538 Riddler: 1 July 2022

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1 Express

If r_n is the amount you have remaining to complete with n days remaining, then the amount you complete on this day is given by $\frac{2}{n} \cdot r_n$. You complete the task if and only if the amount you complete is at least the amount remaining:

$$\frac{2}{n} \cdot r_n \geq r_n \iff n \leq 2$$

The task would be completed on the second-to-last day of the year, December 30.

2 Classic

This riddle doesn't have a definitive answer, so we will examine the general behavior and approximate the correct answer in the limit of a very large planet.

To see that there is no one answer, imagine that one of the cities is polar and the other is equatorial, meaning that the radius of the planet is $R = 100(1 + \sqrt{2})$ meters (small planet!). In this case, you can never climb high enough to see your friend on the ground floor. In the other limit, the planet is nearly locally flat, so that it is not a problem to climb high enough to see your friend (assuming the tower is tall enough).

If we consider the great circle that contains both of these cities, call the minor arc angle of this great circle θ, and call the radius of the planet R (measured in meters), then we can just barely see our friend if the line joining our two points is tangent to the sphere of the planet at its midpoint, giving:

$$(R + 100) \cos\left(\frac{\theta}{2}\right) = R$$

$$R = \frac{100 \cos\left(\frac{\theta}{2}\right)}{1 - \cos\left(\frac{\theta}{2}\right)}$$

Once our friend climbs down to the bottom floor of his tower, let h be the required height in meters in the tower for us to just barely make out our friend again. Then the line joining us will be tangent to the sphere of the planet at our friend's location, giving:

$$
\begin{aligned}
(R+h)\cos\theta &= R \\
h &= R\left(\frac{1-\cos\theta}{\cos\theta}\right) \\
&= \frac{100\cos\left(\frac{\theta}{2}\right)(1-\cos\theta)}{\cos\theta(1-\cos\left(\frac{\theta}{2}\right))} \\
&= \frac{400R^2+20000R}{R^2-200R-10000}, \text{ using the relation between } R \text{ and } \theta \text{ above.}
\end{aligned}
$$

As expected, this last expression has a vertical asymptote at $R = 100(1+\sqrt{2})$, and this expression is able to easily give limits:

$$\lim_{\theta\to 0} h = \lim_{R\to\infty} h = \lim_{R\to\infty} \frac{400R^2+20000R}{R^2-200R-10000} = 400$$

This is a plausible answer, because we are told the cities are neighboring, meaning that θ should be small. However, we do have a precise possibility for the radius of the planet – what if it wasn't a dream, and the radius of the planet is exactly 8 megameters?

Substituting $R = 8000000$ gives $\theta \approx 0.01$ radians and $h \approx 400.0125$ meters, showing that 400 meters is indeed a very reasonable estimation.