

Notes on Canonical Singular Dynamics

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1 Canonical formalism

Lagrangian with velocity

$$L^\vee := L|_{\dot{q}=v} \quad (1.1)$$

Equations of motion

$$\sum_j M_{ij} \dot{v}_j = K_i^\vee, \quad \dot{q}_i = v_i. \quad (1.2)$$

where

$$M_{ij}(q, v) := \frac{\partial^2 L^\vee}{\partial v_i \partial v_j}. \quad (1.3)$$

Adding

$$p_i := \frac{\partial L^\vee}{\partial v_i}. \quad (1.4)$$

Variation of

$$S[q, p; v] := \int dt \left[L^\vee + \sum_i p_i (\dot{q}_i - v_i) \right]. \quad (1.5)$$

gives the *extended Euler–Lagrange equations*

$$\dot{q}_i = v_i, \quad \dot{p}_i = \frac{\partial L^\vee}{\partial q_i}, \quad p_i = \frac{\partial L^\vee}{\partial v_i}. \quad (1.6)$$

Extended Hamiltonian

$$H^\vee(q, p; v) := \sum_i p_i v_i - L^\vee. \quad (1.7)$$

Identities

$$\frac{\partial H^\vee}{\partial q_i} \equiv -\frac{\partial L^\vee}{\partial q_i}, \quad \frac{\partial H^\vee}{\partial p_i} \equiv v_i, \quad \frac{\partial H^\vee}{\partial v_i} \equiv p_i - \frac{\partial L^\vee}{\partial v_i}. \quad (1.8)$$

Variation of

$$S[q, p; v] := \int dt \left[\sum_i p_i \dot{q}_i - H^\vee \right] \quad (1.9)$$

gives the *extended canonical equations*

$$\dot{q}_i = [q_i, H^v]_p, \quad \dot{p}_i = [p_i, H^v]_p, \quad \frac{\partial H^v}{\partial v_i} = 0, \quad (1.10)$$

where the *Poisson bracket* is defined as

$$[f^v, g^v]_p := \sum_i \left(\frac{\partial f^v}{\partial q_i} \frac{\partial g^v}{\partial p_i} - \frac{\partial f^v}{\partial p_i} \frac{\partial g^v}{\partial q_i} \right). \quad (1.11)$$

$v_a = \bar{v}_a(q, p; \{v_\alpha\})$ can be solved, $a = 1, 2, \dots, r_M$; v_α cannot be solved, $\alpha = r_M + 1, \dots, n$, where $r_M = \text{rank } M$.

(need to show $v_a = \bar{v}_a(q, p_a)$)

Primary constraints in the standard form

$$\Phi_\alpha(q, p) := \left. \frac{\partial H^v}{\partial v_\alpha} \right|_{\{v_\alpha = \bar{v}_\alpha\}} \equiv p_\alpha - \bar{p}_\alpha(q, \{p_a\}), \quad (1.12)$$

where

$$\bar{p}_\alpha(q, \{p_a\}) := \left. \frac{\partial L^v}{\partial v_\alpha} \right|_{\{v_a = \bar{v}_a\}}. \quad (1.13)$$

Total Hamiltonian

$$H^t := H^v|_{\{v_a = \bar{v}_a\}} \equiv H^v(q, p; \{\bar{v}^a(q, p_a; \{v_\alpha\}), v_\alpha\}). \quad (1.14)$$

Subspace of primary constraints

$$\Gamma_P = \{(q, p) \mid \Phi_\alpha(q, p) = 0, \forall \alpha\} \quad (1.15)$$

Since

$$\frac{\partial H^t}{\partial v_\alpha} = \left. \frac{\partial H^v}{\partial v_\alpha} \right|_{\{v_a = \bar{v}_a\}} = \Phi_\alpha \equiv p_\alpha - \bar{p}_\alpha(q, \{p_a\}), \quad (1.16)$$

H^t is linear in v_α . One writes

$$H^t(q, \{p_a\}; \{p_\alpha\}, \{v_\alpha\}) = H(q, \{p_a\}) + \sum_\alpha v_\alpha \Phi_\alpha, \quad (1.17)$$

where H is the *canonical Hamiltonian* or simply *Hamiltonian*.

Proposition H is independent of $\{p_\alpha\}$.

Proposition Canonical equations with primary constraints

$$\dot{q}_i = [q_i, H]_p + \sum_\beta v_\beta [q_i, \phi_\beta]_p, \quad (1.18)$$

$$\dot{p}_i = [p_i, H]_p + \sum_\beta v_\beta [p_i, \phi_\beta]_p, \quad (1.19)$$

$$\Phi_\alpha(q, p) = 0, \quad (1.20)$$

where v_β 's are undetermined. Note that eq. (1.18) for $i = \alpha$ holds identically: $\dot{q}_\alpha = \dot{q}_\alpha$.

Weak equality: $f_1 \approx f_2$ iff $f_1|_{\Gamma_P} = f_2|_{\Gamma_P}$.

Proposition if f and g are two functions over the phase space Γ , and $f \approx h$, then

$$\frac{\partial}{\partial q_i} \left(f - \sum_{\beta} \phi_{\beta} \frac{\partial f}{\partial p_{\beta}} \right) \approx \frac{\partial}{\partial q_i} \left(h - \sum_{\beta} \phi_{\beta} \frac{\partial h}{\partial p_{\beta}} \right), \quad (1.21)$$

$$\frac{\partial}{\partial p_i} \left(f - \sum_{\beta} \phi_{\beta} \frac{\partial f}{\partial p_{\beta}} \right) \approx \frac{\partial}{\partial p_i} \left(h - \sum_{\beta} \phi_{\beta} \frac{\partial h}{\partial p_{\beta}} \right). \quad (1.22)$$

Corollary $\forall H_1 \approx H$,

$$\dot{q}_i \approx [q_i, H]_{\text{p}}, \quad \dot{p}_i \approx [p_i, H]_{\text{p}}. \quad (1.23)$$

Primary and second constraints $\phi_{\mu}^{(1,)}, \phi_{\omega}^{(2,)}$; first and second class constraints $\phi_u^{(,1)}, \phi_w^{(,2)}$.

2 Examples

2.1 Toy examples

Example 0

Gitman and Tyutin 1990, sec. 1.2

$$L = \frac{1}{2}(\dot{x} - y)^2 \quad (2.1)$$

Example 1

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x - y)^2. \quad (2.2)$$

One has

$$L^v = \frac{1}{2}v_x^2 + v_x y - \frac{1}{2}(x - y)^2, \quad (2.3)$$

so that

$$p_x = \frac{\partial L^v}{\partial v_x} = v_x + y, \quad p_y = 0, \quad (2.4)$$

thus

$$\bar{v}_x = p_x - y. \quad (2.5)$$

So that v_y is the primary inexpressible velocity.

The extended Hamiltonian reads

$$H^v(q, p; v) = v_x p_x + v_y p_y - \frac{1}{2}v_x^2 - v_x y + \frac{1}{2}(x - y)^2, \quad (2.6)$$

whilst the total Hamiltonian is

$$H^t(q, p; \bar{v}_x, v_y) = \frac{1}{2}(p_x - y)^2 + \frac{1}{2}(x - y)^2 + v_y p_y. \quad (2.7)$$