Integrable Cosmological Models with Liouville Fields

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Outline

1. Introduction
2. Classical model and the implicitised trajectories
3. Dirac quantisation and the wave functions

Introduction

Introduction

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The Friedmann-Lemaître model
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- Flat Robertson–Walker metric $\mathbb{d}s^2 = -N^2(t)\,\mathbb{d}t^2 + \varkappa^{-1/2}\mathbb{e}^{2\alpha(t)}\,\mathbb{d}\Omega_3^2$, where $\varkappa := 8\pi G$, $\mathbb{d}\Omega_3^2$ dimensionless spacial metric.
- Homogeneous real Klein–Gordon field with potential (dubbed Liouville) $V\mathbb{e}^{\lambda\phi}$, $\lambda, V \in \mathbb{R}$, and kinetic term with sign $\ell = \pm 1$ (quintessence / phantom model).
- Total action $\mathcal{S} = S_{\mathsf{EH}} + S_{\mathsf{GHY}} + S_{\mathsf{L}} = \int \mathbb{d}\Omega_3^2 \int \mathbb{d}t\, L$,

$$L := \varkappa^{3/2} N \mathbb{e}^{3\alpha}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{N^2} + \ell\,\frac{\dot{\phi}^2}{2N^2} - V\mathbb{e}^{\lambda\phi}\right), \tag{1}$$

in which dot means $\mathbb{d}/\mathbb{d}t$ and $\ell = \pm 1$.

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Decoupling the variables
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- Choosing $\overline{N} := N\mathbb{e}^{-3\alpha}$, the effective Lagrangian transforms to

$$L = \varkappa^{3/2}\overline{N}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{\overline{N}^2} + \ell\,\frac{\dot{\phi}^2}{2\overline{N}^2} - V\mathbb{e}^{\lambda\phi+6\alpha}\right) \tag{2}$$

- Defining $\Delta := \lambda^2 - 6\ell\varkappa$, $\jmath := \operatorname{sgn}\Delta$ and $g := \jmath\sqrt{|\Delta|} \equiv \jmath\sqrt{\jmath\Delta}$, the *rescaled special orthogonal transformation*

$$\begin{pmatrix}\alpha\\ \phi\end{pmatrix} = \frac{\jmath}{g}\begin{pmatrix}\lambda & -\ell\kappa\\ -6 & \lambda\end{pmatrix}\begin{pmatrix}\jmath_\beta\beta\\ \jmath_\chi\chi\end{pmatrix} \qquad \text{where } \jmath_\beta, \jmath_\chi = \pm 1 \tag{3}$$

gives the decoupled Lagrangian

$$L = \varkappa^{3/2}\overline{N}\left(-\jmath\frac{3}{\varkappa}\frac{\dot{\beta}^2}{\overline{N}^2} + \ell\,\jmath\,\frac{\dot{\chi}^2}{2\overline{N}^2} - V\mathbb{e}^{\jmath_\chi g\chi}\right). \tag{4}$$

- The Euler–Lagrange equations w.r.t. $\overline{N}$, β and χ will be called the trsfed. first, second Friedmann eqs. and the Klein–Gordon eq., respectively.

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Implicitised integration

$p_\beta \neq 0$

- Since β is cyclic in eq. (4), the second Friedmann equation can be integrated

$$\text{const.} \equiv p_\beta := \frac{\partial L}{\partial \dot\beta} = -6\mathscr{s}\varkappa^{1/2}\frac{\dot\beta}{\overline{N}} \equiv -6\mathscr{s}\mathscr{s}_\beta \frac{\varkappa^{1/2}}{g}\frac{\lambda\dot\alpha + \ell\varkappa\dot\phi}{\overline{N}}. \tag{5}$$

- For $p_\beta \neq 0$, fixing the *implicitising gauge* $\overline{N} = -6\mathscr{s}\sqrt{\varkappa}\dot\beta/p_\beta$, the trsfed. first Friedmann equation can be integrated

$$\mathbb{e}^{\mathscr{s}_\chi g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|}S^2\left(\mathscr{s}_\beta\sqrt{\frac{3}{2\varkappa}}g\beta\right), \tag{6}$$

in which $\upsilon := \operatorname{sgn} V$, and

$$S(\gamma) := \begin{cases} \operatorname{sech}(\gamma + C_{++}) & (\ell, \mathscr{s}\upsilon) = (+,+), \\ \operatorname{csch}(\gamma + C_{+-}) & (\ell, \mathscr{s}\upsilon) = (+,-), \\ \sec(\gamma + C_{-+}) & (\ell, \mathscr{s}\upsilon) = (-,+), \\ \mathbb{i}\csc(\gamma + C_{--}) & (\ell, \mathscr{s}\upsilon) = (-,-). \end{cases} \tag{7}$$

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Integration
Discussions

- The integrals are consistent with the trsfed. Klein–Gordon equation.
- The integral for $(+,+)$
  - has two asymptotes
- The implicitised integral for $(+,-)$
  - contains two distinct solutions
  - has three asymptotes
- The implicitised integral for $(-,+)$
  - contains infinite distinct solutions
  - has infinite asymptotes, which are pairwise parallel
- The integral for $(-,-)$
  - is not real
- The implicitised integral enables one to compare trajectories with wave functions, see below.

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Implicitised integration

$p_\beta = 0$

- For $p_\beta = 0$, one has $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell\lambda\alpha/\kappa$, which is the familiar power-law special solution[1].
- Further integrating the first Friedmann equation demands $(+,-)$ or $(-,+)$ to guarantee $\overline{N} > 0$, and the result is automatically consistent with the trsfed. Klein–Gordon equation.
- Fixing $\overline{N} = \left(2\varkappa^2|V|\right)^{-1/2}$ yields

$$\mathrm{e}^{g\jmath_\chi \chi} = \left(\frac{2\kappa}{g(t-t_0)}\right)^2. \tag{8}$$

[1] Mariusz P. Dąbrowski, Claus Kiefer, and Barbara Sandhöfer. “Quantum phantom cosmology”. In: *Physical Review D* 74.4 (Aug. 2006). DOI: 10.1103/PhysRevD.74.044022.

Introduction

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Dirac quantisation
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- The primary Hamiltonian and the Hamiltonian constraint reads

$$H^{\mathsf{p}} = \overline{N} H_{\perp} + p_{\overline{N}} v^{\overline{N}}, \tag{9}$$

$$H_{\perp} = -\jmath \frac{p_{\beta}^2}{12\varkappa^{1/2}} + \ell \jmath \frac{p_{\chi}^2}{2\varkappa^{3/2}} + \varkappa^{3/2} V \mathrm{e}^{g \jmath_{\chi} \chi}. \tag{10}$$

- Applying the Dirac quantisation rules with the Laplace–Beltrami opeator for the generalised momenta, one gets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(\jmath \frac{\hbar^2}{12\varkappa^{1/2}} \partial_{\beta}^2 - \ell \jmath \frac{\hbar^2}{2\varkappa^{3/2}} \partial_{\chi}^2 + \varkappa^{3/2} V \mathrm{e}^{g \jmath_{\chi} \chi} \right) \Psi. \tag{11}$$

- Equation (11) is a Klein–Gordon-like equation, hyperbolic for $\ell = +1$ and elliptic for $\ell = -1$.

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Separation of the variables and mode functions

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- Inserting the separating Ansatz $\Psi(\beta,\chi) = \mathbb{e}^{-\mathbb{i} k_\beta s_\beta \beta}\psi(\chi)$, the remaining equation turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta,\chi) := \mathbb{e}^{-\mathbb{i}\nu\gamma}\psi_\nu(\chi) := \mathbb{e}^{-\mathbb{i}\nu\gamma}\Big(C_1 \mathrm{B}_\nu^{(1)}(\sigma) + C_2 \mathrm{B}_\nu^{(2)}(\sigma)\Big), \tag{12}$$

in which

$$\nu := \sqrt{\frac{2\varkappa}{3}}\frac{k_\beta}{g}, \quad \gamma := \sqrt{\frac{3}{2\varkappa}} g\beta, \quad \sigma^2 := \frac{8\varkappa^3 |V| \mathbb{e}^{g s_\chi \chi}}{\hbar^2 g^2}, \tag{13}$$

$$\mathrm{B}_\nu^{(i)}(\sigma) := \begin{cases} \mathrm{K} \text{ or } \mathrm{I}_{\mathbb{i}\nu}(\sigma) & (\ell, s v) = (+,+), \\ \mathrm{F} \text{ or } \mathrm{G}_{\mathbb{i}\nu}(\sigma) & (\ell, s v) = (+,-), \\ \mathrm{J} \text{ or } \mathrm{Y}_{\nu}(\sigma) & (\ell, s v) = (-,+), \\ \mathrm{K} \text{ or } \mathrm{I}_{\nu}(\sigma) & (\ell, s v) = (-,-). \end{cases} \tag{14}$$

- The $\mathrm{F}_{\mathbb{i}\nu}(\sigma)$ and $\mathrm{G}_{\mathbb{i}\nu}(\sigma)$ are defined in[2].

[2] T. M. Dunster. "Bessel Functions of Purely Imaginary Order, with an Application to Second-Order Linear Differential Equations Having a Large Parameter". In: *SIAM Journal on Mathematical Analysis* 21.4 (July 1990), pp. 995–1018. DOI: 10.1137/0521055.

Physical mode functions

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- Physical mode functions are expected to be regular on the boundary.
- $(+,+)$: $|\mathrm{I}_{\mathbb{i}\nu}(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- $(-,+)$: $|\mathrm{Y}_n(\sigma)| \to +\infty$ as $\alpha \to -\infty$, $\forall n \in \mathbb{Z}$.
  - for $\nu \in \mathbb{R}\backslash\mathbb{Z}$, one can choose $\mathrm{J}_{-\nu}$ as another base instead of $\mathrm{Y}_{+\nu}$, since $\mathrm{J}_{\pm\nu}$ are also linearly independent.
- $(-,-)$: $|\mathrm{K}_\nu(\sigma)| \to +\infty$ as $\alpha \to -\infty$; $|\mathrm{I}_\nu(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- These are not to included in the space of physical wave functions.

Matching quantum number with classical first integral

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- Baustelle
- In order to match the quantum number k_β (or linearly, ν) with the classical first integral p_β, one may apply the principle of constructive interference.
- Writing the mode function in the WKB form

$$\Psi_{k_\beta}(\beta,\chi) \sim \sqrt{\rho}\,\mathrm{e}^{\mathrm{i}S/\hbar}, \qquad S/\hbar \gg 1 \text{ and } k_\beta \gg 1, \tag{15}$$

the principle demands that $\partial S/\partial k_\beta = 0$ be equivalent to the classical trajectory.

Matching quantum number with classical first integral

$(+,+)$ as exemplar

- Fixing $\nu/\sigma > 1$, the asymptotic expansion reads

$$\mathrm{K}_{\mathrm{i}\nu}(\sigma) \sim \frac{\sqrt{2\pi}\cos\left(\sqrt{\nu^2-\sigma^2} - \nu \arccos\frac{\nu}{\sigma} - \frac{\pi}{4}\right)}{\left(\nu^2-\sigma^2\right)^{1/4} \mathrm{e}^{\pi\nu/2}} + O\left(\frac{1}{\sigma}\right). \tag{16}$$

  There are two phases with opposite signs. Applying the principle to the full mode function Ψ_ν, one has $\sigma/\nu = \operatorname{sech}(s_\beta \gamma)$, which match the trajectory if

$$\hbar k_\beta \equiv \hbar\sqrt{\frac{3}{2\varkappa} g\hbar\nu} = p_\beta, \tag{17}$$

  as expected.

- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; hence no WKB approximation works.
- The conclusions also hold for $\mathrm{F}_{\mathrm{i}\nu}(\sigma)$ and $\mathrm{G}_{\mathrm{i}\nu}(\sigma)$ for $(+,-)$, as well as $\mathrm{J}_\nu(\sigma)$ for $(-,+)$.

Introducint norm for the wave functions
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- To make sense of and construct a wave packet, defining a norm is necessary
- A common starting point of most approaches is the Schrödinger norm

$$(\Psi_1, \Psi_2)_{\mathsf{S}} := \int \mathbb{d}\chi\, \rho_{\mathsf{S}}(\beta, \chi), \qquad \rho_{\mathsf{S}} := \Psi_1^* \Psi_2. \tag{18}$$

- Manifestly $\rho_{\mathsf{S}} \geq 0$. However, there is no continuity equation $\dot{\rho}_{\mathsf{S}} + \nabla \cdot \vec{j}_{\mathsf{S}} = 0$ for ρ_{S}, because eq. (11) is KG-like and not heat-like.

Self-adjointness of operators for the phantom model

$(+,-)$

- Imposing self-adjointness for $\hat{p}_\chi = -\mathbb{i}\hbar\partial_\chi$, one finds that $\mathrm{J}_{-|\nu|}$ leads to divergent $\left(\mathrm{J}_{-|\nu|}, \hat{p}_\chi \mathrm{J}_{-|\nu|}\right)_{\mathrm{S}}$, since[3]

$$\mathrm{J}_\nu(z) = \left(\frac{z}{2}\right)^\nu \left(\frac{1}{\Gamma(\nu+1)} + O(z^2)\right). \tag{19}$$

- Imposing self-adjointness for $\hat{p}_\chi^2$, one finds that

[3] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.16 of 2017-09-18. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds. URL: http://dlmf.nist.gov/, eq. (10.2.2).

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Integration of the transformed first Friedmann equation

$p_\beta \neq 0$

In order to integrate the equation under the implicitising gauge

$$\jmath \frac{p_\beta^2}{12}\left(-\ell \frac{\varkappa^{1/2}}{6}\left(\frac{\mathrm{d}\chi}{\mathrm{d}\beta}\right)^2 + \varkappa^{-1/2}\right) - \varkappa^{3/2} V \mathrm{e}^{g\jmath_\chi \chi} = 0, \tag{20}$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa}} g\beta, \qquad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} \mathrm{e}^{-g\jmath_\chi \chi}, \tag{21}$$

to get

$$\left(\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\gamma}\right)^2 + \ell\,(\jmath v - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell\,(-\jmath v + \tilde{\sigma}^2)}}, \tag{22}$$

which is of the standard inverse hyperbolic / trigonometric form for $(+,+)$, $(+,-)$ and $(-,+)$.

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Integration of the separated mss. Wheeler–DeWitt equation
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In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{e}^{-\mathbb{i}k_\beta \text{ʃ}_\beta \beta}\left(-\ell\,\text{ʃ}\,\frac{\hbar^2}{2\varkappa^{3/2}}\psi''(\chi) - \text{ʃ}\,\frac{\hbar^2 k_\beta^2}{12\varkappa^{1/2}}\psi(\chi) + \varkappa^{3/2} V \mathbb{e}^{g \text{ʃ}_\chi \chi}\psi(\chi)\right) = 0, \qquad (23)$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}}\frac{k_\beta}{g}, \qquad \sigma^2 := \frac{8\varkappa^3 |V| \mathbb{e}^{g \text{ʃ}_\chi \chi}}{\hbar^2 g^2}, \qquad (14 \text{ rev.})$$

to get

$$\sigma^2\psi''(\sigma) + \sigma\psi'(\sigma) + \ell\left(\nu^2 - \text{ʃ}\,v\sigma^2\right)\psi(\sigma) = 0, \qquad (24)$$

which is of the standard Besselian form.

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Allgemeines

- Mit diesem *beamer theme* ist es möglich, Präsentationen in LaTeX mit der Beamer-Klasse zu erstellen, die dem Corporate Design der Universität zu Köln entsprechen
- Auf die Beamer-Klasse wird in diesem Dokument nicht näher eingegangen, nähere Informationen finden Sie unter `http://latex-beamer.sourceforge.net/`

Laden des Themes

Das Theme kann mit den folgenden Optionen geladen werden

```
\usetheme[%
% uk,      %% Farben aller Fakultaeten
wiso,     %% Wiso-Fakultaet
% jura,    %% Rechtswissenschaftliche Fakultaet
% medizin, %% Medizinische Fakultaet
% philo,   %% Philosophische Fakultaet
% matnat,  %% Mathematisch-Naturwissenschaftliche Fakultaet
% human,   %% Humanwissenschaftliche Fakultaet
% verw,    %% Universitaetsverwaltung
]{UzK}
```

Die Fußzeile

- Es stehen verschiedene Fußzeilen zur Auswahl, die als Option beim Laden des *themes* übergeben werden:
  - Balken mit allen Fakultätsfarben (Option `uk`)
  - Balken in jeweils einer Fakultätsfarbe (Optionen `wiso`, `jura`, `medizin`, `philo`, `matnat`, `human`, `verw`)[4]
- "'Universität zu Köln"' sowie der Name der Fakultät sind im Theme definiert, das Institut oder Seminar kann mit dem Befehl `\institute{}` festgelegt werden
- Die Optionen sind im Quellcode dieser Präsentation dokumentiert

[4] Es werden die offiziellen RGB-Werte aus dem 2-D Handbuch Corporate Design verwendet.

Englische Präsentationen

- Der Universitäts- sowie die Fakultätsnamen werden standardmäßig auf Deutsch angezeigt.
- Übergeben Sie dem Paket `babel` die Option `english`, so werden diese Namen entsprechen angepasst.
- Die Übersetzungen können in der Theme-Datei `beamerthemeUzK.sty` geändert werden

`block`-Umgebungen

Standard (`block`)

Verwendet die Farbe "'Blaugrau Mittel"' als Blocktitel-Hintergrund

`exampleblock`

Bei Verwendung der Fußzeile mit allen Fakultätsfarben Titelhintergrund in Wiso-Grün, sonst in der jeweiligen Fakultätsfarbe

`alertblock`

Verwendet das Rot der Folientitel

Installation

- Das Theme besteht aus den Dateien `beamerthemeUzK.sty` und `beamercolorthemeUzK.sty` sowie den Grafikdateien `logo.pdf` und `logo-small.pdf`.
- Das Theme kann auf zwei Arten verwendet werden:
  1. Die vier Dateien werden in den selben Ordner wie die zu erstellende Präsentation gelegt
  2. Die vier Dateien werden im lokalen *texmf*-Baum abgelegt
- Die zweite Variante ist der ersten vorzuziehen, da das Theme so an einem zentralen Ort vorliegt

ToDo

Was noch zu tun ist...

- Erstellen einer eigenen Titelseite
- ...