

Coframe formalisms

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1 $U(1)$ –regular theory

1.1 Differential geometry

Interior product Let z be a vector, ω be a 1-form, χ be a k -form. The *interior product* is defined inductively as the bilinear map satisfying

$$z \lrcorner \omega := \omega(z), \quad (1)$$

$$z \lrcorner (\omega \wedge \chi) := (z \lrcorner \omega) \wedge \chi - \omega \wedge (z \lrcorner \chi). \quad (2)$$

Equation (2) is also known as the anti-product rule.

By induction one can show that for a p -form ϕ ,

$$z \lrcorner (\phi \wedge \chi) = (z \lrcorner \phi) \wedge \chi + (-)^p \chi \wedge (z \lrcorner \phi). \quad (3)$$

Hodge star Let ω be an 1-form, χ be a k -form. The Hodge star \star is defined inductively as the linear map [4, sec. 24]

$$\star 1 := \text{vol}, \quad (4)$$

$$\star(\chi \wedge \omega) := \omega^\sharp \lrcorner \star \chi. \quad (5)$$

Non-gravitational theories features $[\delta, \star] = 0$, which means [7, sec. 3.2]

$$\delta g_{\mu\nu} = -2\omega_{(\mu\nu)}, \quad \delta \vartheta^\mu = \omega_\nu{}^\mu \vartheta^\nu; \quad (6)$$

for an orthonormal coframe, the allowed variations are $\omega_{(\alpha\beta)} = 0$.

Codifferential The *codifferential* \mathfrak{d}^\dagger is the adjoint of the exterior derivative \mathfrak{d} in the following sense. Let ϕ be a k -form, χ be a $(k-1)$ -form.

$$\int \mathfrak{d}(\chi^* \wedge \star \phi) \equiv \int \mathfrak{d}\chi^* \wedge \star \phi - (-)^k \chi^* \wedge \mathfrak{d} \star \phi =: \int \mathfrak{d}\chi^* \wedge \star \phi - \chi^* \wedge \star \mathfrak{d}^\dagger \phi \quad (7)$$

$$= \int \mathfrak{d}\chi^* \wedge \star \phi - \chi^* \wedge \star (-)^k \star^{-1} \mathfrak{d} \star \phi. \quad (8)$$

$$\boxed{\mathfrak{d}^\dagger \phi = (-)^k \star^{-1} \mathfrak{d} \star \phi.} \quad (9)$$

1.2 Complex scalar field

The action reads

$$S = \int -\mathfrak{d}\phi^* \wedge \star \mathfrak{d}\phi - m^2 \phi^* \wedge \star \phi. \quad (10)$$

Variation commutes with exterior derivative

$$[\delta, \mathfrak{d}]\phi = 0. \quad (11)$$

A generic variation of the kinetic terms reads

$$\begin{aligned} \delta(\mathfrak{d}\phi^* \wedge \star \mathfrak{d}\phi) &= \mathfrak{d}(\delta\phi^* \wedge \star \mathfrak{d}\phi + \delta\phi \wedge \star \mathfrak{d}\phi^*) \\ &\quad + \delta\phi^* \wedge \star \mathfrak{d}^\dagger \mathfrak{d}\phi + \delta\phi \wedge \star \mathfrak{d}^\dagger \mathfrak{d}\phi^*. \end{aligned} \quad (12)$$

A generic variation of the action reads

$$\begin{aligned} \delta S &= \int \mathfrak{d}(-\delta\phi^* \wedge \star \mathfrak{d}\phi - \delta\phi \wedge \star \mathfrak{d}\phi^*) \\ &\quad - \delta\phi^* \wedge \star (\mathfrak{d}^\dagger \mathfrak{d} + m^2)\phi - \delta\phi \wedge \star (\mathfrak{d}^\dagger \mathfrak{d} + m^2)\phi^*. \end{aligned} \quad (13)$$

1.2.1 Noether current

The action is invariant

$$\delta_\Lambda S \equiv 0 \quad (14)$$

under the rigid transformation

$$\phi \rightarrow e^{-ie\Lambda} \phi, \quad \phi^* \rightarrow e^{+ie\Lambda} \phi^*. \quad (15)$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$0 = \int \Lambda \mathfrak{d}\mathfrak{J}_0, \quad \mathfrak{J}_0 := ie(-\phi \wedge \star \mathfrak{d}\phi^* + \phi^* \wedge \star \mathfrak{d}\phi), \quad (16)$$

which is the Noether current, a twisted 3-form, satisfying the continuity equation

$$\mathfrak{d}\mathfrak{J}_0 = 0. \quad (17)$$

2 U(1) gauge theory

2.1 Connection on the principal bundle

Exterior covariant derivative Let χ be a \mathbb{C} -valued k -form. The *exterior covariant derivative* of χ reads

$$\mathbb{D}\chi := (\mathfrak{d} - ieA)\chi, \quad \mathbb{D}\chi^* := (\mathfrak{d} + ieA)\chi^*, \quad (18)$$

where A is a $\mathfrak{u}(1)$ -valued *connection form*.

Covariant codifferential The *covariant codifferential* \mathbb{D}^\dagger is the adjoint of the exterior covariant derivative \mathbb{D} in the following sense. Let ϕ be a \mathbb{C} -valued k -form, χ be a \mathbb{C} -valued $(k-1)$ -form.

$$\begin{aligned} \int \mathfrak{d}(\chi^* \wedge \star \phi) &\equiv \int \mathfrak{d}\chi^* \wedge \star \phi - (-)^k \chi^* \wedge \mathfrak{d} \star \phi =: \int \mathbb{D}\chi^* \wedge \star \phi - \chi^* \wedge \star \mathbb{D}^\dagger \phi \quad (19) \\ &= \int \mathbb{D}\chi^* \wedge \star \phi - ieA \wedge \chi^* \wedge \star \phi - (-)^k \chi^* \wedge \mathfrak{d} \star \phi \\ &= \int \mathbb{D}\chi^* \wedge \star \phi + \chi^* \wedge (-)^k ieA \star \phi - (-)^k \chi^* \wedge \mathfrak{d} \star \phi \\ &= \int \mathbb{D}\chi^* \wedge \star \phi - \chi^* \wedge \star (-)^k \star^{-1} (\mathfrak{d} - ieA) \star \phi. \end{aligned} \quad (20)$$

$$\boxed{\mathbb{D}^\dagger \phi = (-)^k \star^{-1} (\mathfrak{d} - ieA) \star \phi.} \quad (21)$$

2.2 U(1)-gauged complex scalar field theory

$$S = \int -\mathbb{D}\phi^* \wedge \star \mathbb{D}\phi - m^2 \phi^* \wedge \star \phi - \frac{1}{2} F \wedge \star F. \quad (22)$$

Variation does not commute with exterior covariant derivative.

$$[\delta, \mathbb{D}]\phi = -ie\delta A \phi. \quad (23)$$

$$\begin{aligned} \delta(\mathbb{D}\phi^* \wedge \star \mathbb{D}\phi) &= \mathfrak{d}(\delta\phi^* \star \mathbb{D}\phi + \delta\phi \star \mathbb{D}\phi^*) + \delta\phi^* \wedge \star \mathbb{D}^\dagger \mathbb{D}\phi + \delta\phi \wedge \star \mathbb{D}^\dagger \mathbb{D}\phi^* \\ &\quad + \delta A \wedge \mathfrak{J}_A, \quad \mathfrak{J}_A := ie(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*), \end{aligned} \quad (24)$$

$$\delta(F \wedge \star F) = 2\delta F \wedge \star F \quad (25)$$

$$= 2\mathfrak{d}(\delta A \wedge \star F) + 2\delta A \wedge \mathfrak{d}\star F. \quad (26)$$

Variation of the action

$$\begin{aligned}\delta S = \int & \mathfrak{d}(-\delta\phi^* \wedge \star \mathbb{D}\phi - \delta\phi \wedge \star \mathbb{D}\phi^* - \delta A \wedge \star F) \\ & - \delta\phi^* \wedge \star(\mathbb{D}^\dagger \mathbb{D} + m^2)\phi - \delta\phi \wedge \star(\mathbb{D}^\dagger \mathbb{D} + m^2)\phi^* \\ & - \delta A \wedge (\mathfrak{d}\star F - \mathfrak{J}_A).\end{aligned}\tag{27}$$

2.2.1 Lorenz gauge

The Laplace–de Rham operator, or in our Lorentzian metric signature the d'Alembertian

$$\square^2 := (\mathfrak{d} + \mathfrak{d}^\dagger)^2 = \mathfrak{d}\mathfrak{d}^\dagger + \mathfrak{d}^\dagger\mathfrak{d}.\tag{28}$$

$$\mathfrak{d}\star F = \mathfrak{d}\star\mathfrak{d}A = \star(-)^2\star^{-1}\mathfrak{d}\star\mathfrak{d}A = \star\mathfrak{d}^\dagger\mathfrak{d}A = \star(\square^2 - \mathfrak{d}\mathfrak{d}^\dagger)A.\tag{29}$$

One would like to have $\mathfrak{d}\mathfrak{d}^\dagger A = 0$, or $\mathfrak{d}^\dagger A = \text{const.}$ This would be fulfilled if

$$\mathfrak{d}^\dagger A = 0,\tag{30}$$

which is the Lorenz gauge[2, 3, 6].

2.2.2 Noether's invariances

The action is invariant under the generic transformation

$$\phi \rightarrow \mathfrak{e}^{-ie\Lambda}\phi, \quad \phi^* \rightarrow \mathfrak{e}^{+ie\Lambda}\phi^*, \quad A \rightarrow A - \mathfrak{d}\Lambda.\tag{31}$$

There are two scenarios [1].

If the transformation is rigid, $\mathfrak{d}\Lambda = 0$, one obtains \mathfrak{J}_A as the Noether current from the boundary term as before, which satisfies the continuity equation $\mathfrak{d}\mathfrak{J}_A = 0$.

If the transformation is gauge with a compact support, the boundary term can be dropped, and one obtains the Noether identity

$$123\tag{32}$$

3 Poincaré–regular theory

3.1 Complex scalar field

3.1.1 Noether current

The action is invariant up to a total differential

$$\delta_A S = \int \mathfrak{d}(z \lrcorner \mathfrak{L})\tag{33}$$

under the rigid transformation

$$\phi \rightarrow \mathfrak{e}^{-ie\Lambda}\phi, \quad \phi^* \rightarrow \mathfrak{e}^{+ie\Lambda}\phi^*.\tag{34}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$0 = \int \Lambda \mathfrak{d}\mathfrak{J}_0, \quad \mathfrak{J}_0 := ie(-\phi \wedge \star \mathfrak{d}\phi^* + \phi^* \wedge \star \mathfrak{d}\phi), \quad (35)$$

which is the Noether current, a twisted 3-form, satisfying the continuity equation

$$\mathfrak{d}\mathfrak{J}_0 = 0. \quad (36)$$

4 Poincaré gauge theory

4.1 Differential forms

Untwisted orthonormal k -cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for untwisted k -form is defined inductively as

$$1, \quad (37)$$

$$\vartheta^{\alpha_1 \alpha_2 \dots \alpha_k} := \vartheta^{\alpha_1} \wedge \vartheta^{\alpha_2 \dots \alpha_k}. \quad (38)$$

Upon variation of ϑ^α , $\vartheta^{\alpha_1 \alpha_2 \dots \alpha_k}$ goes under

$$\delta \vartheta^{\alpha_1 \alpha_2 \dots \alpha_k} = \delta \vartheta^\alpha \wedge (\vartheta_\alpha \lrcorner \vartheta^{\alpha_1 \alpha_2 \dots \alpha_k}), \quad (39)$$

which can be proved by induction.

Twisted orthonormal k -cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for twisted $(D - k)$ -form is defined inductively as

$$\epsilon := \text{vol}, \quad (40)$$

$$\epsilon_{\alpha_1 \alpha_2 \dots \alpha_k} := e_{\alpha_k} \lrcorner \epsilon_{\alpha_1 \dots \alpha_{k-1}}. \quad (41)$$

By using eq. (5) and induction, one can show that

$$\epsilon_{\alpha_1 \alpha_2 \dots \alpha_k} = \star \vartheta_{\alpha_1 \alpha_2 \dots \alpha_k}. \quad (42)$$

Upon variation of ϑ^α , $\epsilon_{\alpha_1 \alpha_2 \dots \alpha_k}$ goes under [7, sec. A.2]

$$\delta \epsilon_{\alpha_1 \alpha_2 \dots \alpha_k} = \delta \vartheta^\alpha \wedge (e_\alpha \lrcorner \epsilon_{\alpha_1 \alpha_2 \dots \alpha_k}). \quad (43)$$

Variation of Hodge star In gravitational theories [7, sec. 3.2] with an orthonormal cobasis,

$$[\delta, \star]\phi = \delta \vartheta^\alpha \wedge (e_\alpha \lrcorner \star \phi) - \star(\delta \vartheta^\alpha \wedge (e_\alpha \lrcorner \phi)). \quad (44)$$

Let χ be a p -form, ϕ another form [5, sec. 5].

$$\delta(\chi \wedge \star \phi) = \delta \chi \wedge \star \phi + \delta \phi \wedge \star \chi - \delta \vartheta^\alpha \wedge \Sigma_\alpha, \quad (45)$$

$$\Sigma_\alpha := \chi \wedge \{\star(e_\alpha \lrcorner \phi) - (-)^p(e_\alpha \lrcorner \star \phi)\}. \quad (46)$$

4.2 U(1)-gauged complex scalar field theory

$$\begin{aligned}\Sigma_\alpha = & -\mathbb{D}\phi^* \wedge \{\star(e_\alpha \lrcorner \mathbb{D}\phi) + (e_\alpha \lrcorner \star \mathbb{D}\phi)\} - m^2 \phi^* \phi \epsilon_\alpha \\ & - \frac{1}{2} F \wedge \{\star(e_\alpha \lrcorner F) - (e_\alpha \lrcorner \star F)\}.\end{aligned}\tag{47}$$

4.2.1 Noether's invariances

[1]

Rigid translation

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