Notes on Lagrangian Singular Dynamics

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1 Lagrangian formalism

Lagrangian with velocity

$$L^{\mathrm{v}} := L\big|_{\dot{q}=v} \tag{1.1}$$

Equations of motion

$$\sum_j M_{ij}\dot{v}_j = K_i^{\mathrm{v}}, \quad \dot{q}_i = v_i. \tag{1.2}$$

where

$$M_{ij}(q,v) := \frac{\partial^2 L}{\partial v_i\,\partial v_j}. \tag{1.3}$$

2 Examples

2.1 Toy examples

Example 0

Gitman and Tyutin 1990, sec. 1.2

$$L = \frac{1}{2}(\dot{x} - y)^2 \tag{2.1}$$

Example 1

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x-y)^2. \tag{2.2}$$

Example 2

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x-y)^2 \tag{2.3}$$

Example 3

$$L = \frac{1}{2}(\dot{q}_2 - \mathbb{e}^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2. \tag{2.4}$$

2.2 Parametrised systems

Non-relativistic point particle

Kiefer 2012, sec. 3.1.1

$$S[q(t)] := \int_{t_1}^{t_2} \mathrm{d}t\, L\left(q, \frac{\mathrm{d}q}{\mathrm{d}t}\right) \tag{2.5}$$

Relativistic charged point particle

Landau and Lifshitz 1975, sec. 15, Kiefer 2012, sec. 3.1.2

$$S := \int -m\,\mathrm{d}s + eA_\mu(x)\,\mathrm{d}x^\mu =: \int \mathrm{d}\tau\, L, \tag{2.6}$$

where the Lagrangian reads

$$L = -m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + q\dot{x}^\mu A_\mu(x). \tag{2.7}$$

$$M_{\mu\nu} := \frac{\partial^2 L^{\mathrm{v}}}{\partial\dot{x}^\mu\,\partial\dot{x}^\nu} = m\frac{-\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta}}{\left(-\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma\right)^{3/2}}\dot{x}^\alpha\dot{x}^\beta, \tag{2.8}$$

which has one and only one eigenvector with null eigenvalue

$$\dot{x}^\mu M_{\mu\nu} = 0. \tag{2.9}$$

2.2.1 Neutral scalar field

ibid., sec. 3.3

2.3 Maxwell–Proca theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + \frac{1}{2}F_{\mu\nu}S^{\mu\nu} + A_\mu J^\mu_{\mathrm{f}}, \tag{2.10}$$

where $m > 0$ corresponds to the Proca theory Gitman and Tyutin 1990, sec. 2.3, and $m = 0$ the Maxwell theory H. J. Rothe and K. D. Rothe 2010, sec. 3.3.3 Gitman and Tyutin 1990, sec. 2.4.

2.4 String theories

Relativistic point particle with auxiliary coordinate

Blumenhagen, Lüst, and Theisen 2013, sec. 2.1

$$L := \frac{1}{2}\left(e^{-1}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2 e\right) \tag{2.11}$$

Nambu–Gotō action

Generalising the kinetic part of (2.6), one has

$$S_{\mathrm{NG}} := -T \int_\Sigma \mathbb{d}A =: -T \int_\Sigma \mathbb{d}^2\sigma \mathcal{L}, \tag{2.12}$$

where the Lagrangian density

$$\mathcal{L} = \sqrt{-\Gamma}, \quad \Gamma := \det \Gamma_{\alpha\beta}, \quad \Gamma_{\alpha\beta} := \frac{\partial X^\nu}{\partial \sigma^\alpha} \frac{\partial X_\nu}{\partial \sigma^\alpha}. \tag{2.13}$$

Historically Gotō 1971; Nambu 1970; Reference e.g. Blumenhagen, Lüst, and Theisen 2013 Kiefer 2012, sec. 3.2

Polyakov action

Generalising (2.11)

$$S_{\mathrm{P}}[X^\mu, h\alpha\beta] = -\frac{T}{2} \int_\Sigma \mathcal{L}, \tag{2.14}$$

where

$$\mathcal{L} := \sqrt{-h} h^{\alpha\beta} \Gamma_{\alpha\beta}. \tag{2.15}$$

Historically Brink, Di Vecchia, and Howe 1976; Deser and Zumino 1976; Polyakov 1981; Reference Kiefer 2012, sec. 3.2

References

Blumenhagen, Ralph, Dieter Lüst, and Stefan Theisen (2013). *Basic Concepts of String Theory*. Theoretical and Mathematical Physics. Springer. ISBN: http://id.crossref.org/isbn/978-3-642-29497-6. DOI: `10.1007/978-3-642-29497-6`. URL: `http://dx.doi.org/10.1007/978-3-642-29497-6`.

Brink, L., P. Di Vecchia, and P. Howe (1976). “A locally supersymmetric and reparametrization invariant action for the spinning string”. In: *Physics Letters B* 65.5, pp. 471–474. ISSN: 0370-2693. DOI: `10.1016/0370-2693(76)90445-7`. URL: `http://dx.doi.org/10.1016/0370-2693(76)90445-7`.

Deser, S. and B. Zumino (1976). “A complete action for the spinning string”. In: *Physics Letters B* 65.4, pp. 369–373. ISSN: 0370-2693. DOI: `10.1016/0370-2693(76)90245-8`. URL: `http://dx.doi.org/10.1016/0370-2693(76)90245-8`.

Gitman, Dmitriy M. and Igor V. Tyutin (1990). *Quantization of Fields with Constraints*. Springer Series in Nuclear and Particle Physics. Springer Berlin Heidelberg. ISBN: http://id.crossref.org/isbn/978-3-642-83938-2. DOI: `10.1007/978-3-642-83938-2`. URL: `http://dx.doi.org/10.1007/978-3-642-83938-2`. Дмитрий Максимович Гитман and Игорь Викторович Тютин. *Каноническое квантование полей со связями.* 030077 г. 11овосибирск-77, Станиславского, 25: Наука, 1986.

Gotō, Tetsuo (1971). “Relativistic Quantum Mechanics of One-Dimensional Mechanical Continuum and Subsidiary Condition of Dual Resonance Model”. In: *Progress of Theoretical Physics* 46.5, pp. 1560–1569. ISSN: 0033-068X. DOI: 10.1143/ptp.46.1560. URL: http://dx.doi.org/10.1143/PTP.46.1560.

Kiefer, Claus (2012). *Quantum Gravity*. 3rd. International Series of Monographs on Physics. Oxford University Press. ISBN: http://id.crossref.org/isbn/9780199585205. DOI: 10.1093/acprof:oso/9780199585205.001.0001. URL: http://dx.doi.org/10.1093/acprof:oso/9780199585205.001.0001.

Landau, Lev Davidovich and Evgeny Mikhailovich Lifshitz (1975). *The Classical Theory of Fields*. 4th. Vol. 2. Course of Theoretical Physics. Butterworth-Heinemann. ISBN: http://id.crossref.org/isbn/9780080250724. DOI: 10.1016/c2009-0-14608-1. URL: http://dx.doi.org/10.1016/c2009-0-14608-1.

Nambu, Yoichiro (1970). “Quark model and the factorization of the Veneziano amplitude”. In: *Symmetries and Quark Models*. Ed. by R. Chand. International Conference on Symmetries and Quark Models, Wayne State U., Detroit Detroit, Mich., USA, June 18-20, 1969, pp. 269–278.

Polyakov, A.M. (1981). “Quantum geometry of bosonic strings”. In: *Physics Letters B* 103.3, pp. 207–210. ISSN: 0370-2693. DOI: 10.1016/0370-2693(81)90743-7. URL: http://dx.doi.org/10.1016/0370-2693(81)90743-7.

Rothe, Heinz J and Klaus D Rothe (2010). *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*. World Scientific Lecture Notes in Physics. World Scientific. ISBN: http://id.crossref.org/isbn/978-981-4299-65-7. DOI: 10.1142/7689. URL: http://dx.doi.org/10.1142/7689.