Notes on Canonical Singular Dynamics

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1 Canonical formalism

Lagrangian with velocity
$$L^{\mathrm{v}} := L\big|_{\dot{q}=v} \tag{1.1}$$
Equations of motion
$$\sum_j M_{ij}\dot{v}_j = K^{\mathrm{v}}_i, \quad \dot{q}_i = v_i. \tag{1.2}$$
where
$$M_{ij}(q,v) := \frac{\partial^2 L^{\mathrm{v}}}{\partial v_i\,\partial v_j}. \tag{1.3}$$

Adding
$$p_i := \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.4}$$
Variation of
$$S[q,p;v] := \int \mathrm{d}t \left[L^{\mathrm{v}} + \sum_i p_i(\dot{q}_i - v_i)\right]. \tag{1.5}$$
gives the *extended Euler–Lagrange equations*
$$\dot{q}_i = v_i, \quad \dot{p}_i = \frac{\partial L^{\mathrm{v}}}{\partial q_i}, \quad p_i = \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.6}$$

Extended Hamiltonian
$$H^{\mathrm{v}}(q,p;v) := \sum_i p_i v_i - L^{\mathrm{v}}. \tag{1.7}$$
Identities
$$\frac{\partial H^{\mathrm{v}}}{\partial q_i} \equiv -\frac{\partial L^{\mathrm{v}}}{\partial q_i}, \quad \frac{\partial H^{\mathrm{v}}}{\partial p_i} \equiv v_i, \quad \frac{\partial H^{\mathrm{v}}}{\partial v_i} \equiv p_i - \frac{\partial L^{\mathrm{v}}}{\partial v_i}. \tag{1.8}$$
Variation of
$$S[q,p;v] := \int \mathrm{d}t \left[\sum_i p_i \dot{q}_i - H^{\mathrm{v}}\right] \tag{1.9}$$

gives the *extended canonical equations*

$$\dot{q}_i = [q_i, H^{\mathrm{v}}]_{\mathrm{P}}, \quad \dot{p}_i = [p_i, H^{\mathrm{v}}]_{\mathrm{P}}, \quad \frac{\partial H^{\mathrm{v}}}{\partial v_i} = 0, \tag{1.10}$$

where the Poison bracket is defined as

$$[f^{\mathrm{v}}, g^{\mathrm{v}}]_{\mathrm{P}} := \sum_i \left(\frac{\partial f^{\mathrm{v}}}{\partial q_i} \frac{\partial g^{\mathrm{v}}}{\partial p_i} - \frac{\partial f^{\mathrm{v}}}{\partial p_i} \frac{\partial g^{\mathrm{v}}}{\partial q_i} \right). \tag{1.11}$$

$v_a = \overline{v}_a(q,p)$ can be solved, $a = 1, 2, \dots, r_M$; v^α cannot be solved, $\alpha = r_M + 1, \dots, n$, where $r_M = \operatorname{rank} M$.

(need to show $v_a = \overline{v}_a(q, p_a)$)

Primary constraints in the standard form

$$\phi^{(0)}_\alpha(q,p) := \left. \frac{\partial H^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_\alpha = \overline{v}_\alpha\}} \equiv p_\alpha - \overline{p}_\alpha(q, \{p_a\}), \tag{1.12}$$

where

$$\overline{p}_\alpha(q, \{p_a\}) := \left. \frac{\partial L^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_a = \overline{v}_a\}}. \tag{1.13}$$

Total Hamiltonian

$$H^{\mathrm{t}} := \left. H^{\mathrm{v}} \right|_{\{v_a = \overline{v}_a\}} \equiv H^{\mathrm{v}}(q, p; \{\overline{v}^a(q, p_a), v_\alpha\}). \tag{1.14}$$

Subspace of primary constraints

$$\Gamma_{\mathrm{P}} = \left\{ (q,p) \,|\, \phi^0_\alpha(q,p) = 0, \forall \alpha \right\} \tag{1.15}$$

Since

$$\frac{\partial H^{\mathrm{t}}}{\partial v_\alpha} = \left. \frac{\partial H^{\mathrm{v}}}{\partial v_\alpha} \right|_{\{v_a = \overline{v}_a\}} = \phi^{(0)}_\alpha \equiv p_\alpha - \overline{p}_\alpha(q, \{p_a\}), \tag{1.16}$$

H^{t} is linear in v_α. One writes

$$H^{\mathrm{t}}(q, \{p_a\}; \{p_\alpha\}, \{v_\alpha\}) = H(q, \{p_a\}) + \sum_\alpha v_\alpha \phi^{(0)}_\alpha, \tag{1.17}$$

where H is the *canonical Hamiltonian* or simply *Hamiltonian.*

Proposition show H is independent of $\{p_\alpha\}$.

Proposition Canonical equations with primary constraints

$$\dot{q}_i = [q_i, H]_{\mathrm{P}} + \sum_\beta v_\beta [q_i, \phi_\beta]_{\mathrm{P}}, \tag{1.18}$$

$$\dot{p}_i = [p_i, H]_{\mathrm{P}} + \sum_\beta v_\beta [p_i, \phi_\beta]_{\mathrm{P}}, \tag{1.19}$$

$$\phi_\alpha^{(0)}(q, p) = 0, \tag{1.20}$$

where v_β's are undetermined. Note that eq. (1.18) for $i = \alpha$ holds identically: $\dot{q}_\alpha = \dot{q}_\alpha$.

Weak equality: $f_1 \approx f_2$ iff $f_1|_{\Gamma_{\mathrm{P}}} = f_2|_{\Gamma_{\mathrm{P}}}$.

Proposition if f and g are two functions over the phase space Γ, and $f \approx h$, then

$$\frac{\partial}{\partial q_i}\left(f - \sum_\beta \phi_\beta \frac{\partial f}{\partial p_\beta}\right) \approx \frac{\partial}{\partial q_i}\left(h - \sum_\beta \phi_\beta \frac{\partial h}{\partial p_\beta}\right), \tag{1.21}$$

$$\frac{\partial}{\partial p_i}\left(f - \sum_\beta \phi_\beta \frac{\partial f}{\partial p_\beta}\right) \approx \frac{\partial}{\partial p_i}\left(h - \sum_\beta \phi_\beta \frac{\partial h}{\partial p_\beta}\right). \tag{1.22}$$

Corollary $\forall H_1 \approx H$,

$$\dot{q}_i \approx [q_i, H]_{\mathrm{P}}, \qquad \dot{p}_i \approx [p_i, H]_{\mathrm{P}}. \tag{1.23}$$

2 Examples

2.1 Toy examples

Example 1

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x-y)^2. \tag{2.1}$$

One has

$$L^{\mathrm{v}} = \frac{1}{2}v_x^2 + v_x y - \frac{1}{2}(x-y)^2, \tag{2.2}$$

so that

$$p_x = \frac{\partial L^{\mathrm{v}}}{\partial v_x} = v_x + y, \qquad p_y = 0, \tag{2.3}$$

thus

$$\overline{v}_x = p_x - y. \tag{2.4}$$

So that v_y is the primary inexpressible velocity.

The extended Hamiltonian reads

$$H^{\mathrm{v}}(q, p; v) = v_x p_x + v_y p_y - \frac{1}{2}v_x^2 - v_x y + \frac{1}{2}(x-y)^2, \tag{2.5}$$

whilst the total Hamiltonian is

$$H^{\mathrm{t}}\big(q, p; \overline{v}_x, v_y\big) = \frac{1}{2}(p_x - y)^2 + \frac{1}{2}(x - y)^2 + v_y p_y. \tag{2.6}$$

Example 2

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x - y)^2 \tag{2.7}$$

Primary constraint

$$p_y = 0; \tag{2.8}$$

total Hamiltonian

$$H^{\mathrm{t}} = \frac{1}{2}p_x^2 - p_x y - \frac{1}{2}x^2 + xy + v_y p_y. \tag{2.9}$$

Example 3

$$L = \frac{1}{2}(\dot{q}_2 - \mathbb{e}^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2. \tag{2.10}$$

2.2 Parametrised systems

2.2.1 Non-relativistic point particle

2.2.2 Relativistic charged point particle

2.2.3 Neutral scalar field

2.3 Proca action

2.4 Dirac field

2.5 Gauge theories

2.5.1 Maxwell theory

$$\mathcal{L} = -\frac{1}{4}F_{\boxtimes\boxtimes}F^{\mu\nu} + \frac{1}{2}F_{\mu\nu}\eta^{\mu\nu} + A_\mu J_{\mathrm{f}}^{\mu}. \tag{2.11}$$

2.5.2 Spinor electrodynamics

2.5.3 Yang–Mills theory

2.5.4 Yang–Mills–Higgs theory

2.6 Gravitation theories

2.6.1 Einstein–Hilbert action