538 Riddler: 6 May 2022

Colin Parker

May 8, 2022

1 Express

Let E_n be the expected number of stops from floor n to floor 1, and we will calculate E_{10}. Assuming a uniform distribution of floor probabilities, we are equally likely to stop at each floor on the way down, which will add 1 to our total floor count. This gives the recursion

$$\begin{aligned} E_n &= 1 + \sum_{k=1}^{n-1} \frac{E_k}{n-1} \\ &= 1 + \frac{1}{n-1} \sum_{k=1}^{n-1} E_k;\ E_1 = 0 \end{aligned}$$

Rearranging gives

$$\begin{aligned} (n-1)(E_n - 1) &= \sum_{k=1}^{n-1} E_k \\ &= E_{n-1} + \sum_{k=1}^{n-2} E_k \\ &= E_{n-1} + (n-2)(E_{n-1} - 1) \\ &= (n-1)(E_{n-1} - 1) + 1 \\ \iff \quad E_n &= E_{n-1} + \frac{1}{n-1} \\ &= E_{n-2} + \frac{1}{n-2} + \frac{1}{n-1} \\ &= E_0 + \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n-1} \\ E_n &= \sum_{k=1}^{n-1} \frac{1}{k} \end{aligned}$$

The number we're looking for is $E_{10} = 1 + \frac{1}{2} + \ldots + \frac{1}{9} = \frac{7129}{2520} \approx 2.829$ floors.

In general, the expected number of stopping floors, starting at floor n, is called the $(n-1)^{\text{th}}$ *harmonic number* H_{n-1}. The asymptotic expansion allowing for easy approximation is

$$H_n \sim \ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}}$$

$$E_n = H_{n-1} \approx \ln(n-1) + \gamma + \frac{1}{2n-2}$$

In this expression, $\gamma \approx 0.5772156649$ is the Euler-Mascheroni constant and B_k is the k^{th} Bernoulli number, which can be used to add more terms to ensure more accurate approximation or faster convergence.

2 Classic

After generating lists of all three-digit (and four-, five-, and six-digit primes), I created a graph with connections between the primes that formed adjacent rungs of a prime ladder (e.g. between 101 and 181). From there, repeated use of Dijkstra's algorithm allowed me to find the shortest path (i.e. optimal prime ladder) between every pair of primes. Sorting these by length, the longest chains are:

1-digit primes: 2 rungs
2-digit primes: 4 rungs
3-digit primes: 7 rungs
4-digit primes: 9 rungs
5-digit primes: 11 rungs

Unfortunately, there are 68906 6-digit primes, which means there are up to 2.3 billion connections to check and values to store. This was too taxing on my RAM, so I couldn't find the answer in this case.