Notes on Canonical Singular Dynamics

Yi-Fan Wang (王 一帆)

May 23, 2017

1 Classical formalism

Lagrangian with velocity

$$L^{\mathrm{v}} := L\big|_{\dot{q}=v} \tag{1.1}$$

Equations of motion

$$\sum_j M_{ij}\dot{v}_j = K_i^{\mathrm{v}}, \quad \dot{q}_i = v_i. \tag{1.2}$$

where

$$M_{ij}(q,v) := \frac{\partial^2 L^{\mathrm{v}}}{\partial v_i\,\partial v_j}. \tag{1.3}$$

Adding

$$p_i := \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.4}$$

Variation of

$$S[q,p;v] := \int \mathrm{d}t \left[L^{\mathrm{v}} + \sum_i p_i(\dot{q}_i - v_i)\right]. \tag{1.5}$$

gives the *Euler–Lagrange equations with velocities*

$$\dot{q}_i = v_i, \quad \dot{p}_i = \frac{\partial L^{\mathrm{v}}}{\partial q_i}, \quad p_i = \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.6}$$

Hamiltonian with velocity

$$H^{\mathrm{v}}(q,p;v) := \sum_i p_i v_i - L^{\mathrm{v}}. \tag{1.7}$$

Identities

$$\frac{\partial H^{\mathrm{v}}}{\partial q_i} \equiv -\frac{\partial L^{\mathrm{v}}}{\partial q_i}, \quad \frac{\partial H^{\mathrm{v}}}{\partial p_i} \equiv v_i, \quad \frac{\partial H^{\mathrm{v}}}{\partial v_i} \equiv p_i - \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.8}$$

Variation of

$$S[q,p;v] := \int \mathrm{d}t \left[\sum_i p_i\dot{q}_i - H^{\mathrm{v}}\right] \tag{1.9}$$

gives the *canonical equations with velocities*

$$\dot{q}_i = [q_i, H^{\mathrm{v}}]_{\mathrm{P}}, \quad \dot{p}_i = [p_i, H^{\mathrm{v}}]_{\mathrm{P}}, \quad \frac{\partial H^{\mathrm{v}}}{\partial v_i} = 0, \tag{1.10}$$

where the *Poisson bracket* is defined as

$$[f^{\mathrm{v}}, g^{\mathrm{v}}]_{\mathrm{P}} := \sum_i \left(\frac{\partial f^{\mathrm{v}}}{\partial q_i} \frac{\partial g^{\mathrm{v}}}{\partial p_i} - \frac{\partial f^{\mathrm{v}}}{\partial p_i} \frac{\partial g^{\mathrm{v}}}{\partial q_i} \right). \tag{1.11}$$

$v_a = \overline{v}_a(q, p; \{v_\alpha\})$ can be solved, $a = 1, 2, \dots, r_M$; v_α cannot be solved, $\alpha = r_M + 1, \dots, n$, where $r_M = \operatorname{rank} M$.

(need to show $v_a = \overline{v}_a(q, p_a)$)

Primary constraints in the standard form

$$\Phi_\alpha(q, p) := \left. \frac{\partial H^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_\alpha = \overline{v}_\alpha\}} \equiv p_\alpha - \overline{p}_\alpha(q, \{p_a\}), \tag{1.12}$$

where

$$\overline{p}_\alpha(q, \{p_a\}) := \left. \frac{\partial L^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_a = \overline{v}_a\}}. \tag{1.13}$$

Hamiltonian with primary constraint

$$H^{\mathrm{p}} := \left. H^{\mathrm{v}} \right|_{\{v_a = \overline{v}_a\}} \equiv H^{\mathrm{v}}(q, p; \{\overline{v}^a(q, p_a; \{v_\alpha\}), v_\alpha\}). \tag{1.14}$$

Subspace of primary constraints

$$\Gamma_{\mathrm{P}} = \{(q, p) \mid \Phi_\alpha(q, p) = 0, \forall \alpha\} \tag{1.15}$$

Since

$$\frac{\partial H^{\mathrm{p}}}{\partial v_\alpha} = \left. \frac{\partial H^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_a = \overline{v}_a\}} = \Phi_\alpha \equiv p_\alpha - \overline{p}_\alpha(q, \{p_a\}), \tag{1.16}$$

H^{p} is linear in v_α. One writes

$$H^{\mathrm{p}}(q, \{p_a\}; \{p_\alpha\}, \{v_\alpha\}) = H(q, \{p_a\}) + \sum_\alpha v_\alpha \Phi_\alpha, \tag{1.17}$$

where H^{c} is the *canonical Hamiltonian* or simply *Hamiltonian.*

Proposition H^{c} is independent of $\{p_\alpha\}$.

Proposition Canonical equations with primary constraints

$$\dot{q}_i = [q_i, H]_{\mathrm{P}} + \sum_\beta v_\beta [q_i, \phi_\beta]_{\mathrm{P}}, \tag{1.18}$$

$$\dot{p}_i = [p_i, H]_{\mathrm{P}} + \sum_\beta v_\beta [p_i, \phi_\beta]_{\mathrm{P}}, \tag{1.19}$$

$$\Phi_\alpha(q, p) = 0, \tag{1.20}$$

where v_β's are undetermined. Note that eq. (1.18) for $i = \alpha$ holds identically: $\dot{q}_\alpha = \dot{q}_\alpha$.

Weak equality: $f_1 \approx f_2$ iff $\left. f_1 \right|_{\Gamma_{\mathrm{P}}} = \left. f_2 \right|_{\Gamma_{\mathrm{P}}}$.

Proposition if f and g are two functions over the phase space Γ, and $f \approx h$, then

$$\frac{\partial}{\partial q_i}\left(f - \sum_\beta \phi_\beta \frac{\partial f}{\partial p_\beta}\right) \approx \frac{\partial}{\partial q_i}\left(h - \sum_\beta \phi_\beta \frac{\partial h}{\partial p_\beta}\right), \tag{1.21}$$

$$\frac{\partial}{\partial p_i}\left(f - \sum_\beta \phi_\beta \frac{\partial f}{\partial p_\beta}\right) \approx \frac{\partial}{\partial p_i}\left(h - \sum_\beta \phi_\beta \frac{\partial h}{\partial p_\beta}\right). \tag{1.22}$$

Corollary $\forall H_1 \approx H$,

$$\dot{q}_i \approx [q_i, H]_{\mathrm{p}}, \qquad \dot{p}_i \approx [p_i, H]_{\mathrm{p}}. \tag{1.23}$$

Primary and second constraints $\phi_\mu^{(1,)}$, $\phi_\omega^{(2,)}$; first and second class constraints $\phi_u^{(,1)}$, $\phi_w^{(,2)}$.

2 Examples

2.1 Toy examples

Example 0

Gitman and Tyutin 1990, sec. 1.2

$$L = \frac{1}{2}(\dot{x} - y)^2 \tag{2.1}$$

Example 1

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x-y)^2. \tag{2.2}$$

One has

$$L^{\mathrm{v}} = \frac{1}{2}v_x^2 + v_x y - \frac{1}{2}(x-y)^2, \tag{2.3}$$

so that

$$p_x = \frac{\partial L^{\mathrm{v}}}{\partial v_x} = v_x + y, \qquad p_y = 0, \tag{2.4}$$

thus

$$\overline{v}_x = p_x - y. \tag{2.5}$$

So that v_y is the primary inexpressible velocity.

The Hamiltonian with velocity reads

$$H^{\mathrm{v}}(q,p;v) = v_x p_x + v_y p_y - \frac{1}{2}v_x^2 - v_x y + \frac{1}{2}(x-y)^2, \tag{2.6}$$

whilst the Hamiltonians are

$$H^{\mathrm{p}}\big(q,p;\overline{v}_x,v_y\big) = H^{\mathrm{c}} + v_y\Phi_1, \tag{2.7}$$

$$H^{\mathrm{c}}(q,p) = \frac{1}{2}(p_x - y)^2 + \frac{1}{2}(x-y)^2, \tag{2.8}$$

where the only primary constraint $\Phi_1 = p_y$.

Persistence condition of Φ_1 leads to

$$0 \approx [\Phi_1, H^{\mathrm{p}}]_{\mathrm{P}} = p_x + x - 2y =: \Phi_2. \tag{2.9}$$

Note that $[\Phi_1, \Phi_2] = p_x - x$ does not vanish on Γ, thus $\Phi_{1,2}$ are second-class constraints, and no more constraint can be generated. To solve for v_y one evaluates

$$0 =: [\Phi_1, H^{\mathrm{p}}]_{\mathrm{P}} = p_x - x - 2v_y, \tag{2.10}$$

so that $v_y := (p_x - x)/2$ solves the constraint.

One also has

$$Q_{\alpha\beta} = \big[\Phi_\alpha, \Phi_\beta\big]_{\mathrm{P}} = \begin{pmatrix} 0 & +2 \\ -2 & 0 \end{pmatrix}, \qquad \big(Q^{-1}\big)_{\alpha\beta} = \begin{pmatrix} 0 & -1/2 \\ +1/2 & 0 \end{pmatrix}, \tag{2.11}$$

so that the Dirac brackets are defined as

$$[f,g]_{\mathrm{D}} := [f,g]_{\mathrm{P}} + \frac{1}{2}\Big(\big[f, p_y\big]_{\mathrm{P}}[p_x + x - 2y, g]_{\mathrm{P}} - [f, p_x + x - 2y]_{\mathrm{P}}\big[p_y, g\big]_{\mathrm{P}} \Big). \tag{2.12}$$

The fundamental ones different from Possion brackets are

$$[x,y]_{\mathrm{D}} = [y, p_x]_{\mathrm{D}} = \frac{1}{2}, \qquad \big[y, p_y\big]_{\mathrm{D}} = 0. \tag{2.13}$$

Last one different from book?

Example 2

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x-y)^2 \tag{2.14}$$

Primary constraint

$$p_y = 0; \tag{2.15}$$

Hamiltonian with primary constraint

$$H^{\mathrm{p}} = \frac{1}{2}p_x^2 - p_x y - \frac{1}{2}x^2 + xy + v_y p_y. \tag{2.16}$$

Example 3

$$L = \frac{1}{2}(\dot{q}_2 - \mathbb{e}^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2. \tag{2.17}$$

2.2 Parametrised systems

Non-relativistic point particle

Kiefer 2012, sec. 3.1.1

$$S[q(t)] := \int_{t_1}^{t_2} \mathrm{d}t\, L\Big(q, \frac{\mathrm{d}q}{\mathrm{d}t}\Big) \tag{2.18}$$

Relativistic charged point particle

Landau and Lifshitz 1975, sec. 16, Kiefer 2012, sec. 3.1.2

$$S := \int -m\,\mathrm{d}s + eA_\mu(x)\,\mathrm{d}x^\mu =: \int \mathrm{d}\tau\, L, \tag{2.19}$$

where the Lagrangian reads

$$L = -m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + q\dot{x}^\mu A_\mu(x). \tag{2.20}$$

$$M_{\mu\nu} := \frac{\partial^2 L^{\mathrm{v}}}{\partial v^\mu\,\partial v^\nu} = m\frac{-\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta}}{\big(-\eta_{\rho\sigma}v^\rho v^\sigma\big)^{3/2}} v^\alpha v^\beta, \tag{2.21}$$

which has one and only one eigenvector with null eigenvalue

$$v^\mu M_{\mu\nu} = 0. \tag{2.22}$$

Momenta

$$p_\mu = \frac{\partial L^{\mathrm{v}}}{\partial v^\mu} = \frac{m\eta_{\mu\nu}v^\nu}{\sqrt{-\eta_{\rho\sigma}v^\rho v^\sigma}} + qA_\mu. \tag{2.23}$$

If one chooses v^0 to be the primary inexpressible velocity, then eliminating p_0 in eq. (2.23) yields

$$v^i = \frac{\xi\eta^{ij}(p_j - qA_j)v^0}{\sqrt{m^2 + \eta^{kl}(p_k - qA_k)(p_l - qA_l)}}, \tag{2.24}$$

where $\xi = \operatorname{sgn} v^0$. In the following $\xi = +1$ will be chosen.

Inserting eq. (2.24) into the Hamiltonian with velocity

$$H^{\mathrm{v}} = v^\mu p_\mu - L^{\mathrm{v}} = m\sqrt{-\eta_{\mu\nu}v^\mu v^\nu} + v^\mu\big(p_\mu - qA_\mu(x)\big), \tag{2.25}$$

one obtains the Hamiltonian with primary constraint

$$H^{\mathrm{p}} = v^0\Big(p_0 - qA_0 + \sqrt{m^2 + \eta^{kl}(p_k - qA_k)(p_l - qA_l)}\Big), \tag{2.26}$$

where only a primary constraint survives, which is obviously a first-class constraint

$$\phi^{(1,1)} = p_0 - qA_0 + \sqrt{m^2 + \eta^{kl}(p_k - qA_k)(p_l - qA_l)}, \tag{2.27}$$

and the canonical Hamiltonian vanishes

$$H^{\mathrm{c}} = 0. \tag{2.28}$$

To compare, note in the non-covariant formalism (Landau and Lifshitz 1975, sec. 8)

$$S = \int \mathrm{d}t\, L, \qquad L = -m\sqrt{1 - \dot{\vec{x}}^2} - q\phi + q\dot{\vec{x}} \cdot \vec{A}, \tag{2.29}$$

the system is regular, and the canonical Hamiltonian reads

$$H^{\mathrm{c}} = \sqrt{m^2 + \left(\vec{p} - q\vec{A}\right)^2} + q\phi, \tag{2.30}$$

which corresponds to setting $\phi^{(1,1)} = 0$, $p_0 \to -H^{\mathrm{c}}$ ($p_\mu = (-E, \vec{p})$), and noting $A_\mu = \left(-\phi, \vec{A}\right)$.

Relativistic point particle with einbein

Blumenhagen, Lüst, and Theisen 2013, sec. 2.1

$$L := \frac{1}{2}\left(e^{-1}\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - m^2 e\right) \tag{2.31}$$

$$p_\mu = \frac{\partial L^{\mathrm{v}}}{\partial v^\mu} = e^{-1}\eta_{\mu\nu}v^\nu, \qquad p_e = 0. \tag{2.32}$$

Choosing v^e to be the primary inexpressible velocity, one has

$$v^\mu = e\eta^{\mu\nu}p_\nu. \tag{2.33}$$

Hamiltonian with velocity

$$H^{\mathrm{v}} = v^\mu p_\mu + v^e p_e + \frac{1}{2}\left(-e^{-1}\eta_{\mu\nu}v^\mu v^\nu + m^2 e\right); \tag{2.34}$$

Hamiltonian with primary constraint

$$H^{\mathrm{p}} = \frac{e}{2}\left(\eta^{\mu\nu}p_\mu p_\nu + m^2\right) + v^e p_e; \tag{2.35}$$

canonical Hamiltonian

$$H^{\mathrm{c}} = \frac{e}{2}\left(\eta^{\mu\nu}p_\mu p_\nu + m^2\right). \tag{2.36}$$

The only primary constraint

$$\Phi^{(1,)} = p_e; \tag{2.37}$$

its time evolution

$$\left[\Phi^{(1,)}, H^{\mathrm{p}}\right]_{\mathrm{P}} = [p_e, e]_{\mathrm{P}} \frac{1}{2}\left(\eta^{\mu\nu} p_\mu p_\nu + m^2\right)$$
$$= -\frac{1}{2}\left(\eta^{\mu\nu} p_\mu p_\nu + m^2\right). \tag{2.38}$$

Choose

$$\Phi^{(2,)} = \eta^{\mu\nu} p_\mu p_\nu + m^2, \tag{2.39}$$

whose Possion bracket with H^{p} vanishes; furthermore,

$$\left[\Phi^{(1,)}, \Phi^{(2,)}\right]_{\mathrm{P}} \equiv 0. \tag{2.40}$$

Thus one ends up with two first-class constraints.

2.2.1 Neutral scalar field

Kiefer 2012, sec. 3.3

2.3 Maxwell–Proca theory

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu + A_\mu J^\mu, \tag{2.41}$$

where $m > 0$ corresponds to the Proca theory Gitman and Tyutin 1990, sec. 2.3, and $m = 0$ the Maxwell theory H. J. Rothe and K. D. Rothe 2010, sec. 3.3.3, Gitman and Tyutin 1990, sec. 2.4.

Lagrangian density with velocity

$$\mathcal{L}^{\mathrm{v}} = \frac{1}{2}(V_i - \partial_i A_0)^2 - \frac{1}{4} F_{ij}^2 + \frac{m^2}{2}(A_0^2 - A_i^2) + A_0 J^0 + A_i J^i; \tag{2.42}$$

momenta density

$$B^0 := \frac{\partial \mathcal{L}^{\mathrm{v}}}{\partial V_0} = 0, \qquad B^i := \frac{\partial \mathcal{L}^{\mathrm{v}}}{\partial V_i} = V^i - \partial^i A_0; \tag{2.43}$$

Hamiltonians

$$\mathcal{H}^{\mathrm{p}} = \mathcal{H}^{\mathrm{c}} + V_0 \Phi_1, \tag{2.44}$$
$$\mathcal{H}^{\mathrm{c}} = \frac{1}{2}(B^i)^2 + B^i \partial_i A_0 + \frac{1}{4} F_{ij}^2 + \frac{m^2}{2}(-A_0^2 + A_i^2) - A_0 J^0 - A_i J^i, \tag{2.45}$$
$$\tag{2.46}$$

where

$$\Phi_1 = B^0 \tag{2.47}$$

is the only primary constraint.

$$
\begin{aligned}
[\Phi_1(\vec{x}_1), \mathcal{H}^{\mathrm{p}}(\vec{x}_2)]_{\mathrm{P}} &= \left[\left(B^0\right)_1, \left(B^i \partial_i A_0 - \frac{m^2}{2} A_0^2 - A_0 J^0\right)_2\right]_{\mathrm{P}} \\
&= \left(-B^i \partial_i + m^2 A_0 - J_0\right)_2 \delta(\vec{x}_1 - \vec{x}_2),
\end{aligned}
\tag{2.48}
$$

where $J_0 = -J^0$. Integration with $\mathbb{d}^d x_2$ yields the secondary constraint

$$
[\Phi_1, H^{\mathrm{p}}]_{\mathrm{P}} = \partial_i B^i + m^2 A_0 - J_0 =: \Phi_2, \tag{2.49}
$$

so that

$$
[\Phi_1(\vec{x}_1), \Phi_2(\vec{x}_2)]_{\mathrm{P}} = -m^2 \delta(\vec{x}_1 - \vec{x}_2). \tag{2.50}
$$

One may further compute

$$
[\Phi_2(\vec{x}_1), \mathcal{H}^{\mathrm{c}}(\vec{x}_2)]_{\mathrm{P}} = \left[\left(\partial_i B^i\right)_1, \left(\frac{1}{4} F_{jk}^2 + \frac{m^2}{2} A_j^2 - A_j J^j\right)_2\right]_{\mathrm{P}}, \tag{2.51}
$$

in which

$$
\begin{aligned}
\left[\left(\partial_i B^i\right)_1, \left(\frac{1}{4} F_{jk}^2\right)_2\right]_{\mathrm{P}} &= \left(\partial_j A_k - \partial_k A_j\right)_2 (\partial_i)_1 \left[\left(B^i\right)_1, \left(\partial^j A^k\right)_2\right]_{\mathrm{P}} \\
&= -\left(F^{ij} \partial_j\right)_2 (\partial_i)_1 \delta(\vec{x}_1 - \vec{x}_2).
\end{aligned}
\tag{2.52}
$$

The Poisson bracket can be evaluated to be

$$
[\Phi_2(\vec{x}_1), \mathcal{H}^{\mathrm{c}}(\vec{x}_2)]_{\mathrm{P}} = -\left(F^{ij} \partial_j + m^2 A^i + J^i\right)_2 (\partial_i)_1 \delta(\vec{x}_1 - \vec{x}_2). \tag{2.53}
$$

Integration with $\mathbb{d}^3 x_2$ yields

$$
[\Phi_2, H^{\mathrm{c}}]_{\mathrm{P}} = -\partial_i (m^2 A^i + J^i). \tag{2.54}
$$

For Proca theory $m > 0$, then the algorithm terminates, and one obtains a pure second-class system.

$$
\mathbf{Q} = \begin{pmatrix} 0 & -m^2 \\ +m^2 & 0 \end{pmatrix}, \qquad \mathbf{Q}^{-1} = \begin{pmatrix} 0 & +m^{-2} \\ -m^{-2} & 0 \end{pmatrix}. \tag{2.55}
$$

Dirac bracket

$$
\begin{aligned}
[f(\vec{x}_1), g(\vec{x}_2)]_{\mathrm{D}} &= \left[(f)_1, (g)_2\right]_{\mathrm{P}} + \int \mathbb{d}^d x_3 \\
\Big(&- \left[(f)_1, \left(B^0\right)_3\right]_{\mathrm{P}} \left[\left(m^{-2} \partial_i B^i + A_0\right)_3, (g)_2\right]_{\mathrm{P}} \\
&+ \left[(f)_1, \left(m^{-2} \partial_i B^i + A_0\right)_3\right]_{\mathrm{P}} \left[\left(B^0\right)_3, (g)_2\right]_{\mathrm{P}} \Big).
\end{aligned}
\tag{2.56}
$$

The fundamental ones different from Poisson brackets are

$$[A_0(\vec{x}_1), A_i(\vec{x}_2)]_{\mathrm{D}} = m^{-2}(\partial_i)_1 \delta(\vec{x}_1 - \vec{x}_2), \qquad [A_0(\vec{x}_1), B^0(\vec{x}_2)]_{\mathrm{D}} = 0. \tag{2.57}$$

Introducing the regularising coordinates

$$\alpha_i = A_i + m^{-2}(\partial_i B^0 - J_i), \qquad \beta^i = B^i; \tag{2.58}$$
$$\alpha_0 = A_0 + m^{-2}(\partial_i B^i - J_0), \qquad \beta^0 = B^0. \tag{2.59}$$

Note that $J^i = J_i$. It is easy to show that

$$\left[\alpha_i(\vec{x}_1), \beta^j(\vec{x}_2)\right]_{\mathrm{D}} = \delta^i_j \delta(\vec{x}_1, \vec{x}_2), \tag{2.60}$$
$$\left[\alpha_i(\vec{x}_1), \alpha_j(\vec{x}_2)\right]_{\mathrm{D}} = 0 = \left[\beta^i(\vec{x}_1), \beta^j(\vec{x}_2)\right]_{\mathrm{D}}. \tag{2.61}$$

Furthermore, one has

$$\mathcal{H}^{\mathrm{p}} = \mathcal{H}^{\mathrm{phy}} + \mathcal{H}^{\mathrm{con}} + \mathcal{H}^{\mathrm{irr}}, \tag{2.62}$$

where

$$\begin{aligned}\mathcal{H}^{\mathrm{phy}} &= \frac{1}{2}(\beta^i)^2 + \frac{m^2}{2}\alpha_i^2 + \frac{1}{4}(\partial_i \alpha_j - \partial_j \alpha_i)^2 + \frac{1}{2m^2}(\partial_i \beta^i)^2 \\ &\quad + \frac{1}{m^2} J^0 \partial_i \beta^i,\end{aligned} \tag{2.63}$$

$$\mathcal{H}^{\mathrm{con}} = -\frac{m^2}{2}\alpha_0^2 - \frac{1}{2m^2}(\partial_i \beta^0)^2, \tag{2.64}$$

$$\begin{aligned}\mathcal{H}^{\mathrm{irr}} &= \partial_i \left(\alpha_0 \beta^i - \beta^0 \alpha_i + \frac{1}{m^2}(\beta^0 \partial_i \beta^0 - \beta^i \partial_j \beta^j - J^0 \beta^i)\right) \\ &\quad + \frac{1}{2m^2}\left((J^0)^2 - (J^i)^2\right).\end{aligned} \tag{2.65}$$

Further more,

$$\Phi_1 = \beta^0, \qquad \Phi_2 = m^2 \alpha_0 \propto \alpha_0. \tag{2.66}$$

Thus the (α_i, β^i) are regular pairs of canonical variables, whereas (α_0, β^0) are the singular variables as constraints. The canonical dynamics of the physical (α_i, β^i)'s are determined by $\mathcal{H}^{\mathrm{phy}}$ as a regular system.

Should one compute $\mathcal{H}^{\mathrm{a}}$ here?

For Maxwell theory $m = 0$.

2.4 String theories

Nambu–Gotō action

Generalising the kinetic part of (2.19), one has

$$S_{\mathrm{NG}} := -T \int_\Sigma \mathbb{d}A =: -T \int_\Sigma \mathbb{d}^2\sigma \mathcal{L}, \tag{2.67}$$

where the Lagrangian density

$$\mathcal{L} = \sqrt{-\Gamma}, \quad \Gamma := \det \Gamma_{\alpha\beta}, \quad \Gamma_{\alpha\beta} := \frac{\partial X^\nu}{\partial \sigma^\alpha} \frac{\partial X_\nu}{\partial \sigma^\alpha}. \tag{2.68}$$

Historically Gotō 1971; Nambu 1970; Reference e.g. Blumenhagen, Lüst, and Theisen 2013 Kiefer 2012, sec. 3.2

Polyakov action

Generalising (2.31)

$$S_\mathrm{P}\left[X^\mu, h_{\alpha\beta}\right] = -\frac{T}{2} \int_\Sigma \mathcal{L}, \tag{2.69}$$

where

$$\mathcal{L} := \sqrt{-h} h^{\alpha\beta} \Gamma_{\alpha\beta}. \tag{2.70}$$

Historically Brink, Di Vecchia, and Howe 1976; Deser and Zumino 1976; Polyakov 1981; Reference Kiefer 2012, sec. 3.2

2.5 Gravitation theories

Closed Friedmann universe

This part adapts ibid., sec. 8.1.2.

The total action reads

$$S := S_\mathrm{EG} + S_\phi, \tag{2.71}$$

where S_EG follows (2.88), and

$$S_\phi := \int_\mathcal{M} \mathbb{d}^4 x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} (\nabla_\mu \phi)(\nabla_\nu \phi) - m^2 \phi^2 \right). \tag{2.72}$$

Adapting

$$\mathbb{d}s^2 = -N^2(t)\, \mathbb{d}t^2 + a^2(t)\, \mathbb{d}\Omega_3^2, \tag{2.73}$$

where

$$\Omega_3^2 = \mathbb{d}\chi^2 + \sin^2\chi \left(\mathbb{d}\theta^2 + \sin^2\theta\, \mathbb{d}\phi^2\right). \tag{2.74}$$

One has

$$\sqrt{-g} = N a^3 \sin^2\chi \sin\theta, \qquad \sqrt{h} = a^3 \sin^2\chi \sin\theta; \tag{2.75}$$

whereas

$$R = \frac{6}{N^2} \left(-\frac{\dot{N}\dot{a}}{Na} + \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 \right) + \frac{6}{a^2}, \qquad K = \frac{3\dot{a}}{Na}. \tag{2.76}$$

$$S_\mathrm{EG} = \frac{A_3}{16\pi G} \left(\int_{t_1}^{t_2} \mathbb{d}t N a^3 (R - 2\Lambda) - \left[\frac{6\dot{a}a^2}{N}\right]_{t_1}^{t_2} \right), \tag{2.77}$$

where

$$A_3 = \int \sin^2\chi \, \sin\theta \, \mathrm{d}\chi \, \mathrm{d}\theta \, \mathrm{d}\phi = 2\pi^2. \tag{2.78}$$

The term proportional to $\ddot{a}/a$ in the integrand can be integrated by parts

$$\int_{t_1}^{t_2} \mathrm{d}t \, N a^3 \frac{6}{N^2} \frac{\ddot{a}}{a} = 6 \left(\left[\frac{\dot{a} a^2}{N} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathrm{d}t \, \dot{a} \frac{\mathrm{d}}{\mathrm{d}t} \frac{a^2}{N^2} \right), \tag{2.79}$$

in which the first term cancels the Gibbons–Hawking–York term. One has

$$S_{\mathrm{EG}} = \frac{3\pi}{4G} \int_{t_1}^{t_2} \mathrm{d}t \left(-\frac{a}{N} \dot{a}^2 + N a - \frac{\Lambda}{3} N a^3 \right). \tag{2.80}$$

The matter part of the action reads

$$S_\phi = \pi^2 \int_{t_1}^{t_2} \mathrm{d}t \, a^3 \left(\frac{1}{N} \dot{\phi}^2 - m^2 N \phi^2 \right). \tag{2.81}$$

Lagrangian with velocity

$$L^{\mathrm{v}} = \frac{3\pi}{4G} \left(-\frac{a}{N} {v^a}^2 + N a - \frac{\Lambda}{3} N a^3 \right) + \pi^2 a^3 \left(\frac{1}{N} {v^\phi}^2 - m^2 N \phi^2 \right). \tag{2.82}$$

Canonical momenta

$$p_N := \frac{\partial L^{\mathrm{v}}}{\partial v^N} = 0, \quad p_a := \frac{\partial L^{\mathrm{v}}}{\partial v^a} = -\frac{3\pi}{2G} \frac{a}{N} v^a, \quad p_\phi := \frac{\partial L^{\mathrm{v}}}{\partial v^\phi} = 2\pi^2 \frac{a^3}{N} v^\phi. \tag{2.83}$$

Choosing v^N to be the primary inexpressible velocity, one obtains

$$H^{\mathrm{p}} = -N H_\perp + v^N \Phi, \tag{2.84}$$
$$H^{\mathrm{c}} = -N H_\perp, \tag{2.85}$$

where

$$H_\perp := \frac{G}{3\pi} \frac{p_a^2}{a} - \frac{1}{4\pi^2} \frac{p_\phi^2}{a^3} - \frac{3\pi}{4G} \left(\frac{\Lambda}{3} a^2 - 1 \right) a - \pi^2 m^2 a^3 \phi^2, \tag{2.86}$$
$$\Phi = p_N \tag{2.87}$$

are the *Hamiltonian constraint* and the primary constraint, respectively.

Evaluating the time evolution of Φ yields

$$[\Phi, H^t]_{\mathrm{P}} = H_\perp, \tag{2.88}$$

so that the Hamiltonian constraint is indeed a constraint. There is no further constraint, and $[\Phi, H_\perp]_{\mathrm{P}}$ vanishes identically. Therefore there are two and only two first-class constraints.

2.5.1 Einstein–Hilbert action

$$S_{\text{EG}} = S_{\text{EH}} + S_{\text{GHY}}, \tag{2.89}$$

$$S_{\text{EH}} = \frac{1}{16\pi G} \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}(R - 2\Lambda), \tag{2.90}$$

and

$$S_{\text{GHY}} = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{h}K, \tag{2.91}$$

which is named after Gibbons and Hawking 1977; York 1972 but actually already mentioned in Einstein 1916. See Dyer and Hinterbichler 2009 for a brief review.

References

Blumenhagen, Ralph, Dieter Lüst, and Stefan Theisen (2013). *Basic Concepts of String Theory*. Theoretical and Mathematical Physics. Springer. ISBN: http://id.crossref.org/isbn/978-3-642-29497-6. DOI: 10.1007/978-3-642-29497-6. URL: http://dx.doi.org/10.1007/978-3-642-29497-6.

Brink, L., P. Di Vecchia, and P. Howe (1976). "A locally supersymmetric and reparametrization invariant action for the spinning string". In: *Physics Letters B* 65.5, pp. 471–474. ISSN: 0370-2693. DOI: 10.1016/0370-2693(76)90445-7. URL: http://dx.doi.org/10.1016/0370-2693(76)90445-7.

Deser, S. and B. Zumino (1976). "A complete action for the spinning string". In: *Physics Letters B* 65.4, pp. 369–373. ISSN: 0370-2693. DOI: 10.1016/0370-2693(76)90245-8. URL: http://dx.doi.org/10.1016/0370-2693(76)90245-8.

Dyer, Ethan and Kurt Hinterbichler (2009). "Boundary terms, variational principles, and higher derivative modified gravity". In: *Physical Review D* 79.2. ISSN: 1550-2368. DOI: 10.1103/physrevd.79.024028. URL: http://dx.doi.org/10.1103/PhysRevD.79.024028.

Einstein, Albert (1916). "Hamiltonsches Prinzip und allgemeine Relativitätstheorie". In: *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*. URL: http://adsabs.harvard.edu/abs/1916SPAW.......1111E.

Gibbons, G. W. and S. W. Hawking (1977). "Action integrals and partition functions in quantum gravity". In: *Physical Review D* 15.10, pp. 2752–2756. ISSN: 0556-2821. DOI: 10.1103/physrevd.15.2752. URL: http://dx.doi.org/10.1103/PhysRevD.15.2752.

Gitman, Dmitriy M. and Igor V. Tyutin (1990). *Quantization of Fields with Constraints*. Springer Series in Nuclear and Particle Physics. Springer Berlin Heidelberg. ISBN: http://id.crossref.org/isbn/978-3-642-83938-2. DOI: 10.1007/978-3-642-83938-2. URL: http://dx.doi.org/10.1007/978-3-642-83938-2. Дмитрий Максимович Гитман and Игорь Викторович Тютин. *Каноническое квантование полей со связями*. 030077 г. 11овосибирск-77, Станиславского, 25: Наука, 1986.

Gotō, Tetsuo (1971). "Relativistic Quantum Mechanics of One-Dimensional Mechanical Continuum and Subsidiary Condition of Dual Resonance Model". In: *Progress of Theoretical Physics* 46.5, pp. 1560–1569. ISSN: 0033-068X. DOI: 10.1143/ptp.46.1560. URL: http://dx.doi.org/10.1143/PTP.46.1560.

Kiefer, Claus (2012). *Quantum Gravity*. 3rd. International Series of Monographs on Physics. Oxford University Press. ISBN: http://id.crossref.org/isbn/9780199585205. DOI: 10.1093/acprof:oso/9780199585205.001.0001. URL: http://dx.doi.org/10.1093/acprof:oso/9780199585205.001.0001.

Landau, Lev Davidovich and Evgeny Mikhailovich Lifshitz (1975). *The Classical Theory of Fields*. 4th. Vol. 2. Course of Theoretical Physics. Butterworth-Heinemann. ISBN: http://id.crossref.org/isbn/9780080250724. DOI: 10.1016/c2009-0-14608-1. URL: http://dx.doi.org/10.1016/c2009-0-14608-1.

Nambu, Yoichiro (1970). "Quark model and the factorization of the Veneziano amplitude". In: *Symmetries and Quark Models*. Ed. by R. Chand. International Conference on Symmetries and Quark Models, Wayne State U., Detroit Detroit, Mich., USA, June 18-20, 1969, pp. 269–278.

Polyakov, A.M. (1981). "Quantum geometry of bosonic strings". In: *Physics Letters B* 103.3, pp. 207–210. ISSN: 0370-2693. DOI: 10.1016/0370-2693(81)90743-7. URL: http://dx.doi.org/10.1016/0370-2693(81)90743-7.

Rothe, Heinz J and Klaus D Rothe (2010). *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*. World Scientific Lecture Notes in Physics. World Scientific. ISBN: http://id.crossref.org/isbn/978-981-4299-65-7. DOI: 10.1142/7689. URL: http://dx.doi.org/10.1142/7689.

York, James W. (1972). "Role of Conformal Three-Geometry in the Dynamics of Gravitation". In: *Physical Review Letters* 28.16, pp. 1082–1085. ISSN: 0031-9007. DOI: 10.1103/physrevlett.28.1082. URL: http://dx.doi.org/10.1103/PhysRevLett.28.1082.