Differential form

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1 Interior product

Let z be a vector, ω be a 1-form, χ be a k-form. The *interior product* is defined inductively as the bilinear map satisfying

$$z \lrcorner \omega := \omega(z)\,, \tag{1}$$

$$z \lrcorner (\omega \wedge \chi) := (z \lrcorner \omega) \wedge \chi - \omega \wedge (z \lrcorner \chi)\,. \tag{2}$$

Anti-product rule.

For a p-form ϕ,

$$z \lrcorner (\phi \wedge \chi) = (z \lrcorner \phi) \wedge \chi + (-)^p \chi \wedge (z \lrcorner \chi)\,. \tag{3}$$

2 Hodge star

Let ω be an 1-form, χ be a k-form. The Hodge star $\star$ is defined inductively as the linear map [3, sec. 24]

$$\star 1 := \mathrm{vol}\,, \tag{4}$$

$$\star(\chi \wedge \omega) := \omega^\sharp \lrcorner \star\chi\,. \tag{5}$$

3 Covariant differential of U(1)-gauge field

χ a $\mathbb{C}$-valued k-form

$$\mathbb{D}\chi := (\mathbb{d} - \mathrm{i}eA) \wedge \chi, \qquad \mathbb{D}\chi^* := (\mathbb{d} + \mathrm{i}eA) \wedge \chi^*\,, \tag{6}$$

A the $\mathfrak{u}(1)$-valued connection form.

4 Covariant codifferential

Define the covariant codifferential of a $\mathbb{C}$-valued k-form ϕ as follows. Let χ be an arbitrary $\mathbb{C}$-valued $(k-1)$-form.

$$\begin{aligned}\int \mathbb{d}(\chi^* \wedge \star\phi) &\equiv \int \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi =: \int \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{D}^\dagger\phi \quad (7)\\ &= \int \mathbb{D}\chi^* \wedge \star\phi - \mathrm{i}eA \wedge \chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi \\ &= \int \mathbb{D}\chi^* \wedge \star\phi + \chi^* \wedge (-)^k \mathrm{i}eA \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi \\ &= \int \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\phi\,. \quad (8)\end{aligned}$$

$$\boxed{\mathbb{D}^\dagger\phi = (-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\phi\,.} \tag{9}$$

5 Maxwell–Klein–Fock–Gordon theory

$$S = \int -\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi - m^2\phi^* \wedge \star\phi - \frac{1}{2}F \wedge \star F. \tag{10}$$

$$\delta_{\mathrm{m}}\mathbb{D}\phi = -\mathrm{i}e\delta A\phi + \mathbb{D}\delta\phi. \tag{11}$$

$$\begin{aligned}\delta_{\mathrm{m}}(\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi) &= \mathbb{d}(\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^*)\\ &\quad + \delta\phi^* \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi^* \quad (12)\\ &\quad + \delta A \wedge (\mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*))\,,\\ \delta_{\mathrm{m}}(F \wedge \star F) &= 2\mathbb{d}(\delta A \wedge \star F) - 2\delta A \wedge \mathbb{d}\star F\,. \quad (13)\end{aligned}$$

$$\begin{aligned}\delta_{\mathrm{m}}S = \int &-\mathbb{d}(\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^* + \delta A \wedge \star F)\\ &+ \delta\phi^* \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi + \delta\phi \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi^*\\ &+ \delta A \wedge (-\mathbb{d}\star F + \mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)).\end{aligned} \tag{14}$$

5.1 Lorenz gauge

The Laplace–de Rham operator, or in our signature of metric the d'Alembertian

$$\Box^2 := \left(\mathbb{d} + \mathbb{d}^\dagger\right)^2 = \mathbb{d}\mathbb{d}^\dagger + \mathbb{d}^\dagger\mathbb{d}\,. \tag{15}$$

$$\mathbb{d}\star F = \mathbb{d}\star\mathbb{d}A = \star(-)^2\star^{-1}\mathbb{d}\star\mathbb{d}A = \star\mathbb{d}^\dagger\mathbb{d}A = \star(\Box^2 - \mathbb{d}\mathbb{d}^\dagger)A\,. \tag{16}$$

One would like to have $\mathbb{d}\mathbb{d}^\dagger A = 0$, or $\mathbb{d}^\dagger A = \text{const}$. This would be fulfiled if

$$\mathbb{d}^\dagger A = 0\,, \tag{17}$$

which is the Lorenz gauge[1, 2, 4].

6 Untwisted orthonormal k-cobases

Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for untwisted k-form is defined inductively as

$$1\,, \tag{18}$$

$$\vartheta^{\alpha_1\alpha_2\ldots\alpha_k} := \vartheta^{\alpha_1}\wedge\vartheta^{\alpha_2\ldots\alpha_k}\,. \tag{19}$$

Upon variation of ϑ^α, $\vartheta^{\alpha_1\alpha_2\ldots\alpha_k}$ goes under

$$\delta_\vartheta\vartheta^{\alpha_1\alpha_2\ldots\alpha_k} = \delta\vartheta^\alpha\wedge(e_\alpha \lrcorner \vartheta^{\alpha_1\alpha_2\ldots\alpha_k})\,, \tag{20}$$

which can be proved by induction.

7 Twisted orthonormal k-cobases

Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for twisted $(D-k)$-form is defined inductively as

$$\epsilon := \mathrm{vol}\,, \tag{21}$$

$$\epsilon_{\alpha_1\alpha_2\ldots\alpha_k} := e_{\alpha_k}\lrcorner\,\epsilon_{\alpha_1\ldots\alpha_{k-1}}\,. \tag{22}$$

By using eq. (2.0.2) and induction, one can show that

$$\epsilon_{\alpha_1\alpha_2\ldots\alpha_k} = \star\vartheta_{\alpha_1\alpha_2\ldots\alpha_k}\,. \tag{23}$$

Upon variation of ϑ^α, $\epsilon_{\alpha_1\alpha_2\ldots\alpha_k}$ goes under [5, sec. A.2]

$$\delta_\vartheta\epsilon_{\alpha_1\alpha_2\ldots\alpha_k} = \delta\vartheta^\alpha\wedge\left(e_\alpha\lrcorner\,\epsilon_{\alpha_1\alpha_2\ldots\alpha_k}\right). \tag{24}$$

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