Integrable Cosmological Models with Liouville Fields

Alexander A. Andrianov[1,4] Chen Lan[2] Oleg O. Novikov[1]
Yi-Fan Wang[3]

[1]Saint-Petersburg State University, St. Petersburg 198504, Russia

[2]ELI-ALPS, ELI-Hu NKft, Dugonics tér 13, Szeged 6720, Hungary

[3]Institut für Theoretische Physik, Universität zu Köln, Zülpicher Straße 77, 50937 Köln, Germany

[4]Institut de Ciències del Cosmos (ICCUB), Universitat de Barcelona, Spain

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Outline

1. Introduction
2. Classical model and the implicitised trajectories
3. Dirac quantisation and the wave functions

Introduction

Introduction

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The Friedmann-Lemaître model

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- Flat Robertson–Walker metric $\mathbb{d}s^2 = -N^2(t)\,\mathbb{d}t^2 + \varkappa^{-1}\mathbb{e}^{2\alpha(t)}\,\mathbb{d}\Omega_3^2$, where $\varkappa := 8\pi G$, $\mathbb{d}\Omega_3^2$ dimensionless spacial metric.
- Homogeneous real Klein–Gordon field with potential $V\mathbb{e}^{\lambda\phi}$ (Liouville), where $\lambda, V \in \mathbb{R}$, and kinetic term with sign $\ell = \pm 1$ (quintessence / phantom model).
- Total action $\mathcal{S} := S_{\mathsf{EH}} + S_{\mathsf{GHY}} + S_{\mathsf{L}} = \int \mathbb{d}\Omega_3^2 \int \mathbb{d}t\, L$, in which the effective Lagrangian reads

$$L := \varkappa^{3/2} N \mathbb{e}^{3\alpha}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{N^2} + \ell\,\frac{\dot{\phi}^2}{2N^2} - V\mathbb{e}^{\lambda\phi}\right), \tag{1}$$

in which dot means $\mathbb{d}/\mathbb{d}t$ and $\ell = \pm 1$.

Decoupling the variables

Via rescaled special orthogonal transformation

- Setting $\overline{N} := N\mathrm{e}^{-3\alpha}$, eq. (1) transforms to

$$L = \varkappa^{3/2}\overline{N}\left(-\frac{3}{\varkappa}\frac{\dot{\alpha}^2}{\overline{N}^2} + \ell\frac{\dot{\phi}^2}{2\overline{N}^2} - V\mathrm{e}^{\lambda\phi+6\alpha}\right) \tag{2}$$

- Defining $\Delta := \lambda^2 - 6\ell\varkappa$, $\varsigma := \operatorname{sgn}\Delta$ and $g := \varsigma\sqrt{|\Delta|} \equiv \varsigma\sqrt{\varsigma\Delta}$, the *rescaled* special orthogonal transformation

$$\begin{pmatrix}\alpha\\ \phi\end{pmatrix} = \frac{\varsigma}{g}\begin{pmatrix}\lambda & -\ell\kappa\\ -6 & \lambda\end{pmatrix}\begin{pmatrix}\varsigma_\beta\beta\\ \varsigma_\chi\chi\end{pmatrix} \qquad \text{where } \varsigma_\beta, \varsigma_\chi = \pm 1 \tag{3}$$

gives the decoupled Lagrangian

$$L = \varkappa^{3/2}\overline{N}\left(-\varsigma\frac{3}{\varkappa}\frac{\dot{\beta}^2}{\overline{N}^2} + \ell\varsigma\frac{\dot{\chi}^2}{2\overline{N}^2} - V\mathrm{e}^{\varsigma_\chi g\chi}\right). \tag{4}$$

- The Euler–Lagrange equations w.r.t. $\overline{N}$, β and χ will be called the trsfed. 1st, 2nd Friedmann eqs. and the Klein–Gordon eq., respectively.

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Implicitised integration

$p_\beta \neq 0$

- Since β is cyclic in eq. (4), the trsfed. 2nd Friedmann eq. can be integrated[1]

$$\text{const.} \equiv p_\beta := \frac{\partial L}{\partial \dot{\beta}} = -6\varsigma\varkappa^{1/2}\frac{\dot{\beta}}{\overline{N}} \equiv -6\varsigma\varsigma_\beta \frac{\varkappa^{1/2}}{g}\frac{\lambda\dot{\alpha} + \ell\varkappa\dot{\phi}}{\overline{N}}. \tag{5}$$

- For $p_\beta \neq 0$, fixing the implicitising gauge $\overline{N} = -6\varsigma\sqrt{\varkappa}\dot{\beta}/p_\beta$, the trsfed. 1st Friedmann equation can be integrated

$$\mathbb{e}^{\varsigma_\chi g\chi} = \frac{p_\beta^2}{12\varkappa^2|V|} S^2\left(\varsigma_\beta\sqrt{\frac{3}{2\varkappa}}\,g\beta\right), \tag{6}$$

in which $\upsilon := \operatorname{sgn} V$, and

$$S(\gamma) := \begin{cases} \operatorname{sech}(\gamma + C_{++}) & (\ell, \varsigma\upsilon) = (+,+), \\ \operatorname{csch}(\gamma + C_{+-}) & (\ell, \varsigma\upsilon) = (+,-), \\ \sec(\gamma + C_{-+}) & (\ell, \varsigma\upsilon) = (-,+), \\ \mathbb{i}\csc(\gamma + C_{--}) & (\ell, \varsigma\upsilon) = (-,-). \end{cases} \tag{7}$$

[1] Chen Lan. PhD thesis. Saint Petersburg State University, 2016. URL: https://search.rsl.ru/ru/record/01006663434.

- The integrals are consistent with the trsfed. Klein–Gordon equation.
- The integral for $(+,+)$
  - has two asymptotes
- The implicitised integral for $(+,-)$
  - contains two distinct solutions
  - has three asymptotes
- The implicitised integral for $(-,+)$
  - contains infinite distinct solutions
  - has infinite asymptotes, which are pairwise parallel
- The integral for $(-,-)$
  - is not real
- The implicitised integral enables one to compare trajectories with wave functions, see below.

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Implicitised integration

$p_\beta = 0$

- For $p_\beta = 0$, one has $\beta \equiv \beta_0$ or $\phi - \phi_0 = -\ell\lambda\alpha/\kappa$, which is the familiar power-law special solution[2].
- Further integrating the first Friedmann equation demands $(+,-)$ or $(-,+)$ to guarantee $\overline{N} > 0$, and the result is automatically consistent with the trsfed. Klein–Gordon equation.
- Fixing $\overline{N} = \left(2\varkappa^2|V|\right)^{-1/2}$ yields

$$\mathrm{e}^{g\mathfrak{s}_\chi \chi} = \left(\frac{2\kappa}{g(t-t_0)}\right)^2. \tag{8}$$

[2] Mariusz P. Dąbrowski, Claus Kiefer, and Barbara Sandhöfer. In: *Physical Review D* 74.4 (Aug. 2006).

Introduction

Introduction

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Dirac quantisation

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- The primary Hamiltonian and the Hamiltonian constraint reads

$$H^{\mathrm{p}} = \overline{N} H_{\perp} + p_{\overline{N}} v^{\overline{N}}, \tag{9}$$

$$H_{\perp} = -\jmath \frac{p_{\beta}^2}{12\varkappa^{1/2}} + \ell \jmath \frac{p_{\chi}^2}{2\varkappa^{3/2}} + \varkappa^{3/2} V \mathrm{e}^{g\jmath_{\chi}\chi}. \tag{10}$$

- Applying the Dirac quantisation rules with the Laplace–Beltrami opeator for the generalised momenta, one gets the minisuperspace Wheeler–DeWitt equation with (β, χ)

$$0 = \widehat{H}_{\perp} \Psi(\beta, \chi) := \left(\jmath \frac{\hbar^2}{12\varkappa^{1/2}} \partial_{\beta}^2 - \ell \jmath \frac{\hbar^2}{2\varkappa^{3/2}} \partial_{\chi}^2 + \varkappa^{3/2} V \mathrm{e}^{g\jmath_{\chi}\chi} \right) \Psi. \tag{11}$$

- Equation (11) is a Klein–Gordon-like equation, hyperbolic for $\ell = +1$ and elliptic for $\ell = -1$.

Separation of the variables and mode functions
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- Inserting the separating Ansatz $\Psi(\beta,\chi) = \mathbb{e}^{-\mathbb{i}k_\beta \jmath_\beta \beta}\psi(\chi)$, the remaining equation turns out to be Besselian, and the mode functions are

$$\Psi_\nu(\beta,\chi) := \mathbb{e}^{-\mathbb{i}\nu\gamma}\psi_\nu(\chi) := \mathbb{e}^{-\mathbb{i}\nu\gamma}\Big(C_1 \mathrm{B}_\nu^{(1)}(\sigma) + C_2 \mathrm{B}_\nu^{(2)}(\sigma)\Big), \tag{12}$$

in which

$$\nu := \sqrt{\frac{2\varkappa}{3}}\frac{k_\beta}{g}, \quad \gamma := \sqrt{\frac{3}{2\varkappa}}g\beta, \quad \sigma^2 := \frac{8\varkappa^3|V|\mathbb{e}^{g\jmath_\chi\chi}}{\hbar^2 g^2}, \tag{13}$$

$$\mathrm{B}_\nu^{(i)}(\sigma) := \begin{cases} \mathrm{K} \text{ or } \mathrm{I}_{\mathbb{i}\nu}(\sigma) & (\ell, \jmath v) = (+,+), \\ \mathrm{F} \text{ or } \mathrm{G}_{\mathbb{i}\nu}(\sigma) & (\ell, \jmath v) = (+,-), \\ \mathrm{J} \text{ or } \mathrm{Y}_\nu(\sigma) & (\ell, \jmath v) = (-,+), \\ \mathrm{K} \text{ or } \mathrm{I}_\nu(\sigma) & (\ell, \jmath v) = (-,-). \end{cases} \tag{14}$$

- Adapted to imaginary order, $\mathrm{F}_\nu(\sigma)$ and $\mathrm{G}_\nu(\sigma)$ are defined in[3].

[3] T. Mark Dunster. In: *SIAM Journal on Mathematical Analysis* 21.4 (July 1990), pp. 995–1018.

Physical mode functions

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- Physical mode functions are expected to be regular on the boundary.
- $(+,+)$: $|\mathrm{I}_{\mathbb{i}\nu}(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- $(-,+)$:
  - $\forall n \in \mathbb{Z}$, $|\mathrm{Y}_n(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
  - $\forall \nu \in \mathbb{R} \backslash \mathbb{Z}$, choose $\mathrm{J}_{-\nu}$ instead of $\mathrm{Y}_{+\nu}$, since $\mathrm{J}_{\pm\nu}$ are also linearly independent.
  - $\forall \nu \in \mathbb{R}^{+} \backslash \mathbb{Z}$, $|\mathrm{J}_{-\nu}(\sigma)| \to +\infty$ as $\alpha \to -\infty$.
- $(-,-)$: $|\mathrm{K}_\nu(\sigma)| \to +\infty$ as $\alpha \to -\infty$; $|\mathrm{I}_\nu(\sigma)| \to +\infty$ as $\alpha \to +\infty$
- These are not to included in the space of physical wave functions.
- For $(-,+)$, only J_ν with $\nu \geq 0$ survives.

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Matching quantum number with classical first integral

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- Baustelle
- In order to match the quantum number k_β (or linearly, ν) with the classical first integral p_β, one may apply the principle of constructive interference.
- Writing the mode function in the WKB form

$$\Psi_{k_\beta}(\beta,\chi) \sim \sqrt{\rho}\, \mathbb{e}^{\mathbb{i}S/\hbar}, \qquad S/\hbar \gg 1 \text{ and } k_\beta \gg 1, \tag{15}$$

the principle demands that $\partial S/\partial k_\beta = 0$ be equivalent to the classical trajectory.

Matching quantum number with classical first integral

$(+,+)$ as exemplar

- Fixing $\nu/\sigma > 1$, the asymptotic expansion reads

$$\mathrm{K}_{\mathbb{i}\nu}(\sigma) \sim \frac{\sqrt{2\pi}\cos\left(\sqrt{\nu^2-\sigma^2} - \nu \arccos\frac{\nu}{\sigma} - \frac{\pi}{4}\right)}{\left(\nu^2-\sigma^2\right)^{1/4} \mathrm{e}^{\pi\nu/2}} + O\left(\frac{1}{\sigma}\right). \tag{16}$$

  There are two phases with opposite signs. Applying the principle to the full mode function Ψ_ν, one has $\sigma/\nu = \operatorname{sech}(\jmath_\beta \gamma)$, which match the trajectory if

$$\hbar k_\beta \equiv \hbar\sqrt{\frac{3}{2\varkappa}g\hbar\nu} = p_\beta, \tag{17}$$

  as expected.
- Fixing $\nu/\sigma < 1$, the asymptotic expansion is not oscillatory, but exponential; hence no WKB approximation works.
- The conclusions also hold for $\mathrm{F}_{\mathbb{i}\nu}(\sigma)$ and $\mathrm{G}_{\mathbb{i}\nu}(\sigma)$ for $(+,-)$, as well as $\mathrm{J}_\nu(\sigma)$ for $(-,+)$.

Inner product for wave functions

Schrödinger product

- To make sense of amplitude and wave packet, an inner product is necessary
- A common starting point of most approaches is the Schrödinger product

$$(\Psi_1, \Psi_2)_{\mathsf{S}} := \int \mathbb{d}\chi \, \Psi_1^*(\beta, \chi)\Psi_2(\beta, \chi); \tag{18}$$

- In terms of a norm, $(\Psi, \Psi)_{\mathsf{S}} \equiv \int \mathbb{d}\chi \, \rho_{\mathsf{S}}(\beta, \chi)$, in which $\rho_{\mathsf{S}} := \Psi^*\Psi$. Manifestly $\rho_{\mathsf{S}} \geq 0$; one has $(\Psi, \Psi)_{\mathsf{S}} > 0$.
- The corresponding Schrödinger current does not satisfy continuity equation $\dot{\rho}_{\mathsf{S}} + \nabla \cdot \vec{j}_{\mathsf{S}} = 0$, because eq. (11) is KG-like.
- $\mathrm{K}_{\mathbb{i}\nu}$[4], $\mathrm{F}_{\mathbb{i}\nu}$ and $\mathrm{G}_{\mathbb{i}\nu}$ can be normalised, proved to be orthogonal and complete.

[4] Semyon B. Yakubovich. In: *Opuscula Math.* 26.1 (2006), pp. 161–172. URL: http://www.opuscula.agh.edu.pl/vol26/1/art/opuscula_math_2612.pdf, A. Passian et al. In: *Journal of Mathematical Analysis and Applications* 360.2 (Dec. 2009), pp. 380–390, Radosław Szmytkowski and Sebastian Bielski. In: *Journal of Mathematical Analysis and Applications* 365.1 (May 2010), pp. 195–197.

Peculiarity for the phantom model

Orthonormality and completeness for mode functions; self-adjointness for operators

- Imposing self-adjointness for $\hat{p}_\chi^2$, one finds that

Integration of the transformed first Friedmann equation

$p_\beta \neq 0$

In order to integrate the equation under the implicitising gauge

$$\jmath \frac{p_\beta^2}{12}\left(-\ell\frac{\varkappa^{1/2}}{6}\left(\frac{\mathrm{d}\chi}{\mathrm{d}\beta}\right)^2 + \varkappa^{-1/2}\right) - \varkappa^{3/2} V \mathrm{e}^{g\jmath_\chi \chi} = 0, \tag{19}$$

one can substitute

$$\gamma := \sqrt{\frac{3}{2\varkappa}} g\beta, \qquad \tilde{\sigma}^2 := \frac{p_\beta^2}{12\varkappa^2 |V|}\mathrm{e}^{-g\jmath_\chi \chi}, \tag{20}$$

to get

$$\left(\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\gamma}\right)^2 + \ell\left(\jmath v - \tilde{\sigma}^2\right) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\tilde{\sigma}} = \pm\frac{1}{\sqrt{\ell\left(-\jmath v + \tilde{\sigma}^2\right)}}, \tag{21}$$

which is of the standard inverse hyperbolic / trigonometric form for $(+,+)$, $(+,-)$ and $(-,+)$.

Integration of the separated mss. Wheeler–DeWitt equation
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In order to integrate the separated minisuperspace Wheeler–DeWitt equation

$$\mathbb{e}^{-\mathbb{i}k_\beta \mathscr{s}_\beta \beta}\left(-\ell\,\mathscr{s}\,\frac{\hbar^2}{2\varkappa^{3/2}}\psi''(\chi) - \mathscr{s}\,\frac{\hbar^2 k_\beta^2}{12\varkappa^{1/2}}\psi(\chi) + \varkappa^{3/2} V \mathbb{e}^{g \mathscr{s}_\chi \chi}\psi(\chi)\right) = 0, \qquad (22)$$

one can transform

$$\nu := \sqrt{\frac{2\varkappa}{3}}\frac{k_\beta}{g}, \qquad \sigma^2 := \frac{8\varkappa^3 |V| \mathbb{e}^{g \mathscr{s}_\chi \chi}}{\hbar^2 g^2}, \qquad \text{(14 rev.)}$$

to get

$$\sigma^2\psi''(\sigma) + \sigma\psi'(\sigma) + \ell\left(\nu^2 - \mathscr{s}\upsilon\sigma^2\right)\psi(\sigma) = 0, \qquad (23)$$

which is of the standard Besselian form.

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Allgemeines

- Mit diesem *beamer theme* ist es möglich, Präsentationen in LaTeX mit der Beamer-Klasse zu erstellen, die dem Corporate Design der Universität zu Köln entsprechen
- Auf die Beamer-Klasse wird in diesem Dokument nicht näher eingegangen, nähere Informationen finden Sie unter `http://latex-beamer.sourceforge.net/`

Laden des Themes

Das Theme kann mit den folgenden Optionen geladen werden

```
\usetheme[%
% uk,      %% Farben aller Fakultaeten
wiso,     %% Wiso-Fakultaet
% jura,    %% Rechtswissenschaftliche Fakultaet
% medizin, %% Medizinische Fakultaet
% philo,   %% Philosophische Fakultaet
% matnat,  %% Mathematisch-Naturwissenschaftliche Fakultaet
% human,   %% Humanwissenschaftliche Fakultaet
% verw,    %% Universitaetsverwaltung
]{UzK}
```

Die Fußzeile

- Es stehen verschiedene Fußzeilen zur Auswahl, die als Option beim Laden des *themes* übergeben werden:
  - Balken mit allen Fakultätsfarben (Option `uk`)
  - Balken in jeweils einer Fakultätsfarbe (Optionen `wiso`, `jura`, `medizin`, `philo`, `matnat`, `human`, `verw`)[5]
- "'Universität zu Köln"' sowie der Name der Fakultät sind im Theme definiert, das Institut oder Seminar kann mit dem Befehl `\institute{}` festgelegt werden
- Die Optionen sind im Quellcode dieser Präsentation dokumentiert

[5] Es werden die offiziellen RGB-Werte aus dem 2-D Handbuch Corporate Design verwendet.

Englische Präsentationen

- Der Universitäts- sowie die Fakultätsnamen werden standardmäßig auf Deutsch angezeigt.
- Übergeben Sie dem Paket `babel` die Option `english`, so werden diese Namen entsprechen angepasst.
- Die Übersetzungen können in der Theme-Datei `beamerthemeUzK.sty` geändert werden

`block`-Umgebungen

Standard (`block`)

Verwendet die Farbe "'Blaugrau Mittel"' als Blocktitel-Hintergrund

`exampleblock`

Bei Verwendung der Fußzeile mit allen Fakultätsfarben Titelhintergrund in Wiso-Grün, sonst in der jeweiligen Fakultätsfarbe

`alertblock`

Verwendet das Rot der Folientitel

Installation

- Das Theme besteht aus den Dateien `beamerthemeUzK.sty` und `beamercolorthemeUzK.sty` sowie den Grafikdateien `logo.pdf` und `logo-small.pdf`.
- Das Theme kann auf zwei Arten verwendet werden:
    1. Die vier Dateien werden in den selben Ordner wie die zu erstellende Präsentation gelegt
    2. Die vier Dateien werden im lokalen *texmf*-Baum abgelegt
- Die zweite Variante ist der ersten vorzuziehen, da das Theme so an einem zentralen Ort vorliegt

ToDo

Was noch zu tun ist...

- Erstellen einer eigenen Titelseite
- ...