Notes on Lagrangian Singular Dynamics

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1 Classical formalism

Lagrangian with velocity

$$L^{\mathrm{v}} := L\big|_{\dot{q}=v} \tag{1.1}$$

Equations of motion

$$\sum_j M_{ij}\dot{v}_j = K_i^{\mathrm{v}}, \quad \dot{q}_i = v_i. \tag{1.2}$$

where

$$M_{ij}(q,v) := \frac{\partial^2 L}{\partial v_i\,\partial v_j}. \tag{1.3}$$

2 Examples

2.1 Toy examples

Example 0

Gitman and Tyutin 1990, sec. 1.2

$$L = \frac{1}{2}(\dot{x}-y)^2 \tag{2.1}$$

Example 1

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x-y)^2. \tag{2.2}$$

Example 2

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x-y)^2 \tag{2.3}$$

Example 3

$$L = \frac{1}{2}(\dot{q}_2 - \mathbb{e}^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2. \tag{2.4}$$

2.2 Parametrised systems

Non-relativistic point particle

Kiefer 2012, sec. 3.1.1

$$S[q(t)] := \int_{t_0}^{t_1} \mathrm{d}t\, L\Big(q, \frac{\mathrm{d}q}{\mathrm{d}t}\Big) \tag{2.5}$$

Relativistic charged point particle

Landau and Lifshitz 1975, sec. 16, Kiefer 2012, sec. 3.1.2

$$S := \int_\gamma -m\,\mathrm{d}s + eA_\mu(x)\,\mathrm{d}x^\mu =: \int_{\tau_0}^{\tau_1} \mathrm{d}\tau\, L, \tag{2.6}$$

where the Lagrangian reads

$$L = -m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + q\dot{x}^\mu A_\mu(x). \tag{2.7}$$

$$M_{\mu\nu} := \frac{\partial^2 L}{\partial\dot{x}^\mu\,\partial\dot{x}^\nu} = m\frac{-\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta}}{\big(-\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma\big)^{3/2}}\dot{x}^\alpha\dot{x}^\beta, \tag{2.8}$$

which has one and only one zero eigenvector

$$\dot{x}^\mu M_{\mu\nu} = 0. \tag{2.9}$$

Euler–Lagrange derivatives

$$E_\mu = \Big(\frac{\partial}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial}{\partial\dot{x}^\mu}\Big)L \equiv K_\mu - M_{\mu\nu}\ddot{x}^\nu, \tag{2.10}$$

where

$$K_\mu := -qF_{\mu\nu}\dot{x}^\nu, \tag{2.11}$$

and

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.12}$$

Contracting the zero eigenvector with eq. (2.10) yields

$$\dot{x}^\mu E_\mu = \dot{x}^\mu K_\mu - \dot{x}^\mu M_{\mu\nu}\ddot{x}^\nu \equiv 0, \tag{2.13}$$

so that it generates a gauge identity, and no further constraint exists. Thus the system has a symmetry

$$\delta x^\mu = \dot{x}^\mu\delta\lambda. \tag{2.14}$$

Relativistic point particle with einbein

Blumenhagen, Lüst, and Theisen 2013, sec. 2.1

$$L := \frac{1}{2}\big(e^{-1}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2 e\big) \tag{2.15}$$

Euler–Lagrange derivatives

$$E_\mu := \Big(\frac{\partial}{\partial x^\mu} - \frac{\mathbb{d}}{\mathbb{d}\tau}\frac{\partial}{\partial \dot{x}^\mu}\Big)L = e^{-1}\eta_{\mu\nu}\Big(\frac{\dot{e}}{e}\dot{x}^\nu - \ddot{x}^\nu\Big), \tag{2.16}$$

$$E_e := \Big(\frac{\partial}{\partial e} - \frac{\mathbb{d}}{\mathbb{d}\tau}\frac{\partial}{\partial \dot{e}}\Big)L = -\frac{1}{2}\Big(\frac{1}{e^2}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + m^2\Big), \tag{2.17}$$

and collectively $E^{(0)} = \begin{pmatrix} E_\mu & E_e \end{pmatrix}^\top$.

$$M^{(0)} := M = \begin{pmatrix} e^{-1}\eta_{\mu\nu} & 0^\mu \\ 0^\nu & 0 \end{pmatrix}, \tag{2.18}$$

so that the system is singular, with $w^{(0)} = (0^\mu; 1)$.

One can choose $u^{(0)} = (0^\mu; e^2)$, so that

$$\phi^{(0)} := u^{(0)} \cdot E^{(0)} = e^2 E_e \tag{2.19}$$

$$= -\frac{1}{2}\big(\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + m^2 e^2\big), \tag{2.20}$$

and thus

$$E_1^{(1)} := \dot{\phi}^{(0)} = e\big(2\dot{e}E_e + e\dot{E}_e\big) \tag{2.21}$$

$$= -m^2 e\dot{e} - \eta_{\mu\nu}\dot{x}^\mu\ddot{x}^\nu. \tag{2.22}$$

Collectively, $E^{(1)} = \Big(\big(E^{(0)}\big)^\top \quad E_1^{(1)}\Big)^\top$.

Straightforwardly,

$$M^{(1)} = \begin{pmatrix} e^{-1}\eta_{\mu\nu} & 0^\mu \\ 0^\nu & 0 \\ \eta_{\mu\nu}\dot{x}^\mu & 0 \end{pmatrix}, \tag{2.23}$$

and the new zero eigenvector $w^{(1)} = (e\dot{x}^\mu; 0, -1)$.

One finds that

$$w^{(1)} \cdot E^{(1)} = e\dot{x}^\mu E_\mu - E_1^{(1)} = e\big(\dot{x}^\mu E_\mu - 2\dot{e}E_e - e\dot{E}_e\big) \tag{2.24}$$

$$= \eta_{\mu\nu}\frac{\dot{e}}{e}\dot{x}^\mu\dot{x}^\nu + m^2 e\dot{e} = -2e\dot{e}E_e, \tag{2.25}$$

so that a gauge identity

$$G := \dot{x}^\mu E_\mu - e\dot{E}_e \equiv 0 \tag{2.26}$$

is obtained.

$$G\epsilon = E_\mu \dot{x}^\mu \epsilon + E_e(\dot{e}\epsilon + e\dot{\epsilon}) - \frac{\mathbb{d}}{\mathbb{d}\tau}(eE_e\epsilon), \tag{2.27}$$

so that

$$\delta x^\mu = \dot{x}^\mu \epsilon, \tag{2.28}$$

$$\delta e = \dot{e}\epsilon + e\dot{\epsilon}. \tag{2.29}$$

2.2.1 Neutral scalar field

Kiefer 2012, sec. 3.3

2.3 Maxwell–Proca theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + A_\mu J^\mu \tag{2.30}$$

where $m > 0$ corresponds to the Proca theory Gitman and Tyutin 1990, sec. 2.3, and $m = 0$ the Maxwell theory H. J. Rothe and K. D. Rothe 2010, sec. 3.3.3, Gitman and Tyutin 1990, sec. 2.4.

$$\mathcal{L} \equiv \frac{1}{2}(-\eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\alpha\nu}\eta^{\beta\mu})(\partial_\mu A_\alpha)(\partial_\nu A_\beta) - \frac{1}{2}m^2\eta^{\alpha\beta}A_\alpha A_\beta + A_\alpha J^\alpha \tag{2.31}$$

$$\begin{aligned} E^\alpha &= \left(\frac{\partial}{\partial A_\alpha} - \partial_\mu \frac{\partial}{\partial(\partial_\mu A_\alpha)}\right)\mathcal{L} \\ &= -m^2 A_\beta \eta^{\alpha\beta} + J^\alpha - (-\eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\alpha\nu}\eta^{\beta\mu})\partial_\mu\partial_\nu A_\beta. \end{aligned} \tag{2.32}$$

2.4 String theories

Nambu–Gotō action

Generalising the kinetic part of (2.6), one has

$$S_{\text{NG}} := -T\int_\Sigma \mathbb{d}A =: -T\int_\Sigma \mathbb{d}^2\sigma\mathcal{L}, \tag{2.33}$$

where the Lagrangian density

$$\mathcal{L} = \sqrt{-\Gamma}, \quad \Gamma := \det\Gamma_{\alpha\beta}, \quad \Gamma_{\alpha\beta} := \frac{\partial X^\nu}{\partial\sigma^\alpha}\frac{\partial X_\nu}{\partial\sigma^\alpha}. \tag{2.34}$$

Historically Gotō 1971; Nambu 1970; Reference e.g. Blumenhagen, Lüst, and Theisen 2013 Kiefer 2012, sec. 3.2

Polyakov action

Generalising (2.15)

$$S_{\mathrm{P}}[X^\mu, h\alpha\beta] = -\frac{T}{2}\int_\Sigma \mathcal{L}, \tag{2.35}$$

where

$$\mathcal{L} := \sqrt{-h}h^{\alpha\beta}\Gamma_{\alpha\beta}. \tag{2.36}$$

Historically Brink, Di Vecchia, and Howe 1976; Deser and Zumino 1976; Polyakov 1981; Reference Kiefer 2012, sec. 3.2

$$\surd \tag{2.37}$$

2.5 Gravitation theories

Closed Friedmann universe

This part adapts ibid., sec. 8.1.2.

The total action reads

$$S := S_{\mathrm{EG}} + S_\phi, \tag{2.38}$$

where S_{EG} follows (2.56), and

$$S_\phi := \int_{\mathcal{M}} \mathbb{d}^4x \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) - m^2\phi^2\right). \tag{2.39}$$

Adapting

$$\mathbb{d}s^2 = -N^2(t)\,\mathbb{d}t^2 + a^2(t)\,\mathbb{d}\Omega_3^2, \tag{2.40}$$

where

$$\Omega_3^2 = \mathbb{d}\chi^2 + \sin^2\chi\left(\mathbb{d}\theta^2 + \sin^2\theta\,\mathbb{d}\phi^2\right). \tag{2.41}$$

One has

$$\sqrt{-g} = Na^3\sin^2\chi\sin\theta, \qquad \sqrt{h} = a^3\sin^2\chi\sin\theta; \tag{2.42}$$

whereas

$$R = \frac{6}{N^2}\left(-\frac{\dot{N}\dot{a}}{Na} + \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) + \frac{6}{a^2}, \qquad K = \frac{3\dot{a}}{Na}. \tag{2.43}$$

$$S_{\mathrm{EG}} = \frac{A_3}{16\pi G}\left(\int_{t_1}^{t_2}\mathbb{d}tNa^3(R-2\Lambda) - \left[\frac{6\dot{a}a^2}{N}\right]_{t_1}^{t_2}\right), \tag{2.44}$$

where

$$A_3 = \int \sin^2\chi\sin\theta\,\mathbb{d}\chi\,\mathbb{d}\theta\,\mathbb{d}\phi = 2\pi^2. \tag{2.45}$$

The term proportional to $\ddot{a}/a$ in the integrand can be integrated by parts

$$\int_{t_1}^{t_2} \mathbb{d}t\, Na^3 \frac{6}{N^2}\frac{\ddot{a}}{a} = 6\left(\left[\frac{\dot{a}a^2}{N}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathbb{d}t\, \dot{a}\frac{\mathbb{d}}{\mathbb{d}t}\frac{a^2}{N^2}\right), \tag{2.46}$$

in which the first term cancels the Gibbons–Hawking–York term. One has

$$S_{\mathrm{EG}} = \frac{3\pi}{4G}\int_{t_1}^{t_2} \mathbb{d}t\left(-\frac{a}{N}\dot{a}^2 + Na - \frac{\Lambda}{3}Na^3\right). \tag{2.47}$$

The matter part of the action reads

$$S_\phi = \pi^2 \int_{t_1}^{t_2} \mathbb{d}t\, a^3\left(\frac{1}{N}\dot{\phi}^2 - m^2\phi^2\right). \tag{2.48}$$

One derives the Euler–Lagrange derivatives

$$E_N = \frac{3\pi}{4G}\left(\frac{a\dot{a}^2}{N^2} + a - \frac{\Lambda a^3}{3}\right) - \pi^2 a^3 \frac{\dot{\phi}^2}{N^2}, \tag{2.49}$$

$$E_a = \frac{3\pi}{4G}\left(-\frac{\dot{a}^2}{N} + N - \Lambda N a^2 + \frac{2a\ddot{a}}{N} - \frac{2a\dot{a}\dot{N}}{N^2}\right) + 3\pi^2 a^2\left(\frac{\dot{\phi}^2}{N} - m^2\phi^2\right), \tag{2.50}$$

$$E_\phi = 2\pi^2\left(-m^2a^3\phi - \frac{3a^2\dot{a}\dot{\phi}}{N} - \frac{a^3\ddot{\phi}}{N} + \frac{a^3\dot{\phi}\dot{N}}{N^2}\right), \tag{2.51}$$

and the primary mass matrix reads

$$\mathbf{M}^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{3\pi}{2G}\frac{a}{N} & 0 \\ 0 & 0 & 2\pi^2\frac{a^3}{N}, \end{pmatrix} \tag{2.52}$$

so that the system is singular, with $w^{(0)} = \begin{pmatrix}1 & 0 & 0\end{pmatrix} =: u^{(0)}$.

Therefore, the only primary constraint

$$\phi^{(0)} := u^{(0)} \cdot E^{(0)} = E_N, \tag{2.53}$$

and

$$E^{(1)} := \dot{\phi}^{(0)} = \dot{E}_N \tag{2.54}$$

$$= \frac{3\pi}{4G}\dot{a}\left(\frac{\dot{a}^2}{N^2} + 2\frac{a\ddot{a}}{N^2} - \frac{2a\dot{a}\dot{N}}{N^3} + 1 - \Lambda a^2\right) - \pi^2\frac{a^2\dot{\phi}}{N^2}\left(3\dot{a}\dot{\phi} + 2a\ddot{\phi} - 2a\frac{\dot{N}}{N}\right). \tag{2.55}$$

The secondary mass matrix

$$\mathbf{M}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{3\pi}{2G}\frac{a}{N} & 0 \\ 0 & 0 & 2\pi^2\frac{a^3}{N} \\ 0 & -\frac{3\pi}{2G}\frac{a\dot{a}}{N^2} & 2\pi^2\frac{a^3\dot{\phi}}{N^2}, \end{pmatrix} \tag{2.56}$$

and an additional left zero

$$w^{(1)} := \begin{pmatrix} 0 & \frac{\dot{a}}{N} & \frac{\dot{\phi}}{N} & -1 \end{pmatrix} =: v^{(1)} \tag{2.57}$$

is obtained, resulting in a gauge identity

$$0 \equiv G = \frac{\dot{a}}{N}E_a + \frac{\dot{\phi}}{N}E_\phi - \dot{E}_N, \tag{2.58}$$

which terminates the algorithm. The gauge transformation reads

$$\delta N = \dot{\epsilon}, \qquad \delta a = \frac{\dot{a}}{N}\epsilon, \qquad \delta\phi = \frac{\dot{\phi}}{N}\epsilon. \tag{2.59}$$

2.5.1 Einstein gravity

$$S_{\text{EG}} = S_{\text{EH}} + S_{\text{GHY}}, \tag{2.60}$$

where the Einstein–Hilbert action

$$S_{\text{EH}} = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}\,(R - 2\Lambda), \tag{2.61}$$

and the Gibbons–Hawking–York action

$$S_{\text{GHY}} = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^3x\,\sqrt{h}\,K, \tag{2.62}$$

which is named after Gibbons and Hawking 1977; York 1972 but actually already mentioned in Einstein 1916. See Dyer and Hinterbichler 2009 for a brief review.

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