# Notes on Lagrangian Singular Dynamics

Yi-Fan Wang (王 一帆)

April 18, 2017

## 1 Classical formalism

Lagrangian with velocity

$$L^{\mathrm{v}} := L\big|_{\dot{q}=v} \tag{1.1}$$

Equations of motion

$$\sum_j M_{ij}\dot{v}_j = K_i^{\mathrm{v}}, \quad \dot{q}_i = v_i. \tag{1.2}$$

where

$$M_{ij}(q,v) := \frac{\partial^2 L}{\partial v_i \, \partial v_j}. \tag{1.3}$$

## 2 Examples

### 2.1 Toy examples

**Example 0**

Gitman and Tyutin 1990, sec. 1.2

$$L = \frac{1}{2}(\dot{x} - y)^2 \tag{2.1}$$

**Example 1**

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x-y)^2. \tag{2.2}$$

**Example 2**

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x-y)^2 \tag{2.3}$$

**Example 3**

$$L = \frac{1}{2}(\dot{q}_2 - \mathrm{e}^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2. \tag{2.4}$$

## 2.2 Parametrised systems

### Non-relativistic point particle

Kiefer 2012, sec. 3.1.1

$$S[q(t)] := \int_{t_0}^{t_1} \mathrm{d}t\, L\left(q, \frac{\mathrm{d}q}{\mathrm{d}t}\right) \tag{2.5}$$

### Relativistic charged point particle

Landau and Lifshitz 1975, sec. 16, Kiefer 2012, sec. 3.1.2

$$S := \int_\gamma -m\,\mathrm{d}s + eA_\mu(x)\,\mathrm{d}x^\mu =: \int_{\tau_0}^{\tau_1} \mathrm{d}\tau\, L, \tag{2.6}$$

where the Lagrangian reads

$$L = -m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + q\dot{x}^\mu A_\mu(x). \tag{2.7}$$

$$M_{\mu\nu} := \frac{\partial^2 L}{\partial\dot{x}^\mu\,\partial\dot{x}^\nu} = m\frac{-\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta}}{\left(-\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma\right)^{3/2}}\dot{x}^\alpha\dot{x}^\beta, \tag{2.8}$$

which has one and only one zero eigenvector

$$\dot{x}^\mu M_{\mu\nu} = 0. \tag{2.9}$$

Euler–Lagrange derivatives

$$E_\mu = \left(\frac{\partial}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial}{\partial\dot{x}^\mu}\right)L \equiv K_\mu - M_{\mu\nu}\ddot{x}^\nu, \tag{2.10}$$

where

$$K_\mu := -qF_{\mu\nu}\dot{x}^\nu, \tag{2.11}$$

and

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.12}$$

Contracting the zero eigenvector with eq. (2.10) yields

$$\dot{x}^\mu E_\mu = \dot{x}^\mu K_\mu - \dot{x}^\mu M_{\mu\nu}\ddot{x}^\nu \equiv 0, \tag{2.13}$$

so that it generates a gauge identity, and no further constraint exists. Thus the system has a symmetry

$$\delta x^\mu = \dot{x}^\mu\delta\lambda. \tag{2.14}$$

**Relativistic point particle with auxiliary coordinate**

Blumenhagen, Lüst, and Theisen 2013, sec. 2.1

$$L := \frac{1}{2}\big(e^{-1}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2 e\big) \tag{2.15}$$

Euler–Lagrange derivatives

$$E_\mu := \Big(\frac{\partial}{\partial x^\mu} - \frac{\mathbb{d}}{\mathbb{d}\tau}\frac{\partial}{\partial \dot{x}^\mu}\Big)L = e^{-1}\eta_{\mu\nu}\Big(\frac{\dot{e}}{e}\dot{x}^\nu - \ddot{x}^\nu\Big), \tag{2.16}$$

$$E_e := \Big(\frac{\partial}{\partial e} - \frac{\mathbb{d}}{\mathbb{d}\tau}\frac{\partial}{\partial \dot{e}}\Big)L = -\frac{1}{2}\Big(\frac{1}{e^2}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + m^2\Big), \tag{2.17}$$

and collectively $E^{(0)} = \begin{pmatrix} E_\mu & E_e \end{pmatrix}^\top$.

$$M^{(0)} := M = \begin{pmatrix} e^{-1}\eta_{\mu\nu} & 0^\mu \\ 0^\nu & 0 \end{pmatrix}, \tag{2.18}$$

so that the system is singular, with $w^{(0)} = (0^\mu; 1)$.

One can choose $u^{(0)} = (0^\mu; e^2)$, so that

$$\phi^{(0)} := u^{(0)} \cdot E^{(0)} = e^2 E_e \tag{2.19}$$

$$= -\frac{1}{2}\big(\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + m^2 e^2\big), \tag{2.20}$$

and thus

$$E_1^{(1)} := \dot{\phi}^{(0)} = e\big(2\dot{e}E_e + e\dot{E}_e\big) \tag{2.21}$$

$$= -m^2 e\dot{e} - \eta_{\mu\nu}\dot{x}^\mu\ddot{x}^\nu. \tag{2.22}$$

Collectively, $E^{(1)} = \begin{pmatrix} \big(E^{(0)}\big)^\top & E_1^{(1)} \end{pmatrix}^\top$.

Straightforwardly,

$$M^{(1)} = \begin{pmatrix} e^{-1}\eta_{\mu\nu} & 0^\mu \\ 0^\nu & 0 \\ \eta_{\mu\nu}\dot{x}^\mu & 0 \end{pmatrix}, \tag{2.23}$$

and the new zero eigenvector $w^{(1)} = (e\dot{x}^\mu; 0, -1)$.

One finds that

$$w^{(1)} \cdot E^{(1)} = e\dot{x}^\mu E_\mu - E_1^{(1)} = e\big(\dot{x}^\mu E_\mu - 2\dot{e}E_e - e\dot{E}_e\big) \tag{2.24}$$

$$= \eta_{\mu\nu}\frac{\dot{e}}{e}\dot{x}^\mu\dot{x}^\nu + m^2 e\dot{e} = -2e\dot{e}E_e, \tag{2.25}$$

so that a gauge identity

$$G := \dot{x}^\mu E_\mu - e\dot{E}_e \equiv 0 \tag{2.26}$$

is obtained.

$$G\epsilon = E_\mu \dot{x}^\mu \epsilon + E_e(\dot{e}\epsilon + e\dot{\epsilon}) - \frac{\mathbb{d}}{\mathbb{d}\tau}(eE_e\epsilon), \tag{2.27}$$

so that

$$\delta x^\mu = \dot{x}^\mu \epsilon, \tag{2.28}$$

$$\delta e = \dot{e}\epsilon + e\dot{\epsilon}. \tag{2.29}$$

### 2.2.1 Neutral scalar field

Kiefer 2012, sec. 3.3

## 2.3 Maxwell–Proca theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + A_\mu J^\mu \tag{2.30}$$

where $m > 0$ corresponds to the Proca theory Gitman and Tyutin 1990, sec. 2.3, and $m = 0$ the Maxwell theory H. J. Rothe and K. D. Rothe 2010, sec. 3.3.3, Gitman and Tyutin 1990, sec. 2.4.

$$\mathcal{L} \equiv \frac{1}{2}(-\eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\alpha\nu}\eta^{\beta\mu})(\partial_\mu A_\alpha)(\partial_\nu A_\beta) - \frac{1}{2}m^2\eta^{\alpha\beta}A_\alpha A_\beta + A_\alpha J^\alpha \tag{2.31}$$

$$\begin{aligned} E^\alpha &= \left(\frac{\partial}{\partial A_\alpha} - \partial_\mu \frac{\partial}{\partial(\partial_\mu A_\alpha)}\right)\mathcal{L} \\ &= -m^2 A_\beta \eta^{\alpha\beta} + J^\alpha - (-\eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\alpha\nu}\eta^{\beta\mu})\partial_\mu\partial_\nu A_\beta. \end{aligned} \tag{2.32}$$

## 2.4 String theories

### Nambu–Gotō action

Generalising the kinetic part of (2.6), one has

$$S_{\text{NG}} := -T\int_\Sigma \mathbb{d}A =: -T\int_\Sigma \mathbb{d}^2\sigma\mathcal{L}, \tag{2.33}$$

where the Lagrangian density

$$\mathcal{L} = \sqrt{-\Gamma}, \quad \Gamma := \det\Gamma_{\alpha\beta}, \quad \Gamma_{\alpha\beta} := \frac{\partial X^\nu}{\partial\sigma^\alpha}\frac{\partial X_\nu}{\partial\sigma^\alpha}. \tag{2.34}$$

Historically Gotō 1971; Nambu 1970; Reference e.g. Blumenhagen, Lüst, and Theisen 2013 Kiefer 2012, sec. 3.2

**Polyakov action**

Generalising (2.15)

$$S_{\mathrm{P}}[X^{\mu}, h\alpha\beta] = -\frac{T}{2}\int_{\Sigma}\mathcal{L}, \tag{2.35}$$

where

$$\mathcal{L} := \sqrt{-h}h^{\alpha\beta}\Gamma_{\alpha\beta}. \tag{2.36}$$

Historically Brink, Di Vecchia, and Howe 1976; Deser and Zumino 1976; Polyakov 1981; Reference Kiefer 2012, sec. 3.2

## References


Blumenhagen, Ralph, Dieter Lüst, and Stefan Theisen (2013). *Basic Concepts of String Theory*. Theoretical and Mathematical Physics. Springer. ISBN: http://id.crossref.org/isbn/978-3-642-29497-6. DOI: 10.1007/978-3-642-29497-6. URL: http://dx.doi.org/10.1007/978-3-642-29497-6.

Brink, L., P. Di Vecchia, and P. Howe (1976). "A locally supersymmetric and reparametrization invariant action for the spinning string". In: *Physics Letters B* 65.5, pp. 471–474. ISSN: 0370-2693. DOI: 10.1016/0370-2693(76)90445-7. URL: http://dx.doi.org/10.1016/0370-2693(76)90445-7.

Deser, S. and B. Zumino (1976). "A complete action for the spinning string". In: *Physics Letters B* 65.4, pp. 369–373. ISSN: 0370-2693. DOI: 10.1016/0370-2693(76)90245-8. URL: http://dx.doi.org/10.1016/0370-2693(76)90245-8.

Gitman, Dmitriy M. and Igor V. Tyutin (1990). *Quantization of Fields with Constraints*. Springer Series in Nuclear and Particle Physics. Springer Berlin Heidelberg. ISBN: http://id.crossref.org/isbn/978-3-642-83938-2. DOI: 10.1007/978-3-642-83938-2. URL: http://dx.doi.org/10.1007/978-3-642-83938-2. Дмитрий Максимович Гитман and Игорь Викторович Тютин. *Каноническое квантование полей со связями*. 030077 г. 11овосибирск-77, Станиславского, 25: Наука, 1986.

Gotō, Tetsuo (1971). "Relativistic Quantum Mechanics of One-Dimensional Mechanical Continuum and Subsidiary Condition of Dual Resonance Model". In: *Progress of Theoretical Physics* 46.5, pp. 1560–1569. ISSN: 0033-068X. DOI: 10.1143/ptp.46.1560. URL: http://dx.doi.org/10.1143/PTP.46.1560.

Kiefer, Claus (2012). *Quantum Gravity*. 3rd. International Series of Monographs on Physics. Oxford University Press. ISBN: http://id.crossref.org/isbn/9780199585205. DOI: 10.1093/acprof:oso/9780199585205.001.0001. URL: http://dx.doi.org/10.1093/acprof:oso/9780199585205.001.0001.

Landau, Lev Davidovich and Evgeny Mikhailovich Lifshitz (1975). *The Classical Theory of Fields*. 4th. Vol. 2. Course of Theoretical Physics. Butterworth-Heinemann. ISBN: http://id.crossref.org/isbn/9780080250724. DOI: 10.1016/c2009-0-14608-1. URL: http://dx.doi.org/10.1016/c2009-0-14608-1.

Nambu, Yoichiro (1970). “Quark model and the factorization of the Veneziano amplitude”. In: *Symmetries and Quark Models*. Ed. by R. Chand. International Conference on Symmetries and Quark Models, Wayne State U., Detroit Detroit, Mich., USA, June 18-20, 1969, pp. 269–278.

Polyakov, A.M. (1981). “Quantum geometry of bosonic strings”. In: *Physics Letters B* 103.3, pp. 207–210. ISSN: 0370-2693. DOI: 10.1016/0370-2693(81)90743-7. URL: http://dx.doi.org/10.1016/0370-2693(81)90743-7.

Rothe, Heinz J and Klaus D Rothe (2010). *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*. World Scientific Lecture Notes in Physics. World Scientific. ISBN: http://id.crossref.org/isbn/978-981-4299-65-7. DOI: 10.1142/7689. URL: http://dx.doi.org/10.1142/7689.