

Notes on Canonical Singular Dynamics

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May 19, 2017

1 Classical formalism

Lagrangian with velocity

$$L^\vee := L|_{\dot{q}=v} \quad (1.1)$$

Equations of motion

$$\sum_j M_{ij} \dot{v}_j = K_i^\vee, \quad \dot{q}_i = v_i. \quad (1.2)$$

where

$$M_{ij}(q, v) := \frac{\partial^2 L^\vee}{\partial v_i \partial v_j}. \quad (1.3)$$

Adding

$$p_i := \frac{\partial L^\vee}{\partial v_i}. \quad (1.4)$$

Variation of

$$S[q, p; v] := \int dt \left[L^\vee + \sum_i p_i (\dot{q}_i - v_i) \right]. \quad (1.5)$$

gives the *extended Euler–Lagrange equations*

$$\dot{q}_i = v_i, \quad \dot{p}_i = \frac{\partial L^\vee}{\partial q_i}, \quad p_i = \frac{\partial L^\vee}{\partial v_i}. \quad (1.6)$$

Hamiltonian with velocity

$$H^\vee(q, p; v) := \sum_i p_i v_i - L^\vee. \quad (1.7)$$

Identities

$$\frac{\partial H^\vee}{\partial q_i} \equiv -\frac{\partial L^\vee}{\partial q_i}, \quad \frac{\partial H^\vee}{\partial p_i} \equiv v_i, \quad \frac{\partial H^\vee}{\partial v_i} \equiv p_i - \frac{\partial L^\vee}{\partial v_i}. \quad (1.8)$$

Variation of

$$S[q, p; v] := \int dt \left[\sum_i p_i \dot{q}_i - H^\vee \right] \quad (1.9)$$

gives the *extended canonical equations*

$$\dot{q}_i = [q_i, H^v]_p, \quad \dot{p}_i = [p_i, H^v]_p, \quad \frac{\partial H^v}{\partial v_i} = 0, \quad (1.10)$$

where the *Poisson bracket* is defined as

$$[f^v, g^v]_p := \sum_i \left(\frac{\partial f^v}{\partial q_i} \frac{\partial g^v}{\partial p_i} - \frac{\partial f^v}{\partial p_i} \frac{\partial g^v}{\partial q_i} \right). \quad (1.11)$$

$v_a = \bar{v}_a(q, p; \{v_\alpha\})$ can be solved, $a = 1, 2, \dots, r_M$; v_α cannot be solved, $\alpha = r_M + 1, \dots, n$, where $r_M = \text{rank } M$.

(need to show $v_a = \bar{v}_a(q, p_a)$)

Primary constraints in the standard form

$$\Phi_\alpha(q, p) := \left. \frac{\partial H^v}{\partial v_\alpha} \right|_{\{v_\alpha = \bar{v}_\alpha\}} \equiv p_\alpha - \bar{p}_\alpha(q, \{p_a\}), \quad (1.12)$$

where

$$\bar{p}_\alpha(q, \{p_a\}) := \left. \frac{\partial L^v}{\partial v_\alpha} \right|_{\{v_a = \bar{v}_a\}}. \quad (1.13)$$

Total Hamiltonian

$$H^t := H^v|_{\{v_a = \bar{v}_a\}} \equiv H^v(q, p; \{\bar{v}^a(q, p_a; \{v_\alpha\}), v_\alpha\}). \quad (1.14)$$

Subspace of primary constraints

$$\Gamma_P = \{(q, p) \mid \Phi_\alpha(q, p) = 0, \forall \alpha\} \quad (1.15)$$

Since

$$\frac{\partial H^t}{\partial v_\alpha} = \left. \frac{\partial H^v}{\partial v_\alpha} \right|_{\{v_a = \bar{v}_a\}} = \Phi_\alpha \equiv p_\alpha - \bar{p}_\alpha(q, \{p_a\}), \quad (1.16)$$

H^t is linear in v_α . One writes

$$H^t(q, \{p_a\}; \{p_\alpha\}, \{v_\alpha\}) = H(q, \{p_a\}) + \sum_\alpha v_\alpha \Phi_\alpha, \quad (1.17)$$

where H^c is the *canonical Hamiltonian* or simply *Hamiltonian*.

Proposition H^c is independent of $\{p_\alpha\}$.

Proposition Canonical equations with primary constraints

$$\dot{q}_i = [q_i, H]_p + \sum_\beta v_\beta [q_i, \phi_\beta]_p, \quad (1.18)$$

$$\dot{p}_i = [p_i, H]_p + \sum_\beta v_\beta [p_i, \phi_\beta]_p, \quad (1.19)$$

$$\Phi_\alpha(q, p) = 0, \quad (1.20)$$

where v_β 's are undetermined. Note that eq. (1.18) for $i = \alpha$ holds identically: $\dot{q}_\alpha = \dot{q}_\alpha$.

Weak equality: $f_1 \approx f_2$ iff $f_1|_{\Gamma_P} = f_2|_{\Gamma_P}$.

Proposition if f and g are two functions over the phase space Γ , and $f \approx h$, then

$$\frac{\partial}{\partial q_i} \left(f - \sum_{\beta} \phi_{\beta} \frac{\partial f}{\partial p_{\beta}} \right) \approx \frac{\partial}{\partial q_i} \left(h - \sum_{\beta} \phi_{\beta} \frac{\partial h}{\partial p_{\beta}} \right), \quad (1.21)$$

$$\frac{\partial}{\partial p_i} \left(f - \sum_{\beta} \phi_{\beta} \frac{\partial f}{\partial p_{\beta}} \right) \approx \frac{\partial}{\partial p_i} \left(h - \sum_{\beta} \phi_{\beta} \frac{\partial h}{\partial p_{\beta}} \right). \quad (1.22)$$

Corollary $\forall H_1 \approx H$,

$$\dot{q}_i \approx [q_i, H]_{\text{P}}, \quad \dot{p}_i \approx [p_i, H]_{\text{P}}. \quad (1.23)$$

Primary and second constraints $\phi_{\mu}^{(1,)}, \phi_{\omega}^{(2,)}$; first and second class constraints $\phi_u^{(1,)}, \phi_w^{(2)}$.

2 Examples

2.1 Toy examples

Example 0

Gitman and Tyutin 1990, sec. 1.2

$$L = \frac{1}{2}(\dot{x} - y)^2 \quad (2.1)$$

Example 1

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x - y)^2. \quad (2.2)$$

One has

$$L^v = \frac{1}{2}v_x^2 + v_x y - \frac{1}{2}(x - y)^2, \quad (2.3)$$

so that

$$p_x = \frac{\partial L^v}{\partial v_x} = v_x + y, \quad p_y = 0, \quad (2.4)$$

thus

$$\bar{v}_x = p_x - y. \quad (2.5)$$

So that v_y is the primary inexpressible velocity.

The Hamiltonian with velocity reads

$$H^v(q, p; v) = v_x p_x + v_y p_y - \frac{1}{2}v_x^2 - v_x y + \frac{1}{2}(x - y)^2, \quad (2.6)$$

whilst the total and canonical Hamiltonians are

$$H^t(q, p; \bar{v}_x, v_y) = H^c + v_y \Phi_1, \quad (2.7)$$

$$H^c(q, p) = \frac{1}{2}(p_x - y)^2 + \frac{1}{2}(x - y)^2, \quad (2.8)$$

where the only primary constraint $\Phi_1 = p_y$.

Persistence condition of Φ_1 leads to

$$0 \approx [\Phi_1, H^t]_p = p_x + x - 2y =: \Phi_2. \quad (2.9)$$

Note that $[\Phi_1, \Phi_2] = p_x - x$ does not vanish on Γ , thus $\Phi_{1,2}$ are second-class constraints, and no more constraint can be generated. To solve for v_y one evaluates

$$0 =: [\Phi_1, H^t]_p = p_x - x - 2v_y, \quad (2.10)$$

so that $v_y := (p_x - x)/2$ solves the constraint.

One also has

$$Q_{\alpha\beta} = [\Phi_\alpha, \Phi_\beta]_p = \begin{pmatrix} 0 & +2 \\ -2 & 0 \end{pmatrix}, \quad (Q^{-1})_{\alpha\beta} = \begin{pmatrix} 0 & -1/2 \\ +1/2 & 0 \end{pmatrix}, \quad (2.11)$$

so that the Dirac brackets are defined as

$$[f, g]_D := [f, g]_p + \frac{1}{2} \left([f, p_y]_p [p_x + x - 2y, g]_p - [f, p_x + x - 2y]_p [p_y, g]_p \right). \quad (2.12)$$

The fundamental ones different from Poisson brackets are

$$[x, y]_D = [y, p_x]_D = \frac{1}{2}, \quad [y, p_y]_D = 0. \quad (2.13)$$

Last one different from book?

Example 2

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x - y)^2 \quad (2.14)$$

Primary constraint

$$p_y = 0; \quad (2.15)$$

total Hamiltonian

$$H^t = \frac{1}{2}p_x^2 - p_x y - \frac{1}{2}x^2 + xy + v_y p_y. \quad (2.16)$$

Example 3

$$L = \frac{1}{2}(\dot{q}_2 - e^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2. \quad (2.17)$$

2.2 Parametrised systems

Non-relativistic point particle

Kiefer 2012, sec. 3.1.1

$$S[q(t)] := \int_{t_1}^{t_2} \mathbb{d}t L\left(q, \frac{\mathbb{d}q}{\mathbb{d}t}\right) \quad (2.18)$$

Relativistic charged point particle

Landau and Lifshitz 1975, sec. 16, Kiefer 2012, sec. 3.1.2

$$S := \int -m \mathbb{d}s + e A_\mu(x) \mathbb{d}x^\mu =: \int \mathbb{d}\tau L, \quad (2.19)$$

where the Lagrangian reads

$$L = -m \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} + q \dot{x}^\mu A_\mu(x). \quad (2.20)$$

$$M_{\mu\nu} := \frac{\partial^2 L}{\partial v^\mu \partial v^\nu} = m \frac{-\eta_{\mu\nu} \eta_{\alpha\beta} + \eta_{\mu\alpha} \eta_{\nu\beta}}{(-\eta_{\rho\sigma} v^\rho v^\sigma)^{3/2}} v^\alpha v^\beta, \quad (2.21)$$

which has one and only one eigenvector with null eigenvalue

$$v^\mu M_{\mu\nu} = 0. \quad (2.22)$$

Momenta

$$p_\mu = \frac{\partial L}{\partial v^\mu} = \frac{m \eta_{\mu\nu} v^\nu}{\sqrt{-\eta_{\rho\sigma} v^\rho v^\sigma}} + q A_\mu. \quad (2.23)$$

If one chooses v^0 to be the primary inexpressible velocity, then eliminating p_0 in eq. (2.22) yields

$$v^i = \frac{\xi \eta^{ij} (p_j - q A_j) v^0}{\sqrt{m^2 + \eta^{kl} (p_k - q A_k) (p_l - q A_l)}}, \quad (2.24)$$

where $\xi = \text{sgn } v^0$. In the following $\xi = +1$ will be chosen.

Inserting eq. (2.23) into the Hamiltonian with velocity

$$H^\nu = v^\mu p_\mu - L^\nu = m \sqrt{-\eta_{\mu\nu} v^\mu v^\nu} + v^\mu (p_\mu - q A_\mu(x)), \quad (2.25)$$

one obtains the total Hamiltonian

$$H^t = v^0 \left(p_0 - q A_0 + \sqrt{m^2 + \eta^{kl} (p_k - q A_k) (p_l - q A_l)} \right), \quad (2.26)$$

where only a primary constraint survives, which is obviously a first-class constraint

$$\phi^{(1,1)} = p_0 - q A_0 + \sqrt{m^2 + \eta^{kl} (p_k - q A_k) (p_l - q A_l)}, \quad (2.27)$$

and the canonical Hamiltonian vanishes

$$H^c = 0. \quad (2.28)$$

To compare, note in the non-covariant formalism (Landau and Lifshitz 1975, sec. 8)

$$S = \int dt L, \quad L = -m\sqrt{1 - \dot{\vec{x}}^2} - q\phi + q\dot{\vec{x}} \cdot \vec{A}, \quad (2.29)$$

the system is regular, and the canonical Hamiltonian reads

$$H^c = \sqrt{m^2 + (\vec{p} - q\vec{A})^2} + q\phi, \quad (2.30)$$

which corresponds to setting $\phi^{(1,1)} = 0$, $p_0 \rightarrow -H^c$ ($p_\mu = (-E, \vec{p})$), and noting $A_\mu = (-\phi, \vec{A})$.

Relativistic point particle with einbein

Blumenhagen, Lüst, and Theisen 2013, sec. 2.1

$$L := \frac{1}{2}(e^{-1}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2e) \quad (2.31)$$

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = e^{-1}\eta_{\mu\nu}v^\nu, \quad p_e = 0. \quad (2.32)$$

Choosing v^e to be the primary inexpressible velocity, one has

$$v^\mu = e\eta^{\mu\nu}p_\nu. \quad (2.33)$$

Hamiltonian with velocity

$$H^v = v^\mu p_\mu + v^e p_e + \frac{1}{2}(-e^{-1}\eta_{\mu\nu}v^\mu v^\nu + m^2e); \quad (2.34)$$

total Hamiltonian

$$H^t = \frac{e}{2}(\eta^{\mu\nu}p_\mu p_\nu + m^2) + v^e p_e; \quad (2.35)$$

canonical Hamiltonian

$$H^c = \frac{e}{2}(\eta^{\mu\nu}p_\mu p_\nu + m^2). \quad (2.36)$$

The only primary constraint

$$\Phi^{(1,)} = p_e; \quad (2.37)$$

its time evolution

$$\begin{aligned} [\Phi^{(1,)}, H^t]_p &= [p_e, e]_p \frac{1}{2}(\eta^{\mu\nu}p_\mu p_\nu + m^2) \\ &= -\frac{1}{2}(\eta^{\mu\nu}p_\mu p_\nu + m^2). \end{aligned} \quad (2.38)$$

Choose

$$\Phi^{(2,)} = \eta^{\mu\nu} p_\mu p_\nu + m^2, \quad (2.39)$$

whose Poisson bracket with H^t vanishes; furthermore,

$$[\Phi^{(1,)}, \Phi^{(2,)}]_p \equiv 0. \quad (2.40)$$

Thus one ends up with two first-class constraints.

2.2.1 Neutral scalar field

Kiefer 2012, sec. 3.3

2.3 Maxwell–Proca theory

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu + A_\mu J^\mu, \quad (2.41)$$

where $m > 0$ corresponds to the Proca theory Gitman and Tyutin 1990, sec. 2.3, and $m = 0$ the Maxwell theory H. J. Rothe and K. D. Rothe 2010, sec. 3.3.3, Gitman and Tyutin 1990, sec. 2.4.

Lagrangian density with velocity

$$\mathcal{L}^v = \frac{1}{2} (V_i - \partial_i A_0)^2 - \frac{1}{4} F_{ij}^2 + \frac{m^2}{2} (A_0^2 - A_i^2) + A_0 J^0 + A_i J^i; \quad (2.42)$$

momenta density

$$B^0 := \frac{\partial \mathcal{L}^v}{\partial V_0} = 0, \quad B^i := \frac{\partial \mathcal{L}^v}{\partial V_i} = V^i + \partial_i A^0; \quad (2.43)$$

total and canonical Hamiltonian as well as primary constraint

$$\mathcal{H}^t = \mathcal{H}^c + V_0 \Phi^{(1,)}, \quad (2.44)$$

$$\mathcal{H}^c = \frac{1}{2} B_i^2 + B_i \partial_i A_0 + \frac{1}{4} F_{ij}^2 + \frac{m^2}{2} (-A_0^2 + A_i^2) - A_0 J^0 - A_i J^i, \quad (2.45)$$

$$\Phi^{(1,)} = B^0. \quad (2.46)$$

$$\begin{aligned} [\Phi^{(1,)}_1, \mathcal{H}_2^t]_p &= B^i{}_1 \partial_{i2} [B^0{}_2, A_{02}]_p + \frac{m^2}{2} [B^0{}_1, A_{02}^2]_p - [B^0{}_1, A_{02}]_p J^0{}_2 \\ &= [-B^i{}_2 \partial_{i2} - m^2 A_0 + J^0{}_2]_p \delta(x_1 - x_2). \end{aligned} \quad (2.47)$$

$$\Phi^{(2,)} = \partial_i B^i - m^2 A_0 + J^0. \quad (2.48)$$

2.4 String theories

Nambu–Gotō action

Generalising the kinetic part of (2.18), one has

$$S_{\text{NG}} := -T \int_{\Sigma} \mathrm{d}A =: -T \int_{\Sigma} \mathrm{d}^2\sigma \mathcal{L}, \quad (2.49)$$

where the Lagrangian density

$$\mathcal{L} = \sqrt{-\Gamma}, \quad \Gamma := \det \Gamma_{\alpha\beta}, \quad \Gamma_{\alpha\beta} := \frac{\partial X^\nu}{\partial \sigma^\alpha} \frac{\partial X_\nu}{\partial \sigma^\beta}. \quad (2.50)$$

Historically Gotō 1971; Nambu 1970; Reference e.g. Blumenhagen, Lüster, and Theisen 2013 Kiefer 2012, sec. 3.2

Polyakov action

Generalising (2.30)

$$S_{\text{P}}[X^\mu, h_{\alpha\beta}] = -\frac{T}{2} \int_{\Sigma} \mathcal{L}, \quad (2.51)$$

where

$$\mathcal{L} := \sqrt{-h} h^{\alpha\beta} \Gamma_{\alpha\beta}. \quad (2.52)$$

Historically Brink, Di Vecchia, and Howe 1976; Deser and Zumino 1976; Polyakov 1981; Reference Kiefer 2012, sec. 3.2

2.5 Gravitation theories

Closed Friedmann universe

This part adapts *ibid.*, sec. 8.1.2.

The total action reads

$$S := S_{\text{EG}} + S_{\phi}, \quad (2.53)$$

where S_{EG} follows (2.70), and

$$S_{\phi} := \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} (\nabla_{\mu} \phi) (\nabla_{\nu} \phi) - m^2 \phi^2 \right). \quad (2.54)$$

Adapting

$$\mathrm{d}s^2 = -N^2(t) \mathrm{d}t^2 + a^2(t) \mathrm{d}\Omega_3^2, \quad (2.55)$$

where

$$\Omega_3^2 = \mathrm{d}\chi^2 + \sin^2 \chi (\mathrm{d}\theta^2 + \sin^2 \theta \mathrm{d}\phi^2). \quad (2.56)$$

One has

$$\sqrt{-g} = Na^3 \sin^2 \chi \sin \theta, \quad \sqrt{h} = a^3 \sin^2 \chi \sin \theta; \quad (2.57)$$

whereas

$$R = \frac{6}{N^2} \left(-\frac{\dot{N}\dot{a}}{Na} + \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right) + \frac{6}{a^2}, \quad K = \frac{3\dot{a}}{Na}. \quad (2.58)$$

$$S_{\text{EG}} = \frac{A_3}{16\mathbb{W}G} \left(\int_{t_1}^{t_2} \mathbb{d}t Na^3 (R - 2\Lambda) - \left[\frac{6\dot{a}a^2}{N} \right]_{t_1}^{t_2} \right), \quad (2.59)$$

where

$$A_3 = \int \sin^2 \chi \sin \theta \mathbb{d}\chi \mathbb{d}\theta \mathbb{d}\phi = 2\mathbb{W}^2. \quad (2.60)$$

The term proportional to \ddot{a}/a in the integrand can be integrated by parts

$$\int_{t_1}^{t_2} \mathbb{d}t Na^3 \frac{6}{N^2} \frac{\ddot{a}}{a} = 6 \left(\left[\frac{\dot{a}a^2}{N} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathbb{d}t \dot{a} \frac{\mathbb{d}}{\mathbb{d}t} \frac{a^2}{N^2} \right), \quad (2.61)$$

in which the first term cancels the Gibbons–Hawking–York term. One has

$$S_{\text{EG}} = \frac{3\mathbb{W}}{4G} \int_{t_1}^{t_2} \mathbb{d}t \left(-\frac{a}{N} \dot{a}^2 + Na - \frac{\Lambda}{3} Na^3 \right). \quad (2.62)$$

The matter part of the action reads

$$S_\phi = \mathbb{W}^2 \int_{t_1}^{t_2} \mathbb{d}t a^3 \left(\frac{1}{N} \dot{\phi}^2 - m^2 N \phi^2 \right). \quad (2.63)$$

Lagrangian with velocity

$$L^v = \frac{3\mathbb{W}}{4G} \left(-\frac{a}{N} v^{a^2} + Na - \frac{\Lambda}{3} Na^3 \right) + \mathbb{W}^2 a^3 \left(\frac{1}{N} v^{\phi^2} - m^2 N \phi^2 \right). \quad (2.64)$$

Canonical momenta

$$p_N := \frac{\partial L^v}{\partial v^N} = 0, \quad p_a := \frac{\partial L^v}{\partial v^a} = -\frac{3\mathbb{W}}{2G} \frac{a}{N} v^a, \quad p_\phi := \frac{\partial L^v}{\partial v^\phi} = 2\mathbb{W}^2 \frac{a^3}{N} v^\phi. \quad (2.65)$$

Choosing v^N to be the primary inexpressible velocity, one obtains the total and canonical Hamiltonians

$$H^t = H^c + v^N \Phi, \quad (2.66)$$

$$H^c = -NH_\perp, \quad (2.67)$$

where

$$H_\perp := \frac{G}{3\mathbb{W}} \frac{p_a^2}{a} - \frac{1}{4\mathbb{W}^2} \frac{p_\phi^2}{a^3} - \frac{3\mathbb{W}}{4G} \left(\frac{\Lambda}{3} a^2 - 1 \right) a - \mathbb{W}^2 m^2 a^3 \phi^2, \quad (2.68)$$

$$\Phi = p_N \quad (2.69)$$

are the *Hamiltonian constraint* and the primary constraint, respectively.
Evaluating the time evolution of Φ yields

$$[\Phi, H^t]_p = H_\perp, \quad (2.70)$$

so that the Hamiltonian constraint is indeed a constraint. There is no further constraint, and $[\Phi, H_\perp]_p$ vanishes identically. Therefore there are two and only two first-class constraints.

2.5.1 Einstein–Hilbert action

$$S_{\text{EG}} = S_{\text{EH}} + S_{\text{GHY}}, \quad (2.71)$$

$$S_{\text{EH}} = \frac{1}{16\pi G} \int_{\mathcal{M}} \mathbb{d}^4x \sqrt{-g}(R - 2\Lambda), \quad (2.72)$$

and

$$S_{\text{GHY}} = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} \mathbb{d}^3x \sqrt{h}K, \quad (2.73)$$

which is named after Gibbons and Hawking 1977; York 1972 but actually already mentioned in Einstein 1916. See Dyer and Hinterbichler 2009 for a brief review.

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