Wightman function of Klein–Fock–Gordon field

Yi-Fan Wang

May 5, 2019

Contents

1 Covariant differential

ϕ a $\mathbb{C}$-valued 0-form

$$\mathbb{D}\phi := (\mathbb{d} - \mathrm{i}eA) \wedge \phi, \qquad \mathbb{D}\phi^* := (\mathbb{d} + \mathrm{i}eA) \wedge \phi^*, \tag{1.1}$$

A the $\mathfrak{u}(1)$-valued connection form.

2 Covariant codifferential

Define the covariant codifferential of a $\mathbb{C}$-valued k-form ζ as follows. Let η be an arbitrary $\mathbb{C}$-valued $(k-1)$-form.

$$\int \mathbb{d}(\eta^* \wedge \star\zeta) \equiv \int \mathbb{d}\eta^* \wedge \star\zeta - (-)^k \eta^* \wedge \mathbb{d} \star \zeta =: \int \mathbb{D}\eta^* \wedge \star\zeta - \eta^* \wedge \star\mathbb{D}^\dagger\zeta \tag{2.1}$$

$$= \int \mathbb{D}\eta^* \wedge \star\zeta - \mathrm{i}eA \wedge \eta^* \wedge \star\zeta - (-)^k \eta^* \wedge \mathbb{d} \star \zeta$$

$$= \int \mathbb{D}\eta^* \wedge \star\zeta + \eta^* \wedge (-)^k \mathrm{i}eA \wedge \star\zeta - (-)^k \eta^* \wedge \mathbb{d} \star \zeta$$

$$= \int \mathbb{D}\eta^* \wedge \star\zeta - \eta^* \wedge \star(-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\zeta\,. \tag{2.2}$$

$$\boxed{\mathbb{D}^\dagger\zeta = (-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\zeta\,.} \tag{2.3}$$

3 Maxwell–Klein–Fock–Gordon theory

$$S = \int -\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi - m^2\phi^* \wedge \star\phi - \frac{1}{2}F \wedge \star F. \tag{3.1}$$

$$\delta\mathbb{D}\phi = -\mathrm{i}e\delta A\phi + \mathbb{D}\delta\phi. \tag{3.2}$$

$$\begin{aligned}\delta(\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi) &= \mathbb{d}(\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^*) \\ &\quad + \delta\phi^* \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi^* \qquad (3.3)\\ &\quad + \delta A \wedge (\mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*))\,, \\ \delta(F \wedge \star F) &= 2\mathbb{d}(\delta A \wedge \star F) - 2\delta A \wedge \mathbb{d}\star F\,. \qquad (3.4)\end{aligned}$$

$$\begin{aligned}\delta S = \int & -\mathbb{d}(\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^* + \delta A \wedge \star F) \\ & + \delta\phi^* \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi + \delta\phi \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi^* + \\ & + \delta A \wedge (-\mathbb{d}\star F + \mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)).\end{aligned} \tag{3.5}$$

3.1 Lorenz gauge

$$\Box^2 := \left(\mathbb{d} + \mathbb{d}^\dagger\right)^2 = \mathbb{d}\mathbb{d}^\dagger + \mathbb{d}^\dagger\mathbb{d}. \tag{3.6}$$

$$\mathbb{d}\star F = \mathbb{d} \star \mathbb{d}A = \star(-)^2 \star^{-1} \mathbb{d} \star \mathbb{d}A = \star\mathbb{d}^\dagger\mathbb{d}A = \star(\Box^2 - \mathbb{d}\mathbb{d}^\dagger)A\,. \tag{3.7}$$

One would like to have $\mathbb{d}\mathbb{d}^\dagger A = 0$, or $\mathbb{d}^\dagger A = \text{const}$. This would be fulfiled if

$$\mathbb{d}^\dagger A = 0\,, \tag{3.8}$$

which is the Lorenz gauge[1, 2, 4].

4 Free Klein–Fock–Gordon equation in $(d+1)$-dimensions

In the absence of external electromagnetic field and Cartesian coordinates,

$$0 = (\mathbb{D}^\dagger\mathbb{D} - m^2)\phi = (-\partial_t^2 + \partial_x^2 - m^2)\phi\,. \tag{4.1}$$

Linearly independent solutions are

$$\exp\!\Big(\pm\mathrm{i}\Big(-\omega_{\vec{k}}t + \vec{k}\cdot\vec{x}\Big)\Big)\,, \qquad \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}\,. \tag{4.2}$$

One uses the Noether charge to normalise, which reads

$$(\phi_1, \phi_2) = \int \mathbb{d}^d x\, j^0 \tag{4.3}$$

gives

$$\begin{aligned}
\phi_+\left(\omega_{\vec{k}},\vec{k};t,x\right) &= \frac{1}{\sqrt{2\omega_{\vec{k}}}}\exp\left\{+\mathrm{i}\left(-\omega_{\vec{k}}t+\vec{k}\cdot\vec{x}\right)\right\},\\
\phi_-\left(\omega_{\vec{k}},\vec{k};t,x\right) &= \frac{1}{\sqrt{2\omega_{\vec{k}}}}\exp\left\{-\mathrm{i}\left(-\omega_{\vec{k}}t+\vec{k}\cdot\vec{x}\right)\right\},
\end{aligned} \tag{4.4}$$

For $d \geq 2$,

$$\begin{aligned}
&\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{2\omega_{\vec{k}}}\mathrm{e}^{+\mathrm{i}\left(-\omega\,\Delta t+\vec{k}\cdot\Delta\vec{x}\right)}\\
&= \int_0^{+\infty}\frac{\mathrm{d}k}{(2\pi)^d}\frac{k^{d-1}\mathrm{e}^{-\mathrm{i}\omega_{\vec{k}}\,\Delta t}}{2\omega_{\vec{k}}}\int \mathrm{e}^{\mathrm{i}k\,\Delta x\cos\phi_1}\,\mathrm{d}\Omega_{d-1}\,,
\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}
&\int \mathrm{d}\Omega_{d-1}\\
&:= \int_0^{2\pi}\mathrm{d}\phi_{d-1}\int_0^{\pi}\sin^{d-2}\phi_1\sin^{d-3}\phi_2\ldots\sin^1\phi_{n-2}\,\mathrm{d}\phi_1\,\mathrm{d}\phi_2\ldots\mathrm{d}\phi_{d-2}\,.
\end{aligned} \tag{4.6}$$

Equipped with

$$\int_0^{\pi}\sin^{\nu}\phi\,\mathrm{d}\phi = \pi^{\frac{1}{2}}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}+1\right)}\qquad \Re\nu > -1\,, \tag{4.7}$$

one has

$$\int \mathrm{d}\Omega_{d-1} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\int_0^{\pi}\sin^{d-2}\phi_1\mathrm{d}\phi_1\,. \tag{4.8}$$

The inner integral in eq. (4.5) turns to be

$$\begin{aligned}
\int \mathrm{e}^{\mathrm{i}k\,\Delta x\cos\phi_1}\,\mathrm{d}\Omega_{d-1} &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\int_0^{\pi}\mathrm{e}^{\mathrm{i}k\,\Delta x\cos\phi_1}\sin^{d-2}\phi_1\,\mathrm{d}\phi_1\\
&= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\int_0^{\pi}\cos(k\,\Delta x\cos\phi_1)\sin^{d-2}\phi_1\,\mathrm{d}\phi_1\\
&= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{d-1}{2}\right)}{\left(\frac{k\,\Delta x}{2}\right)^{\frac{d}{2}-1}}J_{\frac{d}{2}-1}(k\,\Delta x)\\
&= (2\pi)^{\frac{d}{2}}\frac{J_{\frac{d}{2}-1}(k\,\Delta x)}{(k\,\Delta x)^{\frac{d}{2}-1}}
\end{aligned} \tag{4.9}$$

using [3, eq. (10.9.4)]. Now eq. (4.5) turns to be

$$\begin{aligned}
&\int_0^{+\infty}\frac{\mathrm{d}k}{(2\pi)^{\frac{d}{2}}}\frac{k^{d-1}\mathrm{e}^{-\mathrm{i}\omega_{\vec{k}}\,\Delta t}}{2\omega_{\vec{k}}}\frac{J_{\frac{d}{2}-1}(k\,\Delta x)}{(k\,\Delta x)^{\frac{d}{2}-1}}\\
&= \frac{m^{d-1}}{2(2\pi)^{\frac{d}{2}}}\int_0^{+\infty}\mathrm{d}z\,z^{d-1}\frac{\mathrm{e}^{-\mathrm{i}\sqrt{z^2+1}\,\tau}}{\sqrt{z^2+1}}\frac{J_{\frac{d}{2}-1}(\lambda z)}{(\lambda z)^{\frac{d}{2}-1}}\,,
\end{aligned} \tag{4.10}$$

$z = k/m$, $\lambda = m\,\Delta x$, $\tau = m\,\Delta t$.

For $\Delta t > \Delta x$, choose a reference frame in which $\Delta x = 0$. Using [3, eq. (10.2.2)], one has

$$\lim_{\lambda\to 0} \frac{J_\nu(\lambda z)}{(\lambda z)^\nu} = \frac{1}{2^\nu \Gamma(\nu+1)} \,. \tag{4.11}$$

Equation (4.5) turns to be

$$\begin{aligned} &\frac{m^{d-1}}{4(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_0^{+\infty} \mathrm{d}z\, z^{d-1} \frac{\mathrm{e}^{-\mathrm{i}\sqrt{z^2+1}\,\tau}}{\sqrt{z^2+1}} \\ =& \frac{m^{d-1}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_1^{+\infty} \mathrm{d}x \left(x^2-1\right)^{\frac{d}{2}-1} \mathrm{e}^{-\mathrm{i}\tau x} \,. \end{aligned} \tag{4.12}$$

Using [3, eq. (10.32.8) and (10.27.8)] yields

$$\frac{\pi^{\frac{1}{2}}\left(\frac{\mathrm{i}\tau}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)} \int_1^{+\infty} \mathrm{d}x \left(x^2-1\right)^{\nu-\frac{1}{2}} \mathrm{e}^{-\mathrm{i}\tau x} = K_\nu(\mathrm{i}\tau) = \frac{\pi}{2}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(\nu+1)} H_\nu^{(2)}(\tau) \,, \tag{4.13}$$

which requires $-\pi < \arg\tau < 0$, i.e. τ at least has a small negative imaginary part. Equation (4.5) finally turns to be

$$\frac{1}{4}\left(-\frac{\mathrm{i}m}{2\pi(-\sigma)^{\frac{1}{2}}}\right)^{\frac{d-1}{2}} H^{(2)}_{\frac{d-1}{2}}\left(m(-\sigma)^{\frac{1}{2}}\right), \qquad \sigma = -(\Delta t)^2 + (\Delta\vec{x})^2 \,, \tag{4.14}$$

$$\Re\sigma < 0 \,, \qquad \Im(-\sigma)^{\frac{1}{2}} < 0 \,. \tag{4.15}$$

References

[1] J. van Bladel, “Lorenz or lorentz?”, IEEE Antennas and Propagation Magazine **33**, 69–69 (1991) (cit. on p. 2).

[2] J. van Bladel, “Lorenz or lorentz? [addendum]”, IEEE Antennas and Propagation Magazine **33**, 56–56 (1991) (cit. on p. 2).

[3] *NIST DIGITAL LIBRARY OF MATHEMATICAL FUNCTIONS*, http://dlmf.nist.gov/, Release 1.0.22 of 2019-03-15, F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds., `http://dlmf.nist.gov/` (cit. on pp. 3, 4).

[4] L. Lorenz, “XXXVIII. on the identity of the vibrations of light with electrical currents”, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **34**, 287–301 (1867) (cit. on p. 2).