Notes on Singular Hamiltonian Dynamics

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1 Hamiltonian formalism

Lagrangian with velocity

$$L^{\mathrm{v}}(q,v) := \left. L(q,\dot{q})\right|_{\dot{q}=v}, \tag{1.1}$$

Equations of motion

$$\dot{q}_i = v_i, \quad \sum_j M_{ij}\dot{v}_j = K_i^{\mathrm{v}}. \tag{1.2}$$

Adding

$$p_i := \frac{\partial L^{\mathrm{v}}}{\partial v_i} =: \overline{p}_i(q,v) \tag{1.3}$$

Extended Euler–Lagrange equations

$$\dot{q}_i = v_i, \quad \dot{p}_i = \frac{\partial L^{\mathrm{v}}}{\partial q_i}, \quad p_i = \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.4}$$

Variation with constraint equivalent to

$$S[q] = \int \mathbb{d}t \left(L^{\mathrm{v}} + \sum_i p_i(\dot{q}_i - v_i) \right) \tag{1.5}$$

Hamiltonian with velocities

$$H^{\mathrm{v}} := \sum_i p_i v_i - L^{\mathrm{v}}. \tag{1.6}$$

Identities

$$\frac{\partial H^{\mathrm{v}}}{\partial q_i} \equiv -\frac{\partial L^{\mathrm{v}}}{\partial q_i}, \quad \frac{\partial H^{\mathrm{v}}}{\partial p_i} \equiv v_i, \quad \frac{\partial H^{\mathrm{v}}}{\partial v_i} \equiv p_i - \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.7}$$

Extended canonical equations

$$\dot{q}_i = [q_i, H^{\mathrm{v}}]_{\mathrm{TP}}, \quad \dot{p}_i = [p_i, H^{\mathrm{v}}]_{\mathrm{TP}}, \quad \frac{\partial H^{\mathrm{v}}}{\partial v_i} = 0. \tag{1.8}$$

total Poisson brackets

$$[f^{\mathrm{v}}(q,p,v), g^{\mathrm{v}}(q,p,v)]_{\mathrm{TP}} := \sum_i \left(\frac{\partial f^{\mathrm{v}}}{\partial q_i}\frac{\partial g^{\mathrm{v}}}{\partial p_i} - \frac{\partial f^{\mathrm{v}}}{\partial p_i}\frac{\partial g^{\mathrm{v}}}{\partial q_i} \right). \tag{1.9}$$

$v_a = \overline{v}_a(q,p)$ can be solved, $a = 1, 2, \ldots, r_M$; v_α cannot be solved, $\alpha = r_M + 1, \ldots, n$.

$$M_{ij}(q,v) = \frac{\partial^2 L^{\mathrm{v}}}{\partial v_i \partial v_j} \tag{1.10}$$

$r_M = \operatorname{rank} M$.

$$\phi_\alpha(q,p) := \left.\frac{\partial H^{\mathrm{v}}}{\partial v_\alpha}\right|_{\{v_\alpha = \overline{v}_a\}} \equiv p_\alpha - \left.\frac{\partial L^{\mathrm{v}}}{\partial v_\alpha}\right|_{\{v_\alpha = \overline{v}_a\}}. \tag{1.11}$$

Total Hamiltonian

$$H^{\mathrm{t}} := \tag{1.12}$$

2 Examples

2.1 Point-particle system quadratic in velocity

$$L = \frac{1}{2} M_{ij}(q)\dot{q}^i \dot{q}^j + \eta_i(q)\dot{q}^i - V(q). \tag{2.1}$$

The Lagrangian with velocity reads

$$L^{\mathrm{v}} = \frac{1}{2}\sum_{i,j} M_{ij}(q) v_i v_j + \sum_i \eta_i(q) v_i - V(q). \tag{2.2}$$

$$p_i = \frac{\partial L^{\mathrm{v}}}{\partial v_i} = \sum_j M_{ij} v_j + \eta_i. \tag{2.3}$$

Let

$$\sum_j M_{ij} w_j^{(a)} = \lambda^{(a)} w_i^{(a)} \neq 0, \tag{2.4}$$

$$\sum_j M_{ij} w_j^{(\alpha)} = 0, \tag{2.5}$$

where $a = 1, 2, \dots, r_M$; $\alpha = r_M + 1, \dots, n$. Projecting (2.3) on w^k, one has

$$\sum_j p_j w_j^{(b)} = \lambda^{(b)} \sum_j w_j^{(b)} v_j + \sum_j \eta_j w_j^{(b)}, \tag{2.6}$$

$$\sum_j p_j w_j^{(\beta)} = 0 + \sum_j \eta_j w_j^{(\beta)}. \tag{2.7}$$

One uses eq. (2.6) to solve for v_a,

$$\sum_a v_a W_{ab} = F_b, \tag{2.8}$$

where

$$W_{ab} = w_a^{(b)}, \tag{2.9}$$

$$F_b = \frac{1}{\lambda^{(b)}} \sum_j \left(p_j w_j^{(b)} - \eta_j w_j^{(b)}\right) - \sum_\beta w_\beta^{(b)} v_\beta. \tag{2.10}$$