Differential form

Yi-Fan Wang

May 11, 2019

Contents

1 Non-gravitational theory

1.1 Differential geometry

Interior product Let z be a vector, ω be a 1-form, χ be a k-form. The *interior product* is defined inductively as the bilinear map satisfying

$$z \lrcorner \omega := \omega(z)\,, \tag{1}$$

$$z \lrcorner (\omega \wedge \chi) := (z \lrcorner \omega) \wedge \chi - \omega \wedge (z \lrcorner \chi)\,. \tag{2}$$

Equation (2) is also known as the anti-product rule.

By inductiion one can show that for a p-form ϕ,

$$z \lrcorner (\phi \wedge \chi) = (z \lrcorner \phi) \wedge \chi + (-)^p \chi \wedge (z \lrcorner \chi)\,. \tag{3}$$

Hodge star Let ω be an 1-form, χ be a k-form. The Hodge star $\star$ is defined inductively as the linear map [3, sec. 24]

$$\star 1 := \mathrm{vol}\,, \tag{4}$$

$$\star(\chi \wedge \omega) := \omega^\sharp \lrcorner \star\chi\,. \tag{5}$$

1.2 Connection on the principal bundle

Covariant differential Let χ be a $\mathbb{C}$-valued k-form. The *covariant differential* of χ reads

$$\mathbb{D}\chi := (\mathbb{d} - \mathbb{i}eA) \wedge \chi, \qquad \mathbb{D}\chi^* := (\mathbb{d} + \mathbb{i}eA) \wedge \chi^*\,, \tag{6}$$

where A is a $\mathfrak{u}(1)$-valued *connection form*.

Covariant codifferential The *covariant codifferential* $\mathbb{D}^\dagger$ is the adjoint of the covariant differential $\mathbb{D}$ in the following sense. Let ϕ be a $\mathbb{C}$-valued k-form, χ be a $\mathbb{C}$-valued $(k-1)$-form.

$$\int \mathbb{d}(\chi^* \wedge \star\phi) \equiv \int \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi =: \int \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{D}^\dagger\phi \quad (7)$$

$$= \int \mathbb{D}\chi^* \wedge \star\phi - \mathrm{i}eA \wedge \chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi$$

$$= \int \mathbb{D}\chi^* \wedge \star\phi + \chi^* \wedge (-)^k \mathrm{i}eA \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi$$

$$= \int \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\phi \,. \quad (8)$$

$$\boxed{\mathbb{D}^\dagger\phi = (-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\phi \,.} \quad (9)$$

1.3 Maxwell–Klein–Fock–Gordon theory

$$S = \int -\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi - m^2\phi^* \wedge \star\phi - \frac{1}{2} F \wedge \star F. \quad (10)$$

$$\delta_{\mathrm{m}}\mathbb{D}\phi = -\mathrm{i}e\delta A\phi + \mathbb{D}\delta\phi. \quad (11)$$

$$\begin{aligned} \delta_{\mathrm{m}}(\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi) &= \mathbb{d}(\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^*) \\ &\quad - \delta\phi^* \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi - \delta\phi \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi^* \quad (12) \\ &\quad + \delta A \wedge (\mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)) \,, \\ \delta_{\mathrm{m}}(F \wedge \star F) &= 2\mathbb{d}(\delta A \wedge \star F) - 2\delta A \wedge \mathbb{d}\star F \,. \quad (13) \end{aligned}$$

$$\begin{aligned} \delta_{\mathrm{m}} S = \int & -\mathbb{d}(\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^* + \delta A \wedge \star F) \\ & + \delta\phi^* \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi + \delta\phi \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi^* \\ & + \delta A \wedge (\mathbb{d}\star F - \mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)). \quad (14) \end{aligned}$$

1.3.1 Lorenz gauge

The Laplace–de Rham operator, or in our metric signature the d'Alembertian

$$\Box^2 := \left(\mathbb{d} + \mathbb{d}^\dagger\right)^2 = \mathbb{d}\mathbb{d}^\dagger + \mathbb{d}^\dagger\mathbb{d} \,. \quad (15)$$

$$\mathbb{d}\star F = \mathbb{d} \star \mathbb{d}A = \star(-)^2 \star^{-1} \mathbb{d} \star \mathbb{d}A = \star\mathbb{d}^\dagger\mathbb{d}A = \star(\Box^2 - \mathbb{d}\mathbb{d}^\dagger)A \,. \quad (16)$$

One would like to have $\mathbb{d}\mathbb{d}^\dagger A = 0$, or $\mathbb{d}^\dagger A = \text{const}$. This would be fulfiled if

$$\mathbb{d}^\dagger A = 0 \,, \quad (17)$$

which is the Lorenz gauge[1, 2, 4].

2 Gravitational theory

2.1 Differential forms

Untwisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for untwisted k-form is defined inductively as

$$1\,, \tag{18}$$

$$\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} := \vartheta^{\alpha_1} \wedge \vartheta^{\alpha_2\cdots\alpha_k}\,. \tag{19}$$

Upon variation of ϑ^α, $\vartheta^{\alpha_1\alpha_2\cdots\alpha_k}$ goes under

$$\delta_\vartheta \vartheta^{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge (e_\alpha \,\lrcorner\, \vartheta^{\alpha_1\alpha_2\cdots\alpha_k})\,, \tag{20}$$

which can be proved by induction.

Twisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for twisted $(D-k)$-form is defined inductively as

$$\epsilon := \mathrm{vol}\,, \tag{21}$$

$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} := e_{\alpha_k} \,\lrcorner\, \epsilon_{\alpha_1\cdots\alpha_{k-1}}\,. \tag{22}$$

By using eq. (5) and induction, one can show that

$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \star\vartheta_{\alpha_1\alpha_2\cdots\alpha_k}\,. \tag{23}$$

Upon variation of ϑ^α, $\epsilon_{\alpha_1\alpha_2\cdots\alpha_k}$ goes under [5, sec. A.2]

$$\delta_\vartheta \epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge \left(e_\alpha \,\lrcorner\, \epsilon_{\alpha_1\alpha_2\cdots\alpha_k}\right). \tag{24}$$

One can further deduce

$$\delta_\vartheta \star \phi = \delta\vartheta^\alpha \wedge (e_\alpha \,\lrcorner\, \star\phi) - \star(\delta\vartheta^\alpha \wedge (e_\alpha \,\lrcorner\, \phi))\,. \tag{25}$$

Let χ be an untwisted p-form, ϕ another untwisted form.

$$\delta_\vartheta(\chi \wedge \star\phi) = \delta\vartheta^\alpha \wedge \{e_\alpha \,\lrcorner\, (\chi \wedge \star\phi) - \chi \wedge \star(e_\alpha \,\lrcorner\, \phi)\}\,. \tag{26}$$

3 Maxwell–Klein–Fock–Gordon theory

$$\delta_\vartheta S = \int \delta\vartheta^\alpha \wedge \left\{e_\alpha \,\lrcorner\, \mathfrak{L} + \mathbb{D}\phi^* \wedge \star(e_\alpha \,\lrcorner\, \mathbb{D}\phi) + \frac{1}{2} F \wedge \star(e_\alpha \,\lrcorner\, F)\right\}. \tag{27}$$

References

[1] J. van Bladel, "Lorenz or lorentz?", IEEE Antennas and Propagation Magazine **33**, 69–69 (1991) (cit. on p. 2).

[2] J. van Bladel, "Lorenz or lorentz? [addendum]", IEEE Antennas and Propagation Magazine **33**, 56–56 (1991) (cit. on p. 2).

[3] W. L. Burke, *Applied differential geometry* (Cambridge University Press, 1985) (cit. on p. 1).

[4] L. Lorenz, "XXXVIII. on the identity of the vibrations of light with electrical currents", The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **34**, 287–301 (1867) (cit. on p. 2).

[5] U. Muench, F. Gronwald, and F. W. Hehl, "A brief guide to variations in teleparallel gauge theories of gravity and the kaniel-itin model", General Relativity and Gravitation **30**, 933–961 (1998), arXiv:`gr-qc/9801036 [gr-qc]` (cit. on p. 3).