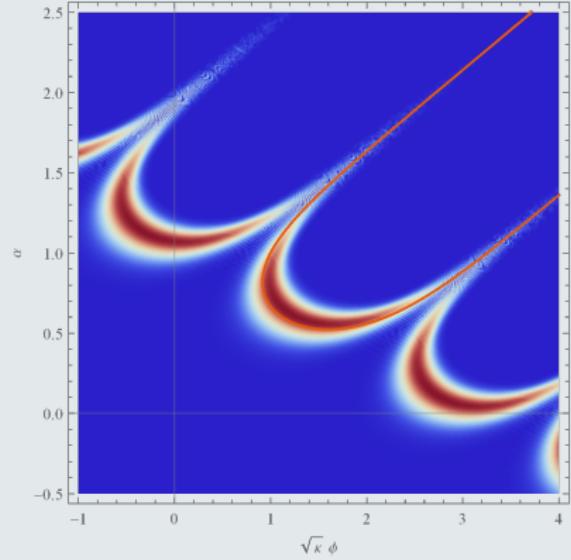
Wave packets of Poissonian amplitude for phantom model

J_{2n+1} , with $\lambda = 2\kappa^{1/2}$, $V = +\kappa^{-2}$ and $\bar{k}_\beta = 8$

Schrödinger



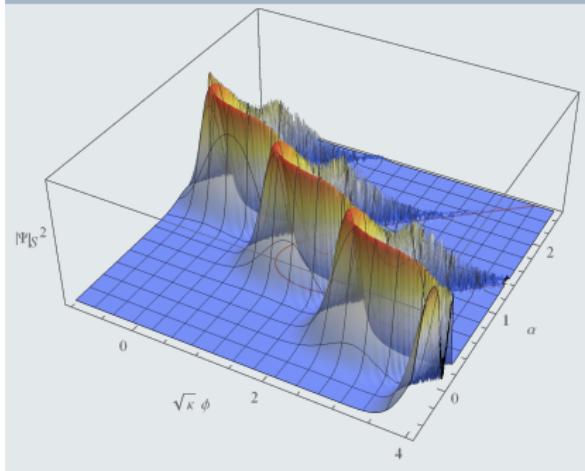
Mostafazadeh



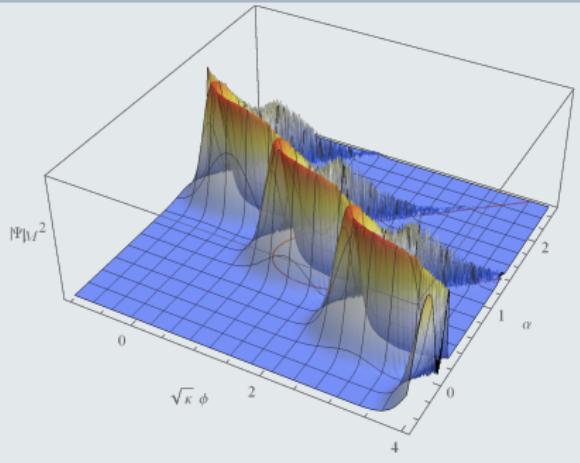
Wave packets of Poissonian amplitude for phantom model

J_{2n+1} , with $\lambda = 2\kappa^{1/2}$, $V = +\kappa^{-2}$ and $\bar{k}_\beta = 8$

Schrödinger



Mostafazadeh



Outline

1. Introduction
2. The classical model and the implicit trajectories
Lagrangian formalism and the first integral

3. Dirac quantisation and self-adjointness of the Hamiltonian
Dirac quantisation and the quantum-mechanical analogy
Self-adjointness of the Hamiltonian

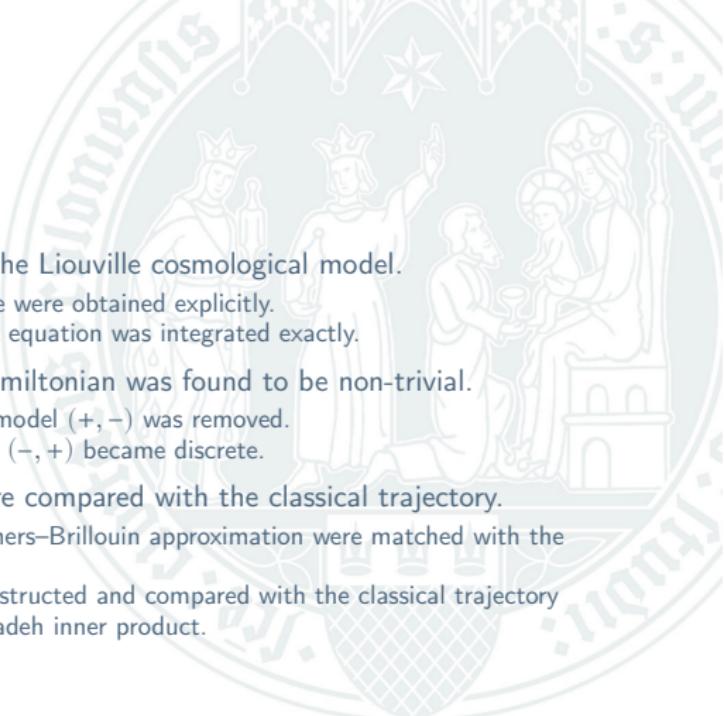
4. The semi-classical wave functions
The Wentzel–Kramers–Brillouin approximation
Inner products and wave packets

5. Conclusions



Highlights

Revisited

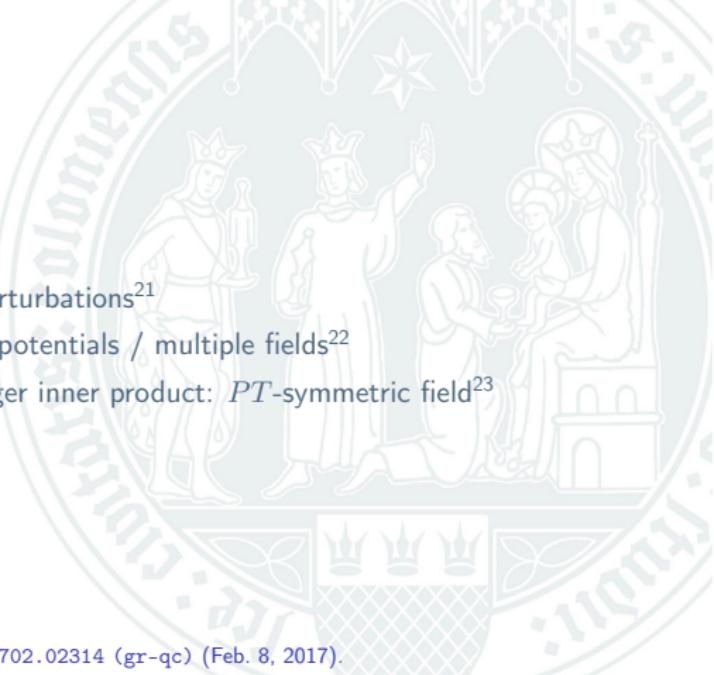


- An **integral of motion** was found for the Liouville cosmological model.
 - Implicit trajectories in minisuperspace were obtained explicitly.
 - The minisuperspace Wheeler–DeWitt equation was integrated exactly.
- The **self-adjointness** of the matter Hamiltonian was found to be non-trivial.
 - The degeneracy of the quintessence model $(+, -)$ was removed.
 - The spectrum of the phantom model $(-, +)$ became discrete.
- The **semi-classical wave functions** were compared with the classical trajectory.
 - Mode functions in the Wentzel–Kramers–Brillouin approximation were matched with the classical Hamilton principal function.
 - Semi-classical wave packets were constructed and compared with the classical trajectory under the Schrödinger and Mostafazadeh inner product.



Outlook

- Beyond isotropy: Bianchi models²⁰
- Beyond homogeneity: cosmological perturbations²¹
- Beyond single Liouville field: multiple potentials / multiple fields²²
- Beyond classical matter and Schrödinger inner product: *PT*-symmetric field²³



²⁰A. Y. Kamenshchik *et al.*, *Phys. Rev. D* **95**, arXiv: 1702.02314 (gr-qc) (Feb. 8, 2017).

²¹D. Brizuela *et al.*, *Phys. Rev. D* **93**, arXiv: 1511.05545 (gr-qc) (Nov. 17, 2015), D. Brizuela *et al.*, *Phys. Rev. D* **94**, arXiv: 1611.02932 (gr-qc) (Nov. 9, 2016), O. O. Novikov, arXiv: 1712.03939 (hep-th) (Dec. 11, 2017), A. Y. Kamenshchik *et al.*, *Class. Quantum Grav.* **35**, 015012 (Sept. 29, 2017), C. Kiefer, D. Wichmann, arXiv: 1802.01422 (gr-qc) (Feb. 5, 2018).

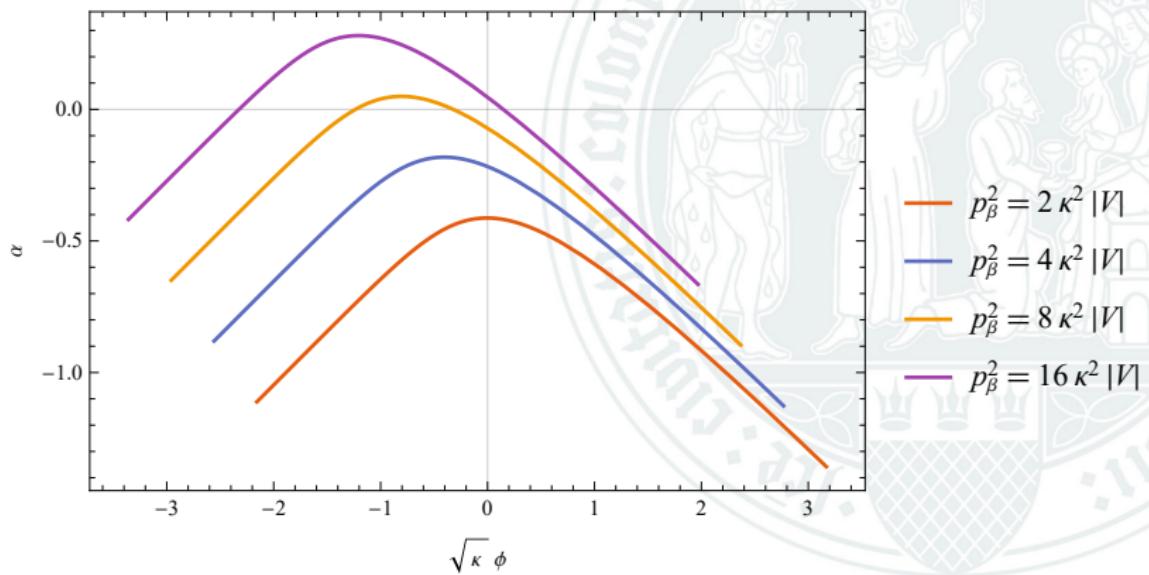
²²A. A. Andrianov *et al.*, *Theor. Math. Phys.* **184**, 1224–1233 (Mar. 18, 2015).

²³A. A. Andrianov *et al.*, *Int. J. Mod. Phys. D* **19**, 97–111 (Jan. 2010), A. A. Andrianov *et al.*, in *Springer Proceedings in Physics* (Springer International Publishing, 2016), pp. 29–44.



Trajectories for quintessence model (+, +)

sech, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $\lambda^2 = 3\kappa$; varying p_β

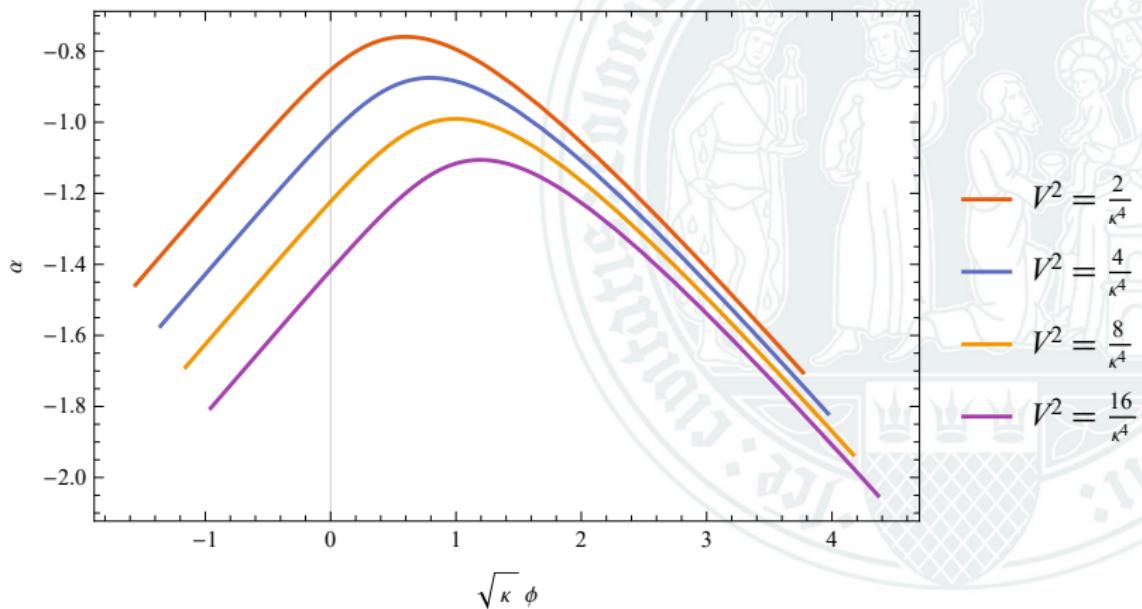


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model (+, +)

sech, with $\beta_0 = 0$, $\lambda^2 = 3\kappa$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying V

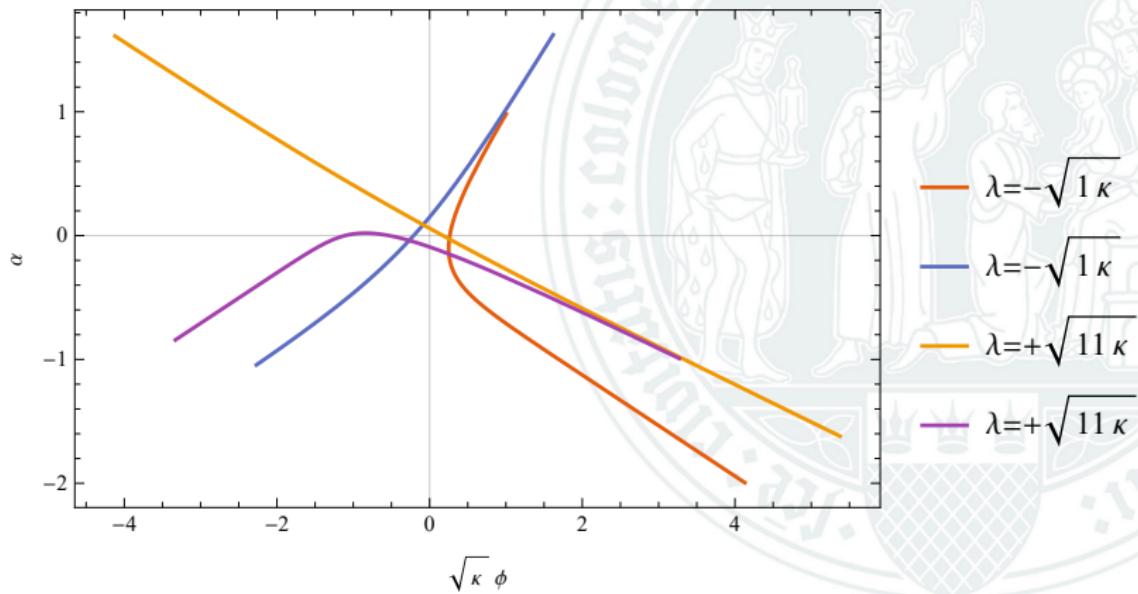


- has two asymptotes $\chi \propto \pm \beta$



Trajectories for quintessence model $(+, -)$: csch

csch, with $\beta_0 = 0$, $|V| = \kappa^{-2}$ and $p_\beta^2 = \kappa^2 \sqrt{|V|}$; varying λ

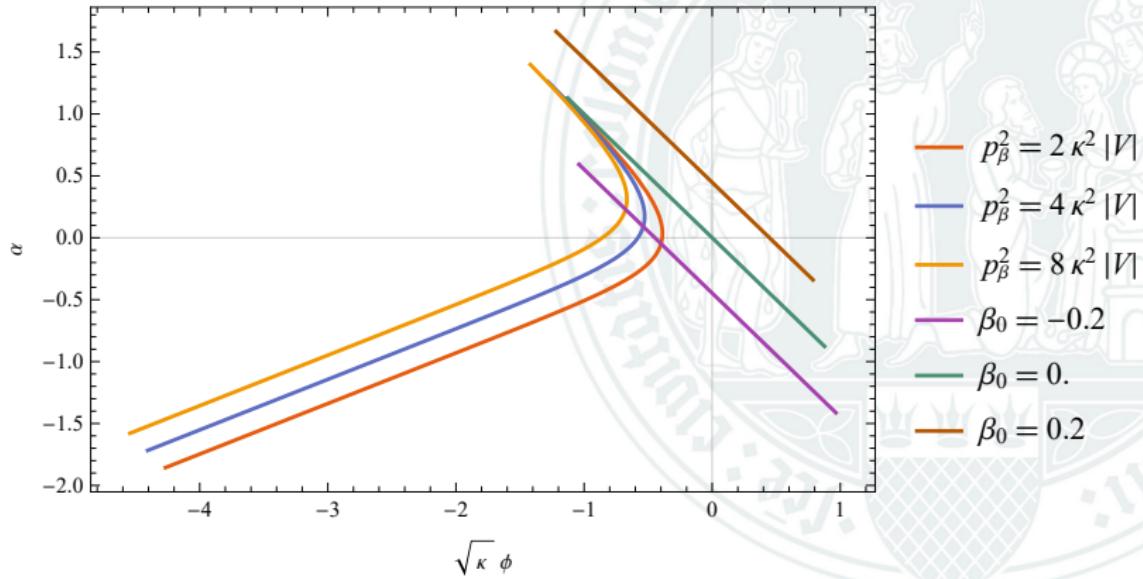


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$



Trajectories for quintessence model (+, -): csch

csch, with $V = \kappa^{-2}$ and $\lambda^2 = \kappa$; varying p_β

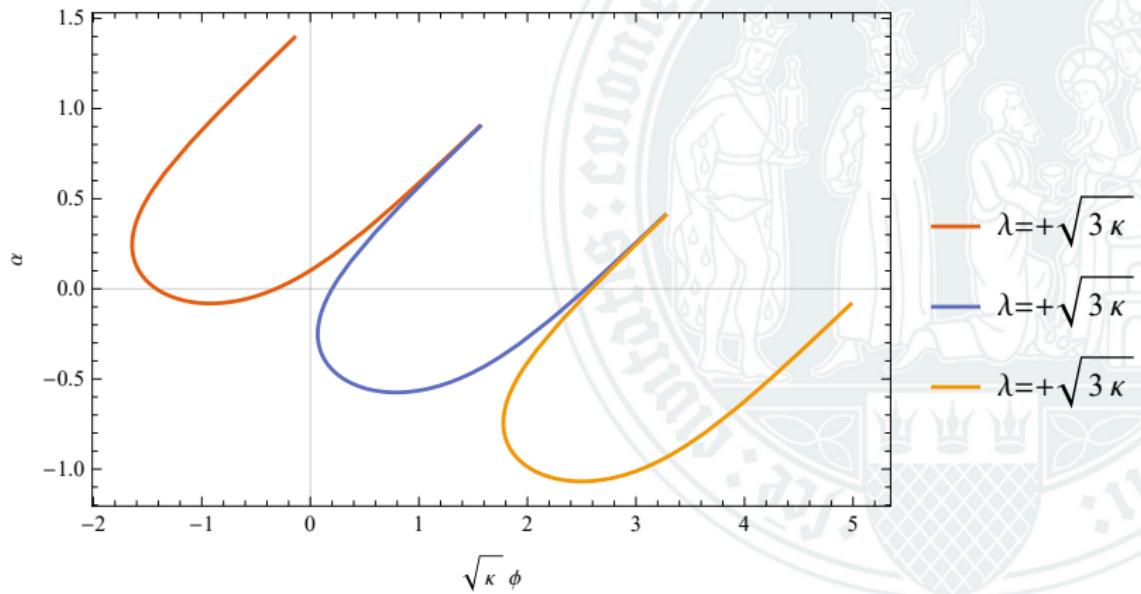


- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm \beta$ and $\beta = 0$



Trajectories for phantom model $(-, +)$

sec, with $V = \varkappa^{-2}$ and $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

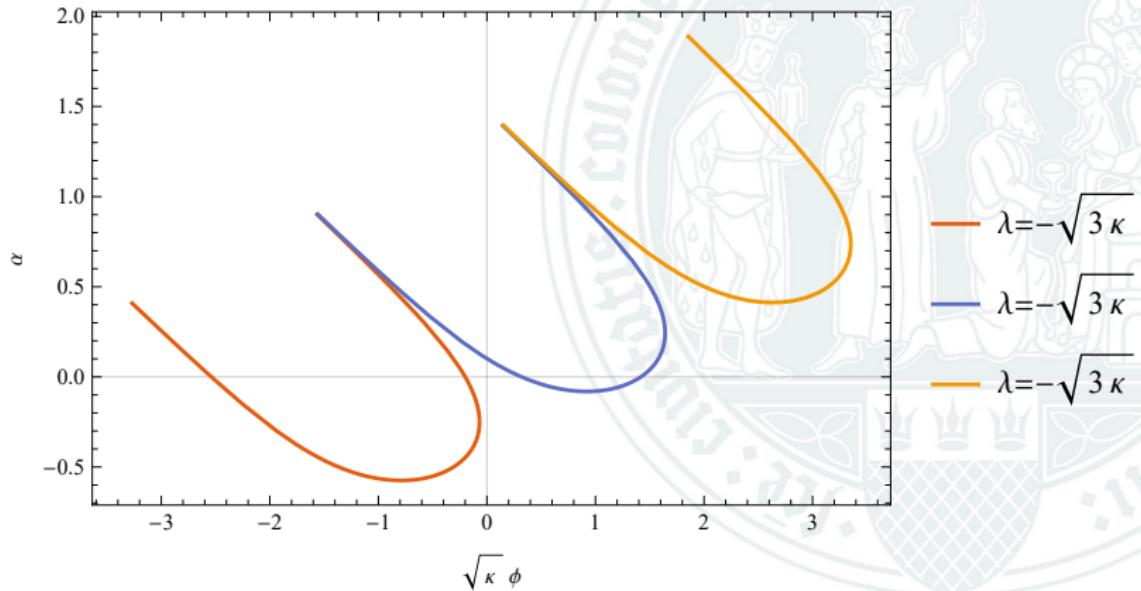


- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Trajectories for phantom model $(-, +)$

csc, with $V = \varkappa^{-2}$ and $p_\beta^2 = 3\varkappa^2 \sqrt{|V|}$

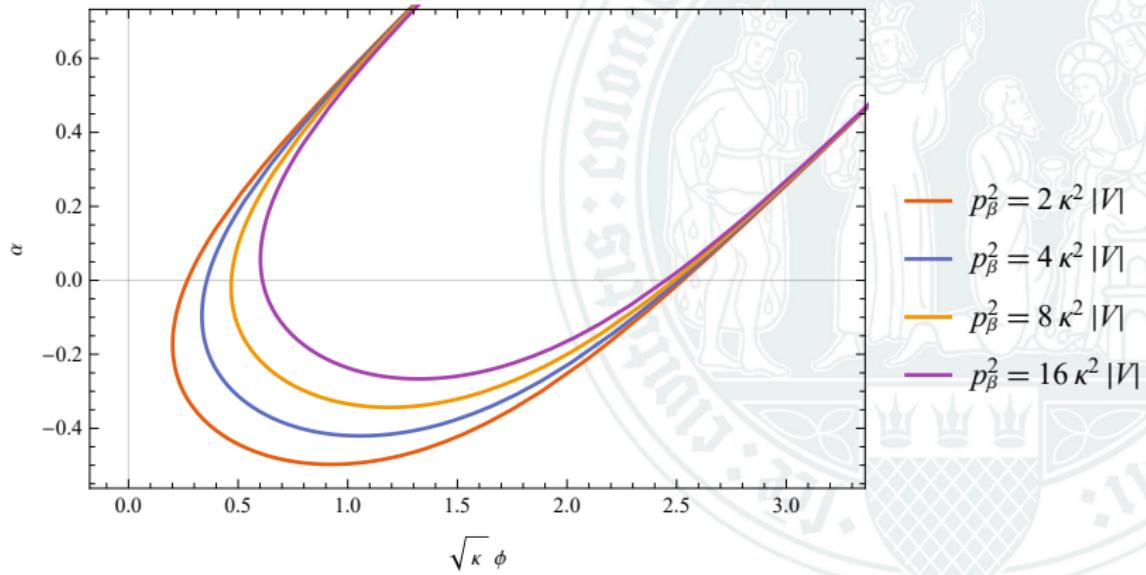


- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Trajectories for phantom model $(-, +)$

csc, with $V = \kappa^{-2}$ and $\lambda^2 = 3\kappa$; varying p_β



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$



Integration of the transformed first Friedmann equation

General integral for $p_\beta \neq 0$

In order to integrate the equation under the implicit gauge

$$s \frac{p_\beta^2}{12} \left(-\ell \frac{\varkappa^{1/2}}{6} \left(\frac{d\chi}{d\beta} \right)^2 + \varkappa^{-1/2} \right) - \varkappa^{3/2} V e^{g\chi} = 0, \quad (19)$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa} g\beta}, \quad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} e^{-g\chi}, \quad (20)$$

to get

$$\left(\frac{d\tilde{\sigma}}{d\gamma} \right)^2 + \ell(s\upsilon - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{d\gamma}{d\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-s\upsilon + \tilde{\sigma}^2)}}, \quad (21)$$

which is of the standard inverse hyperbolic / trigonometric form **except for $(-, -)$** .



Integration of the separated minisuperspace WDW equation

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \mathfrak{s} v \frac{12\varkappa^2 |V|}{\hbar^2}, \quad (22)$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{k_\beta}{g}, \quad \sigma^2 := \frac{8\varkappa^3 |V| e^{g\chi}}{\hbar^2 g^2}, \quad (23)$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma \psi'(\sigma) + \ell(\nu^2 - \mathfrak{s} v \sigma^2) \psi(\sigma) = 0, \quad (24)$$

which is of the standard Besselian form.



Self-adjointness of unbounded operators

General theory

- Even in quantum geometrodynamics, the self-adjointness of the matter Hamiltonian is desirable, which is typically an unbounded operator in the Hilbert space \mathbf{F} endowed with the Schrödinger inner product $(\cdot, \cdot)_S$.
- Mathematically, an *unbounded* operator H is characterised not only by its action on a vector, but also by its domain $\text{Dom}(H) \subsetneq \mathbf{F}^{24}$.
- In addition to the *symmetry* $(H^\dagger \phi_1, \phi_2) \equiv (\phi_1, H\phi_2)$, the self-adjointness of an unbounded operator also requires $\text{Dom}(H^\dagger) = \text{Dom}(H)$.
- For quantum systems not bounded below, not only the self-adjointness, but also the symmetry of the Hamiltonian is not guaranteed automatically.
- Even when one could find a $\text{Dom}_0(H)$ such that H is symmetric, one would still be left with $\text{Dom}_0(H^\dagger) \supsetneq \text{Dom}_0(H)$ in general.
- Sloppily speaking, the process of extending $\text{Dom}(H)$ such that $\text{Dom}(H^\dagger) = \text{Dom}(H)$ is called **self-adjoint extension**²⁵; if the extension is unique, the operator is called *essentially self-adjoint*.

²⁴B. C. Hall, *Quantum Theory for Mathematicians*, (Springer, 2013), vol. 267 , ch. 9.

²⁵D. M. Gitman *et al.*, *Self-adjoint Extensions in Quantum Mechanics*, (Birkhäuser, 2012) , G. Bonneau *et al.*, *Am. J. Phys* **69**, 322–331 (Mar. 28, 2001), V. S. Araujo *et al.*, *Am. J. Phys* **72**, 203–213 (Feb. 2004), A. M. Essin, D. J. Griffiths, *Am. J. Phys* **74**, 109–117 (Feb. 2006).

