538 Riddler: 13 May 2022

Colin Parker

May 13, 2022

1 Express

I first looked at letters used, sounds involved, state name length, and location without finding any kind of pattern. At this point I thought of other unique identifiers for the states (postal code, flag, slogan, bird, flower, etc.), of which postal code is by far the most commonly known and used. This, combined with Sherlock's clue: "it's elementary", led me to try matching up state postal codes with symbols for chemical elements:

$$
\begin{aligned}
\text{Alabama} &= \text{AL} = \text{Al} = \text{Aluminum}\\
\text{Arkansas} &= \text{AR} = \text{Ar} = \text{Argon}\\
\text{California} &= \text{CA} = \text{Ca} = \text{Calcium}\\
\text{Colorado} &= \text{CO} = \text{Al} = \text{Aluminum}\\
\text{Florida} &= \text{FL} = \text{Fl} = \text{Flerovium (or perhaps phonetically Fluorine)}\\
\text{Georgia} &= \text{GA} = \text{Ga} = \text{Gallium}\\
\text{Indiana} &= \text{IN} = \text{In} = \text{Indium}\\
\text{Louisiana} &= \text{LA} = \text{La} = \text{Lanthanum}\\
\text{Maryland} &= \text{MD} = \text{Md} = \text{Mendelevium}\\
\text{Minnesota} &= \text{MN} = \text{Mn} = \text{Manganese}\\
\text{Missouri} &= \text{MO} = \text{Mo} = \text{Molybdenum}\\
\text{Montana} &= \text{MT} = \text{Mt} = \text{Meitnerium}\\
\text{Nebraska} &= \text{NE} = \text{Ne} = \text{Neon}\\
\text{North Dakota} &= \text{ND} = \text{Nd} = \text{Neodymium}\\
\text{Pennsylvania} &= \text{PA} = \text{Pa} = \text{Protactinium}\\
\text{South Carolina} &= \text{SC} = \text{Sc} = \text{Scandium}
\end{aligned}
$$

No other state's postal code is shared with an element's chemical symbol, so more than likely our criminal mastermind is a chemist!

2 Classic

At any point while I'm playing the game, after rolling, I may have 0, 2, 3, or 4 die in the "duplicate" group; having a single die is not possible. In addition, the probabilities of reaching a particular state on the next roll are completely determined by my current state, not by any previous state or how long I've been playing. This is the Markov property, so we can, after many probability calculations, determine the transition matrix and long-run behavior.

Let a_n, b_n, c_n, d_n be the probabilities of getting 0, 1, 3, or 4 duplicates, respectively, after the n^{th} roll. For the first distribution we would typically use $a_0 = 1,\ b_0 = c_0 = d_0 = 0$, but in the subsequent game reaching the state with 0 duplicates ends the game. Therefore we want to use the distribution *after* the first roll:

$$\begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{bmatrix} = \begin{bmatrix} \frac{3}{32} \\ \frac{9}{16} \\ \frac{3}{16} \\ \frac{5}{32} \end{bmatrix}$$

By considering the possible outcomes for rerolling the duplicate die (which I'll not include for brevity's sake), we can find the transition matrix of the Markov chain:

$$\begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{8} & \frac{3}{32} & 0 \\ 0 & \frac{3}{4} & \frac{9}{16} & 0 \\ 0 & \frac{1}{2} & \frac{3}{16} & 0 \\ 0 & \frac{1}{8} & \frac{5}{32} & 1 \end{bmatrix}^n \begin{bmatrix} \frac{3}{32} \\ \frac{9}{16} \\ \frac{3}{16} \\ \frac{5}{32} \end{bmatrix}$$

In any Markov chain, the long-run behavior will be determined by stationary states (eigenvectors of the transition matrix associated with eigenvalues of 1) and cyclic or periodic states (eigenvectors of the transition matrix associated with eigenvalues $|\lambda| = 1$). All other eigenvalues have $|\lambda| < 1$, and therefore $\lim_{n\to\infty} \lambda^n = 0$ means that they will have no effect in the long run.

This Markov process has two stationary states: having 0 or 4 duplicates means the game is over. Therefore, writing our transition matrix eigenvalues as λ_i and their associated eigenvectors as $\vec{v_i}$, we have immediately that

$$\lambda_1 = \lambda_2 = 1;\ \vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix};\ \vec{v_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Then, we can solve for the eigenvalues by finding the other two roots of the characteristic polynomial $\det(M - \lambda I) = 0$. Finding their associated eigenvectors amounts to solving the equation $M\vec{v_i} = \lambda_i \vec{v_i}$. Lastly, writing $\begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{bmatrix} = \sum_{i=1}^{4} c_i \vec{v_i}$, we can solve for the c_i coefficients by solving a system of 4 equations in 4 variables. This results in

$$\begin{aligned}
\begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{bmatrix} &= \frac{3}{7}\vec{v_1} + \frac{4}{7}\vec{v_2} - \frac{57}{136}\vec{v_3} + \frac{15}{3808}\vec{v_4} \\
\implies \begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{bmatrix} &= \frac{3}{7}\vec{v_1} + \frac{4}{7}\vec{v_2} - \frac{57}{136}\left(\frac{3}{4}\right)^n \vec{v_3} + \frac{15}{3808}\left(\frac{-5}{16}\right)^n \vec{v_4} \\
\implies \lim_{n\to\infty} \begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{bmatrix} &= \begin{bmatrix} \frac{3}{7} \\ 0 \\ 0 \\ \frac{4}{7} \end{bmatrix}
\end{aligned}$$

Therefore the probability of winning (i.e. the probability of eventually having 0 duplicate die, 4 unique die) is $\frac{3}{7}$.