Notes on Lagrangian Singular Dynamics

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April 12, 2017

1 Classical formalism

Lagrangian with velocity

$$L^{\mathrm{v}} := L\big|_{\dot{q}=v} \tag{1.1}$$

Equations of motion

$$\sum_j M_{ij}\dot{v}_j = K_i^{\mathrm{v}}, \quad \dot{q}_i = v_i. \tag{1.2}$$

where

$$M_{ij}(q,v) := \frac{\partial^2 L}{\partial v_i \, \partial v_j}. \tag{1.3}$$

2 Examples

2.1 Toy examples

Example 0

Gitman and Tyutin 1990, sec. 1.2

$$L = \frac{1}{2}(\dot{x} - y)^2 \tag{2.1}$$

Example 1

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x-y)^2. \tag{2.2}$$

Example 2

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x-y)^2 \tag{2.3}$$

Example 3

$$L = \frac{1}{2}(\dot{q}_2 - \mathbb{e}^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2. \tag{2.4}$$

2.2 Parametrised systems

Non-relativistic point particle

Kiefer 2012, sec. 3.1.1

$$S[q(t)] := \int_{t_1}^{t_2} \mathbb{d}t \, L\left(q, \frac{\mathbb{d}q}{\mathbb{d}t}\right) \tag{2.5}$$

Relativistic charged point particle

Landau and Lifshitz 1975, sec. 16, Kiefer 2012, sec. 3.1.2

$$S := \int -m \, \mathbb{d}s + eA_\mu(x) \, \mathbb{d}x^\mu =: \int \mathbb{d}\tau \, L, \tag{2.6}$$

where the Lagrangian reads

$$L = -m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + q\dot{x}^\mu A_\mu(x). \tag{2.7}$$

$$M_{\mu\nu} := \frac{\partial^2 L}{\partial \dot{x}^\mu \, \partial \dot{x}^\nu} = m \frac{-\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta}}{\left(-\eta_{\rho\sigma}\dot{x}^\rho \dot{x}^\sigma\right)^{3/2}} \dot{x}^\alpha \dot{x}^\beta, \tag{2.8}$$

which has one and only one zero eigenvector

$$\dot{x}^\mu M_{\mu\nu} = 0. \tag{2.9}$$

Euler–Lagrange derivatives

$$E_\mu = \left(\frac{\partial}{\partial x^\mu} - \frac{\mathbb{d}}{\mathbb{d}\tau}\frac{\partial}{\partial \dot{x}^\mu}\right) L \equiv K_\mu - M_{\mu\nu}\ddot{x}^\nu, \tag{2.10}$$

where

$$K_\mu := -qF_{\mu\nu}\dot{x}^\nu, \tag{2.11}$$

and

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.12}$$

Contracting the zero eigenvector with eq. (2.10) yields

$$\dot{x}^\mu E_\mu = \dot{x}^\mu K_\mu - \dot{x}^\mu M_{\mu\nu}\ddot{x}^\nu \equiv 0, \tag{2.13}$$

so that it generates a gauge identity, and no further constraint exists. Thus the system has a symmetry

$$\delta x^\mu = \dot{x}^\mu \delta\lambda. \tag{2.14}$$

2.2.1 Neutral scalar field

ibid., sec. 3.3

2.3 Maxwell–Proca theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + \frac{1}{2}F_{\mu\nu}S^{\mu\nu} + A_\mu J_{\mathrm{f}}^\mu, \tag{2.15}$$

where $m > 0$ corresponds to the Proca theory Gitman and Tyutin 1990, sec. 2.3, and $m = 0$ the Maxwell theory H. J. Rothe and K. D. Rothe 2010, sec. 3.3.3 Gitman and Tyutin 1990, sec. 2.4.

2.4 String theories

Relativistic point particle with auxiliary coordinate

Blumenhagen, Lüst, and Theisen 2013, sec. 2.1

$$L := \frac{1}{2}\left(e^{-1}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2 e\right) \tag{2.16}$$

Nambu–Gotō action

Generalising the kinetic part of (2.6), one has

$$S_{\mathrm{NG}} := -T\int_\Sigma \mathrm{d}A =: -T\int_\Sigma \mathrm{d}^2\sigma\mathcal{L}, \tag{2.17}$$

where the Lagrangian density

$$\mathcal{L} = \sqrt{-\Gamma}, \quad \Gamma := \det\Gamma_{\alpha\beta}, \quad \Gamma_{\alpha\beta} := \frac{\partial X^\nu}{\partial\sigma^\alpha}\frac{\partial X_\nu}{\partial\sigma^\alpha}. \tag{2.18}$$

Historically Gotō 1971; Nambu 1970; Reference e.g. Blumenhagen, Lüst, and Theisen 2013 Kiefer 2012, sec. 3.2

Polyakov action

Generalising (2.16)

$$S_{\mathrm{P}}[X^\mu, h\alpha\beta] = -\frac{T}{2}\int_\Sigma \mathcal{L}, \tag{2.19}$$

where

$$\mathcal{L} := \sqrt{-h}h^{\alpha\beta}\Gamma_{\alpha\beta}. \tag{2.20}$$

Historically Brink, Di Vecchia, and Howe 1976; Deser and Zumino 1976; Polyakov 1981; Reference Kiefer 2012, sec. 3.2

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