

# Notes on Lagrangian Singular Dynamics

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## 1 Classical formalism

Lagrangian with velocity

$$L^\vee := L|_{\dot{q}=v} \quad (1.1)$$

Equations of motion

$$\sum_j M_{ij} \dot{v}_j = K_i^\vee, \quad \dot{q}_i = v_i. \quad (1.2)$$

where

$$M_{ij}(q, v) := \frac{\partial^2 L}{\partial v_i \partial v_j}. \quad (1.3)$$

## 2 Examples

### 2.1 Toy examples

Example 0

Gitman and Tyutin 1990, sec. 1.2

$$L = \frac{1}{2}(\dot{x} - y)^2 \quad (2.1)$$

Example 1

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x - y)^2. \quad (2.2)$$

Example 2

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x - y)^2 \quad (2.3)$$

Example 3

$$L = \frac{1}{2}(\dot{q}_2 - e^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2. \quad (2.4)$$

## 2.2 Parametrised systems

### Non-relativistic point particle

Kiefer 2012, sec. 3.1.1

$$S[q(t)] := \int_{t_1}^{t_2} \mathbb{d}t L\left(q, \frac{\mathbb{d}q}{\mathbb{d}t}\right) \quad (2.5)$$

### Relativistic charged point particle

Landau and Lifshitz 1975, sec. 16, Kiefer 2012, sec. 3.1.2

$$S := \int -m \mathbb{d}s + e A_\mu(x) \mathbb{d}x^\mu =: \int \mathbb{d}\tau L, \quad (2.6)$$

where the Lagrangian reads

$$L = -m \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} + q \dot{x}^\mu A_\mu(x). \quad (2.7)$$

$$M_{\mu\nu} := \frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu} = m \frac{-\eta_{\mu\nu} \eta_{\alpha\beta} + \eta_{\mu\alpha} \eta_{\nu\beta} \dot{x}^\alpha \dot{x}^\beta}{(-\eta_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma)^{3/2}}, \quad (2.8)$$

which has one and only one zero eigenvector

$$\dot{x}^\mu M_{\mu\nu} = 0. \quad (2.9)$$

Euler–Lagrange derivatives

$$E_\mu = \left( \frac{\partial}{\partial x^\mu} - \frac{\mathbb{d}}{\mathbb{d}\tau} \frac{\partial}{\partial \dot{x}^\mu} \right) L \equiv K_\mu - M_{\mu\nu} \ddot{x}^\nu, \quad (2.10)$$

where

$$K_\mu := -q F_{\mu\nu} \dot{x}^\nu, \quad (2.11)$$

and

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2.12)$$

Contracting the zero eigenvector with eq. (2.10) yields

$$\dot{x}^\mu E_\mu = \dot{x}^\mu K_\mu - \dot{x}^\mu M_{\mu\nu} \ddot{x}^\nu \equiv 0, \quad (2.13)$$

so that it generates a gauge identity, and no further constraint exists. Thus the system has a symmetry

$$\delta x^\mu = \dot{x}^\mu \delta \lambda. \quad (2.14)$$

## Relativistic point particle with auxiliary coordinate

Blumenhagen, Lüst, and Theisen 2013, sec. 2.1

$$L := \frac{1}{2}(e^{-1}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2 e) \quad (2.15)$$

Euler–Lagrange derivatives

$$E_\mu := \left( \frac{\partial}{\partial x^\mu} - \frac{d}{d\tau} \frac{\partial}{\partial \dot{x}^\mu} \right) L = e^{-1}\eta_{\mu\nu} \left( \frac{\dot{e}}{e}\dot{x}^\nu - \ddot{x}^\nu \right), \quad (2.16)$$

$$E_e := \left( \frac{\partial}{\partial e} - \frac{d}{d\tau} \frac{\partial}{\partial \dot{e}} \right) L = -\frac{1}{2} \left( \frac{1}{e^2} \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + m^2 \right), \quad (2.17)$$

and collectively  $E^{(0)} = (E_\mu \quad E_e)^\top$ .

$$M^{(0)} := M = \begin{pmatrix} e^{-1}\eta_{\mu\nu} & 0^\mu \\ 0^\nu & 0 \end{pmatrix}, \quad (2.18)$$

so that the system is singular, with  $w^{(0)} = (0^\mu; 1)$ .

One can choose  $u^{(0)} = (0^\mu; e^2)$ , so that

$$\phi^{(0)} := u^{(0)} \cdot E^{(0)} = e^2 E_e \quad (2.19)$$

$$= -\frac{1}{2}(\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + m^2 e^2), \quad (2.20)$$

and thus

$$E_1^{(1)} := \dot{\phi}^{(0)} = e(2\dot{e}E_e + e\dot{E}_e) \quad (2.21)$$

$$= -m^2 e\dot{e} - \eta_{\mu\nu}\dot{x}^\mu\ddot{x}^\nu. \quad (2.22)$$

Collectively,  $E^{(1)} = ((E^{(0)})^\top \quad E_1^{(1)})^\top$ .

Straightforwardly,

$$M^{(1)} = \begin{pmatrix} e^{-1}\eta_{\mu\nu} & 0^\mu \\ 0^\nu & 0 \\ \eta_{\mu\nu}\dot{x}^\mu & 0 \end{pmatrix}, \quad (2.23)$$

and the new zero eigenvector  $w^{(1)} = (e\dot{x}^\mu; 0, -1)$ .

One finds that

$$w^{(1)} \cdot E^{(1)} = e\dot{x}^\mu E_\mu - E_1^{(1)} = e(\dot{x}^\mu E_\mu - 2\dot{e}E_e - e\dot{E}_e) \quad (2.24)$$

$$= \eta_{\mu\nu} \frac{\dot{e}}{e} \dot{x}^\mu \dot{x}^\nu + m^2 e\dot{e} = -2e\dot{e}E_e, \quad (2.25)$$

so that a gauge identity

$$G := \dot{x}^\mu E_\mu - e \dot{E}_e \equiv 0 \quad (2.26)$$

is obtained.

$$G\epsilon = E_\mu \dot{x}^\mu \epsilon + E_e (\dot{e}\epsilon + e\dot{\epsilon}) - \frac{d}{dt}(eE_e \epsilon), \quad (2.27)$$

so that

$$\delta x^\mu = \dot{x}^\mu \epsilon, \quad (2.28)$$

$$\delta e = \dot{e}\epsilon + e\dot{\epsilon}. \quad (2.29)$$

### 2.2.1 Neutral scalar field

Kiefer 2012, sec. 3.3

## 2.3 Maxwell–Proca theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + \frac{1}{2}F_{\mu\nu}S^{\mu\nu} + A_\mu J_f^\mu, \quad (2.30)$$

where  $m > 0$  corresponds to the Proca theory Gitman and Tyutin 1990, sec. 2.3, and  $m = 0$  the Maxwell theory H. J. Rothe and K. D. Rothe 2010, sec. 3.3.3 Gitman and Tyutin 1990, sec. 2.4.

## 2.4 String theories

### Nambu–Goto action

Generalising the kinetic part of (2.6), one has

$$S_{\text{NG}} := -T \int_\Sigma dA =: -T \int_\Sigma d^2\sigma \mathcal{L}, \quad (2.31)$$

where the Lagrangian density

$$\mathcal{L} = \sqrt{-\Gamma}, \quad \Gamma := \det \Gamma_{\alpha\beta}, \quad \Gamma_{\alpha\beta} := \frac{\partial X^\nu}{\partial \sigma^\alpha} \frac{\partial X_\nu}{\partial \sigma^\beta}. \quad (2.32)$$

Historically Gotō 1971; Nambu 1970; Reference e.g. Blumenhagen, Lüst, and Theisen 2013 Kiefer 2012, sec. 3.2

### Polyakov action

Generalising (2.15)

$$S_{\text{P}}[X^\mu, h_{\alpha\beta}] = -\frac{T}{2} \int_\Sigma \mathcal{L}, \quad (2.33)$$

where

$$\mathcal{L} := \sqrt{-h} h^{\alpha\beta} \Gamma_{\alpha\beta}. \quad (2.34)$$

Historically Brink, Di Vecchia, and Howe 1976; Deser and Zumino 1976; Polyakov 1981; Reference Kiefer 2012, sec. 3.2

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