Differential form

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May 12, 2019

Contents

1 Non-gravitational theories

1.1 Differential geometry

Interior product Let z be a vector, ω be a 1-form, χ be a k-form. The *interior product* is defined inductively as the bilinear map satisfying

$$z \lrcorner \omega := \omega(z)\,, \tag{1}$$
$$z \lrcorner (\omega \wedge \chi) := (z \lrcorner \omega) \wedge \chi - \omega \wedge (z \lrcorner \chi)\,. \tag{2}$$

Equation (2) is also known as the anti-product rule.

By inductiion one can show that for a p-form ϕ,

$$z \lrcorner (\phi \wedge \chi) = (z \lrcorner \phi) \wedge \chi + (-)^p \chi \wedge (z \lrcorner \chi)\,. \tag{3}$$

Hodge star Let ω be an 1-form, χ be a k-form. The Hodge star $\star$ is defined inductively as the linear map [3, sec. 24]

$$\star 1 := \mathrm{vol}\,, \tag{4}$$
$$\star(\chi \wedge \omega) := \omega^\sharp \lrcorner \star\chi\,. \tag{5}$$

Non-gravitational theories features $[\delta, \star] = 0$, which means [6, sec. 3.2]

$$\delta g_{\mu\nu} = -2\omega_{(\mu\nu)}, \qquad \delta\vartheta^\mu = \omega_\nu{}^\mu\, \vartheta^\nu\,; \tag{6}$$

for an orthonormal coframe, the allowed variations are $\omega_{(\alpha\beta)} = 0$.

1.2 Connection on the principal bundle

Covariant differential Let χ be a $\mathbb{C}$-valued k-form. The *covariant differential* of χ reads

$$\mathbb{D}\chi := (\mathbb{d} - \mathrm{i}eA) \wedge \chi, \qquad \mathbb{D}\chi^* := (\mathbb{d} + \mathrm{i}eA) \wedge \chi^* , \tag{7}$$

where A is a $\mathfrak{u}(1)$-valued *connection form.*

Covariant codifferential The *covariant codifferential* $\mathbb{D}^\dagger$ is the adjoint of the covariant differential $\mathbb{D}$ in the following sense. Let ϕ be a $\mathbb{C}$-valued k-form, χ be a $\mathbb{C}$-valued $(k-1)$-form.

$$\begin{aligned}\int \mathbb{d}(\chi^* \wedge \star\phi) &\equiv \int \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi =: \int \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{D}^\dagger\phi \qquad (8)\\
&= \int \mathbb{D}\chi^* \wedge \star\phi - \mathrm{i}eA \wedge \chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi\\
&= \int \mathbb{D}\chi^* \wedge \star\phi + \chi^* \wedge (-)^k \mathrm{i}eA \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi\\
&= \int \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\phi \,. \qquad (9)\end{aligned}$$

$$\boxed{\mathbb{D}^\dagger\phi = (-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\phi \,.} \tag{10}$$

1.3 Maxwell–Klein–Fock–Gordon theory

$$S = \int -\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi - m^2\phi^* \wedge \star\phi - \frac{1}{2}F \wedge \star F. \tag{11}$$

$$\delta\mathbb{D}\phi = -\mathrm{i}e\delta A\phi + \mathbb{D}\delta\phi. \tag{12}$$

$$\begin{aligned}\delta(\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi) &= \mathbb{d}(\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^*)\\
&\quad - \delta\phi^* \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi - \delta\phi \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi^* \qquad (13)\\
&\quad + \delta A \wedge (\mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)) ,\\
\delta(F \wedge \star F) &= 2\mathbb{d}(\delta A \wedge \star F) - 2\delta A \wedge \mathbb{d}\star F \,. \qquad (14)\end{aligned}$$

$$\begin{aligned}\delta S = \int &-\mathbb{d}(\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^* + \delta A \wedge \star F)\\
&+ \delta\phi^* \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi + \delta\phi \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi^*\\
&+ \delta A \wedge (\mathbb{d}\star F - \mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)).\end{aligned} \tag{15}$$

1.3.1 Lorenz gauge

The Laplace–de Rham operator, or in our metric signature the d'Alembertian

$$\Box^2 := \left(\mathbb{d} + \mathbb{d}^\dagger\right)^2 = \mathbb{d}\mathbb{d}^\dagger + \mathbb{d}^\dagger\mathbb{d}\,. \tag{16}$$

$$\mathbb{d}{\star}F = \mathbb{d}\star\mathbb{d}A = \star(-)^2\star^{-1}\mathbb{d}\star\mathbb{d}A = \star\mathbb{d}^\dagger\mathbb{d}A = \star(\Box^2 - \mathbb{d}\mathbb{d}^\dagger)A\,. \tag{17}$$

One would like to have $\mathbb{d}\mathbb{d}^\dagger A = 0$, or $\mathbb{d}^\dagger A = \mathrm{const}$. This would be fulfiled if

$$\mathbb{d}^\dagger A = 0\,, \tag{18}$$

which is the Lorenz gauge[1, 2, 5].

1.3.2 Noether's first theorem

1.3.3 Noether's second theorem

2 Gravitational theory

2.1 Differential forms

Untwisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for untwisted k-form is defined inductively as

$$1\,, \tag{19}$$
$$\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} := \vartheta^{\alpha_1}\wedge\vartheta^{\alpha_2\cdots\alpha_k}\,. \tag{20}$$

Upon variation of ϑ^α, $\vartheta^{\alpha_1\alpha_2\cdots\alpha_k}$ goes under

$$\delta\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge (e_\alpha \lrcorner\, \vartheta^{\alpha_1\alpha_2\cdots\alpha_k})\,, \tag{21}$$

which can be proved by induction.

Twisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for twisted $(D-k)$-form is defined inductively as

$$\epsilon := \mathrm{vol}\,, \tag{22}$$
$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} := e_{\alpha_k} \lrcorner\, \epsilon_{\alpha_1\cdots\alpha_{k-1}}\,. \tag{23}$$

By using eq. (5) and induction, one can show that

$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \star\vartheta_{\alpha_1\alpha_2\cdots\alpha_k}\,. \tag{24}$$

Upon variation of ϑ^α, $\epsilon_{\alpha_1\alpha_2\cdots\alpha_k}$ goes under [6, sec. A.2]

$$\delta\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge \left(e_\alpha \lrcorner\, \epsilon_{\alpha_1\alpha_2\cdots\alpha_k}\right)\,. \tag{25}$$

Variation of Hodge star In gravitational theories [6, sec. 3.2] with an orthonormal cobasis,

$$[\delta, \star]\phi = \delta\vartheta^\alpha \wedge (e_\alpha \lrcorner \star\phi) - \star(\delta\vartheta^\alpha \wedge (e_\alpha \lrcorner\, \phi))\,. \tag{26}$$

Let χ be a p-form, ϕ another form [4, sec. 5].

$$\delta(\chi\wedge\star\phi) = \delta\chi\wedge\star\phi + \delta\phi\wedge\star\chi - \delta\vartheta^\alpha\wedge\Sigma_\alpha\,, \tag{27}$$
$$\Sigma_\alpha := \chi\wedge\left\{\star(e_\alpha\lrcorner\,\phi) - (-)^p(e_\alpha\lrcorner\star\phi)\right\}\,. \tag{28}$$

2.2 Maxwell–Klein–Fock–Gordon theory

$$
\begin{aligned}
\Sigma_\alpha = &-\mathbb{D}\phi^* \wedge \{\star(e_\alpha \rfloor \mathbb{D}\phi) + (e_\alpha \rfloor \star\mathbb{D}\phi)\} - m^2\phi^*\phi\,\epsilon_\alpha \\
&- \frac{1}{2} F \wedge \{\star(e_\alpha \rfloor F) - (e_\alpha \rfloor \star F)\}\,.
\end{aligned} \tag{29}
$$

2.2.1 Noether's first theorem

2.2.2 Noether's second theorem

References

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