

# Notes on Canonical Singular Dynamics

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## 1 Classical formalism

Lagrangian with velocity

$$L^\vee := L|_{\dot{q}=v} \quad (1.1)$$

Equations of motion

$$\sum_j M_{ij} \dot{v}_j = K_i^\vee, \quad \dot{q}_i = v_i. \quad (1.2)$$

where

$$M_{ij}(q, v) := \frac{\partial^2 L^\vee}{\partial v_i \partial v_j}. \quad (1.3)$$

Adding

$$p_i := \frac{\partial L^\vee}{\partial v_i}. \quad (1.4)$$

Variation of

$$S[q, p; v] := \int dt \left[ L^\vee + \sum_i p_i (\dot{q}_i - v_i) \right]. \quad (1.5)$$

gives the *extended Euler–Lagrange equations*

$$\dot{q}_i = v_i, \quad \dot{p}_i = \frac{\partial L^\vee}{\partial q_i}, \quad p_i = \frac{\partial L^\vee}{\partial v_i}. \quad (1.6)$$

Hamiltonian with velocity

$$H^\vee(q, p; v) := \sum_i p_i v_i - L^\vee. \quad (1.7)$$

Identities

$$\frac{\partial H^\vee}{\partial q_i} \equiv -\frac{\partial L^\vee}{\partial q_i}, \quad \frac{\partial H^\vee}{\partial p_i} \equiv v_i, \quad \frac{\partial H^\vee}{\partial v_i} \equiv p_i - \frac{\partial L^\vee}{\partial v_i}. \quad (1.8)$$

Variation of

$$S[q, p; v] := \int dt \left[ \sum_i p_i \dot{q}_i - H^\vee \right] \quad (1.9)$$

gives the *extended canonical equations*

$$\dot{q}_i = [q_i, H^v]_p, \quad \dot{p}_i = [p_i, H^v]_p, \quad \frac{\partial H^v}{\partial v_i} = 0, \quad (1.10)$$

where the *Poisson bracket* is defined as

$$[f^v, g^v]_p := \sum_i \left( \frac{\partial f^v}{\partial q_i} \frac{\partial g^v}{\partial p_i} - \frac{\partial f^v}{\partial p_i} \frac{\partial g^v}{\partial q_i} \right). \quad (1.11)$$

$v_a = \bar{v}_a(q, p; \{v_\alpha\})$  can be solved,  $a = 1, 2, \dots, r_M$ ;  $v_\alpha$  cannot be solved,  $\alpha = r_M + 1, \dots, n$ , where  $r_M = \text{rank } M$ .

(need to show  $v_a = \bar{v}_a(q, p_a)$ )

*Primary constraints in the standard form*

$$\Phi_\alpha(q, p) := \left. \frac{\partial H^v}{\partial v_\alpha} \right|_{\{v_\alpha = \bar{v}_\alpha\}} \equiv p_\alpha - \bar{p}_\alpha(q, \{p_a\}), \quad (1.12)$$

where

$$\bar{p}_\alpha(q, \{p_a\}) := \left. \frac{\partial L^v}{\partial v_\alpha} \right|_{\{v_a = \bar{v}_a\}}. \quad (1.13)$$

*Total Hamiltonian*

$$H^t := H^v|_{\{v_a = \bar{v}_a\}} \equiv H^v(q, p; \{\bar{v}^a(q, p_a; \{v_\alpha\}), v_\alpha\}). \quad (1.14)$$

*Subspace of primary constraints*

$$\Gamma_P = \{(q, p) \mid \Phi_\alpha(q, p) = 0, \forall \alpha\} \quad (1.15)$$

Since

$$\frac{\partial H^t}{\partial v_\alpha} = \left. \frac{\partial H^v}{\partial v_\alpha} \right|_{\{v_a = \bar{v}_a\}} = \Phi_\alpha \equiv p_\alpha - \bar{p}_\alpha(q, \{p_a\}), \quad (1.16)$$

$H^t$  is linear in  $v_\alpha$ . One writes

$$H^t(q, \{p_a\}; \{p_\alpha\}, \{v_\alpha\}) = H(q, \{p_a\}) + \sum_\alpha v_\alpha \Phi_\alpha, \quad (1.17)$$

where  $H^c$  is the *canonical Hamiltonian* or simply *Hamiltonian*.

**Proposition**  $H^c$  is independent of  $\{p_\alpha\}$ .

**Proposition** Canonical equations with primary constraints

$$\dot{q}_i = [q_i, H]_p + \sum_\beta v_\beta [q_i, \phi_\beta]_p, \quad (1.18)$$

$$\dot{p}_i = [p_i, H]_p + \sum_\beta v_\beta [p_i, \phi_\beta]_p, \quad (1.19)$$

$$\Phi_\alpha(q, p) = 0, \quad (1.20)$$

where  $v_\beta$ 's are undetermined. Note that eq. (1.18) for  $i = \alpha$  holds identically:  $\dot{q}_\alpha = \dot{q}_\alpha$ .

Weak equality:  $f_1 \approx f_2$  iff  $f_1|_{\Gamma_P} = f_2|_{\Gamma_P}$ .

**Proposition** if  $f$  and  $g$  are two functions over the phase space  $\Gamma$ , and  $f \approx h$ , then

$$\frac{\partial}{\partial q_i} \left( f - \sum_{\beta} \phi_{\beta} \frac{\partial f}{\partial p_{\beta}} \right) \approx \frac{\partial}{\partial q_i} \left( h - \sum_{\beta} \phi_{\beta} \frac{\partial h}{\partial p_{\beta}} \right), \quad (1.21)$$

$$\frac{\partial}{\partial p_i} \left( f - \sum_{\beta} \phi_{\beta} \frac{\partial f}{\partial p_{\beta}} \right) \approx \frac{\partial}{\partial p_i} \left( h - \sum_{\beta} \phi_{\beta} \frac{\partial h}{\partial p_{\beta}} \right). \quad (1.22)$$

**Corollary**  $\forall H_1 \approx H$ ,

$$\dot{q}_i \approx [q_i, H]_{\text{P}}, \quad \dot{p}_i \approx [p_i, H]_{\text{P}}. \quad (1.23)$$

Primary and second constraints  $\phi_{\mu}^{(1,)}, \phi_{\omega}^{(2,)}$ ; first and second class constraints  $\phi_u^{(,1)}, \phi_w^{(,2)}$ .

## 2 Examples

### 2.1 Toy examples

#### Example 0

Gitman and Tyutin 1990, sec. 1.2

$$L = \frac{1}{2}(\dot{x} - y)^2 \quad (2.1)$$

#### Example 1

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x - y)^2. \quad (2.2)$$

One has

$$L^v = \frac{1}{2}v_x^2 + v_x y - \frac{1}{2}(x - y)^2, \quad (2.3)$$

so that

$$p_x = \frac{\partial L^v}{\partial v_x} = v_x + y, \quad p_y = 0, \quad (2.4)$$

thus

$$\bar{v}_x = p_x - y. \quad (2.5)$$

So that  $v_y$  is the primary inexpressible velocity.

The Hamiltonian with velocity reads

$$H^v(q, p; v) = v_x p_x + v_y p_y - \frac{1}{2}v_x^2 - v_x y + \frac{1}{2}(x - y)^2, \quad (2.6)$$

whilst the total Hamiltonian is

$$H^t(q, p; \bar{v}_x, v_y) = \frac{1}{2}(p_x - y)^2 + \frac{1}{2}(x - y)^2 + v_y p_y. \quad (2.7)$$

### Example 2

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x - y)^2 \quad (2.8)$$

Primary constraint

$$p_y = 0; \quad (2.9)$$

total Hamiltonian

$$H^t = \frac{1}{2}p_x^2 - p_x y - \frac{1}{2}x^2 + xy + v_y p_y. \quad (2.10)$$

### Example 3

$$L = \frac{1}{2}(\dot{q}_2 - \mathfrak{e}^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2. \quad (2.11)$$

## 2.2 Parametrised systems

### Non-relativistic point particle

Kiefer 2012, sec. 3.1.1

$$S[q(t)] := \int_{t_1}^{t_2} \mathfrak{d}t L\left(q, \frac{\mathfrak{d}q}{\mathfrak{d}t}\right) \quad (2.12)$$

### Relativistic charged point particle

Landau and Lifshitz 1975, sec. 16, Kiefer 2012, sec. 3.1.2

$$S := \int -m \mathfrak{d}s + e A_\mu(x) \mathfrak{d}x^\mu =: \int \mathfrak{d}\tau L, \quad (2.13)$$

where the Lagrangian reads

$$L = -m \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} + q \dot{x}^\mu A_\mu(x). \quad (2.14)$$

$$M_{\mu\nu} := \frac{\partial^2 L^\nu}{\partial v^\mu \partial v^\nu} = m \frac{-\eta_{\mu\nu} \eta_{\alpha\beta} + \eta_{\mu\alpha} \eta_{\nu\beta}}{(-\eta_{\rho\sigma} v^\rho v^\sigma)^{3/2}} v^\alpha v^\beta, \quad (2.15)$$

which has one and only one eigenvector with null eigenvalue

$$v^\mu M_{\mu\nu} = 0. \quad (2.16)$$

Momenta

$$p_\mu = \frac{\partial L^\nu}{\partial v^\mu} = \frac{m \eta_{\mu\nu} v^\nu}{\sqrt{-\eta_{\rho\sigma} v^\rho v^\sigma}} + q A_\mu. \quad (2.17)$$

If one chooses  $v^0$  to be the primary inexpressible velocity, then eliminating  $p_0$  in eq. (2.17) yields

$$v^i = \frac{\xi \eta^{ij} (p_j - qA_j) v^0}{\sqrt{m^2 + \eta^{kl} (p_k - qA_k) (p_l - qA_l)}}, \quad (2.18)$$

where  $\xi = \text{sgn } v^0$ . In the following  $\xi = +1$  will be chosen.

Inserting eq. (2.18) into the Hamiltonian with velocity

$$H^v = v^\mu p_\mu - L^v = m \sqrt{-\eta_{\mu\nu} v^\mu v^\nu} + v^\mu (p_\mu - qA_\mu(x)), \quad (2.19)$$

one obtains the total Hamiltonian

$$H^t = v^0 \left( p_0 - qA_0 + \sqrt{m^2 + \eta^{kl} (p_k - qA_k) (p_l - qA_l)} \right), \quad (2.20)$$

where only a primary constraint survives, which is obviously a first-class constraint

$$\phi^{(1,1)} = p_0 - qA_0 + \sqrt{m^2 + \eta^{kl} (p_k - qA_k) (p_l - qA_l)}, \quad (2.21)$$

and the canonical Hamiltonian vanishes

$$H^c = 0. \quad (2.22)$$

To compare, note in the non-covariant formalism (Landau and Lifshitz 1975, sec. 8)

$$S = \int dt L, \quad L = -m \sqrt{1 - \dot{\vec{x}}^2} - q\phi + q\dot{\vec{x}} \cdot \vec{A}, \quad (2.23)$$

the system is regular, and the canonical Hamiltonian reads

$$H^c = \sqrt{m^2 + (\vec{p} - q\vec{A})^2} + q\phi, \quad (2.24)$$

which corresponds to setting  $\phi^{(1,1)} = 0$ ,  $p_0 \rightarrow -H^c$  ( $p_\mu = (-E, \vec{p})$ ), and noting  $A_\mu = (-\phi, \vec{A})$ .

### Relativistic point particle with einbein

Blumenhagen, Lüst, and Theisen 2013, sec. 2.1

$$L := \frac{1}{2} (e^{-1} \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - m^2 e) \quad (2.25)$$

$$p_\mu = \frac{\partial L^v}{\partial v^\mu} = e^{-1} \eta_{\mu\nu} v^\nu, \quad p_e = 0. \quad (2.26)$$

Choosing  $v^e$  to be the primary inexpressible velocity, one has

$$v^\mu = e \eta^{\mu\nu} p_\nu. \quad (2.27)$$

Hamiltonian with velocity

$$H^v = v^\mu p_\mu + v^e p_e + \frac{1}{2}(-e^{-1}\eta_{\mu\nu}v^\mu v^\nu + m^2 e); \quad (2.28)$$

total Hamiltonian

$$H^t = \frac{e}{2}(\eta^{\mu\nu}p_\mu p_\nu + m^2) + v^e p_e; \quad (2.29)$$

canonical Hamiltonian

$$H^c = \frac{e}{2}(\eta^{\mu\nu}p_\mu p_\nu + m^2). \quad (2.30)$$

The only primary constraint

$$\Phi^{(1,)} = p_e; \quad (2.31)$$

its time evolution

$$\begin{aligned} [\Phi^{(1,)}, H^t]_p &= [p_e, e]_p \frac{1}{2}(\eta^{\mu\nu}p_\mu p_\nu + m^2) \\ &= -\frac{1}{2}(\eta^{\mu\nu}p_\mu p_\nu + m^2). \end{aligned} \quad (2.32)$$

Choose

$$\Phi^{(2,)} = \eta^{\mu\nu}p_\mu p_\nu + m^2, \quad (2.33)$$

whose Poisson bracket with  $H^t$  vanishes; furthermore,

$$[\Phi^{(1,)}, \Phi^{(2,)}]_p \equiv 0. \quad (2.34)$$

Thus one ends up with two first-class constraints.

### 2.2.1 Neutral scalar field

Kiefer 2012, sec. 3.3

## 2.3 Maxwell–Proca theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + A_\mu J^\mu, \quad (2.35)$$

where  $m > 0$  corresponds to the Proca theory Gitman and Tyutin 1990, sec. 2.3, and  $m = 0$  the Maxwell theory H. J. Rothe and K. D. Rothe 2010, sec. 3.3.3, Gitman and Tyutin 1990, sec. 2.4.

Lagrangian density with velocity

$$\mathcal{L}^v = \frac{1}{2}(V_i - \partial_i A_0)^2 - \frac{1}{4}F_{ij}^2 + \frac{m^2}{2}(A_0^2 - A_i^2) + A_0 J^0 + A_i J^i; \quad (2.36)$$

momenta density

$$B^0 := \frac{\partial \mathcal{L}^v}{\partial V_0} = 0, \quad B^i := \frac{\partial \mathcal{L}^v}{\partial V_i} = V^i + \partial_i A^0; \quad (2.37)$$

total and canonical Hamiltonian as well as primary constraint

$$\mathcal{H}^t = \mathcal{H}^c + V_0 \Phi^{(1,)}, \quad (2.38)$$

$$\mathcal{H}^c = \frac{1}{2} B_i^2 + B_i \partial_i A_0 + \frac{1}{4} F_{ij}^2 + \frac{m^2}{2} (-A_0^2 + A_i^2) - A_0 J^0 - A_i J^i, \quad (2.39)$$

$$\Phi^{(1,)} = B^0. \quad (2.40)$$

$$\begin{aligned} [\Phi^{(1,)}_1, \mathcal{H}_2^t]_p &= B^i{}_1 \partial_{i2} [B^0{}_2, A_{02}]_p + \frac{m^2}{2} [B^0{}_1, A_{02}^2]_p - [B^0{}_1, A_{02}]_p J^0{}_2 \\ &= [-B^i{}_2 \partial_{i2} - m^2 A_0 + J^0{}_2]_p \delta(x_1 - x_2). \end{aligned} \quad (2.41)$$

$$\Phi^{(2,)} = \partial_i B^i - m^2 A_0 + J^0{}_2. \quad (2.42)$$

## 2.4 String theories

### Nambu–Gotō action

Generalising the kinetic part of (2.13), one has

$$S_{\text{NG}} := -T \int_{\Sigma} \mathrm{d}A =: -T \int_{\Sigma} \mathrm{d}^2 \sigma \mathcal{L}, \quad (2.43)$$

where the Lagrangian density

$$\mathcal{L} = \sqrt{-\Gamma}, \quad \Gamma := \det \Gamma_{\alpha\beta}, \quad \Gamma_{\alpha\beta} := \frac{\partial X^\nu}{\partial \sigma^\alpha} \frac{\partial X_\nu}{\partial \sigma^\beta}. \quad (2.44)$$

Historically Gotō 1971; Nambu 1970; Reference e.g. Blumenhagen, Lüst, and Theisen 2013 Kiefer 2012, sec. 3.2

### Polyakov action

Generalising (2.25)

$$S_p[X^\mu, h_{\alpha\beta}] = -\frac{T}{2} \int_{\Sigma} \mathcal{L}, \quad (2.45)$$

where

$$\mathcal{L} := \sqrt{-h} h^{\alpha\beta} \Gamma_{\alpha\beta}. \quad (2.46)$$

Historically Brink, Di Vecchia, and Howe 1976; Deser and Zumino 1976; Polyakov 1981; Reference Kiefer 2012, sec. 3.2

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