Coframe formalisms

Yi-Fan Wang

May 12, 2019

Contents

1 $\mathsf{U}(1)$–regular theory

1.1 Differential geometry

Interior product Let z be a vector, ω be a 1-form, χ be a k-form. The *interior product* is defined inductively as the bilinear map satisfying

$$z \lrcorner \omega := \omega(z)\,, \tag{1}$$

$$z \lrcorner (\omega \wedge \chi) := (z \lrcorner \omega) \wedge \chi - \omega \wedge (z \lrcorner \chi)\,. \tag{2}$$

Equation (2) is also known as the anti-product rule.

By inductiion one can show that for a p-form ϕ,

$$z \lrcorner (\phi \wedge \chi) = (z \lrcorner \phi) \wedge \chi + (-)^p \chi \wedge (z \lrcorner \chi)\,. \tag{3}$$

Hodge star Let ω be an 1-form, χ be a k-form. The Hodge star $\star$ is defined inductively as the linear map [4, sec. 24]

$$\star 1 := \mathrm{vol}\,, \tag{4}$$

$$\star(\chi\wedge\omega) := \omega^\sharp \lrcorner \star\chi\,. \tag{5}$$

Non-gravitational theories features $[\delta,\star]=0$, which means [7, sec. 3.2]

$$\delta g_{\mu\nu} = -2\omega_{(\mu\nu)}, \qquad \delta\vartheta^\mu = \omega_\nu{}^\mu\,\vartheta^\nu\,; \tag{6}$$

for an orthonormal coframe, the allowed variations are $\omega_{(\alpha\beta)}=0$.

1.2 Complex Klein–Fock–Gordon theory

$$S = \int -\mathbb{d}\phi^*\wedge\star\mathbb{d}\phi - m^2\phi^*\wedge\star\phi\,. \tag{7}$$

$$[\delta,\mathbb{D}]\phi = 0\,. \tag{8}$$

$$\begin{aligned}\delta(\mathbb{d}\phi^*\wedge\star\mathbb{d}\phi) = {}&\mathbb{d}(\delta\phi^*\wedge\star\mathbb{d}\phi + \delta\phi\wedge\star\mathbb{d}\phi^*)\\ &-\delta\phi^*\wedge\star\mathbb{d}^\dagger\mathbb{d}\phi - \delta\phi\wedge\star\mathbb{d}^\dagger\mathbb{d}\phi^*\,.\end{aligned} \tag{9}$$

$$\begin{aligned}\delta S = \int &-\mathbb{d}(\delta\phi^*\wedge\star\mathbb{d}\phi + \delta\phi\wedge\star\mathbb{d}\phi^*)\\ &+\delta\phi^*\wedge\star(\mathbb{D}^\dagger\mathbb{D}-m^2)\phi + \delta\phi\wedge\star(\mathbb{D}^\dagger\mathbb{D}-m^2)\phi^*\,.\end{aligned} \tag{10}$$

1.2.1 Noether current

The theory is invariant under the rigid transformation

$$\phi \to \mathsf{e}^{-\mathrm{i}e\Lambda}\phi\,, \qquad \phi^* \to \mathsf{e}^{+\mathrm{i}e\Lambda}\phi^*\,, \tag{11}$$

i.e. $\delta S \equiv 0$. When the equations of motion are satisfied, infinitesimal transformation leads to

$$\begin{aligned}0 &= \int -\Lambda\,\mathbb{d}\mathfrak{J}\,,\\ \mathfrak{J} &:= \mathrm{i}e(\phi^*\wedge\star\mathbb{d}\phi + \phi\wedge\star\mathbb{d}\phi^*)\end{aligned} \tag{12}$$

which is the Noether current, satisfying the continuity equation

$$\mathbb{d}\mathfrak{J} = 0\,. \tag{13}$$

2 $\mathsf{U}(1)$ gauge theory

2.1 Connection on the principal bundle

Exterior covariant derivative Let χ be a $\mathbb{C}$-valued k-form. The *exterior covariant derivative* of χ reads

$$\mathbb{D}\chi := (\mathbb{d}-\mathrm{i}eA)\chi, \qquad \mathbb{D}\chi^* := (\mathbb{d}+\mathrm{i}eA)\chi^*\,, \tag{14}$$

where A is a $\mathfrak{u}(1)$-valued *connection form*.

Covariant codifferential The *covariant codifferential* $\mathbb{D}^\dagger$ is the adjoint of the exterior covariant derivative $\mathbb{D}$ in the following sense. Let ϕ be a $\mathbb{C}$-valued k-form, χ be a $\mathbb{C}$-valued $(k-1)$-form.

$$\begin{aligned}\int \mathbb{d}(\chi^* \wedge \star\phi) &\equiv \int \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi =: \int \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{D}^\dagger\phi \quad (15)\\ &= \int \mathbb{D}\chi^* \wedge \star\phi - \mathrm{i}eA \wedge \chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi\\ &= \int \mathbb{D}\chi^* \wedge \star\phi + \chi^* \wedge (-)^k \mathrm{i}eA \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi\\ &= \int \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\phi \,. \quad (16)\end{aligned}$$

$$\boxed{\mathbb{D}^\dagger\phi = (-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\phi \,.} \quad (17)$$

2.2 Maxwell–Klein–Fock–Gordon theory

$$S = \int -\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi - m^2\phi^* \wedge \star\phi - \frac{1}{2}F \wedge \star F. \quad (18)$$

$$\delta\mathbb{D}\phi = -\mathrm{i}e\delta A\phi + \mathbb{D}\delta\phi. \quad (19)$$

$$\begin{aligned}\delta(\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi) &= \mathbb{d}(\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^*)\\ &\quad - \delta\phi^* \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi - \delta\phi \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi^* \quad (20)\\ &\quad + \delta A \wedge (\mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)) \,,\\ \delta(F \wedge \star F) &= 2\mathbb{d}(\delta A \wedge \star F) - 2\delta A \wedge \mathbb{d}\star F \,. \quad (21)\end{aligned}$$

$$\begin{aligned}\delta S = \int &-\mathbb{d}(\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^* + \delta A \wedge \star F)\\ &+ \delta\phi^* \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi + \delta\phi \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi^*\\ &+ \delta A \wedge (\mathbb{d}\star F - \mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)). \quad (22)\end{aligned}$$

2.2.1 Lorenz gauge

The Laplace–de Rham operator, or in our metric signature the d'Alembertian

$$\Box^2 := \left(\mathbb{d} + \mathbb{d}^\dagger\right)^2 = \mathbb{d}\mathbb{d}^\dagger + \mathbb{d}^\dagger\mathbb{d} \,. \quad (23)$$

$$\mathbb{d}\star F = \mathbb{d} \star \mathbb{d}A = \star(-)^2 \star^{-1} \mathbb{d} \star \mathbb{d}A = \star\mathbb{d}^\dagger\mathbb{d}A = \star(\Box^2 - \mathbb{d}\mathbb{d}^\dagger)A \,. \quad (24)$$

One would like to have $\mathbb{d}\mathbb{d}^\dagger A = 0$, or $\mathbb{d}^\dagger A = \mathrm{const}$. This would be fulfiled if

$$\mathbb{d}^\dagger A = 0 \,, \quad (25)$$

which is the Lorenz gauge[2, 3, 6].

2.2.2 Noether's invariances

[1]

3 Poincaré–regular theory

4 Poincaré gauge theory

4.1 Differential forms

Untwisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for untwisted k-form is defined inductively as

$$1\,, \tag{26}$$

$$\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} := \vartheta^{\alpha_1} \wedge \vartheta^{\alpha_2\cdots\alpha_k}\,. \tag{27}$$

Upon variation of ϑ^α, $\vartheta^{\alpha_1\alpha_2\cdots\alpha_k}$ goes under

$$\delta\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge (e_\alpha \rfloor \vartheta^{\alpha_1\alpha_2\cdots\alpha_k})\,, \tag{28}$$

which can be proved by induction.

Twisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for twisted $(D-k)$-form is defined inductively as

$$\epsilon := \mathrm{vol}\,, \tag{29}$$

$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} := e_{\alpha_k} \rfloor \epsilon_{\alpha_1\cdots\alpha_{k-1}}\,. \tag{30}$$

By using eq. (5) and induction, one can show that

$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \star\vartheta_{\alpha_1\alpha_2\cdots\alpha_k}\,. \tag{31}$$

Upon variation of ϑ^α, $\epsilon_{\alpha_1\alpha_2\cdots\alpha_k}$ goes under [7, sec. A.2]

$$\delta\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge \left(e_\alpha \rfloor \epsilon_{\alpha_1\alpha_2\cdots\alpha_k}\right). \tag{32}$$

Variation of Hodge star In gravitational theories [7, sec. 3.2] with an orthonormal cobasis,

$$[\delta,\star]\phi = \delta\vartheta^\alpha \wedge (e_\alpha \rfloor \star\phi) - \star(\delta\vartheta^\alpha \wedge (e_\alpha \rfloor \phi))\,. \tag{33}$$

Let χ be a p-form, ϕ another form [5, sec. 5].

$$\delta(\chi\wedge\star\phi) = \delta\chi\wedge\star\phi + \delta\phi\wedge\star\chi - \delta\vartheta^\alpha\wedge\Sigma_\alpha\,, \tag{34}$$

$$\Sigma_\alpha := \chi\wedge\left\{\star(e_\alpha\rfloor\phi) - (-)^p(e_\alpha\rfloor\star\phi)\right\}. \tag{35}$$

4.2 Maxwell–Klein–Fock–Gordon theory

$$\begin{aligned}\Sigma_\alpha &= -\mathbb{D}\phi^*\wedge\{\star(e_\alpha\rfloor\mathbb{D}\phi) + (e_\alpha\rfloor\star\mathbb{D}\phi)\} - m^2\phi^*\phi\,\epsilon_\alpha\\ &\quad -\frac{1}{2}F\wedge\{\star(e_\alpha\rfloor F) - (e_\alpha\rfloor\star F)\}\,.\end{aligned} \tag{36}$$

4.2.1 Noether's invariances

[1]

Rigid translation

References

[1] S. G. Avery and B. U. W. Schwab, "Noether's second theorem and ward identities for gauge symmetries", Journal of High Energy Physics **2016** (2015), arXiv:1510.07038v2 [hep-th] (cit. on pp. 4, 5).

[2] J. van Bladel, "Lorenz or lorentz?", IEEE Antennas and Propagation Magazine **33**, 69–69 (1991) (cit. on p. 3).

[3] J. van Bladel, "Lorenz or lorentz? [addendum]", IEEE Antennas and Propagation Magazine **33**, 56–56 (1991) (cit. on p. 3).

[4] W. L. Burke, *Applied differential geometry* (Cambridge University Press, 1985) (cit. on p. 1).

[5] Y. Itin, "On variations in teleparallelism theories", (1999), arXiv:gr-qc/9904030 [gr-qc] (cit. on p. 4).

[6] L. Lorenz, "XXXVIII. on the identity of the vibrations of light with electrical currents", The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **34**, 287–301 (1867) (cit. on p. 3).

[7] U. Muench, F. Gronwald, and F. W. Hehl, "A brief guide to variations in teleparallel gauge theories of gravity and the kaniel-itin model", General Relativity and Gravitation **30**, 933–961 (1998), arXiv:gr-qc/9801036 [gr-qc] (cit. on pp. 2, 4).