

# Differential form

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## 1 Covariant differential

$\phi$  a  $\mathbb{C}$ -valued 1-form

$$\mathbb{D}\phi := (\mathfrak{d} - ieA) \wedge \phi, \quad \mathbb{D}\phi^* := (\mathfrak{d} + ieA) \wedge \phi^*, \quad (1.0.1)$$

$A$  the  $\mathfrak{u}(1)$ -valued connection form.

## 2 Covariant codifferential

Define the covariant codifferential of a  $\mathbb{C}$ -valued  $k$ -form  $\zeta$  as follows. Let  $\eta$  be an arbitrary  $\mathbb{C}$ -valued  $(k-1)$ -form.

$$\int \mathfrak{d}(\eta^* \wedge \star \zeta) \equiv \int \mathfrak{d}\eta^* \wedge \star \zeta - (-)^k \eta^* \wedge \mathfrak{d} \star \zeta =: \int \mathbb{D}\eta^* \wedge \star \zeta - \eta^* \wedge \star \mathbb{D}^\dagger \zeta \quad (2.0.1)$$

$$\begin{aligned} &= \int \mathbb{D}\eta^* \wedge \star \zeta - ieA \wedge \eta^* \wedge \star \zeta - (-)^k \eta^* \wedge \mathfrak{d} \star \zeta \\ &= \int \mathbb{D}\eta^* \wedge \star \zeta + \eta^* \wedge (-)^k ieA \wedge \star \zeta - (-)^k \eta^* \wedge \mathfrak{d} \star \zeta \\ &= \int \mathbb{D}\eta^* \wedge \star \zeta - \eta^* \wedge \star (-)^k \star^{-1} (\mathfrak{d} - ieA) \wedge \star \zeta. \end{aligned} \quad (2.0.2)$$

$$\boxed{\mathbb{D}^\dagger \zeta = (-)^k \star^{-1} (\mathfrak{d} - ieA) \wedge \star \zeta.} \quad (2.0.3)$$

## 3 Maxwell–Klein–Fock–Gordon theory

$$S = \int -\mathbb{D}\phi^* \wedge \star \mathbb{D}\phi - m^2 \phi^* \wedge \star \phi - \frac{1}{2} F \wedge \star F. \quad (3.0.1)$$

$$\delta \mathbb{D}\phi = -ie\delta A\phi + \mathbb{D}\delta\phi. \quad (3.0.2)$$

$$\begin{aligned} \delta(\mathbb{D}\phi^* \wedge \star \mathbb{D}\phi) &= \mathfrak{d}(\delta\phi^* \wedge \star \mathbb{D}\phi + \delta\phi \wedge \star \mathbb{D}\phi^*) \\ &\quad + \delta\phi^* \wedge \star \mathbb{D}^\dagger \mathbb{D}\phi + \delta\phi \wedge \star \mathbb{D}^\dagger \mathbb{D}\phi^* \\ &\quad + \delta A \wedge (ie(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)), \end{aligned} \quad (3.0.3)$$

$$\delta(F \wedge \star F) = 2\mathfrak{d}(\delta A \wedge \star F) - 2\delta A \wedge \mathfrak{d}\star F. \quad (3.0.4)$$

$$\begin{aligned} \delta S &= \int -\mathfrak{d}(\delta\phi^* \wedge \star \mathbb{D}\phi + \delta\phi \wedge \star \mathbb{D}\phi^* + \delta A \wedge \star F) \\ &\quad + \delta\phi^* \wedge \star (\mathbb{D}^\dagger \mathbb{D} - m^2)\phi + \delta\phi \wedge \star (\mathbb{D}^\dagger \mathbb{D} - m^2)\phi^* + \\ &\quad + \delta A \wedge (-\mathfrak{d}\star F + ie(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)). \end{aligned} \quad (3.0.5)$$

### 3.1 Lorenz gauge

$$\square^2 := (\mathfrak{d} + \mathfrak{d}^\dagger)^2 = \mathfrak{d}\mathfrak{d}^\dagger + \mathfrak{d}^\dagger\mathfrak{d}. \quad (3.1.1)$$

$$\mathfrak{d}\star F = \mathfrak{d}\star \mathfrak{d}A = \star(-)^2 \star^{-1} \mathfrak{d}\star \mathfrak{d}A = \star \mathfrak{d}^\dagger \mathfrak{d}A = \star(\square^2 - \mathfrak{d}\mathfrak{d}^\dagger)A. \quad (3.1.2)$$

One would like to have  $\mathfrak{d}\mathfrak{d}^\dagger A = 0$ , or  $\mathfrak{d}^\dagger A = \text{const.}$  This would be fulfilled if

$$\mathfrak{d}^\dagger A = 0, \quad (3.1.3)$$

which is the Lorenz gauge[1–3].