Coframe formalisms

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1 U(1)–regular theory

1.1 Differential geometry

Interior product Let z be a vector, ω be a 1-form, χ be a k-form. The *interior product* is defined inductively as the bilinear map satisfying

$$z \lrcorner \omega := \omega(z)\,, \tag{1}$$

$$z \lrcorner (\omega \wedge \chi) := (z \lrcorner \omega) \wedge \chi - \omega \wedge (z \lrcorner \chi)\,. \tag{2}$$

Equation (2) is also known as the anti-product rule.

By inductiion one can show that for a p-form ϕ,

$$z \lrcorner (\phi \wedge \chi) = (z \lrcorner \phi) \wedge \chi + (-)^p \chi \wedge (z \lrcorner \chi)\,. \tag{3}$$

Hodge star Let ω be an 1-form, χ be a k-form. The Hodge star $\star$ is defined inductively as the linear map [4, sec. 24]

$$\star 1 := \mathrm{vol}\,, \tag{4}$$

$$\star(\chi \wedge \omega) := \omega^\sharp \lrcorner \star\chi\,. \tag{5}$$

Non-gravitational theories features $[\delta, \star] = 0$, which means [7, sec. 3.2]

$$\delta g_{\mu\nu} = -2\omega_{(\mu\nu)}, \qquad \delta\vartheta^\mu = \omega_\nu{}^\mu\,\vartheta^\nu\,; \tag{6}$$

for an orthonormal coframe, the allowed variations are $\omega_{(\alpha\beta)} = 0$.

Codifferential The *codifferential* $\mathbb{d}^\dagger$ is the adjoint of the exterior derivative $\mathbb{d}$ in the following sense. Let ϕ be a k-form, χ be a $(k-1)$-form.

$$\mathbb{d}(\chi^* \wedge \star\phi) \equiv \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d}\star\phi =: \mathbb{d}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{d}^\dagger\phi \tag{7}$$

$$= \mathbb{d}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} \mathbb{d}\star\phi\,. \tag{8}$$

$$\boxed{\mathbb{d}^\dagger\phi = (-)^k \star^{-1} \mathbb{d}\star\phi\,.} \tag{9}$$

1.2 Complex scalar field

The action reads

$$S = \int -\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi - m^2\phi^* \wedge \star\phi \tag{10}$$

$$= \int -\frac{1}{2}(\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi + \mathbb{d}\phi \wedge \star\mathbb{d}\phi^*) - \frac{1}{2}m^2(\phi^* \wedge \star\phi + \phi \wedge \star\phi^*) \tag{11}$$

Variation commutes with exterior derivative

$$[\delta, \mathbb{d}]\phi = 0\,. \tag{12}$$

A generic variation of the kinetic terms reads

$$\begin{aligned}\delta(\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi) &= \mathbb{d}(\delta\phi^* \wedge \star\mathbb{d}\phi + \delta\phi \wedge \star\mathbb{d}\phi^*) \\ &\quad + \delta\phi^* \wedge \star\mathbb{d}^\dagger\mathbb{d}\phi + \delta\phi \wedge \star\mathbb{d}^\dagger\mathbb{d}\phi^*\,.\end{aligned} \tag{13}$$

A generic variation of the actionreads

$$\begin{aligned}\delta S = \int &\mathbb{d}(-\delta\phi^* \wedge \star\mathbb{d}\phi - \delta\phi \wedge \star\mathbb{d}\phi^*) \\ &- \delta\phi^* \wedge \star(\mathbb{d}^\dagger\mathbb{d} + m^2)\phi - \delta\phi \wedge \star(\mathbb{d}^\dagger\mathbb{d} + m^2)\phi^*\,.\end{aligned} \tag{14}$$

1.2.1 Noether current

The action is invariant

$$\delta_\lambda S \equiv 0 \tag{15}$$

under the rigid transformation

$$\phi \to \mathsf{e}^{-\mathrm{i}e\lambda}\phi\,, \qquad \phi^* \to \mathsf{e}^{+\mathrm{i}e\lambda}\phi^*\,. \tag{16}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$
\begin{aligned}
0 &= \int \lambda\, \mathbb{d}\mathfrak{J}_0\,, \\
\mathfrak{J}_0 &:= \mathrm{i}e(\phi^* \wedge \star\mathbb{d}\phi - \phi \wedge \star\mathbb{d}\phi^*)\,,
\end{aligned} \tag{17}
$$

which is the Noether current, a twisted 3-form, satisfying the continuity equation

$$
\mathbb{d}\mathfrak{J}_0 = 0\,. \tag{18}
$$

2 $\mathsf{U}(1)$ gauge theory

2.1 Connection on the principal bundle

Exterior covariant derivative Let χ be a $\mathbb{C}$-valued k-form. The *exterior covariant derivative* of χ reads

$$
\mathbb{D}\chi := (\mathbb{d} - \mathrm{i}eA)\chi, \qquad \mathbb{D}\chi^* := (\mathbb{d} + \mathrm{i}eA)\chi^*\,, \tag{19}
$$

where A is a $\mathfrak{u}(1)$-valued *connection form.*

Covariant codifferential The *covariant codifferential* $\mathbb{D}^\dagger$ is the adjoint of the exterior covariant derivative $\mathbb{D}$ in the following sense. Let ϕ be a $\mathbb{C}$-valued k-form, χ be a $\mathbb{C}$-valued $(k-1)$-form.

$$
\begin{aligned}
\mathbb{d}(\chi^* \wedge \star\phi) &\equiv \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi =: \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{D}^\dagger\phi \qquad (20)\\
&= \mathbb{D}\chi^* \wedge \star\phi - \mathrm{i}eA \wedge \chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi \\
&= \mathbb{D}\chi^* \wedge \star\phi + \chi^* \wedge (-)^k \mathrm{i}eA \star \phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi \\
&= \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \star \phi\,. \qquad (21)
\end{aligned}
$$

$$
\boxed{\mathbb{D}^\dagger\phi = (-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \star \phi\,.} \tag{22}
$$

2.2 $\mathsf{U}(1)$-gauged complex scalar field theory

$$
S = \int -\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi - m^2\phi^* \wedge \star\phi - \frac{1}{2}F \wedge \star F. \tag{23}
$$

Variation does not commute with exterior covariant derivative.

$$
[\delta, \mathbb{D}]\phi = -\mathrm{i}e\delta A\phi\,. \tag{24}
$$

$$
\begin{aligned}
\delta(\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi) &= \mathbb{d}(\delta\phi^* \star \mathbb{D}\phi + \delta\phi \star \mathbb{D}\phi^*) + \delta\phi^* \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi^* \\
&\quad + \delta A \wedge \mathfrak{J}_A\,, \qquad \mathfrak{J}_A := \mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)\,, \qquad (25)\\
\delta(F \wedge \star F) &= 2\,\delta F \wedge \star F \qquad (26)\\
&= 2\,\mathbb{d}(\delta A \wedge \star F) + 2\,\delta A \wedge \mathbb{d}\star F\,. \qquad (27)
\end{aligned}
$$

Variation of the action

$$\begin{aligned}\delta S = \int & \mathbb{d}(-\delta\phi^* \wedge \star\mathbb{D}\phi - \delta\phi \wedge \star\mathbb{D}\phi^* - \delta A \wedge \star F) \\ & - \delta\phi^* \wedge \star(\mathbb{D}^\dagger\mathbb{D} + m^2)\phi - \delta\phi \wedge \star(\mathbb{D}^\dagger\mathbb{D} + m^2)\phi^* \\ & - \delta A \wedge (\mathbb{d}\star F - \mathfrak{J}_A).\end{aligned} \tag{28}$$

2.2.1 Lorenz gauge

The Laplace–de Rham operator, or in our Lorentzian metric signature the d'Alembertian

$$\Box^2 := \left(\mathbb{d} + \mathbb{d}^\dagger\right)^2 = \mathbb{d}\mathbb{d}^\dagger + \mathbb{d}^\dagger\mathbb{d}\,. \tag{29}$$

$$\mathbb{d}\star F = \mathbb{d}\star\mathbb{d}A = \star(-)^2\star^{-1}\mathbb{d}\star\mathbb{d}A = \star\mathbb{d}^\dagger\mathbb{d}A = \star(\Box^2 - \mathbb{d}\mathbb{d}^\dagger)A\,. \tag{30}$$

One would like to have $\mathbb{d}\mathbb{d}^\dagger A = 0$, or $\mathbb{d}^\dagger A = \text{const}$. This would be fulfilled if

$$\mathbb{d}^\dagger A = 0\,, \tag{31}$$

which is the Lorenz gauge[2, 3, 6].

2.2.2 Noether's invariances

The action is invariant under the generic transformation

$$\phi \to \mathsf{e}^{-\mathrm{i}e\Lambda}\phi\,, \quad \phi^* \to \mathsf{e}^{+\mathrm{i}e\Lambda}\phi^*\,, \quad A \to A - \mathbb{d}\Lambda\,. \tag{32}$$

There are two scenarios [1].

If the transformation is rigid, $\mathbb{d}\Lambda = 0$, one obtains $\mathfrak{J}_A$ as the Noether current from the boundary term as before, which satisfies the continuity equation $\mathbb{d}\mathfrak{J}_A = 0$.

If the transformation is gauge with a compact support, the boundary term can be dropped, and one obtains the Noether identity

$$123 \tag{33}$$

3 Poincaré–regular theory

3.1 Differential geometry

Untwisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for untwisted k-form is defined inductively as

$$1\,, \tag{34}$$

$$\vartheta^{\alpha_1\alpha_2\ldots\alpha_k} := \vartheta^{\alpha_1} \wedge \vartheta^{\alpha_2\ldots\alpha_k}\,. \tag{35}$$

Twisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for twisted $(D-k)$-form is defined inductively as

$$\epsilon := \mathrm{vol}\,, \tag{36}$$

$$\epsilon_{\alpha_1\alpha_2\ldots\alpha_k} := e_{\alpha_k} \lrcorner\, \epsilon_{\alpha_1\ldots\alpha_{k-1}}\,. \tag{37}$$

By using eq. (5) and induction, one can show that

$$\epsilon_{\alpha_1\alpha_2\ldots\alpha_k} = \star\vartheta_{\alpha_1\alpha_2\ldots\alpha_k}\,. \tag{38}$$

Identities Inspired by [8, eq. (3.167)], for a 1-form ω, $(D-k)$-form χ, one can derive

$$\begin{aligned}\omega \wedge \star\chi &= (-)^{D-k} \star \chi \wedge \omega = (-)^{D-k} \star^{-1} \left(\omega^\sharp \lrcorner \star \star \chi\right) \\ &= (-)^{D-k} (-)^{(D-k-1)(k+1)+s} \star \left(\omega^\sharp \lrcorner (-)^{k(D-k)+s} \star^{-1} \star\chi\right) \\ &= (-)^{k+1} \star \left(\omega^\sharp \lrcorner \chi\right). \end{aligned} \tag{39}$$

Specifically,

$$\vartheta^\alpha \wedge \epsilon_{\beta\gamma} = -\delta^\alpha_\beta \epsilon_\gamma + \delta^\alpha_\gamma \epsilon_\beta \,. \tag{40}$$

Translation Consider the rigid infinitesimal transformation

$$\delta_\lambda \chi = \lambda\, £_z \chi = \lambda[z \lrcorner \mathrm{d}\chi + \mathrm{d}(z \lrcorner \chi)]\,; \tag{41}$$

'rigid' means

$$\delta_\lambda \vartheta^\alpha = 0\,. \tag{42}$$

3.2 Complex scalar field

3.2.1 Noether current

The action is invariant up to a total differential

$$\begin{aligned}\delta_\lambda S &= \int \lambda\, \mathrm{d}(z \lrcorner \mathfrak{L}) \\ &= \int \lambda \frac{1}{2}\, \mathrm{d}\{-z \lrcorner \left[\mathrm{d}\phi^* \wedge \star\mathrm{d}\phi + \mathrm{d}\phi \wedge \star\mathrm{d}\phi^* + m^2(\phi^* \star \phi + \phi \star \phi^*)\right]\}\,, \end{aligned} \tag{43}$$

under the rigid infinitesimal transformation

$$\delta\phi = \lambda\, £_z \phi = \lambda\, z \lrcorner \mathrm{d}\phi\,, \qquad \delta\phi^* = \lambda\, £_z \phi^* = \lambda\, z \lrcorner \mathrm{d}\phi^*\,. \tag{44}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$\delta_\lambda S = \int \lambda\, \mathrm{d}\{-(z \lrcorner \mathrm{d}\phi^*) \wedge \star\mathrm{d}\phi - (z \lrcorner \mathrm{d}\phi) \wedge \star\mathrm{d}\phi^*\}\,. \tag{45}$$

Taking the difference and setting $z = e_\alpha$ yields

$$0 = \int \lambda\, \mathrm{d}\mathfrak{T}_\alpha, \tag{46}$$

$$\begin{aligned}2\mathfrak{T}_\alpha := {}& (e_\alpha \lrcorner \mathrm{d}\phi^*) \wedge \star\mathrm{d}\phi + \mathrm{d}\phi^* \wedge (e_\alpha \lrcorner \star\mathrm{d}\phi) \\ & + (e_\alpha \lrcorner \mathrm{d}\phi) \wedge \star\mathrm{d}\phi^* + \mathrm{d}\phi \wedge (e_\alpha \lrcorner \star\mathrm{d}\phi^*) \\ & - m^2(\phi^*\, e_\alpha \lrcorner \star\phi + \phi\, e_\alpha \star \phi^*)\,. \end{aligned} \tag{47}$$

The continuity equation reads

$$\mathrm{d}\mathfrak{T}_\alpha = 0\,. \tag{48}$$

In components,

$$\mathfrak{T}_\alpha = \left(e_\alpha(\phi^*) e^\beta(\phi) + e_\alpha(\phi) e^\beta(\phi^*)\right) \epsilon_\beta - e^\beta(\phi^*) e_\beta(\phi^*)\, \epsilon_\alpha - m^2 \phi^* \phi\, \epsilon_\alpha\,. \tag{49}$$

3.3 Pure electromagnetic field

3.3.1 Noether current

The action is invariant up to a total differential

$$\delta_\lambda S = \int \lambda\, \mathbb{d}(z \lrcorner \mathfrak{L}) = \int \lambda\, \frac{1}{2}\, \mathbb{d}\{-z \lrcorner (F \wedge \star F)\}\,, \tag{50}$$

under the rigid infinitesimal transformation combined with a gauge transformation

$$\delta A = \lambda\{z \lrcorner \mathbb{d}(A - \mathbb{d}\Lambda) + \mathbb{d}[z \lrcorner (A - \mathbb{d}\Lambda)]\}\,. \tag{51}$$

Choosing

$$\mathbb{d}\Lambda = A(z)\, z^\flat \tag{52}$$

makes the second term vanish, yielding

$$\delta A = \lambda\, z \lrcorner \mathbb{d}A = \lambda\, z \lrcorner F\,. \tag{53}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$\delta_\lambda S = \int \lambda\, \mathbb{d}\{-(z \lrcorner F) \wedge \star F\}\,. \tag{54}$$

Taking the difference and setting $z = e_\alpha$ yields

$$0 = \int \lambda\, \mathbb{d}\mathfrak{T}_\alpha, \tag{55}$$

$$2\mathfrak{T}_\alpha := -(e_\alpha \lrcorner F) \wedge \star F + F \wedge (e_\alpha \lrcorner \star F)\,. \tag{56}$$

The continuity equation reads

$$\mathbb{d}\mathfrak{T}_\alpha = 0\,. \tag{57}$$

In components,

$$\mathfrak{T}_\alpha = \,. \tag{58}$$

4 Poincaré gauge theory

4.1 Differential forms

Upon variation of ϑ^α, $\vartheta^{\alpha_1\alpha_2\cdots\alpha_k}$ goes under

$$\delta\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge (e_\alpha \lrcorner \vartheta^{\alpha_1\alpha_2\cdots\alpha_k})\,, \tag{59}$$

which can be proved by induction.

Upon variation of ϑ^α, $\epsilon_{\alpha_1\alpha_2\cdots\alpha_k}$ goes under [7, sec. A.2]

$$\delta\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge \left(e_\alpha \lrcorner \epsilon_{\alpha_1\alpha_2\cdots\alpha_k}\right)\,. \tag{60}$$

Variation of Hodge star In gravitational theories [7, sec. 3.2] with an orthonormal cobasis,

$$[\delta, \star]\phi = \delta\vartheta^\alpha \wedge (e_\alpha \lrcorner \star\phi) - \star(\delta\vartheta^\alpha \wedge (e_\alpha \lrcorner \phi)) \,. \tag{61}$$

Let χ be a p-form, ϕ another form [5, sec. 5].

$$\delta(\chi \wedge \star\phi) = \delta\chi \wedge \star\phi + \delta\phi \wedge \star\chi - \delta\vartheta^\alpha \wedge \Sigma_\alpha \,, \tag{62}$$

$$\Sigma_\alpha := \chi \wedge \left\{\star(e_\alpha \lrcorner \phi) - (-)^p (e_\alpha \lrcorner \star\phi)\right\} . \tag{63}$$

4.2 $\mathsf{U}(1)$-gauged complex scalar field theory

$$\begin{aligned}\Sigma_\alpha = &-\mathbb{D}\phi^* \wedge \{\star(e_\alpha \lrcorner \mathbb{D}\phi) + (e_\alpha \lrcorner \star\mathbb{D}\phi)\} - m^2\phi^*\phi\,\epsilon_\alpha \\ &- \frac{1}{2} F \wedge \{\star(e_\alpha \lrcorner F) - (e_\alpha \lrcorner \star F)\} \,.\end{aligned} \tag{64}$$

4.2.1 Noether's invariances

[1]

Rigid translation

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