GENERAL RELATIVITY

office hours 8.40 (20 mins before lecture)

Prof Claude Warnick (Printed notes by Prof Harvey Reall)
↳ also look at Tong notes (although slightly diff.)

General relativity is our best theory of gravitation on the largest scales.
It is:

- CLASSICAL: No quantum effects
- GEOMETRICAL: space + time are combined in a curved spacetime
- DYNAMICAL: In contrast to Newton's theory of gravity, Einstein's gravitational field has its own non-trivial dynamics

Differentiable Manifolds

The basic object of study in differential geometry is the (differentiable) manifold. This is an object which locally looks like $\mathbb{R}^n$ and has enough structure to let us do calculus.

DEF: A differentiable manifold of dimension n is a set M, together with a collection of coordinate charts (O_α, ϕ_α) where

- $O_\alpha \subset M$ are subsets of M such that $\bigcup_\alpha O_\alpha = M$
- ϕ_α is a bijective map (one-to-one and onto) from O_α to U_α, an open subset of $\mathbb{R}^n$
- If $O_\alpha \cap O_\beta \neq \emptyset$, then $\phi_\beta \circ \phi_\alpha^{-1}$ is a smooth (infinitely differentiable) map from $\phi_\alpha(O_\alpha \cap O_\beta) \subset U_\alpha$ to $\phi_\beta(O_\alpha \cap O_\beta) \subset U_\beta$

their intersection is not empty

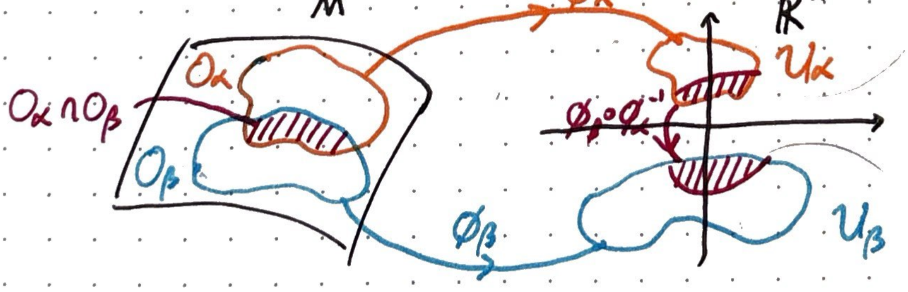


image of $O_\alpha \cap O_\beta$ when mapped with ϕ_α

image of $O_\alpha \cap O_\beta$ when I apply ϕ_β

condition: map between these regions ϕ_α inverse then ϕ_β I want to be smooth (shortcut the loop)

one-to-one and onto: each point from LHS maps to exactly one point on RHS

REMARKS:

- Could replace smooth with finite differentiability (e.g. k-times differentiable)
- the charts define a topology on the original manifold M: $\mathcal{R} \subset M$ is open iff $\phi_\alpha(\mathcal{R} \cap O_\alpha)$ is open in $\mathbb{R}^n$ for all α

curly R

(don't worry too much about this) ↳ think this is maybe just point of interest for those who have done topology

- Every open subset of M is itself a manifold (Restrict charts to $\mathcal{R}$)

The collection $\{(O_\alpha, \phi_\alpha)\}$ is called an <u>atlas</u>. Two atlases are <u>compatible</u> if their union is an atlas.

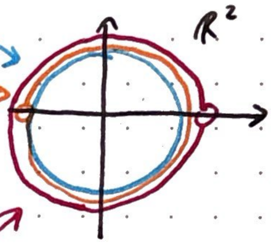
An atlas A is <u>maximal</u> if there exists no atlas B with $A \subsetneq B$ (subset but not equal to). Every atlas is contained to a maximal atlas (consider the union of all compatible atlases). We can assume without loss of generality that we work with a maximal atlas.

<u>EXAMPLES</u>

1. If $\mathcal{U} \subset \mathbb{R}^n$ is open, we can take $O = \mathcal{U}$, $\phi(x^1, \dots, x^n) = (x^1, \dots, x^n)$ $\quad \phi: O \to \mathcal{U} \quad$ $\{(\mathcal{U}, \phi)\}$ is an atlas

(2D sphere?)

2. $S^1 = \{ p \in \mathbb{R}^2 \mid |p| = 1 \}$



(the circles are on top of eachother)

diff. coordinate charts using angles around the circle

if $p \in S^1 \setminus \{(-1,0)\} = O_1$, there is a unique $\theta_1 \in (-\pi, \pi)$ s.t. $p = (\cos\theta_1, \sin\theta_1)$

open interval (excluding $\pi, -\pi$)

open interval

If $p \in S^1 \setminus \{(1,0)\} = O_2$, then there is a unique $\theta_2 \in (0, 2\pi)$ s.t. $p = (\cos\theta_2, \sin\theta_2)$

(minus/not including that point)

$\phi_1 : p \to \theta_1 \quad p \in O_1 \quad \mathcal{U}_1 = (-\pi, \pi)$

$\phi_2 : p \to \theta_2 \quad p \in O_2 \quad \mathcal{U}_2 = (0, 2\pi)$

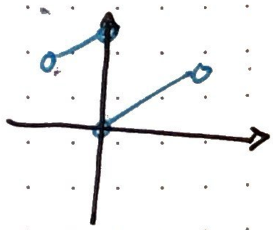
exercise

$\phi_1(O_1 \cap O_2) = (-\pi, 0) \cup (0, \pi)$ (open set)

transition defined on this set

$$\phi_2 \circ \phi_1^{-1}(\theta) = \begin{cases} \theta & \theta \in (0, \pi) \\ \theta + 2\pi & \theta \in (-\pi, 0) \end{cases}$$

(composite function? 1 then 2?)

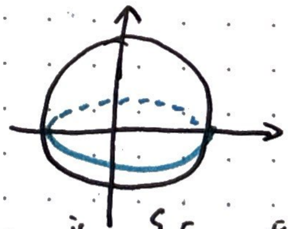


Smooth where defined similarly for $\phi_1 \circ \phi_2^{-1}$

S^1 is a manifold.

higher dimensional sphere

3. $S^n = \{ \underline{p} \in \mathbb{R}^{n+1} \mid |\underline{p}| = 1 \}$



Define charts by stereographic projection, if $\{\underline{E}_1, \dots, \underline{E}_{n+1}\}$ is a standard basis for $\mathbb{R}^{n+1}$ and $\{\underline{e}_1, \dots, \underline{e}_n\}$ the basis for $\mathbb{R}^n$, write

$\underline{p} = p^1 \underline{E}_1 + \dots + p^{n+1} \underline{E}_{n+1}$

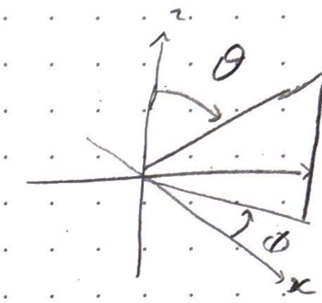
point on surface of sphere

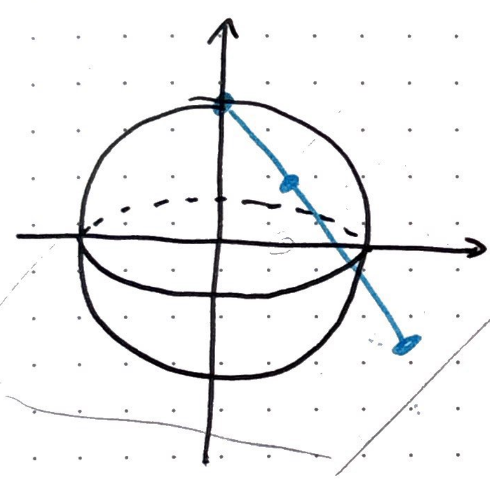
set $O_1 = S^n \setminus \{\underline{E}_{n+1}\}$ (sphere minus north pole)

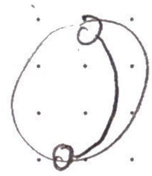
$$\varphi_1(\underline{p}) = \frac{1}{1-p^{n+1}}\left(p^1\underline{e}_1 + \dots + p^n\underline{e}_n\right)$$

$O_2 = S^n \setminus \{-\underline{E}_{n+1}\}$

$$\varphi_2(\underline{p}) = \frac{1}{1+p^{n+1}}\left(p^1\underline{e}_1 + \dots + p^n\underline{e}_n\right)$$







1st map takes point & projects onto plane transverse to $n+1$ direction through the north pole

Other map does the same through the south pole

'excluding the origin'

Claim $\phi_1(O_1 \cap O_2) = \mathbb{R}^n \setminus \{\underline{0}\}$ and $\phi_2 \circ \phi_1^{-1}(\underline{x}) = \dfrac{\underline{x}}{|\underline{x}|^2}$

smooth on $\mathbb{R}^n \setminus \{\underline{0}\}$

similar for $\phi_1 \circ \phi_2^{-1}$, S^n is an n-manifold

well $\Phi_1(\underline{p}) = \underline{x}$, then we see $\Phi_2 \circ \Phi_1^{-1}(\underline{x}) = \dfrac{1-p^{n+1}}{1+p^{n+1}}\underline{x}$

then since point p on n-sphere has unit norm $\Rightarrow (p^1)^2 + \dots + (p^n)^2 = 1-(p^{n+1})^2$.

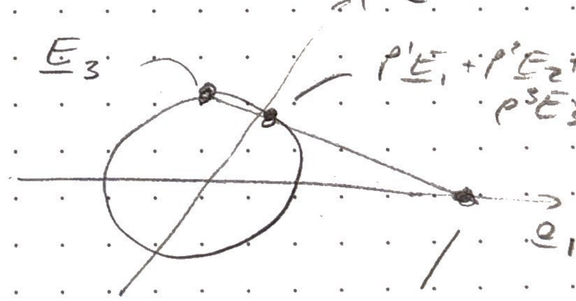
and so $|\underline{x}| = \dfrac{(p^1)^2 + \dots + (p^n)^2}{(1-p^{n+1})^2} = \dfrac{1-(p^{n+1})^2}{(1-p^{n+1})^2} = \dfrac{1+p^{n+1}}{1-p^{n+1}}$

so $\Phi_2 \circ \Phi_1^{-1}(\underline{x}) = \dfrac{\underline{x}}{|\underline{x}|^2}$

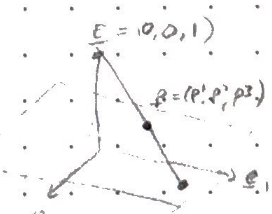
which is smooth

$\underline{p} = p^1\underline{E}_1 + p^2\underline{E}_2 + p^3\underline{E}_3$

$$\psi_1(\underline{p}) = \frac{1}{1-p^3}\left(p^1\underline{e}_1 + p^2\underline{e}_2\right)$$



$\frac{1}{1-p^3}(p^1\underline{e}_1 + p^2 \underline{e}$



$(x, y, z) = (0,0,1) + \lambda(p^1, p^2, p^3 - 1)$

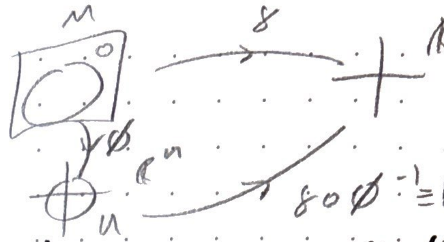
$z = 0$

$\Rightarrow x = \frac{1}{1-p^3}p^1, \; y = \frac{1}{1-p^3}p^2$

so $\psi(p) = \frac{1}{1-p^3}(p^1\underline{e}_1 + p^2\underline{e}_2)$

Lecture 2

14.10.24



Smooth functions on manifolds

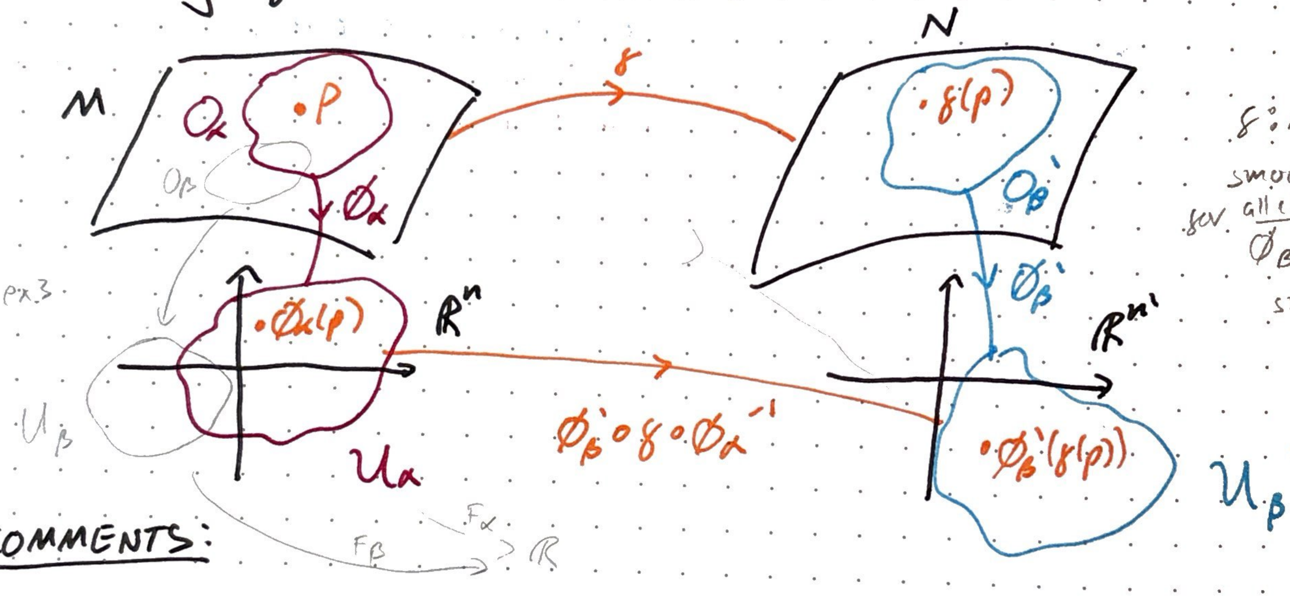
Suppose M, N are manifolds of dim n, n' respectively

let $f : M \to N$

Then let $p \in M$ and pick charts (O_α, ϕ_α) for M and (O_β', ϕ_β') for N with $p \in O_\alpha$, $f(p) \in O_\beta'$. Then $\phi_\beta' \circ f \circ \phi_\alpha^{-1}$ maps an open neighbourhood of $\phi_\alpha(p)$ in $\mathcal{U}_\alpha \subset \mathbb{R}^n$ to $\mathcal{U}_\beta' \subset \mathbb{R}^{n'}$

If this function is smooth for all possible choices of chart, and p we say $f : M \to N$ is smooth

$f: M \to \mathbb{R}$ smooth iff for all charts ϕ says $F \equiv f \circ \phi^{-1} : \mathcal{U} \to \mathbb{R}$ is smooth



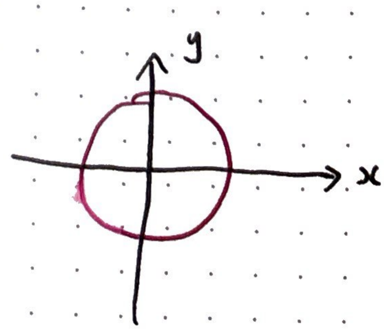
$f : M \to N$ smooth if for all choices of chart $\phi_\beta' \circ f \circ \phi_\alpha$ smooth

COMMENTS:

- A smooth map $\psi : M \to N$ which has a smooth inverse is called a <u>diffeomorphism</u> $(n = n')$
- If $N = \mathbb{R} / \mathbb{C}$ we sometimes call f a scalar field
  open interval on the reals
- If $M = I \subset \mathbb{R}$ an open interval then $f : I \to N$ is a smooth curve in N
- If f is smooth in one atlas, it is smooth in all compatible atlases

EXAMPLES:

1. Recall $S^1 = \{ \underline{x} \in \mathbb{R}^2 \mid |\underline{x}| = 1 \}$
   let $f(x,y) = x$ $\quad f : S^1 \to \mathbb{R}$



Using previous charts

$$f \circ \phi_1^{-1} : (-\pi, \pi) \to \mathbb{R}$$

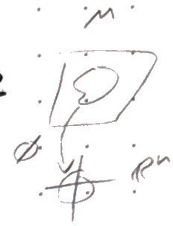
similarly $f \circ \phi_1^{-1}(\theta_1) = \cos\theta_1$

$f \circ \phi_2^{-1} : (0, 2\pi) \to \mathbb{R}$

$f \circ \phi_2^{-1}(\theta_2) = \cos\theta_2 \quad \therefore\ f$ is smooth

notes:
if Φ is any other chart, then $f \circ \Phi^{-1} = (f \circ \phi_i^{-1}) \circ (\phi_i \circ \Phi^{-1})$ which is smooth since we have shown $f \circ \phi_i^{-1}$ is smooth, & $\phi_i \circ \Phi^{-1}$ is smooth by defining of a manifold

2. If (O, ϕ) is a coordinate chart on M, write



$$\phi(p) = (x^1(p), x^2(p), \ldots, x^n(p)) \qquad p \in O$$

Then $x^i(p)$ defines a map from O to $\mathbb{R}$. This is a smooth for each $i = 1, \ldots, n$. If (O', ϕ') is another overlapping coordinate chart then $x^i \circ \phi'^{-1}$ is the ith component of $\phi \circ (\phi')^{-1}$, hence smooth

↳ $f \circ \phi'^{-1}$ is smooth because it's the first component of $\phi \circ \phi'^{-1}$ which is smooth by definition of a manifold

often convenient to def. a fn. by specifying F instead of f:

3. We can define a smooth function chart-by-chart. For simplicity $N = \mathbb{R}$, let $\{(O_\alpha, \phi_\alpha)\}$ be an atlas on M. Define smooth functions

coordinate chart representations of f $\quad F_\alpha : U_\alpha \to \mathbb{R}$

and suppose $F_\alpha \circ \phi_\alpha = F_\beta \circ \phi_\beta$ on $O_\alpha \cap O_\beta$ for all α, β.

Then for some point $p \in M$ we can define $f(p) = F_\alpha \circ \phi_\alpha(p)$ where (O_α, ϕ_α) is any chart with $p \in O_\alpha$. f is smooth as

$$f \circ \phi_\beta^{-1} = F_\alpha \circ \underbrace{\phi_\alpha \circ \phi_\beta^{-1}}_{\text{smooth}}$$

smooth since we defined it to be smooth

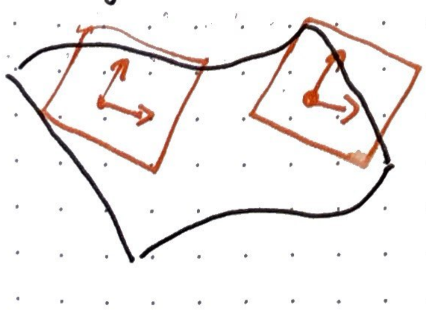
$\leftarrow$ if $O_\alpha \cap O_\beta \neq \emptyset$, $\phi_\alpha \circ \phi_\beta^{-1}$ smooth

$f(p) = F_\alpha \circ \phi_\alpha(p)$ since we have specified the region for the chart overlaps, this definition is chart independent in those regions

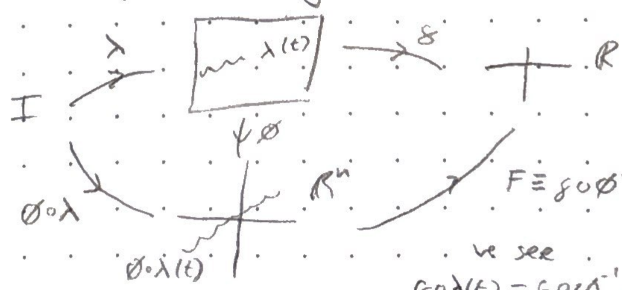
In practice, often don't distinguish between f and its coordinate chart representations F_α. (i.e. will say $f(x)$ when we mean $F(x)$)

CURVES AND VECTORS

For a surface in $\mathbb{R}^3$, we have a notion of 'tangent space' at a point consisting of all vectors tangent to the surface



useful picture generally for this topic!



we see

$$f \circ \lambda(t) = \underbrace{f \circ \phi^{-1}}_{F} \circ \underbrace{\phi \circ \lambda(t)}_{\text{curve in } \mathbb{R}^n}$$

think? $\begin{cases} \lambda \text{ is a map } \lambda : I \to M \\ \lambda(t) \text{ is a curve in } M \\ \phi \text{ is a map } \phi : M \to \mathbb{R} \\ \phi(p) \text{ is a point in } \mathbb{R}^n \end{cases}$

so we move out of the manifold to $\mathbb{R}^n$ where we can do differentiation and eval. $\frac{d}{dt}(f(\lambda(t)))$

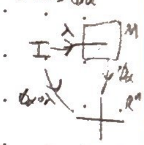
The tangent spaces are vector spaces (copies of $\mathbb{R}^2$).
Different points have different tangent spaces

In order to define the tangent space for a manifold, we first consider tangent vectors of a curve.

<u>Recall</u> $\lambda : \underset{\substack{\wedge \\ \mathbb{R} \\ \text{interval}}}{I} \to M$ a smooth map is a smooth curve in M

I is open interval in $\mathbb{R}$ e.g. $(0,1)$ or $(-1,\infty)$

(a smooth curve in a manifold M is a smooth function $\lambda : I \to M$) → by smooth function mean $\phi_\alpha \circ \lambda$ is a smooth map from I to $\mathbb{R}^n$ for all charts ϕ_α



If $\lambda(t)$ is a smooth curve in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function. The chain rule gives

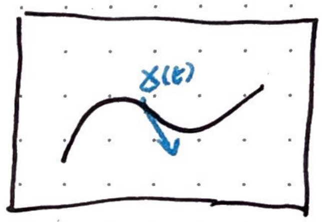
$$\frac{d}{dt}[f(\lambda(t))] = \underline{X}(t) \cdot \underline{\nabla} f(\underline{\lambda}(t))$$

where $\underline{X}(t) = \frac{d\underline{\lambda}(t)}{dt}$ is the tangent vector to $\underline{\lambda}$ at t.

<u>IDEA</u>: treat as though equivalent?

this map takes $f \to \underline{X} \cdot \nabla f$

identify $\underline{X}(t)$ with $\underline{X}(t) \cdot \underline{\nabla}$



<u>DEF</u> Let $\lambda : I \to M$ be a smooth curve with (wlog) $\lambda(0) = p$.
The tangent vector to λ at p is the linear map X_p from the space of smooth functions $f : M \to \mathbb{R}$ given by

$$X_p(f) := \frac{d}{dt} f(\lambda(t))\Big|_{t=0}$$

X_p is the vector, $X_p(f)$ specifies how the vector acts on a function, it a map

map that takes f to $\frac{d}{dt}f = \underline{X} \cdot \nabla f$

<u>We observe</u>:

i) X_p is linear: $X_p(f + ag) = X_p f + a X_p g$ $\quad f, g$ smooth, $a \in \mathbb{R}$

ii) X_p satisfies Leibniz rule $X_p(fg) = (X_p f) g(p) + f(p) X_p g$

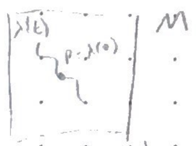
If (O, ϕ) is a chart $p \in O$, write $\phi(p) = (x^1(p), \dots, x^n(p))$
Let $F = f \circ \phi^{-1}$ and $x^i(t) = x^i(\lambda(t))$
Then $f \circ \lambda(t) = f \circ \phi^{-1} \circ \phi \circ \lambda(t) = F \circ x(t)$

f $\mathbb{R}$ to itself

and $\frac{d}{dt}(f(\lambda(t)))\Big|_{t=0} = \underbrace{\frac{\partial F}{\partial x^\mu}(x)\Big|_{\phi(p)}}_{\text{depends on } f, \phi} \underbrace{\frac{dx^\mu}{dt}\Big|_{t=0}}_{\text{depends on } \lambda, \phi}$

(Einstein summation convention: sum over repeated $\mu = 1, \dots, n$)

(see preceeding page)



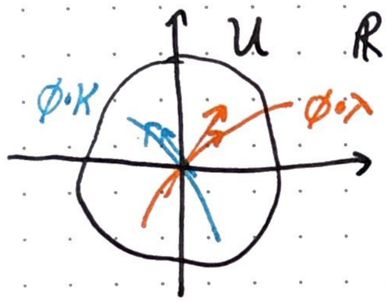
X_p is tangent vector to λ at p.

<u>Lecture 3</u>   <u>16.10.24</u> $(*)\ X_p(g) = \frac{\partial F}{\partial x^\mu}(\phi(p)) \cdot \frac{dx^\mu(t)}{dt}\Big|_{t=0}$

Prop the set of tangent vectors to curves at p forms a vector space T_pM of dimension $n = \dim M$. We call T_pM the tangent space to M at p

dim of manifold

Pf/ Given X_p, Y_p tangent vectors, we need to show $\alpha X_p + \beta Y_p$ is a tangent vector for $\alpha, \beta \in \mathbb{R}$.
Let λ, κ be smooth curves with $\lambda(0) = \kappa(0) = p$ and whose tangent vectors at p are X_p, Y_p resp.
Let (O, ϕ) be a chart with $p \in O$, $\phi(p) = 0$ (chart centred at p)



Let $\nu(t) = \phi^{-1}[\alpha\phi(\lambda(t)) + \beta\phi(\kappa(t))]$

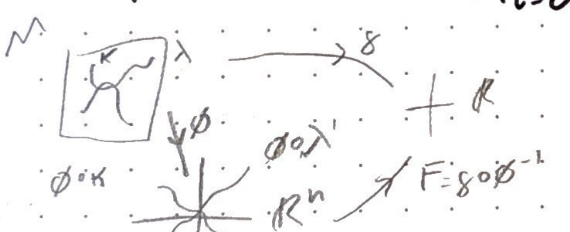
$\nu(0) = \phi^{-1}(0) = p$

From $(*)$ we have that if Z_p is the tangent to ν at p:

$$Z_p(g) = \frac{d}{dt}(g(\nu(t)))\Big|_{t=0} = \frac{\partial F}{\partial x^\mu}\Big|_0 \frac{d}{dt}\left[\alpha x^\mu(\lambda(t)) + \beta x^\mu(\kappa(t))\right]\Big|_{t=0}$$

$$= \alpha \frac{\partial F}{\partial x^\mu}\Big|_0 \frac{d}{dt} x^\mu(\lambda(t)) + \beta \frac{\partial F}{\partial x^\mu}\Big|_0 \frac{d}{dt} x^\mu(\kappa(t))\Big|_{t=0}$$

$$= \alpha X_p(g) + \beta X_p(g)$$



Thus T_pM is a vector space.

To see T_pM is n-dimensional consider the curves

μ component

$$\lambda_\mu(t) = \phi^{-1}(0, \dots, 0, t, 0, \dots, 0)$$

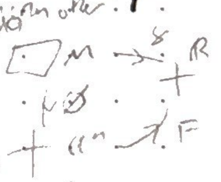
think of it like "basis"? vector in $\mathbb{R}^n$ getting moved to M (see pic below)

We denote the tangent vector to λ_μ at p by $\left(\frac{\partial}{\partial x^\mu}\right)_p$
To see why, note that $(*)$

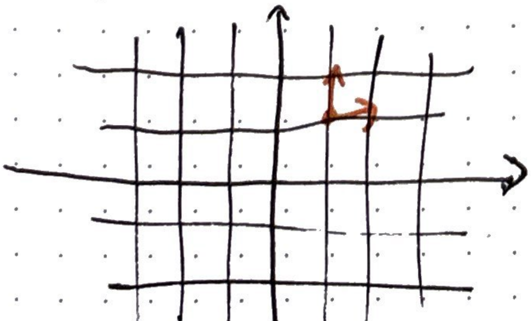
$F(\phi(p)) = g \circ \phi^{-1}(\phi(p)) = g \circ p = g(p)$

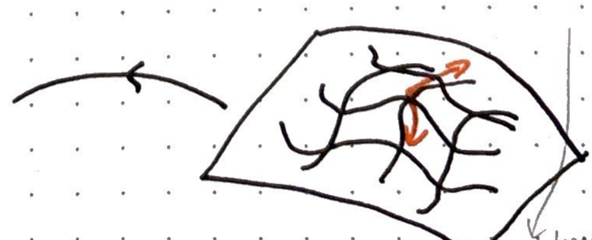
$$\left(\frac{\partial}{\partial x^\mu}\right)_p g = \frac{\partial F}{\partial x^\mu}\Big|_{\phi(p)=0}$$

(approach on other end)
tong: don't know how to diff g on M but do know how to diff $g \circ \phi^{-1}$ on $\mathbb{R}^n$



$\mathbb{R}^n$





since λ_μ

$X_p(g) = \frac{\partial F}{\partial x^\mu} \cdot 1$

call it $= \frac{\partial}{\partial x^\mu}\Big|_p$?

tong says: $\frac{\partial g}{\partial x^\mu}\Big|_p := \frac{\partial(g \circ \phi^{-1})}{\partial x^\mu}\Big|_{\phi(p)}$

The vectors $\left(\frac{\partial}{\partial x^\mu}\right)_p$ are linearly independent. Otherwise $\exists\, \alpha^\mu \in \mathbb{R}$ s.t. not all zero s.t.

$$\alpha^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p = 0$$

$\Rightarrow \alpha^\mu \left.\frac{\partial F}{\partial x^\mu}\right|_0 = 0 \quad \forall F$. setting $F = x^\nu$ gives $\alpha^\nu = 0$

Further, $\left(\frac{\partial}{\partial x^\mu}\right)_p$ form a basis for T_pM, since if λ is any curve with tangent X_p at p (*) gives

$$X_p(f) = \left.\frac{\partial F}{\partial x^\mu}\right|_0 \left.\frac{d}{dt} x^\mu(\lambda(t))\right|_{t=0} = X^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p f$$

where $X^\mu = \left.\frac{d}{dt} x^\mu(\lambda(t))\right|_{t=0}$ are the components of X_p w.r.t. the basis $\left\{\left(\frac{\partial}{\partial x^\mu}\right)_p\right\}_{\mu=1}^n$ for T_pM // completing proof

true for any f so $X_p = X^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p$ ← can express any vector $X_p \in T_pM$ as a linear combination of basis vectors $\left(\frac{\partial}{\partial x^\mu}\right)_p$

Notice that $\left\{\left(\frac{\partial}{\partial x^\mu}\right)_p\right\}_{\mu=1}^n$ depends on the coordinate chart ϕ.

$\Rightarrow$ for g: $\left.\frac{\partial g}{\partial x^\mu}\right|_p := \left.\frac{\partial (g \circ \phi^{-1})}{\partial x^\mu}\right|_{\phi(p)}$

Suppose we choose another chart (O', ϕ'), again centered at p. Write $\phi' = (x'^1, \dots, x'^n)$. Then if $F' = f \circ \phi'^{-1}$ we have

p mapped to origin by ϕ'

$$F(x) = f \circ \phi^{-1}(x) = f \circ \phi'^{-1} \circ \phi' \circ \phi^{-1}(x)$$
$$= F'(x'(x))$$

summation convention sum over ν

$$\text{so } \left(\frac{\partial}{\partial x^\mu}\right)_p f = \left.\frac{\partial F}{\partial x^\mu}\right|_{\phi(p)} = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial F'}{\partial x'^\nu}\right)_{\phi'(p)} = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \cdot \left(\frac{\partial}{\partial x'^\nu}\right)_p f$$

We deduce that

jacobian? (linearity rem' dx's

$$\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\nu}\right)_p$$

Let X^μ be components of X_p w.r.t. $\left\{\left(\frac{\partial}{\partial x^\mu}\right)_p\right\}_{\mu=1}^n$ and X'^μ be components of X_p w.r.t. $\left\{\left(\frac{\partial}{\partial x'^\mu}\right)_p\right\}_{\mu=1}^n$

i.e. $X_p = X^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p = X'^\mu \left(\frac{\partial}{\partial x'^\mu}\right)_p$

$$= X^\mu \left(\frac{\partial x'^\sigma}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\sigma}\right)_p$$

so $X'^{\mu} = \left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)_{\phi(p)} X^{\nu}$

We do not have to choose a coordinate basis such as $\left\{\left(\frac{\partial}{\partial x^{\mu}}\right)_p\right\}_{\mu=1}^{n}$ with respect to a general basis $\{e_\mu\}_{\mu=1}^{n}$ for T_pM we write $X_p = X^{\mu} e_\mu$ for $X^{\mu} \in \mathbb{R}$ are components w.r.t. $\{e_\mu\}_{\mu=1}^{n}$

We always use summation convention: we always contract one upstairs and one downstairs index. The index on $\frac{\partial}{\partial x^{\mu}}$ counts as downstairs.

(side note: how does this relate to X? ... a general vector in V can be written as $X = X^{\mu} e_{\mu}$)

COVECTORS

Recall that if V is a vector space over $\mathbb{R}$, the dual space V^* is the space of linear maps from V to $\mathbb{R}$. If V is n-dimensional so is V^*. Given a basis $\{e_\mu\}_{\mu=1}^{n}$ for V, we define the dual basis $\{f^{\mu}\}_{\mu=1}^{n}$ for V^* by requiring $f^{\mu}(e_\nu) = \delta^{\mu}_{\nu} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}$

think of matrix multiplication row & column vector gives real number, for usual basis vectors ~~other~~ dual is the transpose

this is actually what the note in the lecture? i'm lost

theorem?

If V is finite dimensional then $V^{**} = (V^*)^*$ is isomorphic to V: to an element X of V we the linear map

$$\Lambda_X : V^* \to \mathbb{R} \qquad \Lambda_X(\omega) = \omega(X) \qquad \omega \in V^*$$

= there is a bijection between them?

Def the dual space of T_pM is denoted T_p^*M and called the cotangent space to M at p. An element of T_p^*M is a covector at p. If $\{e_\mu\}_{\mu=1}^{n}$ is a basis for T_pM and $\{f^{\mu}\}_{\mu=1}^{n}$ the dual basis for T_p^*M, we can ~~extend~~ expand a covector η as

$$\eta = \eta_\mu f^{\mu}$$ for $\eta_\mu \in \mathbb{R}$ the components of η.

notes: (isomorphic means there is a bijection between V^{**} and V i think)
the bijection (aka the isomorphism) is $\phi : V \to V^{**}$, where $\phi(X)(\omega) = \omega(X)$ for all $\omega \in V^*$.

Lecture 4

18.10.24

Recap:
- Defined T_pM: space of tangent vectors at p. basis $\{e_\mu\}_{\mu=1}^{n}$
- Coord basis $\left\{\left(\frac{\partial}{\partial x^\mu}\right)_p\right\}_{\mu=1}^{n}$
- change of basis $\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\nu}\right)_p$, $\quad x'(x) = \phi' \circ \phi^{-1}(x)$
- Dual space T_p^*M: space of covectors
  - linear maps η from T_pM to $\mathbb{R}$
  - dual basis $\{f^\mu\}_{\mu=1}^{n}$ satisfying $f^\mu(e_\nu) = \delta^\mu_\nu$; $\eta = \eta_\mu f^\mu$ ($\eta_\mu \in \mathbb{R}$)

general components

(linear map that eats a vector)

NOTE

$$* \ \eta(e_\nu) = \eta_\mu f^\mu(e_\nu) = \eta_\mu \delta^\mu_\nu = \eta_\nu$$

zero except η_ν component

general action on X

$$* \ \eta(X) = \eta(X^\mu e_\mu) = X^\mu \eta(e_\mu) = X^\mu \eta_\mu$$

contraction between components

DEF differentials of smooth fn: natural set of covectors

If $f: M \to \mathbb{R}$ is a smooth function, define $(df)_p \in T_p^*M$, the differential of f at p by (aka gradient in real notes)

$$(df)_p(X) = X(f) \quad \text{for any } X \in T_pM$$

? $(df)_p$ is a covector that acts on vectors X to give $X(f)$ (X is acting on ?)

$(df)_p$ is sometimes also called the gradient of f at p.

$*$ If f is constant $X(f) = 0 \Rightarrow (df)_p = 0$

$*$ If (O, ϕ) is a coord chart with $p \in O$ and $\phi = (x^1, \dots, x^n)$ then we can set $f = x^\mu$ to find $(dx^\mu)_p$ now

same as basis vector acting on fn x^μ — wrong

$$(dx^\mu)_p \left(\frac{\partial x^\mu}{\partial x^\nu}\right)_{\phi(p)} = \delta^\mu_\nu$$

use $(df)_p(X) = X(f)$

(condition for basis of cotangent space to be dual to basis of tangent space)

- Hence $\{(dx^\mu)_p\}_{\mu=1}^{n}$ is the dual basis to $\left\{\left(\frac{\partial}{\partial x^\mu}\right)_p\right\}_{\mu=1}^{n}$
- In this basis we can compute

using $\eta(e_\nu) = \eta_\nu$!

$$\text{components } [(df)_p]_\mu = (df)_p\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial}{\partial x^\mu}\right)_p f = \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)} \quad (F = f \circ \phi^{-1})$$

Justifying the language 'GRADIENT'

why we call it gradient of f at p

Exercise: show that if (O', ϕ') is another chart with $p \in O'$, then

evaluated at p.

$$(dx^\mu)_p = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)_{\phi'(p)} (dx'^\nu)_p, \quad x(x') = \phi \circ \phi'^{-1}$$

and hence if η_μ, η'_μ are components w.r.t. these bases

$$\eta'_\mu = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right)_{\phi'(p)} \eta_\nu$$

(simple; just plug into $(dx^\mu)_p \eta_\mu = (dx'^\nu)_p \eta'_\nu$)

essentially show that each side gives the same for general vector X then remove X (like we did with f before):

$$(dx^\mu)_p(X) = (dx^\mu)_p\left(X^\sigma \left(\frac{\partial}{\partial x^\sigma}\right)_p\right)$$

use transf. law for the basis $\frac{\partial}{\partial x'^\nu}$!

$$= (dx^\mu)_p\left(X^\sigma \frac{dx'^\nu}{dx^\sigma}\left(\frac{\partial}{\partial x'^\nu}\right)_p\right)$$

$$= \frac{dx^\mu}{dx'^\nu}(dx'^\nu)_p\left(\underbrace{X^\sigma \frac{\partial}{\partial x^\sigma}}_{X}\right)$$

tangent space: vectors tangent to manifold

cotangent space: less clear, vectors annihilated by cotangent vector

glue these objects together as the point varies: cotangent bundle

The (co)tangent bundle

TM is 2n dimensional if M is dim. n because element of TM is specified by a point on the manifold along with a vector in the corresponding tangent space T_pM } loca

We can glue together the tangent spaces T_pM as p varies to get a new 2n dimensional manifold TM, the tangent bundle. We get

(union over all points in the manifold) $$TM = \bigcup_{p \in M} \{p\} \times T_pM$$ TM is the collection of all the tangent spaces for all points in a manifold

The set of ordered pairs (p, X), with $p \in M$, $X \in T_pM$

If $\{(O_\alpha, \phi_\alpha)\}$ is an atlas on M, we obtain an atlas for TM by setting

patch of original atlas and at each point corresponding tangent space $$\tilde{O}_\alpha = \bigcup_{p \in O_\alpha} \{p\} \times T_pM$$

and

$$\hat{\phi}_\alpha((p, X)) = (\phi_\alpha(p), X^\mu) \in \mathcal{U}_\alpha \times \mathbb{R}^n = \hat{\mathcal{U}}_\alpha$$

where

X^μ are the components of X w.r.t. the coord bases of ϕ_α

work out transition fns for this atlas

<u>Exercise</u>: If (O, ϕ) and (O', ϕ') are two charts on M, show that on $\tilde{\mathcal{U}} \cap \tilde{\mathcal{U}}$, if we write $\phi' \circ \phi^{-1}(x) = x'(t)$ then

$$\hat{\phi}' \circ \hat{\phi}^{-1}(x, X^\mu) = \left(x'(x), \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)_x X^\nu\right)$$

deduce TM is a manifold

"don't really need this for most of course but may as well state it!"

A similar construction permits us to define the cotangent bundle

$$T^*M = \bigcup_{p \in M} \{p\} \times T_p^*M$$

<u>Exercise</u>: Show that the map $\pi\ TM \to M$ which takes $(p, X) \mapsto p$ is smooth

"for the keener geometers"

["mostly forget the last 10 mins of what I was saying → bundle construction is not going to play much of a role in the rest of the course"]

ABSTRACT INDEX NOTATION

(capturing the value of index notation without having to assume we've chosen a coord chart or basis.)

We've used greek letters μ, ν etc. to label components of vectors (or covectors) w.r.t. the basis $\{e_\mu\}_{\mu=1}^n$ (resp $\{f^\mu\}_{\mu=1}^n$). Equations involving these quantities refer to the specific basis.

E.g. if we write $X^\mu = \delta^\mu_1$ ← (no longer true if change to diff basis)

This says X only has one non-zero component in current basis. This won't be true in other bases.

We know some equations hold in all bases, e.g.

$$\eta(X) = X^\mu \eta_\mu$$

(abstract index promotes this kind of statement to latin indices)

To capture this, we can use abstract index notation (AIN).

We denote a vector by X^a where the latin index a does not denote a component, rather it tells us X^a is a vector.

(if eqn. true in all basis we're allowed to write it w/ latin indices replacing greek ones)

downstairs

Similarly we denote a covector η by η_a. If an equation is true in all bases we can replace greek indices by latin indices.

(can do this because switch out a for μ in any basis this is true)

i.e. $\eta(X) = X^a \eta_a = \eta_a X^a$

or $X(f) = X^a (df)_a$

An equation in AIN can always be turned into an equation for components by picking a basis and changing $a \to \mu$, $b \to \nu$ etc.

TENSORS

some quantities not described by either a scalar or a vector, even in Newtonian physics e.g. moment of inertia – need higher rank object

In Newtonian physics, we know some quantities are described by higher rank objects (e.g. inertia tensor of a body).

DEF: A tensor of type (r,s) is a multilinear map

$$T: \underbrace{T_p^*(M) \times \dots \times T_p^*(M)}_{r \text{ factors}} \times \underbrace{T_pM \times \dots \times T_pM}_{s \text{ factors}} \to \mathbb{R}$$

Multilinear means linear in each argument

Examples

1. A tensor of type $(0,1)$ is a linear map $T_pM \to \mathbb{R}$ i.e. a covector.

2. A tensor of type $(1,0)$ is a linear map $T_p^*M \to \mathbb{R}$ i.e. an element $(T_p^*M)^* \simeq T_pM$ a vector.

   dual to the cotangent space at p — 'isomorphic to' ?

3. We can define a $(1,1)$ tensor, δ, by $\delta(\omega, X) = \omega(X)$, $\omega \in T_p^*M$, $X \in T_pM$

   define it to be the map where the covector eats the vector

(components are what we get if we feed basis vectors into every one of the slots of T)

If $\{e_\mu\}$ is a basis for T_pM and $\{f^\mu\}$ the dual basis, the components of an (r,s) tensor T are

$$\boxed{T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} := T(f^{\mu_1}, \dots, f^{\mu_r}, e_{\nu_1}, \dots, e_{\nu_s})}$$

(by definition)

In AIN we denote T by $T^{a_1 \dots a_r}{}_{b_1 \dots b_s}$. Tensors at p form a vector space over $\mathbb{R}$ of DIM n^{r+s}.

(not going to prove this here but can check)

EXAMPLES

1. consider δ above

$$\delta^\mu{}_\nu := \delta(f^\mu, e_\nu) = f^\mu(e_\nu) = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

we can write δ as $\delta^a{}_b$ in AIN

kronecker delta defines a $(1,1)$ tensor

2. consider a $(2,1)$ tensor T, let $\omega, \eta \in T_p^*M$, $X \in T_pM$

$$T(\omega, \eta, X) = T(\omega_\mu f^\mu, \eta_\nu f^\nu, X^\sigma e_\sigma)$$

follows because multilinear

$$= \omega_\mu \eta_\nu X^\sigma \, T(f^\mu, f^\nu, e_\sigma)$$

defining components

$$= \omega_\mu \eta_\nu X^\sigma T^{\mu\nu}{}_\sigma$$

in AIN $T(\omega, \eta, X) = \omega_a \eta_b X^c T^{ab}{}_c$

generalisation to higher ranks

basically given r covectors & s vectors, a tensor type (r,s) will give me a real number

Lecture 5

21.10.24

CHANGE OF BASES

We've seen how components of X or η w.r.t. a coordinate basis (X^μ, η_ν resp.) change under a change of coordinates.

We don't have to only consider coordinate bases. — i.e. can also examine how they transform under an arbitrary change in basis

Suppose $\{e_\mu\}_{\mu=1}^n$ and $\{e'_\mu\}_{\mu=1}^n$ are two bases for T_pM with dual bases $\{f^\mu\}_{\mu=1}^n$ and $\{f'^\mu\}_{\mu=1}^n$.

We can expand $f'^\mu = A^\mu{}_\nu f^\nu$ and $e'_\mu = B^\nu{}_\mu e_\nu$ for some $A^\mu{}_\nu, B^\nu{}_\mu \in \mathbb{R}$

but
$$\delta^\mu_\nu = f'^\mu(e'_\nu) = A^\mu{}_\tau f^\tau(B^\sigma{}_\nu e_\sigma)$$
f linear map
$$= A^\mu{}_\tau B^\sigma{}_\nu f^\tau(e_\sigma) = A^\mu{}_\tau B^\sigma{}_\nu \delta^\tau_\sigma$$
$$= A^\mu{}_\sigma B^\sigma{}_\nu = \delta^\mu_\nu \text{ identity matrix}$$

answering the question: how do general basis vectors transform?

Thus $B^\mu{}_\nu = (A^{-1})^\mu{}_\nu$.

we can see our previous results for coord. basis transforms indeed satisfy this (?)

If $e_\mu = \left(\frac{\partial}{\partial x^\mu}\right)_p$ and $e'_\mu = \left(\frac{\partial}{\partial x'^\mu}\right)_p$

We've already seen $A^\mu{}_\nu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)_{\phi(p)}$, $B^\mu{}_\nu = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)_{\phi(p)}$

Which indeed satisfy $A^\mu{}_\sigma B^\sigma{}_\nu = \delta^\mu{}_\nu$ by the chain rule.

A change of bases induces a transformation of tensor components.

note also easy to show how components wrt general coord bases transform (e.g. plug into $X^\mu e_\mu = X'^\nu e'_\nu$ as before) we find
$$X'^\mu = A^\mu{}_\nu X^\nu$$
$$\eta'_\mu = (A^{-1})^\nu{}_\mu \eta_\nu$$
↑ how components of (1,0)-vector transform under an arbitrary change of basis

E.g. if T is a $(1,1)$-tensor
$$T^\mu{}_\nu = T(f^\mu, e_\nu)$$
$$T'^\mu{}_\nu = T(f'^\mu, e'_\nu) = T(A^\mu{}_\sigma f^\sigma, (A^{-1})^\tau{}_\nu e_\tau)$$
$$= A^\mu{}_\sigma (A^{-1})^\tau{}_\nu T(f^\sigma, e_\tau) = A^\mu{}_\sigma (A^{-1})^\tau{}_\nu T^\sigma{}_\tau$$

(is also easy to show components of (2,1) tensor transform as $T'^{\mu\nu}{}_\rho = A^\mu{}_\sigma A^\nu{}_\tau (A^{-1})^\lambda{}_\rho T^{\sigma\tau}{}_\lambda$)

TENSOR OPERATIONS

Given an (r,s)-tensor, we can form an $(r-1,s-1)$-tensor by contraction.

i.e. show how it acts on vector & covector

For simplicity assume T is a $(2,2)$-tensor. Define a $(1,1)$-tensor S by
$$S(\omega, X) = T(\omega, f^\mu, X, e_\mu) \quad (*)$$

To see this is independent of the choice of basis:
$$T(\omega, f'^\mu, X, e'_\mu) = T(\omega, A^\mu{}_\sigma f^\sigma, X, (A^{-1})^\tau{}_\mu e_\tau)$$
$$= A^\mu{}_\sigma (A^{-1})^\tau{}_\mu T(\omega, f^\sigma, X, e_\tau)$$
$$= \delta^\tau_\sigma T(\omega, f^\sigma, X, e_\tau) = T(\omega, f^\tau, X, e_\tau) = S(\omega, X)$$

So (*) does not depend on the choice of basis. S and T have components related by

$$S^{\mu}{}_{\nu} = T^{\mu\sigma}{}_{\nu\sigma}$$

note to self: def. S (a (1,1) tensor) as a (?,2) tensor w/ sh upstairs & downstairs index contracted. This def holds in any basis we can write $S^a{}_b = T^{ac}{}_{bc}$

In any basis in AIN we write $S^a{}_b = T^{ac}{}_{bc}$

note to self: → b/c upstairs gives factor A which cancels w/ factor A^{-1} from downstairs)

Generalise to contract any upstairs index with any downstairs index in a general (r,s)-tensor.

→ (note e.g. $S^{bc}{}_e = T^{abc}{}_{de}$, $S^{ab}{}_c = T^{abd}{}_{cd}$ } there are diff ways to contract $T^{abc}{}_{de}$)

Another way to make new tensors from old is to form the $\boxed{\text{tensor product.}}$

If S is a (p,q) tensor and T is an (r,s)-tensor then $S \otimes T$ is a $(p+r, q+s)$ tensor:

outer product

$$S \otimes T(\omega^1, \dots, \omega^p, \eta^1, \dots, \eta^r, X_1, \dots, X_q, Y_1, \dots, Y_s)$$

$\eta^i, \omega^i \in T_p^*M$
$X_i, Y_i \in T_pM$

equivalent definitions

$$= S(\omega^1, \dots, \omega^p, X_1, \dots, X_q)\, T(\eta^1, \dots, \eta^r, Y_1, \dots, Y_s)$$

In ~~AIN~~ $(S \otimes T)^{a_1, \dots, a_p, b_1, \dots, b_r}{}_{c_1, \dots, c_q, d_1, \dots, d_s} = S^{a_1, \dots, a_p}{}_{c_1, \dots, c_q}\, T^{b_1, \dots, b_r}{}_{d_1, \dots, d_s}$

show that they're equivalent?: need to show that this def does not depend on choice of basis I think

Exercise: for any $(1,1)$-tensor, T, in a basis we have $T = T^{\mu}{}_{\nu}\, e_{\mu} \otimes f^{\nu}$

(→ [show both sides acting on (η, X) give the same thing?)

The final tensor operations we require are $\boxed{\text{(anti-)symmetrisation}}$

If T is a $(0,2)$-tensor, we can define two new tensors

$$S(X,Y) := \tfrac{1}{2}\big(T(X,Y) + T(Y,X)\big)$$

$$A(X,Y) := \tfrac{1}{2}\big(T(X,Y) - T(Y,X)\big)$$

In AIN $S_{ab} = \tfrac{1}{2}(T_{ab} + T_{ba})$ we write $S_{ab} = T_{(ab)}$

$A_{ab} = \tfrac{1}{2}(T_{ab} - T_{ba})$ $A_{ab} = T_{[a,b]}$

These operations can be applied to any pair of matching indices in a more general tensor

e.g. $$T^{a(bc)}{}_{de} := \tfrac{1}{2}\left(T^{abc}{}_{de} + T^{acb}{}_{de}\right) \quad \text{etc.}$$

We can (anti-)symmetrise over more than two indices

* To symmetrise over n indices, sum over all permutations of the indices and divide by n!
* To anti-symmetrise over n indices sum over all permutations weighted by sign $\begin{pmatrix} \text{even} = + \\ \text{odd} = - \end{pmatrix}$ and divide by n!

e.g. $T^{(abc)} := \frac{1}{3!}\left(T^{abc} + T^{bca} + T^{cab} + T^{acb} + T^{cba} + T^{bac}\right)$

$T^{[abc]} := \frac{1}{3!}\left(T^{abc} + T^{bca} + T^{cab} - T^{acb} - T^{cba} - T^{bac}\right)$

To exclude indices from (anti-)symmetrisation, use vertical lines

e.g. $T^{(a|b|c)} = \frac{1}{2}\left(T^{abc} + T^{cba}\right)$

→ notes: exercise: show that $T^{(ab)}X_{[a|cd|b]} = 0$
↳ attempt $T^{(ab)}X_{[a|cd|b]} = \frac{1}{2}T^{(ab)}X_{acdb} - \frac{1}{2}T^{(ab)}X_{bcda}$
since $T^{(ab)}$ symmetric $= \frac{1}{2}T^{(ab)}X_{acdb} - \frac{1}{2}T^{(ba)}X_{bcda}$
then relabel dummy indices in 2nd term $= \frac{1}{2}T^{(ab)}X_{acdb} - \frac{1}{2}T^{(ab)}X_{acdb}$
$= 0$ //

TENSOR BUNDLES

(not super relevant)

The space of (r,s)-tensors at a point p is the vector space $(T^r_s)_p M$, these can be glued together to form the bundle of (r,s)-tensors

$$T^r_s M = \bigcup_{p \in M} \{p\} \times (T^r_s)_p M$$

If (O, ϕ) is a coordinate chart on M, set

$$\tilde{O} = \bigcup_{p \in O} \{p\} \times (T^r_s)_p M \subset T^r_s M$$

$$\tilde{\phi}(p, S_p) = (\phi(p), \underbrace{S^{\mu_1, \dots, \mu_r}{}_{\nu_1, \dots, \nu_s}}_{\text{components of } S \text{ w.r.t. coordinate basis}})$$

(this stuff isn't in notes)

$T^r_s M$ is a manifold, with a natural smooth map $\pi : T^r_s M \to M$ such that $\pi(p, S_p) = p$.

so far looked at stuff defined at a point but in physics we want to consider how stuff varies in spacetime → introduce concept of a field

An (r,s)-TENSOR FIELD is a smooth map $T : M \to T^r_s M$ such that $\pi \circ T = \mathrm{id}$.

If (O, ϕ) is a coordinate chart on M then

$$\tilde{\phi} \circ T \circ \phi^{-1}(x) = (x, T^{\mu_1, \dots, \mu_r}{}_{\nu_1, \dots, \nu_s}(x))$$

which is smooth provided the components $T^{\mu_1, \dots, \mu_r}{}_{\nu_1, \dots, \nu_s}(x)$ are smooth functions of x.

$\boxed{\text{SPECIAL CASE}}$ If $T^r_s M = T^1_0 M \cong TM$ ← the tangent bundle → the tangent spaces T_pM glued together as p varies)

→ vector field is map from M to TM!

The tensor field is called a vector field in a local coord. patch, if X is a vector field, we can write

$$X(p) = (p, X_p) \quad \text{with} \quad X_p = \underbrace{X^\mu(x)}_{\text{smooth}} \left(\frac{\partial}{\partial x^\nu}\right)_p$$

↑ point on M ; ↑ vector at point p ; (p, X_p) is an element of TM!

← (we this / require X_p to be smooth by way of / be smooth, by requiring vector field so require $X^\mu(x)$ smooth (since $\frac{\partial}{\partial x^\nu}$ always smooth))

slight abuse of notation

In particular $\frac{\partial}{\partial x^\mu}$ are always smooth (but only defined locally)

(long) a vector field is a smooth assignment of a tangent vector X_p to each point $p \in M$
$(X(f))(p) = X_p(f)$
so if you give it a fn, it spits back another fn.
maps $p \mapsto X_p(f)$
is smooth if $X(f)$ smooth for any f

tensor bundle is gluing together the spaces of tensors at each point p as p varies
a tensor field is constructed by picking a member of

TM is a set of ordered pairs (p, X_p) with $p \in M$, $X_p \in T_pM$
so vector field $X(p) = (p, X_p)$ maps $M \to TM$ (provided $X^\mu(x)$ is smooth)

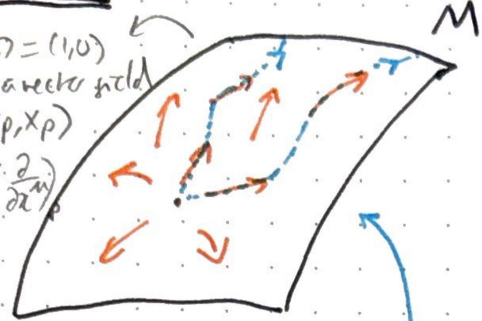
aka tangent vector space T_pM is the set of all tangent vectors to curves at p
vector bundle (tangent bundle) is gluing together all these spaces T_pM as p varies
vector field picks out a member of T_pM for each point in a smooth manner

Lecture 6

23.10.24

RECAP
$I_g(r,s) = (1,0)$
we get a vector field $X(p) = (p, X_p)$
$X = X^\mu(x)\left(\frac{\partial}{\partial x^\mu}\right)_p$
w/ this smooth

assign to every point a 'velocity' like a fluid velocity field



A vector field can act on a function $f : M \to \mathbb{R}$ ~~by~~ to give a new function Xf by

$$Xf(p) = X_p(f)$$

In coordinates

$$Xf(p) = X^\mu(\phi(p)) \left.\frac{\partial F}{\partial x^\mu}\right|_{\phi(p)}$$

INTEGRAL CURVES

Given a vector field X on M, we say a curve $\lambda : I \to M$ (interval) is an integral curve of X if its tangent at every point is X. i.e. denote the tangent vector to λ at t by

$\frac{d\lambda}{dt}(t)$ then, $(\dagger)\ \frac{d\lambda}{dt}(t) = X_{\lambda(t)} \quad \forall t \in I$

note to self: remember tangent vector at p was $X_p(f) = \left.\frac{dx^\mu}{dt}\right|_{t=0} \left(\frac{\partial}{\partial x^\mu}\right)_p f$ (components) where $\phi(p) = (x^1(p), \dots, x^n(p))$ and $x^\mu(t) = x^\mu(\lambda(t))$.

Through each point p, an integral curve passes, unique up to extension / shift of the parameter.

justify:

To see this, pick a chart ϕ with $\phi = (x^1, \dots, x^n)$ and $\phi(p) = 0$. In this chart $(\dagger)$ becomes

note to self: our vector field has $X_p = X^\mu(x)\left(\frac{\partial}{\partial x^\mu}\right)_p$

$$(*)\ \frac{dx^\mu}{dt}(t) = X^\mu(x(t)) \qquad x^\mu(t) = x^\mu(\lambda(t))$$

So arrive at system of diff eqns with initial condition ⇒ solvable

the components of the vector field in a given coord system X^μ are given by the eq. RHS fn. sensible fn of unknowns then solvable

Assuming wlog that $\lambda(0) = p$ we ~~see that~~ get an initial condition $(**)\ x^\mu(0) = 0$

don't do details here but basically

Standard ODE theory gives that $(*)$ with $(**)$ has a solution unique up to extension.

therefore there is a unique integral curve of X through any point p (up to extension)

examples notes: in chart $\phi = (x^1, \dots, x^n)$; consider $X = \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}$ & let $p = (0, \dots, 0)$
components $\frac{dx^1}{dt} = 1$, $\frac{dx^2}{dt} = x^1$ solve these & ICs
$x^1 = t$, $x^2 = t^2/2$ ⇒ integral curve is $t \mapsto \phi^{-1}(t, t^2/2, 0, \dots)$

COMMUTATORS

(see tony notes for more examples of integral curves)

Suppose X and Y are two vector fields and $f : M \to \mathbb{R}$ is smooth then $X(Y(f))$ is a smooth fn. Is it of the form $K(f)$ for some vector field K? No, because

$$X(Y(fg)) = X(fY(g) + gY(f)) = X(fY(g)) + X(gY(f))$$
$$= fX(Y(g)) + gX(Y(f)) + X(f)Y(g) + X(g)Y(f)$$

(leibniz requires $X(Y(fg)) = fX(Y(g)) + X(Y(f))g$)

So leibniz doesn't hold. But if we consider

$[X,Y](f) := X(Y(f)) - Y(X(f))$ then Leibniz does hold.

cancel by this commutator

In fact $[X,Y]$ defines a vector field

we can build a new vector by taking the commutator $[X,Y]$ which acts on functions as

the commutator aka the lie bracket

To see this use coordinates

vector fields act on $C^\infty(M)$

$$[X,Y](f) = X\left(Y^\nu \frac{\partial F}{\partial x^\nu}\right) - Y\left(X^\mu \frac{\partial F}{\partial x^\mu}\right)$$

since $X(f)(p) = X_p(f) = X^\mu \frac{\partial F}{\partial x^\mu}$ components

$$= X^\mu \frac{\partial}{\partial x^\mu}\left(Y^\nu \frac{\partial F}{\partial x^\nu}\right) - Y^\nu \frac{\partial}{\partial x^\nu}\left(X^\mu \frac{\partial F}{\partial x^\mu}\right)$$

terms cancel (use Leibniz rule of X, Y)

$$= \cancel{X^\mu Y^\nu \frac{\partial^2 F}{\partial x^\mu \partial x^\nu}} - \cancel{Y^\nu X^\mu \frac{\partial^2 F}{\partial x^\nu \partial x^\mu}} + X^\mu \frac{\partial Y^\nu}{\partial x^\mu}\frac{\partial F}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\frac{\partial F}{\partial x^\mu}$$

equals?

$$= \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\right)\frac{\partial F}{\partial x^\nu} = [X,Y]^\nu \frac{\partial F}{\partial x^\nu}$$

Where $[X,Y]^\nu = X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}$ are the components of the commutator. (in this basis) (notes: the RHS is a vector field hence $[X,Y]$ is a vector field) (note typo: see real notes for example)

Since f arbitrary: (we can drop it & write:) $[X,Y] = [X,Y]^\nu \frac{\partial}{\partial x^\nu}$, valid only in a coordinate basis. (i.e. can't write this in gen) (i.e. have found a way of generating a vector field from two other vector fields)

METRIC TENSOR

• talked about manifolds so far but not geometry stuff e.g. notions of length & angle → do this now

We're familiar from Euclidean geometry (and special relativity) with the fact that the fundamental object when talking about distance and angles (time intervals / rapidity) is an inner product between vectors.

E.g. * $\underline{x}\cdot\underline{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$ $\mathbb{R}^3$ w/ Euclidean geometry

* $X\cdot Y = -X^0 Y^0 + X^1 Y^1 + X^2 Y^2 + X^3 Y^3$ $\mathbb{R}^{3+1}$ w/ Minkowski geometry (familiar from special rel.) $(c=1)$

→ want to generalise this to curved spacetime: (multilinear map that eats 2 vectors) (scalar product: maps a pair of vectors to a scalar $T_pM \times T_pM \to \mathbb{R}$)

DEF: A metric tensor at $p \in M$ is a $(0,2)$-tensor g satisfying
i) g is symmetric $g(X,Y) = g(Y,X)$ $\forall X,Y \in T_pM$ $(g_{ab} = g_{ba})$
ii) g is non-degenerate $g(X,Y) = 0$ for all $Y \in T_pM$ iff $X = 0$

NOTATION: sometimes write $g(X,Y) = \langle X,Y\rangle = \langle X,Y\rangle_g = X\cdot Y$

"if you don't know what that is you can just believe that this is true"

By adapting the Gram-Schmidt algorithm we can always find a basis $\{e_\mu\}_{\mu=1}^n$ for T_pM such that

$$g(e_\mu, e_\nu) = \begin{cases} 0 & \mu \neq \nu \\ +1 \text{ or } -1 & \mu = \nu \end{cases}$$ ← orthonormal basis

i.e. $g_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ & & +1 & \\ 0 & & & +1 \end{pmatrix}$ The number of -1's and $+1$'s appearing does not depend on choice of basis (Sylvestre's law of inertia) and is called the signature

- If g has signature $++\dots+$ we say it is **RIEMANNIAN**
- If g has signature $-++\dots+$ we say it is **LORENTZIAN**

DEF: A Riemannian (resp. Lorentzian) manifold is a pair (M,g) where M is a manifold and g is a Riemannian (resp. Lorentzian) metric tensor field

(on riemannian, use metric to def. lengths & angles)

notes: Lorentzian manifold sometimes also called a spacetime

remarks:

* On a Riemannian manifold the norm of a vector $X \in T_pM$ is

$$|X| = \sqrt{g(X,X)}$$

* The angle between $X, Y \in T_pM$ is given by

$$\cos\theta = \frac{g(X,Y)}{|X||Y|}$$

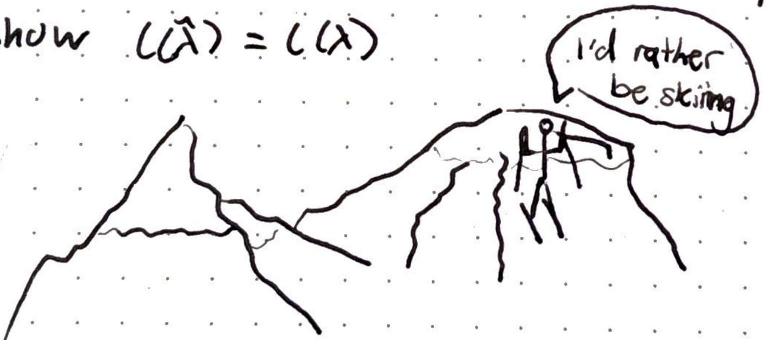
* The length of a curve $\lambda: (a,b) \to M$ is given by

$$l(\lambda) = \int_a^b \left| \frac{d\lambda}{dt}(t) \right| dt$$

EXERCISE! If $\tau: (c,d) \to (a,b)$ with $\frac{d\tau}{du} > 0$ (u's a?) $\tau(c)=a$ $\tau(d)=b$ then $\hat{\lambda} = \lambda \circ \tau : (c,d) \to M$ is a representation of λ

(if reparametrise curve do i get the same ans?)

show $l(\hat{\lambda}) = l(\lambda)$



i.e. our def. of length is independent of parametrisation

attempt: $l(\lambda) = \int_a^b \left|\frac{d\lambda}{dt}\right| dt$

$l(\hat{\lambda}) = \int_c^d \frac{d\hat{\lambda}}{dt} dt$

use $\hat{\lambda} = \lambda(\tau(t)) \Rightarrow l(\hat{\lambda}) = \int_c^d \left| \frac{d\lambda}{d\tau} \frac{d\tau}{dt} \right| dt$

$= \int_a^b \left| \frac{d\lambda}{d\tau} \right| d\tau = l(\lambda)$ // ?

so our def. of length is indep. of parametrisation

→ notes: can also show that $Y^a = \frac{dt}{d\tau} X^a$ if Y is tangent vec to $\hat{\lambda}$ and X to λ.

attempt: components $Y^\mu = \frac{dx^\mu}{d\tau}$, $X^\mu = \frac{dx^\mu}{dt}$

$\Rightarrow Y^\mu = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \frac{dt}{d\tau} X^\mu$ // ?

true for all bases so $\mu \to a$?

Lecture 7

25.10.24

(reminder: can write (0,2) tensor as $T = T_{\mu\nu} f^\mu \otimes f^\nu$ (show this as exercise) In coordinates nice basis $f^\mu = (dx^\mu)$)

In a coordinate basis, $g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$. We often write $dx^\mu dx^\nu := \frac{1}{2}(dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu)$ and by convention write $g = ds^2$ so that

$$g = ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

(note to self: we defined g to be symmetric anyway so what's the point of this?)

(notes just say $g_{\mu\nu} dx^\mu d^\nu$ is an abbreviation $g_{\mu\nu} dx^\mu \otimes dx^\nu$)

Examples:

i) $\mathbb{R}^n$ $= \{(x^1, \dots, x^n)\}$ with

$$g = ds^2 = (dx^1)^2 + (dx^2)^2 + \dots + (dx^n)^2 = \delta_{\mu\nu}\, dx^\mu dx^\nu$$

flat (Riemannian example)

$(\mathbb{R}^n, g)$ is called **Euclidean** space. Any chart covering $\mathbb{R}^n$ in which the metric takes this form (i.e. diag(1,1,...,1)) is called CARTESIAN.

ii) $\mathbb{R}^{1+3} = \{(x^0, x^1, x^2, x^3)\}$ with

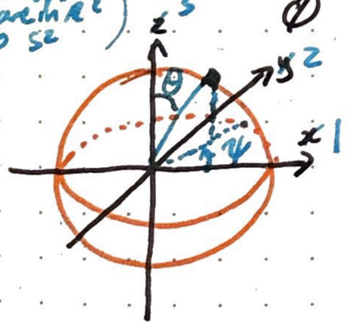
flat (Lorentz example)

$$g = ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$
$$= \eta_{\mu\nu}\, dx^\mu dx^\nu \qquad \left(\eta_{\mu\nu} = \begin{cases} -1 & \mu = \nu = 0 \\ 1 & \mu = \nu \neq 0 \\ 0 & \text{otherwise} \end{cases} \right)$$

$(\mathbb{R}^{3+1}, \eta)$ Is **Minkowski** space. A coordinate chart covering $\mathbb{R}^{1+3}$ in which the metric takes this form (i.e. diag(-1,1,1,1)) is an **inertial frame** such that

iii) On $S^2 = \{\underline{x} \in \mathbb{R}^3 \mid |\underline{x}| = 1\}$ define a chart by (curved example)

(map an open square in $\mathbb{R}^2$ into S^2)



$$\phi^{-1} : (0,\pi) \times (-\pi, \pi) \to S^2$$

$$(\theta, \psi) \longmapsto (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$$

(no? don't think so / other way)

In this chart the round metric is

$$g = ds^2 = d\theta^2 + \sin^2\theta\, d\psi^2$$

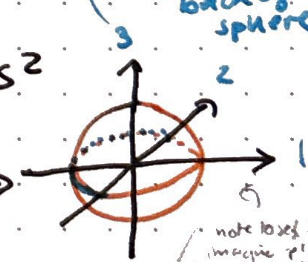
set of points where all satisfied (:)

This covers $S^2 \setminus \{ \underbrace{|\underline{x}| = 1}_{\text{surface}}, \underbrace{x^2 = 0}_{\text{plane } x^2=0}, \underbrace{x^1 \leq 0}_{x^1 \text{ -ve}} \}$

(misses poles & cuts out of back of sphere)

To cover the rest let $\tilde{\phi}^{-1} : (0,\pi) \times (-\pi, \pi) \to S^2$

$$(\theta', \psi') \mapsto (-\sin\theta'\cos\psi', \cos\theta', \sin\theta'\sin\psi')$$



which covers $S^2 \setminus \{ |\underline{x}| = 1 \text{ and } x^3 = 0 \text{ and } x^1 \geq 0 \}$

(note to self: imagine picture before but relabel axes $x \to -x$ and $y \leftrightarrow z$)

setting $g = d\theta'^2 + \sin^2\theta'\, d\psi'^2$ defines a metric on all of S^2. (Check this by finding coordinate transform from unprimed to primed then on region of overlap the metric defined in each set of coords is the same tensor)

("painful" to do this)

... to do this?

(b/c upstairs indices transform w/ inv. transformation basis to down stairs)

Since g_{ab} is non-degenerate, it is invertible as a matrix in any basis. We can check the inverse defines a symmetric $(2,0)$-tensor $\boxed{g^{ab}}$ satisfying

(tong: $\hat{g} = g^{\mu\nu}\partial_\mu \otimes \partial_\nu$)

$$g^{ab} g_{bc} = \delta^a_c$$

Example: In the ϕ coordinates of the S^2 example

(tong: physicists say that we use the metric to lower the index from ... to ... in their heart, ...)

$$g^{\mu\nu} = \left(1, \frac{1}{\sin^2\theta}\right)$$

("a metric determines a natural isomorphism between vectors & covectors")

An important property of the metric is that it induces a canonical identification of T_pM and T_p^*M.

(i.e. given X^a we can get X_a & vice versa)

(covector i get by lowering w/ metric)

* Given $X^a \in T_pM$ (vector) we define a covector $g_{ab}X^b = X_a$ and
* given $\eta_a \in T_p^*M$ (covector) we define a vector $g^{ab}\eta_b = \eta^a$

In $(\mathbb{R}^3, \delta)$ ("standard inner product") Euclidean space we often do this without realising. (metric & its inverse are the identity in these coordinates: don't really distinguish covectors & vectors)

More generally, this allows us to raise tensor indices with g^{ab} and lower with g_{ab}.

Example: If $T^{ab}{}_c$ is a $(2,1)$-tensor then $T_a{}^{bc}$ is the $(2,1)$-tensor given by

$$T_a{}^{bc} = g_{ad}\, g^{ce}\, T^{db}{}_e \qquad \text{etc.}$$

LORENTZIAN SIGNATURE

→ "relevant b/c this is what we assume spacetime to have" (manifold n+1 dim.)

(notes: on Lorentzian manifold, we take basis indices μ, ν to run from 0 to n-1)

In Lorentzian signature, indices $0, 1, \ldots, n$, at any point p in a Lorentzian manifold we can find a basis (orthonormal)

(tong: at any point on general Lorentzian manifold we can find an orthonormal basis $\{e_\mu\}$ for T_pM s.t. locally, the metric looks like the Minkowski metric)

$$\{e_\mu\}_{\mu=0}^{n} \quad \text{s.t.} \quad g(e_\mu, e_\nu) = \eta_{\mu\nu} \equiv \mathrm{diag}(-1, 1, \ldots, 1)$$

(can always diagonalise at a point s.t. looks like Minkowski space)

This basis is not unique, if $e_\mu' = (A^{-1})^\nu{}_\mu e_\nu$ is another such basis then (at point p)

$$\eta_{\mu\nu} = g(e_\mu', e_\nu') = (A^{-1})^\sigma{}_\mu (A^{-1})^\tau{}_\nu\, g(e_\sigma, e_\tau) = (A^{-1})^\sigma{}_\mu (A^{-1})^\tau{}_\nu\, \eta_{\sigma\tau}$$

$$\Rightarrow A^\mu{}_\kappa A^\nu{}_\rho\, \eta_{\mu\nu} = \eta_{\kappa\rho}$$

(defining eq. of a Lorentz transform)

which is the condition that $A^\mu{}_\nu$ is a **LORENTZ TRANSFORMATION** (cf. special relativity)

(notes: different orthonormal bases at p are related by Lorentz transformations)

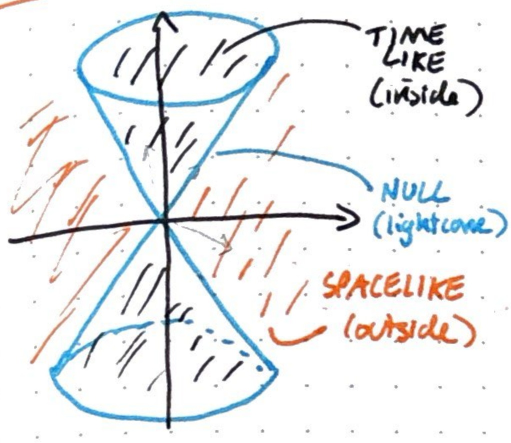
(also reminder: we saw vectors transform as $X^\mu = A^\mu{}_\nu X^\nu$ which is the same as the transformation law of components in special relativity)

The tangent space at p has $\eta_{\mu\nu}$ as metric tensor (in this basis)

(i.e. viewed locally we recover structure of special relativity) so has the structure of Minkowski space in particular

on a Lorentzian manifold (M,g)

DEF: $X \in T_pM$ is SPACELIKE / NULL/LIGHTLIKE / TIMELIKE IF $g(X,X) > 0$ / $g(X,X) = 0$ / $g(X,X) < 0$



(every
whilst a vector has to be one of these a curve can be none → start off one become another)

notes: a causal vector is timelike or null

note the causal vectors split into two disconnected sets

A curve $\lambda : I \to M$ in a Lorentzian manifold is spacelike or timelike or null if the tangent vector is everywhere spacelike or timelike or null resp.

A spacelike curve has a well-defined LENGTH (given by the same formula as in Riemannian case.

For a timelike curve $\lambda : (a,b) \to M$, the relevent quantity is the PROPER TIME

$$\tau(\lambda) = \int_a^b \sqrt{-g_{ab} \frac{d\lambda^a}{du} \frac{d\lambda^b}{du}}\, du$$

(u is parameter along curve)

If $g_{ab} \frac{d\lambda^a}{du} \frac{d\lambda^b}{du} = -1$ for all u, then λ is parametrised by proper time.

In this case we call the tangent vector $u^a := \frac{d\lambda^a}{du}$ the 4-VELOCITY of λ.

CURVES OF EXTREMAL PROPER TIME

Suppose $\lambda : (0,1) \to M$ is timelike, satisfies $\lambda(0) = p$, $\lambda(1) = q$, and extremises proper time among all such curves. This is a variational problem associated to (in a coordinate chart)

$$\tau[\lambda] = \int_0^1 G(x^\mu(u), \dot{x}^\mu(u))\, du \qquad \left(\dot{} \equiv \frac{d}{du} \text{ here} \right)$$

with $G(x^\mu(u), \dot{x}^\mu(u)) = \sqrt{-g_{\mu\nu}(x(u))\, \dot{x}^\mu(u)\, \dot{x}^\nu(u)}$

EULER LAGRANGE EQN is (†) $\left[\frac{d}{du}\left(\frac{\partial G}{\partial \dot{x}^\mu} \right) = \frac{\partial G}{\partial x^\mu} \right]$, we can compute:

$$\frac{\partial G}{\partial \dot{x}^\mu} = -\frac{1}{G} g_{\mu\nu} \dot{x}^\nu \qquad \frac{\partial G}{\partial x^\mu} = -\frac{1}{2G} \underbrace{\frac{\partial}{\partial x^\mu}(g_{\sigma\tau})}_{= g_{\sigma\tau,\mu}} \dot{x}^\sigma \dot{x}^\tau = -\frac{1}{2G} g_{\sigma\tau,\mu} \dot{x}^\sigma \dot{x}^\tau$$

Lecture 8

28.10.24

CURVES OF EXTREMAL PROPER TIME cont.

(before had arbitrary parametrisation u)

* Now fix parametrisation so curve is parametrised by proper time τ. Doing this

$$\frac{dx^\mu}{d\tau} = \dot{x}^\mu \frac{du}{d\tau} \quad \text{and} \quad -1 = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

chain rule — condition τ is proper time along curve

alternatively: from definition in notes $\frac{d\tau}{du} = \sqrt{-(g_{ab}\dot{x}^a\dot{x}^b)}$; write this in coordinates: $\left(\frac{d\tau}{du}\right)^2 = -g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} = G^2$ $\Rightarrow \frac{d\tau}{du} = G$, so $\frac{d}{du} = G\frac{d}{d\tau}$

* Deduce $-1 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\left(\frac{du}{d\tau}\right)^2 \Rightarrow \frac{du}{d\tau} = 1/G \Rightarrow \frac{1}{G}\frac{d}{du} = \frac{d}{d\tau}$

* Returning to (†) (EL) we find

(use $\frac{d}{d\tau}g_{\mu\nu} = \frac{dx^\rho}{d\tau}\frac{d}{dx^\rho}g_{\mu\nu}$)

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) = \frac{1}{2}g_{\nu\rho,\mu}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$$

which we can solve

$$\Rightarrow g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + g_{\mu(\nu,\rho)}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} - \frac{1}{2}g_{\sigma\rho,\mu}\frac{dx^\sigma}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

multiplied by $\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$ symmetric in ν,ρ

note to self: → reminder: when we contract a symmetric tensor with an antisymmetric in the same indices we get zero since the antisymmetric part of $g_{\mu\nu,\rho}$ is zero; we can replace it with just the symmetrised part $g_{\mu(\nu,\rho)}$.

symmetrise & contract w/ inverse metric

* Thus $$\boxed{\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\mu\rho}\frac{dx^\mu}{d\tau}\frac{dx^\rho}{d\tau} = 0} \quad (*)$$

← the geodesic equation

* where $\Gamma^\nu_{\mu\sigma} := \frac{1}{2}g^{\nu\sigma}\left(g_{\mu\sigma,\rho} + g_{\sigma\rho,\mu} - g_{\nu\rho,\sigma}\right)$ are the

μ?

CHRISTOFFEL SYMBOLS of g

comments

NOTE:

- $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$ (symmetric in downstairs indices)
- $\Gamma^\mu_{\nu\rho}$ are not tensor components (cf exercise 2.1)
- We can solve (*) with standard ODE theory, solutions are called GEODESICS
- The same equation governs curves of extremal length in a Riemannian manifold (or spacelike curves in a Lorentzian manifold) parametrised by arc length

↳ i.e. extremise $L(\lambda) = \int du\sqrt{g_{ab}\dot{x}^a\dot{x}^b}$ vs $\tau(\lambda) = \int du\sqrt{-g_{ab}\dot{x}^a\dot{x}^b}$ (minus)

EXERCISE! Show that (*) can be obtained as the Euler-Lagrange equation for the Lagrangian

$$L = -g_{\mu\nu}(x(\tau))\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)$$

$$\frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial\dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0$$

↖ an easier way to derive the geodesic eqn or christoffel symbols

strongly recommend doing this exercise!!

(mnemonic? for finding geodesic eq / christoffel symbols?)

(cf. it says we want to extremise τ we can just use the Euler-Lagrange for L which is that gives the same eqns)

EXAMPLES:

(i.e. the components of the metric are constant)

1) In Minkowski space in an inertial frame $g_{\mu\nu} = \eta_{\mu\nu}$ so $\Gamma^{\nu}_{\mu\rho} = 0$ and geodesic equation is

$$\frac{d^2 x^\mu}{d\tau^2} = 0$$

easy to solve → (solutions are straight lines)

geodesics are straight lines

notes: in Minkowski spacetime, timelike curves of extremal proper time are straight lines. it can be shown that these lines maximise the proper time between two points (in a general spacetime this is only true locally). only in the neighbourhood of p

2) The Schwarzschild metric in Schwarzschild coords. is a metric on $M = \mathbb{R}_t \times (2m, \infty)_r \times S^2_{\theta,\phi}$ given by (t, r, θ, ϕ)

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

i.e. this means its defined on the reals parametrised by t, on the interval (2m,∞) parametrised by r & on the 2-sphere parametrised by θ, φ

metric does not depend on t explicitly

$$f = 1 - \frac{2m}{r} \quad \text{then}$$

(emphasize t is just a (timelike) coordinate in spacetime, τ is a parameter along curve)

$$L = f\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{f}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2$$

(note to self: 4 eqns, one for each $x^\mu = t, r, \theta, \phi$. consider t:)

E-L equation for $t(\tau)$ is $\frac{d}{d\tau}\left(\frac{\partial L}{\partial t'}\right) = \frac{\partial L}{\partial t}$ $\quad t' = \frac{dt}{d\tau}$

$$\Rightarrow 2\frac{d}{d\tau}\left(f\frac{dt}{d\tau}\right) = 0$$

$$\Rightarrow f\frac{d^2t}{d\tau^2} + \frac{df}{dr}\left(\frac{dr}{d\tau}\right)\left(\frac{dt}{d\tau}\right) = 0$$

$$x^\mu = (\overset{0}{t}, \overset{1}{r}, \overset{2}{\theta}, \overset{3}{\phi})$$

remember symmetry! $(\Gamma^0_{10} + \Gamma^0_{01}) = \frac{1}{f}\frac{df}{dr}$

Compare to (*) to see $\Gamma^0_{1\,0} = \Gamma^0_{0\,1} = \frac{1}{2} \cdot \frac{1}{f}\frac{df}{dr}$

$$\Gamma^0_{\mu\nu} = 0 \quad \text{otherwise}$$

note doing this is often more efficient than using formula using metric & doing explicit computation

Rest of symbols (done on examples sheet) can be found from other EL equations

→ just consider each of the 4 el eq. and read off the Γ symbols. remember symmetry $\Gamma^0_{10} = \Gamma^0_{01}$ gives factor ½ and all symbols not seen are zero. (& remembering $g^{\mu\nu}$ is g^{-1})

COVARIANT DERIVATIVE

For a function $f : M \to \mathbb{R}$, we know that $\frac{df}{dx^\mu}$ are components of a covector $(df)_\mu$

For a vector field, we can't just differentiate components

b/c partial deriv of tensor field does not give another tensor field

EXERCISE: show that if V is a vector field then

$$T^\mu{}_\nu := \frac{dV^\mu}{d\tau^\nu}$$ are not components of a (1,1)-tensor

doing this you will find it is relatively to new coordinates I also get a term involving derivative of jacobian

(note: here $\frac{dV^\mu}{dx^\nu}$?) → components of tensor transform as $T'^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu} T^\rho{}_\sigma$

finding $T'^\mu{}_\nu = T(f'^\mu, e'_\nu)$ ... how f'^μ, e'_ν transform

components of vector transform as $V'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x)$

so find $T'^\mu{}_\nu = \frac{\partial}{\partial x'^\nu} V'^\mu$

$$\left[T'^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu} T^\rho{}_\sigma + \left(\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x'^\nu}\right) V^\rho\right]$$

(extra term: does not transf. as tensor!)

(I think)

DEF: A covariant derivative ∇ on a manifold M is a map sending X, Y smooth vector fields to a vector field $\nabla_X Y$ satisfying (X, Y, Z smooth v.fields, f, g functions)

want it to satisfy

i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$

note to self, i.e. for scalars covariant deriv is just partial deriv $\nabla_\mu g = e_\mu(g) = \frac{\partial g}{\partial x^\mu}$

ii) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$

note to self: $(1,1)$ tensor is a multilinear map $T: T_pM^* \times T_pM \to \mathbb{R}$ equivalently we can think of it as a linear map $T: T_pM \to T_pM$ i.e. $V^\mu \mapsto T^\mu{}_\nu V^\nu$ (?) eg?

leibniz type rule.

iii) $\nabla_X (fY) = f\nabla_X Y + (\nabla_X f) Y$ where $\nabla_X f := X(f)$ eg acting ∇ on functions by

since $\nabla_X Y$ is linear in X^μ so $\nabla_X Y$ is a vector $\in T_pM$ so map $T_pM \to T_pM \Rightarrow \nabla Y$ is $(1,1)$ tensor. (?)

Note i) implies that $\nabla Y : X \longmapsto \nabla_X Y$ is a linear map of T_pM to itself, so defines a $(1,1)$-tensor, the COVARIANT DERIVATIVE of Y $\equiv$ AFFINE CONNECTION

In AIN $(\nabla Y)^a{}_b = \nabla_b Y^a$ or $Y^a{}_{;b}$

notation: $\nabla_\nu Y$ shorthand for $\nabla_{e_\nu} Y$ and $\nabla_\mu Y^\nu$ shorthand for $(\nabla_\mu Y)^\nu$

tony says the semi colon notation is stupid (I agree.)

DEF: In a basis $\{e_\mu\}$ the CONNECTION COMPONENTS $\Gamma^\mu{}_{\nu\rho}$ are defined by

assuming basis $\equiv$ basis of smooth vector fields

$$(\nabla_\rho e_\nu =)\quad \nabla_{e_\rho} e_\nu = \Gamma^\mu{}_{\nu\rho}\, e_\mu$$

(note to self remember swaps $\rho \nu \leftrightarrow \nu\rho$)

These determine ∇

$$\nabla_X Y = \nabla_{X^\mu e_\mu}(Y^\nu e_\nu) \overset{(i)}{=} X^\mu \nabla_{e_\mu}(Y^\nu e_\nu) \overset{(ii),(iii)}{=} X^\mu \big(e_\mu(Y^\nu) e_\nu + Y^\sigma \nabla_{e_\mu} e_\sigma\big)$$

property

("skipped a couple of lines")

$$= (X^\mu e_\mu(Y^\nu) + \Gamma^\nu{}_{\sigma\mu} Y^\sigma X^\mu) e_\nu$$

Hence

$$(\nabla_X Y)^\nu = X^\mu (e_\mu(Y^\nu) + \Gamma^\nu{}_{\sigma\mu} Y^\sigma)$$

the νth component of $\nabla_X Y$

components not tensors but they have transformation law that compensates for fact $Y_{,\mu}$ is not a tensor

and $(\nabla_\mu Y)^\nu = Y^\nu{}_{;\mu} = e_\mu(Y^\nu) + \Gamma^\nu{}_{\sigma\mu} Y^\sigma$

In a coord basis $e_\mu = \frac{\partial}{\partial x^\mu}$ then $Y^\nu{}_{;\mu} = Y^\nu{}_{,\mu} + \Gamma^\nu{}_{\sigma\mu} Y^\sigma$

notes? $Y^\nu{}_{,\mu} = \frac{dY^\nu}{dx^\mu}$

but notice $\Gamma - \Gamma$ transforms as a tensor

$\Gamma^\nu{}_{\mu\sigma}$ are not components of a tensor

if $e_\nu = \Gamma^\mu{}_{\nu\rho} e_\mu$, $(1,2)$ tensor transforms as $T'^\mu{}_{\nu\rho} = (A^{-1})^\mu{}_\tau A^\lambda{}_\nu A^\sigma{}_\rho T^\tau{}_{\lambda\sigma}$

i.e. Γ not a tensor $\leftarrow$ $(1,2)$ tensor not 1st term is tensor but 2nd term not! $\leftarrow$ we find $\Gamma'^\mu{}_{\nu\rho} = (A^{-1})^\mu{}_\tau A^\sigma{}_\rho A^\lambda{}_\nu \Gamma^\tau{}_{\lambda\sigma} + (A^{-1})^\mu{}_\tau A^\sigma{}_\rho e_\sigma(A^\tau_\nu)$ $\leftarrow$ take $\Gamma'^\mu{}_{\nu\rho} e'_\mu = \nabla_{e'_\rho} e'_\nu$ and plug in e'_ν, e'_ρ in ehs

E.g. for η a tensor field we define $(\nabla_X \eta)(Y) := \nabla_X(\eta(Y)) - \eta(\nabla_X Y)$

(covector field?)

extend def to act on more general tensor fields?

since $\eta = \eta_\rho f^\rho$?

In components $(\nabla_X \eta)Y = X^\mu e_\mu(\eta_\sigma Y^\sigma) - \eta_\sigma (\nabla_X Y)^\sigma$

(since $\nabla_X Y$ is a vector we get f ease & just components of?)

$(\nabla_X \eta)Y = X^\mu(e_\mu(\eta_\nu) - \eta_\sigma \Gamma^\sigma{}_{\nu\mu})Y^\nu$

$$= X^\mu e_\mu(\eta_\sigma)Y^\sigma + \cancel{X^\mu \eta_\sigma e_\mu(Y^\sigma)} - \eta_\sigma(\cancel{X^\nu e_\nu(Y^\sigma)} + X^\nu \Gamma^\sigma{}_{\tau\nu} Y^\tau)$$

linear in $Y^\nu \Rightarrow \nabla_X \eta$ is a covector w components $(\nabla_X \eta)_\nu$

$(\nabla_X \eta)_\nu$ linear in $X^\mu \Rightarrow \nabla_\mu \eta$ is a $(0,2)$ tensor field w components $(\nabla_\mu \eta)_\nu = \eta_{\nu;\mu}$

$= e_\mu(\eta_\nu) - \eta_\sigma \Gamma^\sigma{}_{\nu\mu}$

$$= \underbrace{(e_\mu(\eta_\sigma) - \Gamma^\nu{}_{\sigma\mu}\eta_\nu) X^\mu Y^\sigma}_{\text{linear in } X, Y} \quad \therefore \nabla\eta \text{ is a tensor - } (0,2)$$

$$\nabla_\mu \eta_\sigma = e_\mu(\eta_\sigma) - \Gamma^\nu{}_{\sigma\mu}\eta_\nu =: \eta_{\sigma;\mu}$$

in coord basis $\eta_{\sigma;\mu} = \eta_{\sigma,\mu} - \Gamma^\nu{}_{\sigma\mu}\eta_\nu$

in coord basis

$\eta_{\nu;\mu} = \eta_{\nu,\mu} - \eta_\sigma \Gamma^\sigma{}_{\nu\mu}$

can use leibniz rule to find the basis components of ∇T for a general tensor

Lecture 9 — 30.10.24

(notice X^μ, ω_μ are not scalars since they transform under a change of basis vectors / coords (scalars don't) so $X^\mu{}_{;\nu} \neq X^\mu{}_{,\nu}$)

useful expressions we have shown (in coords):

$$\nabla_\nu X^\mu = X^\mu{}_{;\nu} = X^\mu{}_{,\nu} + \Gamma^\mu{}_{\sigma\nu} X^\sigma$$

$$\nabla_\nu \omega_\mu = \omega_{\mu;\nu} = \omega_{\mu,\nu} - \Gamma^\sigma{}_{\mu\nu}\omega_\sigma$$

$$\left(X^\mu{}_{;\nu} = e_\nu(X^\mu) + \Gamma^\mu{}_{\sigma\nu}X^\sigma \qquad \omega_{\mu;\nu} = e_\nu(\omega_\mu) - \Gamma^\sigma{}_{\mu\nu}\omega_\sigma \right)$$

not in coords

(easy not rigorous check is scalar: has no free indices or basis vectors)

Exercise: in a coordinate basis

$$T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s;\rho} = T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s,\rho} + \Gamma^{\mu_1}{}_{\sigma\rho}T^{\sigma\mu_2\dots\mu_r}{}_{\nu_1\dots\nu_s} + \dots + \Gamma^{\mu_r}{}_{\sigma\rho}T^{\mu_1\dots\mu_{r-1}\sigma}{}_{\nu_1\dots\nu_s} - \Gamma^{\sigma}{}_{\nu_1\rho}T^{\mu_1\dots\mu_r}{}_{\sigma\nu_2\dots\nu_s} - \dots - \Gamma^{\sigma}{}_{\nu_s\rho}T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_{s-1}\sigma}$$

(for every lower index: $-\Gamma$ term; upper index: $+\Gamma$ term)

show for (1,1): write (1,1) tensor as $T(\omega, X) = \omega_\mu X^\nu T^\mu{}_\nu$

→ we want to find $T^\mu{}_{\nu;\rho} = \nabla_\rho T^\mu{}_\nu$

(1) consider Leibniz

$\nabla_\rho T = \nabla_\rho(\omega_\mu X^\nu T^\mu{}_\nu) = (\nabla_\rho\omega_\mu)X^\nu T^\mu{}_\nu + \omega_\mu(\nabla_\rho X^\nu)T^\mu{}_\nu + \omega_\mu X^\nu(\nabla_\rho T^\mu{}_\nu)$

→ plug in our expressions for $\nabla_\rho\omega_\mu$ and $\nabla_\rho X^\nu$

(2) consider scalar deriv

since $\nabla_\rho T = \nabla_\rho(\omega_\mu X^\nu T^\mu{}_\nu)$

$= e_\rho(\ ) \overset{\text{coords}}{=} \partial_\rho(\omega_\mu X^\nu T^\mu{}_\nu) = (\partial_\rho\omega_\mu)X^\nu T^\mu{}_\nu + \omega_\mu(\partial_\rho X^\nu)T^\mu{}_\nu + \omega_\mu X^\nu(\partial_\rho T^\mu{}_\nu)$

[$T(\omega, X)$ is scalar: remember contracting (1,1) tensor w/ vector & covector gives scalar (or less rigorously we have no free indices or basis vectors in the brackets) so this is just covariant deriv of a scalar = normal deriv (in coords)]

(3) compare both expressions, cancel terms & simplify ⇒

$\nabla_\rho T^\mu{}_\nu = T^\mu{}_{\nu;\rho} = T^\mu{}_{\nu,\rho} + \Gamma^\mu{}_{\sigma\rho}T^\sigma{}_\nu - \Gamma^\sigma{}_{\nu\rho}T^\mu{}_\sigma$ //

- **Remark** If $T^a{}_b$ is a (1,1) tensor, then $T^a{}_{b;c}$ is a (1,2) tensor and we can take further covariant derivatives

$$(T^a{}_{b;c})_{;d} =: T^a{}_{b;cd} = \nabla_d\nabla_c T^a{}_b$$

(i.e. two things after semicolon just means take covariant derivative ∇ twice)

(nearest index to ; is first deriv to be taken)

In general $T^a{}_{b;cd} \neq T^a{}_{b;dc}$

- If f is a function $f_{;a} \overset{!}{=} (df)_a$ is a covector. In a coordinate basis

(use this & plug in expression for $\omega_{\mu;\nu}$ with $\omega = df$) (components of (df))

(reminder df: $(df)_p(X) = X(f)$ and so $\nabla_X f := X(f) = (df)_p(X)$?

$\nabla_X f = X(f) = X^\mu e_\mu(f) = X^\mu(\partial_\mu f)$ in coords — but saw before $\partial_\mu f = [df]_\mu$ so $X^\mu \nabla_\mu f = X^\mu[df]_\mu$ ⇒ $\nabla_\mu f = [df]_\mu$ also $= \partial_\mu f$!)

$$f_{;\mu} = f_{,\mu} \implies f_{;\mu\nu} = f_{,\mu\nu} - \Gamma^\sigma{}_{\mu\nu}f_{,\sigma}$$

$$\implies f_{;[\mu\nu]} = -\Gamma^\sigma{}_{[\mu\nu]}f_{,\sigma}$$

($f_{,\mu\nu} = f_{,\nu\mu}$, $\partial_\mu\partial_\nu f = \partial_\nu\partial_\mu f$)

(i.e. matters which order we take derivatives)

DEF: A connection (= covariant derivative) is torsion free or symmetric if

$$\nabla_a\nabla_b f - \nabla_b\nabla_a f = 0$$

For any function f in a coordinate basis this is equivalent to

$$\Gamma^\rho{}_{[\mu\nu]} = 0 \iff \Gamma^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu}$$

(note: only for torsion free!)

(mostly we will work w/ symmetric ones so don't need to be as worried abt ordering bottom indices)

LEMMA: If ∇ is torsion free, then for X, Y vector fields

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

PROOF: In a coordinate basis

(use results from earlier)

$$\begin{aligned}(\nabla_X Y - \nabla_Y X)^\mu &= X^\sigma Y^\mu{}_{;\sigma} - Y^\sigma X^\mu{}_{;\sigma} \\ &= X^\sigma(Y^\mu{}_{,\sigma} + \Gamma^\mu{}_{\rho\sigma}Y^\rho) - Y^\sigma(X^\mu{}_{,\sigma} + \Gamma^\mu{}_{\rho\sigma}X^\rho) \\ &= [X, Y]^\mu + 2X^\sigma Y^\rho\Gamma^\mu{}_{[\rho\sigma]} = [X, Y]^\mu\end{aligned}$$

This is a tensor equation so if true in one basis, true in all. □