Coframe formalisms

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Contents

1 U(1)–regular theory

1.1 Differential geometry

Interior product Let z be a vector, ω be a 1-form, χ be a k-form. The *interior product* is defined inductively as the bilinear map satisfying

$$z \lrcorner \omega := \omega(z)\,, \tag{1}$$

$$z \lrcorner (\omega \wedge \chi) := (z \lrcorner \omega) \wedge \chi - \omega \wedge (z \lrcorner \chi)\,. \tag{2}$$

Equation (2) is also known as the anti-product rule.

By inductiion one can show that for a p-form ϕ,

$$z \lrcorner (\phi \wedge \chi) = (z \lrcorner \phi) \wedge \chi + (-)^p \chi \wedge (z \lrcorner \chi)\,. \tag{3}$$

Hodge star Let ω be an 1-form, χ be a k-form. The Hodge star $\star$ is defined inductively as the linear map [4, sec. 24]

$$\star 1 := \mathrm{vol}\,, \tag{4}$$

$$\star(\chi \wedge \omega) := \omega^\sharp \lrcorner \star\chi\,. \tag{5}$$

Non-gravitational theories features $[\delta, \star] = 0$, which means [7, sec. 3.2]

$$\delta g_{\mu\nu} = -2\omega_{(\mu\nu)}, \qquad \delta\vartheta^\mu = \omega_\nu{}^\mu\, \vartheta^\nu\,; \tag{6}$$

for an orthonormal coframe, the allowed variations are $\omega_{(\alpha\beta)} = 0$.

Codifferential The *codifferential* $\mathbb{d}^\dagger$ is the adjoint of the exterior derivative $\mathbb{d}$ in the following sense. Let ϕ be a k-form, χ be a $(k-1)$-form.

$$\begin{aligned}\mathbb{d}(\chi^* \wedge \star\phi) &\equiv \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi =: \mathbb{d}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{d}^\dagger\phi & (7)\\ &= \mathbb{d}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} \mathbb{d} \star \phi\,. & (8)\end{aligned}$$

$$\boxed{\mathbb{d}^\dagger\phi = (-)^k \star^{-1} \mathbb{d} \star \phi\,.} \tag{9}$$

Untwisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for untwisted k-form is defined inductively as

$$1\,, \tag{10}$$

$$\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} := \vartheta^{\alpha_1} \wedge \vartheta^{\alpha_2\cdots\alpha_k}\,. \tag{11}$$

Twisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for twisted $(D-k)$-form is defined inductively as

$$\epsilon := \mathrm{vol}\,, \tag{12}$$

$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} := e_{\alpha_k} \lrcorner\, \epsilon_{\alpha_1\cdots\alpha_{k-1}}\,. \tag{13}$$

By using eq. (5) and induction, one can show that

$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \star\vartheta_{\alpha_1\alpha_2\cdots\alpha_k}\,. \tag{14}$$

Identities Inspired by [9, eq. (3.167)], for a 1-form ω, k-form χ, one can derive

$$\begin{aligned}\omega \wedge \star\chi &= (-)^{D-k} \star \chi \wedge \omega = (-)^{D-k} \star^{-1} \left(\omega^\sharp \lrcorner \star\star\, \chi\right)\\ &= (-)^{D-k} (-)^{(D-k-1)(k+1)+s} \star \left(\omega^\sharp \lrcorner\, (-)^{k(D-k)+s} \star^{-1} \star\chi\right)\\ &= (-)^{k+1} \star \left(\omega^\sharp \lrcorner\, \chi\right). \end{aligned} \tag{15}$$

Specifically,

$$\vartheta^\alpha \wedge \epsilon_{\beta\gamma} = -\delta^\alpha_\beta\, \epsilon_\gamma + \delta^\alpha_\gamma\, \epsilon_\beta\,, \tag{16}$$

$$\vartheta^\alpha \wedge \epsilon_{\beta\gamma\delta} = \delta^\alpha_\beta\, \epsilon_{\gamma\delta} - \delta^\alpha_\gamma\, \epsilon_{\beta\delta} + \delta^\alpha_\delta\, \epsilon_{\beta\gamma}\,. \tag{17}$$

One can further deduce that

$$\begin{aligned}\vartheta^{\alpha\beta} \wedge \epsilon_{\gamma\delta\epsilon} = &\left(\delta^\alpha_\gamma \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\gamma\right)\epsilon_\epsilon\\ &- \left(\delta^\alpha_\gamma \delta^\beta_\epsilon - \delta^\alpha_\epsilon \delta^\beta_\gamma\right)\epsilon_\delta\\ &+ \left(\delta^\alpha_\delta \delta^\beta_\epsilon - \delta^\alpha_\epsilon \delta^\beta_\delta\right)\epsilon_\gamma\,.\end{aligned} \tag{18}$$

1.2 Complex scalar field

The action reads

$$
\begin{aligned}
S &= \int -\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi - m^2\phi^* \wedge \star\phi && (19)\\
&= \int -\frac{1}{2}(\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi + \mathbb{d}\phi \wedge \star\mathbb{d}\phi^*) - \frac{1}{2}m^2(\phi^* \wedge \star\phi + \phi \wedge \star\phi^*) && (20)
\end{aligned}
$$

Variation commutes with exterior derivative

$$
[\delta, \mathbb{d}]\phi = 0\,. \tag{21}
$$

A generic variation of the kinetic terms reads

$$
\begin{aligned}
\delta(\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi) = \mathbb{d}(&\delta\phi^* \wedge \star\mathbb{d}\phi + \delta\phi \wedge \star\mathbb{d}\phi^*)\\
&+ \delta\phi^* \wedge \star\mathbb{d}^\dagger\mathbb{d}\phi + \delta\phi \wedge \star\mathbb{d}^\dagger\mathbb{d}\phi^*\,.
\end{aligned} \tag{22}
$$

A generic variation of the actionreads

$$
\begin{aligned}
\delta S = \int &\mathbb{d}(-\delta\phi^* \wedge \star\mathbb{d}\phi - \delta\phi \wedge \star\mathbb{d}\phi^*)\\
&- \delta\phi^* \wedge \star(\mathbb{d}^\dagger\mathbb{d} + m^2)\phi - \delta\phi \wedge \star(\mathbb{d}^\dagger\mathbb{d} + m^2)\phi^*\,.
\end{aligned} \tag{23}
$$

1.2.1 Noether current

The action is invariant

$$
\delta_\lambda S \equiv 0 \tag{24}
$$

under the rigid transformation

$$
\phi \to \mathsf{e}^{-\mathrm{i}e\lambda}\phi\,, \qquad \phi^* \to \mathsf{e}^{+\mathrm{i}e\lambda}\phi^*\,. \tag{25}
$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$
\begin{aligned}
0 &= \int \lambda\, \mathbb{d}\mathfrak{J}_0\,,\\
\mathfrak{J}_0 &:= \mathrm{i}e(\phi^* \wedge \star\mathbb{d}\phi - \phi \wedge \star\mathbb{d}\phi^*)\,,
\end{aligned} \tag{26}
$$

which is the Noether current, a twisted 3-form, satisfying the continuity equation

$$
\mathbb{d}\mathfrak{J}_0 = 0\,. \tag{27}
$$

2 $\mathsf{U}(1)$ gauge theory

2.1 Connection on the principal bundle

Exterior covariant derivative Let χ be a $\mathbb{C}$-valued k-form. The *exterior covariant derivative* of χ reads

$$
\mathbb{D}\chi := (\mathbb{d} - \mathrm{i}eA)\chi, \qquad \mathbb{D}\chi^* := (\mathbb{d} + \mathrm{i}eA)\chi^*\,, \tag{28}
$$

where A is a $\mathfrak{u}(1)$-valued *connection form*.

Covariant codifferential The *covariant codifferential* $\mathbb{D}^\dagger$ is the adjoint of the exterior covariant derivative $\mathbb{D}$ in the following sense. Let ϕ be a $\mathbb{C}$-valued k-form, χ be a $\mathbb{C}$-valued $(k-1)$-form.

$$\begin{aligned} \mathrm{d}(\chi^* \wedge \star\phi) &\equiv \mathrm{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathrm{d} \star \phi =: \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{D}^\dagger\phi \qquad (29) \\ &= \mathbb{D}\chi^* \wedge \star\phi - \mathrm{i}eA \wedge \chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathrm{d} \star \phi \\ &= \mathbb{D}\chi^* \wedge \star\phi + \chi^* \wedge (-)^k \mathrm{i}eA \star \phi - (-)^k \chi^* \wedge \mathrm{d} \star \phi \\ &= \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} (\mathrm{d} - \mathrm{i}eA) \star \phi \,. \qquad (30) \end{aligned}$$

$$\boxed{\mathbb{D}^\dagger\phi = (-)^k \star^{-1} (\mathrm{d} - \mathrm{i}eA) \star \phi \,.} \qquad (31)$$

2.2 $\mathsf{U}(1)$-gauged complex scalar field theory

$$S = \int -\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi - m^2\phi^* \wedge \star\phi - \frac{1}{2}F \wedge \star F. \qquad (32)$$

Variation does not commute with exterior covariant derivative.

$$[\delta, \mathbb{D}]\phi = -\mathrm{i}e\delta A\phi \,. \qquad (33)$$

$$\begin{aligned} \delta(\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi) &= \mathrm{d}(\delta\phi^* \star \mathbb{D}\phi + \delta\phi \star \mathbb{D}\phi^*) + \delta\phi^* \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi^* \\ &\quad + \delta A \wedge \mathfrak{J}_A \,, \qquad \mathfrak{J}_A := \mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*) \,, \qquad (34) \\ \delta(F \wedge \star F) &= 2\,\delta F \wedge \star F \qquad (35) \\ &= 2\,\mathrm{d}(\delta A \wedge \star F) + 2\,\delta A \wedge \mathrm{d}\star F \,. \qquad (36) \end{aligned}$$

Variation of the action

$$\begin{aligned} \delta S = \int & \mathrm{d}(-\delta\phi^* \wedge \star\mathbb{D}\phi - \delta\phi \wedge \star\mathbb{D}\phi^* - \delta A \wedge \star F) \\ & - \delta\phi^* \wedge \star(\mathbb{D}^\dagger\mathbb{D} + m^2)\phi - \delta\phi \wedge \star(\mathbb{D}^\dagger\mathbb{D} + m^2)\phi^* \\ & - \delta A \wedge (\mathrm{d}\star F - \mathfrak{J}_A). \qquad (37) \end{aligned}$$

2.2.1 Lorenz gauge

The Laplace–de Rham operator, or in our Lorentzian metric signature the d'Alembertian

$$\Box^2 := \left(\mathrm{d} + \mathrm{d}^\dagger\right)^2 = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d} \,. \qquad (38)$$

$$\mathrm{d}\star F = \mathrm{d} \star \mathrm{d}A = \star(-)^2 \star^{-1} \mathrm{d} \star \mathrm{d}A = \star\mathrm{d}^\dagger\mathrm{d}A = \star(\Box^2 - \mathrm{d}\mathrm{d}^\dagger)A \,. \qquad (39)$$

One would like to have $\mathrm{d}\mathrm{d}^\dagger A = 0$, or $\mathrm{d}^\dagger A = \mathrm{const}$. This would be fulfilled if

$$\mathrm{d}^\dagger A = 0 \,, \qquad (40)$$

which is the Lorenz gauge[2, 3, 6].

2.2.2 Noether's invariances

The action is invariant under the generic transformation

$$\phi \to \mathsf{e}^{-\mathrm{i}e\Lambda}\phi\,, \quad \phi^* \to \mathsf{e}^{+\mathrm{i}e\Lambda}\phi^*\,, \quad A \to A - \mathbb{d}\Lambda\,. \tag{41}$$

There are two scenarios [1].

If the transformation is rigid, $\mathbb{d}\Lambda = 0$, one obtains $\mathfrak{J}_A$ as the Noether current from the boundary term as before, which satisfies the continuity equation $\mathbb{d}\mathfrak{J}_A = 0$.

If the transformation is gauge with a compact support, the boundary term can be dropped, and one obtains the Noether identity

$$123 \tag{42}$$

3 Poincaré–regular theory

3.1 Differential geometry

Translation Consider the rigid infinitesimal transformation

$$\delta_\lambda \chi = \lambda\, \pounds_z \chi = \lambda[z \lrcorner \mathbb{d}\chi + \mathbb{d}(z \lrcorner \chi)]\,; \tag{43}$$

'rigid' means

$$\delta_\lambda \vartheta^\alpha = 0\,. \tag{44}$$

3.2 Complex scalar field

3.2.1 Noether current

The action is invariant up to a total differential

$$\begin{aligned} \delta_\lambda S &= \int \lambda\, \mathbb{d}(z \lrcorner \mathfrak{L}) \\ &= \int \lambda\, \frac{1}{2}\, \mathbb{d}\big\{-z \lrcorner \big[\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi + \mathbb{d}\phi \wedge \star\mathbb{d}\phi^* + m^2(\phi^* \star \phi + \phi \star \phi^*)\big]\big\}\,, \end{aligned} \tag{45}$$

under the rigid infinitesimal transformation

$$\delta\phi = \lambda\, \pounds_z \phi = \lambda\, z \lrcorner \mathbb{d}\phi\,, \qquad \delta\phi^* = \lambda\, \pounds_z \phi^* = \lambda\, z \lrcorner \mathbb{d}\phi^*\,. \tag{46}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$\delta_\lambda S = \int \lambda\, \mathbb{d}\{-(z \lrcorner \mathbb{d}\phi^*) \wedge \star\mathbb{d}\phi - (z \lrcorner \mathbb{d}\phi) \wedge \star\mathbb{d}\phi^*\}\,. \tag{47}$$

Taking the difference and setting $z = e_\alpha$ yields

$$0 = \int \lambda\, \mathbb{d}\mathfrak{T}_\alpha, \tag{48}$$

$$\begin{aligned} 2\mathfrak{T}_\alpha := &-(e_\alpha \lrcorner \mathbb{d}\phi^*) \wedge \star\mathbb{d}\phi - \mathbb{d}\phi^* \wedge (e_\alpha \lrcorner \star\mathbb{d}\phi) \\ &- (e_\alpha \lrcorner \mathbb{d}\phi) \wedge \star\mathbb{d}\phi^* - \mathbb{d}\phi \wedge (e_\alpha \lrcorner \star\mathbb{d}\phi^*) \\ &+ m^2(\phi^*\, e_\alpha \lrcorner \star\phi + \phi\, e_\alpha \star \phi^*)\,. \end{aligned} \tag{49}$$

The continuity equation reads

$$\mathbb{d}\mathfrak{T}_\alpha = 0\,. \tag{50}$$

In components,

$$\mathfrak{T}_\alpha = -\left[e_\alpha(\phi^*) e^\beta(\phi) + e_\alpha(\phi) e^\beta(\phi^*)\right]\epsilon_\beta + e^\beta(\phi^*) e_\beta(\phi^*)\,\epsilon_\alpha + m^2\phi^*\phi\,\epsilon_\alpha\,. \tag{51}$$

3.3 Pure electromagnetic field

3.3.1 Noether current

The action is invariant up to a total differential

$$\delta_\lambda S = \int \lambda\, \mathbb{d}(z \lrcorner \mathfrak{L}) = \int \lambda\, \frac{1}{2}\, \mathbb{d}\{-z \lrcorner (F \wedge \star F)\}\,, \tag{52}$$

under the rigid infinitesimal transformation combined with a gauge transformation [8, eq. 3.46]

$$\delta A = \lambda\{z \lrcorner \mathbb{d}(A - \mathbb{d}\Lambda) + \mathbb{d}[z \lrcorner (A - \mathbb{d}\Lambda)]\}\,. \tag{53}$$

Choosing

$$\mathbb{d}\Lambda = A(z)\, z^\flat \tag{54}$$

makes the second term vanish, yielding

$$\delta A = \lambda\, z \lrcorner \mathbb{d}A = \lambda\, z \lrcorner F\,. \tag{55}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$\delta_\lambda S = \int \lambda\, \mathbb{d}\{-(z \lrcorner F) \wedge \star F\}\,. \tag{56}$$

Taking the difference and setting $z = e_\alpha$ yields

$$0 = \int \lambda\, \mathbb{d}\mathfrak{T}_\alpha, \tag{57}$$

$$2\mathfrak{T}_\alpha := -(e_\alpha \lrcorner F) \wedge \star F + F \wedge (e_\alpha \lrcorner \star F)\,. \tag{58}$$

The continuity equation reads

$$\mathbb{d}\mathfrak{T}_\alpha = 0\,. \tag{59}$$

In components, one needs

$$e_\alpha \lrcorner F = F_{\alpha\beta}\,\vartheta^\beta\,, \tag{60}$$

$$e_\alpha \lrcorner \star F = \frac{1}{2} F^{\beta\gamma}\,\epsilon_{\beta\gamma\alpha}\,; \tag{61}$$

so that

$$(e_\alpha \lrcorner F) \wedge \star F = -F_{\alpha\gamma}F^{\gamma\beta}\,\epsilon_\beta\,, \tag{62}$$

$$F \wedge (e_\alpha \lrcorner \star F) = \frac{1}{2} F_{\beta\gamma}F^{\beta\gamma}\,\epsilon_\alpha + F_{\alpha\gamma}F^{\gamma\beta}\,\epsilon_\beta\,. \tag{63}$$

One finally has

$$\mathfrak{T}_\alpha = \frac{1}{4} F_{\beta\gamma}F^{\beta\gamma}\,\epsilon_\alpha + F_{\alpha\gamma}F^{\gamma\beta}\,\epsilon_\beta\,. \tag{64}$$

4 Poincaré gauge theory

4.1 Differential forms

Upon variation of ϑ^α, $\vartheta^{\alpha_1\alpha_2\cdots\alpha_k}$ goes under

$$\delta\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge (e_\alpha \lrcorner\, \vartheta^{\alpha_1\alpha_2\cdots\alpha_k}) \,, \tag{65}$$

which can be proved by induction.

Upon variation of ϑ^α, $\epsilon_{\alpha_1\alpha_2\cdots\alpha_k}$ goes under [7, sec. A.2]

$$\delta\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge \left(e_\alpha \lrcorner\, \epsilon_{\alpha_1\alpha_2\cdots\alpha_k}\right) . \tag{66}$$

Variation of Hodge star In gravitational theories [7, sec. 3.2] with an orthonormal cobasis,

$$[\delta, \star]\phi = \delta\vartheta^\alpha \wedge (e_\alpha \lrcorner \star\phi) - \star(\delta\vartheta^\alpha \wedge (e_\alpha \lrcorner\, \phi)) \,. \tag{67}$$

Let χ be a p-form, ϕ another form [5, sec. 5].

$$\delta(\chi \wedge \star\phi) = \delta\chi \wedge \star\phi + \delta\phi \wedge \star\chi - \delta\vartheta^\alpha \wedge \Sigma_\alpha \,, \tag{68}$$

$$\Sigma_\alpha := \chi \wedge \left\{\star(e_\alpha \lrcorner\, \phi) - (-)^p (e_\alpha \lrcorner \star\phi)\right\} . \tag{69}$$

4.2 $\mathsf{U}(1)$-gauged complex scalar field theory

$$\begin{aligned}\Sigma_\alpha = &-\mathbb{D}\phi^* \wedge \{\star(e_\alpha \lrcorner\, \mathbb{D}\phi) + (e_\alpha \lrcorner \star\mathbb{D}\phi)\} - m^2\phi^*\phi\,\epsilon_\alpha \\ &- \frac{1}{2}F \wedge \{\star(e_\alpha \lrcorner\, F) - (e_\alpha \lrcorner \star F)\} \,.\end{aligned} \tag{70}$$

4.2.1 Noether's invariances

[1]

Rigid translation

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