Coframe formalisms

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1 U(1)–regular theory

1.1 Differential geometry

Interior product Let z be a vector, ω be a 1-form, χ be a k-form. The *interior product* is defined inductively as the bilinear map satisfying

$$z \lrcorner \omega := \omega(z)\,, \tag{1}$$

$$z \lrcorner (\omega \wedge \chi) := (z \lrcorner \omega) \wedge \chi - \omega \wedge (z \lrcorner \chi)\,. \tag{2}$$

Equation (2) is also known as the anti-product rule.

By inductiion one can show that for a p-form ϕ,

$$z \lrcorner (\phi \wedge \chi) = (z \lrcorner \phi) \wedge \chi + (-)^p \chi \wedge (z \lrcorner \chi)\,. \tag{3}$$

Hodge star Let ω be an 1-form, χ be a k-form. The Hodge star $\star$ is defined inductively as the linear map [4, sec. 24]

$$\star 1 := \mathrm{vol}\,, \tag{4}$$

$$\star(\chi \wedge \omega) := \omega^\sharp \lrcorner \star\chi\,. \tag{5}$$

Non-gravitational theories features $[\delta, \star] = 0$, which means [7, sec. 3.2]

$$\delta g_{\mu\nu} = -2\omega_{(\mu\nu)}\,, \qquad \delta\vartheta^\mu = \omega_\nu{}^\mu\, \vartheta^\nu\,; \tag{6}$$

for an orthonormal coframe, the allowed variations are $\omega_{(\alpha\beta)} = 0$.

Codifferential The *codifferential* $\mathbb{d}^\dagger$ is the adjoint of the exterior derivative $\mathbb{d}$ in the following sense. Let ϕ be a k-form, χ be a $(k-1)$-form.

$$\mathbb{d}(\chi^* \wedge \star\phi) \equiv \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d}\star\phi =: \mathbb{d}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{d}^\dagger\phi \tag{7}$$

$$= \mathbb{d}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} \mathbb{d}\star\phi\,. \tag{8}$$

$$\boxed{\mathbb{d}^\dagger\phi = (-)^k \star^{-1} \mathbb{d}\star\phi\,.} \tag{9}$$

1.2 Complex scalar field

The action reads

$$S = \int -\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi - m^2\phi^* \wedge \star\phi \tag{10}$$

$$= \int -\frac{1}{2}(\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi + \mathbb{d}\phi \wedge \star\mathbb{d}\phi^*) - \frac{1}{2}m^2(\phi^* \wedge \star\phi + \phi \wedge \star\phi^*) \tag{11}$$

Variation commutes with exterior derivative

$$[\delta, \mathbb{d}]\phi = 0\,. \tag{12}$$

A generic variation of the kinetic terms reads

$$\begin{aligned}\delta(\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi) = {}& \mathbb{d}(\delta\phi^* \wedge \star\mathbb{d}\phi + \delta\phi \wedge \star\mathbb{d}\phi^*) \\ & + \delta\phi^* \wedge \star\mathbb{d}^\dagger\mathbb{d}\phi + \delta\phi \wedge \star\mathbb{d}^\dagger\mathbb{d}\phi^*\,.\end{aligned} \tag{13}$$

A generic variation of the actionreads

$$\begin{aligned}\delta S = \int & \mathbb{d}(-\delta\phi^* \wedge \star\mathbb{d}\phi - \delta\phi \wedge \star\mathbb{d}\phi^*) \\ & - \delta\phi^* \wedge \star(\mathbb{d}^\dagger\mathbb{d} + m^2)\phi - \delta\phi \wedge \star(\mathbb{d}^\dagger\mathbb{d} + m^2)\phi^*\,.\end{aligned} \tag{14}$$

1.2.1 Noether current

The action is invariant

$$\delta_\lambda S \equiv 0 \tag{15}$$

under the rigid transformation

$$\phi \to \mathsf{e}^{-\mathrm{i}e\lambda}\phi\,, \qquad \phi^* \to \mathsf{e}^{+\mathrm{i}e\lambda}\phi^*\,. \tag{16}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$\begin{aligned}
0 &= \int \lambda\, \mathbb{d}\mathfrak{J}_0\,,\\
\mathfrak{J}_0 &:= \mathsf{i}e(\phi^* \wedge \star\mathbb{d}\phi - \phi \wedge \star\mathbb{d}\phi^*)\,,
\end{aligned} \tag{17}$$

which is the Noether current, a twisted 3-form, satisfying the continuity equation

$$\mathbb{d}\mathfrak{J}_0 = 0\,. \tag{18}$$

2 $\mathsf{U}(1)$ gauge theory

2.1 Connection on the principal bundle

Exterior covariant derivative Let χ be a $\mathbb{C}$-valued k-form. The *exterior covariant derivative* of χ reads

$$\mathbb{D}\chi := (\mathbb{d} - \mathsf{i}eA)\chi, \qquad \mathbb{D}\chi^* := (\mathbb{d} + \mathsf{i}eA)\chi^*\,, \tag{19}$$

where A is a $\mathfrak{u}(1)$-valued *connection form.*

Covariant codifferential The *covariant codifferential* $\mathbb{D}^\dagger$ is the adjoint of the exterior covariant derivative $\mathbb{D}$ in the following sense. Let ϕ be a $\mathbb{C}$-valued k-form, χ be a $\mathbb{C}$-valued $(k-1)$-form.

$$\begin{aligned}
\mathbb{d}(\chi^* \wedge \star\phi) &\equiv \mathbb{d}\chi^* \wedge \star\phi - (-)^k\chi^* \wedge \mathbb{d}\star\phi =: \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{D}^\dagger\phi && (20)\\
&= \mathbb{D}\chi^* \wedge \star\phi - \mathsf{i}eA \wedge \chi^* \wedge \star\phi - (-)^k\chi^* \wedge \mathbb{d}\star\phi\\
&= \mathbb{D}\chi^* \wedge \star\phi + \chi^* \wedge (-)^k\mathsf{i}eA \star\phi - (-)^k\chi^* \wedge \mathbb{d}\star\phi\\
&= \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1}(\mathbb{d} - \mathsf{i}eA)\star\phi\,. && (21)
\end{aligned}$$

$$\boxed{\mathbb{D}^\dagger\phi = (-)^k \star^{-1}(\mathbb{d} - \mathsf{i}eA)\star\phi\,.} \tag{22}$$

2.2 $\mathsf{U}(1)$-gauged complex scalar field theory

$$S = \int -\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi - m^2\phi^* \wedge \star\phi - \frac{1}{2}F \wedge \star F. \tag{23}$$

Variation does not commute with exterior covariant derivative.

$$[\delta, \mathbb{D}]\phi = -\mathsf{i}e\delta A\phi\,. \tag{24}$$

$$\begin{aligned}
\delta(\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi) &= \mathbb{d}(\delta\phi^* \star \mathbb{D}\phi + \delta\phi \star \mathbb{D}\phi^*) + \delta\phi^* \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi^*\\
&\quad + \delta A \wedge \mathfrak{J}_A\,, \qquad \mathfrak{J}_A := \mathsf{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)\,, && (25)\\
\delta(F \wedge \star F) &= 2\,\delta F \wedge \star F && (26)\\
&= 2\,\mathbb{d}(\delta A \wedge \star F) + 2\,\delta A \wedge \mathbb{d}\star F\,. && (27)
\end{aligned}$$

Variation of the action

$$
\begin{aligned}
\delta S = \int & \mathbb{d}(-\delta\phi^* \wedge \star\mathbb{D}\phi - \delta\phi \wedge \star\mathbb{D}\phi^* - \delta A \wedge \star F) \\
& - \delta\phi^* \wedge \star(\mathbb{D}^\dagger\mathbb{D} + m^2)\phi - \delta\phi \wedge \star(\mathbb{D}^\dagger\mathbb{D} + m^2)\phi^* \\
& - \delta A \wedge (\mathbb{d}\star F - \mathfrak{J}_A). \qquad (28)
\end{aligned}
$$

2.2.1 Lorenz gauge

The Laplace–de Rham operator, or in our Lorentzian metric signature the d'Alembertian

$$\Box^2 := \left(\mathbb{d} + \mathbb{d}^\dagger\right)^2 = \mathbb{d}\mathbb{d}^\dagger + \mathbb{d}^\dagger\mathbb{d}\,. \qquad (29)$$

$$\mathbb{d}\star F = \mathbb{d} \star \mathbb{d}A = \star(-)^2 \star^{-1} \mathbb{d} \star \mathbb{d}A = \star\mathbb{d}^\dagger\mathbb{d}A = \star(\Box^2 - \mathbb{d}\mathbb{d}^\dagger)A\,. \qquad (30)$$

One would like to have $\mathbb{d}\mathbb{d}^\dagger A = 0$, or $\mathbb{d}^\dagger A = \text{const}$. This would be fulfilled if

$$\mathbb{d}^\dagger A = 0\,, \qquad (31)$$

which is the Lorenz gauge[2, 3, 6].

2.2.2 Noether's invariances

The action is invariant under the generic transformation

$$\phi \to \mathsf{e}^{-\mathrm{i}e\Lambda}\phi\,, \quad \phi^* \to \mathsf{e}^{+\mathrm{i}e\Lambda}\phi^*\,, \quad A \to A - \mathbb{d}\Lambda\,. \qquad (32)$$

There are two scenarios [1].

If the transformation is rigid, $\mathbb{d}\Lambda = 0$, one obtains $\mathfrak{J}_A$ as the Noether current from the boundary term as before, which satisfies the continuity equation $\mathbb{d}\mathfrak{J}_A = 0$.

If the transformation is gauge with a compact support, the boundary term can be dropped, and one obtains the Noether identity

$$123 \qquad (33)$$

3 Poincaré–regular theory

3.1 Differential geometry

Untwisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for untwisted k-form is defined inductively as

$$1\,, \qquad (34)$$

$$\vartheta^{\alpha_1\alpha_2\ldots\alpha_k} := \vartheta^{\alpha_1} \wedge \vartheta^{\alpha_2\ldots\alpha_k}\,. \qquad (35)$$

Twisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for twisted $(D-k)$-form is defined inductively as

$$\epsilon := \mathrm{vol}\,, \qquad (36)$$

$$\epsilon_{\alpha_1\alpha_2\ldots\alpha_k} := e_{\alpha_k} \lrcorner\, \epsilon_{\alpha_1\ldots\alpha_{k-1}}\,. \qquad (37)$$

By using eq. (5) and induction, one can show that

$$\epsilon_{\alpha_1\alpha_2\ldots\alpha_k} = \star\vartheta_{\alpha_1\alpha_2\ldots\alpha_k}\,. \qquad (38)$$

Identities Inspired by [9, eq. (3.167)], for a 1-form ω, $(D-k)$-form χ, one can derive

$$\begin{aligned}\omega \wedge \star\chi &= (-)^{D-k} \star \chi \wedge \omega = (-)^{D-k} \star^{-1} \left(\omega^\sharp \lrcorner \star \star \chi\right) \\ &= (-)^{D-k} (-)^{(D-k-1)(k+1)+s} \star \left(\omega^\sharp \lrcorner (-)^{k(D-k)+s} \star^{-1} \star\chi\right) \\ &= (-)^{k+1} \star \left(\omega^\sharp \lrcorner \chi\right) .\end{aligned} \tag{39}$$

Specifically,

$$\vartheta^\alpha \wedge \epsilon_{\beta\gamma} = -\delta^\alpha_\beta \epsilon_\gamma + \delta^\alpha_\gamma \epsilon_\beta . \tag{40}$$

Translation Consider the rigid infinitesimal transformation

$$\delta_\lambda \chi = \lambda \pounds_z \chi = \lambda[z \lrcorner \mathrm{d}\chi + \mathrm{d}(z \lrcorner \chi)] ; \tag{41}$$

'rigid' means

$$\delta_\lambda \vartheta^\alpha = 0 . \tag{42}$$

3.2 Complex scalar field

3.2.1 Noether current

The action is invariant up to a total differential

$$\begin{aligned}\delta_\lambda S &= \int \lambda \, \mathrm{d}(z \lrcorner \mathfrak{L}) \\ &= \int \lambda \frac{1}{2} \mathrm{d}\{-z \lrcorner [\mathrm{d}\phi^* \wedge \star\mathrm{d}\phi + \mathrm{d}\phi \wedge \star\mathrm{d}\phi^* + m^2(\phi^* \star \phi + \phi \star \phi^*)]\} ,\end{aligned} \tag{43}$$

under the rigid infinitesimal transformation

$$\delta\phi = \lambda \pounds_z \phi = \lambda \, z \lrcorner \mathrm{d}\phi , \qquad \delta\phi^* = \lambda \pounds_z \phi^* = \lambda \, z \lrcorner \mathrm{d}\phi^* . \tag{44}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$\delta_\lambda S = \int \lambda \, \mathrm{d}\{-(z \lrcorner \mathrm{d}\phi^*) \wedge \star\mathrm{d}\phi - (z \lrcorner \mathrm{d}\phi) \wedge \star\mathrm{d}\phi^*\} . \tag{45}$$

Taking the difference and setting $z = e_\alpha$ yields

$$0 = \int \lambda \, \mathrm{d}\mathfrak{T}_\alpha , \tag{46}$$

$$\begin{aligned}2\mathfrak{T}_\alpha := &(e_\alpha \lrcorner \mathrm{d}\phi^*) \wedge \star\mathrm{d}\phi + \mathrm{d}\phi^* \wedge (e_\alpha \lrcorner \star\mathrm{d}\phi) \\ &+ (e_\alpha \lrcorner \mathrm{d}\phi) \wedge \star\mathrm{d}\phi^* + \mathrm{d}\phi \wedge (e_\alpha \lrcorner \star\mathrm{d}\phi^*) \\ &- m^2(\phi^* e_\alpha \lrcorner \star\phi + \phi \, e_\alpha \star \phi^*) .\end{aligned} \tag{47}$$

The continuity equation reads

$$\mathrm{d}\mathfrak{T}_\alpha = 0 . \tag{48}$$

In components,

$$\mathfrak{T}_\alpha = (e_\alpha(\phi^*) e^\beta(\phi) + e_\alpha(\phi) e^\beta(\phi^*)) \, \epsilon_\beta - e^\beta(\phi^*) e_\beta(\phi^*) \, \epsilon_\alpha - m^2 \phi^* \phi \, \epsilon_\alpha . \tag{49}$$

3.3 Pure electromagnetic field

3.3.1 Noether current

The action is invariant up to a total differential

$$\delta_\lambda S = \int \lambda \, \mathbb{d}(z \lrcorner \mathfrak{L}) = \int \lambda \, \frac{1}{2} \, \mathbb{d}\{-z \lrcorner (F \wedge \star F)\} \, , \tag{50}$$

under the rigid infinitesimal transformation combined with a gauge transformation [8, eq. 3.46]

$$\delta A = \lambda \{z \lrcorner \mathbb{d}(A - \mathbb{d}\Lambda) + \mathbb{d}[z \lrcorner (A - \mathbb{d}\Lambda)]\} \, . \tag{51}$$

Choosing

$$\mathbb{d}\Lambda = A(z) \, z^\flat \tag{52}$$

makes the second term vanish, yielding

$$\delta A = \lambda \, z \lrcorner \mathbb{d}A = \lambda \, z \lrcorner F \, . \tag{53}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$\delta_\lambda S = \int \lambda \, \mathbb{d}\{-(z \lrcorner F) \wedge \star F\} \, . \tag{54}$$

Taking the difference and setting $z = e_\alpha$ yields

$$0 = \int \lambda \, \mathbb{d}\mathfrak{T}_\alpha , \tag{55}$$

$$2\mathfrak{T}_\alpha := -(e_\alpha \lrcorner F) \wedge \star F + F \wedge (e_\alpha \lrcorner \star F) \, . \tag{56}$$

The continuity equation reads

$$\mathbb{d}\mathfrak{T}_\alpha = 0 \, . \tag{57}$$

In components,

$$\mathfrak{T}_\alpha = \ . \tag{58}$$

4 Poincaré gauge theory

4.1 Differential forms

Upon variation of ϑ^α, $\vartheta^{\alpha_1 \alpha_2 \cdots \alpha_k}$ goes under

$$\delta \vartheta^{\alpha_1 \alpha_2 \cdots \alpha_k} = \delta \vartheta^\alpha \wedge (e_\alpha \lrcorner \vartheta^{\alpha_1 \alpha_2 \cdots \alpha_k}) \, , \tag{59}$$

which can be proved by induction.

Upon variation of ϑ^α, $\epsilon_{\alpha_1 \alpha_2 \cdots \alpha_k}$ goes under [7, sec. A.2]

$$\delta \epsilon_{\alpha_1 \alpha_2 \cdots \alpha_k} = \delta \vartheta^\alpha \wedge \left(e_\alpha \lrcorner \epsilon_{\alpha_1 \alpha_2 \cdots \alpha_k} \right) . \tag{60}$$

Variation of Hodge star In gravitational theories [7, sec. 3.2] with an orthonormal cobasis,

$$[\delta, \star]\phi = \delta\vartheta^\alpha \wedge (e_\alpha \rfloor \star\phi) - \star(\delta\vartheta^\alpha \wedge (e_\alpha \rfloor \phi)) \,. \tag{61}$$

Let χ be a p-form, ϕ another form [5, sec. 5].

$$\delta(\chi \wedge \star\phi) = \delta\chi \wedge \star\phi + \delta\phi \wedge \star\chi - \delta\vartheta^\alpha \wedge \Sigma_\alpha \,, \tag{62}$$

$$\Sigma_\alpha := \chi \wedge \left\{ \star(e_\alpha \rfloor \phi) - (-)^p (e_\alpha \rfloor \star\phi) \right\} . \tag{63}$$

4.2 U(1)-gauged complex scalar field theory

$$\begin{aligned} \Sigma_\alpha = {} & -\mathbb{D}\phi^* \wedge \{\star(e_\alpha \rfloor \mathbb{D}\phi) + (e_\alpha \rfloor \star\mathbb{D}\phi)\} - m^2\phi^*\phi\,\epsilon_\alpha \\ & - \frac{1}{2} F \wedge \{\star(e_\alpha \rfloor F) - (e_\alpha \rfloor \star F)\} \,. \end{aligned} \tag{64}$$

4.2.1 Noether's invariances

[1]

Rigid translation

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