Integrable Cosmological Models with Liouville Fields

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Outline

1. Introduction

2. Classical model and the implicitised trajectories

3. Dirac quantisation and the wave functions

Introduction

Introduction

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The Friedmann-Lemaître model

- Flat Robertson–Walker metric $\mathrm{d}s^2 = -N^2(t)\,\mathrm{d}t^2 + \varkappa^{-1}\mathrm{e}^{2\alpha(t)}\,\mathrm{d}\Omega_3^2$, where $\varkappa := 8\pi G$, $\mathrm{d}\Omega_3^2$ dimensionless spacial metric.
- Homogeneous real Klein–Gordon field with potential $V\mathrm{e}^{\lambda\phi}$ (Liouville), where $\lambda, V \in \mathbb{R}$, and kinetic term with sign $\ell = \pm 1$ (quintessence / phantom model).
- Total action $\mathcal{S} := S_{\mathsf{EH}} + S_{\mathsf{GHY}} + S_{\mathsf{L}} = \int \mathrm{d}\Omega_3^2 \int \mathrm{d}t\, L$, in which the effective Lagrangian reads

$$L := \varkappa^{3/2} N \mathrm{e}^{3\alpha}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{N^2} + \ell\,\frac{\dot{\phi}^2}{2N^2} - V\mathrm{e}^{\lambda\phi}\right), \tag{1}$$

in which dot means $\mathrm{d}/\mathrm{d}t$ and $\ell = \pm 1$.

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Decoupling the variables

Via *rescaled* special orthogonal transformation

- Setting $\overline{N} := N\mathrm{e}^{-3\alpha}$, eq. (1) transforms to

$$L = \varkappa^{3/2}\overline{N}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{\overline{N}^2} + \ell\,\frac{\dot{\phi}^2}{2\overline{N}^2} - V\mathrm{e}^{\lambda\phi+6\alpha}\right) \tag{2}$$

- Defining $\varDelta := \lambda^2 - 6\ell\varkappa$, $\mathscr{s} := \operatorname{sgn}\varDelta$ and $g := \mathscr{s}\sqrt{|\varDelta|} \equiv \mathscr{s}\sqrt{\mathscr{s}\varDelta}$, the *rescaled* special orthogonal transformation

$$\begin{pmatrix}\alpha\\ \phi\end{pmatrix} = \frac{\mathscr{s}}{g}\begin{pmatrix}\lambda & -\ell\kappa\\ -6 & \lambda\end{pmatrix}\begin{pmatrix}\mathscr{s}_\beta\beta\\ \mathscr{s}_\chi\chi\end{pmatrix} \qquad \text{where } \mathscr{s}_\beta, \mathscr{s}_\chi = \pm 1 \tag{3}$$

gives the decoupled Lagrangian

$$L = \varkappa^{3/2}\overline{N}\left(-\mathscr{s}\frac{3}{\varkappa}\frac{\dot{\beta}^2}{\overline{N}^2} + \ell\,\mathscr{s}\frac{\dot{\chi}^2}{2\overline{N}^2} - V\mathrm{e}^{\mathscr{s}_\chi g\chi}\right). \tag{4}$$

- The Euler–Lagrange equations w.r.t. $\overline{N}$, β and χ will be called the trsfed. 1st, 2nd Friedmann eqs. and the Klein–Gordon eq., respectively.

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Implicitised integration

$p_\beta \neq 0$

- Since β is cyclic in eq. (4), the trsfed. 2nd Friedmann eq. can be integrated[1]

$$\text{const.} \equiv p_\beta := \frac{\partial L}{\partial \dot{\beta}} = -6\varsigma\varkappa^{1/2}\frac{\dot{\beta}}{\overline{N}} \equiv -6\varsigma\varsigma_\beta \frac{\varkappa^{1/2}}{g}\frac{\lambda\dot{\alpha} + \ell\varkappa\dot{\phi}}{\overline{N}}. \tag{5}$$

- For $p_\beta \neq 0$, fixing the implicitising gauge $\overline{N} = -6\varsigma\sqrt{\varkappa}\dot{\beta}/p_\beta$, the trsfed. 1st Friedmann equation can be integrated

$$\mathbb{e}^{\varsigma_\chi g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|} S^2\left(\varsigma_\beta\sqrt{\frac{3}{2\varkappa}}\, g\beta\right), \tag{6}$$

in which $\upsilon := \operatorname{sgn} V$, and

$$S(\gamma) := \begin{cases} \operatorname{sech}(\gamma + C_{++}) & (\ell, \varsigma\upsilon) = (+,+), \\ \operatorname{csch}(\gamma + C_{+-}) & (\ell, \varsigma\upsilon) = (+,-), \\ \sec(\gamma + C_{-+}) & (\ell, \varsigma\upsilon) = (-,+), \\ \mathbb{i}\csc(\gamma + C_{--}) & (\ell, \varsigma\upsilon) = (-,-). \end{cases} \tag{7}$$

[1] Chen Lan. PhD thesis. Saint Petersburg State University, 2016, Alexander A. Andrianov, Oleg O. Novikov, and Chen Lan. In: *Theoretical and Mathematical Physics* 184.3 (2015), pp. 1224–1233.

Integration

Discussions

- The integrals are consistent with the trsfed. Klein–Gordon equation.
- The integral for $(+,+)$
    - has two asymptotes
- The implicitised integral for $(+,-)$
    - contains two distinct solutions
    - has three asymptotes
- The implicitised integral for $(-,+)$
    - contains infinite distinct solutions
    - has infinite asymptotes, which are pairwise parallel
- The integral for $(-,-)$
    - is not real
- The implicitised integral enables one to compare trajectories with wave functions, see below.

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- For $p_\beta = 0$, one has $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell\lambda\alpha/\kappa$, which is the familiar power-law special solution[2].
- Further integrating the first Friedmann equation demands $(+,-)$ or $(-,+)$ to guarantee $\overline{N} > 0$, and the result is automatically consistent with the trsfed. Klein–Gordon equation.
- Fixing $\overline{N} = \left(2\varkappa^2|V|\right)^{-1/2}$ yields

$$\mathbb{e}^{g\mathscr{s}_\chi\chi} = \left(\frac{2\kappa}{g(t-t_0)}\right)^2. \tag{8}$$

[2] Mariusz P. Dąbrowski, Claus Kiefer, and Barbara Sandhöfer. In: *Physical Review D* 74.4 (2006).

Introduction

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Dirac quantisation

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- The primary Hamiltonian and the Hamiltonian constraint reads

$$H^{\mathsf{p}} = \overline{N} H_\perp + p_{\overline{N}} v^{\overline{N}}, \tag{9}$$

$$H_\perp = -\jmath \frac{p_\beta^2}{12\varkappa^{1/2}} + \ell \jmath \frac{p_\chi^2}{2\varkappa^{3/2}} + \varkappa^{3/2} V \mathrm{e}^{g_\jmath \chi}\chi. \tag{10}$$

- Applying the Dirac quantisation rules with Laplace–Beltrami opeator, one gets the mss. Wheeler–DeWitt eq. with (β, χ)

$$0 = \widehat{H}_\perp \Psi(\beta,\chi) := \left(\jmath \frac{\hbar^2}{12\varkappa^{1/2}} \partial_\beta^2 - \ell \jmath \frac{\hbar^2}{2\varkappa^{3/2}} \partial_\chi^2 + \varkappa^{3/2} V \mathrm{e}^{g_\jmath \chi}\chi \right)\Psi. \tag{11}$$

- Equation (11) is KG-like, hyperbolic for $\ell = +1$ and elliptic for $\ell = -1$.

Separation of the variables and mode functions

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- Inserting $\Psi(\beta,\chi) = \mathbb{e}^{-\mathbb{i}k_\beta s_\beta \beta}\psi(\chi)$, the separated eq.

$$\mathbb{D}\psi(\chi) = k_\beta^2\psi(\chi), \qquad \mathbb{D} := -\ell\frac{6}{\varkappa}\partial_\chi^2 + s_v\frac{12\varkappa^2|V|}{\hbar^2}, \tag{12}$$

turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta,\chi) := \mathbb{e}^{-\mathbb{i}\nu\gamma}\psi_\nu(\chi) := \mathbb{e}^{-\mathbb{i}\nu\gamma}\Big(C_1\mathrm{B}_\nu^{(1)}(\sigma) + C_2\mathrm{B}_\nu^{(2)}(\sigma)\Big), \tag{13}$$

in which

$$\nu := \sqrt{\frac{2\varkappa}{3}\frac{1}{g}}\,k_\beta, \quad \gamma := \sqrt{\frac{3}{2\varkappa}\frac{g}{1}}\,\beta, \qquad \sigma^2 := \frac{8\varkappa^3|V|\mathbb{e}^{g s_\chi \chi}}{\hbar^2 g^2}, \tag{14}$$

$$\mathrm{B}_\nu^{(i)}(\sigma) := \begin{cases} \mathrm{K} \text{ or } \mathrm{I}_{\mathbb{i}\nu}(\sigma) & (\ell, s_v) = (+,+), \\ \mathrm{F} \text{ or } \mathrm{G}_{\mathbb{i}\nu}(\sigma) & (\ell, s_v) = (+,-), \\ \mathrm{J} \text{ or } \mathrm{Y}_\nu(\sigma) & (\ell, s_v) = (-,+), \\ \mathrm{K} \text{ or } \mathrm{I}_\nu(\sigma) & (\ell, s_v) = (-,-). \end{cases} \tag{15}$$

- Adapted to imaginary order, $\mathrm{F}_\nu(\sigma)$ and $\mathrm{G}_\nu(\sigma)$ are defined in[3].

[3] T. Mark Dunster. In: *SIAM Journal on Mathematical Analysis* 21.4 (1990), pp. 995–1018.

Physical mode functions
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- Physical mode functions are expected to be regular on the boundary.
- $(+,+)$: $|\mathrm{I}_{\mathbb{i}\nu}(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- $(-,+)$:
  - $\forall n \in \mathbb{Z}$, $|\mathrm{Y}_n(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
  - $\forall \nu \in \mathbb{R}\backslash\mathbb{Z}$, choose $\mathrm{J}_{-\nu}$ instead of $\mathrm{Y}_{+\nu}$, since $\mathrm{J}_{\pm\nu}$ are also linearly independent.
  - $\forall \nu \in \mathbb{R}^{+}\backslash\mathbb{Z}$, $|\mathrm{J}_{-\nu}(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
- $(-,-)$: $|\mathrm{K}_\nu(\sigma)| \to +\infty$ as $\alpha \to -\infty$; $|\mathrm{I}_\nu(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- These are not to included in the space of physical wave functions.
- For $(-,+)$, only J_ν with $\nu \geq 0$ survives.

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Matching quantum number with classical first integral
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- Baustelle
- In order to match the quantum number k_β (or linearly, ν) with the classical first integral p_β, one may apply the principle of constructive interference.
- Writing the mode function in the WKB form

$$\Psi_{k_\beta}(\beta,\chi) \sim \sqrt{\rho}\, \mathrm{e}^{\mathrm{i}S/\hbar}, \qquad S/\hbar \gg 1 \text{ and } k_\beta \gg 1, \tag{16}$$

the principle demands that $\partial S/\partial k_\beta = 0$ be equivalent to the classical trajectory.

Matching quantum number with classical first integral

$(+,+)$ as exemplar

- Fixing $\nu/\sigma > 1$, the asymptotic expansion reads

$$\mathrm{K}_{\mathrm{i}\nu}(\sigma) \sim \frac{\sqrt{2\pi}\cos\left(\sqrt{\nu^2-\sigma^2}-\nu\arccos\frac{\nu}{\sigma}-\frac{\pi}{4}\right)}{\left(\nu^2-\sigma^2\right)^{1/4}\mathrm{e}^{\pi\nu/2}} + O\left(\frac{1}{\sigma}\right). \tag{17}$$

  There are two phases with opposite signs. Applying the principle to the full mode function Ψ_ν, one has $\sigma/\nu = \operatorname{sech}(\jmath_\beta\gamma)$, which match the trajectory if

$$\hbar k_\beta \equiv \hbar\sqrt{\frac{3}{2\varkappa}g\hbar\nu} = p_\beta, \tag{18}$$

  as expected.

- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; hence no WKB approximation works.
- The conclusions also hold for $\mathrm{F}_{\mathrm{i}\nu}(\sigma)$ and $\mathrm{G}_{\mathrm{i}\nu}(\sigma)$ for $(+,-)$, as well as $\mathrm{J}_\nu(\sigma)$ for $(-,+)$.

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Inner product for wave functions

Schrödinger product

- To make sense of amplitude and wave packet, an inner product is necessary
- A common starting point of most approaches is the Schrödinger product

$$(\Psi_1,\Psi_2)_{\mathsf{S}} := \int \mathrm{d}\chi\, \Psi_1^*(\beta,\chi)\Psi_2(\beta,\chi); \tag{19}$$

- In terms of a norm, $(\Psi,\Psi)_{\mathsf{S}} \equiv \int \mathrm{d}\chi\, \rho_{\mathsf{S}}(\beta,\chi)$, in which $\rho_{\mathsf{S}} := \Psi^*\Psi$. Manifestly $\rho_{\mathsf{S}} \geq 0$; one has $(\Psi,\Psi)_{\mathsf{S}} > 0$.
- The corresponding Schrödinger current does not satisfy continuity equation $\dot{\rho}_{\mathsf{S}} + \nabla\cdot\vec{j}_{\mathsf{S}} = 0$, because eq. (11) is KG-like.
- $\mathrm{K}_{\mathring{\imath}\nu}$[4] for $(+,+)$, $\mathrm{F}_{\mathring{\imath}\nu}$ and $\mathrm{G}_{\mathring{\imath}\nu}$ for $(+,-)$ can be proved to be orthogonal and complete individually, as well as can be normalised.
    - $\mathrm{J}_{\mathring{\imath}\nu}$'s for $(+,-)$ are not orthogonal

4 Semyon B. Yakubovich. In: *Opuscula Math.* 26.1 (2006), pp. 161–172, A. Passian et al. In: *Journal of Mathematical Analysis and Applications* 360.2 (2009), pp. 380–390, Radosław Szmytkowski and Sebastian Bielski. In: *Journal of Mathematical Analysis and Applications* 365.1 (2010), pp. 195–197.

Peculiarity for the phantom model

Orthogonality for mode functions; Hermiticity for operators

- $\mathrm{J}_\nu(\sigma)$'s are not orthogonal under the Schrödinger product

$$(\mathrm{J}_\nu, \mathrm{J}_{\tilde{\nu}})_{\mathsf{S}} \propto \int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{J}_\nu^*(\mathrm{e}^x) \mathrm{J}_{\tilde{\nu}}(\mathrm{e}^x) = \frac{2\sin(\pi(\nu - \tilde{\nu})/2)}{\pi(\nu^2 - \tilde{\nu}^2)}, \tag{20}$$

therefore $\mathbb{D}$ in eq. (12) is not Hermitian (though we do not need it so far)

- $\hat{p}_\chi^2$ is not Hermitian for $\{\mathrm{J}_\nu(\sigma)\}$ under Schrödinger product

$$\int_{-\infty}^{+\infty} \mathrm{d}x \left\{ \mathrm{J}_\nu^*(-\partial_x^2 \mathrm{J}_{\tilde{\nu}}) - (-\partial_x^2 \mathrm{J}_\nu)^* \mathrm{J}_{\tilde{\nu}} \right\} = \frac{2}{\pi} \sin \frac{\pi(\nu - \tilde{\nu})}{2} \tag{21}$$

- Restricting

$$\nu = 2n + \nu_0, \qquad n \in \mathbb{N}, \, \nu_0 \in [0, 2) \tag{22}$$

saves Hermiticity for p_χ^2 and $\mathbb{D}$ and orthogonality of the modes under S-product.

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Integration of the transformed first Friedmann equation

$p_\beta \neq 0$

In order to integrate the equation under the implicitising gauge

$$\jmath \frac{p_\beta^2}{12}\left(-\ell \frac{\varkappa^{1/2}}{6}\left(\frac{\mathbb{d}\chi}{\mathbb{d}\beta}\right)^2 + \varkappa^{-1/2}\right) - \varkappa^{3/2} V \mathbb{e}^{g\jmath_\chi \chi} = 0, \tag{23}$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa}} g\beta, \qquad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} \mathbb{e}^{-g\jmath_\chi \chi}, \tag{24}$$

to get

$$\left(\frac{\mathbb{d}\tilde{\sigma}}{\mathbb{d}\gamma}\right)^2 + \ell\left(\jmath v - \tilde{\sigma}^2\right) = 0 \quad \Rightarrow \quad \frac{\mathbb{d}\gamma}{\mathbb{d}\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell\left(-\jmath v + \tilde{\sigma}^2\right)}}, \tag{25}$$

which is of the standard inverse hyperbolic / trigonometric form for $(+,+)$, $(+,-)$ and $(-,+)$.

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Integration of the separated mss. Wheeler–DeWitt equation 233

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2\psi(\chi), \qquad \mathbb{D} := -\ell\frac{6}{\varkappa}\partial_\chi^2 + \mathfrak{s}v\frac{12\varkappa^2|V|}{\hbar^2}, \tag{12 rev.}$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}}\frac{k_\beta}{g}, \qquad \sigma^2 := \frac{8\varkappa^3|V|\mathbb{e}^{g\mathfrak{s}_\chi\chi}}{\hbar^2 g^2}, \tag{15 rev.}$$

to get

$$\sigma^2\psi''(\sigma) + \sigma\psi'(\sigma) + \ell\left(\nu^2 - \mathfrak{s}v\sigma^2\right)\psi(\sigma) = 0, \tag{26}$$

which is of the standard Besselian form.

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Allgemeines

- Mit diesem *beamer theme* ist es möglich, Präsentationen in LaTeX mit der Beamer-Klasse zu erstellen, die dem Corporate Design der Universität zu Köln entsprechen
- Auf die Beamer-Klasse wird in diesem Dokument nicht näher eingegangen, nähere Informationen finden Sie unter `http://latex-beamer.sourceforge.net/`

Laden des Themes

Das Theme kann mit den folgenden Optionen geladen werden

```
\usetheme[%
% uk,      %% Farben aller Fakultaeten
wiso,     %% Wiso-Fakultaet
% jura,    %% Rechtswissenschaftliche Fakultaet
% medizin, %% Medizinische Fakultaet
% philo,   %% Philosophische Fakultaet
% matnat,  %% Mathematisch-Naturwissenschaftliche Fakultaet
% human,   %% Humanwissenschaftliche Fakultaet
% verw,    %% Universitaetsverwaltung
]{UzK}
```

Die Fußzeile

- Es stehen verschiedene Fußzeilen zur Auswahl, die als Option beim Laden des *themes* übergeben werden:
  - Balken mit allen Fakultätsfarben (Option `uk`)
  - Balken in jeweils einer Fakultätsfarbe (Optionen `wiso`, `jura`, `medizin`, `philo`, `matnat`, `human`, `verw`)[5]
- '"Universität zu Köln"' sowie der Name der Fakultät sind im Theme definiert, das Institut oder Seminar kann mit dem Befehl `\institute{}` festgelegt werden
- Die Optionen sind im Quellcode dieser Präsentation dokumentiert

[5] Es werden die offiziellen RGB-Werte aus dem 2-D Handbuch Corporate Design verwendet.

Englische Präsentationen

- Der Universitäts- sowie die Fakultätsnamen werden standardmäßig auf Deutsch angezeigt.
- Übergeben Sie dem Paket `babel` die Option `english`, so werden diese Namen entsprechen angepasst.
- Die Übersetzungen können in der Theme-Datei `beamerthemeUzK.sty` geändert werden

block-Umgebungen

Standard (block)

Verwendet die Farbe "'Blaugrau Mittel"' als Blocktitel-Hintergrund

exampleblock

Bei Verwendung der Fußzeile mit allen Fakultätsfarben Titelhintergrund in Wiso-Grün, sonst in der jeweiligen Fakultätsfarbe

alertblock

Verwendet das Rot der Folientitel

Installation

- Das Theme besteht aus den Dateien `beamerthemeUzK.sty` und `beamercolorthemeUzK.sty` sowie den Grafikdateien `logo.pdf` und `logo-small.pdf`.
- Das Theme kann auf zwei Arten verwendet werden:
  1. Die vier Dateien werden in den selben Ordner wie die zu erstellende Präsentation gelegt
  2. Die vier Dateien werden im lokalen *texmf*-Baum abgelegt
- Die zweite Variante ist der ersten vorzuziehen, da das Theme so an einem zentralen Ort vorliegt

ToDo

Was noch zu tun ist...

- Erstellen einer eigenen Titelseite
- ...