Integrable Cosmological Models with Liouville Fields

Alexander A. Andrianov[1,4] Chen Lan[2] Oleg O. Novikov[1]
Yi-Fan Wang[3]

[1] Saint-Petersburg State University, St. Petersburg 198504, Russia

[2] ELI-ALPS, ELI-Hu NKft, Dugonics tér 13, Szeged 6720, Hungary

[3] Institut für Theoretische Physik, Universität zu Köln, Zülpicher Straße 77, 50937 Köln, Germany

[4] Institut de Ciències del Cosmos (ICCUB), Universitat de Barcelona, Spain

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Outline

Introduction

Introduction

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The Friedmann-Lemaître model
123

- Flat Robertson–Walker metric $\mathbb{d}s^2 = -N^2(t)\,\mathbb{d}t^2 + \varkappa^{-1/2}\mathbb{e}^{2\alpha(t)}\,\mathbb{d}\Omega_3^2$, where $\varkappa := 8\pi G$, $\mathbb{d}\Omega_3^2$ dimensionless spacial metric.
- Homogeneous real Klein–Gordon field with potential (dubbed *Liouville*) $V\mathbb{e}^{\lambda\phi}$, $\lambda, V \in \mathbb{R}$, and kinetic term with sign $\ell = \pm 1$ (quintessence / phantom model).
- Total action $\mathcal{S} = S_{\mathsf{EH}} + S_{\mathsf{GHY}} + S_{\mathsf{L}} = \int \mathbb{d}\Omega_3^2 \int \mathbb{d}t\, L$,

$$L := \varkappa^{3/2} N \mathbb{e}^{3\alpha}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{N^2} + \ell\frac{\dot{\phi}^2}{2N^2} - V\mathbb{e}^{\lambda\phi}\right), \tag{1}$$

in which dot means $\mathbb{d}/\mathbb{d}t$ and $\ell = \pm 1$.

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Decoupling the variables

123

- Choosing $\overline{N} := N \mathbb{e}^{-3\alpha}$, the effective Lagrangian transforms to

$$L = \varkappa^{3/2}\overline{N}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{\overline{N}^2} + \ell\,\frac{\dot{\phi}^2}{2\overline{N}^2} - V\mathbb{e}^{\lambda\phi+6\alpha}\right) \tag{2}$$

- Defining $\Delta := \lambda^2 - 6\ell\varkappa$, $\mathfrak{s} := \operatorname{sgn}\Delta$ and $g := \mathfrak{s}\sqrt{|\Delta|} \equiv \mathfrak{s}\sqrt{\mathfrak{s}\Delta}$, the *rescaled special orthogonal transformation*

$$\begin{pmatrix}\alpha \\ \phi\end{pmatrix} = \frac{\mathfrak{s}}{g}\begin{pmatrix}\lambda & -\ell\kappa \\ -6 & \lambda\end{pmatrix}\begin{pmatrix}\mathfrak{s}_\beta\beta \\ \mathfrak{s}_\chi\chi\end{pmatrix} \qquad \text{where } \mathfrak{s}_\beta, \mathfrak{s}_\chi = \pm 1 \tag{3}$$

gives the decoupled Lagrangian

$$L = \varkappa^{3/2}\overline{N}\left(-\mathfrak{s}\frac{3}{\varkappa}\frac{\dot{\beta}^2}{\overline{N}^2} + \ell\,\mathfrak{s}\,\frac{\dot{\chi}^2}{2\overline{N}^2} - V\mathbb{e}^{\mathfrak{s}_\chi g\chi}\right). \tag{4}$$

- The Euler–Lagrange equations w.r.t. $\overline{N}$, β and χ will be called the trsfed. first, second Friedmann eqs. and the Klein–Gordon eq., respectively.

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Implicitised integration

$p_\beta \neq 0$

- Since β is cyclic in eq. (4), the second Friedmann equation can be integrated

$$\text{const.} \equiv p_\beta := \frac{\partial L}{\partial \dot{\beta}} = -6 \mathscr{s} \varkappa^{1/2} \frac{\dot{\beta}}{\overline{N}} \equiv -6 \mathscr{s} \mathscr{s}_\beta \frac{\varkappa^{1/2}}{g} \frac{\lambda \dot{\alpha} + \ell \varkappa \dot{\phi}}{\overline{N}}. \tag{5}$$

- For $p_\beta \neq 0$, fixing the *implicitising gauge* $\overline{N} = -6 \mathscr{s} \sqrt{\varkappa} \dot{\beta} / p_\beta$, the trsfed. first Friedmann equation can be integrated

$$\mathbb{e}^{\mathscr{s}} \chi^{g\chi} = \frac{p_\beta^2}{12 \varkappa^2 |V|} S^2 \left(\mathscr{s}_\beta \sqrt{\frac{3}{2\varkappa}} g\beta \right), \tag{6}$$

in which $v := \operatorname{sgn} V$, and

$$S(\gamma) := \begin{cases} \operatorname{sech}(\gamma + C_{++}) & (\ell, \mathscr{s} v) = (+,+), \\ \operatorname{csch}(\gamma + C_{+-}) & (\ell, \mathscr{s} v) = (+,-), \\ \sec(\gamma + C_{-+}) & (\ell, \mathscr{s} v) = (-,+), \\ \mathbb{i} \csc(\gamma + C_{--}) & (\ell, \mathscr{s} v) = (-,-). \end{cases} \tag{7}$$

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Integration

Discussions

- The integrals are consistent with the trsfed. Klein–Gordon equation.
- The integral for $(+,+)$
  - has two asymptotes
- The implicitised integral for $(+,-)$
  - contains two distinct solutions
  - has three asymptotes
- The implicitised integral for $(-,+)$
  - contains infinite distinct solutions
  - has infinite asymptotes, which are pairwise parallel
- The integral for $(-,-)$
  - is not real
- The implicitised integral enables one to compare trajectories with wave functions, see below.

Implicitised integration

$p_\beta = 0$

- For $p_\beta = 0$, one has $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell\lambda\alpha/\kappa$, which is the familiar power-law special solution[1].
- Further integrating the first Friedmann equation demands $(+,-)$ or $(-,+)$ to guarantee $\overline{N} > 0$, and the result is automatically consistent with the trsfed. Klein–Gordon equation.
- Fixing $\overline{N} = (2\varkappa^2|V|)^{-1/2}$ yields

$$\mathrm{e}^{g\mathfrak{s}_\chi \chi} = \left(\frac{2\kappa}{g(t-t_0)}\right)^2. \tag{8}$$

[1] Mariusz P. Dąbrowski, Claus Kiefer, and Barbara Sandhöfer. “Quantum phantom cosmology”. In: *Physical Review D* 74.4 (Aug. 2006). DOI: 10.1103/physrevd.74.044022.

Introduction

Introduction

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Dirac quantisation

233

- The primary Hamiltonian and the Hamiltonian constraint reads

$$H^{\mathrm{p}} = \overline{N} H_{\perp} + p_{\overline{N}} v^{\overline{N}}, \tag{9}$$

$$H_{\perp} = -s \frac{p_{\beta}^2}{12\varkappa^{1/2}} + \ell s \frac{p_{\chi}^2}{2\varkappa^{3/2}} + \varkappa^{3/2} V \mathrm{e}^{g s_{\chi} \chi}. \tag{10}$$

- Applying the Dirac quantisation rules with the Laplace–Beltrami opeator for the generalised momenta, one gets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(s \frac{\hbar^2}{12\varkappa^{1/2}} \partial_{\beta}^2 - \ell s \frac{\hbar^2}{2\varkappa^{3/2}} \partial_{\chi}^2 + \varkappa^{3/2} V \mathrm{e}^{g s_{\chi} \chi} \right) \Psi. \tag{11}$$

Separation of the variables and mode functions
233

- Inserting the separating Ansatz $\Psi(\beta, \chi) = \mathbb{e}^{-\mathbb{i} k_\beta \jmath_\beta \beta} \psi(\chi)$, the remaining equation turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta, \chi) := \mathbb{e}^{-\mathbb{i}\nu\gamma} \psi_\nu(\chi) := \mathbb{e}^{-\mathbb{i}\nu\gamma} \Big(C_1 \mathrm{B}_\nu^{(1)}(\sigma) + C_2 \mathrm{B}_\nu^{(2)}(\sigma) \Big), \tag{12}$$

in which

$$\nu := \sqrt{\frac{2\varkappa}{3}} \frac{k_\beta}{g}, \quad \gamma := \sqrt{\frac{3}{2\varkappa}} g\beta, \quad \sigma^2 := \frac{8\varkappa^3 |V| \mathbb{e}^{g \jmath_\chi \chi}}{\hbar^2 g^2}, \tag{13}$$

$$\mathrm{B}_\nu^{(i)}(\sigma) := \begin{cases} \mathrm{K} \text{ or } \mathrm{I}_{\mathbb{i}\nu}(\sigma) & (\ell, \jmath v) = (+,+), \\ \mathrm{F} \text{ or } \mathrm{G}_{\mathbb{i}\nu}(\sigma) & (\ell, \jmath v) = (+,-), \\ \mathrm{J} \text{ or } \mathrm{Y}_\nu(\sigma) & (\ell, \jmath v) = (-,+), \\ \mathrm{K} \text{ or } \mathrm{I}_\nu(\sigma) & (\ell, \jmath v) = (-,-). \end{cases} \tag{14}$$

- The $\mathrm{F}_{\mathbb{i}\nu}(\sigma)$ and $\mathrm{G}_{\mathbb{i}\nu}(\sigma)$ are defined in[2].

[2] T. M. Dunster. "Bessel Functions of Purely Imaginary Order, with an Application to Second-Order Linear Differential Equations Having a Large Parameter". In: *SIAM Journal on Mathematical Analysis* 21.4 (July 1990), pp. 995–1018. DOI: 10.1137/0521055.

Physical mode functions

233

- Physical mode functions are expected to be regular on the boundary.
- $(+,+)$: $|\mathrm{I}_{i\nu}(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- $(-,+)$: $|\mathrm{Y}_n(\sigma)| \to +\infty$ as $\alpha \to -\infty$, $\forall n \in \mathbb{Z}$.
- $(-,-)$: $|\mathrm{K}_\nu(\sigma)| \to +\infty$ as $\alpha \to -\infty$; $|\mathrm{I}_\nu(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- These are not to included in the space of physical wave functions.

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Matching quantum number with classical first integral

233

- In order to match the quantum number k_β (or *linearly*, ν) with the classical first integral p_β, one may apply the *principle of constructive interference*.
- Writing the mode function in the WKB form

$$\Psi_{k_\beta}(\beta,\chi) \sim \sqrt{\rho}\, \mathbb{e}^{\mathbb{i}S/\hbar}, \qquad S/\hbar \gg 1 \text{ and } k_\beta \gg 1, \tag{15}$$

  the principle demands that $\partial S/\partial k_\beta = 0$ be equivalent to the classical trajectory.

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Matching quantum number with classical first integral

$(+,+)$ as exemplar

- Fixing $\nu/\sigma > 1$, the asymptotic expansion reads

$$\mathrm{K}_{\mathbb{i}\nu}(\sigma) \sim \frac{\sqrt{2\pi}\cos\left(\sqrt{\nu^2-\sigma^2} - \nu\arccos\frac{\nu}{\sigma} - \frac{\pi}{4}\right)}{\left(\nu^2-\sigma^2\right)^{1/4} \mathrm{e}^{\pi\nu/2}} + O\left(\frac{1}{\sigma}\right). \tag{16}$$

  There are two phases with opposite signs $\theta_\nu^\pm(\sigma)$. Applying the principle to the full mode function Ψ_ν, one has $\sigma/\nu = \operatorname{sech}(s_\beta \gamma)$, which match the trajectory if

$$\hbar k_\beta \equiv \hbar\sqrt{\frac{3}{2\varkappa}} g\hbar\nu = p_\beta, \tag{17}$$

  as expected.
- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; hence no WKB approximation applies.
- The conclusions also work for $\mathrm{F}_{\mathbb{i}\nu}(\sigma)$ and $\mathrm{G}_{\mathbb{i}\nu}(\sigma)$ for $(+,-)$, as well as $\mathrm{J}_\nu(\sigma)$ for $(-,+)$.

Integration of the transformed first Friedmann equation

$p_\beta \neq 0$

In order to integrate the equation under the implicitising gauge

$$\jmath \frac{p_\beta^2}{12}\left(-\ell \frac{\varkappa^{1/2}}{6}\left(\frac{\mathrm{d}\chi}{\mathrm{d}\beta}\right)^2 + \varkappa^{-1/2}\right) - \varkappa^{3/2} V \mathrm{e}^{g \jmath_\chi \chi} = 0, \tag{18}$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa}} g\beta, \qquad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|} \mathrm{e}^{-g \jmath_\chi \chi}, \tag{19}$$

to get

$$\left(\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\gamma}\right)^2 + \ell(\jmath v - \tilde{\sigma}^2) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\tilde{\sigma}} = \pm \frac{1}{\sqrt{\ell(\jmath v - \tilde{\sigma}^2)}}, \tag{20}$$

which is of the standard inverse hyperbolic / trigonometric form.

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Integration of the separated mss. Wheeler–DeWitt equation
233

In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{e}^{-\mathbb{i}k_\beta \varsigma_\beta \beta}\left(-\ell\,\varsigma\,\frac{\hbar^2}{2\varkappa^{3/2}}\psi''(\chi) - \varsigma\,\frac{\hbar^2 k_\beta^2}{12\varkappa^{1/2}}\psi(\chi) + \varkappa^{3/2} V \mathbb{e}^{g\varsigma_\chi \chi}\psi(\chi)\right) = 0, \qquad (21)$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}}\frac{k_\beta}{g}, \qquad \sigma^2 := \frac{8\varkappa^3 |V| \mathbb{e}^{g\varsigma_\chi \chi}}{\hbar^2 g^2}, \qquad (14\text{ rev.})$$

to get

$$\sigma^2\psi''(\sigma) + \sigma\psi'(\sigma) + \ell\left(\nu^2 - \varsigma\upsilon\sigma^2\right)\psi(\sigma) = 0, \qquad (22)$$

which is of the standard Besselian form.

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Allgemeines

- Mit diesem *beamer theme* ist es möglich, Präsentationen in LaTeX mit der Beamer-Klasse zu erstellen, die dem Corporate Design der Universität zu Köln entsprechen
- Auf die Beamer-Klasse wird in diesem Dokument nicht näher eingegangen, nähere Informationen finden Sie unter `http://latex-beamer.sourceforge.net/`

Laden des Themes

Das Theme kann mit den folgenden Optionen geladen werden

```
\usetheme[%
% uk,       %% Farben aller Fakultaeten
wiso,      %% Wiso-Fakultaet
% jura,     %% Rechtswissenschaftliche Fakultaet
% medizin, %% Medizinische Fakultaet
% philo,    %% Philosophische Fakultaet
% matnat,   %% Mathematisch-Naturwissenschaftliche Fakultaet
% human,    %% Humanwissenschaftliche Fakultaet
% verw,     %% Universitaetsverwaltung
]{UzK}
```

Die Fußzeile

- Es stehen verschiedene Fußzeilen zur Auswahl, die als Option beim Laden des *themes* übergeben werden:
  - Balken mit allen Fakultätsfarben (Option `uk`)
  - Balken in jeweils einer Fakultätsfarbe (Optionen `wiso`, `jura`, `medizin`, `philo`, `matnat`, `human`, `verw`)[3]
- "'Universität zu Köln"' sowie der Name der Fakultät sind im Theme definiert, das Institut oder Seminar kann mit dem Befehl `\institute{}` festgelegt werden
- Die Optionen sind im Quellcode dieser Präsentation dokumentiert

[3] Es werden die offiziellen RGB-Werte aus dem 2-D Handbuch Corporate Design verwendet.

Englische Präsentationen

- Der Universitäts- sowie die Fakultätsnamen werden standardmäßig auf Deutsch angezeigt.
- Übergeben Sie dem Paket `babel` die Option `english`, so werden diese Namen entsprechen angepasst.
- Die Übersetzungen können in der Theme-Datei `beamerthemeUzK.sty` geändert werden

block-Umgebungen

Standard (block)

Verwendet die Farbe "'Blaugrau Mittel"' als Blocktitel-Hintergrund

exampleblock

Bei Verwendung der Fußzeile mit allen Fakultätsfarben Titelhintergrund in Wiso-Grün, sonst in der jeweiligen Fakultätsfarbe

alertblock

Verwendet das Rot der Folientitel

Installation

- Das Theme besteht aus den Dateien `beamerthemeUzK.sty` und `beamercolorthemeUzK.sty` sowie den Grafikdateien `logo.pdf` und `logo-small.pdf`.
- Das Theme kann auf zwei Arten verwendet werden:
  1. Die vier Dateien werden in den selben Ordner wie die zu erstellende Präsentation gelegt
  2. Die vier Dateien werden im lokalen *texmf*-Baum abgelegt
- Die zweite Variante ist der ersten vorzuziehen, da das Theme so an einem zentralen Ort vorliegt

ToDo

Was noch zu tun ist...

- Erstellen einer eigenen Titelseite
- ...