Notes on Canonical Singular Dynamics

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1 Canonical formalism

Lagrangian with velocity

$$L^{\mathrm{v}} := L\big|_{\dot{q}=v} \tag{1.1}$$

Equations of motion

$$\sum_j M_{ij}\dot{v}_j = K^{\mathrm{v}}_i, \quad \dot{q}_i = v_i. \tag{1.2}$$

where

$$M_{ij}(q,v) := \frac{\partial^2 L^{\mathrm{v}}}{\partial v_i \partial v_j}. \tag{1.3}$$

Adding

$$p_i := \frac{\partial L^{\mathrm{v}}}{\partial v_i} =: \overline{p}_i(q,v). \tag{1.4}$$

Variation of

$$S[q,p,v] := \int \mathrm{d}t \left[L^{\mathrm{v}} + \sum_i p_i(\dot{q}_i - v_i) \right]. \tag{1.5}$$

gives the *extended Euler–Lagrange equations*

$$\dot{q}_i = v_i, \quad \dot{p}_i = \frac{\partial L^{\mathrm{v}}}{\partial q_i}, \quad p_i = \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.6}$$

Extended Hamiltonian

$$H^{\mathrm{v}}(q,p,v) := \sum_i p_i v_i - L^{\mathrm{v}}. \tag{1.7}$$

Identities

$$\frac{\partial H^{\mathrm{v}}}{\partial q_i} \equiv -\frac{\partial L^{\mathrm{v}}}{\partial q_i}, \quad \frac{\partial H^{\mathrm{v}}}{\partial p_i} \equiv v_i, \quad \frac{\partial H^{\mathrm{v}}}{\partial v_i} \equiv p_i - \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.8}$$

Variation of

$$S[q,p,v] := \int \mathrm{d}t \left[\sum_i p_i \dot{q}_i H^{\mathrm{v}} + \right] \tag{1.9}$$

gives the *extended canonical equations*

$$\dot{q}_i = [q_i, H^{\mathrm{v}}]_{\mathrm{TP}}, \quad \dot{p}_i = [p_i, H^{\mathrm{v}}]_{\mathrm{TP}}, \quad \frac{\partial H^{\mathrm{v}}}{\partial v_i} = 0, \tag{1.10}$$

where the total Poison bracket is defined as

$$[f^{\mathrm{v}}, g^{\mathrm{v}}]_{\mathrm{TP}} := \sum_i \left(\frac{\partial f^{\mathrm{v}}}{\partial q_i} \frac{\partial g^{\mathrm{v}}}{\partial p_i} - \frac{\partial f^{\mathrm{v}}}{\partial p_i} \frac{\partial g^{\mathrm{v}}}{\partial q_i} \right). \tag{1.11}$$

$v^a = \overline{v}^a(q,p)$ can be solved, $a = 1, 2, \dots, r_M$; v^α cannot be solved, $\alpha = r_M + 1, \dots, n$, where $r_M = \operatorname{rank} M$.

Primary constraints in the standard form

$$\phi_\alpha^{(0)}(q,p) := \left. \frac{\partial H^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_\alpha = \overline{v}_\alpha\}} \equiv p_\alpha - \left. \frac{\partial L^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_\alpha = \overline{v}_\alpha\}}. \tag{1.12}$$

Total Hamiltonian

$$H^{\mathrm{t}} := \left. H^{\mathrm{v}} \right|_{\{v_\alpha = \overline{v}_\alpha\}}. \tag{1.13}$$

2 Examples

$$L^{\mathrm{v}} = \frac{1}{2} \sum_{i,j} W_{ij}(q) v_i v_j + \sum_i \eta_i(q) v_i - V(q). \tag{2.1}$$

$$p_i = \frac{\partial L^{\mathrm{v}}}{\partial v_i} = \sum_{i,j} W_{ij} v_j + \eta_i. \tag{2.2}$$

Let

$$\sum_j W_{ij} e_j^{(a)} = \lambda^{(a)} e_i \neq 0, \tag{2.3}$$

$$\sum_j W_{ij} e_j^{(\alpha)} = 0. \tag{2.4}$$