Tool-kit for the coframe formalism

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Contents

1 Introduction

This note is intended to be pragmatic, collecting a set of toolkits for the calculation of field theories without and with gravitation. The physical motivation will be discussed elsewhere.

2 Complex scalar field

The action reads

$$S = \int -\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi - m^2\phi^* \wedge \star\phi \tag{1}$$

$$= \int -\frac{1}{2}(\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi + \mathbb{d}\phi \wedge \star\mathbb{d}\phi^*) - \frac{1}{2}m^2(\phi^* \wedge \star\phi + \phi \wedge \star\phi^*) \tag{2}$$

Variation commutes with exterior derivative

$$[\delta, \mathbb{d}]\phi = 0\,. \tag{3}$$

For simplicity, one assumes that variation also commutes with Hodge star

$$[\delta, \star]\phi = 0\,. \tag{4}$$

In the absence of gravitation, variation also commutes with Hodge star. A generic variation of the kinetic terms therefore reads

$$\begin{aligned} \delta(\mathrm{d}\phi^* \wedge \star\mathrm{d}\phi) = \mathrm{d}(\delta\phi^* \wedge \star\mathrm{d}\phi + \delta\phi \wedge \star\mathrm{d}\phi^*) \\ + \delta\phi^* \wedge \star\mathrm{d}^\dagger\mathrm{d}\phi + \delta\phi \wedge \star\mathrm{d}^\dagger\mathrm{d}\phi^*\,, \end{aligned} \tag{5}$$

where the codifferential is defined in eq. (53). A generic variation of the action reads

$$\begin{aligned} \delta S = \int & \mathrm{d}(-\delta\phi^* \wedge \star\mathrm{d}\phi - \delta\phi \wedge \star\mathrm{d}\phi^*) \\ & - \delta\phi^* \wedge \star(\mathrm{d}^\dagger\mathrm{d} + m^2)\phi - \delta\phi \wedge \star(\mathrm{d}^\dagger\mathrm{d} + m^2)\phi^*\,. \end{aligned} \tag{6}$$

2.1 $\mathsf{U}(1)$ Noether current

The action is invariant

$$\delta_\lambda S \equiv 0 \tag{7}$$

under the rigid transformation

$$\phi \to \mathrm{e}^{-\mathrm{i}e\lambda}\phi\,, \qquad \phi^* \to \mathrm{e}^{+\mathrm{i}e\lambda}\phi^*\,. \tag{8}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$\begin{aligned} 0 &= \int \lambda\,\mathrm{d}\mathfrak{J}_0\,, \\ \mathfrak{J}_0 &:= \mathrm{i}e(\phi^* \wedge \star\mathrm{d}\phi - \phi \wedge \star\mathrm{d}\phi^*)\,, \end{aligned} \tag{9}$$

which is the Noether current, a twisted 3-form, satisfying the continuity equation

$$\mathrm{d}\mathfrak{J}_0 = 0\,. \tag{10}$$

2.2 $\mathsf{T}(D)$ Noether current

The action is invariant up to a total differential

$$\begin{aligned} \delta_\lambda S &= \int -\lambda\,\mathrm{d}(z \lrcorner \mathfrak{L}) \\ &= \int \lambda\,\frac{1}{2}\,\mathrm{d}\big\{+z \lrcorner \big[\mathrm{d}\phi^* \wedge \star\mathrm{d}\phi + \mathrm{d}\phi \wedge \star\mathrm{d}\phi^* + m^2(\phi^* \star \phi + \phi \star \phi^*)\big]\big\}\,, \end{aligned} \tag{11}$$

under the rigid infinitesimal transformation

$$\delta\phi = -\lambda\,\pounds_z\phi = -\lambda\,z \lrcorner \mathrm{d}\phi\,, \qquad \delta\phi^* = -\lambda\,\pounds_z\phi^* = -\lambda\,z \lrcorner \mathrm{d}\phi^*\,. \tag{12}$$

Note the minus sign, which has to do with our mostly-positive metric convention $(-, +, \dots, +)$.

When the equations of motion are satisfied, infinitesimal transformation leads to

$$\delta_\lambda S = \int \lambda\, \mathbb{d}\{+(z \lrcorner \mathbb{d}\phi^*) \wedge \star\mathbb{d}\phi + (z \lrcorner \mathbb{d}\phi) \wedge \star\mathbb{d}\phi^*\}\,. \tag{13}$$

Taking the difference with eq. (11) and setting $z = e_\alpha$ yield

$$0 = \int \lambda\, \mathbb{d}\mathfrak{T}_\alpha, \tag{14}$$

$$\begin{aligned} 2\mathfrak{T}_\alpha := &+(e_\alpha \lrcorner \mathbb{d}\phi^*) \wedge \star\mathbb{d}\phi + \mathbb{d}\phi^* \wedge (e_\alpha \lrcorner \star\mathbb{d}\phi) \\ &+ (e_\alpha \lrcorner \mathbb{d}\phi) \wedge \star\mathbb{d}\phi^* + \mathbb{d}\phi \wedge (e_\alpha \lrcorner \star\mathbb{d}\phi^*) \\ &- m^2(\phi^*\, e_\alpha \lrcorner \star\phi + \phi\, e_\alpha \star \phi^*)\,. \end{aligned} \tag{15}$$

The continuity equation reads

$$\mathbb{d}\mathfrak{T}_\alpha = 0\,. \tag{16}$$

In components,

$$\mathfrak{T}_\alpha = +\left[e_\alpha(\phi^*)e^\beta(\phi) + e_\alpha(\phi)e^\beta(\phi^*)\right]\epsilon_\beta - \left[e^\beta(\phi^*)e_\beta(\phi^*) + m^2\phi^*\phi\right]\epsilon_\alpha\,. \tag{17}$$

3 Pure electromagnetic field

The action reads

$$S = \int -\frac{1}{2} F \wedge \star F\,. \tag{18}$$

Variation commutes with exterior derivative

$$[\delta, \mathbb{d}]\phi = 0\,. \tag{3 rev.}$$

For simplicity, one assumes that variation also commutes with Hodge star

$$[\delta, \star]\phi = 0\,. \tag{4 rev.}$$

Variation of the Lagrangian reads

$$\delta(F \wedge \star F) = 2\,\delta F \wedge \star F \tag{19}$$

$$= 2\,\mathbb{d}(\delta A \wedge \star F) + 2\,\delta A \wedge \mathbb{d}\star F\,. \tag{20}$$

Note that this is not gauge invariant.

3.1 Lorenz gauge

The Laplace–de Rham operator, or in our Lorentzian metric signature the d'Alembertian

$$\Box^2 := \left(\mathbb{d} + \mathbb{d}^\dagger\right)^2 = \mathbb{d}\mathbb{d}^\dagger + \mathbb{d}^\dagger\mathbb{d}\,. \tag{21}$$

$$\mathbb{d}\star F = \mathbb{d} \star \mathbb{d}A = \star(-)^2 \star^{-1} \mathbb{d} \star \mathbb{d}A = \star\mathbb{d}^\dagger\mathbb{d}A = \star(\Box^2 - \mathbb{d}\mathbb{d}^\dagger)A\,. \tag{22}$$

One would like to have $\mathbb{d}\mathbb{d}^\dagger A = 0$, or $\mathbb{d}^\dagger A = \text{const}$. This would be fulfilled if

$$\mathbb{d}^\dagger A = 0\,, \tag{23}$$

which is the Lorenz gauge[2, 3, 6].

3.2 $\mathsf{T}(D)$ Noether current

The action is invariant up to a total differential

$$\delta_\lambda S = -\int \lambda\, \mathbb{d}(z \lrcorner \mathfrak{L}) = \int \lambda\, \frac{1}{2}\, \mathbb{d}\{+z \lrcorner (F \wedge \star F)\}\,, \tag{24}$$

under the rigid infinitesimal transformation combined with a gauge transformation [8, eq. 3.46]

$$\delta A = -\lambda\{z \lrcorner \mathbb{d}(A - \mathbb{d}\Lambda) + \mathbb{d}[z \lrcorner (A - \mathbb{d}\Lambda)]\}\,. \tag{25}$$

Choosing

$$\mathbb{d}\Lambda = A(z)\, z^\flat \tag{26}$$

makes the second term vanish, yielding

$$\delta A = -\lambda\, z \lrcorner \mathbb{d}A = -\lambda\, z \lrcorner F\,. \tag{27}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$\delta_\lambda S = \int \lambda\, \mathbb{d}\{+(z \lrcorner F) \wedge \star F\}\,. \tag{28}$$

Taking the difference with eq. (24) and setting $z = e_\alpha$ yield

$$0 = \int \lambda\, \mathbb{d}\mathfrak{T}_\alpha, \tag{29}$$

$$2\mathfrak{T}_\alpha := +(e_\alpha \lrcorner F) \wedge \star F - F \wedge (e_\alpha \lrcorner \star F)\,. \tag{30}$$

The continuity equation reads

$$\mathbb{d}\mathfrak{T}_\alpha = 0\,. \tag{31}$$

In components, one needs

$$e_\alpha \lrcorner F = F_{\alpha\beta}\, \vartheta^\beta\,, \tag{32}$$

$$e_\alpha \lrcorner \star F = \frac{1}{2} F^{\beta\gamma}\, \epsilon_{\beta\gamma\alpha}\,; \tag{33}$$

equipped with eqs. (59) to (59), one arrives at

$$(e_\alpha \lrcorner F) \wedge \star F = -F_{\alpha\gamma} F^{\gamma\beta}\, \epsilon_\beta\,, \tag{34}$$

$$F \wedge (e_\alpha \lrcorner \star F) = \frac{1}{2} F_{\beta\gamma} F^{\beta\gamma}\, \epsilon_\alpha + F_{\alpha\gamma} F^{\gamma\beta}\, \epsilon_\beta\,. \tag{35}$$

One finally has

$$\mathfrak{T}_\alpha = -F_{\alpha\gamma} F^{\gamma\beta}\, \epsilon_\beta - \frac{1}{4} F_{\beta\gamma} F^{\beta\gamma}\, \epsilon_\alpha\,. \tag{36}$$

4 $\mathsf{U}(1)$-gauged complex scalar field theory

The action reads

$$S = \int -\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi - m^2\phi^* \wedge \star\phi - \frac{1}{2}F \wedge \star F\,, \tag{37}$$

where the exterior covariant derivative is defined in eq. (64).

Variation does *not* commute with exterior covariant derivative.

$$[\delta, \mathbb{D}]\phi = -\mathrm{i}e\delta A\phi\,. \tag{38}$$

For simplicity, one assumes that variation also commutes with Hodge star

$$[\delta, \star]\phi = 0\,. \tag{4 rev.}$$

Variation of the covariant kinetic term reads

$$\begin{aligned}\delta(\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi) = \mathbb{d}(&\delta\phi^* \star \mathbb{D}\phi + \delta\phi \star \mathbb{D}\phi^*) \\ &+ \delta\phi^* \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi^* + \delta A \wedge \mathfrak{J}_A\,,\end{aligned} \tag{39}$$

where the covariant codifferential is defined in eq. (67), and

$$\mathfrak{J}_A \coloneqq \mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)\,. \tag{40}$$

A generic variation of the action then reads

$$\begin{aligned}\delta S = \int &\mathbb{d}(-\delta\phi^* \wedge \star\mathbb{D}\phi - \delta\phi \wedge \star\mathbb{D}\phi^* - \delta A \wedge \star F) \\ &- \delta\phi^* \wedge \star(\mathbb{D}^\dagger\mathbb{D} + m^2)\phi - \delta\phi \wedge \star(\mathbb{D}^\dagger\mathbb{D} + m^2)\phi^* \\ &- \delta A \wedge (\mathbb{d}\star F - \mathfrak{J}_A).\end{aligned} \tag{41}$$

4.1 Noether's invariances

The action is invariant under the generic transformation

$$\phi \to \mathrm{e}^{-\mathrm{i}e\Lambda}\phi\,, \quad \phi^* \to \mathrm{e}^{+\mathrm{i}e\Lambda}\phi^*\,, \quad A \to A - \mathbb{d}\Lambda\,. \tag{42}$$

There are two scenarios [1].

If the transformation is rigid, $\mathbb{d}\Lambda = 0$, one obtains $\mathfrak{J}_A$ as the Noether current from the boundary term as before, which satisfies the continuity equation $\mathbb{d}\mathfrak{J}_A = 0$.

If the transformation is gauge with a compact support, the boundary term can be dropped, and one obtains a Noether identity, which is under construction.

[1]

A Differential geometry

A.1 Basic operations

Following the style of [4], it is attempted to define the operations *inductively*, instead of giving explicit expressions, which is much less effective than the former in practice.

Interior product Let z be a vector, ω be a 1-form, χ be a k-form. The *interior product* is defined inductively as the bilinear map satisfying

$$z \lrcorner 1 = 0\,, \tag{43}$$

$$z \lrcorner \omega := \omega(z)\,, \tag{44}$$

$$z \lrcorner (\omega \wedge \chi) := (z \lrcorner \omega) \wedge \chi - \omega \wedge (z \lrcorner \chi)\,. \tag{45}$$

Equation (45) is also known as the anti-product rule.

By induction one can show that for a p-form ϕ,

$$z \lrcorner (\phi \wedge \chi) = (z \lrcorner \phi) \wedge \chi + (-)^p \chi \wedge (z \lrcorner \chi)\,. \tag{46}$$

Hodge star Let ω be an 1-form, χ be a k-form. The Hodge star $\star$ is defined inductively as the linear map

$$\star 1 := \mathrm{vol}\,, \tag{47}$$

$$\star(\chi \wedge \omega) := \omega^\sharp \lrcorner \star\chi\,. \tag{48}$$

Non-gravitational theories features $[\updelta, \star] = 0$, which means [7, sec. 3.2]

$$\updelta g_{\mu\nu} = -2\omega_{(\mu\nu)}, \qquad \updelta\vartheta^\mu = \omega_\nu{}^\mu\, \vartheta^\nu\,; \tag{49}$$

for an orthonormal coframe, the allowed variations are $\omega_{(\alpha\beta)} = 0$.

Identity Inspired by [9, eq. (3.167)], for a 1-form ω, k-form χ, one can derive

$$\begin{aligned}\omega \wedge \star\chi &= (-)^{D-k} \star \chi \wedge \omega = (-)^{D-k} \star^{-1} \left(\omega^\sharp \lrcorner \star\star\chi\right) \\ &= (-)^{D-k} (-)^{(D-k-1)(k+1)+s} \star \left(\omega^\sharp \lrcorner (-)^{k(D-k)+s} \star^{-1} \star\chi\right) \\ &= (-)^{k+1} \star \left(\omega^\sharp \lrcorner \chi\right).\end{aligned} \tag{50}$$

This will be useful in eqs. (59) to (61).

Codifferential The *codifferential* $\mathbb{d}^\dagger$ is defined as follows. Let ϕ be a k-form, χ be a $(k-1)$-form.

$$\mathbb{d}(\chi^* \wedge \star\phi) \equiv \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d}\star\phi =: \mathbb{d}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{d}^\dagger\phi \tag{51}$$

$$= \mathbb{d}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} \mathbb{d}\star\phi\,. \tag{52}$$

In other words,

$$\boxed{\mathbb{d}^\dagger\phi = (-)^k \star^{-1} \mathbb{d}\star\phi\,.} \tag{53}$$

If $\int \mathbb{d}(\chi^* \wedge \star\phi) = 0$, then $\mathbb{d}^\dagger$ is the adjoint of the exterior derivative $\mathbb{d}$ in functional sense.

A.2 Coframe bases

Coframes come into play if an explicit expression in components is wanted, or if gravitation comes into play.

For a detailed discussion about the meaning of untwisted (describing extensions) and twisted (describing densities) differential forms, see e.g. [4, sec. 22, 28].

Untwisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for untwisted k-form is defined inductively as

$$1\,, \tag{54}$$

$$\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} := \vartheta^{\alpha_1} \wedge \vartheta^{\alpha_2\cdots\alpha_k}\,. \tag{55}$$

Twisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for twisted $(D-k)$-form is defined inductively as

$$\epsilon := \mathrm{vol}\,, \tag{56}$$

$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} := e_{\alpha_k} \lrcorner\, \epsilon_{\alpha_1\cdots\alpha_{k-1}}\,. \tag{57}$$

By using eq. (48) and induction, one can show that

$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \star\vartheta_{\alpha_1\alpha_2\cdots\alpha_k}\,. \tag{58}$$

Identities The following identities come from eq. (50)

$$\vartheta^\alpha \wedge \epsilon_{\beta\gamma} = -\delta^\alpha_\beta\,\epsilon_\gamma + \delta^\alpha_\gamma\,\epsilon_\beta\,, \tag{59}$$

$$\vartheta^\alpha \wedge \epsilon_{\beta\gamma\delta} = \delta^\alpha_\beta\,\epsilon_{\gamma\delta} - \delta^\alpha_\gamma\,\epsilon_{\beta\delta} + \delta^\alpha_\delta\,\epsilon_{\beta\gamma}\,. \tag{60}$$

One can further deduce that

$$\begin{aligned}\vartheta^{\alpha\beta} \wedge \epsilon_{\gamma\delta\epsilon} = &+\Big(\delta^\alpha_\gamma\delta^\beta_\delta - \delta^\alpha_\delta\delta^\beta_\gamma\Big)\epsilon_\epsilon \\ &-\big(\delta^\alpha_\gamma\delta^\beta_\epsilon - \delta^\alpha_\epsilon\delta^\beta_\gamma\big)\epsilon_\delta \\ &+\Big(\delta^\alpha_\delta\delta^\beta_\epsilon - \delta^\alpha_\epsilon\delta^\beta_\delta\Big)\epsilon_\gamma\,.\end{aligned} \tag{61}$$

Equations (59) to (61) are useful in constructing explicit expressions in components.

Translation Consider the rigid infinitesimal transformation

$$\delta_\lambda\chi = \lambda\,£_z\chi = \lambda[z \lrcorner\, \mathbb{d}\chi + \mathbb{d}(z \lrcorner\, \chi)]\,; \tag{62}$$

'rigid' means

$$\delta_\lambda\vartheta^\alpha = 0\,. \tag{63}$$

A.3 Connection on the principal bundle

Exterior covariant derivative Let χ be a $\mathbb{C}$-valued k-form. The *exterior covariant derivative* of χ reads

$$\mathbb{D}\chi := (\mathbb{d} - \mathbb{i}eA)\chi, \qquad \mathbb{D}\chi^* := (\mathbb{d} + \mathbb{i}eA)\chi^*\,, \tag{64}$$

where A is a $\mathfrak{u}(1)$-valued *connection form.*

Covariant codifferential The *covariant codifferential* $\mathbb{D}^\dagger$ is defined as follows. Let ϕ be a $\mathbb{C}$-valued k-form, χ be a $\mathbb{C}$-valued $(k-1)$-form.

$$\begin{aligned} \mathbb{d}(\chi^* \wedge \star\phi) &\equiv \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi =: \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{D}^\dagger\phi \qquad (65) \\ &= \mathbb{D}\chi^* \wedge \star\phi - \mathrm{i}eA \wedge \chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi \\ &= \mathbb{D}\chi^* \wedge \star\phi + \chi^* \wedge (-)^k \mathrm{i}eA \star \phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi \\ &= \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \star \phi \, . \qquad (66) \end{aligned}$$

In other words,

$$\boxed{\mathbb{D}^\dagger \phi = (-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \star \phi \, .} \qquad (67)$$

If $\int \mathbb{d}(\chi^* \wedge \star\phi) = 0$, $\mathbb{D}^\dagger$ becomes the adjoint of the exterior covariant derivative $\mathbb{D}$ in the functional sense.

A.4 Variation of coframe bases

Upon variation of ϑ^α, $\vartheta^{\alpha_1\alpha_2\cdots\alpha_k}$ goes under

$$\delta\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge (e_\alpha \rfloor \vartheta^{\alpha_1\alpha_2\cdots\alpha_k}) \, , \qquad (68)$$

which can be proved by induction.

Upon variation of ϑ^α, $\epsilon_{\alpha_1\alpha_2\cdots\alpha_k}$ goes under [7, sec. A.2]

$$\delta\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge \left(e_\alpha \rfloor \epsilon_{\alpha_1\alpha_2\cdots\alpha_k}\right) . \qquad (69)$$

Variation of Hodge star In gravitational theories [7, sec. 3.2] with an orthonormal cobasis,

$$[\delta, \star]\phi = \delta\vartheta^\alpha \wedge (e_\alpha \rfloor \star\phi) - \star(\delta\vartheta^\alpha \wedge (e_\alpha \rfloor \phi)) \, . \qquad (70)$$

Let χ be a p-form, ϕ another form [5, sec. 5].

$$\begin{aligned} \delta(\chi \wedge \star\phi) &= \delta\chi \wedge \star\phi + \delta\phi \wedge \star\chi - \delta\vartheta^\alpha \wedge \Sigma_\alpha \, , \qquad (71) \\ \Sigma_\alpha &:= \chi \wedge \left\{\star(e_\alpha \rfloor \phi) - (-)^p (e_\alpha \rfloor \star\phi)\right\} . \qquad (72) \end{aligned}$$

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