Differential form

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April 29, 2019

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1 Covariant differential

ϕ a $\mathbb{C}$-valued 0-form

$$\mathbb{D}\phi := (\mathbb{d} - \mathrm{i}eA) \wedge \phi, \qquad \mathbb{D}\phi^* := (\mathbb{d} + \mathrm{i}eA) \wedge \phi^*, \tag{1.0.1}$$

A the $\mathfrak{u}(1)$-valued connection form.

2 Covariant codifferential

Define the covariant codifferential of a $\mathbb{C}$-valued k-form ζ as follows. Let η be an arbitrary $\mathbb{C}$-valued $(k-1)$-form.

$$\int \mathbb{d}(\eta^* \wedge \star\zeta) \equiv \int \mathbb{d}\eta^* \wedge \star\zeta - (-)^k \eta^* \wedge \mathbb{d} \star \zeta =: \int \mathbb{D}\eta^* \wedge \star\zeta - \eta^* \wedge \star\mathbb{D}^\dagger\zeta \tag{2.0.1}$$

$$\begin{aligned} &= \int \mathbb{D}\eta^* \wedge \star\zeta - \mathrm{i}eA \wedge \eta^* \wedge \star\zeta - (-)^k \eta^* \wedge \mathbb{d} \star \zeta \\ &= \int \mathbb{D}\eta^* \wedge \star\zeta + \eta^* \wedge (-)^k \mathrm{i}eA \wedge \star\zeta - (-)^k \eta^* \wedge \mathbb{d} \star \zeta \\ &= \int \mathbb{D}\eta^* \wedge \star\zeta - \eta^* \wedge \star(-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\zeta \,. \end{aligned} \tag{2.0.2}$$

$$\boxed{\mathbb{D}^\dagger\zeta = (-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\zeta \,.} \tag{2.0.3}$$

3 Maxwell–Klein–Fock–Gordon theory

$$S = \int -\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi - m^2 \phi^* \wedge \star\phi - \frac{1}{2} F \wedge \star F. \tag{3.0.1}$$

$$\delta\mathbb{D}\phi = -\mathrm{i}e\delta A\phi + \mathbb{D}\delta\phi. \tag{3.0.2}$$

$$\begin{aligned}\delta(\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi) = \mathbb{d}(&\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^*) \\ &+ \delta\phi^* \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi^* \\ &+ \delta A \wedge \left(\mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)\right), \end{aligned} \tag{3.0.3}$$

$$\delta(F \wedge \star F) = 2\mathbb{d}(\delta A \wedge \star F) - 2\delta A \wedge \mathbb{d}\star F\,. \tag{3.0.4}$$

$$\begin{aligned}\delta S = \int &-\mathbb{d}(\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^* + \delta A \wedge \star F) \\ &+ \delta\phi^* \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi + \delta\phi \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi^* + \\ &+ \delta A \wedge (-\mathbb{d}\star F + \mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)). \end{aligned} \tag{3.0.5}$$

3.1 Lorenz gauge

$$\Box^2 := \left(\mathbb{d} + \mathbb{d}^\dagger\right)^2 = \mathbb{d}\mathbb{d}^\dagger + \mathbb{d}^\dagger\mathbb{d}. \tag{3.1.1}$$

$$\mathbb{d}\star F = \mathbb{d}\star\mathbb{d}A = \star(-)^2 \star^{-1} \mathbb{d}\star\mathbb{d}A = \star\mathbb{d}^\dagger\mathbb{d}A = \star(\Box^2 - \mathbb{d}\mathbb{d}^\dagger)A\,. \tag{3.1.2}$$

One would like to have $\mathbb{d}\mathbb{d}^\dagger A = 0$, or $\mathbb{d}^\dagger A = \text{const}$. This would be fulfiled if

$$\mathbb{d}^\dagger A = 0\,, \tag{3.1.3}$$

which is the Lorenz gauge[1–3].