Coframe formalisms

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1 U(1)–regular theory

1.1 Differential geometry

Interior product Let z be a vector, ω be a 1-form, χ be a k-form. The *interior product* is defined inductively as the bilinear map satisfying

$$z \lrcorner \omega := \omega(z)\,, \tag{1}$$

$$z \lrcorner (\omega \wedge \chi) := (z \lrcorner \omega) \wedge \chi - \omega \wedge (z \lrcorner \chi)\,. \tag{2}$$

Equation (2) is also known as the anti-product rule.

By inductiion one can show that for a p-form ϕ,

$$z \lrcorner (\phi \wedge \chi) = (z \lrcorner \phi) \wedge \chi + (-)^p \chi \wedge (z \lrcorner \chi)\,. \tag{3}$$

Hodge star Let ω be an 1-form, χ be a k-form. The Hodge star $\star$ is defined inductively as the linear map [4, sec. 24]

$$\star 1 := \mathrm{vol}\,, \tag{4}$$
$$\star(\chi \wedge \omega) := \omega^\sharp \lrcorner \star\chi\,. \tag{5}$$

Non-gravitational theories features $[\delta, \star] = 0$, which means [7, sec. 3.2]

$$\delta g_{\mu\nu} = -2\omega_{(\mu\nu)}, \qquad \delta\vartheta^\mu = \omega_\nu{}^\mu\,\vartheta^\nu\,; \tag{6}$$

for an orthonormal coframe, the allowed variations are $\omega_{(\alpha\beta)} = 0$.

Codifferential The *codifferential* $\mathbb{d}^\dagger$ is the adjoint of the exterior derivative $\mathbb{d}$ in the following sense. Let ϕ be a k-form, χ be a $(k-1)$-form.

$$\int \mathbb{d}(\chi^* \wedge \star\phi) \equiv \int \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d}\star\phi =: \int \mathbb{d}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{d}^\dagger\phi \tag{7}$$
$$= \int \mathbb{d}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} \mathbb{d}\star\phi\,. \tag{8}$$

$$\boxed{\mathbb{d}^\dagger\phi = (-)^k \star^{-1} \mathbb{d}\star\phi\,.} \tag{9}$$

1.2 Complex Klein–Fock–Gordon theory

The action reads

$$S = \int -\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi - m^2\phi^* \wedge \star\phi\,. \tag{10}$$

Variation commutes with exterior derivative

$$[\delta, \mathbb{d}]\phi = 0\,. \tag{11}$$

A generic variation of the kinetic terms reads

$$\begin{aligned}\delta(\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi) = {} & \mathbb{d}(\delta\phi^* \wedge \star\mathbb{d}\phi + \delta\phi \wedge \star\mathbb{d}\phi^*) \\ & + \delta\phi^* \wedge \star\mathbb{d}^\dagger\mathbb{d}\phi + \delta\phi \wedge \star\mathbb{d}^\dagger\mathbb{d}\phi^*\,.\end{aligned} \tag{12}$$

A generic variation of the actionreads

$$\begin{aligned}\delta S = \int & \mathbb{d}(-\delta\phi^* \wedge \star\mathbb{d}\phi - \delta\phi \wedge \star\mathbb{d}\phi^*) \\ & - \delta\phi^* \wedge \star(\mathbb{d}^\dagger\mathbb{d} + m^2)\phi - \delta\phi \wedge \star(\mathbb{d}^\dagger\mathbb{d} + m^2)\phi^*\,.\end{aligned} \tag{13}$$

1.2.1 Noether current

The action is invariant ($\delta_\Lambda S \equiv 0$) under the rigid transformation

$$\phi \to \mathsf{e}^{-\mathrm{i}e\Lambda}\phi\,, \qquad \phi^* \to \mathsf{e}^{+\mathrm{i}e\Lambda}\phi^*\,. \tag{14}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$0 = \int \Lambda \,\mathbb{d}\mathfrak{J}_0 \,,$$
$$\mathfrak{J}_0 := \mathrm{i}e(-\phi \wedge \star \mathbb{d}\phi^* + \phi^* \wedge \star \mathbb{d}\phi)\,, \tag{15}$$

which is the Noether current, a twisted 3-form, satisfying the continuity equation

$$\mathbb{d}\mathfrak{J}_0 = 0\,. \tag{16}$$

2 $\mathsf{U}(1)$ gauge theory

2.1 Connection on the principal bundle

Exterior covariant derivative Let χ be a $\mathbb{C}$-valued k-form. The *exterior covariant derivative* of χ reads

$$\mathbb{D}\chi := (\mathbb{d} - \mathrm{i}eA)\chi, \qquad \mathbb{D}\chi^* := (\mathbb{d} + \mathrm{i}eA)\chi^*\,, \tag{17}$$

where A is a $\mathfrak{u}(1)$-valued *connection form.*

Covariant codifferential The *covariant codifferential* $\mathbb{D}^\dagger$ is the adjoint of the exterior covariant derivative $\mathbb{D}$ in the following sense. Let ϕ be a $\mathbb{C}$-valued k-form, χ be a $\mathbb{C}$-valued $(k-1)$-form.

$$\int \mathbb{d}(\chi^* \wedge \star\phi) \equiv \int \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi =: \int \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star \mathbb{D}^\dagger \phi \tag{18}$$
$$\begin{aligned} &= \int \mathbb{D}\chi^* \wedge \star\phi - \mathrm{i}eA \wedge \chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi \\ &= \int \mathbb{D}\chi^* \wedge \star\phi + \chi^* \wedge (-)^k \mathrm{i}eA \star \phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi \\ &= \int \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \star \phi\,. \end{aligned} \tag{19}$$

$$\boxed{\mathbb{D}^\dagger \phi = (-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \star \phi\,.} \tag{20}$$

2.2 Maxwell–Klein–Fock–Gordon theory

$$S = \int -\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi - m^2 \phi^* \wedge \star\phi - \frac{1}{2} F \wedge \star F. \tag{21}$$

Variation does not commute with exterior covariant derivative.

$$[\delta, \mathbb{D}]\phi = -\mathrm{i}e\delta A\phi\,. \tag{22}$$

$$\begin{aligned} \delta(\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi) &= \mathbb{d}(\delta\phi^* \star \mathbb{D}\phi + \delta\phi \star \mathbb{D}\phi^*) + \delta\phi^* \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi^* \\ &\quad + \delta A \wedge \mathfrak{J}_A\,, \qquad \mathfrak{J}_A := \mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)\,, \end{aligned} \tag{23}$$
$$\delta(F \wedge \star F) = 2\,\delta F \wedge \star F \tag{24}$$
$$= 2\,\mathbb{d}(\delta A \wedge \star F) + 2\,\delta A \wedge \mathbb{d}\star F\,. \tag{25}$$

Variation of the action

$$\begin{aligned}\delta S = \int & \mathbb{d}(-\delta\phi^* \wedge \star\mathbb{D}\phi - \delta\phi \wedge \star\mathbb{D}\phi^* - \delta A \wedge \star F) \\ & - \delta\phi^* \wedge \star(\mathbb{D}^\dagger\mathbb{D} + m^2)\phi - \delta\phi \wedge \star(\mathbb{D}^\dagger\mathbb{D} + m^2)\phi^* \\ & - \delta A \wedge (\mathbb{d}\star F - \mathfrak{J}_A). \end{aligned} \tag{26}$$

2.2.1 Lorenz gauge

The Laplace–de Rham operator, or in our Lorentzian metric signature the d'Alembertian

$$\Box^2 := \left(\mathbb{d} + \mathbb{d}^\dagger\right)^2 = \mathbb{d}\mathbb{d}^\dagger + \mathbb{d}^\dagger\mathbb{d}\,. \tag{27}$$

$$\mathbb{d}\star F = \mathbb{d}\star\mathbb{d}A = \star(-)^2 \star^{-1} \mathbb{d}\star\mathbb{d}A = \star\mathbb{d}^\dagger\mathbb{d}A = \star(\Box^2 - \mathbb{d}\mathbb{d}^\dagger)A\,. \tag{28}$$

One would like to have $\mathbb{d}\mathbb{d}^\dagger A = 0$, or $\mathbb{d}^\dagger A = \text{const}$. This would be fulfilled if

$$\mathbb{d}^\dagger A = 0\,, \tag{29}$$

which is the Lorenz gauge[2, 3, 6].

2.2.2 Noether's invariances

The action is invariant under the generic transformation

$$\phi \to \mathsf{e}^{-\mathrm{i}e\Lambda}\phi\,, \quad \phi^* \to \mathsf{e}^{+\mathrm{i}e\Lambda}\phi^*\,, \quad A \to A - \mathbb{d}\Lambda\,. \tag{30}$$

There are two scenarios [1].

If the transformation is rigid, $\mathbb{d}\Lambda = 0$, one obtains $\mathfrak{J}_A$ as the Noether current from the boundary term as before, which satisfies the continuity equation $\mathbb{d}\mathfrak{J}_A = 0$.

If the transformation is gauge with a compact support, the boundary term can be dropped, and one obtains the Noether identity identity

$$123 \tag{31}$$

3 Poincaré–regular theory

4 Poincaré gauge theory

4.1 Differential forms

Untwisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for untwisted k-form is defined inductively as

$$1\,, \tag{32}$$
$$\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} := \vartheta^{\alpha_1} \wedge \vartheta^{\alpha_2\cdots\alpha_k}\,. \tag{33}$$

Upon variation of ϑ^α, $\vartheta^{\alpha_1\alpha_2\cdots\alpha_k}$ goes under

$$\delta\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge (e_\alpha \rfloor \vartheta^{\alpha_1\alpha_2\cdots\alpha_k})\,, \tag{34}$$

which can be proved by induction.

Twisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for twisted $(D-k)$-form is defined inductively as

$$\epsilon := \mathrm{vol}\,, \tag{35}$$

$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} := e_{\alpha_k} \rfloor \epsilon_{\alpha_1\cdots\alpha_{k-1}}\,. \tag{36}$$

By using eq. (5) and induction, one can show that

$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \star\vartheta_{\alpha_1\alpha_2\cdots\alpha_k}\,. \tag{37}$$

Upon variation of ϑ^α, $\epsilon_{\alpha_1\alpha_2\cdots\alpha_k}$ goes under [7, sec. A.2]

$$\delta\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge \left(e_\alpha \rfloor \epsilon_{\alpha_1\alpha_2\cdots\alpha_k}\right). \tag{38}$$

Variation of Hodge star In gravitational theories [7, sec. 3.2] with an orthonormal cobasis,

$$[\delta,\star]\phi = \delta\vartheta^\alpha \wedge (e_\alpha \rfloor \star\phi) - \star(\delta\vartheta^\alpha \wedge (e_\alpha \rfloor \phi))\,. \tag{39}$$

Let χ be a p-form, ϕ another form [5, sec. 5].

$$\delta(\chi\wedge\star\phi) = \delta\chi\wedge\star\phi + \delta\phi\wedge\star\chi - \delta\vartheta^\alpha \wedge \Sigma_\alpha\,, \tag{40}$$

$$\Sigma_\alpha := \chi \wedge \left\{\star(e_\alpha \rfloor \phi) - (-)^p (e_\alpha \rfloor \star\phi)\right\}. \tag{41}$$

4.2 Maxwell–Klein–Fock–Gordon theory

$$\begin{aligned}\Sigma_\alpha = &-\mathbb{D}\phi^* \wedge \{\star(e_\alpha \rfloor \mathbb{D}\phi) + (e_\alpha \rfloor \star\mathbb{D}\phi)\} - m^2\phi^*\phi\,\epsilon_\alpha \\ &- \frac{1}{2}F\wedge\{\star(e_\alpha \rfloor F) - (e_\alpha \rfloor \star F)\}\,.\end{aligned} \tag{42}$$

4.2.1 Noether's invariances

[1]

Rigid translation

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