

# Notes on Lagrangian Singular Dynamics

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## 1 Classical formalism

Lagrangian with velocity

$$L^\vee := L|_{\dot{q}=v} \quad (1.1)$$

Equations of motion

$$\sum_j M_{ij} \dot{v}_j = K_i^\vee, \quad \dot{q}_i = v_i. \quad (1.2)$$

where

$$M_{ij}(q, v) := \frac{\partial^2 L}{\partial v_i \partial v_j}. \quad (1.3)$$

## 2 Examples

### 2.1 Toy examples

Example 0

Gitman and Tyutin 1990, sec. 1.2

$$L = \frac{1}{2}(\dot{x} - y)^2 \quad (2.1)$$

Example 1

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x - y)^2. \quad (2.2)$$

Example 2

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x - y)^2 \quad (2.3)$$

Example 3

$$L = \frac{1}{2}(\dot{q}_2 - e^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2. \quad (2.4)$$

## 2.2 Parametrised systems

### Non-relativistic point particle

Kiefer 2012, sec. 3.1.1

$$S[q(t)] := \int_{t_0}^{t_1} dt L\left(q, \frac{dq}{dt}\right) \quad (2.5)$$

### Relativistic charged point particle

Landau and Lifshitz 1975, sec. 16, Kiefer 2012, sec. 3.1.2

$$S := \int_{\gamma} -m ds + e A_{\mu}(x) dx^{\mu} =: \int_{\tau_0}^{\tau_1} d\tau L, \quad (2.6)$$

where the Lagrangian reads

$$L = -m \sqrt{-\eta_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}} + q \dot{x}^{\mu} A_{\mu}(x). \quad (2.7)$$

$$M_{\mu\nu} := \frac{\partial^2 L}{\partial \dot{x}^{\mu} \partial \dot{x}^{\nu}} = m \frac{-\eta_{\mu\nu} \eta_{\alpha\beta} + \eta_{\mu\alpha} \eta_{\nu\beta}}{(-\eta_{\rho\sigma} \dot{x}^{\rho} \dot{x}^{\sigma})^{3/2}} \dot{x}^{\alpha} \dot{x}^{\beta}, \quad (2.8)$$

which has one and only one zero eigenvector

$$\dot{x}^{\mu} M_{\mu\nu} = 0. \quad (2.9)$$

Euler–Lagrange derivatives

$$E_{\mu} = \left( \frac{\partial}{\partial x^{\mu}} - \frac{d}{d\tau} \frac{\partial}{\partial \dot{x}^{\mu}} \right) L \equiv K_{\mu} - M_{\mu\nu} \ddot{x}^{\nu}, \quad (2.10)$$

where

$$K_{\mu} := -q F_{\mu\nu} \dot{x}^{\nu}, \quad (2.11)$$

and

$$F_{\mu\nu} := \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}. \quad (2.12)$$

Contracting the zero eigenvector with eq. (2.10) yields

$$\dot{x}^{\mu} E_{\mu} = \dot{x}^{\mu} K_{\mu} - \dot{x}^{\mu} M_{\mu\nu} \ddot{x}^{\nu} \equiv 0, \quad (2.13)$$

so that it generates a gauge identity, and no further constraint exists. Thus the system has a symmetry

$$\delta x^{\mu} = \dot{x}^{\mu} \delta \lambda. \quad (2.14)$$

### Relativistic point particle with einbein

Blumenhagen, Lüst, and Theisen 2013, sec. 2.1

$$L := \frac{1}{2}(e^{-1}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2 e) \quad (2.15)$$

Euler–Lagrange derivatives

$$E_\mu := \left( \frac{\partial}{\partial x^\mu} - \frac{d}{d\tau} \frac{\partial}{\partial \dot{x}^\mu} \right) L = e^{-1}\eta_{\mu\nu} \left( \frac{\dot{e}}{e}\dot{x}^\nu - \ddot{x}^\nu \right), \quad (2.16)$$

$$E_e := \left( \frac{\partial}{\partial e} - \frac{d}{d\tau} \frac{\partial}{\partial \dot{e}} \right) L = -\frac{1}{2} \left( \frac{1}{e^2} \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + m^2 \right), \quad (2.17)$$

and collectively  $E^{(0)} = (E_\mu \quad E_e)^\top$ .

$$M^{(0)} := M = \begin{pmatrix} e^{-1}\eta_{\mu\nu} & 0^\mu \\ 0^\nu & 0 \end{pmatrix}, \quad (2.18)$$

so that the system is singular, with  $w^{(0)} = (0^\mu; 1)$ .

One can choose  $u^{(0)} = (0^\mu; e^2)$ , so that

$$\phi^{(0)} := u^{(0)} \cdot E^{(0)} = e^2 E_e \quad (2.19)$$

$$= -\frac{1}{2}(\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + m^2 e^2), \quad (2.20)$$

and thus

$$E_1^{(1)} := \dot{\phi}^{(0)} = e(2\dot{e}E_e + e\dot{E}_e) \quad (2.21)$$

$$= -m^2 e\dot{e} - \eta_{\mu\nu}\dot{x}^\mu\ddot{x}^\nu. \quad (2.22)$$

Collectively,  $E^{(1)} = ((E^{(0)})^\top \quad E_1^{(1)})^\top$ .

Straightforwardly,

$$M^{(1)} = \begin{pmatrix} e^{-1}\eta_{\mu\nu} & 0^\mu \\ 0^\nu & 0 \\ \eta_{\mu\nu}\dot{x}^\mu & 0 \end{pmatrix}, \quad (2.23)$$

and the new zero eigenvector  $w^{(1)} = (e\dot{x}^\mu; 0, -1)$ .

One finds that

$$w^{(1)} \cdot E^{(1)} = e\dot{x}^\mu E_\mu - E_1^{(1)} = e(\dot{x}^\mu E_\mu - 2\dot{e}E_e - e\dot{E}_e) \quad (2.24)$$

$$= \eta_{\mu\nu} \frac{\dot{e}}{e} \dot{x}^\mu \dot{x}^\nu + m^2 e\dot{e} = -2e\dot{e}E_e, \quad (2.25)$$

so that a gauge identity

$$G := \dot{x}^\mu E_\mu - e\dot{E}_e \equiv 0 \quad (2.26)$$

is obtained.

$$G\epsilon = E_\mu \dot{x}^\mu \epsilon + E_e (\dot{e}\epsilon + e\dot{\epsilon}) - \frac{d}{dt}(eE_e \epsilon), \quad (2.27)$$

so that

$$\delta x^\mu = \dot{x}^\mu \epsilon, \quad (2.28)$$

$$\delta e = \dot{e}\epsilon + e\dot{\epsilon}. \quad (2.29)$$

### 2.2.1 Neutral scalar field

Kiefer 2012, sec. 3.3

## 2.3 Maxwell–Proca theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + A_\mu J^\mu \quad (2.30)$$

where  $m > 0$  corresponds to the Proca theory Gitman and Tyutin 1990, sec. 2.3, and  $m = 0$  the Maxwell theory H. J. Rothe and K. D. Rothe 2010, sec. 3.3.3, Gitman and Tyutin 1990, sec. 2.4.

$$\mathcal{L} \equiv \frac{1}{2}(-\eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\alpha\nu}\eta^{\beta\mu})(\partial_\mu A_\alpha)(\partial_\nu A_\beta) - \frac{1}{2}m^2\eta^{\alpha\beta}A_\alpha A_\beta + A_\alpha J^\alpha \quad (2.31)$$

$$\begin{aligned} E^\alpha &= \left( \frac{\partial}{\partial A_\alpha} - \partial_\mu \frac{\partial}{\partial(\partial_\mu A_\alpha)} \right) \mathcal{L} \\ &= -m^2 A_\beta \eta^{\alpha\beta} + J^\alpha - (-\eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\alpha\nu}\eta^{\beta\mu})\partial_\mu \partial_\nu A_\beta. \end{aligned} \quad (2.32)$$

## 2.4 String theories

### Nambu–Goto action

Generalising the kinetic part of (2.6), one has

$$S_{\text{NG}} := -T \int_\Sigma dA =: -T \int_\Sigma d^2\sigma \mathcal{L}, \quad (2.33)$$

where the Lagrangian density

$$\mathcal{L} = \sqrt{-\Gamma}, \quad \Gamma := \det \Gamma_{\alpha\beta}, \quad \Gamma_{\alpha\beta} := \frac{\partial X^\nu}{\partial \sigma^\alpha} \frac{\partial X_\nu}{\partial \sigma^\beta}. \quad (2.34)$$

Historically Gotō 1971; Nambu 1970; Reference e.g. Blumenhagen, Lüst, and Theisen 2013 Kiefer 2012, sec. 3.2

## Polyakov action

Generalising (2.15)

$$S_{\text{P}}[X^\mu, h_{\alpha\beta}] = -\frac{T}{2} \int_{\Sigma} \mathcal{L}, \quad (2.35)$$

where

$$\mathcal{L} := \sqrt{-h} h^{\alpha\beta} \Gamma_{\alpha\beta}. \quad (2.36)$$

Historically Brink, Di Vecchia, and Howe 1976; Deser and Zumino 1976; Polyakov 1981; Reference Kiefer 2012, sec. 3.2

$$\sqrt{\phantom{x}} \quad (2.37)$$

## 2.5 Gravitation theories

### Closed Friedmann universe

This part adapts *ibid.*, sec. 8.1.2.

The total action reads

$$S := S_{\text{EG}} + S_{\phi}, \quad (2.38)$$

where  $S_{\text{EG}}$  follows (2.60), and

$$S_{\phi} := \int_{\mathcal{M}} \mathbb{d}^4 x \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} (\nabla_{\mu} \phi) (\nabla_{\nu} \phi) - m^2 \phi^2 \right). \quad (2.39)$$

Adapting

$$\mathbb{d}s^2 = -N^2(t) \mathbb{d}t^2 + a^2(t) \mathbb{d}\Omega_3^2, \quad (2.40)$$

where

$$\Omega_3^2 = \mathbb{d}\chi^2 + \sin^2 \chi (\mathbb{d}\theta^2 + \sin^2 \theta \mathbb{d}\phi^2). \quad (2.41)$$

One has

$$\sqrt{-g} = Na^3 \sin^2 \chi \sin \theta, \quad \sqrt{h} = a^3 \sin^2 \chi \sin \theta; \quad (2.42)$$

whereas

$$R = \frac{6}{N^2} \left( -\frac{\dot{N}\dot{a}}{Na} + \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right) + \frac{6}{a^2}, \quad K = \frac{3\dot{a}}{Na}. \quad (2.43)$$

$$S_{\text{EG}} = \frac{A_3}{16\pi G} \left( \int_{t_1}^{t_2} \mathbb{d}t Na^3 (R - 2\Lambda) - \left[ \frac{6\dot{a}a^2}{N} \right]_{t_1}^{t_2} \right), \quad (2.44)$$

where

$$A_3 = \int \sin^2 \chi \sin \theta \mathbb{d}\chi \mathbb{d}\theta \mathbb{d}\phi = 2\pi^2. \quad (2.45)$$

The term proportional to  $\ddot{a}/a$  in the integrand can be integrated by parts

$$\int_{t_1}^{t_2} dt N a^3 \frac{6}{N^2} \frac{\ddot{a}}{a} = 6 \left( \left[ \frac{\dot{a} a^2}{N} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \dot{a} \frac{d}{dt} \frac{a^2}{N^2} \right), \quad (2.46)$$

in which the first term cancels the Gibbons–Hawking–York term. One has

$$S_{\text{EG}} = \frac{3\mathbb{W}}{4G} \int_{t_1}^{t_2} dt \left( -\frac{a}{N} \dot{a}^2 + Na - \frac{\Lambda}{3} Na^3 \right). \quad (2.47)$$

The matter part of the action reads

$$S_\phi = \mathbb{W}^2 \int_{t_1}^{t_2} dt a^3 \left( \frac{1}{N} \dot{\phi}^2 - m^2 N \phi^2 \right). \quad (2.48)$$

One derives the Euler–Lagrange derivatives

$$E_N = \frac{3\mathbb{W}}{4G} \left( \frac{a \dot{a}^2}{N^2} + a - \frac{\Lambda a^3}{3} \right) - \mathbb{W}^2 a^3 \left( \frac{\dot{\phi}^2}{N^2} - m^2 N \phi^2 \right), \quad (2.49)$$

$$E_a = \frac{3\mathbb{W}}{4G} \left( -\frac{\dot{a}^2}{N} + N - \Lambda Na^2 + \frac{2a\ddot{a}}{N} - \frac{2a\dot{a}\dot{N}}{N^2} \right) + 3\mathbb{W}^2 a^2 \left( \frac{\dot{\phi}^2}{N} - m^2 N \phi^2 \right), \quad (2.50)$$

$$E_\phi = 2\mathbb{W}^2 \left( -m^2 Na^3 \phi - \frac{3a^2 \dot{a} \dot{\phi}}{N} - \frac{a^3 \ddot{\phi}}{N} + \frac{a^3 \dot{\phi} \dot{N}}{N^2} \right), \quad (2.51)$$

and the primary mass matrix reads

$$\mathbf{M}^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{3\mathbb{W}}{2G} \frac{a}{N} & 0 \\ 0 & 0 & 2\mathbb{W}^2 \frac{a^3}{N} \end{pmatrix} \quad (2.52)$$

so that the system is singular, with  $w^{(0)} = (1 \ 0 \ 0) =: u^{(0)}$ .

Therefore, the only primary constraint

$$\phi^{(0)} := u^{(0)} \cdot E^{(0)} = E_N, \quad (2.53)$$

and

$$E^{(1)} := \dot{\phi}^{(0)} = \dot{E}_N \quad (2.54)$$

$$\begin{aligned} &= \frac{3\mathbb{W}}{4G} \dot{a} \left( \frac{\dot{a}^2}{N^2} + 2 \frac{a\ddot{a}}{N^2} - \frac{2a\dot{a}\dot{N}}{N^3} + 1 - \Lambda a^2 \right) \\ &\quad - \mathbb{W}^2 a^2 \left( 3 \frac{\dot{a} \dot{\phi}^2}{N^2} + 2 \frac{a \dot{\phi} \ddot{\phi}}{N^2} - 2 \frac{a \dot{N} \dot{\phi}^2}{N^3} - m^2 \phi (3N \phi \dot{a} + a \phi \dot{N} + 2aN \dot{\phi}) \right). \end{aligned} \quad (2.55)$$

The secondary mass matrix

$$\mathbf{M}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{3\mathbb{I}}{2G} \frac{a}{N} & 0 \\ 0 & 0 & 2\mathbb{I}^2 \frac{a^3}{N^3} \\ 0 & -\frac{3\mathbb{I}}{2G} \frac{a\dot{a}}{N^2} & 2\mathbb{I}^2 \frac{a^3\dot{\phi}}{N^2} \end{pmatrix} \quad (2.56)$$

and an additional left zero

$$w^{(1)} := \begin{pmatrix} 0 & \dot{a} & \dot{\phi} & -N \end{pmatrix} =: v^{(1)} \quad (2.57)$$

is obtained, resulting in a gauge identity

$$0 \equiv G = \dot{a}E_a + \dot{\phi}E_\phi - N\dot{E}_N, \quad (2.58)$$

which terminates the algorithm. The gauge transformation reads

$$\delta N = -\dot{N}\epsilon - N\dot{\epsilon}, \quad \delta a = \dot{a}\epsilon, \quad \delta\phi = \dot{\phi}\epsilon. \quad (2.59)$$

### 2.5.1 Einstein gravity

$$S_{\text{EG}} = S_{\text{EH}} + S_{\text{GHY}}, \quad (2.60)$$

where the Einstein–Hilbert action

$$S_{\text{EH}} = \frac{1}{16\mathbb{I}G} \int_{\mathcal{M}} \mathbb{I}^4 x \sqrt{-g} (R - 2\Lambda), \quad (2.61)$$

and the Gibbons–Hawking–York action

$$S_{\text{GHY}} = -\frac{1}{8\mathbb{I}G} \int_{\partial\mathcal{M}} \mathbb{I}^3 x \sqrt{h} K, \quad (2.62)$$

which is named after Gibbons and Hawking 1977; York 1972 but actually already mentioned in Einstein 1916. See Dyer and Hinterbichler 2009 for a brief review.

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