Tool-kit for the coframe formalism

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Contents

1 Complex scalar field

The action reads

$$S = \int -\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi - m^2\phi^* \wedge \star\phi \tag{1}$$

$$= \int -\frac{1}{2}(\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi + \mathbb{d}\phi \wedge \star\mathbb{d}\phi^*) - \frac{1}{2}m^2(\phi^* \wedge \star\phi + \phi \wedge \star\phi^*) \tag{2}$$

Variation commutes with exterior derivative

$$[\delta, \mathbb{d}]\phi = 0\,. \tag{3}$$

A generic variation of the kinetic terms reads

$$\begin{aligned}\delta(\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi) = \mathbb{d}(&\delta\phi^* \wedge \star\mathbb{d}\phi + \delta\phi \wedge \star\mathbb{d}\phi^*) \\ &+ \delta\phi^* \wedge \star\mathbb{d}^\dagger\mathbb{d}\phi + \delta\phi \wedge \star\mathbb{d}^\dagger\mathbb{d}\phi^*\,.\end{aligned} \tag{4}$$

A generic variation of the actionreads

$$\delta S = \int \mathbb{d}(-\delta\phi^* \wedge \star\mathbb{d}\phi - \delta\phi \wedge \star\mathbb{d}\phi^*) \\ - \delta\phi^* \wedge \star(\mathbb{d}^\dagger\mathbb{d} + m^2)\phi - \delta\phi \wedge \star(\mathbb{d}^\dagger\mathbb{d} + m^2)\phi^* \,. \tag{5}$$

1.1 $\mathsf{U}(1)$ Noether current

The action is invariant

$$\delta_\lambda S \equiv 0 \tag{6}$$

under the rigid transformation

$$\phi \to \mathsf{e}^{-\mathrm{i}e\lambda}\phi\,, \qquad \phi^* \to \mathsf{e}^{+\mathrm{i}e\lambda}\phi^*\,. \tag{7}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$0 = \int \lambda\, \mathbb{d}\mathfrak{J}_0\,, \\ \mathfrak{J}_0 := \mathrm{i}e(\phi^* \wedge \star\mathbb{d}\phi - \phi \wedge \star\mathbb{d}\phi^*)\,, \tag{8}$$

which is the Noether current, a twisted 3-form, satisfying the continuity equation

$$\mathbb{d}\mathfrak{J}_0 = 0\,. \tag{9}$$

1.2 $\mathsf{T}(D)$ Noether current

The action is invariant up to a total differential

$$\delta_\lambda S = \int \lambda\, \mathbb{d}(z \lrcorner \mathfrak{L}) \\ = \int \lambda\, \frac{1}{2}\, \mathbb{d}\{-z \lrcorner [\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi + \mathbb{d}\phi \wedge \star\mathbb{d}\phi^* + m^2(\phi^* \star \phi + \phi \star \phi^*)]\}\,, \tag{10}$$

under the rigid infinitesimal transformation

$$\delta\phi = \lambda\, \pounds_z\phi = \lambda\, z \lrcorner \mathbb{d}\phi\,, \qquad \delta\phi^* = \lambda\, \pounds_z\phi^* = \lambda\, z \lrcorner \mathbb{d}\phi^*\,. \tag{11}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$\delta_\lambda S = \int \lambda\, \mathbb{d}\{-(z \lrcorner \mathbb{d}\phi^*) \wedge \star\mathbb{d}\phi - (z \lrcorner \mathbb{d}\phi) \wedge \star\mathbb{d}\phi^*\}\,. \tag{12}$$

Taking the difference and setting $z = e_\alpha$ yields

$$0 = \int \lambda\, \mathbb{d}\mathfrak{T}_\alpha, \tag{13}$$

$$2\mathfrak{T}_\alpha := -(e_\alpha \lrcorner \mathbb{d}\phi^*) \wedge \star\mathbb{d}\phi - \mathbb{d}\phi^* \wedge (e_\alpha \lrcorner \star\mathbb{d}\phi) \\ - (e_\alpha \lrcorner \mathbb{d}\phi) \wedge \star\mathbb{d}\phi^* - \mathbb{d}\phi \wedge (e_\alpha \lrcorner \star\mathbb{d}\phi^*) \\ + m^2(\phi^*\, e_\alpha \lrcorner \star\phi + \phi\, e_\alpha \star \phi^*)\,. \tag{14}$$

The continuity equation reads

$$\mathbb{d}\mathfrak{T}_\alpha = 0\,. \tag{15}$$

In components,

$$\mathfrak{T}_\alpha = -\left[e_\alpha(\phi^*) e^\beta(\phi) + e_\alpha(\phi) e^\beta(\phi^*)\right] \epsilon_\beta + \left[e^\beta(\phi^*) e_\beta(\phi^*) + m^2 \phi^* \phi\right] \epsilon_\alpha\,. \tag{16}$$

2 Pure electromagnetic field

2.1 Lorenz gauge

The Laplace–de Rham operator, or in our Lorentzian metric signature the d'Alembertian

$$\Box^2 := \left(\mathbb{d} + \mathbb{d}^\dagger\right)^2 = \mathbb{d}\mathbb{d}^\dagger + \mathbb{d}^\dagger\mathbb{d}\,. \tag{17}$$

$$\mathbb{d}{\star}F = \mathbb{d} \star \mathbb{d}A = \star(-)^2 \star^{-1} \mathbb{d} \star \mathbb{d}A = \star\mathbb{d}^\dagger\mathbb{d}A = \star(\Box^2 - \mathbb{d}\mathbb{d}^\dagger)A\,. \tag{18}$$

One would like to have $\mathbb{d}\mathbb{d}^\dagger A = 0$, or $\mathbb{d}^\dagger A = \text{const}$. This would be fulfilled if

$$\mathbb{d}^\dagger A = 0\,, \tag{19}$$

which is the Lorenz gauge[2, 3, 6].

2.2 $\mathsf{T}(D)$ Noether current

The action is invariant up to a total differential

$$\delta_\lambda S = \int \lambda\, \mathbb{d}(z \lrcorner \mathfrak{L}) = \int \lambda\, \frac{1}{2}\, \mathbb{d}\{-z \lrcorner (F \wedge \star F)\}\,, \tag{20}$$

under the rigid infinitesimal transformation combined with a gauge transformation [8, eq. 3.46]

$$\delta A = \lambda\{z \lrcorner \mathbb{d}(A - \mathbb{d}\Lambda) + \mathbb{d}[z \lrcorner (A - \mathbb{d}\Lambda)]\}\,. \tag{21}$$

Choosing

$$\mathbb{d}\Lambda = A(z)\, z^\flat \tag{22}$$

makes the second term vanish, yielding

$$\delta A = \lambda\, z \lrcorner \mathbb{d}A = \lambda\, z \lrcorner F\,. \tag{23}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$\delta_\lambda S = \int \lambda\, \mathbb{d}\{-(z \lrcorner F) \wedge \star F\}\,. \tag{24}$$

Taking the difference and setting $z = e_\alpha$ yields

$$0 = \int \lambda \, \mathrm{d}\mathfrak{T}_\alpha, \tag{25}$$

$$2\mathfrak{T}_\alpha := -(e_\alpha \lrcorner F) \wedge \star F + F \wedge (e_\alpha \lrcorner \star F) \,. \tag{26}$$

The continuity equation reads

$$\mathrm{d}\mathfrak{T}_\alpha = 0 \,. \tag{27}$$

In components, one needs

$$e_\alpha \lrcorner F = F_{\alpha\beta} \, \vartheta^\beta \,, \tag{28}$$

$$e_\alpha \lrcorner \star F = \frac{1}{2} F^{\beta\gamma} \, \epsilon_{\beta\gamma\alpha} \,; \tag{29}$$

so that

$$(e_\alpha \lrcorner F) \wedge \star F = -F_{\alpha\gamma} F^{\gamma\beta} \, \epsilon_\beta \,, \tag{30}$$

$$F \wedge (e_\alpha \lrcorner \star F) = \frac{1}{2} F_{\beta\gamma} F^{\beta\gamma} \, \epsilon_\alpha + F_{\alpha\gamma} F^{\gamma\beta} \, \epsilon_\beta \,. \tag{31}$$

One finally has

$$\mathfrak{T}_\alpha = \frac{1}{4} F_{\beta\gamma} F^{\beta\gamma} \, \epsilon_\alpha + F_{\alpha\gamma} F^{\gamma\beta} \, \epsilon_\beta \,. \tag{32}$$

3 $\mathsf{U}(1)$-gauged complex scalar field theory

$$S = \int -\mathbb{D}\phi^* \wedge \star \mathbb{D}\phi - m^2 \phi^* \wedge \star \phi - \frac{1}{2} F \wedge \star F. \tag{33}$$

Variation does not commute with exterior covariant derivative.

$$[\delta, \mathbb{D}]\phi = -\mathrm{i}e\delta A\phi \,. \tag{34}$$

$$\begin{aligned} \delta(\mathbb{D}\phi^* \wedge \star \mathbb{D}\phi) = {} & \mathrm{d}(\delta\phi^* \star \mathbb{D}\phi + \delta\phi \star \mathbb{D}\phi^*) + \delta\phi^* \wedge \star \mathbb{D}^\dagger \mathbb{D}\phi + \delta\phi \wedge \star \mathbb{D}^\dagger \mathbb{D}\phi^* \\ & + \delta A \wedge \mathfrak{J}_A \,, \qquad \mathfrak{J}_A := \mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*) \,, \end{aligned} \tag{35}$$

$$\delta(F \wedge \star F) = 2\, \delta F \wedge \star F \tag{36}$$

$$= 2\, \mathrm{d}(\delta A \wedge \star F) + 2\, \delta A \wedge \mathrm{d}\star F \,. \tag{37}$$

Variation of the action

$$\begin{aligned} \delta S = \int & \mathrm{d}(-\delta\phi^* \wedge \star \mathbb{D}\phi - \delta\phi \wedge \star \mathbb{D}\phi^* - \delta A \wedge \star F) \\ & - \delta\phi^* \wedge \star(\mathbb{D}^\dagger \mathbb{D} + m^2)\phi - \delta\phi \wedge \star(\mathbb{D}^\dagger \mathbb{D} + m^2)\phi^* \\ & - \delta A \wedge (\mathrm{d}\star F - \mathfrak{J}_A). \end{aligned} \tag{38}$$

3.1 Noether's invariances

The action is invariant under the generic transformation

$$\phi \to \mathsf{e}^{-\mathrm{i}e\Lambda}\phi\,, \quad \phi^* \to \mathsf{e}^{+\mathrm{i}e\Lambda}\phi^*\,, \quad A \to A - \mathbb{d}\Lambda\,. \tag{39}$$

There are two scenarios [1].

If the transformation is rigid, $\mathbb{d}\Lambda = 0$, one obtains $\mathfrak{J}_A$ as the Noether current from the boundary term as before, which satisfies the continuity equation $\mathbb{d}\mathfrak{J}_A = 0$.

If the transformation is gauge with a compact support, the boundary term can be dropped, and one obtains the Noether identity

$$123 \tag{40}$$

$$\begin{aligned} \Sigma_\alpha = &-\mathbb{D}\phi^* \wedge \{\star(e_\alpha \lrcorner \mathbb{D}\phi) + (e_\alpha \lrcorner \star\mathbb{D}\phi)\} - m^2\phi^*\phi\,\epsilon_\alpha \\ &- \frac{1}{2}F \wedge \{\star(e_\alpha \lrcorner F) - (e_\alpha \lrcorner \star F)\}\,. \end{aligned} \tag{41}$$

3.2 $\mathsf{T}(D)$ Noether current

[1]

A Differential geometry

A.1 Basic operations

Following the style of [4, sec. 24], it is attempted to define the operations inductively, instead of giving explicit expressions, which is much less effective than the former in practice.

Interior product Let z be a vector, ω be a 1-form, χ be a k-form. The *interior product* is defined inductively as the bilinear map satisfying

$$z \lrcorner \omega := \omega(z)\,, \tag{42}$$

$$z \lrcorner (\omega \wedge \chi) := (z \lrcorner \omega) \wedge \chi - \omega \wedge (z \lrcorner \chi)\,. \tag{43}$$

Equation (43) is also known as the anti-product rule.

By induction one can show that for a p-form ϕ,

$$z \lrcorner (\phi \wedge \chi) = (z \lrcorner \phi) \wedge \chi + (-)^p \chi \wedge (z \lrcorner \chi)\,. \tag{44}$$

Hodge star Let ω be an 1-form, χ be a k-form. The Hodge star $\star$ is defined inductively as the linear map

$$\star 1 := \mathrm{vol}\,, \tag{45}$$

$$\star(\chi \wedge \omega) := \omega^\sharp \lrcorner \star\chi\,. \tag{46}$$

Non-gravitational theories features $[\delta, \star] = 0$, which means [7, sec. 3.2]

$$\delta g_{\mu\nu} = -2\omega_{(\mu\nu)}, \qquad \delta\vartheta^\mu = \omega_\nu{}^\mu\,\vartheta^\nu\,; \tag{47}$$

for an orthonormal coframe, the allowed variations are $\omega_{(\alpha\beta)} = 0$.

Identities Inspired by [9, eq. (3.167)], for a 1-form ω, k-form χ, one can derive

$$\begin{aligned}\omega \wedge \star\chi &= (-)^{D-k} \star \chi \wedge \omega = (-)^{D-k} \star^{-1} \left(\omega^\sharp \lrcorner \star\star\chi\right) \\ &= (-)^{D-k} (-)^{(D-k-1)(k+1)+s} \star \left(\omega^\sharp \lrcorner (-)^{k(D-k)+s} \star^{-1} \star\chi\right) \\ &= (-)^{k+1} \star \left(\omega^\sharp \lrcorner \chi\right) . \end{aligned} \tag{48}$$

Codifferential The *codifferential* $\mathbb{d}^\dagger$ is the adjoint of the exterior derivative $\mathbb{d}$ in the following sense. Let ϕ be a k-form, χ be a $(k-1)$-form.

$$\mathbb{d}(\chi^* \wedge \star\phi) \equiv \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi =: \mathbb{d}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{d}^\dagger\phi \tag{49}$$

$$= \mathbb{d}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} \mathbb{d} \star \phi . \tag{50}$$

$$\boxed{\mathbb{d}^\dagger \phi = (-)^k \star^{-1} \mathbb{d} \star \phi .} \tag{51}$$

A.2 Coframe bases

Untwisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for untwisted k-form is defined inductively as

$$1 , \tag{52}$$

$$\vartheta^{\alpha_1 \alpha_2 \cdots \alpha_k} := \vartheta^{\alpha_1} \wedge \vartheta^{\alpha_2 \cdots \alpha_k} . \tag{53}$$

Twisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for twisted $(D-k)$-form is defined inductively as

$$\epsilon := \mathrm{vol} , \tag{54}$$

$$\epsilon_{\alpha_1 \alpha_2 \cdots \alpha_k} := e_{\alpha_k} \lrcorner \epsilon_{\alpha_1 \cdots \alpha_{k-1}} . \tag{55}$$

By using eq. (46) and induction, one can show that

$$\epsilon_{\alpha_1 \alpha_2 \cdots \alpha_k} = \star\vartheta_{\alpha_1 \alpha_2 \cdots \alpha_k} . \tag{56}$$

Identities

$$\vartheta^\alpha \wedge \epsilon_{\beta\gamma} = -\delta^\alpha_\beta \, \epsilon_\gamma + \delta^\alpha_\gamma \, \epsilon_\beta , \tag{57}$$

$$\vartheta^\alpha \wedge \epsilon_{\beta\gamma\delta} = \delta^\alpha_\beta \, \epsilon_{\gamma\delta} - \delta^\alpha_\gamma \, \epsilon_{\beta\delta} + \delta^\alpha_\delta \, \epsilon_{\beta\gamma} . \tag{58}$$

One can further deduce that

$$\begin{aligned}\vartheta^{\alpha\beta} \wedge \epsilon_{\gamma\delta\epsilon} = &\left(\delta^\alpha_\gamma \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\gamma\right)\epsilon_\epsilon \\ &- \left(\delta^\alpha_\gamma \delta^\beta_\epsilon - \delta^\alpha_\epsilon \delta^\beta_\gamma\right)\epsilon_\delta \\ &+ \left(\delta^\alpha_\delta \delta^\beta_\epsilon - \delta^\alpha_\epsilon \delta^\beta_\delta\right)\epsilon_\gamma . \end{aligned} \tag{59}$$

A.3 Connection on the principal bundle

Exterior covariant derivative Let χ be a $\mathbb{C}$-valued k-form. The *exterior covariant derivative* of χ reads

$$\mathbb{D}\chi := (\mathbb{d} - \mathbb{i}eA)\chi, \qquad \mathbb{D}\chi^* := (\mathbb{d} + \mathbb{i}eA)\chi^* , \tag{60}$$

where A is a $\mathfrak{u}(1)$-valued *connection form*.

Covariant codifferential The *covariant codifferential* $\mathbb{D}^\dagger$ is the adjoint of the exterior covariant derivative $\mathbb{D}$ in the following sense. Let ϕ be a $\mathbb{C}$-valued k-form, χ be a $\mathbb{C}$-valued $(k-1)$-form.

$$\mathrm{d}(\chi^* \wedge \star\phi) \equiv \mathrm{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathrm{d} \star \phi =: \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{D}^\dagger\phi \tag{61}$$

$$\begin{aligned} &= \mathbb{D}\chi^* \wedge \star\phi - \mathrm{i}eA \wedge \chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathrm{d} \star \phi \\ &= \mathbb{D}\chi^* \wedge \star\phi + \chi^* \wedge (-)^k \mathrm{i}eA \star \phi - (-)^k \chi^* \wedge \mathrm{d} \star \phi \\ &= \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} (\mathrm{d} - \mathrm{i}eA) \star \phi\,. \end{aligned} \tag{62}$$

$$\boxed{\mathbb{D}^\dagger\phi = (-)^k \star^{-1} (\mathrm{d} - \mathrm{i}eA) \star \phi\,.} \tag{63}$$

Translation Consider the rigid infinitesimal transformation

$$\delta_\lambda \chi = \lambda\, \pounds_z \chi = \lambda[z \lrcorner \mathrm{d}\chi + \mathrm{d}(z \lrcorner \chi)]\,; \tag{64}$$

'rigid' means

$$\delta_\lambda \vartheta^\alpha = 0\,. \tag{65}$$

A.4 Variation of coframe bases

Upon variation of ϑ^α, $\vartheta^{\alpha_1\alpha_2\cdots\alpha_k}$ goes under

$$\delta\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge (e_\alpha \lrcorner \vartheta^{\alpha_1\alpha_2\cdots\alpha_k})\,, \tag{66}$$

which can be proved by induction.

Upon variation of ϑ^α, $\epsilon_{\alpha_1\alpha_2\cdots\alpha_k}$ goes under [7, sec. A.2]

$$\delta\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge \left(e_\alpha \lrcorner \epsilon_{\alpha_1\alpha_2\cdots\alpha_k}\right). \tag{67}$$

Variation of Hodge star In gravitational theories [7, sec. 3.2] with an orthonormal cobasis,

$$[\delta, \star]\phi = \delta\vartheta^\alpha \wedge (e_\alpha \lrcorner \star\phi) - \star(\delta\vartheta^\alpha \wedge (e_\alpha \lrcorner \phi))\,. \tag{68}$$

Let χ be a p-form, ϕ another form [5, sec. 5].

$$\delta(\chi \wedge \star\phi) = \delta\chi \wedge \star\phi + \delta\phi \wedge \star\chi - \delta\vartheta^\alpha \wedge \Sigma_\alpha\,, \tag{69}$$

$$\Sigma_\alpha := \chi \wedge \left\{\star(e_\alpha \lrcorner \phi) - (-)^p (e_\alpha \lrcorner \star\phi)\right\}. \tag{70}$$

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