Coframe formalisms

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1 Regular theories

1.1 Differential geometry

Interior product Let z be a vector, ω be a 1-form, χ be a k-form. The *interior product* is defined inductively as the bilinear map satisfying

$$z \lrcorner \omega := \omega(z)\,, \tag{1}$$

$$z \lrcorner (\omega \wedge \chi) := (z \lrcorner \omega) \wedge \chi - \omega \wedge (z \lrcorner \chi)\,. \tag{2}$$

Equation (2) is also known as the anti-product rule.

By inductiion one can show that for a p-form ϕ,

$$z \lrcorner (\phi \wedge \chi) = (z \lrcorner \phi) \wedge \chi + (-)^p \chi \wedge (z \lrcorner \chi)\,. \tag{3}$$

Hodge star Let ω be an 1-form, χ be a k-form. The Hodge star $\star$ is defined inductively as the linear map [4, sec. 24]

$$\star 1 := \mathrm{vol}\,, \tag{4}$$

$$\star(\chi \wedge \omega) := \omega^\sharp \lrcorner \star\chi\,. \tag{5}$$

Non-gravitational theories features $[\delta, \star] = 0$, which means [7, sec. 3.2]

$$\delta g_{\mu\nu} = -2\omega_{(\mu\nu)}, \qquad \delta\vartheta^\mu = \omega_\nu{}^\mu\, \vartheta^\nu\,; \tag{6}$$

for an orthonormal coframe, the allowed variations are $\omega_{(\alpha\beta)} = 0$.

1.2 Complex Klein–Fock–Gordon theory

$$S = \int -\mathrm{d}\phi^* \wedge \star\mathrm{d}\phi - m^2\phi^* \wedge \star\phi\,. \tag{7}$$

$$[\delta\mathbb{D} - \mathbb{D}\delta]\phi = 0\,. \tag{8}$$

$$\begin{aligned}\delta(\mathrm{d}\phi^* \wedge \star\mathrm{d}\phi) = \mathrm{d}(\delta\phi^* \wedge \star\mathrm{d}\phi + \delta\phi \wedge \star\mathrm{d}\phi^*)\\ - \delta\phi^* \wedge \star\mathrm{d}^\dagger\mathrm{d}\phi - \delta\phi \wedge \star\mathrm{d}^\dagger\mathrm{d}\phi^*\,.\end{aligned} \tag{9}$$

$$\begin{aligned}\delta S = \int -\mathrm{d}(\delta\phi^* \wedge \star\mathrm{d}\phi + \delta\phi \wedge \star\mathrm{d}\phi^*)\\ + \delta\phi^* \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi + \delta\phi \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi^*\,.\end{aligned} \tag{10}$$

1.2.1 Noether current

The theory is invariant under the rigid transformation

$$\phi \to \mathrm{e}^{-\mathrm{i}e\Lambda}\phi\,, \qquad \phi^* \to \mathrm{e}^{+\mathrm{i}e\Lambda}\phi^*\,, \tag{11}$$

i.e. $\delta S = 0$. When the equations of motion are satisfied, infinitesimal transformation leads to

$$0 = \mathrm{d}\mathfrak{J}\,, \tag{12}$$

$$\mathfrak{J} := e(\phi^* \wedge \star\mathrm{d}\phi + \phi \wedge \star\mathrm{d}\phi^*)\,. \tag{13}$$

2 Differential geometry

2.1 Connection on the principal bundle

Covariant differential Let χ be a $\mathbb{C}$-valued k-form. The *covariant differential* of χ reads

$$\mathbb{D}\chi := (\mathrm{d} - \mathrm{i}eA) \wedge \chi, \qquad \mathbb{D}\chi^* := (\mathrm{d} + \mathrm{i}eA) \wedge \chi^*\,, \tag{14}$$

where A is a $\mathfrak{u}(1)$-valued *connection form.*

Covariant codifferential The *covariant codifferential* $\mathbb{D}^\dagger$ is the adjoint of the covariant differential $\mathbb{D}$ in the following sense. Let ϕ be a $\mathbb{C}$-valued k-form, χ be a $\mathbb{C}$-valued $(k-1)$-form.

$$\begin{aligned}
\int \mathbb{d}(\chi^* \wedge \star\phi) &\equiv \int \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi =: \int \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{D}^\dagger\phi \qquad (15)\\
&= \int \mathbb{D}\chi^* \wedge \star\phi - \mathrm{i}eA \wedge \chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi\\
&= \int \mathbb{D}\chi^* \wedge \star\phi + \chi^* \wedge (-)^k \mathrm{i}eA \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi\\
&= \int \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\phi\,. \qquad (16)
\end{aligned}$$

$$\boxed{\mathbb{D}^\dagger\phi = (-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\phi\,.} \tag{17}$$

2.2 Maxwell–Klein–Fock–Gordon theory

$$S = \int -\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi - m^2\phi^* \wedge \star\phi - \frac{1}{2}F \wedge \star F. \tag{18}$$

$$\delta\mathbb{D}\phi = -\mathrm{i}e\delta A\phi + \mathbb{D}\delta\phi. \tag{19}$$

$$\begin{aligned}
\delta(\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi) &= \mathbb{d}(\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^*)\\
&\quad - \delta\phi^* \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi - \delta\phi \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi^* \qquad (20)\\
&\quad + \delta A \wedge (\mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*))\,,\\
\delta(F \wedge \star F) &= 2\mathbb{d}(\delta A \wedge \star F) - 2\delta A \wedge \mathbb{d}\star F\,. \qquad (21)
\end{aligned}$$

$$\begin{aligned}
\delta S = \int & -\mathbb{d}(\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^* + \delta A \wedge \star F)\\
&+ \delta\phi^* \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi + \delta\phi \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi^*\\
&+ \delta A \wedge (\mathbb{d}\star F - \mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)). \qquad (22)
\end{aligned}$$

2.2.1 Lorenz gauge

The Laplace–de Rham operator, or in our metric signature the d'Alembertian

$$\Box^2 := \left(\mathbb{d} + \mathbb{d}^\dagger\right)^2 = \mathbb{d}\mathbb{d}^\dagger + \mathbb{d}^\dagger\mathbb{d}\,. \tag{23}$$

$$\mathbb{d}\star F = \mathbb{d} \star \mathbb{d}A = \star(-)^2 \star^{-1} \mathbb{d} \star \mathbb{d}A = \star\mathbb{d}^\dagger\mathbb{d}A = \star(\Box^2 - \mathbb{d}\mathbb{d}^\dagger)A\,. \tag{24}$$

One would like to have $\mathbb{d}\mathbb{d}^\dagger A = 0$, or $\mathbb{d}^\dagger A = \mathrm{const}$. This would be fulfiled if

$$\mathbb{d}^\dagger A = 0\,, \tag{25}$$

which is the Lorenz gauge[2, 3, 6].

2.2.2 Noether's invariances

[1]

3 Gravitational theory

3.1 Differential forms

Untwisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for untwisted k-form is defined inductively as

$$1\,, \tag{26}$$
$$\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} := \vartheta^{\alpha_1} \wedge \vartheta^{\alpha_2\cdots\alpha_k}\,. \tag{27}$$

Upon variation of ϑ^α, $\vartheta^{\alpha_1\alpha_2\cdots\alpha_k}$ goes under

$$\delta\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge (e_\alpha \rfloor \vartheta^{\alpha_1\alpha_2\cdots\alpha_k})\,, \tag{28}$$

which can be proved by induction.

Twisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for twisted $(D-k)$-form is defined inductively as

$$\epsilon := \mathrm{vol}\,, \tag{29}$$
$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} := e_{\alpha_k} \rfloor \epsilon_{\alpha_1\cdots\alpha_{k-1}}\,. \tag{30}$$

By using eq. (5) and induction, one can show that

$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \star\vartheta_{\alpha_1\alpha_2\cdots\alpha_k}\,. \tag{31}$$

Upon variation of ϑ^α, $\epsilon_{\alpha_1\alpha_2\cdots\alpha_k}$ goes under [7, sec. A.2]

$$\delta\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge \left(e_\alpha \rfloor \epsilon_{\alpha_1\alpha_2\cdots\alpha_k}\right)\,. \tag{32}$$

Variation of Hodge star In gravitational theories [7, sec. 3.2] with an orthonormal cobasis,

$$[\delta, \star]\phi = \delta\vartheta^\alpha \wedge (e_\alpha \rfloor \star\phi) - \star(\delta\vartheta^\alpha \wedge (e_\alpha \rfloor \phi))\,. \tag{33}$$

Let χ be a p-form, ϕ another form [5, sec. 5].

$$\delta(\chi \wedge \star\phi) = \delta\chi \wedge \star\phi + \delta\phi \wedge \star\chi - \delta\vartheta^\alpha \wedge \Sigma_\alpha\,, \tag{34}$$
$$\Sigma_\alpha := \chi \wedge \left\{\star(e_\alpha \rfloor \phi) - (-)^p (e_\alpha \rfloor \star\phi)\right\}\,. \tag{35}$$

3.2 Maxwell–Klein–Fock–Gordon theory

$$\begin{aligned}\Sigma_\alpha = &-\mathbb{D}\phi^* \wedge \{\star(e_\alpha \rfloor \mathbb{D}\phi) + (e_\alpha \rfloor \star\mathbb{D}\phi)\} - m^2\phi^*\phi\,\epsilon_\alpha \\ &- \frac{1}{2}F \wedge \{\star(e_\alpha \rfloor F) - (e_\alpha \rfloor \star F)\}\,.\end{aligned} \tag{36}$$

3.2.1 Noether's invariances

[1]

Rigid translation

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