Integrable Cosmological Models with Liouville Fields

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Outline

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Introduction

Quintessence and phantom Liouville field

- Observed accelerated expansion can be explained by a cosmological constant[1] as stationary Dark Energy, but its origin has yet to be understood.
- Dynamical Dark Energy has been modeled by quintessence[2] and phantom[3] matter, with barotropic index[4] $w > -1$ and $w < -1$, respectively.
- They can be realised by minimally-coupled real scalar fields with $\ell = \pm 1$[5]

$$\mathcal{S} = \int \mathrm{d}^4 x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \tag{1}$$

- $\mathcal{V}(\phi) = V \mathrm{e}^{\lambda\phi}$, $\lambda, V \in \mathbb{R}$ is of interest: Liouville field[6].

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[1] E. J. Copeland, M. Sami, and S. Tsujikawa. In: *International Journal of Modern Physics D* 15.11 (2006), pp. 1753–1935, K. Bamba et al. In: *Astrophysics and Space Science* 342.1 (2012), pp. 155–228.

[2] R. R. Caldwell, R. Dave, and P. J. Steinhardt. In: *Physical Review Letters* 80.8 (1998), pp. 1582–1585.

[3] R. R. Caldwell. In: *Physics Letters B* 545.1-2 (2002), pp. 23–29.

[4] Barotropic index is the w in equation of state $\rho = wp$.

[5] The signature of metric is mostly positive.

[6] Y. Nakayama. In: *International Journal of Modern Physics A* 19.17n18 (2004), pp. 2771–2930.

Introduction

Friedmann–Lemaître model

- Assume flat Robertson–Walker metric for a homog. and isotr. model

$$g_{\mu\nu}\,\mathbb{d}x^\mu\,\mathbb{d}x^\nu = -N^2(t)\,\mathbb{d}t^2 + \varkappa \mathbb{e}^{2\alpha(t)}\,\mathbb{d}\Omega_{3\mathsf{F}}^2 \tag{2}$$

  w/ $\varkappa := 8\pi G$, $\mathbb{d}\Omega_{3\mathsf{F}}^2 := \mathbb{d}\chi^2 + \chi^2(\mathbb{d}\theta^2 + \sin^2\theta\,\mathbb{d}\varphi^2)$, N lapse function.

- Combined with the Liouville field, the total action reads
  $\mathcal{S} := \mathcal{S}_{\mathsf{EH}} + \mathcal{S}_{\mathsf{GHY}} + \mathcal{S}_{\mathsf{L}} = \int \mathbb{d}\Omega_{3\mathsf{F}}^2 \int \mathbb{d}t\, L$, where

$$L := \varkappa^{3/2} N \mathbb{e}^{3\alpha}\left(-\frac{3}{\varkappa}\frac{\dot\alpha^2}{N^2} + \ell\,\frac{\dot\phi^2}{2N^2} - V\mathbb{e}^{\lambda\phi}\right), \tag{3}$$

  in which dot means $\mathbb{d}/\mathbb{d}t$.

- The model turns out to be integrable, both classically and quantum-mechanically, enabling one to study its full physical properties, e.g. the relation between its classical and quantum theory.

Outline

Decoupling the variables

Via *rescaled* special orthogonal transformation

- Setting $\overline{N} := N \mathrm{e}^{-3\alpha}$, eq. (3) transforms to

$$L = \varkappa^{3/2}\overline{N}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{\overline{N}^2} + \ell\,\frac{\dot{\phi}^2}{2\overline{N}^2} - V\mathrm{e}^{\lambda\phi+6\alpha}\right) \tag{4}$$

- Defining $\Delta := \lambda^2 - 6\ell\varkappa$, $\mathfrak{s} := \operatorname{sgn}\Delta$ and $g := \mathfrak{s}\sqrt{|\Delta|} \equiv \mathfrak{s}\sqrt{\mathfrak{s}\Delta}$, the *rescaled* special orthogonal transformation

$$\begin{pmatrix}\alpha\\ \phi\end{pmatrix} = \frac{\mathfrak{s}}{g}\begin{pmatrix}\lambda & -\ell\varkappa\\ -6 & \lambda\end{pmatrix}\begin{pmatrix}\mathfrak{s}_\beta\beta\\ \mathfrak{s}_\chi\chi\end{pmatrix} \qquad \text{where } \mathfrak{s}_\beta, \mathfrak{s}_\chi = \pm 1 \tag{5}$$

gives the decoupled Lagrangian

$$L = \varkappa^{3/2}\overline{N}\left(-\mathfrak{s}\frac{3}{\varkappa}\frac{\dot{\beta}^2}{\overline{N}^2} + \ell\,\mathfrak{s}\,\frac{\dot{\chi}^2}{2\overline{N}^2} - V\mathrm{e}^{\mathfrak{s}_\chi g\chi}\right). \tag{6}$$

- The Euler–Lagrange equations w.r.t. $\overline{N}$, β and χ will be called the trsfed. 1st, 2nd Friedmann eqs. and the Klein–Gordon eq., respectively.

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Implicit integration

General integral for $p_\beta \neq 0$

- Since β is cyclic in eq. (6), the trsfed. 2nd Friedmann eq. can be integrated[7]

$$\text{const.} \equiv p_\beta := \frac{\partial L}{\partial \dot{\beta}} = -6\jmath \varkappa^{1/2} \frac{\dot{\beta}}{\overline{N}} \equiv -6\jmath\jmath_\beta \frac{\varkappa^{1/2}}{g} \frac{\lambda\dot{\alpha} + \ell\varkappa\dot{\phi}}{\overline{N}}. \tag{7}$$

- For $p_\beta \neq 0$, fixing the *implicit gauge* $\overline{N} = -6\jmath\sqrt{\varkappa}\dot{\beta}/p_\beta$, the trsfed. 1st Friedmann equation can be integrated to get the trajectory

$$\mathrm{e}^{\jmath_\chi g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|} S^2\left(\jmath_\beta\sqrt{\frac{3}{2\varkappa}}\, g\beta + C\right), \tag{8}$$

in which $\upsilon := \operatorname{sgn} V$, $(\operatorname{sgn}, \operatorname{sgn})$ means $(\ell, \jmath\upsilon)$, and

$$\begin{aligned} {}_{(+,+)}S(\gamma) &:= \operatorname{sech}(\gamma), & {}_{(+,-)}S(\gamma) &:= \operatorname{csch}(\gamma), \\ {}_{(-,+)}S(\gamma) &:= \sec(\gamma), & {}_{(-,-)}S(\gamma) &:= \mathbb{i}\csc(\gamma). \end{aligned} \tag{9}$$

[7] The same first integral has been found in C. Lan. PhD thesis. Saint Petersburg State University, 2016, A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theoretical and Mathematical Physics* 184.3 (2015), pp. 1224–1233, in canonical formalism.

Implicit integration

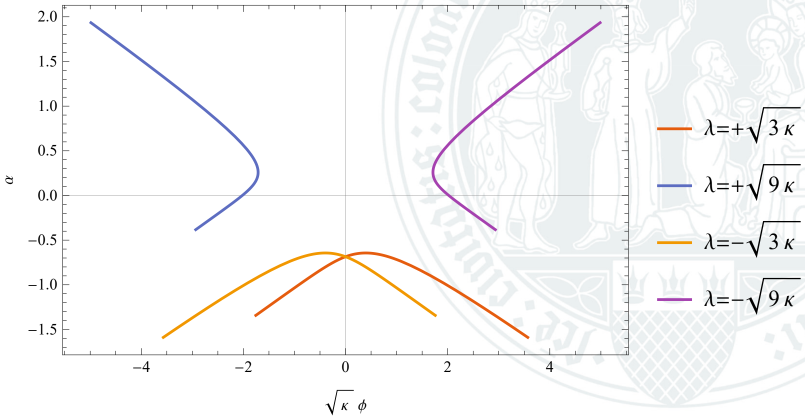
Specific integral for $p_\beta = 0$

- For $p_\beta = 0$, integrating the transformed second Friedmann equation yields $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell\lambda\alpha/\varkappa$, which is the well-known power-law special solution[8].
- Further integrating the transformed first Friedmann equation demands $(+,-)$ or $(-,+)$ to guarantee $\overline{N} > 0$.

[8] For instance, A. R. Liddle and D. H. Lyth. Cambridge University Press, 2000, ch. 3.

Trajectories for quintessence model $(+,+)$

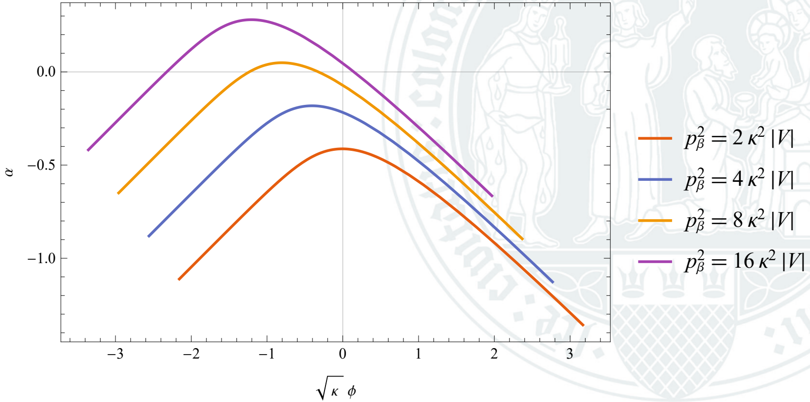
sech, with $C=0$, $|V| = \varkappa^{-2}$ and $p_\beta^2 = \varkappa^2\sqrt{|V|}$; varying λ



- has two asymptotes $\chi \propto \pm\beta$

Trajectories for quintessence model $(+,+)$

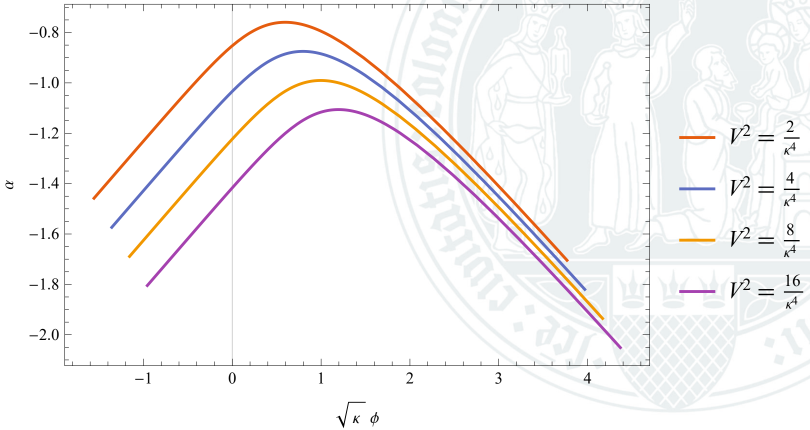
sech, **with** $C=0$, $|V|=\varkappa^{-2}$ **and** $\lambda^2=3\varkappa$; **varying** p_β



- has two asymptotes $\chi \propto \pm\beta$

Trajectories for quintessence model $(+,+)$

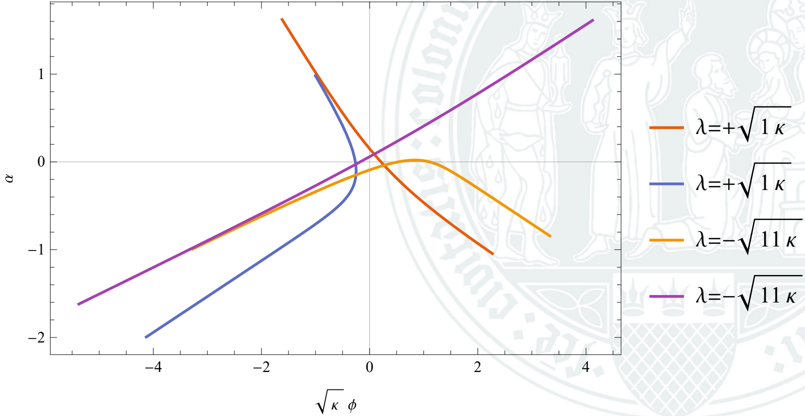
sech, **with** $C=0$, $\lambda^2 = 3\varkappa$ **and** $p_\beta^2 = \varkappa^2\sqrt{|V|}$; **varying** V



- has two asymptotes $\chi \propto \pm\beta$

Trajectories for quintessence model $(+,-)$

csch, with $C=0$, $|V|=\varkappa^{-2}$ and $p_\beta^2=\varkappa^2\sqrt{|V|}$; varying λ



- contains two distinct solutions, separated by $\beta=0$
- has three asymptotes $\chi\propto\pm\beta$ and $\beta=0$

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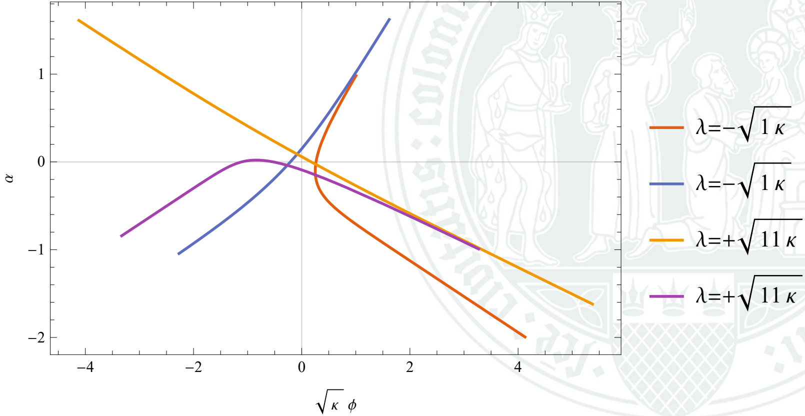
Trajectories for quintessence model $(+,-)$: csch

csch, with $C=0$, $|V|=\varkappa^{-2}$ and $p_\beta^2=\varkappa^2\sqrt{|V|}$; varying λ



- contains two distinct solutions, separated by $\beta=0$
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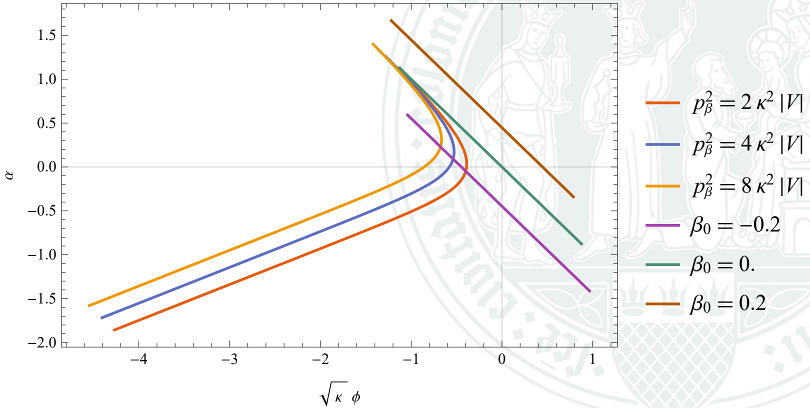
Trajectories for quintessence model $(+,-)$: csch

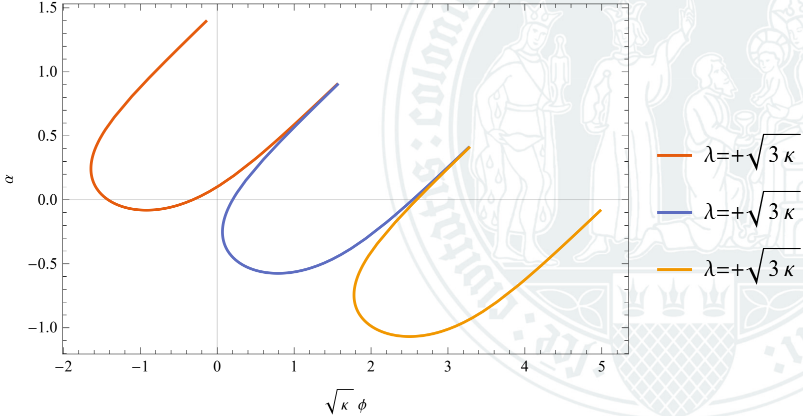
csch, with $V = \varkappa^{-2}$ and $\lambda^2 = \varkappa$; varying p_β



- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$

Trajectories for phantom model $(-,+)$

csc, **with** $V = \varkappa^{-2}$ **and** $p_\beta^2 = 3\varkappa^2\sqrt{|V|}$; **varying** λ



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$

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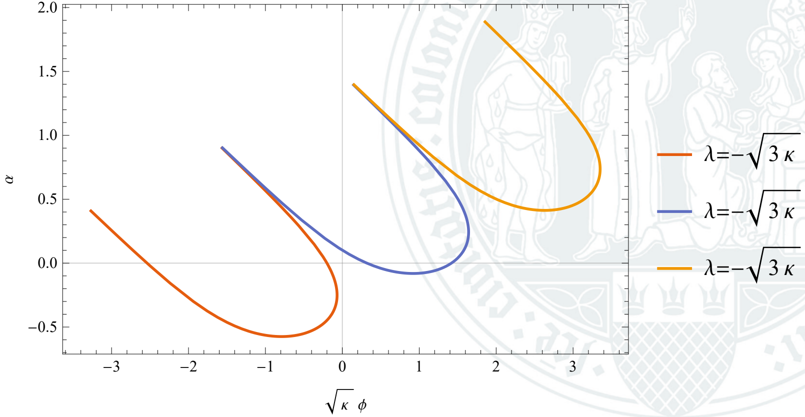
Trajectories for phantom model $(-,+)$

csc, **with** $V = \varkappa^{-2}$ **and** $p_\beta^2 = 3\varkappa^2\sqrt{|V|}$; **varying** λ



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$

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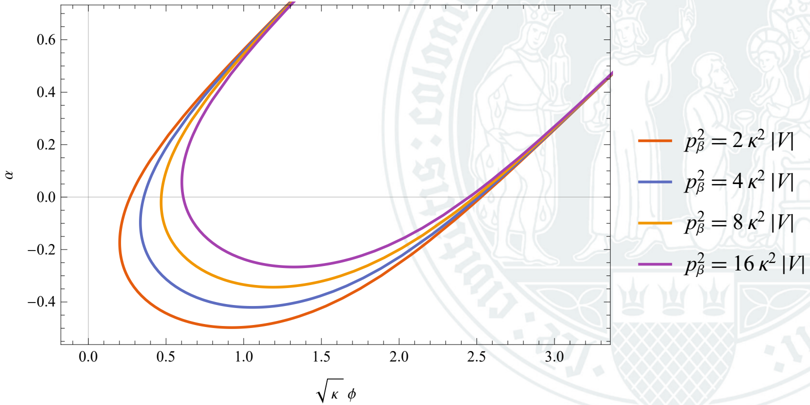
Trajectories for phantom model $(-,+)$

csc, with $V = \varkappa^{-2}$ and $\lambda^2 = 3\varkappa$; varying p_β



- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n + 1/2)\pi$

Integration

Further discussions

- The integral for $(-,-)$ is not real.
- The trajectories can be parametrised by β, inspiring recognising β as a 'time variable'.
- The implicit integration enables one to compare trajectories with wave functions, see below.

Outline

Dirac quantisation and the mode functions

- The primary Hamiltonian and the Hamiltonian constraint[9] reads

$$H^{\mathrm{p}} = \overline{N} H_{\perp} + v^{\overline{N}} p_{\overline{N}}, \tag{10}$$

$$H_{\perp} := -\mathfrak{s} \frac{p_{\beta}^2}{12\varkappa^{1/2}} + \ell \mathfrak{s} \frac{p_{\chi}^2}{2\varkappa^{3/2}} + \varkappa^{3/2} V \mathrm{e}^{g \mathfrak{s}_{\chi} \chi}. \tag{11}$$

- Applying the Dirac quantisation rules with Laplace–Beltrami opeator[10], one gets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(\mathfrak{s} \frac{\hbar^2}{12\varkappa^{1/2}} \partial_{\beta}^2 - \ell \mathfrak{s} \frac{\hbar^2}{2\varkappa^{3/2}} \partial_{\chi}^2 + \varkappa^{3/2} V \mathrm{e}^{g \mathfrak{s}_{\chi} \chi} \right) \Psi. \tag{12}$$

- Equation (12) is KG-like, hyperbolic for $\ell = +1$ and *elliptic* for $\ell = -1$.

[9] D. M. Gitman and I. V. Tyutin. Springer Berlin Heidelberg, 1990, H. J. Rothe and K. D. Rothe. World Scientific, 2010.

[10] C. Kiefer. 3rd ed. Oxford University Press, 2012, ch. 8.

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Separation of the variables and mode functions

- Writing $\Psi(\beta,\chi) = \varphi(\beta)\psi(\chi)$, eq. (12) can be separated into

$$\partial_\beta^2 \varphi(\beta) = k_\beta^2 \varphi(\beta); \tag{13}$$

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \qquad \mathbb{D} := -\ell \frac{6}{\varkappa}\partial_\chi^2 + \jmath v \frac{12\varkappa^2 |V|}{\hbar^2}. \tag{14}$$

- Equation (14) turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta,\chi) := \sum_{i=1}^{2} c_i \varphi_\nu^{(i)}(\gamma) \sum_{j=1}^{2} a_j \mathrm{B}_\nu^{(j)}(\sigma), \quad \nu \geq 0; \tag{15}$$

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{1}{g} k_\beta, \quad \gamma := \sqrt{\frac{3}{2\varkappa}} \frac{g}{1} \beta, \qquad \sigma^2 := \frac{8\varkappa^3 |V| \mathrm{e}^{g\jmath_\chi \chi}}{\hbar^2 g^2}, \tag{16}$$

$$_{(+,+)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{K} \text{ and } \mathrm{I}_{\mathbb{i}\nu}(\sigma), \qquad _{(+,-)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{F} \text{ and } \mathrm{G}_{\mathbb{i}\nu}(\sigma),$$

$$_{(-,+)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{J} \text{ and } \mathrm{Y}_\nu(\sigma), \qquad _{(-,-)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{K} \text{ and } \mathrm{I}_\nu(\sigma).$$

- Adapted to imaginary order, $\mathrm{F}_\nu(\sigma)$ and $\mathrm{G}_\nu(\sigma)$ are defined in[11].

[11] T. M. Dunster. In: *SIAM Journal on Mathematical Analysis* 21.4 (1990), pp. 995–1018.

Physical mode functions

- Physical mode functions are expected to be regular on the boundary.
- $(+,+)$: $|\mathrm{I}_{\mathbb{i}\nu}(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- $(-,+)$:
    - $\forall n \in \mathbb{N}$, $|\mathrm{Y}_n(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
    - $\forall \nu \in \mathbb{R}^{+}\backslash\mathbb{N}$, choose $\mathrm{J}_{-\nu}$ instead of Y_ν, since $\mathrm{J}_{\pm\nu}$ are also linearly independent.
    - $\forall \nu \in \mathbb{R}^{+}\backslash\mathbb{N}$, $|\mathrm{J}_{-\nu}(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
- $(-,-)$: $|\mathrm{K}_\nu(\sigma)| \to +\infty$ as $\alpha \to -\infty$; $|\mathrm{I}_\nu(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- These are not to be included in the space of physical wave functions. $\forall \nu \geq 0$,
    - $(+,+)$: $\mathrm{K}_{\mathbb{i}\nu}(\sigma)$ survives
    - $(+,-)$: $\mathrm{F}_{\mathbb{i}\nu}(\sigma)$ and $\mathrm{G}_{\mathbb{i}\nu}(\sigma)$ survives
    - $(-,+)$: $\mathrm{J}_\nu(\sigma)$ survives
    - $(-,-)$: drops out

Matching quantum number with classical first integral

Principle of constructive interference

- Write the mode function in the WKB form

$$\Psi_{k_\beta}(\beta,\chi) \sim \sqrt{\rho(\beta,\chi)}\exp\left\{\frac{\mathrm{i}}{\hbar}S(\beta,\chi)\right\}. \tag{17}$$

- For $S/\hbar \gg 1$ and $k_\beta \gg 1$[12], $S(\beta,\chi)$ becomes the Hamilton principle function in the leading-order approximation.
- A Hamilton principle function is stationary with respect to variation of integral constants[13]

$$\frac{\partial S}{\partial k_\beta} = 0. \tag{18}$$

- Demanding eq. (18) matching the classical trajectory, k_β can be related to p_β.

[12] The common form is $\hbar \to 0^+$.

[13] U. H. Gerlach. In: *Physical Review* 177.5 (1969), pp. 1929–1941, L. D. Landau and E. M. Lifshitz. 3rd ed. Elsevier, 1976.

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Matching quantum number with classical first integral

$(+,+)$ as exemplar

- Fixing $\nu/\sigma > 1$, the asymptotic expansion reads

$$\mathrm{K}_{\mathbb{i}\nu}(\sigma) \sim \frac{\sqrt{2\pi}\cos\left(\sqrt{\nu^2-\sigma^2}-\nu\arccos\frac{\nu}{\sigma}-\frac{\pi}{4}\right)}{\left(\nu^2-\sigma^2\right)^{1/4}\mathrm{e}^{\pi\nu/2}} + O\left(\frac{1}{\sigma}\right). \tag{19}$$

  - There are two phases with opposite signs. Assuming c_i, a_j's are real and applying *the principle* to $\Psi_\nu(\sigma)$, one has$\sigma/\nu = \mathrm{sech}(\jmath_\beta\gamma)$, which matches the trajectory with $C = 0$ if

$$\hbar k_\beta \equiv \hbar\sqrt{\frac{3}{2\varkappa}g\hbar\nu} = p_\beta, \tag{20}$$

  - Non-vanishing C can be compensated by the phase of c_i and a_j's.

- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; it is not within the WKB regime.

- The conclusions also hold for $\mathrm{F}_{\mathbb{i}\nu}(\sigma)$, $\mathrm{G}_{\mathbb{i}\nu}(\sigma)$ for $(+,-)$, and $\mathrm{J}_\nu(\sigma)$ for $(-,+)$.

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Inner product for wave functions

Schrödinger product

- To make sense of amplitude and wave packet, an inner product is necessary
- In terms of a Klein–Gordon equation, call β the "temporal" variable, and χ the "spacial" variable.
- A common starting point is the *Schrödinger product*[14]

$$(\Psi_1, \Psi_2)_{\mathsf{S}} := \int \mathrm{d}\chi\, \Psi_1^*(\beta, \chi)\Psi_2(\beta, \chi); \tag{21}$$

- $(\Psi, \Psi)_{\mathsf{S}}$ is positive-definite, and the integrand $\rho_{\mathsf{S}}(\beta, \chi)$ is non-negative
- The corresponding Schrödinger current is real but does not satisfy continuity equation $\dot{\rho}_{\mathsf{S}} + \nabla \cdot \vec{j}_{\mathsf{S}} = 0$, because eq. (12) is KG-like.
- $\mathrm{K}_{\mathring{\imath}\nu}$[15] for $(+,+)$, $\mathrm{F}_{\mathring{\imath}\nu}$ and $\mathrm{G}_{\mathring{\imath}\nu}$ for $(+,-)$ can be proved to be orthogonal and complete among themselves, as well as can be normalised.
  - $\mathrm{J}_{\mathring{\imath}\nu}$'s for $(+,-)$ are not orthogonal

14 C. Kiefer. 3rd ed. Oxford University Press, 2012, ch. 5.

15 S. B. Yakubovich. In: *Opuscula Math.* 26.1 (2006), pp. 161–172, A. Passian et al. In: *Journal of Mathematical Analysis and Applications* 360.2 (2009), pp. 380–390, R. Szymtkowski and S. Bielski. In: *Journal of Mathematical Analysis and Applications* 365.1 (2010), pp. 195–197.

Peculiarity of the phantom model $(-,+)$

Orthogonality for mode functions; Hermiticity for operators

- $\mathrm{J}_\nu(\sigma)$'s are not orthogonal under the Schrödinger product

$$(\mathrm{J}_\nu, \mathrm{J}_{\tilde{\nu}})_{\mathsf{S}} \propto \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{J}_\nu^*(\mathrm{e}^x)\mathrm{J}_{\tilde{\nu}}(\mathrm{e}^x) = \frac{2\sin(\pi(\nu-\tilde{\nu})/2)}{\pi(\nu^2-\tilde{\nu}^2)}, \tag{22}$$

  therefore $\mathbb{D}$ in eq. (14) is not Hermitian (though we do not need it so far)
- $\hat{p}_\chi^2$ is not Hermitian for $\{\mathrm{J}_\nu(\sigma)\}$ under the Schrödinger product

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{J}_\nu^*(-\partial_x^2 \mathrm{J}_{\tilde{\nu}}) - \int_{-\infty}^{+\infty} \mathrm{d}x\, (-\partial_x^2 \mathrm{J}_\nu)^* \mathrm{J}_{\tilde{\nu}} = \frac{2}{\pi}\sin\frac{\pi(\nu-\tilde{\nu})}{2}. \tag{23}$$

- In order to save Hermiticity for p_χ^2 and $\mathbb{D}$ and orthogonality of the modes under Schrödinger product, one can restrict

$$\nu = 2n + \nu_0, \qquad n \in \mathbb{N}, \quad \nu_0 \in [0,2). \tag{24}$$

- Using classical singularities as boundary condition, one can fix $\nu_0 = 1$.

Discritisation of the phantom model $(-,+)$

Levels of the phantom model are discretised if one imposes Hermiticity of squared momenta under the Schrödinger product.

- This kind of subtlety on Hermiticity is named as *self-adjoint extension*, which arises already for infinite square well[16].
- It also applies to x^{-2} potentials[17], which is of cosmological relevance[18].

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16 G. Bonneau, J. Faraut, and G. Valent. In: *American Journal of Physics* 69.3 (2001), pp. 322–331.

17 A. M. Essin and D. J. Griffiths. In: *American Journal of Physics* 74.2 (2006), pp. 109–117, V. S. Araujo, F. A. B. Coutinho, and J. F. Perez. In: *American Journal of Physics* 72.2 (2004), pp. 203–213.

18 M. Bouhmadi-López et al. In: *Physical Review D* 79.12 (2009).

Further inner products for wave functions

Klein–Gordon and Mostafazadeh product

- Since eq. (12) is KG-like, another popular choice is the KG product

$$(\Psi_1, \Psi_2)^g_{\mathsf{KG}} := \mathbb{i} g\Big\{\big(\Psi_1, \partial_\beta \Psi_2\big)_{\mathsf{S}} - \big(\partial_\beta \Psi_1, \Psi_2\big)_{\mathsf{S}}\Big\}, \qquad g > 0. \tag{25}$$

  - $(\Psi, \Psi)^g_{\mathsf{KG}}$ is real but not positive-definite, so does the integrand ρ_{KG};
  - The corresponding $\vec{J}_{\mathsf{KG}}$ is conserved $\dot{\rho}_{\mathsf{KG}} + \nabla \cdot \vec{J}_{\mathsf{KG}} = 0$ and real.

- Mostafazadeh[19] found a product *for Hermitian* $\mathbb{D}$ *with positive spectrum*:

$$(\Psi_1, \Psi_2)^\kappa_{\mathsf{M}} := \kappa\Big\{\big(\Psi_1, \mathbb{D}^{+1/2}\Psi_2\big)_{\mathsf{S}} + \big(\partial_\beta \Psi_1, \mathbb{D}^{-1/2}\partial_\beta \Psi_2\big)_{\mathsf{S}}\Big\}, \qquad \kappa > 0. \tag{26}$$

  - $(\Psi, \Psi)^\kappa_{\mathsf{M}}$ is positive-definite, but the integrand ρ^κ_{M} is complex
  - The corresponding $\vec{J}^\kappa_{\mathsf{M}}$ is conserved $\dot{\rho}^\kappa_{\mathsf{M}} + \nabla \cdot \vec{J}^\kappa_{\mathsf{M}} = 0$ but also complex.

[19] A. Mostafazadeh. In: *Classical and Quantum Gravity* 20.1 (2002), pp. 155–171.

Mostafazadeh inner product and the corresponding density

- Real power of $\mathbb{D}$ is defined by spectral decomposition $\mathbb{D}^\gamma := \sum_\nu \nu^{2\gamma}\mathbf{P}_\nu$, $\mathbf{P}_\nu\Psi := \Psi_\nu(\Psi_\nu, \Psi)_{\mathsf{S}}$.

- It can be shown[20] that *the density*

$$\varrho^\kappa_{\mathsf{M}} := \kappa\left\{\left|\mathbb{D}^{+1/4}\Psi\right|^2 + \left|\mathbb{D}^{-1/4}\partial_\beta\Psi\right|^2\right\} \tag{27}$$

  - is equivalent to ρ^κ_{M} up to a boundary term

$$\int \mathbb{d}\chi\, \varrho^\kappa_{\mathsf{M}} = \int \mathbb{d}\chi\, \rho^\kappa_{\mathsf{M}} \equiv (\Psi_1, \Psi_2)^\kappa_{\mathsf{M}}; \tag{28}$$

  - is non-negative.
  - The corresponding current $\vec{\mathcal{J}}^\kappa_{\mathsf{M}}$ is real but not conserved[21].

[20] A. Mostafazadeh and F. Zamani. In: *Annals of Physics* 321.9 (2006), pp. 2183–2209.

[21] B. Rosenstein and L. P. Horwitz. In: *Journal of Physics A: Mathematical and General* 18.11 (1985), pp. 2115–2121.

Wave packets of Gaussian amplitude for continuous spectrum

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu;\overline{\nu},\sigma) := \left(\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(\nu-\overline{\nu})^2}{2\sigma^2}\right)\right)^{1/2} \tag{29}$$

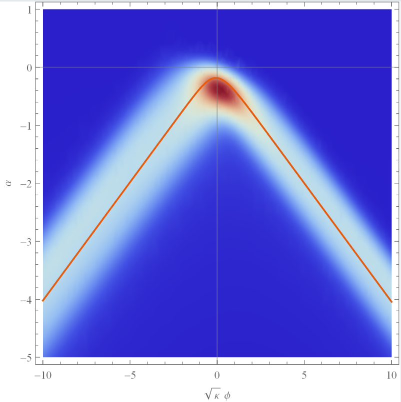
- In[22], $A\left(\nu;\overline{\nu},\sigma/\sqrt{2}\right)$ was chosen.

[22] M. P. Dąbrowski, C. Kiefer, and B. Sandhöfer. In: *Physical Review D* 74.4 (2006).

Wave packets of Gaussian amplitude for quintessence model $(+,+)$

$\mathrm{K}_{\mathbb{i}\nu}$, **with** $\lambda = \varkappa^{1/2}/2$, $V = -\varkappa^{-2}$, $\overline{k}_\beta = -2$ **and** $\sigma_\beta = 5/4$

Schödinger



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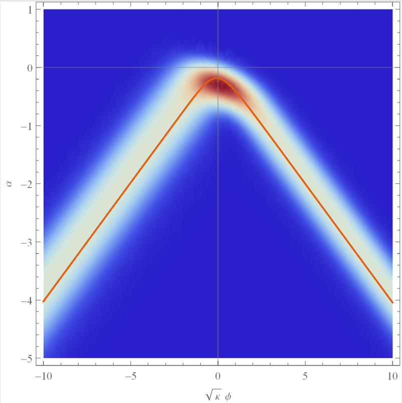

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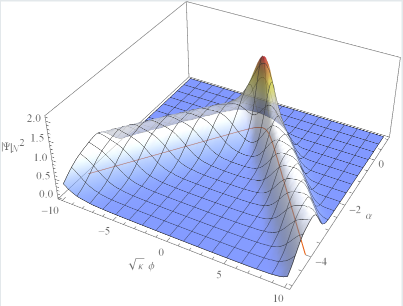
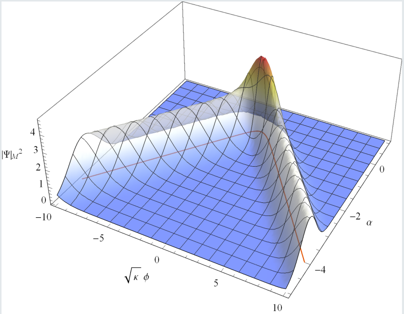


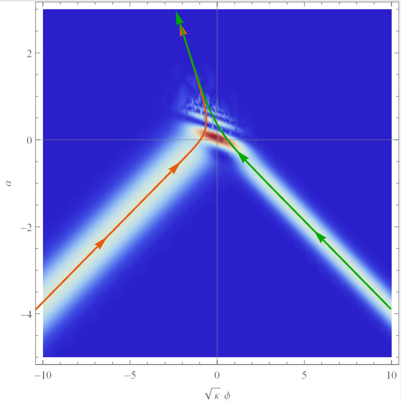
Mostafazadeh



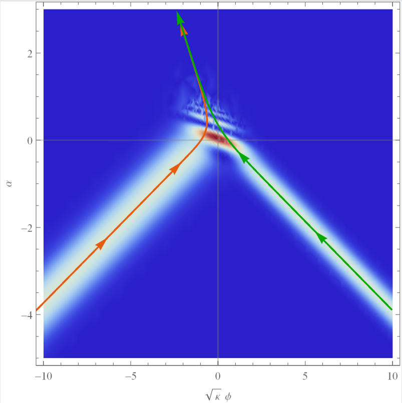
Wave packets of Gaussian amplitude for quintessence model $(+,-)$

$\mathrm{F}_{\mathbb{i}\nu}$, **with** $\lambda = 4\varkappa^{1/2}/5$, $V = +\varkappa^{-2}$, $\overline{k}_\beta = -7/2$ **and** $\sigma_\beta = 7/5$

Schödinger



Mostafazadeh



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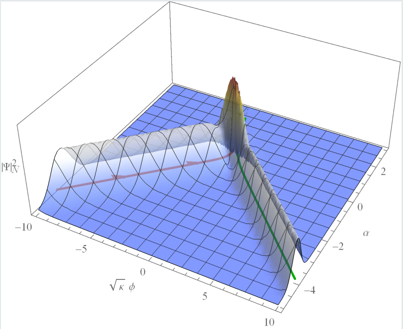

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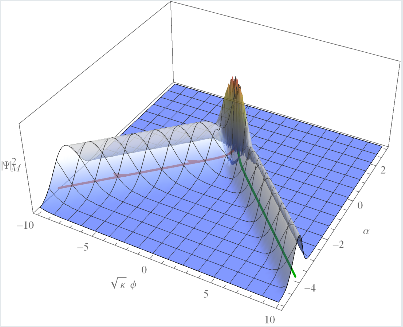

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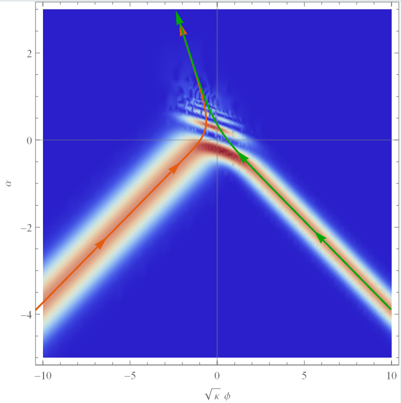

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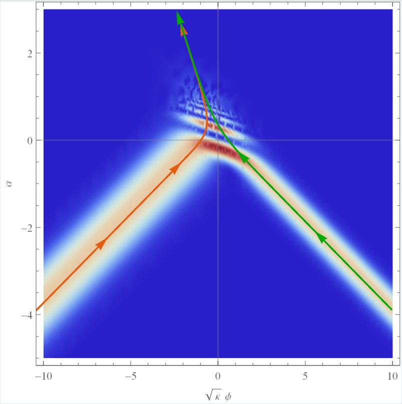

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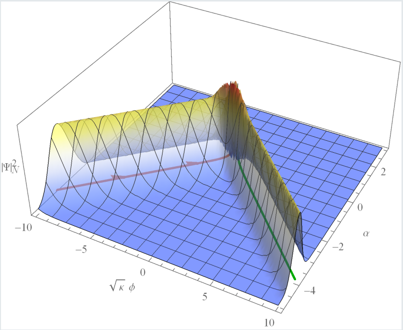
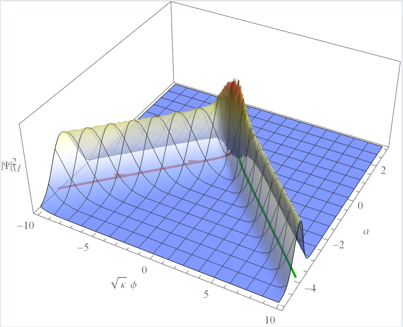


Mostafazadeh



Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model $(-,+)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\overline{n}) := \left(\mathbb{e}^{-\overline{n}} \frac{\overline{n}^n}{n!} \right)^{1/2} \tag{30}$$

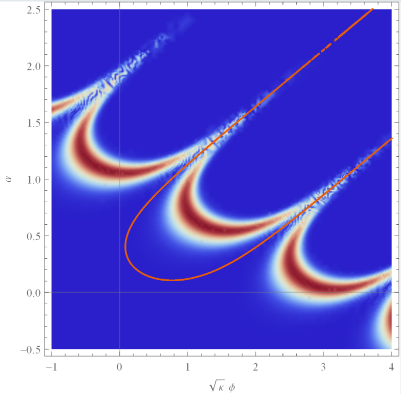
- In[23], $A_n\left(\overline{n}/\sqrt{2}\right)$ was chosen.

[23] C. Kiefer. In: *Nuclear Physics B* 341.1 (1990), pp. 273–293.

Wave packets of Poissonian amplitude for phantom model

J_{2n+1}, with $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$ and $\overline{k}_\beta = 8$

Schödinger



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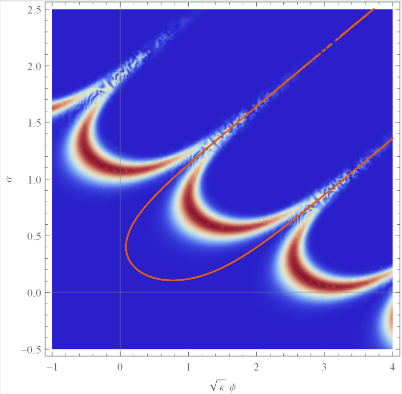

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Wave packets of Poissonian amplitude for phantom model

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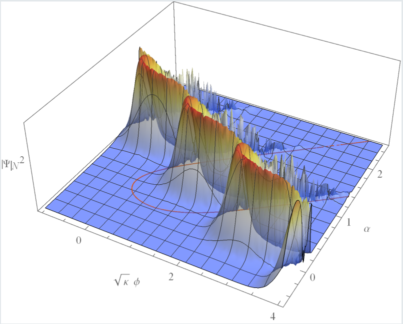
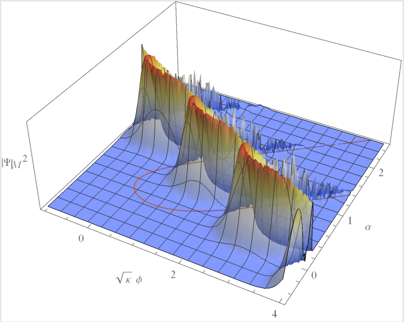


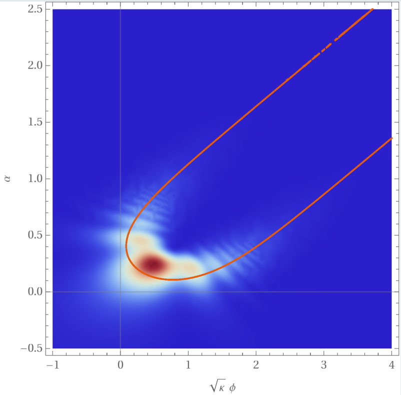
Mostafazadeh



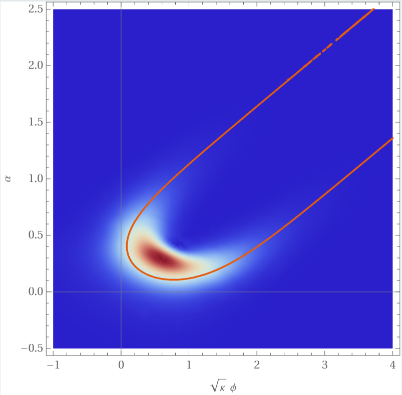
Wave packets of Gaussian amplitude for phantom model

J_ν, with $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$, $\overline{k}_\beta = 8$ and $\sigma_\beta = 11/2$

Schödinger



Mostafazadeh



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Wave packets of Gaussian amplitude for phantom model

J_ν, **with** $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$, $\overline{k}_\beta = 8$ **and** $\sigma_\beta = 11/2$

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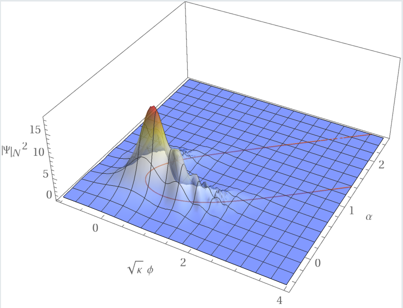

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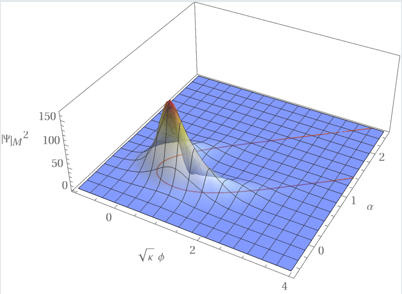

Outline

Highlights

- An integral of motion was found for Liouville cosmological models.
- Implicit trajectories in minisuperspace were obtained.
- The levels of phantom Liouville model were found to be discrete due to the Hermiticity requirement of observables.
- The semi-classical quantum mode functions and wave packets was compared directly with the classical trajectory.

Issues

- In $(+,-)$ and $(-,+)$, wave packets contain multiple branches; however, the classical universe runs only on one trajectory.
- Quantum-corrected $\overline{k}_\beta$ is to be understood.
- Wave packets with Gaussian profile are to be constructed, instead of inserting Gaussian / Possonian amplitude artificially.
- A normalising κ for $(\cdot,\cdot)^\kappa_{\mathsf{M}}$ is to be evaluated, otherwise a quantitative comparison of $(\cdot,\cdot)_{\mathsf{S}}$ and $(\cdot,\cdot)^\kappa_{\mathsf{M}}$ is not possible.

Outlook

- Beyond isotropy: generalise to Bianchi models
  - Bianchi Type-I: a natural extension, under investigation
- Beyond homogeneity: cosmological perturbation, under investigation
- Beyond single field
  - Two exponential potentials: $|V_1| = |V_2|$ and special λ_i
  - Multiple Liouville fields: mixing kinetic terms needed[24]
- Beyond classic matter
  - PT-symmetric Liouville field[25]: may cross the phantom divide $w = -1$.

[24] A. A. Andrianov, O. O. Novikov, and C. Lan. In: *Theoretical and Mathematical Physics* 184.3 (2015), pp. 1224–1233.

[25] A. A. Andrianov et al. In: *International Journal of Modern Physics D* 19.01 (2010), pp. 97–111, A. A. Andrianov, C. Lan, and O. O. Novikov. In: *Springer Proceedings in Physics*. Springer International Publishing, 2016, pp. 29–44.

Integration of the transformed first Friedmann equation

General integral for $p_\beta \neq 0$

In order to integrate the equation under the implicit gauge

$$\jmath \frac{p_\beta^2}{12}\left(-\ell \frac{\varkappa^{1/2}}{6}\left(\frac{\mathrm{d}\chi}{\mathrm{d}\beta}\right)^2 + \varkappa^{-1/2}\right) - \varkappa^{3/2} V \mathrm{e}^{g\jmath_\chi \chi} = 0, \tag{31}$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa}} g\beta, \qquad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} \mathrm{e}^{-g\jmath_\chi \chi}, \tag{32}$$

to get

$$\left(\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\gamma}\right)^2 + \ell\left(\jmath v - \tilde{\sigma}^2\right) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell\left(-\jmath v + \tilde{\sigma}^2\right)}}, \tag{33}$$

which is of the standard inverse hyperbolic / trigonometric form except for $(-,-)$.

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Integration of the separated minisuperspace WDW equation

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \qquad \mathbb{D} := -\ell \frac{6}{\varkappa}\partial_\chi^2 + \jmath \upsilon \frac{12\varkappa^2 |V|}{\hbar^2}, \tag{14 rev.}$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{k_\beta}{g}, \qquad \sigma^2 := \frac{8\varkappa^3 |V| \mathbb{e}^{g \jmath_\chi \chi}}{\hbar^2 g^2}, \tag{16 rev.}$$

to get

$$\sigma^2 \psi''(\sigma) + \sigma \psi'(\sigma) + \ell(\nu^2 - \jmath \upsilon \sigma^2)\psi(\sigma) = 0, \tag{34}$$

which is of the standard Besselian form.