Notes on Canonical Singular Dynamics

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1 Classical formalism

Lagrangian with velocity

$$L^{\mathrm{v}} := L\big|_{\dot{q}=v} \tag{1.1}$$

Equations of motion

$$\sum_j M_{ij}\dot{v}_j = K^{\mathrm{v}}_i, \quad \dot{q}_i = v_i. \tag{1.2}$$

where

$$M_{ij}(q,v) := \frac{\partial^2 L^{\mathrm{v}}}{\partial v_i \, \partial v_j}. \tag{1.3}$$

Adding

$$p_i := \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.4}$$

Variation of

$$S[q,p;v] := \int \mathrm{d}t \left[L^{\mathrm{v}} + \sum_i p_i(\dot{q}_i - v_i) \right]. \tag{1.5}$$

gives the *extended Euler–Lagrange equations*

$$\dot{q}_i = v_i, \quad \dot{p}_i = \frac{\partial L^{\mathrm{v}}}{\partial q_i}, \quad p_i = \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.6}$$

Hamiltonian with velocity

$$H^{\mathrm{v}}(q,p;v) := \sum_i p_i v_i - L^{\mathrm{v}}. \tag{1.7}$$

Identities

$$\frac{\partial H^{\mathrm{v}}}{\partial q_i} \equiv -\frac{\partial L^{\mathrm{v}}}{\partial q_i}, \quad \frac{\partial H^{\mathrm{v}}}{\partial p_i} \equiv v_i, \quad \frac{\partial H^{\mathrm{v}}}{\partial v_i} \equiv p_i - \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.8}$$

Variation of

$$S[q,p;v] := \int \mathrm{d}t \left[\sum_i p_i \dot{q}_i - H^{\mathrm{v}} \right] \tag{1.9}$$

gives the *extended canonical equations*

$$\dot{q}_i = [q_i, H^{\mathrm{v}}]_{\mathrm{P}}, \quad \dot{p}_i = [p_i, H^{\mathrm{v}}]_{\mathrm{P}}, \quad \frac{\partial H^{\mathrm{v}}}{\partial v_i} = 0, \tag{1.10}$$

where the *Poisson bracket* is defined as

$$[f^{\mathrm{v}}, g^{\mathrm{v}}]_{\mathrm{P}} := \sum_i \left(\frac{\partial f^{\mathrm{v}}}{\partial q_i} \frac{\partial g^{\mathrm{v}}}{\partial p_i} - \frac{\partial f^{\mathrm{v}}}{\partial p_i} \frac{\partial g^{\mathrm{v}}}{\partial q_i} \right). \tag{1.11}$$

$v_a = \overline{v}_a(q, p; \{v_\alpha\})$ can be solved, $a = 1, 2, \dots, r_M$; v_α cannot be solved, $\alpha = r_M + 1, \dots, n$, where $r_M = \operatorname{rank} M$.

(need to show $v_a = \overline{v}_a(q, p_a)$)

Primary constraints in the standard form

$$\Phi_\alpha(q, p) := \left. \frac{\partial H^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_\alpha = \overline{v}_\alpha\}} \equiv p_\alpha - \overline{p}_\alpha(q, \{p_a\}), \tag{1.12}$$

where

$$\overline{p}_\alpha(q, \{p_a\}) := \left. \frac{\partial L^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_a = \overline{v}_a\}}. \tag{1.13}$$

Total Hamiltonian

$$H^{\mathrm{t}} := H^{\mathrm{v}}|_{\{v_a = \overline{v}_a\}} \equiv H^{\mathrm{v}}(q, p; \{\overline{v}^a(q, p_a; \{v_\alpha\}), v_\alpha\}). \tag{1.14}$$

Subspace of primary constraints

$$\Gamma_{\mathrm{P}} = \{(q, p) \mid \Phi_\alpha(q, p) = 0, \forall \alpha\} \tag{1.15}$$

Since

$$\frac{\partial H^{\mathrm{t}}}{\partial v_\alpha} = \left. \frac{\partial H^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_a = \overline{v}_a\}} = \Phi_\alpha \equiv p_\alpha - \overline{p}_\alpha(q, \{p_a\}), \tag{1.16}$$

H^{t} is linear in v_α. One writes

$$H^{\mathrm{t}}(q, \{p_a\}; \{p_\alpha\}, \{v_\alpha\}) = H(q, \{p_a\}) + \sum_\alpha v_\alpha \Phi_\alpha, \tag{1.17}$$

where H^{c} is the *canonical Hamiltonian* or simply *Hamiltonian*.

Proposition H^{c} is independent of $\{p_\alpha\}$.

Proposition Canonical equations with primary constraints

$$\dot{q}_i = [q_i, H]_{\mathrm{P}} + \sum_\beta v_\beta [q_i, \phi_\beta]_{\mathrm{P}}, \tag{1.18}$$

$$\dot{p}_i = [p_i, H]_{\mathrm{P}} + \sum_\beta v_\beta [p_i, \phi_\beta]_{\mathrm{P}}, \tag{1.19}$$

$$\Phi_\alpha(q, p) = 0, \tag{1.20}$$

where v_β's are undetermined. Note that eq. (1.18) for $i = \alpha$ holds identically: $\dot{q}_\alpha = \dot{q}_\alpha$.

Weak equality: $f_1 \approx f_2$ iff $f_1|_{\Gamma_{\mathrm{P}}} = f_2|_{\Gamma_{\mathrm{P}}}$.

Proposition if f and g are two functions over the phase space Γ, and $f \approx h$, then

$$\frac{\partial}{\partial q_i}\left(f - \sum_\beta \phi_\beta \frac{\partial f}{\partial p_\beta}\right) \approx \frac{\partial}{\partial q_i}\left(h - \sum_\beta \phi_\beta \frac{\partial h}{\partial p_\beta}\right), \tag{1.21}$$

$$\frac{\partial}{\partial p_i}\left(f - \sum_\beta \phi_\beta \frac{\partial f}{\partial p_\beta}\right) \approx \frac{\partial}{\partial p_i}\left(h - \sum_\beta \phi_\beta \frac{\partial h}{\partial p_\beta}\right). \tag{1.22}$$

Corollary $\forall H_1 \approx H$,

$$\dot{q}_i \approx [q_i, H]_{\mathrm{P}}, \qquad \dot{p}_i \approx [p_i, H]_{\mathrm{P}}. \tag{1.23}$$

Primary and second constraints $\phi_\mu^{(1,)}$, $\phi_\omega^{(2,)}$; first and second class constraints $\phi_u^{(,1)}$, $\phi_w^{(,2)}$.

2 Examples

2.1 Toy examples

Example 0

Gitman and Tyutin 1990, sec. 1.2

$$L = \frac{1}{2}(\dot{x} - y)^2 \tag{2.1}$$

Example 1

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x - y)^2. \tag{2.2}$$

One has

$$L^{\mathrm{v}} = \frac{1}{2}v_x^2 + v_x y - \frac{1}{2}(x - y)^2, \tag{2.3}$$

so that

$$p_x = \frac{\partial L^{\mathrm{v}}}{\partial v_x} = v_x + y, \qquad p_y = 0, \tag{2.4}$$

thus

$$\overline{v}_x = p_x - y. \tag{2.5}$$

So that v_y is the primary inexpressible velocity.

The Hamiltonian with velocity reads

$$H^{\mathrm{v}}(q, p; v) = v_x p_x + v_y p_y - \frac{1}{2}v_x^2 - v_x y + \frac{1}{2}(x - y)^2, \tag{2.6}$$

whilst the total Hamiltonian is

$$H^{\mathrm{t}}(q, p; \overline{v}_x, v_y) = \frac{1}{2}(p_x - y)^2 + \frac{1}{2}(x - y)^2 + v_y p_y. \tag{2.7}$$

Example 2

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x-y)^2 \tag{2.8}$$

Primary constraint

$$p_y = 0; \tag{2.9}$$

total Hamiltonian

$$H^{\mathrm{t}} = \frac{1}{2}p_x^2 - p_x y - \frac{1}{2}x^2 + xy + v_y p_y. \tag{2.10}$$

Example 3

$$L = \frac{1}{2}(\dot{q}_2 - \mathbb{e}^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2. \tag{2.11}$$

2.2 Parametrised systems

Non-relativistic point particle

Kiefer 2012, sec. 3.1.1

$$S[q(t)] := \int_{t_1}^{t_2} \mathbb{d}t\, L\left(q, \frac{\mathbb{d}q}{\mathbb{d}t}\right) \tag{2.12}$$

Relativistic charged point particle

Landau and Lifshitz 1975, sec. 16, Kiefer 2012, sec. 3.1.2

$$S := \int -m\,\mathbb{d}s + eA_\mu(x)\,\mathbb{d}x^\mu =: \int \mathbb{d}\tau\, L, \tag{2.13}$$

where the Lagrangian reads

$$L = -m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + q\dot{x}^\mu A_\mu(x). \tag{2.14}$$

$$M_{\mu\nu} := \frac{\partial^2 L^{\mathrm{v}}}{\partial v^\mu\,\partial v^\nu} = m\frac{-\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta}}{\left(-\eta_{\rho\sigma}v^\rho v^\sigma\right)^{3/2}}v^\alpha v^\beta, \tag{2.15}$$

which has one and only one eigenvector with null eigenvalue

$$v^\mu M_{\mu\nu} = 0. \tag{2.16}$$

Momenta

$$p_\mu = \frac{\partial L^{\mathrm{v}}}{\partial v^\mu} = \frac{m\eta_{\mu\nu}v^\nu}{\sqrt{-\eta_{\rho\sigma}v^\rho v^\sigma}} + qA_\mu. \tag{2.17}$$

If one chooses v^0 to be the primary inexpressible velocity, then eliminating p_0 in eq. (2.17) yields

$$v^i = \frac{\xi \eta^{ij}(p_j - qA_j)v^0}{\sqrt{m^2 + \eta^{kl}(p_k - qA_k)(p_l - qA_l)}}, \tag{2.18}$$

where $\xi = \operatorname{sgn} v^0$. In the following $\xi = +1$ will be chosen.

Inserting eq. (2.18) into the Hamiltonian with velocity

$$H^{\mathrm{v}} = v^\mu p_\mu - L^{\mathrm{v}} = m\sqrt{-\eta_{\mu\nu}v^\mu v^\nu} + v^\mu\big(p_\mu - qA_\mu(x)\big), \tag{2.19}$$

one obtains the total Hamiltonian

$$H^{\mathrm{t}} = v^0\left(p_0 - qA_0 + \sqrt{m^2 + \eta^{kl}(p_k - qA_k)(p_l - qA_l)}\right), \tag{2.20}$$

where only a primary constraint survives, which is obviously a first-class constraint

$$\phi^{(1,1)} = p_0 - qA_0 + \sqrt{m^2 + \eta^{kl}(p_k - qA_k)(p_l - qA_l)}, \tag{2.21}$$

and the canonical Hamiltonian vanishes

$$H^{\mathrm{c}} = 0. \tag{2.22}$$

To compare, note in the non-covariant formalism (Landau and Lifshitz 1975, sec. 8)

$$S = \int \mathrm{d}t\, L, \qquad L = -m\sqrt{1 - \dot{\vec{x}}^2} - q\phi + q\dot{\vec{x}} \cdot \vec{A}, \tag{2.23}$$

the system is regular, and the canonical Hamiltonian reads

$$H^{\mathrm{c}} = \sqrt{m^2 + \left(\vec{p} - q\vec{A}\right)^2} + q\phi, \tag{2.24}$$

which corresponds to setting $\phi^{(1,1)} = 0$, $p_0 \to -H^{\mathrm{c}}$ ($p_\mu = (-E, \vec{p})$), and noting $A_\mu = \left(-\phi, \vec{A}\right)$.

Relativistic point particle with einbein

Blumenhagen, Lüst, and Theisen 2013, sec. 2.1

$$L := \frac{1}{2}\left(e^{-1}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2 e\right) \tag{2.25}$$

$$p_\mu = \frac{\partial L^{\mathrm{v}}}{\partial v^\mu} = e^{-1}\eta_{\mu\nu}v^\nu, \qquad p_e = 0. \tag{2.26}$$

Choosing v^e to be the primary inexpressible velocity, one has

$$v^\mu = e\eta^{\mu\nu}p_\nu. \tag{2.27}$$

Hamiltonian with velocity

$$H^{\mathrm{v}} = v^\mu p_\mu + v^e p_e + \frac{1}{2}\bigl(-e^{-1}\eta_{\mu\nu}v^\mu v^\nu + m^2 e\bigr); \tag{2.28}$$

total Hamiltonian

$$H^{\mathrm{t}} = \frac{e}{2}\bigl(\eta^{\mu\nu}p_\mu p_\nu + m^2\bigr) + v^e p_e; \tag{2.29}$$

canonical Hamiltonian

$$H^{\mathrm{c}} = \frac{e}{2}\bigl(\eta^{\mu\nu}p_\mu p_\nu + m^2\bigr). \tag{2.30}$$

The only primary constraint

$$\Phi^{(1,)} = p_e; \tag{2.31}$$

its time evolution

$$\begin{aligned}\bigl[\Phi^{(1,)}, H^{\mathrm{t}}\bigr]_{\mathrm{P}} &= [p_e, e]_{\mathrm{P}}\frac{1}{2}\bigl(\eta^{\mu\nu}p_\mu p_\nu + m^2\bigr) \\ &= -\frac{1}{2}\bigl(\eta^{\mu\nu}p_\mu p_\nu + m^2\bigr).\end{aligned} \tag{2.32}$$

Choose

$$\Phi^{(2,)} = \eta^{\mu\nu}p_\mu p_\nu + m^2, \tag{2.33}$$

whose Possion bracket with H^{t} vanishes; furthermore,

$$\bigl[\Phi^{(1,)}, \Phi^{(2,)}\bigr]_{\mathrm{P}} \equiv 0. \tag{2.34}$$

Thus one ends up with two first-class constraints.

2.2.1 Neutral scalar field

Kiefer 2012, sec. 3.3

2.3 Maxwell–Proca theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + A_\mu J^\mu, \tag{2.35}$$

where $m > 0$ corresponds to the Proca theory Gitman and Tyutin 1990, sec. 2.3, and $m = 0$ the Maxwell theory H. J. Rothe and K. D. Rothe 2010, sec. 3.3.3, Gitman and Tyutin 1990, sec. 2.4.

Lagrangian density with velocity

$$\mathcal{L}^{\mathrm{v}} = \frac{1}{2}(V_i - \partial_i A_0)^2 - \frac{1}{4}F_{ij}^2 + \frac{m^2}{2}(A_0^2 - A_i^2) + A_0 J^0 + A_i J^i; \tag{2.36}$$

momenta density

$$B^0 := \frac{\partial \mathcal{L}^{\mathrm{v}}}{\partial V_0} = 0, \qquad B^i := \frac{\partial \mathcal{L}^{\mathrm{v}}}{\partial V_i} = V^i + \partial_i A^0; \tag{2.37}$$

total and canonical Hamiltonian as well as primary constraint

$$\mathcal{H}^{\mathrm{t}} = \mathcal{H}^{\mathrm{c}} + V_0 \Phi^{(1,)}, \tag{2.38}$$

$$\mathcal{H}^{\mathrm{c}} = \frac{1}{2} B_i^2 + B_i \partial_i A_0 + \frac{1}{4} F_{ij}^2 + \frac{m^2}{2} (-A_0^2 + A_i^2) - A_0 J^0 - A_i J^i, \tag{2.39}$$

$$\Phi^{(1,)} = B^0. \tag{2.40}$$

$$\begin{aligned} \left[\Phi^{(1,)}{}_1, \mathcal{H}^{\mathrm{t}}_2\right]_{\mathrm{P}} &= B^i{}_1 \partial_{i_2} \left[B^0{}_2, A_{0_2}\right]_{\mathrm{P}} + \frac{m^2}{2} \left[B^0{}_1, A_{0_2}^2\right]_{\mathrm{P}} - \left[B^0{}_1, A_{0_2}\right]_{\mathrm{P}} J^0{}_2 \\ &= \left[-B^i{}_2 \partial_{i_2} - m^2 A_0 + J^0{}_2\right]_{\mathrm{P}} \delta(x_1 - x_2). \end{aligned} \tag{2.41}$$

$$\Phi^{(2,)} = \partial_i B^i - m^2 A_0 + J^0. \tag{2.42}$$

2.4 String theories

Nambu–Gotō action

Generalising the kinetic part of (2.13), one has

$$S_{\mathrm{NG}} := -T \int_\Sigma \mathbb{d} A =: -T \int_\Sigma \mathbb{d}^2 \sigma \mathcal{L}, \tag{2.43}$$

where the Lagrangian density

$$\mathcal{L} = \sqrt{-\Gamma}, \quad \Gamma := \det \Gamma_{\alpha\beta}, \quad \Gamma_{\alpha\beta} := \frac{\partial X^\nu}{\partial \sigma^\alpha} \frac{\partial X_\nu}{\partial \sigma^\alpha}. \tag{2.44}$$

Historically Gotō 1971; Nambu 1970; Reference e.g. Blumenhagen, Lüst, and Theisen 2013 Kiefer 2012, sec. 3.2

Polyakov action

Generalising (2.25)

$$S_{\mathrm{P}}[X^\mu, h\alpha\beta] = -\frac{T}{2} \int_\Sigma \mathcal{L}, \tag{2.45}$$

where

$$\mathcal{L} := \sqrt{-h} h^{\alpha\beta} \Gamma_{\alpha\beta}. \tag{2.46}$$

Historically Brink, Di Vecchia, and Howe 1976; Deser and Zumino 1976; Polyakov 1981; Reference Kiefer 2012, sec. 3.2

2.5 Gravitation theories

Closed Friedmann universe

Kiefer 2012, sec. 8.1.2 Adapting

$$\mathbb{d}s^2 = -N^2(t)\,\mathbb{d}t^2 + a^2(t)\,\mathbb{d}\Omega_3^2, \tag{2.47}$$

where

$$\Omega_3^2 = \mathbb{d}\chi^2 + \sin^2\chi\left(\mathbb{d}\theta^2 + \sin^2\theta\,\mathbb{d}\phi^2\right). \tag{2.48}$$

One has

$$\sqrt{-g} = Na^3\sin^2\chi\,\sin\theta, \qquad \sqrt{h} = a^3\sin^2\chi\,\sin\theta; \tag{2.49}$$

whereas

$$R = \frac{6}{N^2}\left(-\frac{\dot{N}\dot{a}}{Na} + \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) + \frac{6}{a^2}, \qquad K = \frac{3\dot{a}}{Na}. \tag{2.50}$$

$$L^{\mathrm{v}} = \frac{3\pi}{4G}\left(-\frac{a}{N}{v^a}^2 + Na - \frac{\Lambda}{3}Na^3\right) + \pi^2 a^3\left(\frac{1}{N^2}{v^\phi}^2 - m^2\phi^2\right). \tag{2.51}$$

2.5.1 Einstein–Hilbert action

$$S_{\mathrm{EG}} = S_{\mathrm{EH}} + S_{\mathrm{GHY}}, \tag{2.52}$$

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathbb{d}^4x\,\sqrt{-g}(R - 2\Lambda), \tag{2.53}$$

and

$$S_{\mathrm{GHY}} = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} \mathbb{d}^3x\,\sqrt{h}K, \tag{2.54}$$

which is named after Gibbons and Hawking 1977; York 1972 but actually already mentioned in Einstein 1916. See Dyer and Hinterbichler 2009 for a brief review.

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