Boundary Actions in Geometrodynamics

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Outline

Outline

Introduction

Quintessence and phantom Liouville field

- Observed accelerated expansion can be explained by a cosmological constantits origin has yet to be understood.
- Dynamical Dark Energy has been modeled by quintessencematter, with barotropic index[1] $w > -1$ and $w < -1$, respectively.
- They can be realised by minimally-coupled real scalar fields with $\ell = \pm 1$[2]

$$\mathcal{S} = \int \mathrm{d}^4 x \sqrt{-g} \left\{ -\ell \frac{g^{\mu\nu}}{2} (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{V}(\phi) \right\}. \tag{1}$$

- $\mathcal{V}(\phi) = V \mathrm{e}^{\lambda\phi}$, $\lambda, V \in \mathbb{R}$ is of interest: Liouville field

[1] Barotropic index is the w in equation of state $\rho = wp$.
[2] The signature of metric is mostly positive.

Introduction

Friedmann–Lemaître model

- Assume flat Robertson–Walker metric for a homog. and isotr. model

$$g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -N^2(t)\,\mathrm{d}t^2 + \varkappa \mathrm{e}^{2\alpha(t)}\,\mathrm{d}\Omega_{3\mathrm{F}}^2 \tag{2}$$

  w/ $\varkappa := 8\pi G$, $\mathrm{d}\Omega_{3\mathrm{F}}^2 := \mathrm{d}\chi^2 + \chi^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)$, N lapse function.

- Combined with the Liouville field, the total action reads $\mathcal{S} := \mathcal{S}_{\mathrm{EH}} + \mathcal{S}_{\mathrm{GHY}} + \mathcal{S}_{\mathrm{L}} = \int \mathrm{d}\Omega_{3\mathrm{F}}^2 \int \mathrm{d}t\,L$, where

$$L := \varkappa^{3/2} N \mathrm{e}^{3\alpha}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{N^2} + \ell\,\frac{\dot{\phi}^2}{2N^2} - V\mathrm{e}^{\lambda\phi}\right), \tag{3}$$

  in which dot means $\mathrm{d}/\mathrm{d}t$.

- The model turns out to be integrable, both classically and quantum-mechanically, enabling one to study its full physical properties, e.g. the relation between its classical and quantum theory.

Outline

Decoupling the variables

Via *rescaled* special orthogonal transformation

- Setting $\overline{N} := N\mathbb{e}^{-3\alpha}$, eq. (3) transforms to

$$L = \varkappa^{3/2}\overline{N}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{\overline{N}^2} + \ell\,\frac{\dot{\phi}^2}{2\overline{N}^2} - V\mathbb{e}^{\lambda\phi+6\alpha}\right) \tag{4}$$

- Defining $\Delta := \lambda^2 - 6\ell\varkappa$, $\mathfrak{s} := \operatorname{sgn}\Delta$ and $g := \mathfrak{s}\sqrt{|\Delta|} \equiv \mathfrak{s}\sqrt{\mathfrak{s}\Delta}$, the *rescaled* special orthogonal transformation

$$\begin{pmatrix}\alpha\\ \phi\end{pmatrix} = \frac{\mathfrak{s}}{g}\begin{pmatrix}\lambda & -\ell\varkappa\\ -6 & \lambda\end{pmatrix}\begin{pmatrix}\mathfrak{s}_\beta\beta\\ \mathfrak{s}_\chi\chi\end{pmatrix} \qquad \text{where } \mathfrak{s}_\beta, \mathfrak{s}_\chi = \pm 1 \tag{5}$$

gives the decoupled Lagrangian

$$L = \varkappa^{3/2}\overline{N}\left(-\mathfrak{s}\frac{3}{\varkappa}\frac{\dot{\beta}^2}{\overline{N}^2} + \ell\,\mathfrak{s}\frac{\dot{\chi}^2}{2\overline{N}^2} - V\mathbb{e}^{\mathfrak{s}_\chi g\chi}\right). \tag{6}$$

- The Euler–Lagrange equations w.r.t. $\overline{N}$, β and χ will be called the trsfed. 1st, 2nd Friedmann eqs. and the Klein–Gordon eq., respectively.

Implicit integration

General integral for $p_\beta \neq 0$

- Since β is cyclic in eq. (6), the trsfed. 2nd Friedmann eq. can be integrated
- For $p_\beta \neq 0$, fixing the *implicit gauge* $\overline{N} = -6\mathfrak{s}\sqrt{\varkappa}\dot{\beta}/p_\beta$, the trsfed. 1st Friedmann equation can be integrated to get the trajectory
  in which $\mathfrak{v} := \operatorname{sgn} V$, $(\operatorname{sgn}, \operatorname{sgn})$ means $(\ell, \mathfrak{s}\mathfrak{v})$, and

Implicit integration

Specific integral for $p_\beta = 0$

- For $p_\beta = 0$, integrating the transformed second Friedmann equation yields $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell\lambda\alpha/\varkappa$, which is the well-known power-law special solution
- Further integrating the transformed first Friedmann equation demands $(+,-)$ or $(-,+)$ to guarantee $\overline{N} > 0$.

Trajectories for quintessence model $(+,+)$

sech, with $\beta_0 = 0$, $|V| = \varkappa^{-2}$ and $p_\beta^2 = \varkappa^2\sqrt{|V|}$; varying λ

- has two asymptotes $\chi \propto \pm\beta$

Trajectories for quintessence model $(+,+)$

sech, with $\beta_0 = 0$, $|V| = \varkappa^{-2}$ and $\lambda^2 = 3\varkappa$; varying p_β

- has two asymptotes $\chi \propto \pm\beta$

Trajectories for quintessence model $(+,+)$

sech, with $\beta_0 = 0$, $\lambda^2 = 3\varkappa$ and $p_\beta^2 = \varkappa^2\sqrt{|V|}$; varying V

- has two asymptotes $\chi \propto \pm\beta$

Trajectories for quintessence model $(+,-)$

csch, with $\beta_0 = 0$, $|V| = \varkappa^{-2}$ and $p_\beta^2 = \varkappa^2\sqrt{|V|}$; varying λ

- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$

Trajectories for quintessence model $(+,-)$: csch

csch, with $\beta_0 = 0$, $|V| = \varkappa^{-2}$ and $p_\beta^2 = \varkappa^2\sqrt{|V|}$; varying λ

- contains two distinct solutions, separated by $\beta = 0$
- has three asymptotes $\chi \propto \pm\beta$ and $\beta = 0$

Trajectories for quintessence model $(+,-)$: csch

csch, **with $V=\varkappa^{-2}$ and $\lambda^2=\varkappa$; varying p_β**

- contains two distinct solutions, separated by $\beta=0$
- has three asymptotes $\chi\propto\pm\beta$ and $\beta=0$

Trajectories for phantom model $(-,+)$

$\csc$, **with** $V = \varkappa^{-2}$ **and** $p_\beta^2 = 3\varkappa^2\sqrt{|V|}$

- contains infinite distinct solutions
- has infinite parallel asymptotes $\beta \propto (n+1/2)\pi$

Trajectories for phantom model $(-,+)$

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Trajectories for phantom model $(-,+)$

$\csc$, with $V = \varkappa^{-2}$ and $\lambda^2 = 3\varkappa$; varying p_β

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Integration
Further discussions

- The integral for $(-,-)$ is not real.
- The trajectories can be parametrised by β, inspiring recognising β as a 'time variable'.
- The implicit integration enables one to compare trajectories with wave functions, see below.

Outline

Dirac quantisation and the mode functions

- The primary Hamiltonian and the Hamiltonian constraint

$$H^{\mathsf{p}} = \overline{N} H_{\perp} + v^{\overline{N}} p_{\overline{N}}, \tag{7}$$

$$H_{\perp} := -\jmath \frac{p_{\beta}^2}{12\varkappa^{1/2}} + \ell \jmath \frac{p_{\chi}^2}{2\varkappa^{3/2}} + \varkappa^{3/2} V \mathbb{e}^{g\jmath_{\chi}\chi}. \tag{8}$$

- Applying the Dirac quantisation rules with Laplace–Beltrami operatorgets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(\jmath \frac{\hbar^2}{12\varkappa^{1/2}} \partial_{\beta}^2 - \ell \jmath \frac{\hbar^2}{2\varkappa^{3/2}} \partial_{\chi}^2 + \varkappa^{3/2} V \mathbb{e}^{g\jmath_{\chi}\chi} \right) \Psi. \tag{9}$$

- Equation (9) is KG-like, hyperbolic for $\ell = +1$ and *elliptic* for $\ell = -1$.

Separation of the variables and mode functions

- Writing $\Psi(\beta, \chi) = \varphi(\beta)\psi(\chi)$, eq. (9) can be separated into

$$\partial_\beta^2 \varphi(\beta) = k_\beta^2 \varphi(\beta); \tag{10}$$

$$\mathbb{D}\psi(\chi) = k_\beta^2 \psi(\chi), \qquad \mathbb{D} := -\ell \frac{6}{\varkappa} \partial_\chi^2 + \mathscr{s}\mathscr{v} \frac{12\varkappa^2 |V|}{\hbar^2}. \tag{11}$$

- Equation (11) turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta, \chi) := \sum_{i=1}^{2} c_i \varphi_\nu^{(i)}(\gamma) \sum_{j=1}^{2} a_j \mathrm{B}_\nu^{(j)}(\sigma), \quad \nu \geq 0; \tag{12}$$

$$\nu := \sqrt{\frac{2\varkappa}{3} \frac{1}{g}}\, k_\beta, \quad \gamma := \sqrt{\frac{3}{2\varkappa} \frac{g}{1}}\, \beta, \qquad \sigma^2 := \frac{8\varkappa^3 |V| \mathbb{e}^{g \mathscr{s}_\chi \chi}}{\hbar^2 g^2}, \tag{13}$$

$${}_{(+,+)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{K} \text{ and } \mathrm{I}_{\mathbb{i}\nu}(\sigma), \qquad {}_{(+,-)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{F} \text{ and } \mathrm{G}_{\mathbb{i}\nu}(\sigma),$$

$${}_{(-,+)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{J} \text{ and } \mathrm{Y}_\nu(\sigma), \qquad {}_{(-,-)}\mathrm{B}_\nu^{(i)}(\sigma) := \mathrm{K} \text{ and } \mathrm{I}_\nu(\sigma).$$

- Adapted to imaginary order, $\mathrm{F}_\nu(\sigma)$ and $\mathrm{G}_\nu(\sigma)$ are defined in

Physical mode functions

- Physical mode functions are expected to be regular on the boundary.
- $(+,+)$: $|\mathrm{I}_{\mathbb{i}\nu}(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- $(-,+)$:
  - $\forall n \in \mathbb{N}$, $|\mathrm{Y}_n(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
  - $\forall \nu \in \mathbb{R}^+ \backslash \mathbb{N}$, choose $\mathrm{J}_{-\nu}$ instead of Y_ν, since $\mathrm{J}_{\pm\nu}$ are also linearly independent.
  - $\forall \nu \in \mathbb{R}^+ \backslash \mathbb{N}$, $|\mathrm{J}_{-\nu}(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
- $(-,-)$: $|\mathrm{K}_\nu(\sigma)| \to +\infty$ as $\alpha \to -\infty$; $|\mathrm{I}_\nu(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- These are not to be included in the space of physical wave functions. $\forall \nu \geq 0$,
  - $(+,+)$: $\mathrm{K}_{\mathbb{i}\nu}(\sigma)$ survives
  - $(+,-)$: $\mathrm{F}_{\mathbb{i}\nu}(\sigma)$ and $\mathrm{G}_{\mathbb{i}\nu}(\sigma)$ survives
  - $(-,+)$: $\mathrm{J}_\nu(\sigma)$ survives
  - $(-,-)$: drops out

Matching quantum number with classical first integral

Principle of constructive interference

- Write the mode function in the WKB form

$$\Psi_{k_\beta}(\beta,\chi) \sim \sqrt{\rho(\beta,\chi)}\exp\left\{\frac{\mathbb{i}}{\hbar}S(\beta,\chi)\right\}. \tag{14}$$

- For $S/\hbar \gg 1$ and $k_\beta \gg 1$[3], $S(\beta,\chi)$ becomes the Hamilton principle function in the leading-order approximation.
- A Hamilton principle function is stationary with respect to variation of integral constants

$$\frac{\partial S}{\partial k_\beta} = 0. \tag{15}$$

- Demanding eq. (15) matching the classical trajectory, k_β can be related to p_β.

[3] The common form is $\hbar \to 0^+$.

Matching quantum number with classical first integral

$(+,+)$ as exemplar

- Fixing $\nu/\sigma > 1$, the asymptotic expansion reads

$$\mathrm{K}_{\mathbb{i}\nu}(\sigma) \sim \frac{\sqrt{2\mathbb{\pi}}\cos\left(\sqrt{\nu^2-\sigma^2}-\nu\arccos\frac{\nu}{\sigma}-\frac{\mathbb{\pi}}{4}\right)}{\left(\nu^2-\sigma^2\right)^{1/4}\mathbb{e}^{\mathbb{\pi}\nu/2}} + O\left(\frac{1}{\sigma}\right). \tag{16}$$

  - There are two phases with opposite signs. Assuming c_i, a_j's are real and applying *the principle* to $\Psi_\nu(\sigma)$, one has$\sigma/\nu = \operatorname{sech}(\mathcal{s}_\beta\gamma)$, which matches the trajectory with $\beta_0 = 0$ if

$$\hbar k_\beta \equiv \hbar\sqrt{\frac{3}{2\varkappa}g\hbar\nu} = p_\beta, \tag{17}$$

  - Non-vanishing C can be compensated by the phase of c_i and a_j's.

- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; it is not within the WKB regime.

- The conclusions also hold for $\mathrm{F}_{\mathbb{i}\nu}(\sigma)$, $\mathrm{G}_{\mathbb{i}\nu}(\sigma)$ for $(+,-)$, and $\mathrm{J}_\nu(\sigma)$ for $(-,+)$.

Inner product for wave functions

Schrödinger product

- To make sense of amplitude and wave packet, an inner product is necessary
- In terms of a Klein–Gordon equation, call β the "temporal" variable, and χ the "spacial" variable.
- A common starting point is the *Schrödinger product*

$$(\Psi_1, \Psi_2)_{\mathrm{S}} := \int \mathrm{d}\chi\, \Psi_1^*(\beta, \chi)\Psi_2(\beta, \chi); \tag{18}$$

- $(\Psi, \Psi)_{\mathrm{S}}$ is positive-definite, and the integrand $\rho_{\mathrm{S}}(\beta, \chi)$ is non-negative
- The corresponding Schrödinger current is real but does not satisfy continuity equation $\dot{\rho}_{\mathrm{S}} + \nabla \cdot \vec{j}_{\mathrm{S}} = 0$, because eq. (9) is KG-like.
- $\mathrm{K}_{\mathbb{i}\nu}\mathrm{F}_{\mathbb{i}\nu}$ and $\mathrm{G}_{\mathbb{i}\nu}$ for $(+,-)$ can be proved to be orthogonal and complete among themselves, as well as can be normalised.
  - $\mathrm{J}_{\mathbb{i}\nu}$'s for $(+,-)$ are not orthogonal

Peculiarity of the phantom model $(-,+)$

Orthogonality for mode functions; Hermiticity for operators

- $\mathrm{J}_\nu(\sigma)$'s are not orthogonal under the Schrödinger product

$$(\mathrm{J}_\nu, \mathrm{J}_{\tilde{\nu}})_\mathsf{S} \propto \int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{J}_\nu^*(\mathrm{e}^x)\mathrm{J}_{\tilde{\nu}}(\mathrm{e}^x) = \frac{2\sin(\pi(\nu-\tilde{\nu})/2)}{\pi(\nu^2-\tilde{\nu}^2)}, \tag{19}$$

  therefore $\mathbb{D}$ in eq. (11) is not Hermitian (though we do not need it so far)

- $\hat{p}_\chi^2$ is not Hermitian for $\{\mathrm{J}_\nu(\sigma)\}$ under the Schrödinger product

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{J}_\nu^*(-\partial_x^2 \mathrm{J}_{\tilde{\nu}}) - \int_{-\infty}^{+\infty} \mathrm{d}x \, (-\partial_x^2 \mathrm{J}_\nu)^* \mathrm{J}_{\tilde{\nu}} = \frac{2}{\pi} \sin \frac{\pi(\nu-\tilde{\nu})}{2}. \tag{20}$$

- In order to save Hermiticity for p_χ^2 and $\mathbb{D}$ and orthogonality of the modes under Schrödinger product, one can restrict

$$\nu = 2n + \nu_0, \qquad n \in \mathbb{N}, \quad \nu_0 \in [0,2)\,. \tag{21}$$

- Using classical singularities as boundary condition, one can fix $\nu_0 = 1$.

Discretisation of the phantom model $(-,+)$

Levels of the phantom model are discretised if one imposes Hermiticity of squared momenta under the Schrödinger product.

- This kind of subtlety on Hermiticity is named as *self-adjoint extension*, which arises already for infinite square well
- It also applies to x^{-2} potentials

Further inner products for wave functions

Klein–Gordon and Mostafazadeh product

- Since eq. (9) is KG-like, another popular choice is the KG product

$$(\Psi_1, \Psi_2)^g_{\mathsf{KG}} := \mathbb{i} g \Big\{ \big(\Psi_1, \partial_\beta \Psi_2\big)_{\mathsf{S}} - \big(\partial_\beta \Psi_1, \Psi_2\big)_{\mathsf{S}} \Big\}, \qquad g > 0. \tag{22}$$

  - $(\Psi, \Psi)^g_{\mathsf{KG}}$ is real but not positive-definite, so does the integrand ρ_{KG};
  - The corresponding $\vec{J}_{\mathsf{KG}}$ is conserved $\dot{\rho}_{\mathsf{KG}} + \nabla \cdot \vec{J}_{\mathsf{KG}} = 0$ and real.

- Mostafazadeha product *for Hermitian* $\mathbb{D}$ *with positive spectrum*:

$$(\Psi_1, \Psi_2)^\kappa_{\mathsf{M}} := \kappa \Big\{ \big(\Psi_1, \mathbb{D}^{+1/2} \Psi_2\big)_{\mathsf{S}} + \big(\partial_\beta \Psi_1, \mathbb{D}^{-1/2} \partial_\beta \Psi_2\big)_{\mathsf{S}} \Big\}, \qquad \kappa > 0. \tag{23}$$

  - $(\Psi, \Psi)^\kappa_{\mathsf{M}}$ is positive-definite, but the integrand ρ^κ_{M} is complex
  - The corresponding $\vec{J}^\kappa_{\mathsf{M}}$ is conserved $\dot{\rho}^\kappa_{\mathsf{M}} + \nabla \cdot \vec{J}^\kappa_{\mathsf{M}} = 0$ but also complex.

Mostafazadeh inner product and the corresponding density

- Real power of $\mathbb{D}$ is defined by spectral decomposition $\mathbb{D}^{\gamma} := \sum_{\nu} \nu^{2\gamma} \mathbf{P}_{\nu}$, $\mathbf{P}_{\nu} \Psi := \Psi_{\nu} (\Psi_{\nu}, \Psi)_{\mathsf{S}}$.

- It can be shown

$$\varrho_{\mathsf{M}}^{\kappa} := \kappa \left\{ \left| \mathbb{D}^{+1/4} \Psi \right|^2 + \left| \mathbb{D}^{-1/4} \partial_{\beta} \Psi \right|^2 \right\} \tag{24}$$

  - is equivalent to $\rho_{\mathsf{M}}^{\kappa}$ up to a boundary term

$$\int \mathrm{d}\chi \, \varrho_{\mathsf{M}}^{\kappa} = \int \mathrm{d}\chi \, \rho_{\mathsf{M}}^{\kappa} \equiv (\Psi_1, \Psi_2)_{\mathsf{M}}^{\kappa}; \tag{25}$$

  - is non-negative.
  - The corresponding current $\vec{\mathcal{J}}_{\mathsf{M}}^{\kappa}$ is real but not conserved

Wave packets of Gaussian amplitude for continuous spectrum

Quintessence models

- It is difficult to find an amplitude such that the wave packet is Gaussian
- Instead, one can choose a Gaussian amplitude

$$A(\nu;\overline{\nu},\sigma) := \left(\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(\nu-\overline{\nu})^2}{2\sigma^2}\right)\right)^{1/2} \tag{26}$$

- In $A\left(\nu;\overline{\nu},\sigma/\sqrt{2}\right)$ was chosen.

Wave packets of Gaussian amplitude for quintessence model $(+,+)$

$\mathrm{K}_{\mathrm{i}\nu}$, **with** $\lambda = \varkappa^{1/2}/2$, $V = -\varkappa^{-2}$, $\overline{k}_{\beta} = -2$ **and** $\sigma_{\beta} = 5/4$

Schrödinger

Mostafazadeh

Wave packets of Gaussian amplitude for quintessence model $(+,-)$

$\mathrm{F}_{\mathbb{i}\nu}$, **with** $\lambda = 4\varkappa^{1/2}/5$, $V = +\varkappa^{-2}$, $\overline{k}_\beta = -7/2$ **and** $\sigma_\beta = 7/5$

Schrödinger

Mostafazadeh

Wave packets of Gaussian amplitude for quintessence model $(+,-)$

$G_{\mathbb{i}\nu}$, **with** $\lambda = 4\varkappa^{1/2}/5$, $V = +\varkappa^{-2}$, $\overline{k}_\beta = -7/2$ **and** $\sigma_\beta = 7/5$

Schrödinger

Mostafazadeh

Wave packets with Poissonian amplitude for discrete spectrum

Discrete phantom model $(-,+)$

- Gaussian distribution works for continuous variable
- For discrete spectrum, one can choose a Poissonian amplitude

$$A_n(\overline{n}) := \left(\mathrm{e}^{-\overline{n}} \frac{\overline{n}^n}{n!} \right)^{1/2} \tag{27}$$

- In $A_n\left(\overline{n}/\sqrt{2}\right)$ was chosen.

Wave packets of Poissonian amplitude for phantom model

J_{2n+1}, **with** $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$ **and** $\overline{k}_\beta = 8$

Schrödinger

Mostafazadeh

Wave packets of Gaussian amplitude for phantom model

J_ν, with $\lambda = 2\varkappa^{1/2}$, $V = +\varkappa^{-2}$, $\overline{k}_\beta = 8$ and $\sigma_\beta = 11/2$

Schrödinger

Mostafazadeh

Outline

Highlights

- An integral of motion was found for Liouville cosmological models.
- Implicit trajectories in minisuperspace were obtained.
- The levels of phantom Liouville model were found to be discrete due to the Hermiticity requirement of observables.
- The semi-classical quantum mode functions and wave packets was compared directly with the classical trajectory.

Issues

- In $(+,-)$ and $(-,+)$, wave packets contain multiple branches; however, the classical universe runs only on one trajectory.
- Quantum-corrected $\overline{k}_\beta$ is to be understood.
- Wave packets with Gaussian profile are to be constructed, instead of inserting Gaussian / Poissonian amplitude artificially.
- A normalising κ for $(\cdot,\cdot)^\kappa_{\mathsf{M}}$ is to be evaluated, otherwise a quantitative comparison of $(\cdot,\cdot)_{\mathsf{S}}$ and $(\cdot,\cdot)^\kappa_{\mathsf{M}}$ is not possible.

Outlook

- Beyond isotropy: generalise to Bianchi models
  - Bianchi Type-I: a natural extension, under investigation
- Beyond homogeneity: cosmological perturbation, under investigation
- Beyond single field
  - Two exponential potentials: $|V_1| = |V_2|$ and special λ_i
  - Multiple Liouville fields: mixing kinetic terms needed
- Beyond classic matter
  - PT-symmetric Liouville fieldmay cross the phantom divide $w = -1$.