Wightman function of Klein–Fock–Gordon field

Yi-Fan Wang

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Contents

1 Covariant differential

ϕ a $\mathbb{C}$-valued 0-form

$$\mathbb{D}\phi := (\mathbb{d} - \mathrm{i}eA) \wedge \phi, \qquad \mathbb{D}\phi^* := (\mathbb{d} + \mathrm{i}eA) \wedge \phi^*, \tag{1.0.1}$$

A the $\mathfrak{u}(1)$-valued connection form.

2 Covariant codifferential

Define the covariant codifferential of a $\mathbb{C}$-valued k-form ζ as follows. Let η be an arbitrary $\mathbb{C}$-valued $(k-1)$-form.

$$\begin{aligned}\int \mathbb{d}(\eta^* \wedge \star\zeta) &\equiv \int \mathbb{d}\eta^* \wedge \star\zeta - (-)^k \eta^* \wedge \mathbb{d} \star \zeta =: \int \mathbb{D}\eta^* \wedge \star\zeta - \eta^* \wedge \star\mathbb{D}^\dagger\zeta & (2.0.1)\\ &= \int \mathbb{D}\eta^* \wedge \star\zeta - \mathrm{i}eA \wedge \eta^* \wedge \star\zeta - (-)^k \eta^* \wedge \mathbb{d} \star \zeta \\ &= \int \mathbb{D}\eta^* \wedge \star\zeta + \eta^* \wedge (-)^k \mathrm{i}eA \wedge \star\zeta - (-)^k \eta^* \wedge \mathbb{d} \star \zeta \\ &= \int \mathbb{D}\eta^* \wedge \star\zeta - \eta^* \wedge \star(-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\zeta \,. & (2.0.2)\end{aligned}$$

$$\boxed{\mathbb{D}^\dagger\zeta = (-)^k \star^{-1} (\mathbb{d} - \mathrm{i}eA) \wedge \star\zeta \,.} \tag{2.0.3}$$

3 Maxwell–Klein–Fock–Gordon theory

$$S = \int -\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi - m^2\phi^* \wedge \star\phi - \frac{1}{2}F \wedge \star F. \tag{3.0.1}$$

$$\delta\mathbb{D}\phi = -\mathrm{i}e\delta A\phi + \mathbb{D}\delta\phi. \tag{3.0.2}$$

$$\begin{aligned}\delta(\mathbb{D}\phi^* \wedge \star\mathbb{D}\phi) = \mathbb{d}(&\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^*)\\ &+ \delta\phi^* \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}^\dagger\mathbb{D}\phi^* \\ &+ \delta A \wedge (\mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*))\,,\end{aligned} \tag{3.0.3}$$

$$\delta(F \wedge \star F) = 2\mathbb{d}(\delta A \wedge \star F) - 2\delta A \wedge \mathbb{d}\star F\,. \tag{3.0.4}$$

$$\begin{aligned}\delta S = \int &-\mathbb{d}(\delta\phi^* \wedge \star\mathbb{D}\phi + \delta\phi \wedge \star\mathbb{D}\phi^* + \delta A \wedge \star F)\\ &+ \delta\phi^* \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi + \delta\phi \wedge \star(\mathbb{D}^\dagger\mathbb{D} - m^2)\phi^* + \\ &+ \delta A \wedge (-\mathbb{d}\star F + \mathrm{i}e(\phi^* \star \mathbb{D}\phi - \phi \star \mathbb{D}\phi^*)).\end{aligned} \tag{3.0.5}$$

3.1 Lorenz gauge

$$\Box^2 := \left(\mathbb{d} + \mathbb{d}^\dagger\right)^2 = \mathbb{d}\mathbb{d}^\dagger + \mathbb{d}^\dagger\mathbb{d}. \tag{3.1.1}$$

$$\mathbb{d}\star F = \mathbb{d} \star \mathbb{d}A = \star(-)^2 \star^{-1} \mathbb{d} \star \mathbb{d}A = \star\mathbb{d}^\dagger\mathbb{d}A = \star(\Box^2 - \mathbb{d}\mathbb{d}^\dagger)A\,. \tag{3.1.2}$$

One would like to have $\mathbb{d}\mathbb{d}^\dagger A = 0$, or $\mathbb{d}^\dagger A = \mathrm{const}$. This would be fulfiled if

$$\mathbb{d}^\dagger A = 0\,, \tag{3.1.3}$$

which is the Lorenz gauge[1, 2, 4].

4 Free Klein–Fock–Gordon equation in $(d+1)$-dimensions

In the absence of external electromagnetic field and Cartesian coordinates,

$$0 = (\mathbb{D}^\dagger\mathbb{D} - m^2)\phi = (-\partial_t^2 + \partial_x^2 - m^2)\phi\,. \tag{4.0.1}$$

Linearly independent solutions are

$$\exp\!\Big(\pm\mathrm{i}\Big(-\omega_{\vec{k}}t + \vec{k}\cdot\vec{x}\Big)\Big)\,, \qquad \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}\,. \tag{4.0.2}$$

One uses the Noether charge to normalise, which reads

$$(\phi_1, \phi_2) = \int \mathbb{d}^d x\, j^0 \tag{4.0.3}$$

gives

$$\begin{aligned}
\phi_+\Big(\omega_{\vec{k}},\vec{k};t,x\Big) &= \frac{1}{\sqrt{2\omega_{\vec{k}}}}\exp\Big\{+\mathsf{i}\Big(-\omega_{\vec{k}}t+\vec{k}\cdot\vec{x}\Big)\Big\}\,,\\
\phi_-\Big(\omega_{\vec{k}},\vec{k};t,x\Big) &= \frac{1}{\sqrt{2\omega_{\vec{k}}}}\exp\Big\{-\mathsf{i}\Big(-\omega_{\vec{k}}t+\vec{k}\cdot\vec{x}\Big)\Big\}\,,
\end{aligned} \tag{4.0.4}$$

For $d \geq 2$,

$$\int \frac{\mathbb{d}^d k}{(2\pi)^d}\frac{1}{2\omega_{\vec{k}}}\mathsf{e}^{+\mathsf{i}\left(-\omega\,\Delta t+\vec{k}\cdot\Delta\vec{x}\right)} = \int_0^{+\infty}\frac{\mathbb{d}k}{(2\pi)^d}\frac{k^{d-1}\mathsf{e}^{-\mathsf{i}\omega_{\vec{k}}\,\Delta t}}{2\omega_{\vec{k}}}\int \mathsf{e}^{\mathsf{i}k\,\Delta x\cos\phi_1}\,\mathbb{d}\Omega_{d-1}\,, \tag{4.0.5}$$

where

$$\begin{aligned}
&\int \mathbb{d}\Omega_{d-1}\\
:=&\int_0^{2\pi}\mathbb{d}\phi_{d-1}\int_0^{\pi}\sin^{d-2}\phi_1\sin^{d-3}\phi_2\ldots\sin^1\phi_{n-2}\,\mathbb{d}\phi_1\,\mathbb{d}\phi_2\ldots\mathbb{d}\phi_{d-2}\,.
\end{aligned} \tag{4.0.6}$$

Equipped with

$$\int_0^{\pi}\sin^{\nu}\phi\,\mathbb{d}\phi = \pi^{\frac{1}{2}}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}+1\right)}\qquad \Re\nu > -1\,, \tag{4.0.7}$$

one has

$$\int \mathbb{d}\Omega_{d-1} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\int_0^{\pi}\sin^{d-2}\phi_1\mathbb{d}\phi_1\,. \tag{4.0.8}$$

The inner integral in eq. (4.0.5) turns to be

$$\begin{aligned}
\int \mathsf{e}^{\mathsf{i}k\,\Delta x\cos\phi_1}\,\mathbb{d}\Omega_{d-1} &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\int_0^{\pi}\mathsf{e}^{\mathsf{i}k\,\Delta x\cos\phi_1}\sin^{d-2}\phi_1\,\mathbb{d}\phi_1\\
&= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\int_0^{\pi}\cos(k\,\Delta x\cos\phi_1)\sin^{d-2}\phi_1\,\mathbb{d}\phi_1\\
&= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{d-1}{2}\right)}{\left(\frac{k\,\Delta x}{2}\right)^{\frac{d}{2}-1}}J_{\frac{d}{2}-1}(k\,\Delta x)\\
&= (2\pi)^{\frac{d}{2}}\frac{J_{\frac{d}{2}-1}(k\,\Delta x)}{(k\,\Delta x)^{\frac{d}{2}-1}}
\end{aligned} \tag{4.0.9}$$

using [3, eq. (10.9.4)]. Now eq. (4.0.5) turns to be

$$\int_0^{+\infty}\frac{\mathbb{d}k}{(2\pi)^{\frac{d}{2}}}\frac{k^{d-1}\mathsf{e}^{-\mathsf{i}\omega_{\vec{k}}\,\Delta t}}{2\omega_{\vec{k}}}\frac{J_{\frac{d}{2}-1}(k\,\Delta x)}{(k\,\Delta x)^{\frac{d}{2}-1}}\,. \tag{4.0.10}$$

For $\Delta t > \Delta x$, choose a frame in which $\Delta x = 0$. Using [3, eq. (10.2.2)], one has

$$\lim_{x\to 0}\frac{J_\nu(k\,\Delta x)}{(k\,\Delta x)^{\nu}} = \frac{1}{2^{\nu}\Gamma(\nu+1)} \tag{4.0.11}$$

References

References

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