Integrable Cosmological Models with Liouville Fields

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Outline

1. Introduction
2. Classical model and the implicitised trajectories
3. Dirac quantisation and the wave functions

Introduction

Introduction

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The Friedmann-Lemaître model
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- Flat Robertson–Walker metric $\mathbb{d}s^2 = -N^2(t)\,\mathbb{d}t^2 + \varkappa^{-1}\mathbb{e}^{2\alpha(t)}\,\mathbb{d}\Omega_3^2$, where $\varkappa := 8\pi G$, $\mathbb{d}\Omega_3^2$ dimensionless spacial metric.
- Homogeneous real Klein–Gordon field with potential $V\mathbb{e}^{\lambda\phi}$ (Liouville), where $\lambda, V \in \mathbb{R}$, and kinetic term with sign $\ell = \pm 1$ (quintessence / phantom model).
- Total action $\mathcal{S} := S_{\mathsf{EH}} + S_{\mathsf{GHY}} + S_{\mathsf{L}} = \int \mathbb{d}\Omega_3^2 \int \mathbb{d}t\, L$, in which the effective Lagrangian reads

$$L := \varkappa^{3/2} N \mathbb{e}^{3\alpha}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{N^2} + \ell\,\frac{\dot{\phi}^2}{2N^2} - V\mathbb{e}^{\lambda\phi}\right), \tag{1}$$

in which dot means $\mathbb{d}/\mathbb{d}t$ and $\ell = \pm 1$.

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Decoupling the variables

Via *rescaled* special orthogonal transformation

- Setting $\overline{N} := N\mathrm{e}^{-3\alpha}$, eq. (1) transforms to

$$L = \varkappa^{3/2}\overline{N}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{\overline{N}^2} + \ell\,\frac{\dot{\phi}^2}{2\overline{N}^2} - V\mathrm{e}^{\lambda\phi+6\alpha}\right) \tag{2}$$

- Defining $\varDelta := \lambda^2 - 6\ell\varkappa$, $\jmath := \operatorname{sgn}\varDelta$ and $g := \jmath\sqrt{|\varDelta|} \equiv \jmath\sqrt{\jmath\varDelta}$, the *rescaled* special orthogonal transformation

$$\begin{pmatrix}\alpha\\ \phi\end{pmatrix} = \frac{\jmath}{g}\begin{pmatrix}\lambda & -\ell\kappa\\ -6 & \lambda\end{pmatrix}\begin{pmatrix}\jmath_\beta\beta\\ \jmath_\chi\chi\end{pmatrix} \qquad \text{where } \jmath_\beta, \jmath_\chi = \pm 1 \tag{3}$$

gives the decoupled Lagrangian

$$L = \varkappa^{3/2}\overline{N}\left(-\jmath\frac{3}{\varkappa}\frac{\dot{\beta}^2}{\overline{N}^2} + \ell\jmath\,\frac{\dot{\chi}^2}{2\overline{N}^2} - V\mathrm{e}^{\jmath_\chi g\chi}\right). \tag{4}$$

- The Euler–Lagrange equations w.r.t. $\overline{N}$, β and χ will be called the trsfed. 1st, 2nd Friedmann eqs. and the Klein–Gordon eq., respectively.

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Implicitised integration

$p_\beta \neq 0$

- Since β is cyclic in eq. (4), the trsfed. 2nd Friedmann eq. can be integrated[1]

$$\text{const.} \equiv p_\beta := \frac{\partial L}{\partial \dot\beta} = -6\mathscr{s}\varkappa^{1/2}\frac{\dot\beta}{\overline{N}} \equiv -6\mathscr{s}\mathscr{s}_\beta \frac{\varkappa^{1/2}}{g}\frac{\lambda\dot\alpha + \ell\varkappa\dot\phi}{\overline{N}}. \tag{5}$$

- For $p_\beta \neq 0$, fixing the implicitising gauge $\overline{N} = -6\mathscr{s}\sqrt{\varkappa}\dot\beta/p_\beta$, the trsfed. 1st Friedmann equation can be integrated

$$\mathbb{e}^{\mathscr{s}_\chi g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|}S^2\left(\mathscr{s}_\beta\sqrt{\frac{3}{2\varkappa}}\,g\beta\right), \tag{6}$$

in which $\mathscr{v} := \operatorname{sgn} V$, and

$$S(\gamma) := \begin{cases} \operatorname{sech}(\gamma + C_{++}) & (\ell, \mathscr{s}\mathscr{v}) = (+,+), \\ \operatorname{csch}(\gamma + C_{+-}) & (\ell, \mathscr{s}\mathscr{v}) = (+,-), \\ \sec(\gamma + C_{-+}) & (\ell, \mathscr{s}\mathscr{v}) = (-,+), \\ \mathbb{i}\csc(\gamma + C_{--}) & (\ell, \mathscr{s}\mathscr{v}) = (-,-). \end{cases} \tag{7}$$

[1] Chen Lan. PhD thesis. Saint Petersburg State University, 2016. URL: https://search.rsl.ru/ru/record/01006663434.

- The integrals are consistent with the trsfed. Klein–Gordon equation.
- The integral for $(+,+)$
    - has two asymptotes
- The implicitised integral for $(+,-)$
    - contains two distinct solutions
    - has three asymptotes
- The implicitised integral for $(-,+)$
    - contains infinite distinct solutions
    - has infinite asymptotes, which are pairwise parallel
- The integral for $(-,-)$
    - is not real
- The implicitised integral enables one to compare trajectories with wave functions, see below.

Implicitised integration

$p_\beta = 0$

- For $p_\beta = 0$, one has $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell\lambda\alpha/\kappa$, which is the familiar power-law special solution[2].
- Further integrating the first Friedmann equation demands $(+,-)$ or $(-,+)$ to guarantee $\overline{N} > 0$, and the result is automatically consistent with the trsfed. Klein–Gordon equation.
- Fixing $\overline{N} = \left(2\varkappa^2|V|\right)^{-1/2}$ yields

$$\mathrm{e}^{g\mathfrak{s}_\chi \chi} = \left(\frac{2\kappa}{g(t-t_0)}\right)^2 . \tag{8}$$

[2] Mariusz P. Dąbrowski, Claus Kiefer, and Barbara Sandhöfer. In: *Physical Review D* 74.4 (Aug. 2006).

Introduction

Introduction

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Dirac quantisation
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- The primary Hamiltonian and the Hamiltonian constraint reads

$$H^{\mathrm{p}} = \overline{N} H_\perp + p_{\overline{N}} v^{\overline{N}}, \tag{9}$$

$$H_\perp = -\jmath \frac{p_\beta^2}{12\varkappa^{1/2}} + \ell \jmath \frac{p_\chi^2}{2\varkappa^{3/2}} + \varkappa^{3/2} V \mathbb{e}^{g\jmath_\chi \chi}. \tag{10}$$

- Applying the Dirac quantisation rules with the Laplace–Beltrami opeator, one gets the mss. Wheeler–DeWitt eq. with (β, χ) $\widehat{H}_\perp \Psi(\beta, \chi) = 0$,

$$-\jmath \frac{12\varkappa^{1/2}}{\hbar^2} \widehat{H}_\perp \Psi =: -\partial_\beta^2 \Psi + \mathbb{D}\Psi, \qquad \mathbb{d} = \ell \frac{6}{\varkappa} \partial_\chi^2 - \jmath \frac{12\varkappa^2 V \mathbb{e}^{g\jmath_\chi \chi}}{\hbar^2}. \tag{11}$$

- Equation (11) is KG-like, hyperbolic for $\ell = +1$ and elliptic for $\ell = -1$.

Separation of the variables and mode functions
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- Inserting the separating Ansatz $\Psi(\beta,\chi) = \mathrm{e}^{-\mathrm{i}k_\beta \jmath_\beta \beta}\psi(\chi)$, the remaining equation turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta,\chi) := \mathrm{e}^{-\mathrm{i}\nu\gamma}\psi_\nu(\chi) := \mathrm{e}^{-\mathrm{i}\nu\gamma}\Big(C_1 \mathrm{B}_\nu^{(1)}(\sigma) + C_2 \mathrm{B}_\nu^{(2)}(\sigma)\Big), \tag{12}$$

in which

$$\nu := \sqrt{\frac{2\varkappa}{3}}\frac{k_\beta}{g}, \quad \gamma := \sqrt{\frac{3}{2\varkappa}} g\beta, \quad \sigma^2 := \frac{8\varkappa^3 |V| \mathrm{e}^{g \jmath_\chi \chi}}{\hbar^2 g^2}, \tag{13}$$

$$\mathrm{B}_\nu^{(i)}(\sigma) := \begin{cases} \mathrm{K} \text{ or } \mathrm{I}_{\mathrm{i}\nu}(\sigma) & (\ell, \jmath v) = (+,+), \\ \mathrm{F} \text{ or } \mathrm{G}_{\mathrm{i}\nu}(\sigma) & (\ell, \jmath v) = (+,-), \\ \mathrm{J} \text{ or } \mathrm{Y}_\nu(\sigma) & (\ell, \jmath v) = (-,+), \\ \mathrm{K} \text{ or } \mathrm{I}_\nu(\sigma) & (\ell, \jmath v) = (-,-). \end{cases} \tag{14}$$

- Adapted to imaginary order, $\mathrm{F}_\nu(\sigma)$ and $\mathrm{G}_\nu(\sigma)$ are defined in[3].

[3] T. Mark Dunster. In: *SIAM Journal on Mathematical Analysis* 21.4 (July 1990), pp. 995–1018.

Physical mode functions
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- Physical mode functions are expected to be regular on the boundary.
- $(+,+)$: $|\mathrm{I}_{\mathbb{i}\nu}(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- $(-,+)$:
    - $\forall n \in \mathbb{Z}$, $|\mathrm{Y}_n(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
    - $\forall \nu \in \mathbb{R}\backslash\mathbb{Z}$, choose $\mathrm{J}_{-\nu}$ instead of $\mathrm{Y}_{+\nu}$, since $\mathrm{J}_{\pm\nu}$ are also linearly independent.
    - $\forall \nu \in \mathbb{R}^{+}\backslash\mathbb{Z}$, $|\mathrm{J}_{-\nu}(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
- $(-,-)$: $|\mathrm{K}_\nu(\sigma)| \to +\infty$ as $\alpha \to -\infty$; $|\mathrm{I}_\nu(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- These are not to included in the space of physical wave functions.
- For $(-,+)$, only J_ν with $\nu \geq 0$ survives.

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Matching quantum number with classical first integral

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- Baustelle
- In order to match the quantum number k_β (or linearly, ν) with the classical first integral p_β, one may apply the principle of constructive interference.
- Writing the mode function in the WKB form

$$\Psi_{k_\beta}(\beta, \chi) \sim \sqrt{\rho}\, \mathbb{e}^{\mathbb{i} S/\hbar}, \qquad S/\hbar \gg 1 \text{ and } k_\beta \gg 1, \tag{15}$$

the principle demands that $\partial S/\partial k_\beta = 0$ be equivalent to the classical trajectory.

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Matching quantum number with classical first integral

$(+,+)$ as exemplar

- Fixing $\nu/\sigma > 1$, the asymptotic expansion reads

$$\mathrm{K}_{\mathbb{i}\nu}(\sigma) \sim \frac{\sqrt{2\mathbb{\pi}}\cos\left(\sqrt{\nu^2-\sigma^2} - \nu \arccos\frac{\nu}{\sigma} - \frac{\mathbb{\pi}}{4}\right)}{\left(\nu^2-\sigma^2\right)^{1/4}\mathbb{e}^{\mathbb{\pi}\nu/2}} + O\left(\frac{1}{\sigma}\right). \tag{16}$$

  There are two phases with opposite signs. Applying the principle to the full mode function Ψ_ν, one has $\sigma/\nu = \operatorname{sech}(\jmath_\beta\gamma)$, which match the trajectory if

$$\hbar k_\beta \equiv \hbar\sqrt{\frac{3}{2\varkappa}g\hbar\nu} = p_\beta, \tag{17}$$

  as expected.

- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; hence no WKB approximation works.
- The conclusions also hold for $\mathrm{F}_{\mathbb{i}\nu}(\sigma)$ and $\mathrm{G}_{\mathbb{i}\nu}(\sigma)$ for $(+,-)$, as well as $\mathrm{J}_\nu(\sigma)$ for $(-,+)$.

Inner product for wave functions

Schrödinger product

- To make sense of amplitude and wave packet, an inner product is necessary
- A common starting point of most approaches is the Schrödinger product

$$(\Psi_1, \Psi_2)_{\mathsf{S}} := \int \mathbb{d}\chi \, \Psi_1^*(\beta, \chi) \Psi_2(\beta, \chi); \tag{18}$$

- In terms of a norm, $(\Psi, \Psi)_{\mathsf{S}} \equiv \int \mathbb{d}\chi \, \rho_{\mathsf{S}}(\beta, \chi)$, in which $\rho_{\mathsf{S}} := \Psi^* \Psi$. Manifestly $\rho_{\mathsf{S}} \geq 0$; one has $(\Psi, \Psi)_{\mathsf{S}} > 0$.
- The corresponding Schrödinger current does not satisfy continuity equation $\dot{\rho}_{\mathsf{S}} + \nabla \cdot \vec{j}_{\mathsf{S}} = 0$, because eq. (11) is KG-like.
- $\mathrm{K}_{\mathbb{i}\nu}$[4] for $(+,+)$, $\mathrm{F}_{\mathbb{i}\nu}$ and $\mathrm{G}_{\mathbb{i}\nu}$ (not $\mathrm{J}_{\mathbb{i}\nu}$!) for $+,-$ can be proved to be orthogonal and complete individually, as well as normalised.

[4] Semyon B. Yakubovich. In: *Opuscula Math.* 26.1 (2006), pp. 161–172. URL: http://www.opuscula.agh.edu.pl/vol26/1/art/opuscula_math_2612.pdf, A. Passian et al. In: *Journal of Mathematical Analysis and Applications* 360.2 (Dec. 2009), pp. 380–390, Radosław Szmytkowski and Sebastian Bielski. In: *Journal of Mathematical Analysis and Applications* 365.1 (May 2010), pp. 195–197.

Peculiarity for the phantom model

Orthonormality and completeness for mode functions; self-adjointness for operators

- $\{\mathrm{J}_\nu(\sigma)\}$'s are not orthogonal

$$\int_0^{+\infty} \frac{\mathrm{d}\sigma}{\sigma} \mathrm{J}_\nu(\sigma) \mathrm{J}_{\tilde{\nu}}(\sigma) = \frac{2\sin(\pi(\tilde{-})/2)}{\pi(\nu^2 - \tilde{\nu}^2)}. \tag{19}$$

- Imposing self-adjointness for $\hat{p}_\chi^2$, one finds that

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Integration of the transformed first Friedmann equation

$p_\beta \neq 0$

In order to integrate the equation under the implicitising gauge

$$\jmath \frac{p_\beta^2}{12}\left(-\ell\frac{\varkappa^{1/2}}{6}\left(\frac{\mathbb{d}\chi}{\mathbb{d}\beta}\right)^2+\varkappa^{-1/2}\right)-\varkappa^{3/2}V\mathbb{e}^{g\jmath_\chi\chi}=0, \tag{20}$$

one can substitute

$$\gamma:=\sqrt{\frac{3}{2\varkappa}}g\beta, \qquad \tilde{\sigma}^2:=\frac{p_\beta^2}{12\varkappa^2|V|}\mathbb{e}^{-g\jmath_\chi\chi}, \tag{21}$$

to get

$$\left(\frac{\mathbb{d}\tilde{\sigma}}{\mathbb{d}\gamma}\right)^2+\ell\left(\jmath v-\tilde{\sigma}^2\right)=0 \quad \Rightarrow \quad \frac{\mathbb{d}\gamma}{\mathbb{d}\tilde{\sigma}}=\pm\frac{1}{\sqrt{\ell\left(-\jmath v+\tilde{\sigma}^2\right)}}, \tag{22}$$

which is of the standard inverse hyperbolic / trigonometric form for $(+,+)$, $(+,-)$ and $(-,+)$.

Integration of the separated mss. Wheeler–DeWitt equation 233

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{e}^{-\mathbb{i}k_\beta \jmath_\beta \beta}\left(-\ell\,\jmath\,\frac{\hbar^2}{2\varkappa^{3/2}}\psi''(\chi) - \jmath\,\frac{\hbar^2 k_\beta^2}{12\varkappa^{1/2}}\psi(\chi) + \varkappa^{3/2} V \mathbb{e}^{g\jmath_\chi \chi}\psi(\chi)\right) = 0, \qquad (23)$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}}\frac{k_\beta}{g}, \qquad \sigma^2 := \frac{8\varkappa^3 |V| \mathbb{e}^{g\jmath_\chi \chi}}{\hbar^2 g^2}, \qquad (14\text{ rev.})$$

to get

$$\sigma^2\psi''(\sigma) + \sigma\psi'(\sigma) + \ell\,(\nu^2 - \jmath v \sigma^2)\psi(\sigma) = 0, \qquad (24)$$

which is of the standard Besselian form.

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Allgemeines

- Mit diesem *beamer theme* ist es möglich, Präsentationen in LaTeX mit der Beamer-Klasse zu erstellen, die dem Corporate Design der Universität zu Köln entsprechen
- Auf die Beamer-Klasse wird in diesem Dokument nicht näher eingegangen, nähere Informationen finden Sie unter `http://latex-beamer.sourceforge.net/`

Laden des Themes

Das Theme kann mit den folgenden Optionen geladen werden

```
\usetheme[%
% uk,      %% Farben aller Fakultaeten
wiso,     %% Wiso-Fakultaet
% jura,    %% Rechtswissenschaftliche Fakultaet
% medizin, %% Medizinische Fakultaet
% philo,   %% Philosophische Fakultaet
% matnat,  %% Mathematisch-Naturwissenschaftliche Fakultaet
% human,   %% Humanwissenschaftliche Fakultaet
% verw,    %% Universitaetsverwaltung
]{UzK}
```

Die Fußzeile

- Es stehen verschiedene Fußzeilen zur Auswahl, die als Option beim Laden des *themes* übergeben werden:
  - Balken mit allen Fakultätsfarben (Option `uk`)
  - Balken in jeweils einer Fakultätsfarbe (Optionen `wiso`, `jura`, `medizin`, `philo`, `matnat`, `human`, `verw`)[5]
- "'Universität zu Köln"' sowie der Name der Fakultät sind im Theme definiert, das Institut oder Seminar kann mit dem Befehl `\institute{}` festgelegt werden
- Die Optionen sind im Quellcode dieser Präsentation dokumentiert

[5] Es werden die offiziellen RGB-Werte aus dem 2-D Handbuch Corporate Design verwendet.

Englische Präsentationen

- Der Universitäts- sowie die Fakultätsnamen werden standardmäßig auf Deutsch angezeigt.
- Übergeben Sie dem Paket `babel` die Option `english`, so werden diese Namen entsprechen angepasst.
- Die Übersetzungen können in der Theme-Datei `beamerthemeUzK.sty` geändert werden

`block`-Umgebungen

Standard (`block`)

Verwendet die Farbe "'Blaugrau Mittel"' als Blocktitel-Hintergrund

`exampleblock`

Bei Verwendung der Fußzeile mit allen Fakultätsfarben Titelhintergrund in Wiso-Grün, sonst in der jeweiligen Fakultätsfarbe

`alertblock`

Verwendet das Rot der Folientitel

Installation

- Das Theme besteht aus den Dateien `beamerthemeUzK.sty` und `beamercolorthemeUzK.sty` sowie den Grafikdateien `logo.pdf` und `logo-small.pdf`.
- Das Theme kann auf zwei Arten verwendet werden:
  1. Die vier Dateien werden in den selben Ordner wie die zu erstellende Präsentation gelegt
  2. Die vier Dateien werden im lokalen *texmf*-Baum abgelegt
- Die zweite Variante ist der ersten vorzuziehen, da das Theme so an einem zentralen Ort vorliegt

ToDo

Was noch zu tun ist…

- Erstellen einer eigenen Titelseite
- …