Coframe formalisms

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Contents

1 U(1)–regular theory

1.1 Differential geometry

Interior product Let z be a vector, ω be a 1-form, χ be a k-form. The *interior product* is defined inductively as the bilinear map satisfying

$$z \lrcorner\, \omega := \omega(z)\,, \tag{1}$$

$$z \lrcorner\, (\omega \wedge \chi) := (z \lrcorner\, \omega) \wedge \chi - \omega \wedge (z \lrcorner\, \chi)\,. \tag{2}$$

Equation (2) is also known as the anti-product rule.

By inductiion one can show that for a p-form ϕ,

$$z \lrcorner\, (\phi \wedge \chi) = (z \lrcorner\, \phi) \wedge \chi + (-)^p \chi \wedge (z \lrcorner\, \chi)\,. \tag{3}$$

Hodge star Let ω be an 1-form, χ be a k-form. The Hodge star $\star$ is defined inductively as the linear map [4, sec. 24]

$$\star 1 := \mathrm{vol}\,, \tag{4}$$

$$\star(\chi \wedge \omega) := \omega^\sharp \lrcorner \star\chi\,. \tag{5}$$

Non-gravitational theories features $[\delta, \star] = 0$, which means [7, sec. 3.2]

$$\delta g_{\mu\nu} = -2\omega_{(\mu\nu)}, \qquad \delta\vartheta^\mu = \omega_\nu{}^\mu\,\vartheta^\nu\,; \tag{6}$$

for an orthonormal coframe, the allowed variations are $\omega_{(\alpha\beta)} = 0$.

Codifferential The *codifferential* $\mathbb{d}^\dagger$ is the adjoint of the exterior derivative $\mathbb{d}$ in the following sense. Let ϕ be a k-form, χ be a $(k-1)$-form.

$$\mathbb{d}(\chi^* \wedge \star\phi) \equiv \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi =: \mathbb{d}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{d}^\dagger\phi \tag{7}$$

$$= \mathbb{d}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} \mathbb{d} \star \phi\,. \tag{8}$$

$$\boxed{\mathbb{d}^\dagger\phi = (-)^k \star^{-1} \mathbb{d} \star \phi\,.} \tag{9}$$

1.2 Complex scalar field

The action reads

$$S = \int -\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi - m^2\phi^* \wedge \star\phi \tag{10}$$

$$= \int -\frac{1}{2}(\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi + \mathbb{d}\phi \wedge \star\mathbb{d}\phi^*) - \frac{1}{2}m^2(\phi^* \wedge \star\phi + \phi \wedge \star\phi^*) \tag{11}$$

Variation commutes with exterior derivative

$$[\delta, \mathbb{d}]\phi = 0\,. \tag{12}$$

A generic variation of the kinetic terms reads

$$\begin{aligned}\delta(\mathbb{d}\phi^* \wedge \star\mathbb{d}\phi) &= \mathbb{d}(\delta\phi^* \wedge \star\mathbb{d}\phi + \delta\phi \wedge \star\mathbb{d}\phi^*) \\ &\quad + \delta\phi^* \wedge \star\mathbb{d}^\dagger\mathbb{d}\phi + \delta\phi \wedge \star\mathbb{d}^\dagger\mathbb{d}\phi^*\,.\end{aligned} \tag{13}$$

A generic variation of the actionreads

$$\begin{aligned}\delta S = \int &\mathbb{d}(-\delta\phi^* \wedge \star\mathbb{d}\phi - \delta\phi \wedge \star\mathbb{d}\phi^*) \\ &- \delta\phi^* \wedge \star(\mathbb{d}^\dagger\mathbb{d} + m^2)\phi - \delta\phi \wedge \star(\mathbb{d}^\dagger\mathbb{d} + m^2)\phi^*\,.\end{aligned} \tag{14}$$

1.2.1 Noether current

The action is invariant

$$\delta_\lambda S \equiv 0 \tag{15}$$

under the rigid transformation

$$\phi \to \mathrm{e}^{-\mathrm{i}e\lambda}\phi\,, \qquad \phi^* \to \mathrm{e}^{+\mathrm{i}e\lambda}\phi^*\,. \tag{16}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$
\begin{aligned}
0 &= \int \lambda \, \mathbb{d}\mathfrak{J}_0 \,, \\
\mathfrak{J}_0 &:= \mathsf{i}e(\phi^* \wedge \star\mathbb{d}\phi - \phi \wedge \star\mathbb{d}\phi^*) \,,
\end{aligned} \tag{17}
$$

which is the Noether current, a twisted 3-form, satisfying the continuity equation

$$
\mathbb{d}\mathfrak{J}_0 = 0 \,. \tag{18}
$$

2 $\mathsf{U}(1)$ gauge theory

2.1 Connection on the principal bundle

Exterior covariant derivative Let χ be a $\mathbb{C}$-valued k-form. The *exterior covariant derivative* of χ reads

$$
\mathbb{D}\chi := (\mathbb{d} - \mathsf{i}eA)\chi, \qquad \mathbb{D}\chi^* := (\mathbb{d} + \mathsf{i}eA)\chi^* \,, \tag{19}
$$

where A is a $\mathfrak{u}(1)$-valued *connection form.*

Covariant codifferential The *covariant codifferential* $\mathbb{D}^\dagger$ is the adjoint of the exterior covariant derivative $\mathbb{D}$ in the following sense. Let ϕ be a $\mathbb{C}$-valued k-form, χ be a $\mathbb{C}$-valued $(k-1)$-form.

$$
\begin{aligned}
\mathbb{d}(\chi^* \wedge \star\phi) &\equiv \mathbb{d}\chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi =: \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star\mathbb{D}^\dagger\phi && (20)\\
&= \mathbb{D}\chi^* \wedge \star\phi - \mathsf{i}eA \wedge \chi^* \wedge \star\phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi \\
&= \mathbb{D}\chi^* \wedge \star\phi + \chi^* \wedge (-)^k \mathsf{i}eA \star \phi - (-)^k \chi^* \wedge \mathbb{d} \star \phi \\
&= \mathbb{D}\chi^* \wedge \star\phi - \chi^* \wedge \star(-)^k \star^{-1} (\mathbb{d} - \mathsf{i}eA) \star \phi \,. && (21)
\end{aligned}
$$

$$
\boxed{\mathbb{D}^\dagger\phi = (-)^k \star^{-1} (\mathbb{d} - \mathsf{i}eA) \star \phi \,.} \tag{22}
$$

2.2 $\mathsf{U}(1)$-gauged complex scalar field theory

$$
S = \int -\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi - m^2 \phi^* \wedge \star\phi - \frac{1}{2} F \wedge \star F. \tag{23}
$$

Variation does not commute with exterior covariant derivative.

$$
[\delta, \mathbb{D}]\phi = -\mathsf{i}e\delta A\phi \,. \tag{24}
$$

$$
\begin{aligned}
\delta(\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi) &= \mathbb{d}(\delta\phi^* \star \mathbb{D}\phi + \delta\phi \star \mathbb{D}\phi^*) + \delta\phi^* \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi^* \\
&\quad + \delta A \wedge \mathfrak{J}_A \,, \qquad \mathfrak{J}_A := \mathsf{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*) \,, && (25)\\
\delta(F \wedge \star F) &= 2\, \delta F \wedge \star F && (26)\\
&= 2\, \mathbb{d}(\delta A \wedge \star F) + 2\, \delta A \wedge \mathbb{d}\star F \,. && (27)
\end{aligned}
$$

Variation of the action

$$\begin{aligned}\delta S = \int & \mathbb{d}(-\delta\phi^* \wedge \star\mathbb{D}\phi - \delta\phi \wedge \star\mathbb{D}\phi^* - \delta A \wedge \star F) \\ & - \delta\phi^* \wedge \star(\mathbb{D}^\dagger\mathbb{D} + m^2)\phi - \delta\phi \wedge \star(\mathbb{D}^\dagger\mathbb{D} + m^2)\phi^* \\ & - \delta A \wedge (\mathbb{d}\star F - \mathfrak{J}_A). \end{aligned} \tag{28}$$

2.2.1 Lorenz gauge

The Laplace–de Rham operator, or in our Lorentzian metric signature the d'Alembertian

$$\Box^2 := \left(\mathbb{d} + \mathbb{d}^\dagger\right)^2 = \mathbb{d}\mathbb{d}^\dagger + \mathbb{d}^\dagger\mathbb{d}\,. \tag{29}$$

$$\mathbb{d}\star F = \mathbb{d} \star \mathbb{d}A = \star(-)^2 \star^{-1} \mathbb{d} \star \mathbb{d}A = \star\mathbb{d}^\dagger\mathbb{d}A = \star(\Box^2 - \mathbb{d}\mathbb{d}^\dagger)A\,. \tag{30}$$

One would like to have $\mathbb{d}\mathbb{d}^\dagger A = 0$, or $\mathbb{d}^\dagger A = \text{const}$. This would be fulfilled if

$$\mathbb{d}^\dagger A = 0\,, \tag{31}$$

which is the Lorenz gauge[2, 3, 6].

2.2.2 Noether's invariances

The action is invariant under the generic transformation

$$\phi \to \mathsf{e}^{-\mathrm{i}e\Lambda}\phi\,, \quad \phi^* \to \mathsf{e}^{+\mathrm{i}e\Lambda}\phi^*\,, \quad A \to A - \mathbb{d}\Lambda\,. \tag{32}$$

There are two scenarios [1].

If the transformation is rigid, $\mathbb{d}\Lambda = 0$, one obtains $\mathfrak{J}_A$ as the Noether current from the boundary term as before, which satisfies the continuity equation $\mathbb{d}\mathfrak{J}_A = 0$.

If the transformation is gauge with a compact support, the boundary term can be dropped, and one obtains the Noether identity

$$123 \tag{33}$$

3 Poincaré–regular theory

3.1 Differential geometry

Untwisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for untwisted k-form is defined inductively as

$$1\,, \tag{34}$$

$$\vartheta^{\alpha_1\alpha_2\ldots\alpha_k} := \vartheta^{\alpha_1} \wedge \vartheta^{\alpha_2\ldots\alpha_k}\,. \tag{35}$$

Twisted orthonormal k-cobases Let $\{\vartheta^\alpha\}$ be an orthonormal coframe. The orthonormal basis for twisted $(D-k)$-form is defined inductively as

$$\epsilon := \mathrm{vol}\,, \tag{36}$$

$$\epsilon_{\alpha_1\alpha_2\ldots\alpha_k} := e_{\alpha_k} \lrcorner\, \epsilon_{\alpha_1\ldots\alpha_{k-1}}\,. \tag{37}$$

By using eq. (5) and induction, one can show that

$$\epsilon_{\alpha_1\alpha_2\ldots\alpha_k} = \star\vartheta_{\alpha_1\alpha_2\ldots\alpha_k}\,. \tag{38}$$

Identities Inspired by [8, eq. (3.167)], for a 1-form ω, $(D-k)$-form χ, one can derive

$$\begin{aligned}\omega\wedge\star\chi &= (-)^{D-k}\star\chi\wedge\omega = (-)^{D-k}\star^{-1}\left(\omega^\sharp\lrcorner\star\star\chi\right)\\ &= (-)^{D-k}(-)^{(D-k-1)(k+1)+s}\star\left(\omega^\sharp\lrcorner(-)^{k(D-k)+s}\star^{-1}\star\chi\right)\\ &= (-)^{k+1}\star\left(\omega^\sharp\lrcorner\chi\right).\end{aligned} \tag{39}$$

Specifically,

$$\vartheta^\alpha\wedge\epsilon_{\beta\gamma} = -\delta^\alpha_\beta\epsilon_\gamma + \delta^\alpha_\gamma\epsilon_\beta\,. \tag{40}$$

Translation Consider the rigid infinitesimal transformation

$$\delta_\lambda\chi = \lambda\,\pounds_z\chi = \lambda[z\lrcorner\mathbb{d}\chi + \mathbb{d}(z\lrcorner\chi)]\,; \tag{41}$$

'rigid' means

$$\delta_\lambda\vartheta^\alpha = 0\,. \tag{42}$$

3.2 Complex scalar field

3.2.1 Noether current

The action is invariant up to a total differential

$$\begin{aligned}\delta_\lambda S &= \int\lambda\,\mathbb{d}(z\lrcorner\mathfrak{L})\\ &= \int\lambda\,\frac{1}{2}\,\mathbb{d}\{-z\lrcorner\left[\mathbb{d}\phi^*\wedge\star\mathbb{d}\phi + \mathbb{d}\phi\wedge\star\mathbb{d}\phi^* + m^2(\phi^*\star\phi + \phi\star\phi^*)\right]\}\,,\end{aligned} \tag{43}$$

under the rigid infinitesimal transformation

$$\delta\phi = \lambda\,\pounds_z\phi = \lambda\,z\lrcorner\mathbb{d}\phi\,,\qquad \delta\phi^* = \lambda\,\pounds_z\phi^* = \lambda\,z\lrcorner\mathbb{d}\phi^*\,. \tag{44}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$\delta_\lambda S = \int\lambda\,\mathbb{d}\{-(z\lrcorner\mathbb{d}\phi^*)\wedge\star\mathbb{d}\phi - (z\lrcorner\mathbb{d}\phi)\wedge\star\mathbb{d}\phi^*\}\,. \tag{45}$$

Taking the difference and setting $z = e_\alpha$ yields

$$0 = \int\lambda\,\mathbb{d}\mathfrak{T}_\alpha, \tag{46}$$

$$\begin{aligned}2\mathfrak{T}_\alpha &\coloneqq (e_\alpha\lrcorner\mathbb{d}\phi^*)\wedge\star\mathbb{d}\phi + \mathbb{d}\phi^*\wedge(e_\alpha\lrcorner\star\mathbb{d}\phi)\\ &\quad + (e_\alpha\lrcorner\mathbb{d}\phi)\wedge\star\mathbb{d}\phi^* + \mathbb{d}\phi\wedge(e_\alpha\lrcorner\star\mathbb{d}\phi^*)\\ &\quad - m^2(\phi^*\,e_\alpha\lrcorner\star\phi + \phi\,e_\alpha\star\phi^*)\,.\end{aligned} \tag{47}$$

The continuity equation reads

$$\mathbb{d}\mathfrak{T}_\alpha = 0\,. \tag{48}$$

In components,

$$\mathfrak{T}_\alpha = \left(e_\alpha(\phi^*)e^\beta(\phi) + e_\alpha(\phi)e^\beta(\phi^*)\right)\epsilon_\beta - e^\beta(\phi^*)e_\beta(\phi^*)\,\epsilon_\alpha - m^2\phi^*\phi\,\epsilon_\alpha\,. \tag{49}$$

3.3 Pure electromagnetic field

3.3.1 Noether current

The action is invariant up to a total differential

$$\begin{aligned}\delta_\lambda S &= \int \lambda \, \mathrm{d}(z \lrcorner \mathfrak{L}) \\ &= \int \lambda \frac{1}{2} \mathrm{d}\{-z \lrcorner F \wedge \star F\} \,,\end{aligned} \tag{50}$$

under the rigid infinitesimal transformation

$$\delta A = \tag{51}$$

When the equations of motion are satisfied, infinitesimal transformation leads to

$$\delta_\lambda S = \int \lambda \, \mathrm{d} \,. \tag{52}$$

Taking the difference and setting $z = e_\alpha$ yields

$$0 = \int \lambda \, \mathrm{d}\mathfrak{T}_\alpha, \tag{53}$$

$$2\mathfrak{T}_\alpha := \,. \tag{54}$$

The continuity equation reads

$$\mathrm{d}\mathfrak{T}_\alpha = 0 \,. \tag{55}$$

In components,

$$\mathfrak{T}_\alpha = \,. \tag{56}$$

4 Poincaré gauge theory

4.1 Differential forms

Upon variation of ϑ^α, $\vartheta^{\alpha_1\alpha_2\cdots\alpha_k}$ goes under

$$\delta\vartheta^{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge (e_\alpha \lrcorner \vartheta^{\alpha_1\alpha_2\cdots\alpha_k}) \,, \tag{57}$$

which can be proved by induction.

Upon variation of ϑ^α, $\epsilon_{\alpha_1\alpha_2\cdots\alpha_k}$ goes under [7, sec. A.2]

$$\delta\epsilon_{\alpha_1\alpha_2\cdots\alpha_k} = \delta\vartheta^\alpha \wedge \left(e_\alpha \lrcorner \epsilon_{\alpha_1\alpha_2\cdots\alpha_k}\right) . \tag{58}$$

Variation of Hodge star In gravitational theories [7, sec. 3.2] with an orthonormal cobasis,

$$[\delta, \star]\phi = \delta\vartheta^\alpha \wedge (e_\alpha \lrcorner \star\phi) - \star(\delta\vartheta^\alpha \wedge (e_\alpha \lrcorner \phi)) \,. \tag{59}$$

Let χ be a p-form, ϕ another form [5, sec. 5].

$$\delta(\chi \wedge \star\phi) = \delta\chi \wedge \star\phi + \delta\phi \wedge \star\chi - \delta\vartheta^\alpha \wedge \Sigma_\alpha \,, \tag{60}$$

$$\Sigma_\alpha := \chi \wedge \left\{\star(e_\alpha \lrcorner \phi) - (-)^p (e_\alpha \lrcorner \star\phi)\right\} . \tag{61}$$

4.2 $\mathsf{U}(1)$-gauged complex scalar field theory

$$\begin{aligned}\Sigma_\alpha = &-\mathbb{D}\phi^* \wedge \{\star(e_\alpha \lrcorner \mathbb{D}\phi) + (e_\alpha \lrcorner \star\mathbb{D}\phi)\} - m^2\phi^*\phi\,\epsilon_\alpha \\ &- \frac{1}{2}F \wedge \{\star(e_\alpha \lrcorner F) - (e_\alpha \lrcorner \star F)\}\,.\end{aligned} \tag{62}$$

4.2.1 Noether's invariances

[1]

Rigid translation

References

[1] S. G. Avery and B. U. W. Schwab, "Noether's second theorem and ward identities for gauge symmetries", Journal of High Energy Physics **2016** (2015) `10.1007/JHEP02(2016)031`, arXiv:`1510.07038v2 [hep-th]` (cit. on pp. 4, 6).

[2] J. van Bladel, "Lorenz or lorentz?", IEEE Antennas and Propagation Magazine **33**, 69–69 (1991) (cit. on p. 4).

[3] J. van Bladel, "Lorenz or lorentz? [addendum]", IEEE Antennas and Propagation Magazine **33**, 56–56 (1991) (cit. on p. 4).

[4] W. L. Burke, *Applied differential geometry* (Cambridge University Press, 1985) (cit. on p. 2).

[5] Y. Itin, "On variations in teleparallelism theories", (1999), arXiv:`gr-qc/9904030 [gr-qc]` (cit. on p. 6).

[6] L. Lorenz, "XXXVIII. on the identity of the vibrations of light with electrical currents", The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **34**, 287–301 (1867) (cit. on p. 4).

[7] U. Muench, F. Gronwald, and F. W. Hehl, "A brief guide to variations in teleparallel gauge theories of gravity and the kaniel-itin model", General Relativity and Gravitation **30**, 933–961 (1998), arXiv:`gr-qc/9801036 [gr-qc]` (cit. on pp. 2, 6).

[8] N. Straumann, *General relativity*, 2nd ed. (Springer, 2013), ISBN: 9789400754096 (cit. on p. 5).