Notes on Canonical Singular Dynamics

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1 Classical formalism

Lagrangian with velocity

$$L^{\mathrm{v}} := L\big|_{\dot{q}=v} \tag{1.1}$$

Equations of motion

$$\sum_j M_{ij}\dot{v}_j = K_i^{\mathrm{v}}, \quad \dot{q}_i = v_i. \tag{1.2}$$

where

$$M_{ij}(q,v) := \frac{\partial^2 L^{\mathrm{v}}}{\partial v_i\, \partial v_j}. \tag{1.3}$$

Adding

$$p_i := \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.4}$$

Variation of

$$S[q,p;v] := \int \mathrm{d}t \left[L^{\mathrm{v}} + \sum_i p_i(\dot{q}_i - v_i) \right]. \tag{1.5}$$

gives the *extended Euler–Lagrange equations*

$$\dot{q}_i = v_i, \quad \dot{p}_i = \frac{\partial L^{\mathrm{v}}}{\partial q_i}, \quad p_i = \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.6}$$

Hamiltonian with velocity

$$H^{\mathrm{v}}(q,p;v) := \sum_i p_i v_i - L^{\mathrm{v}}. \tag{1.7}$$

Identities

$$\frac{\partial H^{\mathrm{v}}}{\partial q_i} \equiv -\frac{\partial L^{\mathrm{v}}}{\partial q_i}, \quad \frac{\partial H^{\mathrm{v}}}{\partial p_i} \equiv v_i, \quad \frac{\partial H^{\mathrm{v}}}{\partial v_i} \equiv p_i - \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.8}$$

Variation of

$$S[q,p;v] := \int \mathrm{d}t \left[\sum_i p_i \dot{q}_i - H^{\mathrm{v}} \right] \tag{1.9}$$

gives the *extended canonical equations*

$$\dot{q}_i = [q_i, H^{\mathrm{v}}]_{\mathrm{P}}, \quad \dot{p}_i = [p_i, H^{\mathrm{v}}]_{\mathrm{P}}, \quad \frac{\partial H^{\mathrm{v}}}{\partial v_i} = 0, \tag{1.10}$$

where the *Poisson bracket* is defined as

$$[f^{\mathrm{v}}, g^{\mathrm{v}}]_{\mathrm{P}} := \sum_i \left(\frac{\partial f^{\mathrm{v}}}{\partial q_i} \frac{\partial g^{\mathrm{v}}}{\partial p_i} - \frac{\partial f^{\mathrm{v}}}{\partial p_i} \frac{\partial g^{\mathrm{v}}}{\partial q_i} \right). \tag{1.11}$$

$v_a = \overline{v}_a(q, p; \{v_\alpha\})$ can be solved, $a = 1, 2, \dots, r_M$; v_α cannot be solved, $\alpha = r_M + 1, \dots, n$, where $r_M = \operatorname{rank} M$.

(need to show $v_a = \overline{v}_a(q, p_a)$)

Primary constraints in the standard form

$$\Phi_\alpha(q, p) := \left. \frac{\partial H^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_\alpha = \overline{v}_\alpha\}} \equiv p_\alpha - \overline{p}_\alpha(q, \{p_a\}), \tag{1.12}$$

where

$$\overline{p}_\alpha(q, \{p_a\}) := \left. \frac{\partial L^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_a = \overline{v}_a\}}. \tag{1.13}$$

Total Hamiltonian

$$H^{\mathrm{t}} := \left. H^{\mathrm{v}} \right|_{\{v_a = \overline{v}_a\}} \equiv H^{\mathrm{v}}(q, p; \{\overline{v}^a(q, p_a; \{v_\alpha\}), v_\alpha\}). \tag{1.14}$$

Subspace of primary constraints

$$\Gamma_{\mathrm{P}} = \{(q, p) \mid \Phi_\alpha(q, p) = 0, \forall \alpha\} \tag{1.15}$$

Since

$$\frac{\partial H^{\mathrm{t}}}{\partial v_\alpha} = \left. \frac{\partial H^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_a = \overline{v}_a\}} = \Phi_\alpha \equiv p_\alpha - \overline{p}_\alpha(q, \{p_a\}), \tag{1.16}$$

H^{t} is linear in v_α. One writes

$$H^{\mathrm{t}}(q, \{p_a\}; \{p_\alpha\}, \{v_\alpha\}) = H(q, \{p_a\}) + \sum_\alpha v_\alpha \Phi_\alpha, \tag{1.17}$$

where H^{c} is the *canonical Hamiltonian* or simply *Hamiltonian.*

Proposition H^{c} is independent of $\{p_\alpha\}$.

Proposition Canonical equations with primary constraints

$$\dot{q}_i = [q_i, H]_{\mathrm{P}} + \sum_\beta v_\beta \left[q_i, \phi_\beta\right]_{\mathrm{P}}, \tag{1.18}$$

$$\dot{p}_i = [p_i, H]_{\mathrm{P}} + \sum_\beta v_\beta \left[p_i, \phi_\beta\right]_{\mathrm{P}}, \tag{1.19}$$

$$\Phi_\alpha(q, p) = 0, \tag{1.20}$$

where v_β's are undetermined. Note that eq. (1.18) for $i = \alpha$ holds identically: $\dot{q}_\alpha = \dot{q}_\alpha$.

Weak equality: $f_1 \approx f_2$ iff $\left. f_1 \right|_{\Gamma_{\mathrm{P}}} = \left. f_2 \right|_{\Gamma_{\mathrm{P}}}$.

Proposition if f and g are two functions over the phase space Γ, and $f \approx h$, then

$$\frac{\partial}{\partial q_i}\left(f - \sum_\beta \phi_\beta \frac{\partial f}{\partial p_\beta}\right) \approx \frac{\partial}{\partial q_i}\left(h - \sum_\beta \phi_\beta \frac{\partial h}{\partial p_\beta}\right), \tag{1.21}$$

$$\frac{\partial}{\partial p_i}\left(f - \sum_\beta \phi_\beta \frac{\partial f}{\partial p_\beta}\right) \approx \frac{\partial}{\partial p_i}\left(h - \sum_\beta \phi_\beta \frac{\partial h}{\partial p_\beta}\right). \tag{1.22}$$

Corollary $\forall H_1 \approx H$,

$$\dot{q}_i \approx [q_i, H]_{\mathrm{P}}, \qquad \dot{p}_i \approx [p_i, H]_{\mathrm{P}}. \tag{1.23}$$

Primary and second constraints $\phi_\mu^{(1,)}$, $\phi_\omega^{(2,)}$; first and second class constraints $\phi_u^{(,1)}$, $\phi_w^{(,2)}$.

2 Examples

2.1 Toy examples

Example 0

Gitman and Tyutin 1990, sec. 1.2

$$L = \frac{1}{2}(\dot{x} - y)^2 \tag{2.1}$$

Example 1

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x - y)^2. \tag{2.2}$$

One has

$$L^{\mathrm{v}} = \frac{1}{2}v_x^2 + v_x y - \frac{1}{2}(x - y)^2, \tag{2.3}$$

so that

$$p_x = \frac{\partial L^{\mathrm{v}}}{\partial v_x} = v_x + y, \qquad p_y = 0, \tag{2.4}$$

thus

$$\overline{v}_x = p_x - y. \tag{2.5}$$

So that v_y is the primary inexpressible velocity.

The Hamiltonian with velocity reads

$$H^{\mathrm{v}}(q, p; v) = v_x p_x + v_y p_y - \frac{1}{2}v_x^2 - v_x y + \frac{1}{2}(x - y)^2, \tag{2.6}$$

whilst the total Hamiltonian is

$$H^{\mathrm{t}}\big(q, p; \overline{v}_x, v_y\big) = \frac{1}{2}(p_x - y)^2 + \frac{1}{2}(x - y)^2 + v_y p_y. \tag{2.7}$$

Example 2

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x-y)^2 \tag{2.8}$$

Primary constraint

$$p_y = 0; \tag{2.9}$$

total Hamiltonian

$$H^{\mathrm{t}} = \frac{1}{2}p_x^2 - p_x y - \frac{1}{2}x^2 + xy + v_y p_y. \tag{2.10}$$

Example 3

$$L = \frac{1}{2}(\dot{q}_2 - \mathbb{e}^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2. \tag{2.11}$$

2.2 Parametrised systems

Non-relativistic point particle

Kiefer 2012, sec. 3.1.1

$$S[q(t)] := \int_{t_1}^{t_2} \mathbb{d}t\, L\Big(q, \frac{\mathbb{d}q}{\mathbb{d}t}\Big) \tag{2.12}$$

Relativistic charged point particle

Landau and Lifshitz 1975, sec. 16, Kiefer 2012, sec. 3.1.2

$$S := \int -m\,\mathbb{d}s + eA_\mu(x)\,\mathbb{d}x^\mu =: \int \mathbb{d}\tau\, L, \tag{2.13}$$

where the Lagrangian reads

$$L = -m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + q\dot{x}^\mu A_\mu(x). \tag{2.14}$$

$$M_{\mu\nu} := \frac{\partial^2 L^{\mathrm{v}}}{\partial v^\mu\,\partial v^\nu} = m\frac{-\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta}}{\big(-\eta_{\rho\sigma}v^\rho v^\sigma\big)^{3/2}}v^\alpha v^\beta, \tag{2.15}$$

which has one and only one eigenvector with null eigenvalue

$$v^\mu M_{\mu\nu} = 0. \tag{2.16}$$

Momenta

$$p_\mu = \frac{\partial L^{\mathrm{v}}}{\partial v^\mu} = \frac{m\eta_{\mu\nu}v^\nu}{\sqrt{-\eta_{\rho\sigma}v^\rho v^\sigma}} + qA_\mu. \tag{2.17}$$

If one chooses v^0 to be the primary inexpressible velocity, then eliminating p_0 in eq. (2.17) yields

$$v^i = \frac{\xi\eta^{ij}(p_j - qA_j)v^0}{\sqrt{m^2 + \eta^{kl}(p_k - qA_k)(p_l - qA_l)}}, \tag{2.18}$$

where $\xi = \operatorname{sgn} v^0$. In the following $\xi = +1$ will be chosen.

Inserting eq. (2.18) into the Hamiltonian with velocity

$$H^{\mathrm{v}} = v^\mu p_\mu - L^{\mathrm{v}} = m\sqrt{-\eta_{\mu\nu}v^\mu v^\nu} + v^\mu\big(p_\mu - qA_\mu(x)\big), \tag{2.19}$$

one obtains the total Hamiltonian

$$H^{\mathrm{t}} = v^0\left(p_0 - qA_0 + \sqrt{m^2 + \eta^{kl}(p_k - qA_k)(p_l - qA_l)}\right), \tag{2.20}$$

where only a primary constraint survives, which is obviously a first-class constraint

$$\phi^{(1,1)} = p_0 - qA_0 + \sqrt{m^2 + \eta^{kl}(p_k - qA_k)(p_l - qA_l)}, \tag{2.21}$$

and the canonical Hamiltonian vanishes

$$H^{\mathrm{c}} = 0. \tag{2.22}$$

To compare, note in the non-covariant formalism (Landau and Lifshitz 1975, sec. 8)

$$S = \int \mathrm{d}t\, L, \qquad L = -m\sqrt{1 - \dot{\vec{x}}^2} - q\phi + q\dot{\vec{x}}\cdot\vec{A}, \tag{2.23}$$

the system is regular, and the canonical Hamiltonian reads

$$H^{\mathrm{c}} = \sqrt{m^2 + \left(\vec{p} - q\vec{A}\right)^2} + q\phi, \tag{2.24}$$

which corresponds to setting $\phi^{(1,1)} = 0$, $p_0 \to -H^{\mathrm{c}}$ ($p_\mu = (-E, \vec{p})$), and noting $A_\mu = \left(-\phi, \vec{A}\right)$.

Relativistic point particle with einbein

Blumenhagen, Lüst, and Theisen 2013, sec. 2.1

$$L := \frac{1}{2}\left(e^{-1}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2 e\right) \tag{2.25}$$

$$p_\mu = \frac{\partial L^{\mathrm{v}}}{\partial v^\mu} = e^{-1}\eta_{\mu\nu}v^\nu, \qquad p_e = 0. \tag{2.26}$$

Choosing v^e to be the primary inexpressible velocity, one has

$$v^\mu = e\eta^{\mu\nu}p_\nu. \tag{2.27}$$

Hamiltonian with velocity

$$H^{\mathrm{v}} = v^\mu p_\mu + v^e p_e + \frac{1}{2}\big(-e^{-1}\eta_{\mu\nu}v^\mu v^\nu + m^2 e\big); \tag{2.28}$$

total Hamiltonian

$$H^{\mathrm{t}} = \frac{e}{2}\big(\eta^{\mu\nu}p_\mu p_\nu + m^2\big) + v^e p_e; \tag{2.29}$$

canonical Hamiltonian

$$H^{\mathrm{c}} = \frac{e}{2}\big(\eta^{\mu\nu}p_\mu p_\nu + m^2\big). \tag{2.30}$$

The only primary constraint

$$\Phi^{(1,)} = p_e; \tag{2.31}$$

its time evolution

$$\begin{aligned}\big[\Phi^{(1,)}, H^{\mathrm{t}}\big]_{\mathrm{P}} &= [p_e, e]_{\mathrm{P}}\frac{1}{2}\big(\eta^{\mu\nu}p_\mu p_\nu + m^2\big)\\ &= -\frac{1}{2}\big(\eta^{\mu\nu}p_\mu p_\nu + m^2\big).\end{aligned} \tag{2.32}$$

Choose

$$\Phi^{(2,)} = \eta^{\mu\nu}p_\mu p_\nu + m^2, \tag{2.33}$$

whose Possion bracket with H^{t} vanishes; furthermore,

$$\big[\Phi^{(1,)}, \Phi^{(2,)}\big]_{\mathrm{P}} \equiv 0. \tag{2.34}$$

Thus one ends up with two first-class constraints.

2.2.1 Neutral scalar field

Kiefer 2012, sec. 3.3

2.3 Maxwell–Proca theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + \frac{1}{2}F_{\mu\nu}S^{\mu\nu} + A_\mu J^\mu_{\mathrm{f}}, \tag{2.35}$$

where $m > 0$ corresponds to the Proca theory Gitman and Tyutin 1990, sec. 2.3, and $m = 0$ the Maxwell theory H. J. Rothe and K. D. Rothe 2010, sec. 3.3.3 Gitman and Tyutin 1990, sec. 2.4.

2.4 String theories

Nambu–Gotō action

Generalising the kinetic part of (2.13), one has

$$S_{\mathrm{NG}} := -T \int_{\Sigma} \mathbb{d}A =: -T \int_{\Sigma} \mathbb{d}^2\sigma \mathcal{L}, \tag{2.36}$$

where the Lagrangian density

$$\mathcal{L} = \sqrt{-\Gamma}, \quad \Gamma := \det \Gamma_{\alpha\beta}, \quad \Gamma_{\alpha\beta} := \frac{\partial X^{\nu}}{\partial\sigma^{\alpha}} \frac{\partial X_{\nu}}{\partial\sigma^{\alpha}}. \tag{2.37}$$

Historically Gotō 1971; Nambu 1970; Reference e.g. Blumenhagen, Lüst, and Theisen 2013 Kiefer 2012, sec. 3.2

Polyakov action

Generalising (2.25)

$$S_{\mathrm{P}}[X^{\mu}, h\alpha\beta] = -\frac{T}{2} \int_{\Sigma} \mathcal{L}, \tag{2.38}$$

where

$$\mathcal{L} := \sqrt{-h} h^{\alpha\beta} \Gamma_{\alpha\beta}. \tag{2.39}$$

Historically Brink, Di Vecchia, and Howe 1976; Deser and Zumino 1976; Polyakov 1981; Reference Kiefer 2012, sec. 3.2

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