

Integrable Cosmological Models with Liouville Fields

Alexander A. Andrianov^{1,4} Chen Lan² Oleg O. Novikov¹
Yi-Fan Wang³

¹ Saint-Petersburg State University, St. Petersburg 198504, Russia

² ELI-ALPS, ELI-Hu NKft, Dugonics tér 13, Szeged 6720, Hungary

³ Institut für Theoretische Physik, Universität zu Köln, Zùlpicher StraÙe 77, 50937 Köln, Germany

⁴ Institut de Ciències del Cosmos (ICCUB), Universitat de Barcelona, Spain

December 8, 2017



Outline

1. Introduction
2. Classical model and the implicitised trajectories
3. Dirac quantisation and the wave functions



Introduction

Introduction



- Flat Robertson–Walker metric $ds^2 = -N^2(t) dt^2 + \varkappa^{-1} e^{2\alpha(t)} d\Omega_3^2$, where $\varkappa := 8\pi G$, $d\Omega_3^2$ dimensionless spacial metric.
- Homogeneous real Klein–Gordon field with potential $V e^{\lambda\phi}$ (Liouville), where $\lambda, V \in \mathbb{R}$, and kinetic term with sign $\ell = \pm 1$ (quintessence / phantom model).
- Total action $\mathcal{S} := S_{\text{EH}} + S_{\text{GHY}} + S_{\text{L}} = \int d\Omega_3^2 \int dt L$, in which the effective Lagrangian reads

$$L := \varkappa^{3/2} N e^{3\alpha} \left(-\frac{3}{\varkappa} \frac{\dot{\alpha}^2}{N^2} + \ell \frac{\dot{\phi}^2}{2N^2} - V e^{\lambda\phi} \right), \quad (1)$$

in which dot means d/dt and $\ell = \pm 1$.



Decoupling the variables

Via *rescaled* special orthogonal transformation

- Setting $\overline{N} := N e^{-3\alpha}$, eq. (1) transforms to

$$L = \kappa^{3/2} \overline{N} \left(-\frac{3}{\kappa} \frac{\dot{\alpha}^2}{\overline{N}^2} + \ell \frac{\dot{\phi}^2}{2\overline{N}^2} - V e^{\lambda\phi+6\alpha} \right) \quad (2)$$

- Defining $\Delta := \lambda^2 - 6\ell\kappa$, $\jmath := \text{sgn } \Delta$ and $g := \jmath\sqrt{|\Delta|} \equiv \jmath\sqrt{\jmath\Delta}$, the *rescaled* special orthogonal transformation

$$\begin{pmatrix} \alpha \\ \phi \end{pmatrix} = \frac{\jmath}{g} \begin{pmatrix} \lambda & -\ell\kappa \\ -6 & \lambda \end{pmatrix} \begin{pmatrix} \jmath_\beta \beta \\ \jmath_\chi \chi \end{pmatrix} \quad \text{where } \jmath_\beta, \jmath_\chi = \pm 1 \quad (3)$$

gives the decoupled Lagrangian

$$L = \kappa^{3/2} \overline{N} \left(-\jmath \frac{3}{\kappa} \frac{\dot{\beta}^2}{\overline{N}^2} + \ell \jmath \frac{\dot{\chi}^2}{2\overline{N}^2} - V e^{\jmath_\chi g \chi} \right). \quad (4)$$

- The Euler–Lagrange equations w.r.t. \overline{N} , β and χ will be called the trsfed. 1st, 2nd Friedmann eqs. and the Klein–Gordon eq., respectively.



Implicitised integration

$$p_\beta \neq 0$$

- Since β is cyclic in eq. (4), the trsfed. 2nd Friedmann eq. can be integrated¹

$$\text{const.} \equiv p_\beta := \frac{\partial L}{\partial \dot{\beta}} = -6\mathfrak{s}\kappa^{1/2} \frac{\dot{\beta}}{\bar{N}} \equiv -6\mathfrak{s}\mathfrak{s}_\beta \frac{\kappa^{1/2}}{g} \frac{\lambda \dot{\alpha} + \ell \kappa \dot{\phi}}{\bar{N}}. \quad (5)$$

- For $p_\beta \neq 0$, fixing the *implicitising gauge* $\bar{N} = -6\mathfrak{s}\sqrt{\kappa}\dot{\beta}/p_\beta$, the trsfed. 1st Friedmann equation can be integrated

$$\mathfrak{e}^{\mathfrak{s}_\times g\chi} = \frac{p_\beta^2}{12\kappa^2|V|} S^2 \left(\mathfrak{s}_\beta \sqrt{\frac{3}{2\kappa}} g\beta + C \right), \quad (6)$$

in which $\mathfrak{v} := \text{sgn } V$, (sgn,sgn) means $(\ell, \mathfrak{s}\mathfrak{v})$, and

$$\begin{aligned} (+,+)S(\gamma) &:= \text{sech}(\gamma), & (+,-)S(\gamma) &:= \text{csch}(\gamma), \\ (-,+)S(\gamma) &:= \sec(\gamma), & (-,-)S(\gamma) &:= \mathfrak{i} \csc(\gamma). \end{aligned} \quad (7)$$

¹ Chen Lan. PhD thesis. Saint Petersburg State University, 2016, Alexander A. Andrianov, Oleg O. Novikov, and Chen Lan. In: *Theoretical and Mathematical Physics* 184.3 (2015), pp. 1224–1233.



Integration

Discussions

- The integrals are consistent with the trsfed. Klein–Gordon equation.
- The integral for $(+, +)$
 - has two asymptotes
- The implicitised integral for $(+, -)$
 - contains two distinct solutions
 - has three asymptotes
- The implicitised integral for $(-, +)$
 - is β -even for $C = 0$
 - contains infinite distinct solutions
 - has infinite asymptotes, which are pairwise parallel
- The integral for $(-, -)$
 - is not real
- The implicitised integral enables one to compare trajectories with wave functions, see below.



Implicitised integration

$$p_\beta = 0$$

- For $p_\beta = 0$, one has $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell\lambda\alpha/\kappa$, which is the familiar power-law special solution².
- Further integrating the first Friedmann equation demands (+, −) or (−, +) to guarantee $\bar{N} > 0$, and the result is automatically consistent with the transferred Klein–Gordon equation.
- Fixing $\bar{N} = (2\kappa^2|V|)^{-1/2}$ yields

$$e^{g_{\beta\chi}\chi} = \left(\frac{2\kappa}{g(t-t_0)} \right)^2. \quad (8)$$

² Mariusz P. Dąbrowski, Claus Kiefer, and Barbara Sandhöfer. In: *Physical Review D* 74.4 (2006).



Introduction

Introduction



- The primary Hamiltonian and the Hamiltonian constraint reads

$$H^p = \bar{N}H_\perp + p_{\bar{N}}v^{\bar{N}}, \quad (9)$$

$$H_\perp = -\imath \frac{p_\beta^2}{12\kappa^{1/2}} + \ell \imath \frac{p_\chi^2}{2\kappa^{3/2}} + \kappa^{3/2} V e^{g_\imath \chi}. \quad (10)$$

- Applying the Dirac quantisation rules with Laplace–Beltrami operator, one gets the mss. Wheeler–DeWitt eq. with (β, χ)

$$0 = \widehat{H}_\perp \Psi(\beta, \chi) := \left(\imath \frac{\hbar^2}{12\kappa^{1/2}} \partial_\beta^2 - \ell \imath \frac{\hbar^2}{2\kappa^{3/2}} \partial_\chi^2 + \kappa^{3/2} V e^{g_\imath \chi} \right) \Psi. \quad (11)$$

- Equation (11) is KG-like, hyperbolic for $\ell = +1$ and *elliptic* for $\ell = -1$.

Separation of the variables and mode functions

233

- Writing $\Psi(\beta, \chi) = \varphi(\beta)\psi(\chi)$, eq. (11) can be separated into

$$\partial_\beta^2 \varphi(\beta) = k_\beta^2 \varphi(\beta); \quad (12)$$

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell \frac{6}{\kappa} \partial_\chi^2 + \imath v \frac{12\kappa^2 |V|}{\hbar^2}. \quad (13)$$

- Equation (13) turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta, \chi) := \left(c_1 \varphi_\nu^{(1)}(\gamma) + c_2 \varphi_\nu^{(2)}(\gamma) \right) \left(a_1 B_\nu^{(1)}(\sigma) + a_2 B_\nu^{(2)}(\sigma) \right), \quad \nu \geq 0, \quad (14)$$

in which

$$\nu := \sqrt{\frac{2\kappa}{3}} \frac{1}{g} k_\beta, \quad \gamma := \sqrt{\frac{3}{2\kappa}} \frac{g}{1} \beta, \quad \sigma^2 := \frac{8\kappa^3 |V| e^{g\imath \chi}}{\hbar^2 g^2}, \quad (15)$$

$$\begin{aligned} (+, +) B_\nu^{(i)}(\sigma) &:= K \text{ and } I_{i\nu}(\sigma), & (+, -) B_\nu^{(i)}(\sigma) &:= F \text{ and } G_{i\nu}(\sigma), \\ (-, +) B_\nu^{(i)}(\sigma) &:= J \text{ and } Y_\nu(\sigma), & (-, -) B_\nu^{(i)}(\sigma) &:= K \text{ and } I_\nu(\sigma). \end{aligned}$$

- Adapted to imaginary order, $F_\nu(\sigma)$ and $G_\nu(\sigma)$ are defined in³.

³ T. Mark Dunster. In: *SIAM Journal on Mathematical Analysis* 21.4 (1990), pp. 995–1018.



- Physical mode functions are expected to be regular on the boundary.
- $(+, +)$: $|I_{i\nu}(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow +\infty$
- $(-, +)$:
 - $\forall n \in \mathbb{N}$, $|Y_n(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow -\infty$.
 - $\forall \nu \in \mathbb{R}^+ \setminus \mathbb{N}$, choose $J_{-\nu}$ instead of Y_ν , since $J_{\pm\nu}$ are also linearly independent.
 - $\forall \nu \in \mathbb{R}^+ \setminus \mathbb{N}$, $|J_{-\nu}(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow -\infty$.
- $(-, -)$: $|K_\nu(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow -\infty$; $|I_\nu(\sigma)| \rightarrow +\infty$ as $\alpha \rightarrow +\infty$
- These are not to be included in the space of physical wave functions.
 $\forall \nu \geq 0$,
 - $(+, +)$: $K_{i\nu}(\sigma)$ survives
 - $(+, -)$: F and $G_{i\nu}(\sigma)$ survives
 - $(-, +)$: $J_\nu(\sigma)$ survives
 - $(-, -)$: drops out



Matching quantum number with classical first integral

233

- Baustelle
- In order to match the quantum number k_β (or linearly, ν) with the classical first integral p_β , one may apply the *principle of constructive interference*.
- Writing the mode function in the WKB form

$$\Psi_{k_\beta}(\beta, \chi) \sim \sqrt{\rho} e^{iS/\hbar}, \quad S/\hbar \gg 1 \text{ and } k_\beta \gg 1, \quad (16)$$

the principle demands that $\partial S / \partial k_\beta = 0$ be equivalent to the classical trajectory.



Matching quantum number with classical first integral

(+, +) as exemplar

- Fixing $\nu/\sigma > 1$, the asymptotic expansion reads

$$K_{i\nu}(\sigma) \sim \frac{\sqrt{2\pi} \cos\left(\sqrt{\nu^2 - \sigma^2} - \nu \arccos \frac{\nu}{\sigma} - \frac{\pi}{4}\right)}{(\nu^2 - \sigma^2)^{1/4} e^{\pi\nu/2}} + O\left(\frac{1}{\sigma}\right). \quad (17)$$

- There are two phases with opposite signs. Assuming c, a are real and applying *the principle* to $\Psi_\nu(\sigma)$, one has $\sigma/\nu = \operatorname{sech}(\beta\gamma)$, which matches the trajectory with $C = 0$ if

$$\hbar k_\beta \equiv \hbar \sqrt{\frac{3}{2\kappa}} g \hbar \nu = p_\beta, \quad (18)$$

- Non-vanishing C can be compensated by the phase of c 's and a 's.
- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; it is not within the WKB regime.
- The conclusions also hold for $F_{i\nu}(\sigma)$, $G_{i\nu}(\sigma)$ for $(+, -)$, and $J_\nu(\sigma)$ for $(-, +)$.



Inner product for wave functions

Schrödinger product

- To make sense of amplitude and wave packet, an inner product is necessary
- A common starting point of most approaches is the *Schrödinger product*

$$(\Psi_1, \Psi_2)_S := \int d\chi \Psi_1^*(\beta, \chi) \Psi_2(\beta, \chi); \quad (19)$$

- In terms of a norm, $(\Psi, \Psi)_S \equiv \int d\chi \rho_S(\beta, \chi)$, in which $\rho_S := \Psi^* \Psi$. Manifestly $\rho_S \geq 0$; one has $(\Psi, \Psi)_S > 0$.
- The corresponding Schrödinger current does not satisfy continuity equation $\dot{\rho}_S + \nabla \cdot \vec{j}_S = 0$, because eq. (11) is KG-like.
- $K_{i\nu}^4$ for $(+, +)$, $F_{i\nu}$ and $G_{i\nu}$ for $(+, -)$ can be proved to be orthogonal and complete individually, as well as can be normalised.
 - $J_{i\nu}$'s for $(+, -)$ are not orthogonal

⁴ Semyon B. Yakubovich. In: *Opuscula Math.* 26.1 (2006), pp. 161–172, A. Passian et al. In: *Journal of Mathematical Analysis and Applications* 360.2 (2009), pp. 380–390, Radosław Szmytkowski and Sebastian Bielski. In: *Journal of Mathematical Analysis and Applications* 365.1 (2010), pp. 195–197.



Peculiarity of the phantom model $(-, +)$

Orthogonality for mode functions; Hermiticity for operators

- $J_\nu(\sigma)$'s are not orthogonal under the Schrödinger product

$$(J_\nu, J_{\tilde{\nu}})_S \propto \int_{-\infty}^{+\infty} dx J_\nu^*(e^x) J_{\tilde{\nu}}(e^x) = \frac{2\sin(\pi(\nu - \tilde{\nu})/2)}{\pi(\nu^2 - \tilde{\nu}^2)}, \quad (20)$$

therefore \mathbb{D} in eq. (13) is not Hermitian (though we do not need it so far)

- \hat{p}_χ^2 is not Hermitian for $\{J_\nu(\sigma)\}$ under Schrödinger product

$$\int_{-\infty}^{+\infty} dx J_\nu^*(-\partial_x^2 J_{\tilde{\nu}}) - \int_{-\infty}^{+\infty} dx (-\partial_x^2 J_\nu)^* J_{\tilde{\nu}} = \frac{2}{\pi} \sin \frac{\pi(\nu - \tilde{\nu})}{2}. \quad (21)$$

- In order to save Hermiticity for p_χ^2 and \mathbb{D} and orthogonality of the modes under Schrödinger product, one can restrict

$$\nu = 2n + \nu_0, \quad n \in \mathbb{N}, \quad \nu_0 \in [0, 2). \quad (22)$$

- The classical trajectory is β -even; imposing the same condition fixes $\nu_0 = 1$.



Levels of the phantom model are discretised

...if one imposes Hermiticity of squared momenta under the Schrödinger product

- This kind of subtlety on Hermiticity is named as *self-adjoint extension*, which arises already for infinite square well⁵.
- It also applies to x^{-2} potentials⁶, which is of cosmological relevance⁷.

⁵ Guy Bonneau, Jacques Faraut, and Galliano Valent. In: *American Journal of Physics* 69.3 (2001), pp. 322–331.

⁶ Andrew M. Essin and David J. Griffiths. In: *American Journal of Physics* 74.2 (2006), pp. 109–117, Vanilse S. Araujo, F. A. B. Coutinho, and J. Fernando Perez. In: *American Journal of Physics* 72.2 (2004), pp. 203–213.

⁷ Mariam Bouhmadi-López et al. In: *Physical Review D* 79.12 (2009).



Further inner products for wave functions

Klein-Gordon and Mostafazadeh product

- Since eq. (11) is KG-like, another popular choice is the KG product

$$(\Psi_1, \Psi_2)_{\text{KG}}^g := \text{i}g \left\{ (\Psi_1, \dot{\Psi}_2)_S - (\dot{\Psi}_1, \Psi_2)_S \right\}, \quad g > 0. \quad (23)$$

- ρ_{KG} is real but may go negative
- Unique \vec{J}_{KG} exists such that $\dot{\rho}_{\text{KG}} + \nabla \cdot \vec{J}_{\text{KG}} = 0$.
- Mostafazadeh⁸ found an inner product for *Hermitian* \mathbb{D} with positive spectrum: $\forall \kappa > 0$,

$$(\Psi_1, \Psi_2)_{\text{M}}^{\kappa} := \kappa \left\{ (\Psi_1, \mathbb{D}^{+1/2} \Psi_2)_S + (\dot{\Psi}_1, \mathbb{D}^{-1/2} \dot{\Psi}_2)_S \right\}, \quad (24)$$

and $-1 < a < 1$.

- ρ_{M}^{κ} may go complex
- Unique $\vec{J}_{\text{M}}^{\kappa}$ exists such that $\dot{\rho}_{\text{M}}^{\kappa} + \nabla \cdot \vec{J}_{\text{M}}^{\kappa} = 0$.

⁸ Ali Mostafazadeh. In: *Classical and Quantum Gravity* 20.1 (2002), pp. 155–171.



- It can be shown⁹ that

$$\varrho_M^\kappa := \kappa \left\{ |\mathbb{D}^{+1/4}\Psi|^2 + |\mathbb{D}^{-1/4}\dot{\Psi}|^2 \right\} \quad (25)$$

satisfies

- ϱ_M^κ is non-negative
- $\int d\chi \varrho_M^\kappa = \int d\chi \rho_M^\kappa \equiv (\Psi_1, \Psi_2)_M^\kappa$: it may be understood as a prob. density; but
- The corresponding conserved $\tilde{\mathcal{J}}_M^\kappa$ does not exist.

⁹ Ali Mostafazadeh and F. Zamani. In: *Annals of Physics* 321.9 (2006), pp. 2183–2209.



Integration of the transformed first Friedmann equation

$$p_\beta \neq 0$$

In order to integrate the equation under the implicitising gauge

$$\mathcal{J} \frac{p_\beta^2}{12} \left(-\ell \frac{\kappa^{1/2}}{6} \left(\frac{d\chi}{d\beta} \right)^2 + \kappa^{-1/2} \right) - \kappa^{3/2} V e^{g\mathcal{J}_\chi \chi} = 0, \quad (26)$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\kappa}} g\beta, \quad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\kappa^2 |V|} e^{-g\mathcal{J}_\chi \chi}, \quad (27)$$

to get

$$\left(\frac{d\tilde{\sigma}}{d\gamma} \right)^2 + \ell(\mathcal{J}v - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{d\gamma}{d\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(-\mathcal{J}v + \tilde{\sigma}^2)}}, \quad (28)$$

which is of the standard inverse hyperbolic / trigonometric form for $(+, +)$, $(+, -)$ and $(-, +)$.



Integration of the separated mss. Wheeler–DeWitt equation

233

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \quad \mathbb{D} := -\ell^2 \frac{6}{\varkappa} \partial_\chi^2 + \imath \nu \frac{12\varkappa^2 |V|}{\hbar^2}, \quad (13 \text{ rev.})$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{k_\beta}{g}, \quad \sigma^2 := \frac{8\varkappa^3 |V| e^{g^2 \chi}}{\hbar^2 g^2}, \quad (15 \text{ rev.})$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma \psi'(\sigma) + \ell(\nu^2 - \imath \nu \sigma^2) \psi(\sigma) = 0, \quad (29)$$

which is of the standard Besselian form.





- Mit diesem *beamer theme* ist es möglich, Präsentationen in \LaTeX mit der Beamer-Klasse zu erstellen, die dem Corporate Design der Universität zu Köln entsprechen
- Auf die Beamer-Klasse wird in diesem Dokument nicht näher eingegangen, nähere Informationen finden Sie unter <http://latex-beamer.sourceforge.net/>



Das Theme kann mit den folgenden Optionen geladen werden

```
\usepackage[%  
% uk,          %% Farben aller Fakultäten  
wiso,          %% Wiso-Fakultät  
% jura,        %% Rechtswissenschaftliche Fakultät  
% medizin,     %% Medizinische Fakultät  
% philo,       %% Philosophische Fakultät  
% matnat,      %% Mathematisch-Naturwissenschaftliche Fakultät  
% human,       %% Humanwissenschaftliche Fakultät  
% verw,        %% Universitätsverwaltung  
{UzK}
```



- Es stehen verschiedene Fußzeilen zur Auswahl, die als Option beim Laden des *themes* übergeben werden:
 - Balken mit allen Fakultätsfarben (Option uk)
 - Balken in jeweils einer Fakultätsfarbe (Optionen wiso, jura, medizin, philo, matnat, human, verw)¹⁰
- "'Universität zu Köln"' sowie der Name der Fakultät sind im Theme definiert, das Institut oder Seminar kann mit dem Befehl `\institute{}` festgelegt werden
- Die Optionen sind im Quellcode dieser Präsentation dokumentiert

¹⁰Es werden die offiziellen RGB-Werte aus dem 2-D Handbuch Corporate Design verwendet.



- Der Universitäts- sowie die Fakultätsnamen werden standardmäßig auf Deutsch angezeigt.
- Übergeben Sie dem Paket babel die Option english, so werden diese Namen entsprechen angepasst.
- Die Übersetzungen können in der Theme-Datei beamerthemeUzK.sty geändert werden

block-Umgebungen

Standard (block)

Verwendet die Farbe "Blaugrau Mittel" als Blocktitel-Hintergrund

exampleblock

Bei Verwendung der Fußzeile mit allen Fakultätsfarben Titelhintergrund in Wiso-Grün, sonst in der jeweiligen Fakultätsfarbe

alertblock

Verwendet das Rot der Folientitel



Installation

- Das Theme besteht aus den Dateien `beamerthemeUzK.sty` und `beamercolorthemeUzK.sty` sowie den Grafikdateien `logo.pdf` und `logo-small.pdf`.
- Das Theme kann auf zwei Arten verwendet werden:
 1. Die vier Dateien werden in den selben Ordner wie die zu erstellende Präsentation gelegt
 2. Die vier Dateien werden im lokalen *texmf*-Baum abgelegt
- Die zweite Variante ist der ersten vorzuziehen, da das Theme so an einem zentralen Ort vorliegt



Was noch zu tun ist...

- Erstellen einer eigenen Titelseite
- ...