Notes on Canonical Singular Dynamics

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1 Canonical formalism

Lagrangian with velocity

$$L^{\mathrm{v}} := L\big|_{\dot{q}=v} \tag{1.1}$$

Equations of motion

$$\sum_j M_{ij}\dot{v}_j = K_i^{\mathrm{v}}, \quad \dot{q}_i = v_i. \tag{1.2}$$

where

$$M_{ij}(q,v) := \frac{\partial^2 L^{\mathrm{v}}}{\partial v_i\,\partial v_j}. \tag{1.3}$$

Adding

$$p_i := \frac{\partial L^{\mathrm{v}}}{\partial v_i} =: \overline{p}_i(q,v). \tag{1.4}$$

Variation of

$$S[q,p;v] := \int \mathrm{d}t \left[L^{\mathrm{v}} + \sum_i p_i(\dot{q}_i - v_i) \right]. \tag{1.5}$$

gives the *extended Euler–Lagrange equations*

$$\dot{q}_i = v_i, \quad \dot{p}_i = \frac{\partial L^{\mathrm{v}}}{\partial q_i}, \quad p_i = \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.6}$$

Extended Hamiltonian

$$H^{\mathrm{v}}(q,p;v) := \sum_i p_i v_i - L^{\mathrm{v}}. \tag{1.7}$$

Identities

$$\frac{\partial H^{\mathrm{v}}}{\partial q_i} \equiv -\frac{\partial L^{\mathrm{v}}}{\partial q_i}, \quad \frac{\partial H^{\mathrm{v}}}{\partial p_i} \equiv v_i, \quad \frac{\partial H^{\mathrm{v}}}{\partial v_i} \equiv p_i - \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.8}$$

Variation of

$$S[q,p;v] := \int \mathrm{d}t \left[\sum_i p_i \dot{q}_i - H^{\mathrm{v}} \right] \tag{1.9}$$

gives the *extended canonical equations*

$$\dot{q}_i = [q_i, H^{\mathrm{v}}]_{\mathrm{TP}}, \quad \dot{p}_i = [p_i, H^{\mathrm{v}}]_{\mathrm{TP}}, \quad \frac{\partial H^{\mathrm{v}}}{\partial v_i} = 0, \tag{1.10}$$

where the total Poison bracket is defined as

$$[f^{\mathrm{v}}, g^{\mathrm{v}}]_{\mathrm{TP}} := \sum_i \left(\frac{\partial f^{\mathrm{v}}}{\partial q_i} \frac{\partial g^{\mathrm{v}}}{\partial p_i} - \frac{\partial f^{\mathrm{v}}}{\partial p_i} \frac{\partial g^{\mathrm{v}}}{\partial q_i} \right). \tag{1.11}$$

$v_a = \overline{v}_a(q,p)$ can be solved, $a = 1, 2, \dots, r_M$; v^α cannot be solved, $\alpha = r_M + 1, \dots, n$, where $r_M = \operatorname{rank} M$.

(need to show $v_a = \overline{v}_a(q, p_a)$)

Primary constraints in the standard form

$$\phi_\alpha^{(0)}(q,p) := \left. \frac{\partial H^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_\alpha = \overline{v}_\alpha\}} \equiv p_\alpha - \left. \frac{\partial L^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_a = \overline{v}_a\}}. \tag{1.12}$$

Total Hamiltonian

$$H^{\mathrm{t}} := \left. H^{\mathrm{v}} \right|_{\{v_a = \overline{v}_a\}} \equiv H^{\mathrm{v}}(q, p; \{\overline{v}^a(q, p_a), v_\alpha\}). \tag{1.13}$$

Subspace of primary constraints

$$\Gamma_{\mathrm{P}} = \{(q,p) | \phi_\alpha^0(q,p) = 0, \forall \alpha\} \tag{1.14}$$

Since

$$\frac{\partial H^{\mathrm{t}}}{\partial v_\alpha} = \left. \frac{\partial H^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_a = \overline{v}_a\}} = \phi_\alpha^{(0)} \equiv p_\alpha - \left. \frac{\partial L^{\mathrm{v}}}{\partial v_\alpha} \right|_{v_a = \overline{v}_a}, \tag{1.15}$$

H^{t} is linear in v_α. One writes

$$H^{\mathrm{t}}(q, \{p_a\}; \{p_\alpha\}, (v_\alpha)) = H(q, \{p_a\}) + \sum_\alpha v_\alpha \phi_\alpha^{(0)}, \tag{1.16}$$

where H is the *canonical Hamiltonian* or simply *Hamiltonian*. need to show H is independent of $\{p_\alpha\}$.

Proposition Canonical equations with primary constraints

$$\dot{q}_a = [q_a, H]_{\mathrm{TP}} + \sum_\beta v_\beta [q_a, \phi_\beta]_{\mathrm{TP}}, \tag{1.17}$$

$$\dot{p}_i = [p_i, H]_{\mathrm{TP}} + \sum_\beta v_\beta [p_i, \phi_\beta]_{\mathrm{TP}}, \tag{1.18}$$

$$\phi_\alpha(q,p) = 0, \tag{1.19}$$

where v_β's are undetermined.

2 Examples

2.1 Toy systems

2.2 Parametrised systems

2.2.1 Non-relativistic point particle

2.2.2 Relativistic charged point particle

2.2.3 Neutral scalar field

2.3 Proca action

2.4 Dirac field

2.5 Gauge theories

2.5.1 Electrodynamics

2.5.2 Spinor electrodynamics

2.5.3 Yang–Mills theory

2.5.4 Yang–Mills–Higgs theory

2.6 Gravitation theories

2.6.1 Einstein–Hilbert action