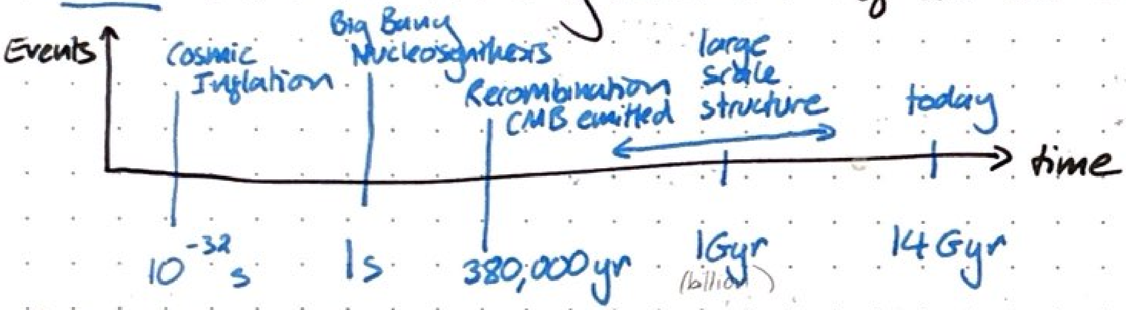
COSMOLOGY

Blake Sherwin   Notes: More detailed notes & lecture scripts on moodle
(enrico Pajer notes for even more detail)

Intro / Metric of the Universe (L1)

- What are the goals of this course?
- Goal: understand history / evolution of our universe



Relics / Observables

$\mapsto$ cosmic microwave background

$\mapsto$ light elements

$\mapsto$ galaxy distribution

$\mapsto$ cosmic neutrino background

- Goal: Understand cosmology as a probe of new physics
  - Probe physics at ultra-high energies ($\gg$ LHC)
    Inflation physics? Phase transitions?
  - + weak interactions (high densities + gravity!)
    New particles? New phenomena?
  - Our clearest indication of beyond-standard model physics.
    Dark matter + dark energy?

I Geometry + Dynamics of the Universe

symmetries crucial to describe our universe
* on largest scales, universe is very uniform

* **Assume** Universe is
  - Isotropic, i.e. the same in all directions
  - Homogeneous, i.e. the same at all positions (our position in the universe is not special)

  → geometry (allows us to derive geometry of universe)

A. Friedmann Robertson Walker metric (comes from/derived from these 2 assumptions)

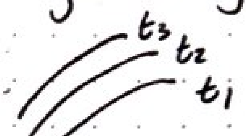
* In General Relativity, invariant line element ds^2 is given by

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu$$

metric ↑ ↑ $X^\mu = (t, \underline{x})$ (time position 4-vector)

c.f. special relativity $g_{\mu\nu} = \eta_{\mu\nu} = \underset{\text{diag}(1,-1,-1,-1)}{(1,-1,-1,-1)}$ (minkowski metric)
GR now $\eta_{\mu\nu} \to g_{\mu\nu}(t, \underline{x})$ function of $t, \underline{x}$

(Imagine slicing into three spaces of const. time)



* Spatial homogeneity + isotropy
  ⇒ can describe spacetime with time ordered sequence of H + I 3-spaces with 3D line element $dl^2 = \gamma_{ij}(t) dx^i dx^j$

  → general 4D metric $ds^2 = \underbrace{dt^2}_{\text{time part}} - \underbrace{\gamma_{ij}(t) dx^i dx^j}_{\text{spatial}}$

* To determine metric of our universe: write down dl^2 of all possible homogeneous + isotropic 3-spaces labelled by curvature.

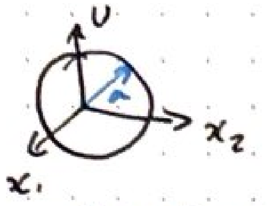
* **1. Flat Space**: line element is simple Euclidean

$$dl^2 = d\underline{x}^2 = \delta_{ij} dx^i dx^j \qquad dl^2(\hat{x}(x)) = dl^2(x)$$

H + I as invariant under rotations / translations (invariant under translation / rotation $\hat{x}$ is e.g. $x + a$)

* **2. Positive Curvature**: three spaces w constant +ve curvature can be represented as a 3-sphere embedded in 4-D euclidean space

e.g. think in 3D ball
what are distances on ball
apply constraint that all points have to lie on 2D sphere



$$dl^2 = d\underline{x}^2 + du^2$$

$$\underline{x}^2 + u^2 = a^2 \leftarrow$$ to get 3D surface on the sphere add constraint all points have to lie on on sphere radius a

(H+I follow from 4D rotations)

* 3. Negative curvature :

once again (just like for positive) can represent it as 3 sphere embedded in 4D euclidean space

3-space with constant -ve curvature can be represented by hyperboloid in 4D ~~cartesian space~~ lorentzian space

(has lorentzian signature like the minkowski space)

$$dl^2 = dx^2 - du^2, \quad \text{where } \underline{x}^2 - u^2 = -a^2$$

Rewrite more elegantly $\underline{x} \to a\underline{x}$, $u \to au$

$$dl^2 = a^2\left[d\underline{x}^2 \pm du^2\right], \quad x^2 \pm u^2 = \pm 1$$

(eq. for positive & negative curvatures)

Now eliminate u differentiate $x^2 \pm u^2 = \pm 1 \to u\,du = \mp \underline{x}\,d\underline{x}$

$$\Rightarrow \quad \left[\, dl^2 = a^2\left[d\underline{x}^2 + k\,\frac{(\underline{x}\,d\underline{x})^2}{1-k\underline{x}^2}\right], \quad k \equiv \begin{cases} 0 & \text{flat} \\ +1 & +\text{ve} \\ -1 & -\text{ve} \end{cases} \right]$$

(includes $d\underline{x}^2 = \delta_{ij}\,dx^i dx^j$)

this expression found just from relying on H+I → we have found there are only 3 possibilities just based on symmetries (instead of deriving from the start)

Lecture 2 : Motion in FRW

14.10.24

recap: homogeneity + isotropy imply metric $ds^2 = dt^2 - dl^2$

$$dl^2 = a^2\left[d\underline{x}^2 + \frac{k(\underline{x}\cdot d\underline{x})}{1-k\underline{x}^2}\right], \quad k = \begin{cases} 0 & \text{flat} \\ +1 & \text{+ve curvature} \\ -1 & \text{-ve curvature} \end{cases}$$

* Rewrite using polar coordinates

$$d\underline{x}^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \qquad \underline{x}\, d\underline{x} = r\,dr \qquad (\underline{x} = \underline{r})$$

* This gives (exercise!)

$$dl^2 = a^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right], \quad d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

* Since a can be different for each slice, take $a(t)$, gives

$$\boxed{ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]}$$

FREIDMANN (-Lemaite) ROBERTSON WALKER METRIC (FRW)

tells us how big univ. is → changes as it expands

if I do these scalings, metric unchanged

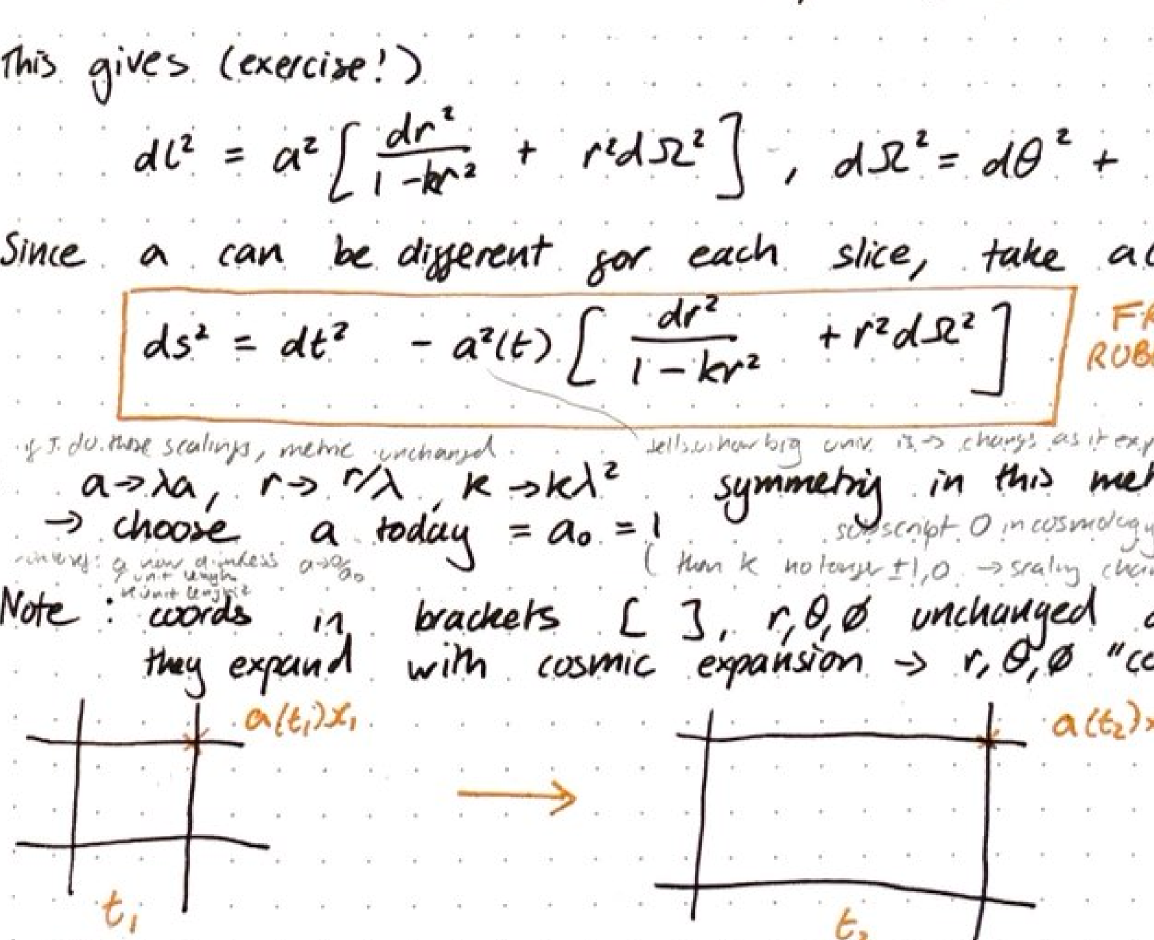
NB $a \to \lambda a$, $r \to r/\lambda$, $k \to k\lambda^2$ symmetry in this metric

→ choose a today $= a_0 = 1$ subscript 0 in cosmology means today

(then k no longer $\pm 1, 0$ → scaling changes it to +ve, 0 or -ve

convention: a now dimless $a \to a/a_0$, unit length, r unit length

* Note : coords in brackets [ ], r, θ, ϕ unchanged as a evolves they expand with cosmic expansion → r, θ, ϕ "comoving coordinates"



physical coord changes, comoving coordinate unchanged (grid changes)

* point comoving coordinate x_i , physical coordinate $a \wedge x_i$

* Physical velocity has two parts

$$V^i_{phys} = \frac{d}{dt}\left(a(t)x^i(t)\right) = \frac{\frac{da}{dt}}{a} \wedge a \wedge x^i + a\frac{dx^i}{dt}$$

$$= H x^i_{phys} + V^i_{pec}$$

two parts to velocity/motion

velocity due to background expansion

"peculiar velocity"

* Have in Hubble parameter

$$\boxed{H(t) \equiv \frac{\frac{da}{dt}}{a}}$$

"how quickly is universe expanding?"

note: we use notation later: usually
x for physical distance
X for comoving distance } $x = aX$

Rewriting the FRW metric

* Taking $d\chi = \frac{dr}{\sqrt{1-kr^2}}$, we get $dl^2 = a^2\left[d\chi^2 + S_k^2(\chi)\,d\Omega^2\right]$

$$S_k(\chi) = \begin{cases} \chi, & k=0 \\ \sin\chi, & k>0 \\ \sinh\chi, & k<0 \end{cases}$$

* If now introduce conformal time τ via $d\tau = dt/a(t)$

=> FRW becomes $$\boxed{ds^2 = a^2(t)\left[d\tau^2 - \left(d\chi^2 + S_k^2(\chi)\,d\Omega^2\right)\right]}$$

Photons have $ds=0$ (travel on null geodesics). Radial trajectory $\Delta\chi = \Delta\tau$

↳ photons don't feel expansion of the universe, they see something like Minkowski space

↳ tells us about trajectories: what about energies/frequencies?

Calculating Motion in FRW

— why do further galaxies move away faster from us than closer galaxies?

* Reminder: In GR, particle velocity on a trajectory $X^\mu(s)$

$$U^\mu(s) = \frac{dX^\mu}{ds}$$

* Particles travel on geodesics — have property that they ~~maximise~~ minimize? extremise proper time (?)

geodesic eqn. $$\frac{dU^\mu}{ds} + \Gamma^\mu_{\alpha\beta}U^\alpha U^\beta$$

christoffel symbol $\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta})$

Note: $$\frac{dU^\mu}{ds} = \frac{dX^\alpha}{ds}\frac{dU^\mu}{dX^\alpha} = U^\alpha\frac{dU^\mu}{dX^\alpha}, \quad P^\mu = mU^\mu$$

* Can rewrite as $$P^\alpha\frac{\partial P^\mu}{\partial X^\alpha} + \Gamma^\mu_{\alpha\beta}P^\alpha P^\beta = 0$$

valid also for massless particles

we have found geodesic eqn for photons/particles

(see printed notes for workings)

* Evaluate $\mu=0$ component for FRW (note: def. amplitude physical momentum as $p^2 = g_{ij}P^iP^j$)

how energy of photon evolves as it travels in expanding univ. → $$E\frac{dE}{dt} = -\frac{\dot a}{a}p^2 \quad \text{with } E^2 - p^2 = m^2,\ E\,dE = p\,dp$$

$$\frac{\dot p}{p} = -\frac{\dot a}{a} \rightarrow p \propto 1/a$$

— how does momentum of particle depend on scale factor?

interpretation: all wavelengths stretched out by expanding univ. incl. de Broglie wavelength so momentum falls

* For massless particles, $E = p \propto 1/a$

* $E = h/\lambda \rightarrow \lambda \propto a$

a photon emitted at t_1 with λ_1 will be measured with $\lambda_0 = \left(\frac{a(t_0)}{a(t_1)}\right)\lambda_1$

* The redshift is defined as $$z = \frac{\lambda_0 - \lambda_1}{\lambda_1}$$

relationship between, light we measure & scale factor when that light was emitted
→ plugging in λ_0 expression. amount of redshift => scale factor when it was emitted.?

$$\Rightarrow \boxed{1+z = \frac{1}{a(t_1)}}$$

* For nearby sources, taylor expand

(assume this small)

$$a(t_1) = a(t_0 + (t_1 - t_0)) = a(t_0) + \dot{a}\,(t_1 - t_0)$$

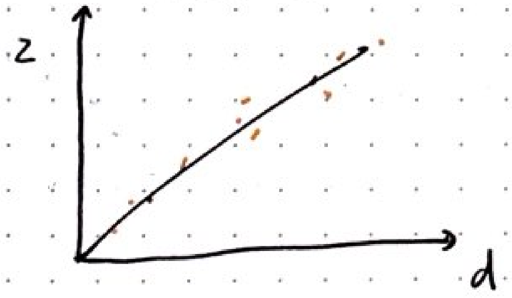
Hubble const.

$$a(t_1) = a(t_0)\,[1 + (t_1 - t_0)H_0 + \dots] \text{ , using } H_0 = \frac{\dot{a}(t_0)}{a(t_0)}$$

$$\Rightarrow z = H_0 \underbrace{(t_0 - t_1)}_{- d/c - d \ [c=1]}$$ redshift related to diff. in time it was emitted vs. today

* $\boxed{z \approx H_0 d + \dots}$ → relates change in wavelength & distance
i.e. relates how fast things move away & distance
further you look the faster things are moving away
is what hubble observed
space between objects is expanding

Hubble's law?



Lecture 3: Dynamics of the Universe

16.10.24

* Recall: FRW only has 1 free function, $a(t)$. want to determine this
* $\Rightarrow$ use **Einstein equations** $G_{\mu\nu} = 8\pi G T_{\mu\nu}$

Relates geometry via $G_{\mu\nu}[g_{\mu\nu}(a(t))]$ = Einstein tensor to energy-momentum tensor $T_{\mu\nu}$

$$T_{\mu\nu} = \begin{bmatrix} T_{00} & T_{0i} \\ T_{j0} & T_{ij} \end{bmatrix} = \begin{bmatrix} \text{Energy density} & \text{Momentum density} \\ \text{Momentum density} & \text{Stress tension} \end{bmatrix}$$

(have seen this from fluid dynamics)

Energy-Momentum + cosmic components

* For a comoving observer, H + I, specify

(energy density can only be fn. of time) (momentum density zero, otherwise violate isotropy) pressure (see notes for argument)

$$T_{00} = \rho(t) \; ; \; T_{0i} = 0 \; \text{(iso)} \; ; \; T_{ij} = -P(t)\, g_{ij}$$

$U^\mu = \frac{dx^\mu}{ds}$

* Write in generally covariant form by noting $U_\mu = (1, \underline{0})$

$$T_{\mu\nu} = (\rho + P) U_\mu U_\nu - P g_{\mu\nu}$$

(plug this in & evaluate components and we get what is written above)

(same components as what we desire but written in generally covariant form)

From Special Relativity — conservation of energy, momentum in Minkowski space specified by:

$$\partial_\mu T^\mu_{\ \nu} = 0 \xrightarrow{\text{now in GR}} \nabla_\mu T^\mu_{\ \nu} = 0$$

(GR version of conservation of E + momentum)

↳ what does applying this look like in GR?

* Applying conservation of $T_{\mu\nu}$:

(christoffel symbols: remember: correction (for curved spacetime))

$$\nabla_\mu T^\mu_{\ \nu} = \partial_\mu T^\mu_{\ \nu} + \Gamma^\mu_{\mu\lambda} T^\lambda_{\ \nu} - \Gamma^\lambda_{\mu\nu} T^\mu_{\ \lambda} = 0$$

(write for completeness, don't need to remember this)

$\Downarrow$ conservation law

(1).

$$\boxed{\dot\rho + 3\frac{\dot a}{a}(\rho + P) = 0}$$

CONTINUITY EQUATION

(how energy density of substance evolves w time as fn of its energy density & pressure)

(analogy w. 1st law of thermo: stuff with pressure does more work)

(i.e. check this explicitly w/ product rule)

* Can be equivalently written as $\frac{d}{dt}(\rho a^3) = -P \frac{d(a^3)}{dt} \leftrightarrow dU = -pdV$

* If we assume constant **equation of state parameter** $\omega = \frac{P}{\rho}$

↳ how does energy density evolve as univ expands?

$$\frac{\dot\rho}{\rho} = -3(1+\omega)\frac{\dot a}{a} \rightarrow \rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+\omega)}$$

(• negligible pressure, dilution of energy, falls as a^{-3}
• but if pressure then falls off faster than a^{-3})

* known components have log 3 e.o.s.

Matter (m)
- cold dark matter (m)
- baryons (b) — nucleons, electrons, atoms? for cosmologists

$w = 0 \rightarrow \rho \propto a^{-3}$

Radiation (r)
- photons (γ)
- neutrinos (ν)

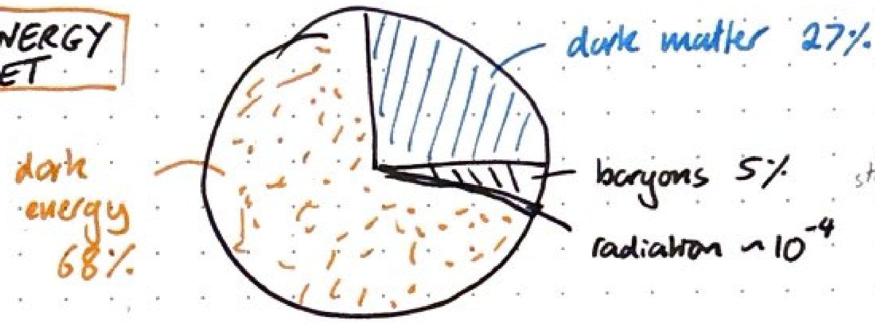
$w = 1/3 \rightarrow \rho \propto a^{-4}$ — mostly g redshift, no. g photons falls as a^{-3} but frequency also falls as a^{-1}

Dark Energy (Λ) (most dominant component of univ.)
- vacuum energy??

$w = -1 \rightarrow \underset{= \text{const.}}{\rho \propto a^0}$ → does not dilute! violate simple energy conservation is strange (obeys conservation law $\nabla_\mu T^\mu_\nu = 0$)

* Note: from observations, we know (extremely well)

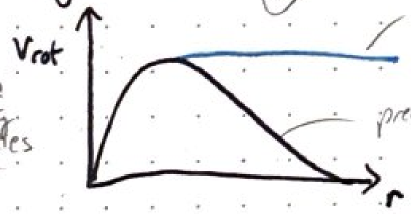
$\boxed{\text{COSMIC ENERGY BUDGET}}$



stuff we can see & feel

aside (background) (Evidence for dark matter (??))

(1)



rotation curve g galaxies

what we observe (realise there has to be some additional source of gravity → dark matter)

prediction based on visible matter → V_{rot} should fall off.

(2) CMB fluctuations small $10^{-4,-5}$ DM → $10^{-1,-2}$ something like need this extra gravity to make structures??

(3) image DM w gravitational lensing

• don't know what it is, → know it does not interact via electromagnetism but does via gravity

Back to dynamics of the universe: Applying Einstein Equations → $a(t)$

(related to E, momentum tensor) → $G_{\mu\nu} = 8\pi G T_{\mu\nu}$

* Recall
$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R$$
remind Ricci tensor complicated fn. of $\Gamma^\lambda_{\alpha\beta}\,(g_{\mu\nu})$ christoffel symbols & metric

* Inserting FRW metric; show:
$$G_{00} = 3\left[\left(\frac{\dot a}{a}\right) + \frac{k}{a^2}\right], \quad \underset{\text{by isotropy}}{G_{0i} = 0}, \quad G_{ij} = \left[2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2}\right] g_{ij}$$

* Einstein equations give
$$G_{00} = 8\pi G T_{00} \rightarrow 3\left[\left(\frac{\dot a}{a}\right)^2 + k/a^2\right] = 8\pi G \rho$$
$$G_{ij} = 8\pi G T_{ij} \rightarrow -\left[2\left(\frac{\dot a}{a}\right)^2 + k/a^2\right] = 8\pi G p$$

* Obtain the Friedmann Equations

REMEMBER THESE

$$\boxed{(F1)\ \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}}$$
$$\boxed{(F2)\ \frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p)}$$

(F1), (F2) ⇒ $\boxed{(F3)\ \dot\rho = -3\frac{\dot a}{a}(\rho + p)}$

• F1, F2 only hold for total $\rho = \Sigma \rho_i$, $p = \Sigma p_i$
• F3 holds for components also

note: since their EM tensors are separately conserved

Lecture 4: Evaluating the Friedmann Equations 18.10.24

* Last time derived ~~Einstein~~ Friedmann eqns.

relative scale factor & its derivatives in evolution of stuff in univ.

$$\text{(F1)} \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - k/a^2$$

$$\text{(F2)} \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$$\text{(F3)} \quad \dot{\rho} = 3\frac{\dot{a}}{a}(\rho + p)$$

(continuity)

Today: evolution of $a(t)$

* Using $H \equiv \left(\frac{\dot{a}}{a}\right) \rightarrow H^2 = \frac{8\pi G}{3}\rho - k/a^2$

$\rightarrow k=0$ defines a critical density (at which univ. is flat)

$$\boxed{\rho_{crit} = \frac{3H^2}{8\pi G}}$$

or $\rho_{crit,0}$ today

N.B. $\rho_{crit,0}$ is $10^{-29}\ g/cm^3$ low: ~ 6 H atoms per m^3

* can use ρ_{crit} to define dimensionless density parameters:

that we use to specify

$$\Omega_x(a) = \frac{\rho_x(a)}{\rho_{crit}(a)}, \quad \text{today} \quad \boxed{\Omega_{x,0} = \frac{\rho_{x,0}}{\rho_{crit,0}}}$$

note on notation: usually write Ω_x when we mean $\Omega_{x,0}$

↳ $\sum_x \Omega_x = 1$ implies flat univ.

* Can relate ρ_i to density parameters today

$$\frac{8\pi G}{3H_0^2}\rho_r = \frac{\rho_r}{\rho_{crit,0}} = \Omega_{r,0}\, a^{-4}$$

$$\frac{8\pi G}{3H_0^2}\rho_m = \Omega_{m,0}\, a^{-3}$$

$$\frac{8\pi G\, \rho_\Lambda}{3H_0^2} = \Omega_{\Lambda,0}$$

* with these expressions in (F1) $\left(H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\right)$, obtain

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_{\Lambda,0} + \Omega_{k,0}\, a^{-2} + \Omega_{m,0}\, a^{-3} + \Omega_{r,0}\, a^{-4}\right],$$

defining $\Omega_k(a) \equiv -\frac{k}{a^2H^2}$

* For any $\Omega_{i,0}$ this can be integrated $\rightarrow a(t)$

collect all a term on one side, use mathematica to eval

$$\int dt = \int da \frac{1}{aH_0\sqrt{\Omega_{\Lambda,0} + \Omega_{k,0}a^{-2} + \Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4}}} \to a(t)$$

how univ. expands as function of time depending on what is in the universe

→ more intuition about solutions

* F1, F2

$$\dot{a}^2 = H_0^2\left(\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0}a^2\right) - k$$

$$\ddot{a} = -H_0^2\left(\frac{\Omega_{r,0}}{a^3} + \frac{\Omega_{m,0}}{2a^2} - \Omega_{\Lambda,0}a\right)$$

→ 2 conclusions

* 1) cannot have a static universe
can set $\dot{a}, \ddot{a} = 0$ for suitable $\Omega_{\Lambda,0}, k$ but unstable to perturbations

→ unsettling to early relativists, einstein didn't like the idea of an expanding univ → didn't include Λ in his original eqns then added it back in to make static
show on 1st example sheet

⇒ Friedmann eqs imply the universe is evolving!
and at early times we inevitably have a point where $a \to 0$

diff components have diff scalings → go as a^{-2}, a^{-3}, a^{0}, etc

* 2) due to different scalings generally one component dominates the dynamics of the univ. → diff epochs

observationally $\Omega_{k,0}$ less than 1%

* Note that current observations → $|\Omega_{k,0}| \equiv \left|\frac{k}{H_0^2}\right| \leq 0.01$
→ assume from now on $k=0$ – based on theoretical observations is our prior
given that it doesn't matter today, conclude it never mattered (?)

Different Epochs

(changeover from radiation to matter occurs when components are equal)

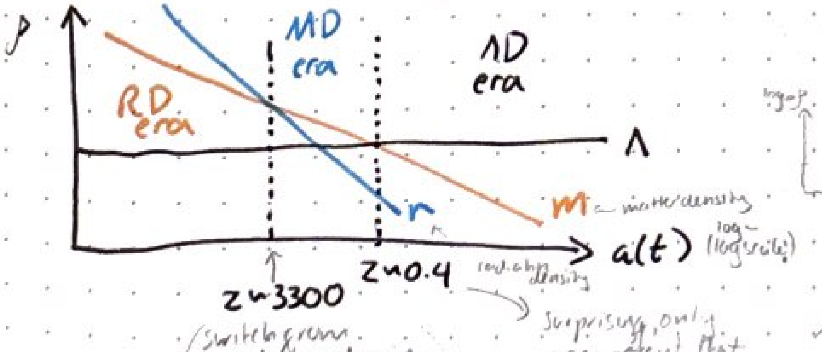
* at early times, radiation domination (RD) when $0 < a < \frac{\Omega_{r,0}}{\Omega_{m,0}}$

* at intermediate times, matter domination (MD), $\frac{\Omega_{r,0}}{\Omega_{m,0}} < a < \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}$

matter dilutes away as a^{-3} which leads to ...

* at late times, dark energy domination (ΛD), $\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} < a$

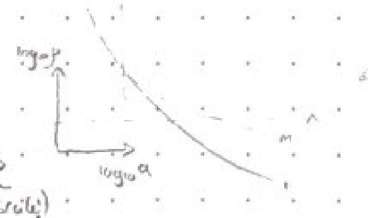
(note look at $\ddot{a}$ equation to give idea of changeover ($\ddot{a}$ gives slightly different?))



(switch from radiation domination to matter domination at redshift 3300)

⟷ 5 orders of mag

surprising, only very recent that Λ becomes dominant parameter



note to self: straight lines on log–log scale but curvy lines for a vs ρ

m: $\rho \propto a^{-3}$
r: $\rho \propto a^{-4}$

coincidence problem
strange Λ & matter happen to be comparable in their contributions today
(in any other eras we wouldn't notice Λ)

Solutions:

Recall: Universe dominated by a single component, e.o.s. ω
$\rightarrow \rho \propto a^{-3(1+\omega)}$

↳ we can work out how univ. expands depending on if dominated by radiation, matter etc.

* In this case, FI becomes $\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 a^{-3(1+\omega)}$ — FI when one component matters ↳ easier to do sep of variables, integral to find

$\Rightarrow a \propto t^{2/3(1+\omega)}$ → see for yourself by sep'g variables

* Evaluating this in different epochs:

useful to remember

$$\boxed{a(t) \propto \begin{cases} t^{1/2} & (RD) \\ t^{2/3} & (MD) \\ e^{H_0 t} & (\Lambda D) \end{cases} \quad \xrightarrow{dt = a d\tau} \quad \text{or } a(\tau) \propto \begin{cases} \tau & (RD) \\ \tau^2 & (MD) \\ -1/\tau & (\Lambda D) \end{cases}}$$

in conformal time ↳ do one more integral

Expansion/distance as a probe of new physics

* $H(z)$ depends on Ω_i, ω_i — what's in the universe & what its properties are — , $H^2(z) = \frac{8\pi G}{3} \sum_i \rho_{i,0} (1+z)^{3(1+\omega_i)}$

hubble parameter sensitive to what stuff (e.o.s) have and how it evolves w/ redshift

everything is felt by H → can use $H(z)$ to find how much stuff is around

* What forms of energy distance $(z) \rightarrow H(z) \rightarrow \Omega_{i,0} / \omega_i$

* E.g. comoving distance for an incoming photon $ds^2 = dt^2 - a^2(t)\left[dx^2 + S_k(x) d\Omega^2\right]$, $ds = 0$, $dx = dt/a$

if I find distance as fn of redshift → hubble parameter as fn of redshift → find stuff

$$X(\tau) = \int_{t_1}^{t_0} \frac{dt}{a(t)} = \int \frac{\frac{dt}{da} \times \frac{da}{dz}}{a} dz = \int_0^{z?} \frac{dz}{H(z)}$$

(see notes for example!)

(office hours on 11.1 think)

Lecture 5 : Inflation : Motivation + Basics — 21.10.24

* Big bang made successful predictions
  - expansion (expanding univ.)
  - CMB radiation (from plasma from hot big bang)
  - Big Bang Nucleosynthesis (can predict abundance of lightest elements)

* BUT : key problems remained

flatness problem of fine-tuning problem would have to be so fine tuned to be as flat as it is

1) Flatness problem

* During radiation + matter domination, curvature grows rapidly. Proof:

$$(F1)\quad H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \rightarrow \frac{3H^2}{8\pi G} - \rho = -\frac{3}{8\pi G}\frac{k}{a^2}$$

$\leftarrow \rho_c$

$$\left(\frac{\rho_c}{\rho} - 1\right)\rho = -\frac{3}{8\pi G}\frac{k}{a^2}$$

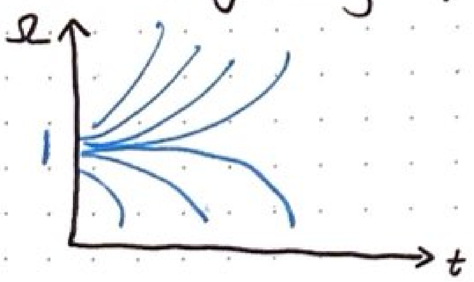
in sense very $\Omega^{-1}-1$ is deviation from flatness

$$\underbrace{(\Omega^{-1} - 1)}_{\text{deviation from flatness}} = -\underbrace{\frac{3k}{8\pi G}}_{\text{const.}}\underbrace{\left[\frac{1}{\rho a^2}\right]}_{} \propto \begin{matrix} a & (MD) \\ a^2 & (RD) \end{matrix} \Rightarrow \text{fast growth!}$$

$\varepsilon\Omega_i$

* Today, we measure $|\Omega - 1| \leq 0.01$ departure from flatness very small

* $|\Omega - 1|$ growing rapidly



by factor $\sim 10^{60}$ from planck scale
Then initial curvature must be strangely small 10^{-62} why?

in order to get <0.01 now ideally find some mechanism to explain

(most important problem for problem of inflation)

2) Horizon problem

(why are diff regions in CMB at same temp even though they never had time to communicate, sth to do with correlations of CMB)

start with some definitions

* Start by defining particle horizon
Recall: radial photon in FRW $\rightarrow \Delta x = \pm\Delta\tau$

* The comoving particle horizon at point P corresponds to the largest distance that could have influenced that point.
It is given by:

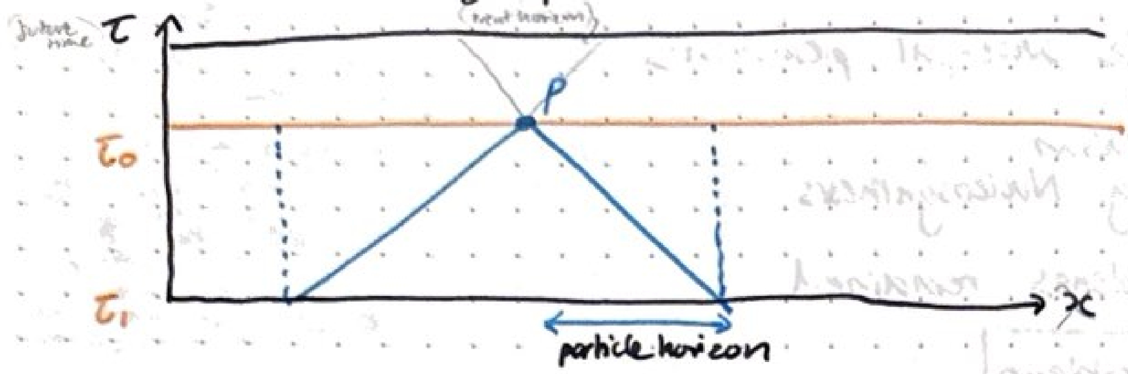
$$X_{ph} = (\tau_0 - \tau_i) = \int_{t_i}^{t_0}\frac{1}{a}dt = \int_{a_i}^{a_0}\frac{da}{aa} = \int_{\ln a_i}^{\ln a_0}\frac{1}{aH}\,d\ln a = \int_{\ln a_i}^{\ln a}\mathcal{H}^{-1}\,d\ln a$$

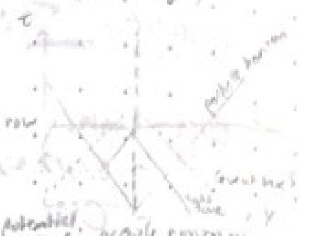
point

Here $\mathcal{H} \equiv aH = \frac{a'}{a}$ $\leftarrow \frac{d}{d\tau} = a\frac{d}{dt}$ is the conformal 'Hubble parameter'

$\boxed{\mathcal{H}^{-1}\ \text{comoving Hubble Radius}}$ comoving distance beyond which particles recede at greater than speed of light, third Hubble flow velocity

* Illustration of particle horizon





potential confusion: particle horizons: distance with which we can see (since light from there has had a chance to reach us). Both diagrams on right represent region of sky that is able to reach us (i.e. causally connected region) (w.r.t. lightcone)

* Consider universe with 1 component e.o.s. ω calculate particle horizon

Note: (FI) $\rightarrow H^2 = H_0^2 a^{-3(1+\omega)} \rightarrow \mathcal{H} = aH = H_0 a^{-(1+3\omega)/2}$; $\mathcal{H}^{-1} = H_0^{-1} a^{(1+3\omega)/2}$

(for "normal" matter $\omega(1+3\omega) > 0$ growing)

* Therefore

$$X_{ph} = \int_{a_i}^{a_0} \mathcal{H}^{-1} \frac{da}{a} = \frac{2H_0^{-1}}{1+3\omega}\left[a_0^{(1+3\omega)/2} - \cancel{a_i^{(1+3\omega)/2}}\right]$$

τ_0 $\qquad$ τ_i

assuming $a_i \rightarrow 0$ → DROP a_i TERM
$(1+3\omega) > 0$, $\mathcal{H}^{-1}$ growing (comoving hubble radius)

$$\approx \frac{2H_0^{-1}}{1+3\omega} a_0^{(1+3\omega)/2} \approx \frac{2}{1+3\omega}\mathcal{H}_0^{-1}$$

* Confusingly, both $\mathcal{H}^{-1}$ and X_{ph} are called "horizon". We will call $\mathcal{H}^{-1}$ horizon.

to be precise, should distinguish ↳ $X_{ph} \sim$ everything ever in causal contact

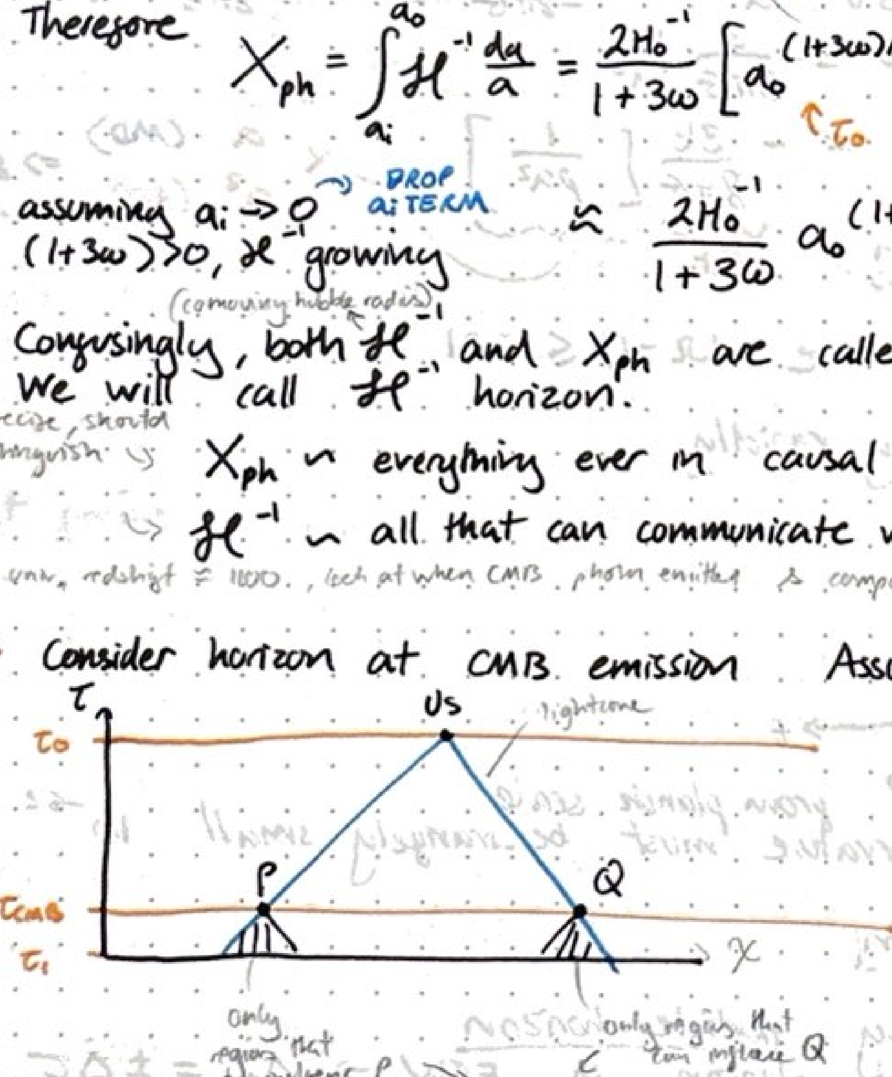
↳ $\mathcal{H}^{-1} \sim$ all that can communicate within 1 expansion timescale

pretty recent

early univ., redshift ≈ 1100, look at when CMB photon emitted & compare w/ horizon today

(when since latest radiation dominate in calculating the horizon)

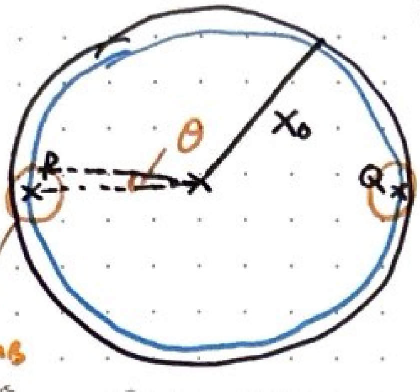
* Consider horizon at CMB emission. Assume MD so $a \propto \tau^2$



look at opposite directions in sky and observe P, Q to have the exact same temperature ↳ strange since particle horizons don't overlap, these points were never in causal contact

Particle horizons $X_0 = \tau_0 - \tau_i$; $X_{CMB} = \tau_{CMB} - \tau_i$

* CMB is made up of many causally disconnected regions, what angle do these subtend? on the sky



simple geometry to find this angle

$$\theta = \frac{X_{CMB}}{X_0 - X_{CMB}} \approx \frac{\tau_{CMB}}{\tau_0 - \tau_{CMB}}$$

$$= \left[\frac{\tau_0}{\tau_{CMB}} - 1\right]^{-1} = \left[\left(\frac{a_0}{a_{CMB}}\right)^{1/2} - 1\right]^{-1}$$ assume MD

$$= \left[(1+z_{CMB})^{1/2} - 1\right]^{-1} \approx \text{2 degrees!}$$

how far do I have to go before it's weird they have the same temp.? is pred angle subtended

any 2 points where angle is greater than this; points will be causally disconnected

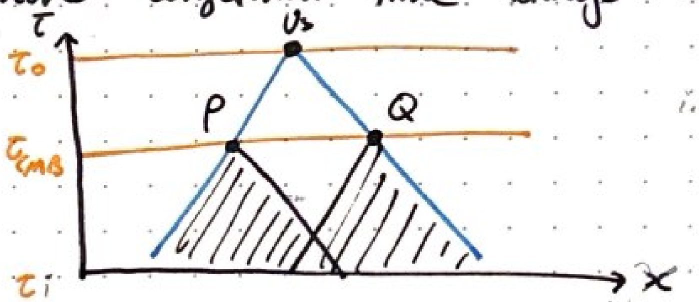
(CMB emitted at redshift 1100 can eval. this expression)

(only regions within 2° of each other on CMB sky were ever able to talk to each other!)

* Problem: CMB uniform across sky >> 2 degrees!

(this is historical statement of problem but more modern) much stronger statement of this puzzle, there are points correlated in CMB across sky much larger than 2 degrees

* Adding more conformal time brings regions into causal contact!



i.e. extend downwards

* How? Propose very early phase $\frac{d}{dt}(\mathcal{H}^{-1}) < 0$

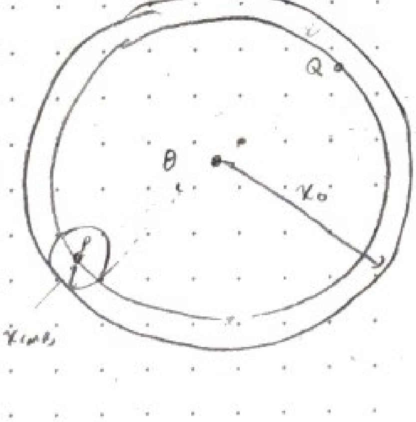
* $\tau_i = \frac{2H_0^{-1}}{1+3w} a_i^{(1+3w)/2} \xrightarrow[a_i \to 0]{} -\infty$, $X_{ph} \to \infty$

what

true if weird substance or shrinking horizon (so X_{ph} gets huge & everything in causal contact) $\rightarrow H(1+3w) < 0$ or $\frac{d}{dt}(\mathcal{H}^{-1}) < 0$

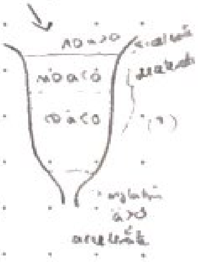
* => A shrinking horizon corresponds to adding lots more conformal time, resolving horizon problem!

my notes:

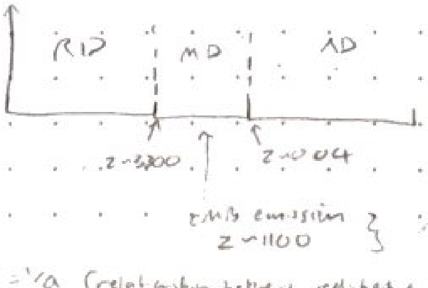


propose period of inflation $\frac{d}{dt}(\mathcal{H}^{-1}) < 0$
→ understand this horizon (thinking of this as region of stuff that can communicate) 'zooms in' on a causally connected region so stuff that looks like it can't communicate now actually could in the past
→ can also think of this as adding more conformal time.
$\frac{d}{dt}(\mathcal{H}^{-1}) < 0 \Rightarrow (1+3w) < 0$
$\Rightarrow \tau_i = \frac{2H_0^{-1}}{1+3w} a_i^{(1+3w)/2}$
$\rightarrow -\infty$ as $a_i \to 0$

$\frac{d}{dt}(\mathcal{H}^{-1}) < 0 \Rightarrow \ddot{a} > 0$
during inflation has accelerating universe!
(also accelerating in current AD)
in MD, AD universe still expanding but $\ddot{a} < 0$ so expansion slows.



note to self
- remember X_{CMB} is not radius from centre to CMB, it is particle horizon at CMB i.e. centred on CMB surface!!



∴ assume MD

and use $1+z = 1/a$ (relationship between redshift of light and scale factor when that light was emitted)

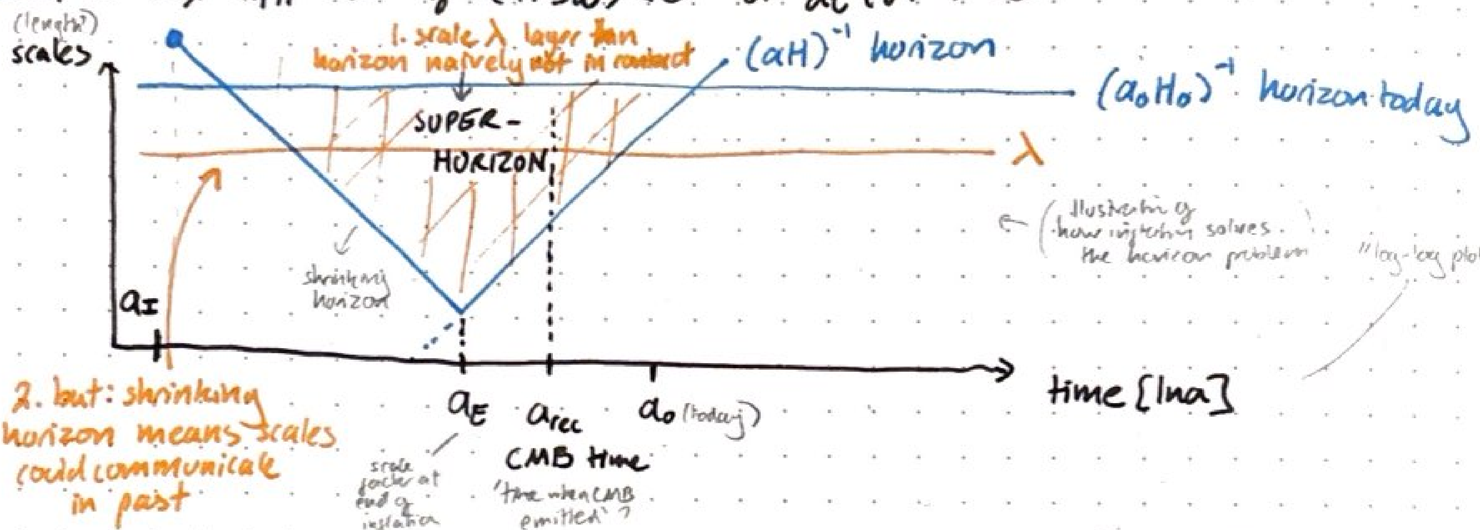
Lecture 6: Basics of Inflation 23.10.24

$\boxed{\text{Last lecture:}}$ Showed that horizon problem can be resolved by adding more $\tau \rightarrow$ can do with phase with $\frac{d}{dt}(\mathcal{H}) < 0 \equiv$ a phase of inflation!

* Prog: since $\frac{d}{dt}(\mathcal{H}^{-1}) < 0 \leftrightarrow (1+3w) < 0$

* $\tau_i \rightarrow -\infty$, $x_{PH} \rightarrow \infty$ if $(1+3w) < 0$ or $\frac{d}{dt}(\mathcal{H}^{-1}) < 0$

so everything in causal contact



then we can explain why everything is so uniform

* How much inflation needed? Need observable scales today to have been subhorizon at start of inflation, i.e.

$$(\mathcal{H}_I)^{-1} > (\mathcal{H}_0)^{-1} \leftarrow \text{all observable scales today}$$

my notes: since observe H+I today $\mathcal{H}_I^{-1}$ must be larger than $\mathcal{H}_0^{-1}$ today before shrinks today H+I inflation

* See notes $\rightarrow \ln\left(\frac{a_E}{a_I}\right) \gtrsim 64 \rightarrow$ need at least 60 e-folds of inflation

increase by factor of e?

(everything we see today must have been in causal contact at one point)

(def. $dN = d\ln a$) $\rightarrow$ intuition: $a_E = a_I \times e^{N}$ ← no of efolds, $N = \ln\left(\frac{a_E}{a_I}\right)$

Definitions of Inflation

* Several definitions, equivalent to $\frac{d}{dt}(\mathcal{H}^{-1}) < 0$: all equivalent

1) Period of acceleration $\ddot{a} > 0$: this follows from $\frac{d\mathcal{H}^{-1}}{dt} = \frac{d(\dot{a}^{-1})}{dt} = -\frac{\ddot{a}}{\dot{a}^2}$

important

2) Slowly varying Hubble: $\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{-d\ln H}{d\ln a} < 1$ (1st hubble slow roll parameter epsilon)

(it's note $\frac{d}{da}\dot{a} = \frac{dt}{da}\frac{d}{dt}\dot{a} = \frac{\ddot{a}}{\dot{a}}$)

step: $\epsilon = -\frac{1}{H}\frac{dH}{dt}\frac{dt}{da}$

what is ϵ? how large fractional change in H is for e-fold change in a?

This follows from $\frac{d}{dt}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\varepsilon) < 0$

3) Negative pressure $(w < -\frac{1}{3})$:

Showed $\mathcal{H} = H_0 a^{-(1+3w)/2} \rightarrow \frac{d}{dt}(\mathcal{H}^{-1}) = \frac{da}{dt}\frac{d}{da}(\mathcal{H}^{-1}) = \frac{(1+3w)}{2a}$

$$\rightarrow (1+3w) < 0$$

"fractional change in ρ per e-fold has to be small"

4) Constant density $\left|\frac{d\ln\rho}{d\ln a}\right| = \frac{2\epsilon}{2} < \frac{2}{2}$ (left as exercise) (1 in notes)

since i can show $\epsilon < 1$ then $2\epsilon < 2$

$\left(\frac{d\ln\rho}{d\ln H}\right)\left(-\frac{d\ln H}{d\ln a}\right)$ use F1 $\frac{H^2}{H_0^2} = \rho/\rho_0$ $2\log H - 2\log H_0 = \log\rho - \log\rho_0$ $\rightarrow \frac{d\ln\rho}{d\ln H} = 2$

Conditions for Inflation to work

* Inflation must occur, i.e. $\epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{d\ln a} < 1$

not enough that just this condition is satisfied => it must also last for large no. of efolds.

* It must last for a large number of e-folds

To quantify, introduce second Hubble slow-roll parameter

(in same way that ϵ says how much hubble parameter changes per e-fold, η says how much ϵ changes per e-fold)

$$\eta \equiv \frac{\dot{\epsilon}}{H\epsilon} = \frac{d\ln\epsilon}{d\ln a}$$

what is η?

want fractional change in H per e-fold ≈ constant. (1)

For inflation to last, fractional change in ϵ, $d\ln\epsilon$, per e-fold $\ln a$, must be small

$$\Rightarrow |\eta| << 1$$

whatever is driving inflation must 'decay' into the standard model

* Inflation must end, so a mechanism is needed

* Inflation must decay (at $T > 100\,GeV$ ← why?) to give the Standard Model particles, reheating the universe

(area of active research -> slightly speculative, we don't know the exact model of inflation but will discuss our most likely model: Inflation with a scalar field)

Many models, but let's discuss the simplest:

Inflation with a single scalar field.

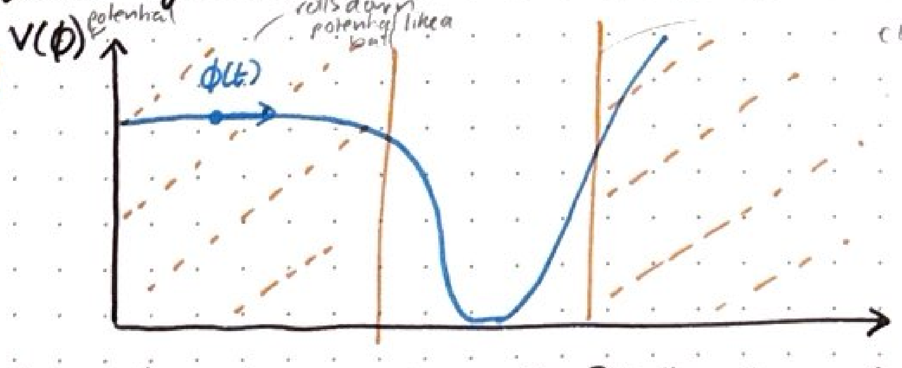
(for now classical field, H+I) but later consider quantum

(want to solve horizon problem w/ this field eventually)

* Consider single scalar field $\phi(\underline{x}, t)$ with a potential $V(\phi)$ which we'll call the inflaton field (pronounced 'inflaton' not inflation)

If inflaton field's $T^{\mu}{}_{\nu}$ dominates drives FRW dynamics

* Example



claim regions (don't know what these was about, didn't mention)

IDEA: rolling slowly so potential energy ≈ constant if energy density const. then hubble parameter const so exponential expansion

* When does ϕ produce inflation? Details to follow, but basic idea: if ϕ evolves slowly then $\rho \sim V \sim$ const. Therefore $H \sim \sqrt{V} \sim$ const.

$$\rightarrow \frac{d\mathcal{H}^{-1}}{dt} < 0$$

from F1 $\frac{H^2}{H_0^2} = \frac{\rho}{\rho_0}$

$H = \frac{\dot{a}}{a}$

=> exponential expansion

$a \propto e^{Ht}$

=> inflation

Detail

maybe familiar from QFT

* Scalar field has energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}\left(\tfrac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi)\right)$$

$\dot\phi^2 - (\tfrac{1}{2}\dot\phi^2 - V(\phi))$

assuming H+I, $\phi = \phi(t)$ only — simplifies considerably

* Therefore

makes sense, familiar just

$$\rho_\phi \equiv T^0_{\ 0} = \tfrac{1}{2}\dot\phi^2 + V(\phi) \qquad (KE + PE)$$

$$P_\phi = -\tfrac{1}{3} T^i_{\ i} = \tfrac{1}{2}\dot\phi^2 - V(\phi) \qquad (KE - PE)$$

* What happens if I plug these into Friedmann equations?

NB note: $T^i_{\ i}$ so add 3 terms $i=1,2,3$!

F1: $$H^2 = \frac{8\pi G}{3}\rho \equiv \frac{\rho}{3M_{pl}^2} = \frac{\tfrac{1}{2}\dot\phi^2 + V(\phi)}{3M_{pl}^2}$$

don't know what this is

F2: $$\dot H = -\frac{1}{2M_{pl}^2}\dot\phi^2$$

$$M_{pl}^2 = \frac{1}{8\pi G}$$

* Taking $(\dot{F1})$, F2:

$(\dot{F1})$: $$2H\dot H = \frac{1}{3M_{pl}^2}\ddot\phi\dot\phi + \frac{1}{3M_{pl}^2}V_{,\phi}\dot\phi$$

F2: $$\dot H = -\frac{1}{2M_{pl}^3}\dot\phi^2$$

$$\underbrace{\ddot\phi}_{\text{acceleration}} + \underbrace{3H\dot\phi}_{\text{friction}} = \underbrace{-V_{,\phi}}_{\text{force}}$$

opposes acceleration

Equation of motion for scalar field!

Lecture 7: Inflationary Trajectories + Potentials

25.10.24

(Recap:) Derived ρ, P for a scalar field

$$\rho_\phi = \tfrac{1}{2}\dot{\phi}^2 + V(\phi) \quad \rightarrow \quad F1 \quad H^2 = \frac{\tfrac{1}{2}\dot{\phi}^2 + V(\phi)}{3M_{pl}^2}$$

$$P_\phi = \tfrac{1}{2}\dot{\phi}^2 - V(\phi) \qquad F2 \quad \dot{H} = -\frac{1}{2M_{pl}^2}\dot{\phi}^2$$

$\rightarrow$ e.o.m. $\ddot{\phi} + 3H\dot{\phi} = -V_{,\phi}$ (force, gradient down the potential)

e.o.m. tells us inflaton rely on potential (inflaton analogous to ball rolling down hill w/ 3H as friction)

* When does this ϕ field inflate? — what potentials further support inflation?

Slow-roll inflation

(trajectories)

* Condition for inflation: $\boxed{\varepsilon \equiv \frac{-\dot{H}}{H^2} < 1}$ (1st slow roll variable param)

Recalling that F2: $\dot{H} = -\frac{1}{2M_{pl}^2}\dot{\phi}^2$

=> implies ~~for~~ that for inflation $\varepsilon = \frac{\tfrac{1}{2}\dot{\phi}^2}{M_{pl}^2 H^2} < 1$

Inflation happens when KE $\tfrac{1}{2}\dot{\phi}^2$ is small compared to total ρ (total energy)

$\rho = 3M_{pl}^2 H^2$ (less than 1/3?)

(call this low KE inflation trajectory = "slow roll")

* We want this low KE state to last (works)

(guess condition is accel small and show that this is equivalent)?

* Let us define a variable giving traditional acceleration per e-fold

$$\delta \equiv -\frac{d\ln\dot{\phi}}{d\ln a} = \frac{-\ddot{\phi}}{\dot{\phi}H}$$

(assume low accel. is condition for inflation to last, show this)

* Let's relate this to η time derivative of ε

$$\dot{\varepsilon} = \frac{\dot{\phi}\ddot{\phi}}{M_{pl}^2 H^2} - \frac{\dot{\phi}^2\dot{H}}{M_{pl}^2 H^2}$$

(fractional derivative of ε)

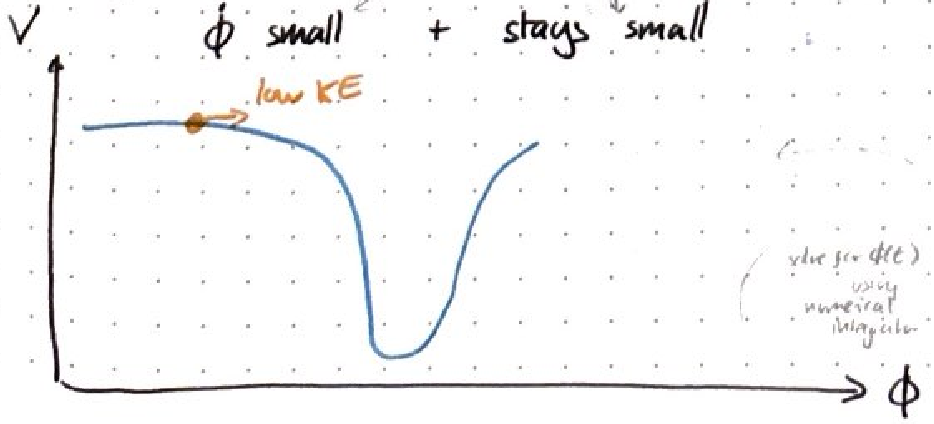
$$\eta \equiv \frac{\dot{\varepsilon}}{H\varepsilon} = 2\frac{\ddot{\phi}}{H\dot{\phi}} - 2\frac{\dot{H}}{H} = \underbrace{2(\delta - \varepsilon)}^{2(\varepsilon-\delta)} ; \Rightarrow \{|\delta| \ll 1, \varepsilon \ll 1\}$$

small acceleration (small δ) and small ε (change in H per e-fold) equivalent to small η (condition for inflation to last)

equivalent to $\{|\eta| \ll 1, \varepsilon \ll 1\}$ (equiv. conditions)

* Lasting inflation, $\varepsilon \ll 1$, $|\eta| \ll 1$, implies a slowly rolling field

$\dot{\phi}$ small + stays small



slowly rolls down this potential

(2) simple useful to get a condition where I can know, given a potential, a condition to tell whether it permits inflation or not → p1-0

solve for $\phi(t)$ using numerical integration

(1) how does inflaton evolve? (solve Klein Gordon eq (e.o.m.) along with F1 and at each point along trajectory see if $\varepsilon < 1$ i.e. if there is inflation at that point but often useful to approximate...)

Trajectories in the slow-roll approximation

we have inflation if it is

plug into klein gordon eq. to get this

* In general, integrate → $\phi(t)$, test $\epsilon < 1$. But often useful to make approximation. eval. trajectory more simply if we assume we are on an inflated one

assume we are on inflated trajectory

make the slow roll approximation

* Assume $\{\epsilon \ll 1, |\delta| \ll 1\}$ to simplify e.o.m. = "slow roll approx."

(how do the e.o.m simplify? → need to consider klein gordon eq & FI (which gives H))

* Since $\{\epsilon \ll 1, |\delta| \ll 1\}$ means KE + acceleration are small, approximate (drop KE term) ↑ $\epsilon \ll 1$ ↑ $|\delta| \ll 1$

simplify FI: expr for H :

$$\left[H^2 = \frac{\cancel{\tfrac{1}{2}\dot{\phi}} + V(\phi)}{3M_{pl}^2} \simeq \frac{V(\phi)}{3M_{pl}^2} \right]$$

* Since $\delta \equiv \frac{-\ddot{\phi}}{H\dot{\phi}} \ll 1$, can drop $\ddot{\phi}$ term in KG eq., gives:

$$\left[3H\dot{\phi} \simeq -V_{,\phi} \right]$$

* These 2 eqns. describe a slowly rolling trajectory, can solve until inflation ends

(intuition: as ball rolls down the hill, it is friction dominated, 2nd eq; 1st eq: potential is only energy density that matters)

slow roll approx 1st reason to apply is to simplify these eqs., 2nd reason is:

* Can V support inflation? Use eqns. to define new slow-roll parameters related to potential. Using these two: $3H\dot{\phi} \simeq -V_{,\phi}$ and $H^2 = V/3M_{pl}^2$, "M planck"

condition on $V_{,\phi}$ & V to give inflating trajectory

$$\text{true } \epsilon = \frac{\tfrac{1}{2}\dot{\phi}^2}{M_{pl}^2 H^2} \overset{\text{plug in sr}}{\approx} \frac{M_{pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \rightarrow \boxed{\epsilon_V = \frac{M_{pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2}$$

~~consistency test~~: assumed ϵ is small to get this so check ϵ small

* ϵ_V is the first "potential slow-roll parameter" cf. ϵ (1st Hubble slow roll parameter

* Similarly, taking time derivative of KG-equation

2nd deriv of V wrt ϕ

divide by $-3H^2\dot{\phi}$

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} = -V_{,\phi\phi}\dot{\phi}$$

$$\rightarrow -\frac{\dot{H}}{H^2} - \frac{\ddot{\phi}}{H\dot{\phi}} = \frac{V_{,\phi\phi}}{3H^2} \quad ; \quad \epsilon + \delta = M_{pl}^2\left(\frac{V_{,\phi\phi}}{V}\right),$$

$$\delta + \epsilon \simeq M_{pl}^2 \frac{V_{,\phi\phi}}{V} \equiv \eta_V$$

define $\boxed{\eta_V \equiv M_{pl}^2 \frac{V_{,\phi\phi}}{V}}$ "second potential slow-roll parameter"

why do this? cs we want to test whether a potential can support slow roll inflation bc can take potential & eval. its derivatives to test

* $\epsilon_V, \eta_V \ll 1$: potential can support slow-roll inflation But not all trajectories must be inflating

e.g. slow-roll inflation may not take place if large initial ϕ

* in contrast, $\epsilon < 1, |\eta| \ll 1$, general condition for inflation to happen

can avoid these potential slow roll params & just do numerical integrate

two sets: hubble slow roll params ϵ,η general condition for inflation whereas potential slow roll params (ϵ_V, η_V) are a convenient test to see if a certain potential can support slow roll inflation)

note: η_V defined slightly differently to 2nd Hubble slow roll parameter $\eta := 2(\epsilon - \delta)$
→ can relate η_V to η assuming slow roll: $\eta_V \simeq 2\epsilon - \tfrac{1}{2}\eta$

$\epsilon_V \simeq \epsilon$

consider the params to derive amount of inflation

* see notes: Amount of inflation, can derive with slow-roll params $N(\epsilon_V, \phi_I, \phi_c)$

def $dN = d\ln a$ no of e-folds of inflation as

$$N = \int_{a_i}^{a_E} d\ln a = \int_{t_i}^{t_E} H dt = \int_{\phi_i}^{\phi_E} \frac{1}{\sqrt{2\epsilon}\, M_{pl}} |d\phi| \approx \int_{\phi_i}^{\phi_E} \frac{1}{\sqrt{2\epsilon_V}\, M_{pl}} |d\phi| \gtrsim 60$$

(inflation ends at t_E when ϵ ...)

EXAMPLE: Slow-roll analysis

simplest potential we can imagine is quadratic

* Let's analyse simple $m^2\phi^2$ inflation, with

$$V(\phi) = \tfrac{1}{2}m^2\phi^2$$

$N_{tot} = \ln \frac{a_E}{a_I}$

* Potential slow-roll params

$$\epsilon_V = \frac{M_{pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 = 2\left(\frac{M_{pl}}{\phi}\right)^2 ; \quad \eta_V = M_{pl}^2 \frac{V_{,\phi\phi}}{V} = 2\left(\frac{M_{pl}}{\phi}\right)^2$$

also use ϵ_V to get idea when inflation ends

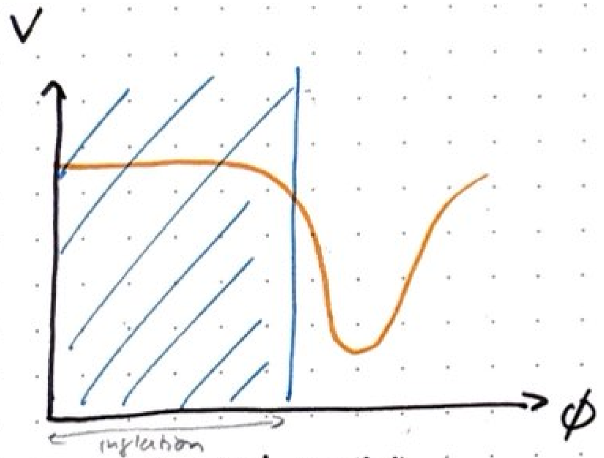
* To inflate require $\epsilon_V, |\eta_V| \ll 1$ i.e. $\phi \gg \sqrt{2} M_{pl} \leftarrow = \phi_E$

inflation ends when these conditions are no longer satisfied

Find no of e-folds until end of inflation $\int_{\phi_i}^{\phi_E} \frac{1}{\sqrt{2\epsilon_V} M_{pl}} |d\phi| = N(\phi_i) = \frac{\phi_i^2}{4M_{pl}^2} - \frac{1}{2}$

we know when a potential supports inflation but what about what kind of potentials can support inflation? → 2 types (based on how they satisfy these conditions)

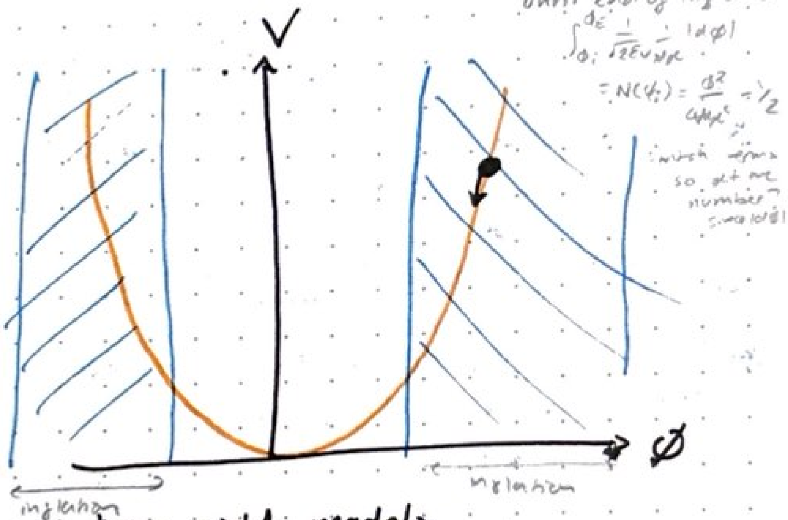
Models of Inflation



small-field models

• ϵ_V, η_V small via flat V — i.e. derivative of V is small

• small r? (lecture records ...)



large-field models

• ϵ_V, η_V small via large V

• large r — parametrises level of gravitational waves

"primordial gravitational waves" → strength depends on energy scale of inflation

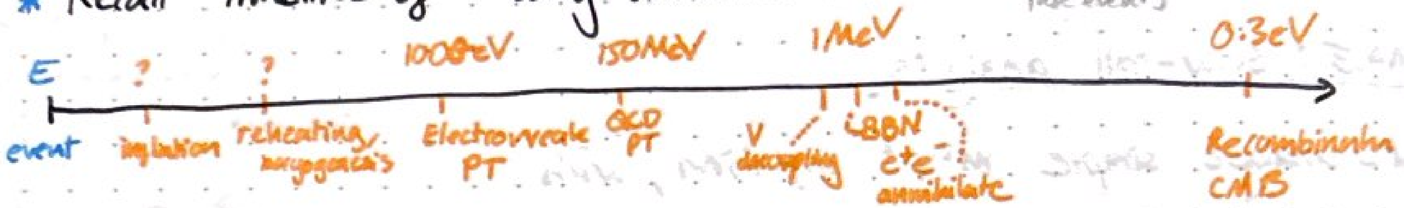
topic
less speculative now
look at timeline

Lecture 8 : Thermal history basics

28.10.24

* Recall timeline of early universe

thermal history gives clues to these events



* key concept for understanding history of universe : Thermal equilibrium

* = stable state where entropy is maximised. Often easily dictates abundances!

we discuss this in more quantitative detail

* Guideline : thermal equilibrium takes place when

$$\underset{\text{interaction rate}}{\Gamma} > \underset{\text{expansion rate (Hubble)}}{H}$$

order of age of the universe

this is sensible condition; need >1 interaction in the age of the universe for the process to work

as universe cools, processes start to fall out of equilibrium when their rate of interaction cannot keep up with the expansion rate $\tau_I > \tau_H$

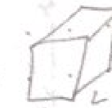
$$\tau_I = 1/\Gamma < \tau_H = 1/H$$

able to reach thermal equilibrium before the effects of expansion matter

* Big picture : considering equilibrium + microscopic states → key microscopic quantities - density, pressure...

STATISTICAL MECHANICS + THERMAL EQUILIBRIUM

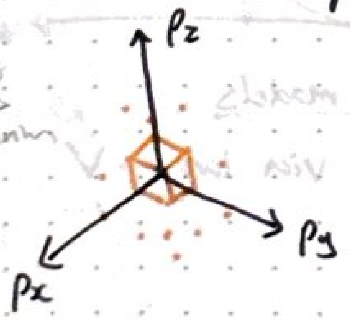
should be vaguely familiar



* in QM, momentum eigenstates of particle in volume $V = L^3$ have a discrete spectrum:

(i.e. momentum of particle in a box is quantised)

grid of states in momentum space



$$k_i = n\frac{2\pi}{L} \quad , \quad p = \hbar k$$

$$\Rightarrow \Delta p = h/L$$

momentum e-states separated by h/L

⇒ density of states in momentum space is

$$\frac{1}{(\Delta p_i)^3} = \left(L/h\right)^3 = \frac{V}{h^3}$$

i.e. this density of states

density of states in phase space is $1/h^3$

(side note this is a nice way to define planck's constant h)

* However, we should also include g internal degrees of freedom e.g. spin to give

$$\text{density of states} = \frac{g}{h^3} \overset{\hbar=1}{\underset{\text{in natural units}}{=}} \frac{g}{(2\pi)^3}$$

figured out how many states per vol. in phase space, now proceed :)

(to proceed, we need to know how often a state contains a particle)

* Now, consider a gas of particles. The average occupation of a state is given by the distribution function

from isotropy + (homogeneity) can only be fn of magnitude of momentum and indep of position

$$f(\underline{x}, \underline{p}, t) = f(p,t) = f(p)$$

→ particle density in phase space $= \frac{g}{(2\pi)^3} \times f(p)$

now we can get quantities of interest

* Can hence integrate over momentum to get

- number density n $\left[n = \frac{g}{(2\pi)^3} \int d^3p \, f(p) \right]$ (can get this if we know the distribution fn. we get to this later)

- energy density ρ $\left[\rho = \frac{g}{(2\pi)^3} \int d^3p \, f(p) E(p) \right]$

(where we assume weak interactions so that $E(p) = \sqrt{m^2+p^2}$)

- pressure p $\left[p = \frac{g}{(2\pi)^3} \int d^3p \, f(p) \frac{p^2}{3E(p)} \right]$ (see notes, go through derivation → to do with particles & walls)

* In thermal equilibrium: (distribution fn. in equilibrium is derived to maximise entropy of the universe)

- the distribution function is

$$f(p) = \frac{1}{e^{\frac{E(p)-\mu}{T}} \pm 1}$$

+ fermions
− bosons

fermi-dirac and bose-einstein distributions
both → maxwell boltzmann at low T

where
here $T = T(t)$ $[k=1]$
μ = chemical potential (can be different for each particle)

- the temperatures of components are equal (kinetic equilibrium)

(all particle species)
typically take T at which they are equal to be the photon temp T_γ which we refer to as "the temp. of the universe"

- chemical potentials of particles that react "balance" i.e.

$$1 + 2 \rightleftharpoons 3 + 4 \quad \rightarrow \quad \mu_1 + \mu_2 = \mu_3 + \mu_4 \quad \text{(chemical equilibrium)}$$

→ (analogous to T but instead of flow of energy relates to flow of particles)

* Reminder chemical potential analogous to T but for flow of particles

tot. internal energy $U(S,V,N)$

$$ds = \frac{dU + p\,dV - \mu\, dN}{T}$$

T tells us how much entropy we get if we add energy to system
in same way $-\mu/T$ is parameter that tells us how much entropy we gain if we add particles to the system.

Particles flow from high to low μ

→ if μ is low on one side of reaction, particles will flow there from the other side in order to maximise entropy

it follows from this
→ photons $\mu_\gamma = 0$ are not conserved $e^- + \gamma \rightleftharpoons e^- + \gamma + \gamma$

2nd consequence
→ antiparticles have $\mu_X = -\mu_{\bar{X}}$ since $X + \bar{X} \overset{\text{annihilation}}{\rightleftharpoons} \gamma + \gamma$

in an expanding univ, T & $\mu(T)$ evolve s.t. continuity equations for particle no. and energy density are still satisfied

Evaluating n, ρ, P in thermal equilibrium

do this by plugging in $f(p)$ & eval integrals

* Wish to find $n(T), \rho(T), P(T)$ at early times $\mu \ll T$ so neglect.

$$n = \frac{g}{2\pi^2}\int_0^\infty dp \, \frac{p^2}{e^{\sqrt{p^2+m^2}/T} \pm 1}$$

(writing d^3p [3D] as $dp\,p^2 \cdot$ const. [1D])

simplify w/ dimless params

$$\rho = \frac{g}{2\pi^2}\int_0^\infty dp \, \frac{p^2\sqrt{p^2+m^2}}{e^{\sqrt{p^2+m^2}/T} \pm 1}$$

, Defining $x \equiv \frac{m}{T}$, $\Sigma \equiv \frac{p}{T}$

$$n = \frac{g}{2\pi^2} T^3 I_\pm(x) \quad , \quad I_\pm(x) \equiv \int_0^\infty d\Sigma \, \frac{\Sigma^2}{e^{\sqrt{\Sigma^2+x^2}} \pm 1}$$

$$\rho = \frac{g}{2\pi^2} T^4 J_\pm(x) \quad , \quad J_\pm(x) \equiv \int_0^\infty d\Sigma \, \frac{\Sigma^2\sqrt{\Sigma^2+x^2}}{e^{\sqrt{\Sigma^2+x^2}} \pm 1}$$

general results

just need to plug in spin etc (g) and we're done!
↳ but first gain some intuition by looking at limits.

* 1) Relativistic limit ($T \gg m$) $\quad x = m/T \to 0$

(note to self: relativistic limit: kinetic energy dominates over mass energy, $T \gg m$; $E = \sqrt{(mc^2)^2 + p^2c^2}$)

now $\quad I_\pm(0) = \int_0^\infty d\Sigma \, \frac{\Sigma^2}{e^\Sigma \pm 1}$

(Riemann zeta fn.) $\Gamma(3)\zeta(3) = 2\zeta(3)$

→ for bosons $(-)$ standard integral $\quad I_-(0) = 2\zeta(3) \simeq 2.4$

→ for fermions $(+)$ → $\quad I_+ = 3/4\, I_-$

$I_\pm(0) = \int \frac{\Sigma^2}{e^\Sigma \pm 1}$

$$\boxed{n = \frac{\zeta(3)}{\pi^2} g T^3 \begin{cases} 1 & \text{bosons} \\ 3/4 & \text{fermions} \end{cases}}$$

powerful expression
↳ how many particles in early univ.
↳ depends on early T & spin

(use $\frac{1}{e^\Sigma+1} = \frac{1}{e^\Sigma-1} - \frac{2}{e^{2\Sigma}-1}$
$\Rightarrow I_+(0) = 3/4\, I_-(0)$)

similarly

familiar ↳ gas of photons $\rho \propto T^4$

$J(0) = \int \frac{\Sigma^3}{e^\Sigma \pm 1}$

$$\boxed{\rho = \frac{\pi^2}{30} g T^4 \begin{cases} 1 & \text{bosons} \\ 7/8 & \text{fermions} \end{cases}}$$

(left as exercise) $\quad \boxed{P = \rho/3}$

show by writing integral for P in terms of integral for ρ

recovers standard $P = \rho$ relationship for radiation!

(i.e. use standard integral $\zeta(s)\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}dx$ (gamma)
same trick as before $\Rightarrow J_+(0) = \frac{7}{8} J_-(0)$
$J_-(0) = \zeta(4)\Gamma(4) = \frac{\pi^4}{90} \times 6$)

* 2) Non-relativistic limit ($m \gg T$) $\quad x = m/T \gg 1$

Integral is $\quad I_\pm(x \gg 1) = \int_0^\infty d\Sigma \, \frac{\Sigma^2}{e^{\sqrt{\Sigma^2+x^2}}}$, largest contributions are from $\Sigma \ll x$, so expand

$$\Rightarrow I_\pm(x) = \sqrt{\frac{\pi}{2}}\, x^{3/2} e^{-x} \qquad \sqrt{\Sigma^2+x^2} = x + \frac{\Sigma^2}{2x}$$

- drop 1 on bottom ($T \ll m$)
- taylor exp exponent for $\Sigma \ll x$
- bring e^{-x} term outside integral
- leaves standard integral $\int_0^\infty \Sigma^n e^{-\Sigma^2} = \frac{1}{2}\Gamma(\frac{1}{2}(n+1))$
- use $\Gamma(3/2) = \sqrt{\pi}/2$

$$\boxed{n = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}}$$

$$\boxed{\rho = mn} \qquad \boxed{p = nT \ll \rho}$$

v. non-rel. $E \approx m$

non-rel gas has (pressure << energy density)

massive particle no. density is exponentially suppressed at low T (recomb)!

interpretation: as T gets below mass, particles disappear, become boltzmann suppressed

interpretation: annihilating particles & anti-particles no longer balanced by creation as T falls below mass of particle

30.10.24

Lecture 9: Entropy + conservation + the cosmic neutrino background

* Last time th. equation implies for relativistic particles:

$$\rho = \frac{\pi^2}{30} T^4 g \begin{cases} 1 & \text{bosons} \\ 7/8 & \text{fermions} \end{cases}$$

densities & pressures of particles in thermal eq. as fn. of T of our expanding univ.

$$n = \frac{\zeta(3)}{\pi^2} T^3 g \begin{cases} 1 & \text{bosons} \\ 3/4 & \text{fermions} \end{cases}$$

radiation density ρ_r = energy density of relativistic particles

matter density ρ_m = energy density of non-rel particles

particle no. density, n = no. of particles per volume

$$p = \rho/3$$

* Today: how does T evolve? properties of relic ν background?

Effective no. of relativistic deg. of freedom $g_*(T)$

* Total radiation energy density is

$$\rho_r = \sum_i \rho_i \equiv \frac{\pi^2}{30} T^4 g_*(T)$$

← total no. of effective deg. of freedom

as we go over particle mass threshold temps, g changes

$$g_* = g_*^{th} + g_*^{dec}$$

g_*^{th}: species in thermal eq. w/ photons; g_*^{dec} ← decoupled

why T dependence? its expr. only summing over relativistic particles. at diff. T, diff. particles are relativistic or not

- species in equilibrium w/ γ: $g_*^{th}(T) = \sum_{i \in b} g_i + \frac{7}{8} \sum_{i \in f} g_i$ ← internal spin deg. of freedom

- decoupled species, not in eqbm. w/ γ:

$$g_*^{dec}(T) = \sum_{i \in b} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i \in f} g_i \left(\frac{T_i}{T}\right)^4$$

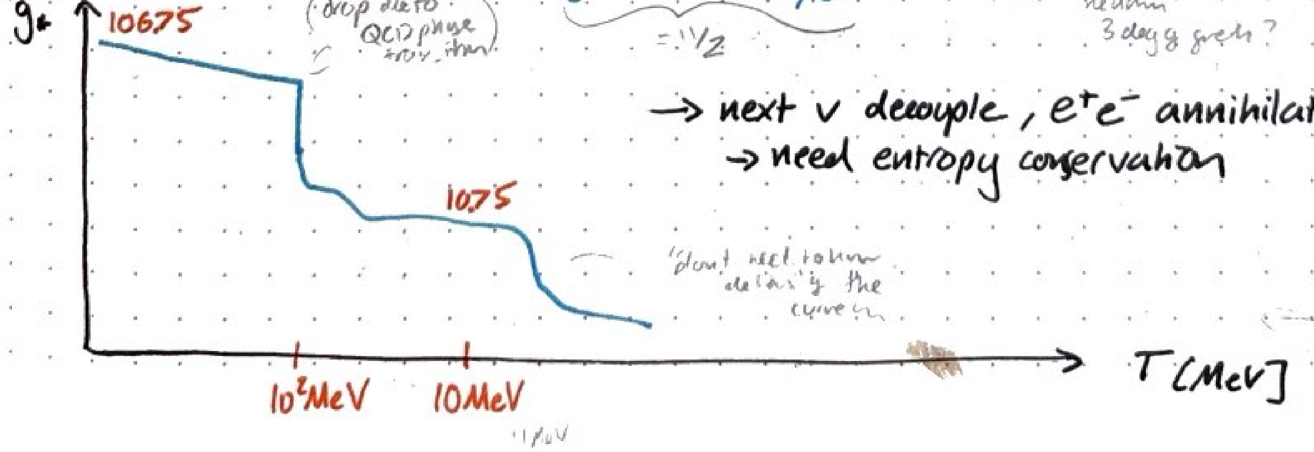
(corrects temp. factor from overall T^4)

* What species are relativistic depends on T (standard model)
  - $T > 100\,\text{GeV}$, all S.M. particles $\Rightarrow g_* = 106.75$ (no. of deg. of freedom if we add everything up)
  - $T \sim 10\,\text{eV}$, (only γ, e^-, e^+, ν relativistic at this T)

$$g_* = \underset{\gamma}{2} + \frac{7}{8}\Big(\underbrace{(2\times 2)}_{e^+,\,e^-} + \underset{\nu}{(3\times 2)}\Big) = 10.75$$

$= 11/2$

neutrino 3 deg. of freedom?



→ next ν decouple, e^+e^- annihilate ~1 MeV

→ need entropy conservation

Conservation of energy

Noether: consequence of symmetry
energy cons from time translation invariant
but in cosmology we don't have the invariance so energy conservation is not guaranteed → eg. can have const. vacuum energy yet univ. is getting bigger
↳ consequence of univ. not being time invariant

approximation?) so instead consider entropy

* Energy conservation tricky ⇒ entropy conservation

prove entropy not conserved ?

* For particles in equilibrium or non-equilibrium with the same distribution function

"show pretty easily" →
$$\frac{dP}{dT} = \frac{\rho + P}{T} \quad (*)$$

show from earlier expressions e.g. since $P = \rho/3$ and $\rho \propto T^4$

consider rel only; since entropy dominated by photon/radiation bath?

* First law : $TdS = dU + PdV$

still assuming $\mu = 0$?

move into derivative & subtract this to keep it consistent

Since $U = \rho V$,
$$dS = \tfrac{1}{T}\left(d[(\rho+p)V] - Vdp\right)$$
$$= \tfrac{1}{T} d[(P+\rho)V] - \tfrac{V}{T^2}(P+\rho)dT \quad (*)$$
$$= d\left[\frac{\rho+P}{T}V\right]$$

S

have shown what entropy is, now show it is conserved w/ time

note: neglect chemical potential since negligible here

* Now take
$$\frac{dS}{dt} = \frac{d}{dt}\left[\frac{\rho+P}{T}V\right]$$
$$= \frac{V}{T}\underbrace{\left[\frac{d\rho}{dt} + \frac{1}{V}\frac{dV}{dt}(\rho+P)\right]}_{=0 \text{ due to (continuity) } \dot\rho + 3H(\rho+P)=0} + \frac{V}{T}\underbrace{\left[\frac{dP}{dt} - \frac{\rho+p}{T}\frac{dT}{dt}\right]}_{=0 \text{ due to } (*)}$$

* Work with (for convenience) entropy density $s = S/V = \dfrac{\rho + P}{T}$

now derive how T of univ. varies w/ time

The entropy for a mix of different particles is

(assuming weak interactions)

$$s = \sum_i \frac{\rho_i + P_i}{T_i} = \frac{2\pi^2}{45} T^3 g_{*s}(T)$$

effective no. of entropy deg of freedom, similar to g_*

similar out like for g_*

$$\Rightarrow g_{*s}(T) = \underbrace{g_{*s}^{th}(T)}_{= g_*(T)} + g_{*s}^{dec}(T)$$

g_{*s}^{th} same as g_*, difference shows up in the decoupled terms

relativistic species at at T as well as ...?

and
$$g_{*s}^{dec}(T) = \sum_{i\in b} g_i \left(\frac{T_i}{T}\right)^3 + \tfrac{7}{8}\sum_{i\in f} g_i\left(\frac{T_i}{T}\right)^3$$

far away from particle mass threshold $T \propto 1/a$

so can read off temp of univ usually $T \propto 1/a$

* Since $S \propto sV \propto sa^3$ is conserved (key! as shown earlier), $\Rightarrow$ $g_{*s}(T)T^3a^3 = \text{const.}$
Usually $T \propto 1/a$ but can be slower $T \propto g_{*s}(T)^{-1/3} a^{-1}$
(intuition: if a particle is becoming non-relativistic & will annihilate into photons → its with energy will heat up the photons so g_* falls; less quickly than $1/a$
(we have found how temperature of universe evolves!)

(end of next page)

* $\Rightarrow T_{\gamma t} = T_{\nu t}\left(\dfrac{g_{*s\gamma}^{th}}{g_{*si}^{th}}\right)^{-1/3} = T_{\nu t}\left(\tfrac{4}{11}\right)^{-1/3} \Rightarrow T_\nu = \left(\tfrac{4}{11}\right)^{1/3} T_\gamma$

(2 over 11/2)

neutrino temp $(4/11)^{1/3} \times$ CMB temp

* A cosmic ν background $n_\nu = 336/cm^3$

↳ $n_\nu = \frac{3\zeta(3)}{\pi^2} T_\nu^3 \times 3/4 \times 2 \times N_{eff}$
$n_\gamma = \frac{3\zeta(3)}{\pi^2} T_\gamma^3 \times 2$

⇒ $n_\nu = \frac{4}{11} n_\gamma \times \frac{3}{4} N_{eff}$

since CMB temp today ~2.73K → $n_\nu = 336/cm^3$

The cosmic ν background

they are interacting w/ everything else in thermal equilibrium

couple "X" to the thermal bath through their weak interaction processes

* Neutrino decoupling: νs coupled at high T

$$\nu_e + \bar{\nu}_e \leftrightarrow e^+ + e^-$$ annihilation
$$e^- + \bar{\nu}_e \leftrightarrow e^- + \bar{\nu}_e$$ scattering

ν interacting w/ everything else

but these interactions are via weak force, stop at low T

just going to give this, you can check it via dimensional analysis

- cross section $\sigma \sim G_F^2 T^2$

$$\rightarrow \Gamma \sim n\sigma v \sim G_F^2 T^5$$

number density?

$n \propto a^{-3} \propto T^3$ $c = 1$

since in early univ, tot energy density ρ is dominated by relativistic $\rho \propto g_* T^4$ (chatgpt)

other rate we need to think about

- drop faster than Hubble, $H \propto a^{-2} \propto T^2$

note: $H \sim T^2/M_{pl}$ from F1 i think

$$\frac{\Gamma}{H} \sim \left(\frac{T}{1\text{MeV}}\right)^3$$

expect at high temp the process work but at low temp, rate of interaction falls below hubble rate (because Γ/H ratio falls below 1)

→ at $T < 1$MeV, neutrinos decouple from thermal bath

- entropy $\propto g_{*s} T^3 a^3$ separately conserved in neutrino & thermal bath

γ, e^-, e^+	ν
$T \propto a^{-1} g_{*s}^{th\,-1/3}(T)$	$T \propto a^{-1}$

note to self (chatgpt): ν decouple → no heating of photon bath. g_{*s} of plasma stays the same so & don't interact w/ bath so both γ, ν evolve separately. e^+e^- decouple → yes heating of photon bath

note to self: At decoupling, temp of bath with e^+, e^-, γ and dec ν are equal. Then e^+, e^- also decouple so T_γ rises above T_ν (which falls as a^{-1})

temp drop from redshift only (?)

Same T at first, but

* e^+, e^- annihilation occurs!

disappear & turn into photons

As T falls below m_e, $e^+, e^- \rightarrow \gamma\gamma$, transferring entropy to γ photons but not ν neutrinos (since they decouple) → expect γ hotter than ν!

* Quantitatively, for e^+, e^-, γ

$$g_{*s}^{th} \begin{cases} 2 + 7/8 \times 2 \times 2 = 11/2 & T > m_e \\ 2 & T < m_e \end{cases}$$

after they annihilate, just 2 deg. of freedom

and hence (i = before decoupling / e^+e^- annih., f after)

1MeV

T

$T_{\gamma_i} = T_{\nu_i}$ decouple

goes up by getting hotter

$$T_{\gamma f} = T_{\gamma i}\left(\frac{a_f}{a_i}\right)^{-1}\left(\frac{g_{*s f}^{th}}{g_{*s i}^{th}}\right)^{-1/3} \qquad T_{\nu f} = T_{\nu i}\left(\frac{a_f}{a_i}\right)^{-1} = T_{\gamma i}\left(\frac{a_f}{a_i}\right)^{-1}$$

since init. in thermal eq.

note to self (chatgpt): at decoupling of ν: $g_{*s}^{th} = 11/2$ (γ, e^+, e^-), $g_{*s}^{dec} = 21/4$ (ν)

at decoupling of e^+, e^-: $g_{*s}^{th} = 2$ (e^+, e^-)

at the moment of decoupling, ν are same T as the thermal bath and g_{*s}^{th} of bath is 11/2. Decoupling of e^+, e^- causes g_{*s}^{th} of bath to drop to 2 so temp of bath increases relative to temp of decoupled neutrinos

summary: ν keep their own entropy but e^+e^- annihilate into photons which heat up plasma (?)

Lecture 10: BBN and N_{eff}, relics and new physics — 1.11.24

* **Reminder:** relic cosmic ~~microwave~~ neutrino background with

$$T_\nu = \left(\frac{g_{*s,\,today}}{g_{*s,\,dec}}\right)^{1/3} T = \left(\frac{4}{11}\right)^{1/3} T$$

keep same dist g_{*}, s after dec (?)

can we do anything else w/ physics we have learned?

→ can calculate n_ν, ρ_ν from relativistic particle formulae

– even though non-rel, decoupled ν → follow relativistic F-D distribution (?)

see notes, also can see that ν only recently non-rel (?) Lecture

* Can show from $\rho_{\gamma_r} = \frac{\pi^2}{30} g_*(T) T^4$

radiation energy density $\nu + \gamma$?

$$\rho_{\gamma_r} = \frac{\pi^2}{30}\Big[\underbrace{2}_{\rho_\gamma} + \overbrace{\tfrac{7}{8}\times 2\times\left(\tfrac{4}{11}\right)^{4/3} \times N_{eff}}^{\rho_\nu}\Big] T^4$$

γ — fermions ⇒ factor 7/8 — ν — 3

learn from why. can constrain no. of ν carry by their energy density (T_γ)

i.e. write: $g_*(T) = g_*^{\gamma}(T) + g_*^{dec}(T)$, with $g_*^\gamma \to 2$, $g_*^{dec} \to \frac{7}{8}\times 2 \times \left(\frac{T_\nu}{T}\right)^4 \times N_{eff}$

* CMB sensitive to early $H = \sqrt{\frac{8\pi G \rho_r}{3}}$ → $N_{eff} = 3.04 \pm 0.18$

– early univ dominated by ρ_r (?)

(indirectly, infer H from CMB then calc ρ_r and N_{eff}?)

so what? we know there are 3 — you are, if particle physicist, you will be unimpressed by this — couplings of ν in S.M.

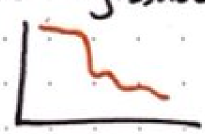
* CMB/H probes <u>all light particles</u> — can tell us if new particles are present in early univ

Can search for new ones parametrised $N_{eff} > 3$

→ why care again? not only get integer values of N_{eff}, we can get smaller g_*

Examples sheet: decoupling when $g_{*s} = g_{*s,dec}$ ⇒ $N_{eff} = 3 + \Delta N_{eff}$ — new light fermion

with $\Delta N_{eff} = \frac{g}{2}\left(\frac{43}{4 g_{*s,dec}}\right)^{4/3}$



← reminder: early time much larger g_*

* ΔN_{eff} can be non integer + small (neutrino – if particle decouples early, it can miss more heating than ν → $\Delta N_{eff} < 1$

misses out on heating so colder

use measurement of CMB to constrain how many particles there were in early univ, doesn't have to be 4, can be e.g. 3.05

* Minimum ΔN_{eff} for a particle ~ 0.04

* If CMB measurements get a constraint $N_{eff} = X \pm 0.01$

→ allows us to rule out or detect any particle that was <u>ever</u> in equilibrium in the early universe!

e.g. can rule out certain particle was <u>ever</u> in equilib

Big Bang Nucleosynthesis

story: now consider abundances of light elements H, He, Li

↳ 1s after big bang: most important energetic contribution is radiation but small no. of matter exists, since conservation of baryon no. no matter how low temp is they won't disappear.

also happens at ~1MeV

* Relic light elements H, He, Li were synthesised in big bang. How?

(complicated problem → modest goal)

Goal: explain 1 number $\boxed{\frac{n_{He}}{n_H} \sim \frac{1}{16}}$

(relative abundance He vs Hydrogen)

Reminder: for non-relativistic particles, eqm. (in equilibrium!)

reminder relativistic ($m \ll T$ (?)

($m \gg T$)

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-(m-\mu)/T}$$

chemical potential makes sure no. of baryons are conserved

restore μ dependence omitted earlier!

now discuss how much He formed:

"He most favourable nucleon energetically so no. of neutrons tells us how much "He produced } so we follow neutron density

· interconvert n & p
· reaction that keeps n, p in equilib.

* **Neutron density** considering $n + \nu_e \leftrightarrow p + e^-$ (and why this is weak interaction that freezes out at ~0.8 MeV)

(for justification) minor assumption (see notes) chemical potential dominated by n, p

Neutron to proton ratio is, assuming μ_ν, μ_e small so that $\mu_n \approx \mu_p$

(or note to self $\mu_n \approx \mu_p$ because in equilibrium? (assume?) w b/c ν_e & e^-?)

$$\left(\frac{n_n}{n_p}\right) = \underbrace{\frac{g_n}{g_p}\left(\frac{m_n}{m_p}\right)^{3/2}}_{\sim 1 \text{ assume this is } 1\,(?)} e^{-Q/T}$$

spin deg. (g) freedom(?) masses

$Q \equiv m_n - m_p = 1.3\,\text{MeV}$

(e.g. so T same? difference $m_n - m_p$ is important in exponential but not sig. mattering in prefactor?)

halloween horror film: neutrons are our hero. Horrible things happen to them then eventually they are safe

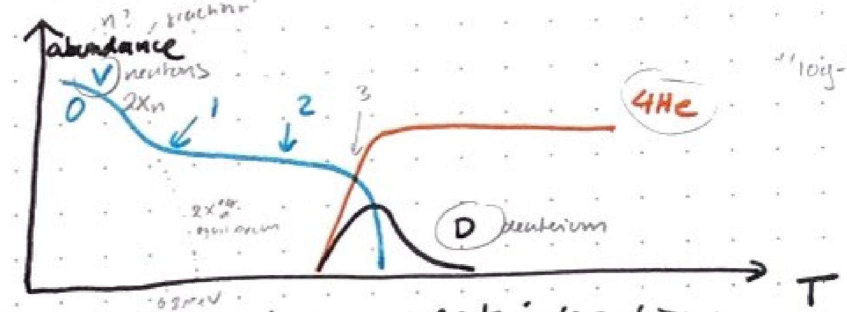
* Starting point: lots of free n, p in equilibrium.
Danger for ν! n?

oh no! they are going to die

Neutron abundance starts falling with T Step 0 (equilibrium ...)

* Now, as universe cools, three steps occur

Step 1 Neutron freeze out
Step 2 neutron decay
Step 3 Helium fusion



– **Step 1**: neutron freeze out occurs when weak interactions become inefficient ~ T ~ 0.8 MeV

Define neutron fraction $X_n = \frac{n_n}{n_n + n_p} = \frac{e^{-Q/T}}{1 + e^{-Q/T}} \sim \frac{1}{6}$ / eval. at $T \sim 0.8\,\text{MeV}$ at freeze out

(saves the n!) but something else horrible happens

– **Step 2**: neutron decay. Free n are unstable (have finite lifetime)
decay $n \to p^+ e^- + \bar{\nu}_e$ with a time constraint $\tau_n \approx 887\,\text{s}$

does our hero survive?

How long does the decay last? Do any n survive? need to explore deeper

↓ answer comes into step 3

– **Step 3**: helium fusion, $4He^{\leftarrow 2n, 2p \text{ (consisting)}}$ has a very large binding energy
All n would 'like' to form 4He (then safe!)

naive: just need 2n to smash w/ 2p but this is unlikely → we actually have reactions that build up to 4He in steps

But, can only be formed by first building D – bottleneck!

Form:

(p)(n) + γ → $n + p \leftrightarrow D + \gamma$ only after enough D is formed we can form 4He

(p)(n)(p) + γ → $D + p \to 3He + \gamma$

(n)(p)(n)(p) + (p) → $D + {}^3He \to {}^4He + p$

consider rate of forming enough D → tells us how long it will take for n to be saved, get an idea of this (details in notes)

$\left(\frac{n_D}{n_p}\right)_{eqm}$

When? Estimate time $\left(\frac{n_p}{n_n}\right)_{eqm} \sim 1$. Exercise (notes) → when $T < 0.06\,\text{MeV}$

Surprising as $B_D \sim 2.2\,\text{MeV}$

temp has to fall much lower than naively expect. since. sorority w/ photons

binding energy of deuterium

Formed when $T < 0.06\,\mathrm{MeV}$ which is surprising as $B_D \sim 2.2\,\mathrm{MeV}$

surprising that: temp has to drop far below binding energy in order for D to form. from notes: turns out this is b/c ratio of baryons to photons & ratio of neutrons to photons is so small $\sim 10^{-9}$

$$\eta \equiv n_b / n_\gamma = 10^{-9}$$

$\rightarrow$ $t \sim 330\,\mathrm{s}$

***** After 330s, enough D builds up that finally ^4He can form

so large no. of photons inhibits production of D until T well below B_D b/c enough photons w/ high energy to dissociate the deuterium

– neutrons saved! $\rightarrow X_n(t=330s) = \frac{1}{6} e^{-330s/\tau_n} = \frac{1}{8}$

$\rightarrow$ all these remaining n processed into ^4He

***** As $2n$ go into $1\ ^4He$, ^4He has $\frac{1}{2}$ n density

(Luca agrees this approx is dodgy)

$$\frac{n_{He}}{n_H} = \frac{\frac{1}{2}n_n}{n_p} = \frac{1}{2}\frac{X_n(330s)}{1 - X_n(330s)} \sim \frac{1}{2} X_n(330s) \sim \frac{1}{16}$$

what we wanted

experimental landscape: generally abundances agree w/ predictions, Li is a little off (beyond standard model)

***** Powerful probe of BSM physics

- if i add more particles these calculations will change
- can use BBN to test new physics at 1s after b.b.
- we are lucky that there are any n left over at all (or He)
  - diff between n & p energies similar to weak timescale is a coincidence means we are able to do BBN calculations at all (we exist)

see notes

Luca's notes good for this lecture

chat gpt explanation

choose frame comoving velocity $U^\mu = (1,0,0,0) \Rightarrow N^\mu = (n,0,0,0)$

now what is $\nabla_\mu N^\mu$?

In GR: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

FRW metric for flat univ: $ds^2 = dt^2 - a^2 dx^2$

so take $g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2)$

$g = |g_{\mu\nu}| = -a^6$

$$\nabla_\mu N^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, N^\mu)$$

$$= a^{-3} \frac{\partial}{\partial t}(a^3 n)$$

$$= 3\frac{\dot{a}}{a} n + \frac{dn}{dt}$$

$\nabla_\mu N^\mu = 0$ equivalent to $\frac{d}{dt}(a^3 n) = 0$!

Lecture 11: Boltzmann + CMB 4.11.24

* Last lectures: described relic particles (ν, n in BBN, ΔN_{eff}) that decoupled or froze out from thermal equilibrium when

$$\Gamma \sim H$$

hubble timescale: expansion timescale

(more rigorous treatment today)

* Today make beyond-equilibrium rigorous w/ Boltzmann, final relic; CMB

starting point: want to know number density of species (no decays)

* With no interactions, number density conserved
$\nabla_\mu N^\mu = 0$, where $N^\mu = n U^\mu$

in special rel: if there is change in number density, it can only happen because there is flux of particles in the region

number density of particle species

$$\frac{dn_i}{dt} + 3\frac{\dot{a}}{a} n_i = 0 \Leftrightarrow n_i \propto a^{-3}$$

equivalent

(note this) equiv to: $\frac{\partial}{\partial t}(a^3 n) = 0$

• number density of particle i given by this eqn. it dilutes as a^{-3}

interpret as: no. of particles in a phys. physical volume $V \propto a^3$ is conserved so density dilutes as the volume expands $n_i \propto a^{-3}$

now what if we add interactions?

* Add interactions by adding a collision term to RHS

BOLTZMANN EQUATION
$$\left[\frac{1}{a^3}\frac{d(n_i a^3)}{dt} = C_i[\{n_j\}]\right]$$

depends on all other particles? n_j

(as we saw before, some interactions rare & require v. high density)

* Whole form is interaction dependent, decays and 2-particle processes are the most common. Discuss the latter

interaction between many particles are unlikely so limit our discussion to decays/2-particle interactions

* Consider $1+2 \Leftrightarrow 3+4$, for n, change given by

$$\frac{1}{a^3}\frac{d}{dt}(n_1 a^3) = \underbrace{-\alpha n_1 n_2}_{\text{destruction of } n_1} + \underbrace{\beta n_3 n_4}_{\text{creation of } n_1}$$

if 1,2 interact → so simplest form is this

if n_3, n_4 interact (simplest form).

$\alpha = \langle \sigma v \rangle$ from QFT

"thermally averaged cross section"

but what is β?

• argue that it has to have a simple form
• 2 contributions - creation of 1 if 3, 4 interact, - destruction of 1, 2

* In eqm. the RHS must vanish $\longrightarrow \beta = \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq} \alpha$

in eq: rate of forward reaction $\alpha n_1 n_2$ = rate of backward reaction $\beta n_3 n_4$

* Hence we obtain
$$\boxed{\frac{1}{a^3}\frac{d}{dt}(n_1 a^3) = -\langle\sigma v\rangle\left[n_1 n_2 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq} n_3 n_4\right]}.$$

rewrite this!

* Now, rewrite in terms of the number of particles in a comoving volume

$$N_i \equiv n_i/s$$

entropy density — rewrite to remove ambiguity

no. particles in comoving vol. does not change in time w/ scale factor.

Note that without interactions, since both $n_i, s \propto a^{-3}$, N_i is conserved, $N_i \propto n_i a^3$.

intuitively: define N_i in terms of entropy, but actually easier to think of it as particles per comoving volume $N_i \propto n_i a^3$

$$\longrightarrow \frac{d\ln N_1}{d\ln a} = -\frac{\Gamma_1}{H}\left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{eq}\frac{N_3 N_4}{N_1 N_2}\right]$$

$\Gamma_1 = n_2 \langle \sigma v \rangle$

prefactor: how efficient is interaction?

term in [ ]: how far is the reaction from equilibrium?

$$\frac{d\ln N_1}{d\ln a} = -\frac{\Gamma_1}{H}\left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{eq} \frac{N_3 N_4}{N_1 N_2}\right]$$

rewrite for

COMMENTS:

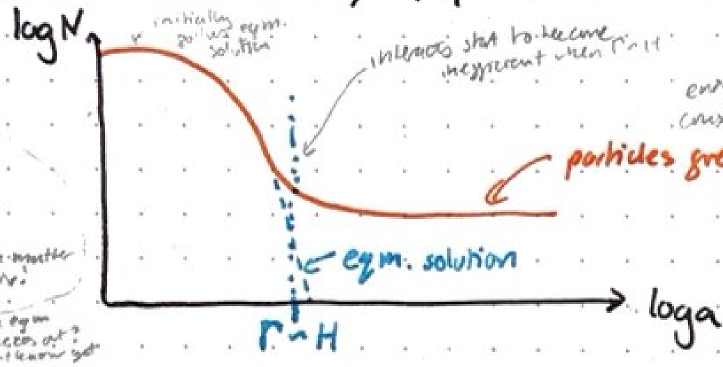
* Consider $\underline{\Gamma_1 \gg H}$, $N_{\neq 1}$ close to eqm. values
  - take $\underline{N_1 \gg N_{eq}}$ (type 1 particles destroyed)
    RHS -ve, $N_1 \downarrow$ to eqm.
  - take $\underline{N_1 \ll N_{eq}}$
    RHS +ve, $N_1 \uparrow$ to eqm.
    (type 1 particles created?)

always drives number density (N_1) towards equilibrium with efficiency determined by Γ/H

* If $\underline{\Gamma_1 \ll H}$, RHS becomes small, $N_1 = \text{const.}$

  soln to boltzmann eq.

* Numerical solution



summary
(reaction rate > hubble rate)
- $\Gamma_1 \gg H$, system will be driven towards equilibrium
- but if $\Gamma_1 \ll H$, N_1 becomes ≈ const.
- this eq is a more accurate way of describing the freeze-out process e.g. for neutrons in BBN, e^- in recombination & even dark matter in early univ.

starts in eqm then freezes out? we don't know yet

maybe move onto most important rel'n

COSMIC MICROWAVE BACKGROUND

~400,000 yr post big bang

[STORY OF THE CMB] plasma: electrons, protons & photons interact rigorously. electrons interact w/ photons frequently, particles highly coupled i.e. freq. interactions. But T falls and suddenly plasma condenses: form first H atoms. call this recombination even though it makes a lot more sense to call it combination since this is first time it happened. univ. becomes neutral so density of free e^- falls quickly and suddenly phone nothing to scatter off so they decouple as the CMB. if we had "microwave eyes" we could look out into the sky and see a boring plasma → cool

* high T / early times, e^-, p, γ scatter
  - T falls, first atoms form "recombination"
    Density of e^- falls fast,
    → photons decouple, stream to us as CMB

More quantitative:

as density of free electrons fell, mean path of photons grew rapidly: univ. became transparent to photons (CMB)

① Saha Equilibrium

* At high T > 1eV, baryons + γ in eqm. from E-M interactions

$$e^- + p \longleftrightarrow H + \gamma$$

do calculation & comment on how boltzmann is different

* Number density (non-rel) $\quad n_i = g_i\left(\frac{m_i T}{2\pi}\right)^{3/2} e^{\mu_i - m_i/T}$, $\mu_e + \mu_p = \mu_H$

  apply same trick as before

  note to self: again like last lecture, remove μ dependence we wanted earlier

* Cancel μ with ratio:

spins: can align have zero spin H; align 1, 0, -1 (?) so 4 in total

$$\left(\frac{n_H}{n_e n_p}\right) = \frac{\overset{4}{g_H}}{\underset{2}{g_e}\,\underset{2}{g_p}}\left(\frac{m_H}{m_e m_p}\,\frac{2\pi}{T}\right)^{3/2} e^{(m_p + m_e - m_H)/T}$$

spin degen.

take $m_p \simeq m_e$ here (m_H?)

charge neutrality require $n_e = n_p$ otherwise would completely dominate gravity & blow univ. apart → strong bound on neutrality of univ.

$$\Rightarrow \left(\frac{n_H}{n_e^2}\right) = \left(\frac{2\pi}{m_e T}\right)^{3/2} e^{B_H/T} \qquad (I)$$

neutrality $n_e = n_p$

* Consider free electron fraction $X_e \equiv \frac{n_e}{n_b}$ ← $n_p + n_H$ (neglect not protons)

* Neglecting nuclei beyond protons

$$n_b = n_p + n_H = n_e + n_H, \quad [n_H = n_b - n_e]$$

(exercise:) $$\frac{1 - X_e}{X_e^2} = \left(\frac{n_H}{n_e^2}\right) n_b$$ ← (I)

* We will also use

$$n_b = \eta_b n_\gamma = \eta_b \frac{2\zeta(3)}{\pi^2} T^3 \quad \text{(II)}$$

n_b/n, eta $= \eta_b$ ↑ $\sim 10^{-4}$

$$\Rightarrow \boxed{\left(\frac{1 - X_e}{X_e^2}\right)_{eq} = \frac{2\zeta(3)}{\pi^2} \eta \left(\frac{2\pi T}{m_e}\right)^{3/2} e^{B_H/T}}$$ SAHA EQUATION

B_H: 13.6eV

* Can evaluate this to determine when first H atoms form! Say recombination when $X_e = 0.1$ ← notes say this is reasonable but somewhat arbitrary, so just accept

→ $T_{rec} \simeq 0.3\text{eV}$

univ becomes transparent at 0.3eV ← why lower than expected?

surprising: due to binding energy H, B_H, why?

↳ small η: interpretation: huge no. of photons compared to b so will always be a rare high energy photon to ionise hydrogen until temp falls way below B_H (?))

(high energy tail of photon dist still has enough energy to ionise H atom even when $T < B_H$)

[illegible]

* $T_{rec} = T_0(1+z) \rightarrow z_{rec} = 1320$

CMB temp today

(since $T a^{-1}$ & $= 1+z$?)

estimate t_{rec}: since recombination occurs in matter, estimate $\frac{a(t)}{a(t_0)} = \left(\frac{t}{t_0}\right)^{2/3}$

$\Rightarrow t_{rec} = \frac{t_0}{(1+z_{rec})^{3/2}} \approx 290000$ yrs

"log-log plot"

[illegible] but boltzmann [illegible]

Saha approximation [illegible] more general [illegible] equation. Saha assumes thermal & chem. equilibrium [illegible]. Saha fails when [illegible] equilibrium [illegible] Boltzmann becomes necessary to track the system [illegible] [illegible] "freeze out"). Saha gives a correct estimate for the onset of recombination

considered e^-, what about protons?

* Photons decouple as X_e falls quickly, happens when reactions

$$e^- + \gamma \leftrightarrow e^- + \gamma$$

fall, so that

(left as exercise - see notes)

interaction rate: $\Gamma_\gamma = n_e \sigma_T$ falls w/ n_e

$$\Gamma_\gamma(t_{dec}) \sim H(t_{dec})$$

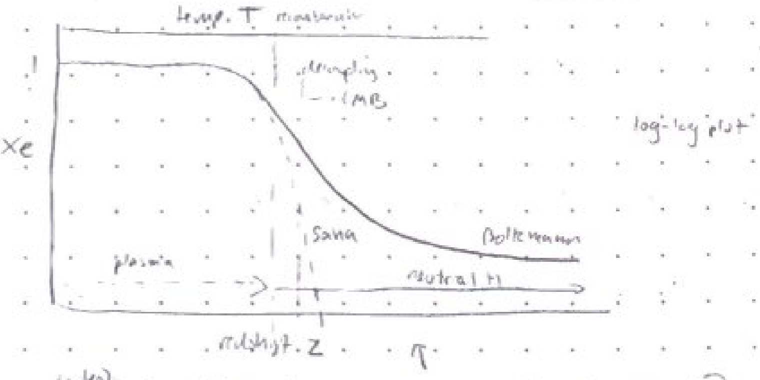
decouple at $\Gamma \sim H$

→ $T_{dec} \simeq 0.27\text{eV}$

($a \propto T$, $\frac{a_0}{a} = 1+z$) ← $z_{dec} \simeq 1100$

(same MD $a \propto t^{2/3}$) ← $t_{dec} \simeq 380{,}000$ yr

CMB photons rapidly decouple.



(notes) evolution of X_e (free electron fraction) in Saha approx and by a more accurate Boltzmann treatment. Saha gives correct estimate of onset of recombination.

6.11.24

Lecture 12: The Inhomogeneous Universe: Introduction + Statistics

halfway through course: let's dramatically change our path: in 1st half of course we examined homogeneous univ., talked about background expansion. But now we look at variations -> on small scales univ. is not homogeneous
how do we square inhomogeneity at small scales with our fundamental assumption of H+I univ.?
↳ the laws of physics are still H+I but there are random fluctuations at small scales (due to quantum stuff)

Cosmological Fields

Interesting: galaxies are not completely uniformly distributed
↳ see web-like structures
↳ "cosmic web"

(1)
* **Large-scale structure**
  - have seen $\sim 10^{11-12}$ galaxies, which are certainly not uniformly distributed.
  - pixelise to get galaxy number density field

define galaxy no. density field after galaxy fractional density contrast

$$\delta_g(\underline{x}) \equiv \frac{n_g(\underline{x}) - \bar{n}_g}{\bar{n}_g}$$

mean density of galaxies
galaxy density at pos. $\underline{x}$

notes: "fractional overdensity of galaxies as a fn of position $\delta_g(\underline{x})$"

(we can't predict exactly where a galaxy will form but we can approximate in large scale)

  - on large scales, good approximation

relate to something we can compute density of

$$\delta_g(\underline{x}) = b \times \delta_m(\underline{x})$$

linear galaxy bias

notes: "matter density perturbation" "fractional matter density contrast"
↳ can predict this theoretically
↳ fortunately it depends on galaxy density -> we can do a taylor expansion

$$\equiv \frac{\rho_m(\underline{x}) - \bar{\rho}_m}{\bar{\rho}_m}$$

matter density

notes: since galaxies form in overdense regions, makes sense but n_g follows n_m (no. density of matter)
we also (?) theoretical motivation, don't go into this course

(2) 2nd major cosmological field:

* **CMB relic** 2.73K blackbody radiation from the early universe

I need microwave eyes to see the observable here: I am interested in measuring variation in intensity of this light as a fn of position

  - we measure intensity field

$$\Delta I(\underline{\hat{n}}) = \frac{I(\underline{\hat{n}}) - \bar{I}}{\bar{I}}$$

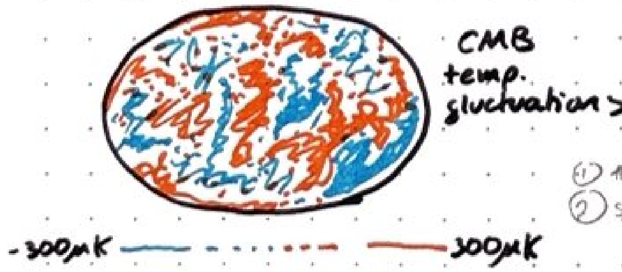
(notes: intensity of CMB radiation on the sky as a fn of direction)
directly observe the state of the univ. ~400,000 yrs after the big bang through CMB

notes: averaged by relative to mean every direction

notes: since CMB near-perfect black body, write in terms of temperature fluctuations $T(\hat{n})$

but usually use $T(\underline{\hat{n}}) = \dfrac{\Delta I(\underline{\hat{n}})}{4}$ as $I \propto T^4$

note to self: see from $\frac{\delta I}{I} = 4\frac{\delta T}{T}$



modern technology: build a thermometer that only recieves light from a certain direction then temp at that direction tells you the intensity of that light.

Important for 2 reasons:
(1) the fact we can see afterglow of big bang is definitive proof for big bang
(2) shows fluctuations in temperature
↳ we can learn from these
↳ shows variation on the last scattering surface: what it looked like in early time
"baby picture of universe"

(can't predict CMB has a cold spot in precisely this direction)

* But: our theories cannot predict $\delta_g(\underline{x})$, $T(\underline{\hat{n}})$, only their variances + statistics! Statistical properties are H+I. (variation arises from random quantum fluctuations in the early univ.)

cannot predict realisations of these fields, only statistical properties

Statistics of Cosmological Fields

(well described by multivariate gaussian & very gaussian, measure departure from gaussian ~10^{-4})

* Cosmological fields on large scales ~ Gaussian

example $\delta(\underline{x}) \rightarrow P[\delta]$ Gaussian — has gaussian probability distribution

general cosmological field (e.g. CMB or large scale structure)
$T(\hat{n})$? $\delta_m(\underline{x})$

introduce general field $\delta(\underline{x})$ to discuss properties of gaussians correlates: do the values rise & fall together? — my intuition: how much does knowing about a field (eg temp or density) at one point tell us about its value at another point? — comes from Fourier series — if spread out over all points (e.g. then every positive is somewhere above mean, somewhere below so avg. should tend to zero if field has zero mean)

can describe gaussian w/ covariance matrix so?

* Fully described by the two-point correlation function

$$\xi(\underline{x},\underline{y}) \equiv \langle \delta(\underline{x})\delta(\underline{y})\rangle$$ ← avg. over universe

ensemble avg over realisations of the univ. ↳ but we only have one univ. so how can we find avg? ↳ only have 1 sample but making use e.g. translation symmetries means we can generate lots more samples (widely separated points ≈ indep. in practice)

notes say Dδ "integral over all field configurations"

$$= \int D\delta\, \delta(\underline{x})\delta(\underline{y})\, P[\delta]$$

probability of field having a given configuration $\delta(\underline{x})$

even though physical properties might not be

* Assume: universe is statistically homogeneous

$$[\mathrm{I}] \quad \xi(\underline{x},\underline{y}) = \xi(\underline{x}+\underline{a}, \underline{y}+\underline{a})$$ — translation

and statistically isotropic — no special direction

$$[\mathrm{II}] \quad \xi(\underline{x},\underline{y}) = \xi(\underline{\underline{R}}\,\underline{x}, \underline{\underline{R}}\,\underline{y})$$ rotation

dep. on which tells us how separated points are

* $[\mathrm{I}]$ implies $\xi(\underline{x},\underline{y}) = \xi(\underline{x}-\underline{y})$ — can only depend on separation of points $\underline{x}$ & $\underline{y}$

since true for all rot it follows

$$[\mathrm{II}] \quad '' \quad \xi(\underline{x},\underline{y}) \overset{\forall \underline{\underline{R}}}{=} \xi(\underline{\underline{R}}(\underline{x}-\underline{y})) \Rightarrow \xi(\underline{x},\underline{y}) = \xi(|\underline{x}-\underline{y}|) = \xi(r)$$

$\underline{y} = \underline{x} + \underline{r}$

* Examine Fourier coefficients

homogeneity

– H
$$\delta(\underline{x}) = \int \frac{d^3k}{(2\pi)^3}\, \delta(\underline{k}) e^{i\underline{k}\cdot\underline{x}}$$ replace $\delta(\underline{x}), \delta(\underline{y})$ w/ sum over fourier modes

use k.v.s: translation in real space ≡ rotation in phase space (k-space)

$$\xi(\underline{x},\underline{y}) = \langle \delta(\underline{x})\delta(\underline{y})\rangle = \int \frac{d^3k\,d^3k'}{(2\pi)^6}\, e^{i\underline{k}\cdot\underline{x} + i\underline{k}'\cdot\underline{y}} \langle \delta(\underline{k})\delta(\underline{k}')\rangle$$

homogeneity requires

$$= \langle \delta(\underline{x}+\underline{a})\delta(\underline{y}+\underline{a})\rangle = \int \frac{d^3k\,d^3k'}{(2\pi)^6}\, e^{i\underline{k}\cdot\underline{x} + i\underline{k}'\cdot\underline{y}} \langle \delta(\underline{k})\delta(\underline{k}')\rangle e^{i(\underline{k}+\underline{k}')\cdot\underline{a}}$$

additional phase!

True if $e^{i(\underline{k}+\underline{k}')\cdot\underline{a}} \langle \delta(\underline{k})\delta(\underline{k}')\rangle = \langle \delta(\underline{k})\delta(\underline{k}')\rangle$ (must be true for all poss. a)

only if $\langle \delta(\underline{k})\delta(\underline{k}')\rangle = (2\pi)^3 P^F(\underline{k})\, \delta^{(3)}(\underline{k}+\underline{k}')$

power spectrum

if i want translation invariance i.e. homogeneity, require it is prop. to a delta fn with const of proportionality related to power spectrum

what about isotropy?

rgd: since require $\underline{k} = -\underline{k}'$ so $\langle\delta(\underline{k})\delta(\underline{k}')\rangle$ is zero unless $\underline{k} = -\underline{k}'$ i.e. it is proportional to delta fn $\delta(\underline{k}-\underline{k}')$

– H + I
$$\langle \delta(\underline{x})\delta(\underline{y})\rangle = \langle \delta(\underline{\underline{R}}\,\underline{x})\delta(\underline{\underline{R}}\,\underline{y})\rangle$$

$$= \int \frac{d^3k\,d^3k'}{(2\pi)^6}\, e^{i\underline{k}\cdot\underline{x} + i\underline{k}'\cdot\underline{y}} \langle \delta(\underline{\underline{R}}^{-1}\underline{k})\delta(\underline{\underline{R}}^{-1}\underline{k}')\rangle$$

think this might be wrong? & k not $\underline{k}'$?

using $\mathrm{FT}[\delta(\underline{\underline{R}}\,\underline{x})] = \delta(\underline{\underline{R}}^{-1}\underline{k})$ property of fourier transforms ↳ show with property of rot matrices: $\underline{\underline{R}}^{-1} = \underline{\underline{R}}^T$ and $|\underline{\underline{R}}| = 1$ (remarkable p196 for full proof)

(show $P^F(\underline{k}) = P^F(\underline{\underline{R}}^{-1}\underline{k})$ ⇒ $P^F(\underline{k}) = P^F(|\underline{k}|)$ if P^F inv under rotations)

Exercise: show this implies

power spectrum

says that: the fourier mode 2 p.f. correlation is prop to delta fn. w/ cnst of proportionality related to power spectrum

$$\boxed{\langle \delta(\underline{k})\delta(\underline{k}')\rangle = (2\pi)^3 P^F(|\underline{k}|)\, \delta(\underline{k}+\underline{k}')} \quad (*)$$

* Simple relation between $\xi(r)$ and $P^F(\underline{k})$

use (*)

subst. $\underline{y} = \underline{x} + \underline{r}$

$$\xi(r) \equiv \xi(\underline{x}, \underline{x}+\underline{r}) = \langle \delta(\underline{x})\delta(\underline{x}+\underline{r})\rangle = \int \frac{d^3k\,d^3k'}{(2\pi)^6}\, e^{i\underline{k}\cdot\underline{x} + i\underline{k}'\cdot\underline{x} + i\underline{k}'\cdot\underline{r}} \langle \delta(\underline{k})\delta(\underline{k}')\rangle$$

nice way of understanding power spectrum ↓

$$= \int \frac{d^3k}{(2\pi)^3}\, P^F(k) e^{i\underline{k}\cdot\underline{r}}$$

the correlation function is the FT of the power spectrum!

rgd: how much power each wavelength contributes

how does that correlate w/ value of distance r?

additional scales & gives $\xi(r)$

my intuition: $\int \frac{d^3k}{(2\pi)^3} P^T(|k|) e^{i k \cdot r} = \xi$

↑ ① take how much power contained at each scale / wavelength

② multiply by a wave of that scale at position r (weight?)

③ add up all contributions from diff scales/wavelengths (integrate) => correlation e.g. if peaks of hills separated by r then we will get high corr.

↳ measure of how repetitive our field is on scale r / how similar are two points sep. by r

Correlation function = fourier transform of power spectrum

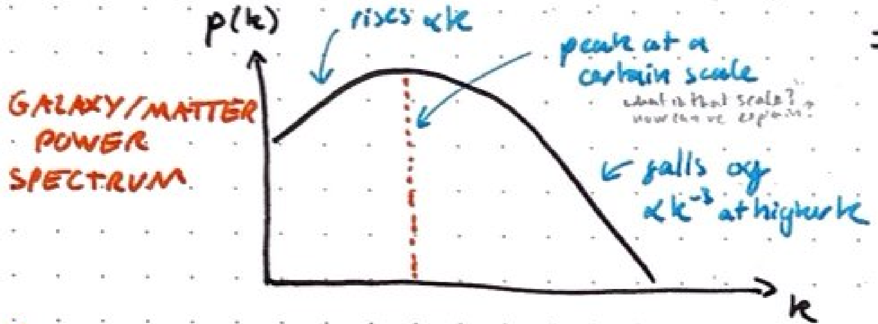
Cosmological Power Spectra

The main spectra we seek to understand:

* **Galaxy power spectrum** $\langle \delta_g(\underline{k}) \delta_g(\underline{k}') \rangle = (2\pi)^3 P^g(\underline{k}) \delta(\underline{k}+\underline{k}')$

$$= (2\pi)^3 b^2 P(\underline{k}) \delta(\underline{k}+\underline{k}')$$

↖ matter power spectrum

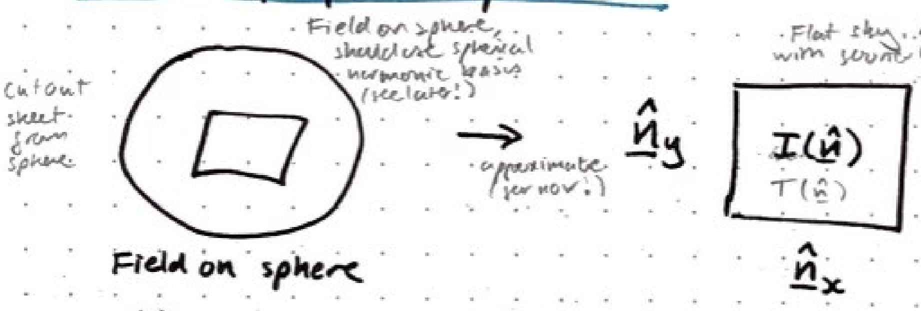


why does it have these properties?

* We want to explain shape + scalings

* **CMB temp. power spectrum**

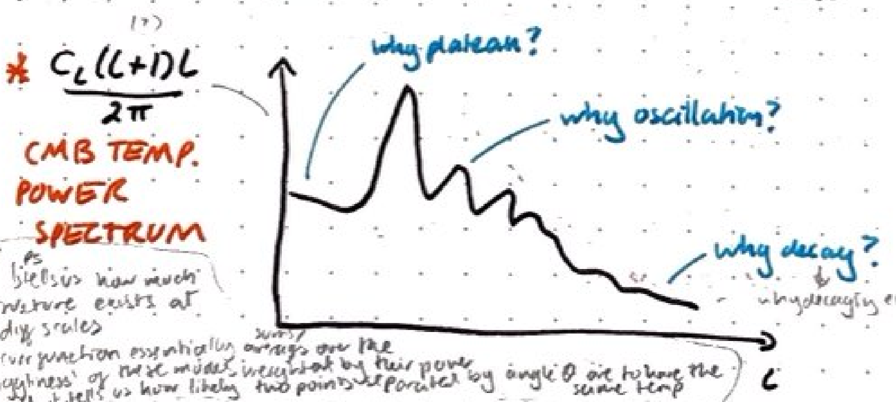
information about PS fully encoded in corr fn & vice versa



measure 2D power spectrum

$P^T(\underline{l}) = C_{\underline{l}}$

↑ label 2D power spectrum C_l



notes:

$\langle T(\underline{l}) T_g(\underline{l}') \rangle = (2\pi)^2 C_l \delta(\underline{l}+\underline{l}')$

↑ 2D FT of $T(\hat{n})$ (temp fluctuation as fn of direction in the sky)

↑ CMB temp power spectrum (2D)

(ps tells us how much structure exists at diff scales
↳ corr function essentially averages over the 'wiggliness' of these modes weighted by their power (sums) and it tells us how likely two points separated by angle θ are to have the same temp)

Assumptions about initial conditions (notes)

Later -> show these observations can be predicted by known physics subject to the following assumptions about initial density fluctuations:

1) their power spectrum is near scale-invariant $P \propto k^{-4+n_s}$ ...

2) they are adiabatic

3) they are gaussian

to be justified at end of course...

(side remark (note)
a gaussian random field is entirely characterised by its two point correlation function or power spectrum)

$\langle \delta_i \delta_j \rangle = \xi(|x_i - x_j|) = \xi_{ij}$

$\Rightarrow P[\delta] \propto \frac{\exp(-\delta_i \xi_{ij} \delta_j)}{\sqrt{\det(\xi_{ij})}}$ (gaussian)

L15 summary (eqn?) (maths?)

* want to understand how small perturbations in the early univ grow over time due to gravity to eventually form large scale structure we see today e.g. galaxies
* to try achieve this we model the univ. as a self-gravitating fluid (that obeys 1),2),3))
* we analyse these equations in an expanding universe & restrict ourselves to linear terms which gives an expression for how a small overdensity will grow over time (on linear scales) in an expanding univ.
* this treatment is a good approx on sub-horizon scales but require full relativistic treatment on larger scales

discuss physics of structure formation

Lecture 13: Gravitational Instability + Newtonian perturbation Theory 8.11.24

* Want to explain structure formation + matter power spectrum

(invoked inflation to explain horizon problem ... gravity fluctuations of inflation is seed from which structure grows)

* Assume initial condition - scale invariant power spectrum of potential fluctuations

(in early universe no particular scale preferred, fluctuations look similar no matter how much we zoom in (right?))

dimensionless power spectrum
$$\Delta(k) \equiv \frac{k^3}{2\pi^2} P(k) \propto k^{n_s - 1} \sim \text{constant}$$

discuss all of this in more detail later

assume power spectrum goes like k^{-3}

Also all $\frac{n_i(\underline{x}) - \bar{n}_i}{\bar{n}_i}$ same for all i → "adiabatic perturbations"

(entropy per particle is const. so if overdensity in e.g. dark matter then all other components also must be overdense.)

basic idea: matter clumps because of gravity

Structure forms through gravitational instability

Overdense region → more gravitational attraction → infall → grows

on small scales non-linear grows even more as more mass

but we restrict to scales where we can do linear perturb. theory

Newtonian Perturbation Theory

(newtonian theory gives good results up to horizon scales; adequate to determine matter evolution on large scales)

(nonlinear problems on small scales but we restrict to large, linear scales)

* Good approximation to GR on subhorizon scales and for non-relativistic matter ($P \ll \rho$) - e.g. CDM, baryons

doesn't work well for e.g. radiation, works on matter

* Consider (non-rel) fluid with mass density ρ, pressure P, velocity $\underline{u}$

Fluid equations

1) Continuity equation $\partial_t \rho + \nabla_r \cdot (\rho \underline{u}) = 0$

2) Euler equation (momentum) $\partial_t \underline{u} + \underline{u} \cdot \nabla_r \underline{u} = -\frac{\nabla_r P}{\rho} - \nabla_r \Phi$

(relative to even stationary fluid, fluid element accelerates?)

reminder: force balance $\rho A dx \times \frac{D u}{D t} = -A \Delta P - \rho A dx \nabla_r \Phi$

AFD $\rho \frac{D\underline{v}}{Dt} = -\nabla p - \rho \nabla \Phi$

$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}$ 'lagrangian derivative'

3) Poisson's equation $\nabla^2 \Phi = 4\pi G \rho$

* Adding expansion

* Comoving observer has $\underline{r}(t) = a(t)\underline{x}$

observer's physical coordinate $\underline{r}$; comoving coordinate $\underline{x}$

→ comoving observer has velocity $\underline{u} = \frac{d\underline{r}}{dt} = \dot{a}\underline{x} = H\underline{r} = Ha\underline{x}$

* Relate physical + comoving derivatives

(see notes, basically chain rule)

$$\left.\frac{\partial}{\partial t}\right|_r = \left.\frac{\partial}{\partial t}\right|_x + \left.\frac{\partial \underline{x}}{\partial t}\right|_r \nabla_x = \partial_t - H(t)\underline{x}\nabla$$

$\left.\frac{\partial \underline{x}}{\partial t}\right|_r = -a^{-1}\underline{r}$ ... $\left.\frac{\partial}{\partial t}\right|_x$

gradient w.r.t. x^i (comoving coord)

my intuition: change at physical coord = change at grid point + change due to motion of grid point itself (due to expansion)

the chain rule: $\left.\frac{\partial}{\partial t}\right|_r = \frac{\partial x}{\partial t}\Big|_r \frac{\partial}{\partial x} + \left.\frac{\partial}{\partial t}\right|_x$?

(think $f(r,t) = \phi(x(r,t),t)$; then $\left.\frac{\partial f}{\partial t}\right|_r = \left.\frac{\partial \phi}{\partial t}\right|_x + \frac{\partial \phi}{\partial x}\left.\frac{\partial x}{\partial t}\right|_r$ (right))

gradient w/o subscript? means at fixed t (comoving coord)

AFD $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v}\cdot\nabla$

Perturbation analysis

Take $\rho \to \bar{\rho} + \delta\rho = \bar{\rho}(1+\delta)$ $\quad \delta \equiv \frac{\delta\rho}{\bar{\rho}}$ fractional density contrast (perturbation)

$\underline{u} = Ha\underline{x} + \underline{v}$

$P \to \bar{P} + \delta P$

$\Phi \to \bar{\Phi} + \phi$

also $\nabla_r = \frac{1}{a}\nabla$

reminders to self:
- remember to convert $\delta_\rho \to \delta$ at end
- background values depend only on time (not $\underline{x}$!)
- $\underline{v}$ is weird → but still keep deriv. wrt. $\underline{x}$ for $Ha\underline{x}$ (but still treat $\underline{v}$ as small perturbation / 1st order term)
- ignore 2nd order terms (remembering $\underline{v}$ is small)
- $(\underline{v}\cdot\nabla)\underline{x} = \underline{v}$ and $\nabla\cdot\underline{x} = 3$!

and use $\partial_t|_r \to \partial_t - H\underline{x}\cdot\nabla$

$\nabla_r \to \frac{1}{a}\nabla$

2nd order $\delta\rho\,\delta p$

Analyse this:

* Continuity equation

proceed order by order

$$[\partial_t - H\underline{x}\nabla](\bar\rho(1+\delta)) + \tfrac{1}{a}\nabla[\bar\rho(1+\delta)(Ha\underline{x}+\underline{v})] = 0$$

0th order: $\partial_t\bar\rho + \frac{1}{a}\bar\rho Ha\underbrace{\nabla\cdot\underline{x}}_{3} = 0$

$$\Rightarrow \underline{\partial_t\bar\rho + 3H\bar\rho = 0} \quad \text{CE}$$

similar: standard continuity eq.) for matter

1st order: $[\partial_t - H\underline{x}\nabla](\bar\rho\delta) + \frac{1}{a}\nabla[\bar\rho\delta Ha\underline{x} + \bar\rho v] = 0$

identity
$\nabla(\psi\underline{A}) = \psi(\nabla\cdot\underline{A}) + \underline{A}(\nabla\psi)$
$\nabla\cdot(\psi\underline{A}) = \psi(\nabla\cdot\underline{A}) + \underline{A}\cdot(\nabla\psi)$

my way: subst. $\delta = \frac{\delta\rho}{\bar\rho}$, into $\partial_t\delta\rho + 3H\delta\rho + \frac{1}{a}\bar\rho\,\nabla\cdot\underline{v} = 0$

$$\Rightarrow [\dot{\bar\rho} + 3\bar\rho H]\delta + \bar\rho[\dot\delta + \tfrac{1}{a}\nabla\cdot\underline{v}] = 0$$

$$\Rightarrow \underline{\dot\delta = -\tfrac{1}{a}\nabla\cdot\underline{v}} \quad (\mathrm{I})$$

proceed similarly for other equations

* Euler equation

note to self: from $(\underline{v}\cdot\nabla)\underline{x} = \underline{v}$

$$\Rightarrow \underline{\dot{\underline{v}} + H\underline{v} = -\frac{\nabla\delta p}{a\bar\rho} - \frac{\nabla\phi}{a}} \quad (\mathrm{II})$$

* Poisson

$$\Rightarrow \underline{\nabla^2\phi = 4\pi G a^2\bar\rho\delta} \quad (\mathrm{III})$$

* Combine : Take $\nabla(\mathrm{II})$, ins (I), (III)

elim v

- take divergence of II
- elim $\underline{v}$ using I and $\dot{\mathrm{I}}$
- elim δ with III

$$(\nabla\cdot\dot{\underline{v}}) + H(\nabla\cdot\underline{v}) = -\frac{\nabla^2\delta p}{a\bar\rho} - \frac{(\nabla^2\phi)}{a}$$

$(\nabla\cdot\dot{\underline{v}}) = -a\dot{\dot\delta}$ ← $-a\ddot\delta - Ha\dot\delta$, $(\nabla\cdot\underline{v}) = -a\dot\delta$

gravity drives growth in univ. gravity: early growth

opposes growth - harder to grow might in an expanding univ.

DESCRIBES STRUCTURE GROWTH (†) in Newtonian theory for non-rel matter

$$\boxed{\ddot\delta + 2H\dot\delta - \frac{1}{a^2\bar\rho}\nabla^2\delta p - 4\pi G\bar\rho\delta = 0}$$

HUBBLE FRICTION — PRESSURE — GRAVITY

press also opposes growth. prevents clumping
gas on small scales pressure force operates efficiently but above Jeans length don't feel pressure, everything collapses so we ignore this term generally. for dark matter not v. important!

how does a matter perturbation grow in diff epochs?

Dark matter evolution ($\delta P = 0$) on sublinear scales

inside Hubble radius (inside horizon?)

derive growth of perturbation

$c_s = 0$

matter EOS, $P = w\rho$, $w = 0 \Rightarrow P = 0$

$\delta P = 0$ since dark matter is collisionless

→ consequence: no pressure - homogeneous dark matter perturbations grow on all scales, not just above the Jeans length (cf. gas)

* Matter domination : neglect non matter perturbations + contributions to H

+ assume Hubble rate only defined by matter density

(†) gives $\ddot\delta_m + 2H\dot\delta_m - 4\pi G\bar\rho_m\delta_m = 0$

since $a \propto t^{2/3}$, $H = \frac{\dot a}{a}$ $\frac{2}{3t}$; (using) $H^2 = \frac{8\pi G\bar\rho_m}{3}$

$$\rightarrow \ddot\delta_m + \frac{4}{3t}\dot\delta_m - \frac{2}{3t^2}\delta_m = 0$$

power law ansatz

try power law ansatz $\delta_m \propto t^c$

$$c(c-1) + \tfrac{4}{3}c - \tfrac{2}{3} = 0 \quad \rightarrow c = \begin{cases} 2/3 \\ -1 \end{cases}$$

$$\Rightarrow \delta_m \propto \begin{cases} t^{2/3} & \text{growing} \\ 1/t & \text{decaying mode} \end{cases} \quad \text{Neglect latter}$$

$$\Rightarrow \boxed{\delta_m \propto a} \quad \text{MD (growing)}$$

(during MD matter perturbations grow like a)

* Radiation domination: total ρ sources Φ — total energy density sources the potential

so $\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G \sum_i \bar{\rho}_i \delta_i = 0$

all components contribute to potential. matter in radiation dom. most important terms should be radiation ones δ_r but not the case: δ_r oscillates very fast so forces produced by this given potential average out: don't care about fast oscillations from radiation, care about slow mode growth → drop radiation term

later we will show δ_r oscillates rapidly → effect averages to zero

also no $\bar{\rho}_\Lambda$ term since $\bar{\rho}_\Lambda \propto a^0 = $ const? so constant, fluct. → $\delta_\Lambda = 0$?

since $a \propto t^{1/2}$, $H = \frac{1}{2t}$,

$$\Rightarrow \ddot{\delta}_m + \frac{1}{t}\dot{\delta}_m - \cancel{4\pi G \bar{\rho}_m \delta_m} = 0$$

drop 3rd term

order of mag estimate — only relevant timescale is H

$$\text{Order of mag}: \ddot{\delta}_m \sim H\dot{\delta}_m \sim H^2 \delta_m \sim \frac{8\pi G \bar{\rho}_r}{3}\delta_m \gg 4\pi G \bar{\rho}_m \delta_m$$

$$\Rightarrow \ddot{\delta}_m + \tfrac{1}{t}\dot{\delta}_m = 0 \rightarrow \boxed{\delta_m \propto \ln a} \quad \text{RD (growing)}$$

sub $x = \dot{\delta}_m$, $\dot{x} = -\frac{x}{t}$ → $\delta_m \propto \ln t$

understand this: universe expands quickly, not balanced by gravity, driving ... slows growth down

* Dark energy domination

since $\bar{\rho}_\Lambda \propto a^0 = $ const.

no fluctuations in Λ so: $-4\pi G \sum_i \bar{\rho}_i \delta_i$ becomes just m term

$$\ddot{\delta}_m + 2H\dot{\delta}_m - \cancel{4\pi G \bar{\rho}_m \delta_m} = 0$$

drop 3rd term

$$\text{again } H^2 \approx \frac{8\pi G \bar{\rho}_\Lambda}{3} \gg 4\pi G \bar{\rho}_m$$

$$\rightarrow \ddot{\delta}_m + 2H\dot{\delta}_m = 0 \longrightarrow \boxed{\delta_m \simeq \text{const.}} \quad \Lambda\text{D (const.)}$$

$\sim$ const. due to Λ ← (ΛD: $a \propto e^{Ht}$)

(as well as decaying solution $\delta_m \propto e^{-2Ht}$ which we ignore)

→ perturbations stop growing in ΛD

we have seen how structure grows in every epoch ... growth just stops

(conclusion)

* Growth of structure: sensitive probe of energy content & perturbation properties.
(non-clustering ρ suppresses structure growth)

→ added by me

* Aside: barotropic fluid $P = P(\rho)$, $\delta P = \frac{\partial P}{\partial \rho}\delta\rho = \frac{\partial P}{\partial \rho}\bar{\rho}\delta \equiv c_s^2 \bar{\rho}\delta$

i.e. treat $\delta\rho$ as δx: $\delta_k e^{i(k\cdot x - \omega t)}$

Fourier transform $\nabla \rightarrow -ik$, $\ddot{\delta} + 2H\dot{\delta} + \left(\frac{c_s^2 k^2}{a^2} - 4\pi G \bar{\rho}\right)\delta = 0$

$k > k_J$ (small scales, $\lambda < \lambda_J$) get damped oscillations; but $k < k_J$ (large scales, $\lambda > \lambda_J$) unstable, leads to collapse

Gas clouds with $\frac{2\pi a}{\lambda} = k < k_J \equiv \frac{\sqrt{4\pi G \bar{\rho} a^2}}{c_s}$ collapse! ⇒ galaxies!

$k_J = \frac{2\pi a}{\lambda_J}$, Jeans length $\lambda_J = c_s\sqrt{\frac{\pi}{G\bar{\rho}}}$

on large scales pressure cannot successfully resist gravitational infall since not enough time for pressure wave ...

Lecture 14 : Cosmological Perturbation Theory

11.11.24

understand details of how small perturbations grow w/ time. last lecture: valid for matter on subhorizon scales (Newtonian), now want more detail - more generality

* Want to develop perturbation theory valid for all scales + also radiation - apply the full GR-PT to derive evolution equations!

not just matter

Perturbed metric

FRW: $ds^2 = a^2(\tau)\left[d\tau^2 - \delta_{ij}\,dx^i dx^j\right]$ describes perfectly uniform (homogeneous) univ. but now we want to consider small perturbations to the metric $g_{\mu\nu} \to \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$

* General perturbation of FRW

A, B, h_{ij} totally general perturbations of space & time

$$ds^2 = a^2(\tau)\left((1+2A)\,d\tau^2 - 2B_i\,dx^i d\tau - (\delta_{ij} + h_{ij})\,dx^i dx^j\right) \quad (\dagger)$$

fn. of space & time $\quad A = A(\tau, \underline{x}),\ B_i = B_i(\tau, \underline{x})$ etc.

don't perturb scale factor, if perturbation small I can absorb it into definition of other factors

* Perform Scalar-vector-decomposition (aka SVT)

analogy to EM Helmholtz decomposition with ϕ & A but more complex b/c dealing w/ tensors

Define

write in terms of scalar, vector, tensor perturbations - most general decomposition

- $A = A$ (scalar)
- $B_i = \partial_i \underline{B} + \underline{B}_i^{\,v}$ ← decomp into scalar + vector
- $h_{ij} = 2C\,\delta_{ij} + 2(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2)E + (\partial_i E_j^{\,v} + \partial_j E_i^{\,v}) + 2E_{ij}^{\,T}$

$\uparrow$ trace scalar $\qquad \uparrow$ trace free scalar

here vector quantities are transverse, i.e. $\partial^i B_i^{\,v} = \partial^i E_i^{\,v} = 0$
and tensors are transverse / traceless $\partial^i E^T_{ij} = \delta^{ij} E^T_{ij} = 0$

* Scalar perturbations only $\ ^v = {}^T = 0$

JUSTIFICATION?

Gauge Problem

* Consider unperturbed universe

$$ds^2 = a^2(\tau)(d\tau^2 - \delta_{ij}\,dx^i dx^j)$$

make small change of coordinates $x^i \to \hat{x}^i = x^i + \Sigma^i(\tau, \underline{x})$

$$dx^i = d\hat{x}^i - \partial_\tau \Sigma^i d\tau - \partial_k \Sigma^i d\hat{x}^k$$

$x, \hat{x}$ differ only by 1st order perturbation, non trivial: replace x by $\hat{x}$ here

then we get → keeping linear terms only!

$$ds^2 = a^2(\tau)(d\tau^2 - 2\Sigma_i'\,d\hat{x}^i d\tau - (\delta_{ij} + 2(\partial_{(i}\Sigma_{j)})\,d\hat{x}^i d\hat{x}^j)$$

just by changing spatial coord by small amount we get a perturbed universe, hence we started w/ unperturbed univ, there are "fake" perturbations

→ have induced "fake" perturbations Σ_i' and $2\partial_{(i}\Sigma_{j)}$

= gauge modes

Redundancy in description different perturbations can describe same physics.

* General question: how do perturbations change under gauge transformation? Consider general gauge transformation with only scalars

$x^\mu \to \hat{x}^\mu = x^\mu + \xi^\mu$ i.e. $\left[\begin{array}{l} \tau \to \tau + T \\ x^i \to x^i + \partial_i L \end{array}\right]$

$$x^\mu \to \hat{x}^\mu + \xi^\mu \; ; \quad \underbrace{\xi^0 \equiv T}_{\text{scalar}}, \; \underbrace{\xi^i = \partial_i L}_{\text{scalar}}$$

general gauge transformation only involving scalar perturbations here (in general would have vector L^T)

also no tensor part ⇒ can immediately deduce tensor part is gauge invariant & not affected by this problem

What is the effect of gauge transformations on metric perturbations?

* From $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \hat{g}_{\alpha\beta}\, d\hat{x}^\alpha d\hat{x}^\beta$ → has to be true: invariant interval same in all coord. systems (line element is invariant)

$$g_{\mu\nu} = \frac{d\hat{x}^\alpha}{dx^\mu}\frac{d\hat{x}^\beta}{dx^\nu}\hat{g}_{\alpha\beta}$$ relate metrics

hence can describe components e.g. 00 in $\hat{x}$ (any other than $\alpha,\beta = 0$ is 2nd order (see notes))

see how this applies to A

$$g_{00} = \frac{d\hat{x}^\alpha}{dx^0}\frac{d\hat{x}^\beta}{dx^0}\hat{g}_{\alpha\beta} = \left(\frac{d\hat{x}^0}{dx^0}\right)^2 \hat{g}_{00} \Longleftrightarrow a^2(\tau)(1+2A) = (1+T')a^2(\tau+T)\cdot(1+2\hat{A})$$

drop $\alpha\beta = 0i, ii$ as ξ^2

(because of $\tau + T$) not multiplied by

conformal Hubble param.

$$a^2(\tau)(1+2A) = (1+T')^2 a^2(\tau+T)(1+2\hat{A})$$
$$= a^2(\tau)(1 + 2\mathcal{H}T + 2T' + 2\hat{A} + \dots)$$

$a^2(\tau+T) = a^2(\tau) + T\frac{da^2}{d\tau} + \dots$; $\frac{d}{d\tau} = a\frac{d}{dt}$

Taylor expand (keep only linear terms)

we have found how A transforms explicitly, B, C, E left as exercise

$$\Rightarrow \left[\begin{array}{l} \hat{A} = A - T' - \mathcal{H}T \\ \hat{B} = B + T - L' \\ \hat{C} = C - \mathcal{H}T - \tfrac{1}{3}\nabla^2 L \\ \hat{E} = E - L \end{array}\right]$$

effect of gauge transf. on general metric perturbations (?)

← I'm so lost on this

what we meant by redundant: we actually only have 2 deg. of freedom not 4 (2 'fake' gauge deg of freedom)

[2 real deg., 2 gauge freedom]

How to solve gauge problem? 1 Work with gauge-invariant quantities e.g. "Bardeen potentials"

can see it applying here in Taylor exp.

$$\Psi_B = A + \mathcal{H}(B - E') + (B - E')'$$
$$\Phi_B = -C - \mathcal{H}(B - E') + \tfrac{1}{3}\nabla^2 E$$

2nd way: say I don't care; any redundancy has to be gone once I measure observables e.g. frequency of light

2 Compute observables

in order to compute observables.

3 Fix the gauge: use freedom in T/L to set

a) Synchronous: $A = B = 0$, unperturbed time

b) Spatially flat: $C = E = 0$, " space

we use c) Longitudinal/Newtonian gauge $B = E = 0$, no off diag. in metric

↳ perturbations ~ potential in Newtonian gravity

in this gauge spatial perturbations are directly analogous to potential in Newtonian gravity

* In Newtonian gauge, renaming $A = \Psi$, $C = -\Phi$

$$\Rightarrow \boxed{ds^2 = a^2(\tau)\left((1+2\Psi)d\tau^2 - (1-2\Phi)\delta_{ij}\,dx^i dx^j\right)}$$

perturbed metric, 2 deg of freedom.

Perturbed Matter

energy-mom tensor for fluid at the background level

* Now consider E-M perturbation of $\bar{T}^{\mu}{}_{\nu} = (\bar{\rho} + \bar{P})\bar{U}^{\mu}\bar{U}_{\nu} - \bar{P}\delta^{\mu}_{\nu}$

$\frac{dx^0}{ds} = \frac{d\tau}{\rho d\tau}$?

Perturbing gives

anything missed

$$\delta T^{\mu}{}_{\nu} = (\delta\rho + \delta P)\bar{U}^{\mu}\bar{U}_{\nu} + (\bar{\rho} + \bar{P})(\delta U^{\mu}\bar{U}_{\nu} + \bar{U}^{\mu}\delta U_{\nu}) - \delta P\delta^{\mu}_{\nu} - \Pi^{\mu}{}_{\nu}$$

stationary? → fluid element at rest wrt cosmic expansion? (rgpt) → $ds^2 = a^2 d\tau^2$?
i.e. moves with expansion, doesn't move wrt comoving coords x?

for our purposes ignore — anisotropic stress

* $\bar{U}^{\mu} = \frac{1}{a}(1, 0)$, also know $g_{\mu\nu}U^{\mu}U^{\nu} = 1$

$\bar{U}^{\mu} = \frac{dx^{\mu}}{ds}$

normalisation condition for any massive particle, not an assumption → automatically follows from defining $U^{\mu} = \frac{dx^{\mu}}{ds}$ (see from plugging into metric) (rgpt)

$$\Rightarrow U^{\mu} = \frac{1}{a}(1 - A, \underline{v}^i)$$

notes

$$\delta T^0{}_0 = \delta\rho, \quad \delta T^i{}_0 = (\bar{\rho} + \bar{P})\delta^i v_i, \quad \delta T^0{}_j = -(\bar{\rho} + \bar{P})\partial_j(v + B)$$

$$\delta T^i{}_j = -\delta P\delta^i{}_j - \Pi^i{}_j$$

note:

* Total E-M tensor is sum of individual tensors so

$$\delta\rho = \sum_I \delta\rho_I, \quad \delta P = \sum \delta P$$

* Gauge invariant quantity $\Delta = \frac{\delta\rho}{\bar{\rho}} + \frac{\bar{\rho}'}{\bar{\rho}}(v + B)$

Linearised Evolution Equations

$$\delta g_{\mu\nu}(\Phi, \Psi), \; \delta T^{\mu}{}_{\nu}(\delta\rho, \delta P, v) \rightarrow \begin{array}{l} \nabla_{\mu} T^{\mu}{}_{\nu} = 0 \\ \delta G_{\mu\nu} = 8\pi G\, \delta T^{\mu}{}_{\nu} \end{array}$$

why $\bar{U}^{\mu} = \frac{1}{a}(1, \underline{0})$?
Given $U^{\mu} = \frac{dx^{\mu}}{ds}$

"tangent 4-vector of a timelike world line"

see by:
$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$
so $g_{00} = a^2$, $x^0 = \tau$ → $\frac{dx^0}{ds} = \frac{d\tau}{a d\tau} = \frac{1}{a}$?
and $ds = a d\tau$

want to find δU^{μ}:
- use $g_{\mu\nu}U^{\mu}U^{\nu} = 1$
- perturb =>

$$\delta g_{\mu\nu}\bar{U}^{\mu}\bar{U}^{\nu} + g_{\mu\nu}\delta U^{\mu}\bar{U}^{\nu} + g_{\mu\nu}\bar{U}^{\mu}\delta U^{\nu} = 0$$
$$\delta g_{\mu\nu}\bar{U}^{\mu}\bar{U}^{\nu} + 2g_{\mu\nu}\delta U^{\mu}\bar{U}^{\nu} = 0$$
$$\delta g_{\mu\nu}\bar{U}^{\mu}\bar{U}^{\nu} + 2\delta U^{\mu}\bar{U}_{\mu} = 0$$
$$2a^2 A \frac{1}{a^2} + 2a\delta U^0 = 0$$
$$\Rightarrow \delta U^0 = -\frac{1}{a}A$$

$\bar{U}_{\mu} = \bar{g}_{\mu\nu}\bar{U}^{\nu}$
$= a(1, \underline{0})$

free to define δU^i so we let
$U^{\mu} = \frac{1}{a}(1 - A, v^i)$ where $v^i \equiv \frac{dx^i}{d\tau}$
$= \partial_i v$ (since have scalars?)

13.11.24

Lecture 15 Cosmological Perturbation Theory - Preliminaries / Initial Conditions

* **Last time**: basics of perturbation theory and subtleties (gauges)
* To get evolution equations, perturb in Newtonian gauge

invoke EM conservation &

$$ds^2 = a^2(\tau)\left[(1+2\phi)d\tau^2 - (1-2\psi)\delta_{ij}dx^i dx^j\right]$$

also evaluate $\delta T^\mu_{\ \nu}$ in terms of $\delta\rho$ δP, $v_i = \partial_i v$

notational: remember $\nabla_\mu \to \partial_\mu + \Gamma^{\cdot}_{\cdot\cdot} + \Gamma^{\cdot}_{\cdot\cdot}$

* Impose Energy-momentum conservation $\nabla_\mu \delta T^\mu_{\ \nu} = 0$

conservation of E-M apply for each component individually, as long as interaction not too strong

1st order $\rightarrow$ continuity equation

modified version of ordinary continuity, to add expanding universe & potentials

$$\boxed{\delta' + 3\mathcal{H}\left(\frac{\delta P}{\delta\rho} - \frac{\bar{P}}{\bar{\rho}}\right)\delta = -\left(1 + \frac{\bar{P}}{\bar{\rho}}\right)(\nabla\cdot\underline{v} - 3\phi')} \quad \text{CE}$$

remember, basically conserving momentum

$\rightarrow$ Euler equation

$$\boxed{\underline{v}' + 3\mathcal{H}\left(\frac{1}{3} - \frac{\bar{P}}{\bar{\rho}}\right)\underline{v} = -\frac{\nabla\delta P}{\bar{\rho}+\bar{P}} - \nabla\phi} \quad \text{EU}$$

(apply for all components individually)

* Evaluate Einstein equations

$$\underbrace{G_{\mu\nu}}_{\bar{G}_{\mu\nu}+\delta G_{\mu\nu}} = 8\pi G \underbrace{T_{\mu\nu}}_{\bar{T}_{\mu\nu}+\delta T_{\mu\nu}}$$

notes: $\underline{v} = \nabla v$

1st order $\rightarrow$

where did ψ go? eq relates ψ, ϕ via anisotropic stress, if zero then $\psi = \phi$, we assume anisotropic stress is 0

$$\boxed{\nabla^2\phi - 3\mathcal{H}(\phi' + \mathcal{H}\phi) = 4\pi G a^2 \delta\rho} \quad \text{PE}$$

Poisson eq.

$$\boxed{\phi'' + \mathcal{H}\phi' = -4\pi G a^2(\bar{\rho} + \bar{P})v} \quad \text{E2}$$

note here, potential is sourced by total matter

$$\boxed{\phi'' + 3\mathcal{H}\phi' + (2\mathcal{H}' + \mathcal{H}^2)\phi = 4\pi G a^2 \delta P} \quad \text{E3}$$

2nd & 3rd potential eqs

* Recall comoving gauge density contrast

$$\Delta = \frac{\delta\rho}{\bar{\rho}} + \frac{\bar{\rho}'}{\bar{\rho}}(v + \cancel{B})$$

in Newtonian gauge $B = 0$. we specify Newtonian gauge outline

gauge invariant quantity

In Newtonian gauge $\Delta = \delta - 3\mathcal{H}\left(1+\frac{\bar{P}}{\bar{\rho}}\right)v$
subst. E2 in PE

$$\boxed{\nabla^2\phi = 4\pi G a^2 \bar{\rho}\,\Delta} \quad P$$

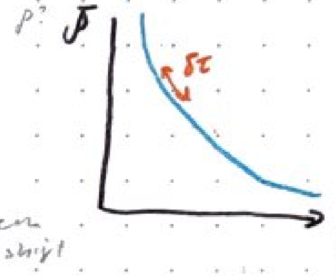
Adiabatic Perturbations - made by inflation

* Perturbations describable by time delay ($\underline{x}$) for all components

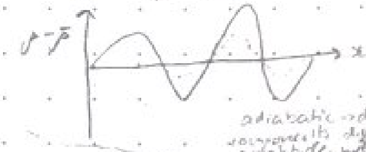
general: $\delta\rho_I = \bar{\rho}_I(\tau+\delta\tau) - \bar{\rho}_I(\tau) = \bar{\rho}_I'\,\delta\tau(\underline{x})$

adiabatic definition: Adiabatic: $\delta\tau(\underline{x})$ same for all I

$$\delta\tau = \frac{\delta\rho_I}{\bar{\rho}_I'} = \frac{\delta\rho_J}{\bar{\rho}_J'}$$

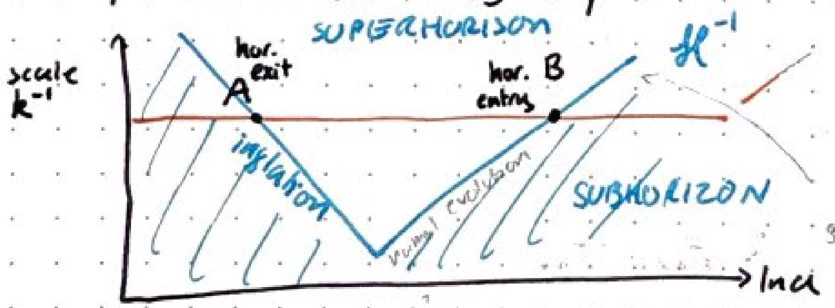


* Using $\bar{\rho}_I' = -3\mathcal{H}(1+\omega_I)\bar{\rho}_I \Rightarrow \frac{\delta_I}{1+\omega_I} = \frac{\delta_J}{1+\omega_J}$, e.g. $\delta_r = \frac{4}{3}\delta_m$

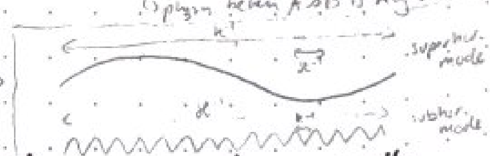


Preliminaries

* Recap: subhorizon vs superhorizon scales



- $k \ll \mathcal{H}$ superhorizon, $k \gg \mathcal{H}$ subhorizon
- only subhorizon modes in causal contact, evolve dynamically



* How to calculate from A to B despite unknown physics?

Introduce variable in which inflation generates perturbations

$$\mathcal{R} \equiv C - \tfrac{1}{3}\nabla^2 E + \mathcal{H}(B + v)$$

$$\boxed{\mathcal{R} = -\phi + \mathcal{H}v}$$
newtonian gauge

$$= -\phi - \frac{\mathcal{H}(\phi' + \mathcal{H}\phi)}{4\pi G a^2(\bar{\rho}+\bar{P})}$$

Evaluate (see notes) for adiabatic modes:

$$\mathcal{R}' \rightarrow \frac{d\ln\mathcal{R}}{d\ln a} \sim \left(\frac{k}{\mathcal{H}}\right)^2 \underset{\substack{\text{superhorizon} \\ k \ll \mathcal{H}}}{\sim} 0$$

→ $\mathcal{R}$ is conserved on superhorizon scales.
assuming adiabatic perturbations (notes)

Recall: initial conditions assumed

$$\Delta^2_{\mathcal{R}} = \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k) \sim \text{constant}$$

power spectrum of comoving curvature perturb. times k^3 is ~ constant. describe inflationary initial conditions by comoving curvature perturbation

now continue discussion of how perturbations evolve
LARGE SCALE STRUCTURE OBSERVABLES FROM COSMOLOGICAL PERTURBATION THEORY:

Evaluate evolution eqs. i) Φ ii) δ_r iii) δ_m

how do potential, radiation & matter evolve?
we now eval evolution of perturbations beyond the validity of the newtonian theory

(after inflation the relevant fluctuations are assumed to be superhorizon (?))

Initial conditions: on superhorizon scales (?)

* Get differential equation for Φ if one component dominates with $\frac{dP}{d\rho} = \omega$ notes say $\omega = \frac{\bar{P}}{\bar{\rho}} \approx c_s^2$

→ write $\delta P = \omega\delta\rho$ in E3, sub in PE.

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = \omega\nabla^2\Phi - \omega 3\mathcal{H}\Phi' - \omega 3\mathcal{H}^2\Phi$$

from Friedmann, $\mathcal{H}^2 + 2\mathcal{H}' + 3\omega\mathcal{H}^2 = 0$.

F1: $\mathcal{H}^2 = \frac{8\pi G \bar{\rho} a^2}{3}$
F2: $\mathcal{H}' = -\frac{4\pi G \bar{\rho} a^2(1+3\omega)}{3}$
$\Rightarrow \mathcal{H}^2(1+3\omega) = -2\mathcal{H}'$

FT, $\nabla \rightarrow -i\underline{k}$

group terms w/ Φ except $\omega\nabla^2\Phi$ & plug in this expr.

$$\boxed{\Phi'' + 3(1+\omega)\mathcal{H}\Phi' + k^2\omega\Phi = 0}$$

evolution of the potential when one component dominates?
(on superhor. scales Φ = const.)

Superhorizon limit ($k \ll \mathcal{H}$)

solution $\ddot{x} + a\dot{x} = 0$, x = const. or $x \propto e^{-at}$
only one growing mode solution $\Rightarrow \Phi$ = const.

* Since $k \ll \mathcal{H}$, we can neglect $k^2\omega\Phi$

$\Phi'' + 3(1+\omega)\mathcal{H}\Phi' = 0$, $\Rightarrow \underline{\Phi = \text{const.}}$ true | when 1 component dominates.

(Φ = const true when one component dominates. But Φ must change during transition from radiation to matter dominated)

* To see Φ changes, recall def of comoving curvature perturbation

$$\mathcal{R} = -\Phi - \frac{\mathcal{H}(\Phi' + \mathcal{H}\Phi)}{4\pi G a^2(\bar{\rho} + \bar{P})} \overset{\mathcal{H}^2 \leftarrow F1}{=} -\frac{5+3\omega}{3+3\omega}\Phi \quad \text{(superhorizon)}$$

reminder ($MD: \omega = 0$, $RD: \omega = \frac{1}{3}$)

$\mathcal{R}$ conserved → $\Phi_{MD} = \frac{9}{10}\Phi_{RD}$ (matter dom, radiation dom), also → $\Phi_{RD}(\underline{k}, \tau=0) = -\frac{2}{3}\mathcal{R}(\underline{k}, \tau=0)$

initial condition for the potential

* Get ICs for δ_m, δ_r

subst. F1 into RHS of PE & fourier trafo.

PE + friedmann: $\delta = \cancel{-\frac{2}{3}\frac{k^2\Phi}{\mathcal{H}^2}}^{\text{superhor}} \cancel{-\frac{2\Phi'}{\mathcal{H}}}^{\Phi = \text{const.}} - 2\Phi$

dominant component → $\delta = -2\Phi = \text{const.}$ (superhorizon)

behaviours of superhorizon perturbations of dominant component (e.g. radiation or matter).

comoving. take deriv. $\mathcal{R}'$ & use evolution equations → $4\pi G a^2(\bar{\rho}+\bar{P})\mathcal{R}' = 4\pi G a^2 \mathcal{H}\delta P_{nad} + \mathcal{H}\frac{\bar{P}'}{\bar{\rho}'}\nabla^2\Phi$ where $P_{nad} = \delta P - \frac{\bar{P}'}{\bar{\rho}'}\delta\rho$. then in fourier space $\nabla \to -ik$ and noting $\Phi \sim \mathcal{R}$ ⇒ $\frac{d\ln\mathcal{R}}{d\ln a} \sim \left(\frac{k}{\mathcal{H}}\right)^2$? I don't get it. adiabatic ⇒ throw away P_{nad} term ⇒ get $\mathcal{R}' \propto \frac{\mathcal{H}}{\bar{\rho}+\bar{P}}\frac{\bar{P}'}{\bar{\rho}'}k^2\Phi$? use $\frac{d\ln\mathcal{R}}{d\ln a} = \frac{\mathcal{R}'}{\mathcal{R}\mathcal{H}}$?

reminder from adiabatic perturbations

- we can relate ϕ, δ_m, δ_r (all const.) on superhor. scales and different eras to each other & also to inflationary initial conditions $R(0)$.

Summary L15
- in inflation relevant fluctuations assumed to be superhorizon
- potential ϕ is const. when one component dominates on superhor. scales.
- we have also seen δ_m, δ_r are const. on superhor. scales

* Radiation domination

$$\delta_r = -2\phi_{RD}$$

$$= \tfrac{4}{3} R(0) = \text{const.}$$

$$\xrightarrow{\delta_m = \frac{3}{4}\delta_r} \quad \delta_m = -\tfrac{3}{2}\phi_{RD} = R(0) = \text{const.}$$

$\delta_r = 2\phi_{RD}$ -> RD -> rad. sources potential

$\delta_m = 2\phi_{MD}$ -> MD -> matter sources potential

* Matter domination

$$\delta_m = -2\phi_{MD} = \tfrac{6}{5} R(0), \qquad \delta_r = -\tfrac{8}{3}\phi_{MD} = \tfrac{8}{5} R(0)$$

connect observables in sky δ_m δ_r to primordial stuff

succeeded in managing to relate things we measure to inflationary initial conditions

added after from notes:

* We have described the (constant) behaviour on superhorizon scales & related all quantities to R, initial condition modes => initial conditions for $\phi / \delta_r / \delta_m$!

of things we measure e.g. δ_m δ_r?

- relate to inflationary initial conditions

next want to examine evolution of potential, radiation & matter fluctuations once they enter the horizon in each case considering evolution in both RD & MD
and finally, we will also look at matter fluctuations in our current era of ΛD

	RD superhor	RD subhor	MD superhor	MD subhor
ϕ	const.	∿	const.	const.
δ_r	const.	osc.	(const.)	osc.
δ_m	const.) *		const.	*

$\phi'' + 3(1+w)\mathcal{H}\phi' + k^2 w\phi$
PE + E ...

$\delta_r = -\tfrac{2}{3}\frac{k^2\phi}{\mathcal{H}^2} - \frac{2\phi'}{\mathcal{H}} - 2\phi$
PE + FI

* CE + EU

* CE + EU

-> reminder we also find δ_m & δ_r evolution on superhor. scales later which is diff

( ) sound from dominant component in ... that era ... adiabatic ... to δ_m & δ_r on superhor. scales

(no lecture on Fri)

Lecture 16: Evolution of Potential, Radiation — 18.11.24

i.e. after end of inflation (?) $\tau = 0$

* Last lecture: discussed superhorizon initial conditions for radiation, potential, matter

* Today: evolution of potential + radiation, during radiation + matter domination. super -+ sub horizon.

Potential Evolution

* Showed that from PE + E3, evolution equations

poisson

($\nabla \to ik$) then FT $\partial \to ik$

we get expr. for ϕ potential

solve to see what happens to potential

can solve this!

$$\boxed{\phi'' + 3(1+\omega)\mathcal{H}\phi' + k^2\omega\phi = 0}$$

* Can solve this equation for both

since RD $a \propto \tau$

- Radiation domination $\omega = 1/3$, $\mathcal{H} = a'/a = 1/\tau$
- Matter domination $\omega = 0$, $\mathcal{H} = 2/\tau$

since MD $a \propto \tau^2$

* ϕ in RD

Eq becomes $\phi'' + \frac{4}{\tau}\phi' + \frac{k^2}{3}\phi = 0$

The solution to this equation is

solution in terms of bessel functions "mathematician would know this I would not"

spherical bessel & neumann fns

$$\phi_{\underline{k}} \equiv \phi(\underline{k},\tau) = A_{\underline{k}} \frac{j_1(k\tau/\sqrt{3})}{k\tau/\sqrt{3}} + B_k \frac{n_1(k\tau/\sqrt{3})}{k\tau/\sqrt{3}}$$

j, n are spherical bessel/neumann functions

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

now relate back to early time BCs: const

* These functions have low-x (early time) expansions

$$j_1(x) \approx \frac{x}{3} + O(x^2), \quad n_1(x) \approx -1/x^2 + O(x^4)$$

neumann fn blows up at low x; can't match to a const!

as $x \to 0$

so set = 0 since we match to initial potential cond.

=> at early times, since B_k term is singular, $\to B_k = 0$.

$\tau = 0$

const. A_k need depends on inflationary initial conditions

Early time limit, $\phi_k = \frac{A_k}{3}$, c.f. $\phi = -\frac{2}{3}R \Rightarrow A_k = -2R(0)$

asymptotes to const at early times $-2/3 R$

behaviour of potential during RD

$$\boxed{\phi(\underline{k},\tau) = -2R(\underline{k},0)\, \frac{\sin(k\tau/\sqrt{3}) - k\tau\cos(k\tau/\sqrt{3})/\sqrt{3}}{(k\tau/\sqrt{3})^3}}$$

* Describe behaviour:

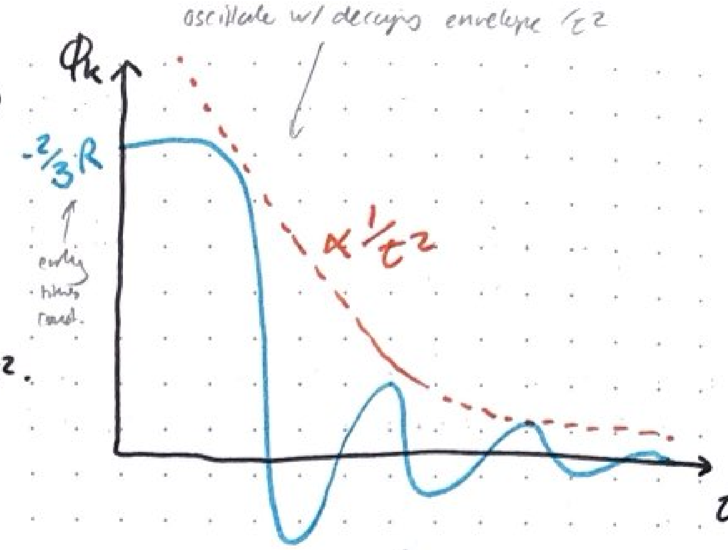
- early times ($\tau = \frac{1}{\mathcal{H}} \ll \frac{1}{k}$), constant $-2/3 R$

mode is still outside the horizon we find it is const

- late times ($\tau = \frac{1}{\mathcal{H}} \gg k$)

mode is well inside the horizon

$$\phi_k = 6 R_k(0) \frac{\cos(k\tau/\sqrt{3})}{(k\tau)^2}$$



Potential mode enters horizon oscillates w/ decaying envelope $\propto 1/\tau^2$.

* ϕ in matter domination

much easier to solve

Can now take $\omega = 0$, evolution equation with $\mathcal{H} = 2/\tau$

$$\phi'' + \frac{6}{\tau}\phi' = 0 \Rightarrow \text{solution } \phi = \text{constant}$$

notes: "growing mode solution is const" (ignore decaying solution?)

critical value of k switch from RD to MD

* Evolution of the potential depends on k relative to $k_{eq} = \mathcal{H}_{eq}$ = the 1/Horizon size at matter, rad. equality.

depends on when becomes subhorizon!

i.e. when becomes larger than H for 1st time

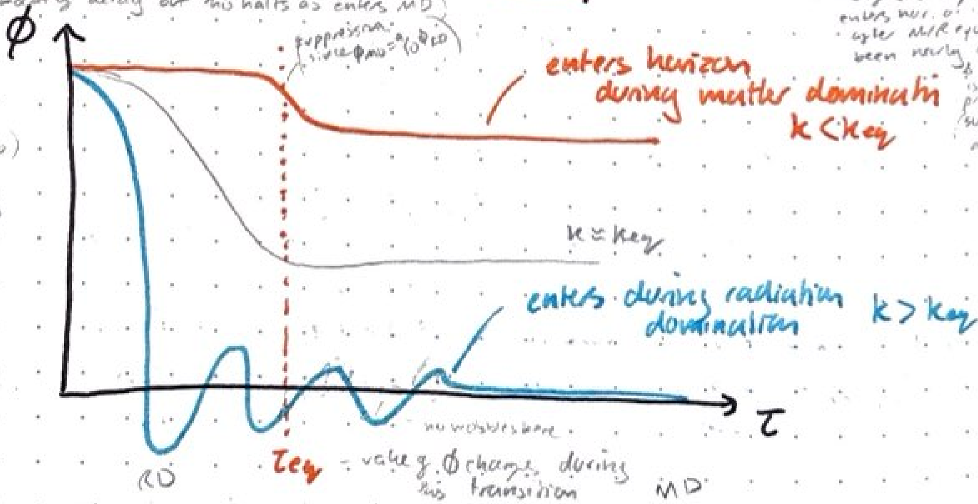
- if $k \gg k_{eq}$, enters horizon during RD, oscillates/decays
- if $k \ll k_{eq}$, " " " MD, ~ const.

$k \simeq k_{eq}$ small amount of decay but this halts as enters MD

small mode enters hor. during RD ⇒ oscillates & decays but small amplitude preserved once enters MD

large mode, only enters hor. after M/R equality, has been nearly const in RD & this amplitude is roughly preserved in MD (suppression $\frac{9}{10}$ at RD/MD transition)

ϕ evolution:



const in diff eras w/ $\phi_{MD} = \frac{9}{10}\phi_{RD}$

superhorizon scales: nothing happens
subhorizon scales: depends on when it enters horizon.
const in MD but value depends on how much it decayed during RD

τ_{eq} - value of ϕ changes during this transition

Radiation Evolution

now discuss evolution of radiation perturbations

* Q: which variable should we use?

reminder: closely related to δ_r introduced to this since we found PE has nice form w/ Δr instead of δr

density contrast $\rightarrow \delta_r$?

$$\Delta_r \equiv \delta_r - 3\mathcal{H}(1+\omega)V$$

comoving gauge density contrast

doesn't matter, choose whatever is most convenient

A: On subhorizon scales, where we make observations, equal!
notes: $\Delta - \delta \simeq \mathcal{H}/k \sim 0$.

(w/ is fourier space $\underline{V} = \nabla V$ so $V \sim \frac{|V|}{k}$)

both give correct results because they are identical on subhor. scales where we can actually make observations

* (1) Radiation domination

$\delta = \delta_r$, dominant component

we shortcut from PE w/ additional steps?

To calculate, begin from [PE]

$$\delta_r = -\frac{2}{3}\frac{k^2\phi}{\mathcal{H}^2} - 2\frac{\phi'}{\mathcal{H}} - 2\phi$$

Hence $\delta_r = -\frac{2}{3}k^2\tau^2\phi - 2\phi'\tau - 2\phi$

which can be evaluated with our previous expressions for Φ.

- On superhorizon scales $k \ll \mathcal{H} \sim 1/\tau$, $k\tau \ll 1$ already know $\delta_r = -2\phi = \text{const}$.
- On subhorizon scales, $k\tau \gg 1$, $\phi'\tau \sim k\tau\phi$

$$\delta_r \approx -\frac{2}{3}(k\tau)^2\phi = -4R(0)\cos\left(\frac{1}{\sqrt{3}}k\tau\right)$$

→ rad. perturbations oscillate around zero.

* ② Matter domination

Return to $\delta_r' + 3\mathcal{H}\left(\frac{\delta P}{\delta\rho} - \frac{\bar{P}}{\bar{\rho}}\right)\delta_r = -\left(1+\frac{\bar{P}}{\bar{\rho}}\right)(\nabla \underline{v}_r - 3\phi')$ [CE]

and $\underline{v}_r' + 3\mathcal{H}\left(\frac{1}{3} - \frac{\bar{P}}{\bar{\rho}}\right)\underline{v}_r = -\frac{\nabla\delta P}{\bar{\rho}+\bar{P}} - \nabla\phi$ [Eu]

Evaluate with $\frac{\delta P}{\delta\rho} = \frac{\bar{P}}{\bar{\rho}} = \frac{1}{3}$, ϕ const.

→ $\delta_r' = -\frac{4}{3}\nabla\cdot\underline{v}_r$, $\underline{v}_r' = -\frac{1}{4}\nabla\delta_r - \nabla\phi$

* Combining these, $(\nabla\cdot\underline{v}_r)' = -\frac{1}{4}\nabla^2\delta_r - \nabla^2\phi$

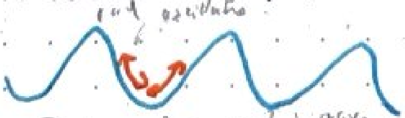
$\Rightarrow \delta_r'' - \frac{1}{3}\nabla^2\delta_r = \frac{4}{3}\nabla^2\phi = \text{const.}$ $\Rightarrow$ $\delta_r'' + \frac{k^2}{3}\delta_r = -\frac{4}{3}k^2\phi$

(adding $\phi''=0$)

$$\Rightarrow \left[\frac{\delta_r(\underline{k},\tau)}{4} + \phi(\underline{k},\tau)\right]'' + \frac{k^2}{3}\left[\frac{\delta_r(k,\tau)}{4} + \phi(k,\tau)\right] = 0$$

~ Forced harmonic oscillator for each k-mode
⇒ oscillations, now about $-4\phi_k$

* Acoustic oscillations

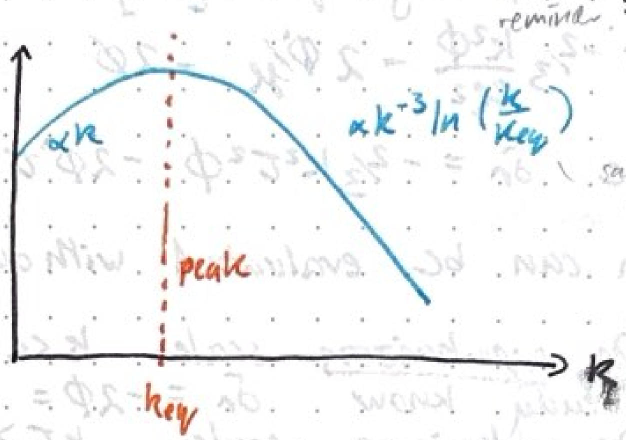


Lecture 17: Matter Evolution 20.11.24

remind

Ultimate goal:
explain matter power spectrum shape + scalings

matter spectrum describes large scale structure evolution of univ.



scales like k^{-3} w/ small logarithmic corrections

Matter Power Spectrum - ICs

(explain initial shape of matter power spectrum)

* Starting point, superhorizon

initial conditions

on superhor. scales potential prop to curvature power spectrum set up by inflation

$$\Phi(k,0) \propto R(k,0) \rightarrow P^{\Phi}(k) \propto P^{R}(k) \propto k^{-3}$$

relate potential to matter perturbations

poisson eq. ≃ [P] $\rightarrow -k^2\Phi(\underline{k},0) = 4\pi G\bar{\rho}_r a^2 \Delta_r(\underline{k},0)$

(initial Δs)

$$P^{i,\Delta m}(k) \propto P^{i,\Delta r} \sim \langle \Delta r(k)\Delta r^*(k)\rangle \propto \langle k^2\Phi(k) k^2\Phi^*(k)\rangle \propto P^{\Phi}(k)k^4$$

$\Delta m \& \Delta r$ adiabatic

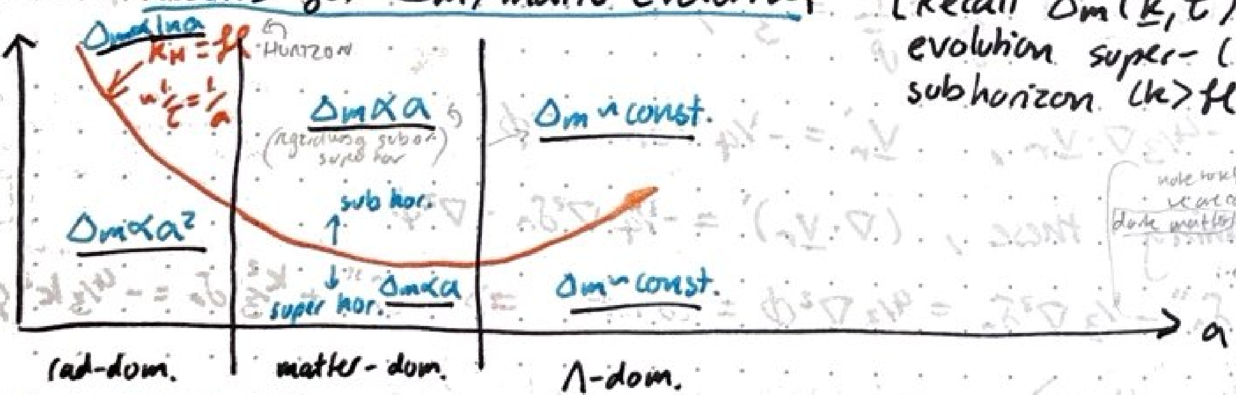
these are initial conditions → now explain evolution ↳ start by previewing result

$$\rightarrow P^{i,\Delta m}(k) \propto k^{-3} \times k^4 \rightarrow \boxed{P^{i,\Delta m}(k) \propto k}$$ initial matter power spectrum

(done!)

Preview: Results for Δm/matter evolution

[Recall $\Delta m(\underline{k},\tau)$, different evolution super- ($k<\mathcal{H}$) vs subhorizon ($k>\mathcal{H}$).]



note to self: we are considering evolution of dark matter fluctuations in RD, MD, ΛD i.e. neglect baryons so take we can set $\Delta = 0$

now derive these results:

Superhorizon ($k \ll \mathcal{H}$) "evolution"

because not evolution in a physical sense?

posses. of for dominant component ... (note on lecture concerning setting)

$\Phi = $ const.

* Poisson equation [P] $\quad \Delta = \dfrac{\nabla^2\Phi}{4\pi G a^2\bar{\rho}}$

[note to self] from PE + EU so does not apply to components individually so have to write it w/ only dominant component of eu considered

[note to self:] before we found Δm ∝ a in MD & RD

(recall $\delta_m = \delta - 3\mathcal{H}(1+w)v$ ... rearrange poisson to find ... then plug in v from E2)

* Matter domination $\quad \bar{\rho} \propto a^{-3} \rightarrow \Delta m \propto \dfrac{1}{a^2\bar{\rho}_m} \rightarrow \Delta m \propto a$

Radiation domination $\quad \bar{\rho} \propto a^{-4} \rightarrow \Delta m \propto \Delta r \propto \dfrac{1}{a^2\bar{\rho}} \rightarrow \Delta m \propto a^2$

follows since adiabatic modes

been superhorizon scales? only on ... condition

Subhorizon evolution

newtonian calc or its approach 'can screw out' if you only care about newtonian

* Calc valid for all scales — since derivs matter

Use [CE] + [Eu] with $p=0$

(more rigorously derive results we derived from newtonian theory)

$\rightarrow \nabla\cdot\underline{v}_m = -\delta_m' + 3\phi'$ (I)

$\underline{v}_m' + \mathcal{H}\underline{v}_m = -\nabla\phi$ (II)

note to self: (take time deriv of continuity "$\delta'' = \ldots$" and subst in Eu & EE again to get eq. ↓)

Take $\nabla\cdot$(II), replace in (I) — gives evolution eq.

$$\delta_m'' + \mathcal{H}\delta_m' = \nabla^2\phi + 3(\phi'' + \mathcal{H}\phi')$$

(linear grav potential is only sourced by matter fluctuations $\phi \propto \delta_m$)

(remember did same in newtonian treatment)

grav. potential can think of being part sourced by which is matter & part sourced by rad. (rapidly oscillating) ↓ so irrelevant to slow growth of matter, drop

* Can take $\phi \simeq \phi_m + \cancel{\phi_r}$ ← rapidly oscillating; $\phi_m \sim$ const.

(split into two parts, one ~ const & one oscillating. As we saw before, $\delta_m \approx$ const & ϕ_{rad} oscillates. I don't think therefore exactly we can show this but by same logic one can say $\phi_m \approx$ const. ?. since $\delta_m \approx$ const)

$\rightarrow$ drop, $\phi \simeq \phi_m \simeq$ const.

(see astro-ph/0207.375) more rigorous reasoning — anyways, just accept this, too tired

from poisson

$\rightarrow \nabla^2\phi \simeq \nabla^2\phi_m \simeq 4\pi G\bar{\rho}_m a^2\Delta_m \simeq \delta_m$

on subhorizon scales $\Delta \simeq \delta$.

$\xrightarrow{\text{subst.}}$ $\underline{\delta_m'' + \mathcal{H}\delta_m' - 4\pi G\bar{\rho}_m a^2\delta_m = 0}$

(note to self: works since poisson scales everything so $\nabla^2\phi = 4\pi G a^2(\bar{\rho}_m\Delta_m + \bar{\rho}_r\Delta_r)$ but $\bar{\rho}_r\Delta_r$'s we drop. Δ_r term since Δ_r oscillates rapidly so each averages to zero, & we did same thing in 'newtonian' case → in MD $\bar{\rho}_r \ll \bar{\rho}_m$ but this eq also holds in RD since there Δ_r ...)

$\Rightarrow$ you'll show (on ex sheet) in terms of a, defining $y \equiv a/a_{eq}$ ← $\rho_m(a_{eq}) = \rho_r(a_{eq})$

$$\delta_m = \begin{cases} 2 + 3y \\ (2+3y)\ln\left(\dfrac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 6\sqrt{1+y} \end{cases}$$

on subhor scales $\delta_m = \Delta_m$ so can conclude results we wrote down

$\Rightarrow \Delta_m \propto \begin{cases} \log a & \text{(RD)} \\ a & \text{(MD)} \end{cases}$

* $\underline{\text{Rad. dom}}$, subhor. $y \ll 1$ $\rightarrow \delta_m \propto \ln y \;\underline{\propto \ln a}$

$\underline{\text{Matter dom}}$, $y \gg 1$ $\rightarrow \underline{\delta_m \propto y \propto a}$

(for MD, can also show straight from $\Delta_m = \dfrac{\nabla^2\phi}{4\pi G a^2\rho}$, since ϕ const in MD and $\rho \propto a^{-3}$ $\Rightarrow \Delta_m \propto a$)

(can't show straight from poisson since don't have $\Delta_m \ll \Delta_r$ on subhor scales) (only worked before since on superhor, $\Delta_m \propto \Delta_r$)

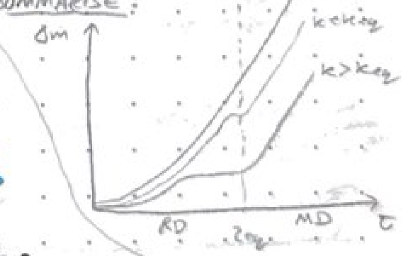
Dark Energy Dom. (DE)

no fluctuations in Λ so: (Δ drop Δ_r term)

* At late times $\nabla^2\phi = 4\pi G a^2\bar{\rho}\Delta \underset{=\bar{\rho}_r\Delta_r + \bar{\rho}_m\Delta_m}{=} 4\pi G a^2\bar{\rho}_m\Delta_m$

Since $\bar{\rho}_m \propto a^{-3}$, $\phi \propto \Delta_m/a$ — holds for each wavenumber

SUMMARISE:



* Recall Einstein eq. for potential evolution [E3]

$$\phi'' + 3\mathcal{H}\phi' + (2\mathcal{H}' + \mathcal{H}^2)\phi = 4\pi G a^2\delta P \;\text{[E3]}, \quad \delta P = 0$$

$$\rightarrow \phi'' + 3\mathcal{H}\phi' + (2\mathcal{H}' + \mathcal{H}^2)\phi = 0$$

trick:

* Inserting $\phi \propto \dfrac{\Delta_m}{a}$

$$(\Delta_m/a)'' + 3\mathcal{H}(\Delta_m/a)' + (2\mathcal{H}' + \mathcal{H}^2)\Delta_m/a = 0$$

(& group Δ_m'', Δ_m', Δ_m terms)

1. expand out $\left(\frac{\Delta_m}{a}\right)''$, $\left(\frac{\Delta_m}{a}\right)'$
2. use $\mathcal{H} = \frac{a'}{a}$, $\mathcal{H}' = \frac{a''}{a} - \frac{a'^2}{a^2}$

$\Rightarrow \Delta_m'' + \Delta_m'\mathcal{H} + \Delta_m(\mathcal{H}' - \mathcal{H}^2) = 0$

3. use F1, F2 to get expression for $\mathcal{H}' - \mathcal{H}^2$

(Ex show implies (using $\mathcal{H}' - \mathcal{H}^2 = -4\pi G\bar{\rho}_m a^2$))

$\Rightarrow \underline{\Delta_m'' + \mathcal{H}\Delta_m' - 4\pi G a^2\bar{\rho}_m\Delta_m = 0}$

(*) confused since here we treat F1, F2 for matter but shouldn't we have ρ & ρ_Λ since in ΛD era as we did for newtonian example?

i.e. (are we are multiplying by δ whatever δ_m? (we consider $\mathcal{H}'\Delta_m - \mathcal{H}^2\Delta_m$)

(I think because we consider so F1: $(\Delta_m + \Delta_r)\mathcal{H}^2 = \frac{8\pi G a^2}{3}(\bar{\rho}_m\Delta_m + \bar{\rho}_r\Delta_r)$ $= \delta\rho_m$, $= \delta\rho_r$ $\Rightarrow \Delta_m\mathcal{H}^2 = \Delta_m\frac{8\pi G a^2\bar{\rho}_m}{3}$ (?)

$\Delta_m = \quad \Delta_r$

Same eqn. as in Newtonian calc. but valid on all scales.

Again, drop 3rd term as

$$\mathcal{H}^2 \underset{\text{eg}}{\overset{\text{3rd Friedmann}}{=}} \frac{8\pi G a^2}{3}\bar{\rho}_\Lambda > 4\pi G a^2 \bar{\rho}_m$$

$\rightarrow \Delta_m'' + \mathcal{H}\Delta_m' = 0$

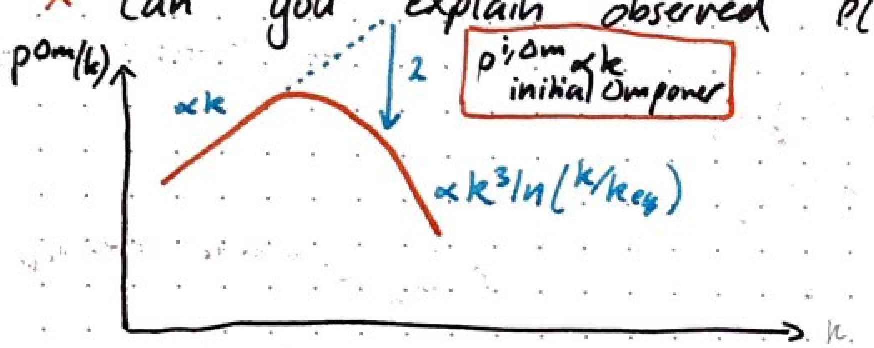
"growing" identify solution that is (roughly) less quickly?

$\rightarrow \boxed{\Delta_m \approx \text{const.}}$ (on all scales)

we have explained how Δ_m evolves on all scales for all times (RD, MD, ΛD)

power spectrum; how clumpy univ is at diff scale packets

* Can you explain observed $P(k)$?



how does pow spectrum that starts $\propto k$ end ups looking like that?

is where in previous drawing do I see high k growing slower than low k

is during rad dom

physical reason: rad pressure prevents growth on super? horizon scales

homework: derive this

Tip: $\left(\dfrac{a_{eq}}{a_{\text{horizon entry}}}\right) = \left(\dfrac{k_{\text{horizon entry}}}{k_{eq}}\right)$

matter pow spectrum v. important key quantity describing structure of univ

or my way:

growth of low k mode:

$A = \left(\frac{a_{eq}}{a_i}\right)^2 \left(\frac{a_\Lambda}{a_{eq}}\right)$

growth of high k mode:

$B = \left(\frac{a_{hor}}{a_i}\right)^2 \ln\left(\frac{a_{eq}}{a_{hor}}\right)\left(\frac{a_\Lambda}{a_{eq}}\right)$

$\Rightarrow B = A\left(\frac{a_i}{a_{eq}}\right)^2\left(\frac{a_{hor}}{a_i}\right)^2 \ln\left(\frac{a_{eq}}{a_{hor}}\right)$

$= A\left(\frac{a_{hor}}{a_{eq}}\right)^2 \ln\left(\frac{a_{eq}}{a_{hor}}\right)$

(more rigorous treatment of log term attempt:)

log term actually also have const. solution so grows as $b\ln a + c$ (for consts b, c)

smooth transition at a_{eq} go from growing as $b\ln a + c$ to growing prop to a say da

smooth $\Rightarrow b\ln a_{eq} + c = d a_{eq}$

& $b/a_{eq} = d$

$\Rightarrow c = b - b\ln a_{eq}$

so factor is $\dfrac{b\ln a_{eq} + b - b\ln a_{eq}}{b\ln a_{hor} + b - b\ln a_{eq}}$

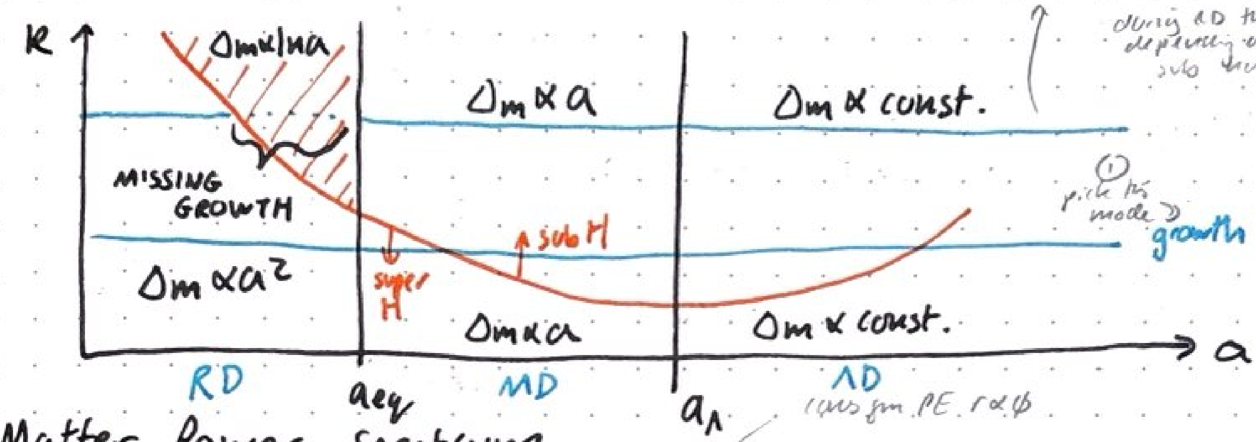
$\Rightarrow 1 + b\log\left(\frac{a_{eq}}{a_{hor}}\right) \sim \log\left(\frac{a_{eq}}{a_{hor}}\right)$

(?!)

Lecture 18: Matter Power spectrum 22.11.24

② key point: as opposed to this mode which has missing growth, the higher k you are at the more growth you miss out on

last time: Δ_m evolution



during RD there is diff. in growth depending on whether I am super/sub horizon

① pick this mode → growth $A = \left(\frac{a_{eq}}{a_i}\right)^2 \left(\frac{a_\Lambda}{a_{eq}}\right)$

from a_{eq} to a_{eq} ... growth until Λ-dom sets in

cause from P.E. r & φ

Matter Power spectrum

* Initial state : $P^{i,\Delta_m}(k) \propto k^4 P(k) \propto \underline{\underline{k}}$

initial pow. spectra for matter prop. to k

* Growth is the same on all scales except in radiation dom. modes. Subhorizon have growth nearly stop ($\propto \ln a$) whereas outside horizon, $\Delta_m \propto a^2$

subsequent · logarithmic only · grows prop to a^2 · emphasize missing growth

* "Missing growth" inside horizon → amplitude of such a mode relative to A (growth factor), therefore growth outside that in RD is changed by a factor

growth that was always outside hor.

$$A \times \frac{1}{\text{growth outside horizon after } H_e} \times \text{slow growth inside horizon}$$

horizon entry · (take this out) · compensate a bit by this · put this in

$$\sim A \times \frac{1}{\left(\frac{a_{eq}}{a_{He}}\right)^2} \times \log\left(\frac{a_{eq}}{a_{He}}\right)$$

notation: general mode k enters horizon at time a_{He} during RD.

← growth I miss out on · ↖ slow growth I experience inside horizon

← (this is handwavey argument, see notes for non-trivial argument for how log term comes about)

statement in terms of k: corrections:

* $k = \mathcal{H}(\tau_{He}) = \frac{1}{\tau_{He}} \propto \frac{1}{a_{He}}$ (RD) $\Big\}$ ⟹ $\left(\frac{a_{eq}}{a_{He}}\right) = \left(\frac{k}{k_{eq}}\right)$

$k_{eq} = \mathcal{H}(\tau_{eq}) \propto \frac{1}{a_{eq}}$

this says k_{eq} is the mode that enters horizon at τ_{eq} ($a_{He} = a_{eq}$) so $k_{eq} \propto \frac{1}{a_{eq}}$

naive from graph: guess the higher k the more growth I miss out on. we verify this ✓

(A)

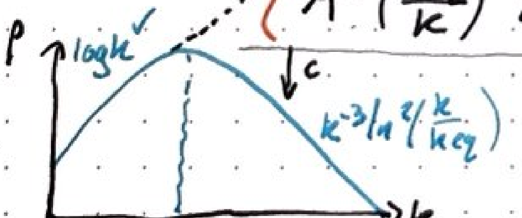
* $\Delta_m(k,\tau) \propto \Delta_m^i \times \begin{cases} A & k \ll k_{eq} \\ A\left(\frac{k_{eq}}{k}\right)^2 \ln\left(\frac{k}{k_{eq}}\right) & k \gg k_{eq} \end{cases}$

← (growth suppressed for high k)

! severe, we'll see pow. spectrum is suppressed

initial pow. spectrum ∝ k · square since $P \propto \langle \Delta_m(k) \Delta_m^*(k) \rangle$? · depend on whether I am super/sub hor. during RD

* $\boxed{P(k) = P^{i,\Delta_m}(k) \times \begin{cases} A^2 \\ A^2\left(\frac{k_{eq}}{k}\right)^4 \ln^2\left(\frac{k}{k_{eq}}\right) \end{cases} \propto \begin{cases} k & k \ll k_{eq} \\ k^{-3} \ln^2\left(\frac{k}{k_{eq}}\right) & k \gg k_{eq} \end{cases}}$



we have explained scalings in matter power spectrum!

intuition: high k mode which were subhorizon during RD missed out on growth b/c radiation pressure prevented the collapse in the growth of modes

explained!

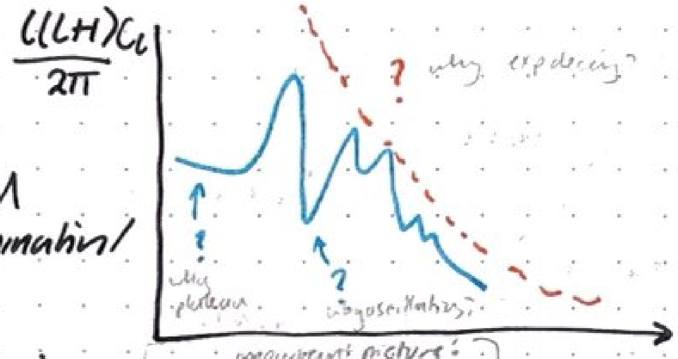
peak corresponds to mode k_{eq} which entered horizon exactly when matter-radiation equality occurred

* N.B. def. transfer function $T(k)$ $\Delta_m(k,\tau) \propto R(k,0) \times T(\ )$

(cont. in next notes) or in this notes?

CMB Anisotropies

* Goal: understand CMB power spectrum

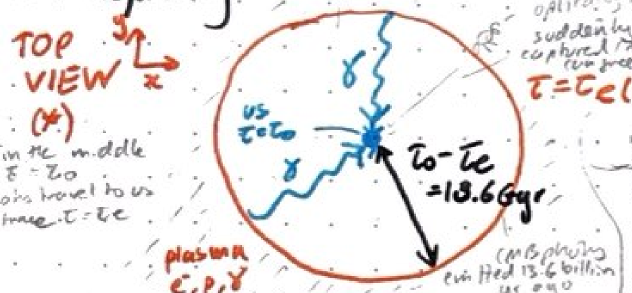


First steps: relate observed $T(\hat{n})$ to δ_r, ϕ on recombination/last-scattering surface

$\Rightarrow$ understand physical origin of CMB anisotropies

* Picture: instant recombination + decoupling

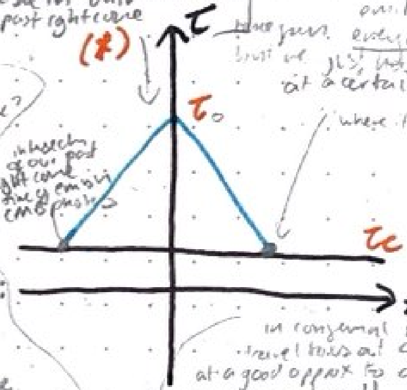
good approx: instantaneous recomb. & decoupling



we are in the middle at $\tau = \tau_0$
CMB photons travel to us from time $\tau = \tau_e$

suddenly free electrons get captured (recomb) & photons can freely travel to us

important picture: our past light cone. everything we see in univ we observe along our past light cone



in conformal time light rays travel at 45°, CMB emitted at a good approx to certain time τ_e (certain temp when H became neutral corresponds to this time)

Background average $\bar{\tau}_e = \tau_*$

average time of emission τ_* (depends on direction we look: $\tau_e(\hat{n})$)

* Consider propagation of photons

we want to find propagation of γ from circle boundary to centre

metric $\quad ds^2 = a^2(\tau)\left[(1+2\phi)d\tau^2 - (1-2\phi)d\underline{r}^2\right]$

$U^\mu U_\mu = 1$

* Observer at rest $U^\nu \propto \delta^0{}_\nu$, correctly normalised $U^\nu = \frac{(1-\phi)}{a}\delta^\nu_0$

$$U_\nu = g_{\nu\mu}U^\mu = a(1+\phi)\delta_{0\nu}$$

* Consider photon 4-momentum $p^\mu = \frac{dx^\mu}{d\lambda}$

$$E = p^\nu U_\nu = a(1+\phi)p^0 \Rightarrow p^0 = E(1-\phi)/a$$

expression of energy measured by free observer

* Evolution of photon 4-momentum is given by geodesic eqn.:

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho}p^\nu p^\rho = 0$$

* Evaluating $\mu = 0$ gives E evolution

show yourself / ex sheet

$$\frac{d\ln[aE(1+\phi)]}{d\tau} = 2\phi'$$

evolution described by this differential equation for E measured by comoving observer

* Integrate from emission, e, to today, 0

$$\ln\left[\frac{a_0 E_0 (1+\Phi_0)}{a_e E_e (1+\Phi_e)}\right] = \int_e^0 d\tau\, 2\Phi'$$

"can't just do the integral because this is a partial derivative"

taylor expand, really we are working in 1st order perturbation theory

* $E_0 = a_e E_e (1+\Phi_0)^{-1}(1+\Phi_e)\exp\left(\int_e^0 d\tau\, 2\Phi'\right)$

scattering surface

taylor exp.

$$E_0 = a(\tau_e)E(\tau_e)\left[1 + (\Phi_e - \Phi_0) + \int_e^0 d\tau\, 2\Phi'\right]$$

notes: redshifting of photons when climbing out of potential wells on their way; integrated Sachs-Wolfe effect: potential well changes while photons in it then potential in & potential out don't cancel

reduces energy photons depends on whether moving away or towards us; increases energy

* Have assumed source at rest, if not, include doppler effect

add this correction by hand

$$E_{obs} = E_0(1 - \hat{\underline{n}}\cdot\underline{v})$$

observation direction; source velocity

(note to self: see this taylor expand again assuming $\hat{n}\cdot v$ is 1st order?)

$$E_{obs} = a(\tau_e)E(\tau_e)\left[1 + (\Phi_e - \Phi_0) - \hat{\underline{n}}\cdot\underline{v} + \int_e^0 2\Phi'\right]$$

whether or not photon is emitted depends on temp around photon! set by thermodynamics; also happens at certain temp

* Since both E and T are $\propto 1/a$, — from earlier in course

(T at recombination)

$$\frac{E_{obs}}{E(\tau_e)} = \frac{T_{obs}}{T_*}$$

local emitted T same everywhere

notes: $E_{obs}/E(\tau_e)$ is frequency independent so photons of all frequencies redshift in the same way => this redshifting is the same that must affect the CMB temp T_{obs}/T_*

potentials at emission, & today; velocity last scattering surface

$$\frac{T_{obs}}{T_*} = a(\tau_e)\left[1 + (\Phi_e - \Phi_0) - \hat{\underline{n}}\cdot\underline{v} + \int_e^0 d\tau\, 2\Phi'\right]$$

↑ ???

know all of RHS but what is $a(\tau_e)$? ← when / scale factor at which recomb. then decoupling happened?

(notes: assume instantaneous recomb/decoupling & use these terms interchangeably for convenience)

* Recombination / decoupling happens at locally fixed temp. T_*, and hence ρ_r, but time can vary

(determined by the atomic physics of recomb & decoupling)

also const.

$$\bar{\rho}_r[\bar{\tau}_*] = c\,T_*^4$$

note to self: $\bar{\rho}_r[\tau] = (\bar{\rho}+\delta\rho)[\bar{\tau}+\delta\tau]$; $\bar{\rho}_r = \bar{\rho}_r + \bar{\rho}'\delta\tau + \delta\rho$; $\Rightarrow \bar{\rho}'\delta\tau = -\delta\rho$ perturbation to time & ρ must cancel to give same as unperturbed

notes have $\bar{\tau}_*$ instead of $\bar{\tau}$ = background recombination time

increase in more radiation I have to wait a bit longer to cool s.t. photons can be released i.e. universe will need to expand longer to reach a sufficiently cold temperature.

$$(\bar{\rho}_r + \delta\rho_r)(\bar{\tau} + \delta\tau) = c\,T_*^4 \rightarrow \bar{\rho}_r'\,\delta\tau + \delta\rho_r = 0$$

(since RD $\rho_r \propto a^{-4}$)

* $$\delta\tau = -\frac{\delta\rho_r}{\bar{\rho}_r'} = \frac{\delta\rho_r}{4\mathcal{H}\bar{\rho}_r} = \frac{\delta_r}{4\mathcal{H}}$$

assume $\bar{\tau} = \tau_e$, $\tau_e = \tau_*$?

rewrite in terms of rad density perturbation

taylor expand → $a(\tau)$

* $$a(\tau_e) = a(\bar{\tau} + \delta\tau) = a_* \times (1 + \mathcal{H}\delta\tau) = a_*\left(1 + \frac{\delta_r}{4}\right)$$

notes: +ve overdense perturbation in rad. density, leads to delaying emission, so at later time when a is larger, so less redshift => +ve Temp fluctuation

plug in for $a(\tau_e) = a_*(1+\delta_r/4)$

Hence $$\frac{T_{obs}}{T_*} = a_*\left[1 + \frac{\delta_r}{4} + (\Phi_e - \Phi_0) - \hat{\underline{n}}\cdot\underline{v} + \int_e^0 d\tau\, 2\Phi'\right]$$

↑ expr for temp of CMB which involves rad density contrast on last scattering surface

CMB temp we observe depends on redshift from potential fluctuation, doppler shift from whether last scattering surface is moving towards you "Sachs-Wolfe effect" and also ... effect we will discuss

added by me:

* temp. anisotropy: $$T(\hat{\underline{n}}) = \frac{T_{obs} - \bar{T}}{\bar{T}} = \frac{T_{obs}}{a_* T_*} - 1$$

— mean CMB temp today

follows since $T \propto 1/a$, $T_0 a_0 = T_* a_*$

CMB perturbations about the observed background temp today ($\bar{T}$): ↓ relates $T(\hat{n})$ observed today to δ_r, Φ on recombination / last scattering surface!

$$\boxed{T(\hat{n}) = \frac{\delta_r}{4} + \Phi_e - \Phi_0 - \hat{\underline{n}}\cdot\underline{v} + \int_e^0 d\tau\, 2\Phi'}$$

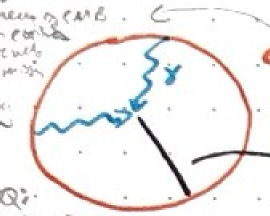
CMB TEMP FLUCTUATION (shorthand for $T(\hat{n})$ = temperature anisotropy)

fluctuations in radiation density; redshift from potential fluctuations; doppler; integrated Sachs-Wolfe

Lecture 19 - CMB Power spectrum

25.11.24

last time: how energy of CMB photon evolves as it travels from emission to us today



Last time

$$T(\hat{\underline{n}}) \equiv \frac{T_{obs}}{a_* \bar{T}_*} - 1 = \frac{\delta_r}{4} + (\Phi_e - \Phi_o) - \hat{\underline{n}}\cdot\underline{v} + \int_e^o d\tau\, 2\Phi'$$

temp we see in direction $\hat{n}$ in terms of change in temp/energy of photons since they were emitted: depends on radiation density, potentials, velocity at emission & also along line of sight

this quantity: equiv of fractional density contrast but for temp. → how much higher than avg is temp in certain direction: call this CMB temp fluctuation (shorthand)

(distance to LSS defined by diff in times) $\chi_* = \tau_0 - \tau_e \equiv \tau_*$

observed

KEY EQ: CMB TEMP FLUCTUATION IN CERTAIN DIRECTION

all eval. at at LS surface (at radial distance χ_* at τ_*)

usually omit potential at the observer Φ_o as this is const. & can be absorbed into background temp.

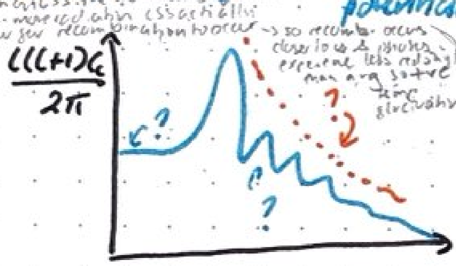
2nd 2 terms less important

$$T(\hat{\underline{n}}) = \frac{\delta_r}{4}(\hat{\underline{n}}\chi_*, \tau_*) + \Phi_e(\hat{\underline{n}}\chi_*, \tau_*) - \hat{\underline{n}}\cdot\underline{v}(\hat{\underline{n}}\chi_*, \tau_*) + \int_e^o d\tau\, 2\frac{\partial\Phi}{\partial\tau}$$

- fluctuations in radiation density — depends on intuition: the more radiation at LSS, the hotter CMB will be in that direction. more radiation at LSS acts like curved, it takes longer for recombination to occur → so recombination occurs closer to us & photons experience less redshift than avg so there are temp fluctuations
- redshift from potential fluctuations — again evaluated on LSS. grav redshift effect. (so photon has to escape potential well & lose energy & temp decrease)
- doppler term — if LSS moving towards us it means some there is a doppler shift & CMB photons gain energy & arrive at higher T
- integrated Sachs-Wolfe — integral over time derivative potential. time varying potential: so if in something falls into potential and comes out again, during which time the potential changes, then there is no net effect of red/blueshift. time varying potential leads to change in energy

* Goal

to describe CMB power spectrum



SPHERICAL HARMONICS RECAP: origin as → solutions to laplace's eqn $\nabla^2 g = 0$ in spherical coords → function defined on the surface of a sphere → form an orthonormal basis so any fn on surface of sphere can be written as a sum of spherical harmonics.

Spherical Harmonics $= Y_{\ell m}(\hat{\underline{n}})$

recap: lol. apparently I should

* Eigenfunctions of $\hat{L}^2 = -\nabla^2$, $\hat{L}_z = -i\partial_\phi$, analogues of $e^{-i\underline{k}\cdot\underline{n}}$ on sphere

in QM spherical harmonics are eigenfunctions of $\hat{L}^2$, $\hat{L}_z$

i.e. analogous to what the e's are on everything

Can expand $T(\hat{\underline{n}}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{m=+\ell} a_{\ell m} Y_{\ell m}(\hat{\underline{n}})$ — can expand in terms of spherical harmonic functions $Y_{\ell m}(\hat{n})$

deltas are useful

(orthonormal)

$Y_{\ell m}$ orthogonal on sphere $\int d\hat{n}\, Y_{\ell m}(\hat{\underline{n}}) Y^*_{\ell' m'}(\hat{\underline{n}}) = \delta_{\ell\ell'}\delta_{mm'}$

useful if we want to find property over surface of sphere

(so can find coefficient $a_{\ell m}$ by) $a_{\ell m} = \int d\hat{\underline{n}}\, T(\hat{\underline{n}}) Y^*_{\ell m}(\hat{\underline{n}})$

spherical harmonic transform take field defined on sphere & return $a_{\ell m}$ coefficients. 'packages' will do this for you

* Power spectrum C_ℓ defined as (note does not depend on m!)

$$\langle a_{\ell m} a^*_{\ell' m'}\rangle = C_\ell\, \delta_{\ell\ell'}\delta_{mm'}$$

deltas arise of fn of statistical isotropy on the sphere → no preferred direction!

"ℓ is kind of like 'angle in radians'"

instead of avg over all univ. use fact that C_ℓ does not depend on m and make approx of avg over ms

for a given value of ℓ there are $2\ell+1$ diff. values of m

Estimate this via average over ms $\hat{C}_\ell = \frac{1}{2\ell+1}\sum_m |a_{\ell m}|^2$.

Calculating the power spectrum

assume CMB emitted instantaneously from spherical LSS at time τ_* & distance $\chi_* = \tau_0 - \tau_*$ in all directions

focus on 1 most important terms

intuitively: the higher PS is defined as of proportionality that comes from isotropic, derived from that 2pt. correlation fn is homogeneous?

here it comes from isotropy on the sphere?

* Focus on 'Sachs-Wolfe' term (largest)

where did observation term Φ_0 go: absorbed into background

to good approx, Temp is SW term evaluated on LSS.

$$T(\hat{\underline{n}}) \simeq \left[\frac{\delta_r}{4} + \Phi_e\right](\underline{x} = \hat{\underline{n}}\chi_*, \tau_*) \equiv S(\hat{\underline{n}}\chi_*, \tau_*)$$

NB. due to linearity, can write

first, how do this is due to linearity & something else?

run FT on χ as const time τ_*

transfer fn; initial condition

$$S(\underline{x}, \tau_*) \xrightarrow{FT} S(\underline{k}, \tau_*) = \left[\frac{\delta_r}{4} + \Phi_e\right](\underline{k}, \tau_*) = T_S(\underline{k}, \tau_*)\mathcal{R}(\underline{k}, 0)$$

i.e. what this is

recap: we stated earlier that both δ_r, Φ can be related to the primordial curvature perturbation $\mathcal{R}$ in the linear theory. hence we can write S in fourier space in terms of a transfer function T_S

not going to prove, you can look it up if you like (not necessary) identity

Rayleigh something relate some more e^{ikx} to spherical harmonics

* Want a_{lm} and C_l

temp at sw term evaluated LSS at time τ_*

Note: $T(\hat{n}) = S(\underline{x} = \hat{n}\chi_*, \tau_*) = \int \frac{d^3k}{(2\pi)^3} e^{-i\underline{k}\cdot\hat{n}\chi_*} S(\underline{k}, \tau_*)$

spherical bessel fns

use $e^{ikx} \equiv 4\pi \sum_{lm} i^l j_l(kx) Y_{lm}^*(\hat{k}) Y_{lm}(\hat{x})$ ← "Rayleigh plane wave expansion identity"

allows us to write e^{ikx} in terms of spherical harmonics?

note to self: idk why we can just just expression for $e^{ik\cdot x}$ in place of $e^{-ik\cdot x}$??

$$T(\hat{n}) = 4\pi \int \frac{d^3k}{(2\pi)^3} T_S(k, \tau_*) R(\underline{k}, 0) \sum_{lm} i^l j_l(k\chi_*) Y_{lm}^*(\hat{k}) Y_{lm}(\hat{n})$$

$\underline{x} = \hat{n}\chi_*$ ∴ $\hat{x} = \hat{n}$

$$(\text{c.f. } T(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n}))$$

works if we defined $T(\hat{n}) = \sum a^*_{lm} Y^*_{lm}(\hat{n})$ instead?

$$\Rightarrow a_{lm} = 4\pi i^l \int \frac{d^3k}{(2\pi)^3} T_S(k, \tau_*) R(\underline{k}, 0) j_l(k\chi_*) Y^*_{lm}(\hat{k})$$

* Calculate power spectrum via $\langle a_{lm} a^*_{l'm'} \rangle$.

not conceptually difficult just a lot of maths

notes definition of the primordial power spectrum

* We obtain, using $\langle R(\underline{k}, 0) R^*(\underline{k}', 0)\rangle = \langle R(k,0) R^*(-k, 0)\rangle$
$= (2\pi)^3 P_R(k) \delta^{(3)}(\underline{k} + \underline{k}')$

+ using orthogonality of spherical harmonics

$$\langle a_{lm} a^*_{l'm'}\rangle = (4\pi)^2 i^{l-l'} \iint \frac{d^3k\, d^3k'}{(2\pi)^3} T_S(k, \tau_*) T_S(k', \tau_*) \langle R(\underline{k}, 0) R^*(\underline{k}', 0)\rangle j_l(k\chi_*)$$

not crossed out

note to self: see this since fns not orthogonal, then whole thing zero but if they are then we get integral over remaining terms

from applying orthogonality

$$\times j_{l'}(k'\chi_*) \cancel{Y^*_{lm}(\hat{k}) Y_{l'm'}(\hat{k}')}$$

or also will give something once we get to this stage? idk

why is this not $(4\pi)^2$??

$$= 4\pi\, \delta_{ll'} \delta_{mm'} \int d\ln k \left[\frac{k^3}{2\pi} P_R(k)\right] [T_S(k, \tau_*)]^2 j_l^2(k\chi_*) \times Y^*_{lm}(\hat{k}) Y_{l'm'}(\hat{k})$$

cryptsags because $d^3k = k^2 dk$ and not $4\pi k^2 dk$ idk why?

* CMB Power spectrum is

depends on 3 parts.

$$\boxed{C_l = 4\pi \int d\ln k \underbrace{\left[\frac{k^3}{2\pi^2} P_R(k)\right]}_{\text{PRIMORDIAL POWER (spectrum)}} \times \underbrace{[T_S(k, \tau_*)]^2}_{\text{TRANSFER FUNCTION}} \times \underbrace{[j_l(k\chi_*)]^2}_{\text{PROJECTION } k\text{-}l}}$$

CMB POWER SPECTRUM (*)

projects position or wavenumber k to angle or multipole l via bessel fns

greater than a few degrees & larger than the horizon at recombination

info about radiation & potential evolution aka plasma processing.

* large scales $l < 100 \Leftrightarrow 2°$?

interesting! looking at CMB on scales larger than 2° we are directly seeing primordial modes, just multiplied by factor −1/5

* Evaluate T_S, superhorizon scales, matter-dominated

from earlier expression for Φ_c on superhorizon scales

assume recombination is fully matter dominated

δ_r in MD on superhorizon scales

$$\Phi_c(\underline{k}, \tau_*) = -3/5\, R(\underline{k}, 0)$$
$$\delta_r(\underline{k}, \tau_*) = +8/5\, R(\underline{k}, 0)$$

$$\rightarrow S(\underline{k}, \tau_*) = \frac{\delta_r}{4} + \Phi_c = -1/5\, R(\underline{k}, 0)$$

T_S

inflationary initial condition note to self: since should R conserved on superhorizon scales, this IC also holds on superhorizon scales?

large scale

if i plot power spectrum, then should get it flat (units of l ℓ stuff)

1) if power spectrum is _______

2) if so so PS is _______

on large scales: modes still superhorizon

* Assume $\frac{k^3}{2\pi^2} P_R(k) \sim \text{const.} \equiv A_S$

$$\rightarrow C_l = \frac{4\pi}{25} A_S \int d\ln k\, j_l^2(k\chi_*) = A_S \frac{4\pi}{25} \frac{1}{2l(l+1)}$$

using a standard integral for bessel fn.

$$\frac{C_l l(l+1)}{2\pi} = \frac{A_S}{25}$$ ← PLATEAU! observed

by convention we plot the power spectrum as $\frac{C_l l(l+1)}{2\pi}$

($k = l/\chi_*$ corresponds to a certain wavelength $\lambda = \frac{2\pi}{k}$ which projects to an angle $\theta = \frac{2\pi}{l}$, by small angle approx $\theta \approx \frac{\lambda}{\chi_*}$)

Smaller Scales

1) Approximate solution from (*) — kernel fn peaks strongly at a certain value of k

at high l → $j_l(k\chi_*)$ peaks strongly for $k = \frac{l}{\chi_*}$ [small angle $\lambda^{-1} = \frac{\theta^{-1}}{\chi_*}$]

⇒ Pull from integral $\left[\frac{k^3}{2\pi^2} P^R\right]^n (k = \frac{l}{\chi_*}) \times T_S^2(k = l/\chi_*)$

(notes: since P_R, T_S vary slowly & j_l peaks sharply, we take P_R, T_S outside the integral & eval at $k = l/\chi_*$, then we use standard integral for bessel fn)

(see acoustic times: the rise from sq, so any features have to do with T_S!)

$$\frac{l(l+1)C_l}{2\pi} = \left[\frac{(l/\chi_*)^3}{2\pi^2} P_R(k = l/\chi_*)\right] \times T_S^2\left(k = \frac{l}{\chi_*}, \tau_*\right)$$

intermediate scales: not superhorizon but enter horizon during MD

2) $k \ll k_{eq}$ matter dominated → enter horizon during MD

initial conditions as argued earlier

IC $S(\underline{k}, \tau = 0) = -1/5\, R(\underline{k}, 0)$

(subhor in MD: saw earlier δ_r oscillates about $-4\phi_m$ & $\phi \approx$ const ⇒ $\frac{\delta_r}{4} + \phi$ oscillates about zero w/ $S \propto \cos(k\tau/\sqrt{3})$ — oscillating term!)

* We saw: oscillations of $\frac{\delta_r}{4} + \phi = S \propto \cos\left(\frac{k\tau}{\sqrt{3}}\right)$

consequence of acoustic oscillations in radiation fluid

$$S(\underline{k}, \tau) = \underbrace{-1/5 \cos\left(\frac{k\tau}{\sqrt{3}}\right)}_{T_S(k,\tau)} R(\underline{k}, 0)$$

oscillates in fns here into oscillating feats in CMB power spectrum

3) $k \gg k_{eq}$, rad. dominated → enter horizon during RD

(subhor in RD: ϕ oscillates & decays, & oscillates about 0, $\delta_r \approx -4R(0)\cos(\frac{k\tau}{\sqrt{3}})$)

* Potentials decay, $\phi \simeq 0$, $S = \delta r/4$

* $\delta_r(\underline{k}, \tau) = -4 \cos\left(\frac{k\tau}{\sqrt{3}}\right) R(\underline{k}, 0)$

$$S(\underline{k}, \tau) = \underbrace{-\cos\left(\frac{k\tau}{\sqrt{3}}\right)}_{T_S(k,\tau)} R(\underline{k}, 0)$$

* $$T_S^2(k, \tau_*) = \begin{cases} 1/25 \cos^2\left(\frac{k\tau_*}{\sqrt{3}}\right) & k \ll k_{eq} \\ \cos^2\left(\frac{k\tau_*}{\sqrt{3}}\right) & k \gg k_{eq} \end{cases}$$

Lecture 20 - CMB Power II

27.11.24

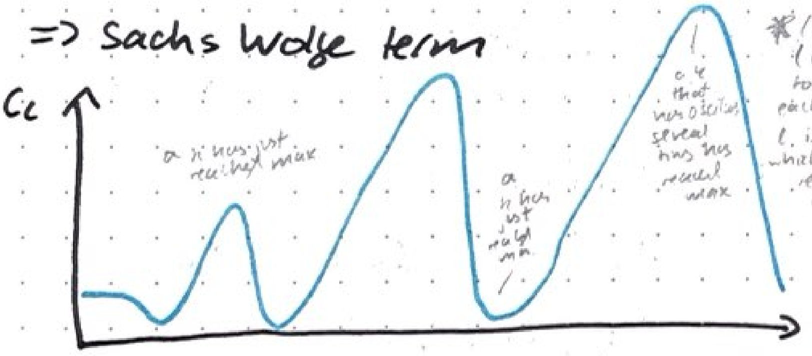
* Last time: derived

$$\frac{l(l+1)C_l}{2\pi} = \left[\frac{(l/\chi_*)^3}{2\pi^2} P^R(l/\chi_*)\right] T_s^2(k = l/\chi_*)$$

* and

$$T_s^2(k = l/\chi_*, \tau_*) = \begin{cases} \frac{1}{25}\cos^2\left[\frac{l}{\chi_*}\frac{\tau_*}{\sqrt{3}}\right] & l/\chi_* \ll k_{eq} \\ \cos^2\left[\frac{l}{\chi_*}\frac{\tau_*}{\sqrt{3}}\right] & l/\chi_* \gg k_{eq} \end{cases}$$

=> Sachs Wolfe term

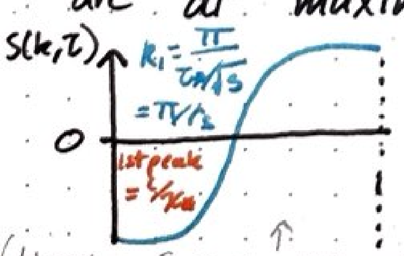


if I eval sachs wolfe term, I get a series of increasing peaks

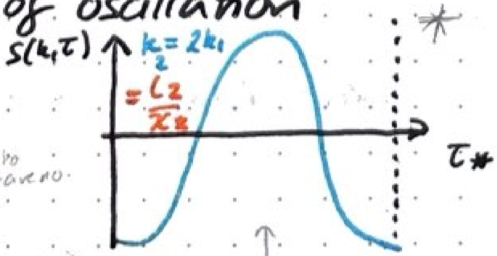
what is intuition why we see peaks?

Intuition

* Acoustic oscillations: radiation pressure provides a restoring force that opposes gravity / Φ, w. inertia, $\propto \cos\left(\frac{k\tau}{\sqrt{3}}\right)$
* Key point: frequency of the oscillations depends on $k = l/\chi_0$; $2k$ oscillates twice as fast
* At a time τ_* (CMB emission), certain frequencies / ks / ls are at maximum of oscillation



1st peak of C_l corresponds to mode w/ wave no. k_1



and so on

$k_n = nk_1$

(1st peak of C_l corresponds to mode that just reaches max of oscillation when CMB is emitted e.g. $S(k,\tau)$ sachs wolfe term (density + potential) consider mode at max at CMB emission)

(2nd peak wave no. $2k_1$ is multiple of 1st peak i.e. also at maximum)

=> Series of peaks evenly spaced in k and hence $l = k/\chi$.

but CMB doesn't actually look like this

Adding Complexity

1) Adding baryons:

* Original calculation of $\delta_r(k,\tau)$ during matter domination

think of what we did before then try to add baryons after

combine continuity $\delta_r' = -\frac{4}{3}(\nabla\cdot\underline{v}_r - 3\phi')$

and Euler equation $\underset{\text{(original eq)}}{\overset{\text{(sum in species)}}{q'}} + 4\mathcal{H}q + \nabla\delta P + (\bar\rho+\bar P)\nabla\phi = 0.$

same as [Ea] but written in terms of momentum density $q = (\bar\rho + \bar P)\underline{v}$

reminder: what we previously derived

* This gives $\underline{\delta_r'' + \frac{k^2}{3}\delta_r = \frac{4}{3}\nabla^2\phi} \rightarrow S \propto \cos\left(\frac{k\tau}{\sqrt{3}}\right)$

w/ this relabling, S becomes

can relabel $\underset{\text{speed of sound}}{C_s} = \frac{1}{\sqrt{3}}$; $r_s(\tau) = \int_0^\tau C_s d\tau$; $S(k,\tau) \propto \cos(kr_s)$

now make more complicated:

* How to add baryons? Assume scattering tightly couples $\gamma + b$, so they move with same $\underline{v}$ as a fluid (single γ+b fluid)

→ modify calc. so γ drag baryons along

* Now q in Euler eq.: $q = (\bar\rho + \bar P)\underline{v}$ (reminder: here $\bar\rho + \bar P$ total ρ, P summed over all components)

$= (\bar\rho_r + \bar P_r)\underline{v}_r + \bar\rho_b \underline{v}_r$ (note: assume baryons non-rel., so $\bar P$ negligible)

$= \frac{4}{3}(1+R)\bar\rho_r \underline{v}_r$

effect of adding b: [→ momentum density enhanced by const. factor 1+R!]

$R \equiv \frac{\bar\rho_b}{\bar\rho_r + \bar P_r} \approx$ const.

note: actually a terrible approximation, $R \propto a$ but only out b/c a doesn't vary a lot

→ continuity unaffected by baryons (momentum scattering of photons off baryons is elastic so no energy transferred)

Plug q in Euler, solve as usual for δ_r

what is physics that arises b/c of γ fluid dragging along b

but now include ϕ? we still want ϕ to be const. ignoring this dependence later (ϕ=0)

$$\delta_r'' + \underbrace{\frac{\mathcal{H}R}{1+R}}_{\text{enhanced drag force (baryon drag)}}\delta_r' - \underbrace{\frac{1}{3(1+R)}}_{\text{speed of sound changes}}\nabla^2\delta_r = 4\phi'' + \frac{4\mathcal{H}R}{1+R}\phi' + \frac{4}{3}\nabla^2\phi$$

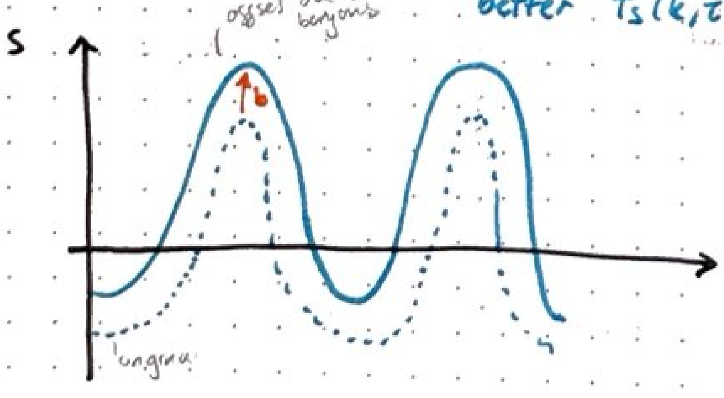
* Solution matching B.C.s ← complicated so I'll just tell you the solution

offset!

↓ effect: peaks of + sign reduced, peaks of − sign enhanced

$$\underbrace{S(k,\tau) = -\frac{1}{5}\left[(1+3R)\cos(kr_s) - 3R\right]}_{\text{better } T_s(k,\tau)\ k \ll k_{eq}}\mathcal{R}(\underline{k},0)$$

better transfer fn.



$C_\ell \sim T_s^2$: odd peaks enhanced

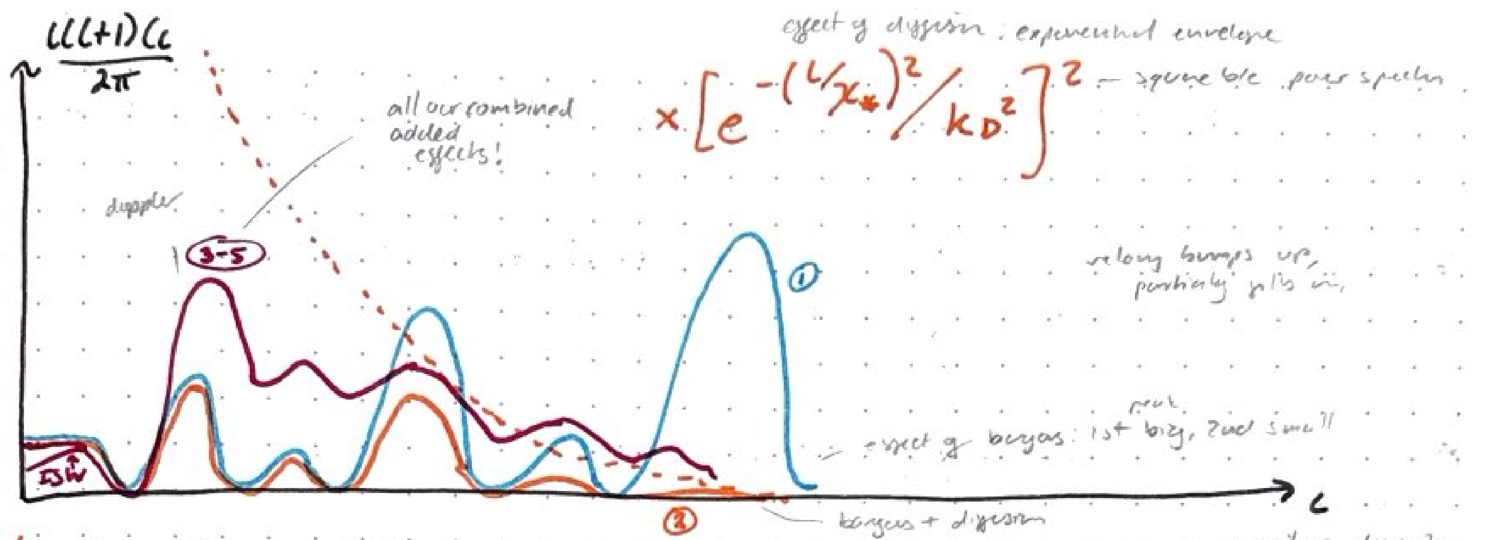
see in overleaf: 1st peak bigger but 2nd smaller

2) Diffusion of γ

so far treat γ + b as a fluid but actually they are not perfectly coupled in the fluid → photons scatter rapidly, diffuse due to scattering, so small perturbations will be 'washed out' due to physics of γ: small perturbations in CMB are washed out

want to understand physics, won't do rigorous derivation here

* Baryons + γ not perfectly coupled in fluid, γ diffuse over small distances, which washes out small features in CMB.



* Quantify: photon mean free path $\lambda_p = \frac{1}{a n \sigma_T}$

scattering is random walk so we can find how far γ travels

similar definition

Diffusion length: $\lambda_D = \sqrt{N}\lambda_p = \sqrt{\frac{d\tau}{\lambda_p}}\lambda_p = \sqrt{\lambda_p d\tau}$

no. of scatterings

diffusion wavenumber $k_D^{-2} \sim D^2 = \int_0^{\tau_*} \lambda_p d\tau$ — total distance diffused from this integral

distance diffused

in real space diffusing is convolution w/ a gaussian width $D = \sigma^2$ which is multiplication in fourier space

handwavy argument! → parameter describing Gaussian smoothing in real space or in fourier space multiplication by e^{-k^2/k_D^2}

(so small features cut off & smoothed)

3) Doppler terms

* have neglected $-\hat{n}\cdot\underline{v}$ term

potential: constant in MD, negligible in RD, so drop

continuity since $\delta_r' = -\frac{4}{3}(\nabla\cdot\underline{V}_r - \cancel{3\phi'})$

we may write $|\underline{V}_r| = \frac{3\delta_r'}{4k}$ in Fourier space

velocity undergoes sin oscillations

$\Rightarrow |V_r| \sim c_s \sin(kr_s)$

→ problem since density is cos oscillating w/ similar amplitude so if we add together & square we get $\cos^2 + \sin^2 = $ const. so no oscillations! no oscillations in CMB? → but this doesn't happen

Doesn't compensate for $\delta_r \sim \cos(kr_s)$ due to projection of v

(so smooth out sin, only partially compensates)

4) Integrated Sachs Wolfe

reminder: fall into potential well, potential decays so when come out of potential well net boost? shift

* $\int d\tau\, 2\phi'$ - decaying potential during ΛD, boost γ energy
→ bit of extra large scale power

(just effects very beginning)

5) Scattering at late times

* Reduces overall power (doesn't really change anything) → our pic is unchanged

conclusion
→ use CMB power spectrum as a tool to figure out composition of univ.
e.g. what if i have more baryons? → would see odd peaks even bigger, even peaks smaller
what about relativistic particle abundance → more time to damp CMB, more time to diffuse so exp envelope will change

Lecture 21: Primordial Density Perturbations from Inflation 29.11.24

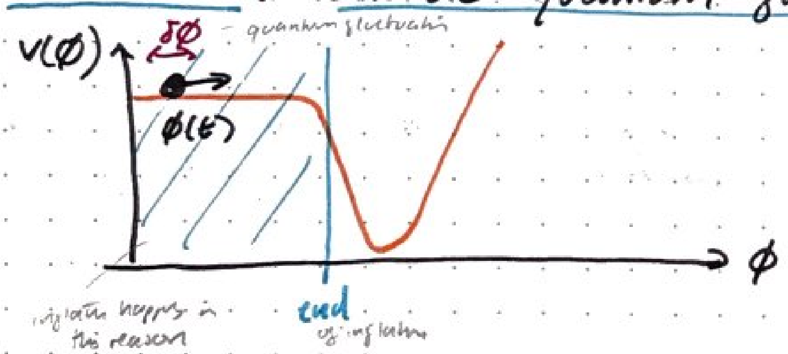
* Recall: Inflation was a mechanism proposed to solve horizon problem, $\frac{d}{dt}(\mathcal{H}^{-1}) < 0$

* Standard mechanism: single scalar field ϕ, slowly rolling down potential $V(\phi)$

* Soon realised inflation "accidentally" also predicted origin of density perturbations!

analyse how quantum mechanical treatment give rise to density perturbations

Basic Idea: Include quantum fluctuations



very slowly rolls down flat potential until suddenly steeps at characteristic time

quantum fluctuations in value of ϕ: in QM we have uncertainty to measurement of ϕ

* Can think of ϕ value as a "clock" parametrising time to end inflation
* Quantum zero-point fluctuations about background value
$$\bar{\Phi} \to \bar{\Phi} + \delta\Phi$$

* This implies that time to end of inflation is
$$\delta t(\underline{x}) = -\frac{\delta\Phi(\underline{x})}{\dot{\Phi}}$$
phi bar

if inflation fluctuates down 'hill' then you have less time for inflation & vice versa

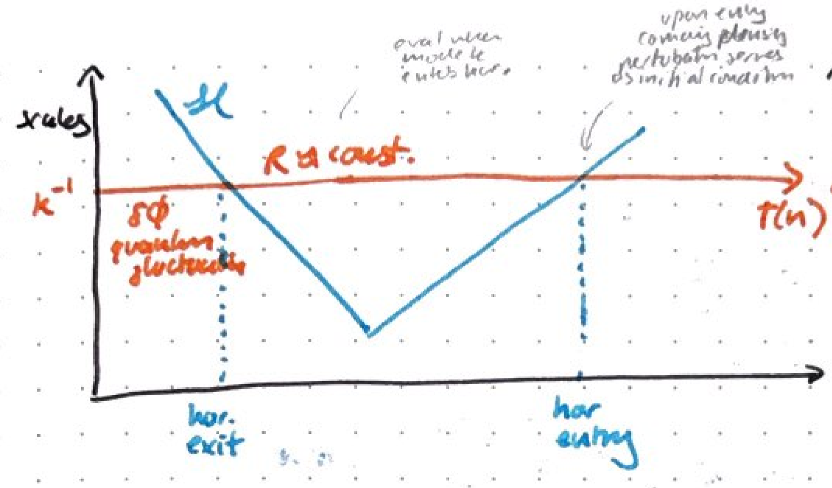
intuition: $\Rightarrow$ different regions expand by different amounts → some regions of inflation have inflation for longer/shorter

↳ why does this introduce curvature perturbation: imagine plastic sheet, expanding some parts & compress others → sheet warps & curves

* Curvature perturbation implied by this is
$$\mathcal{R}(x) \sim H\,\delta t \sim \frac{H}{\dot{\Phi}}\,\delta\Phi(\underline{x})$$

curvature perturbation directly related to fluctuation in inflation value due to quantum fluctuations

* This quantity is evaluated at horizon exit for each mode $\underline{k}$ $\quad k = \mathcal{H} = aH$



$\mathcal{R}(x)$ stays constant outside horizon, has become classical
classical field

Serves as initial conditions for Φ, δ, ... at horizon entry
$\to$ CMB / LSS. (at our location)

profound: we exist due to overdensity why is there overdensity: due to random fluctuation in inflaton field. so we exist due to huge random quantum fluctuation at our loc. in inflaton field

Classical evolution of Inflaton Perturbations

* Now: quantitative discussion
Goal: Show that each mode obeys an equation like an SHO and so can quantize accordingly (as we do a lot in theoretical physics)

* Starting point scalar field action
"not gonna write down gravitational part" just scalar field part
omitting grav. part

$$S = \int dx^4 \sqrt{-g}\left[\tfrac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right].$$

* Will assume metric $ds^2 = dt^2 - a(t)dx^2$
or in conformal time $ds^2 = a^2(\tau)\left[d\tau^2 - d\underline{x}^2\right]$.

usual assumptions for slow roll inflation

Will also assume that during inflation

(slow roll: neglect KE so H given by potential energy via FI) also assume const.

$$a(t) = -\frac{1}{H\tau}\ ;\ H = \sqrt{\frac{V(\phi)}{3M_{Pl}^2}} \simeq \text{const.}\ ;\ 2H^2 \gg \partial^2_{\phi\phi}V(\phi),\ \eta_V \ll 1.$$

* Let's start from the action and vary ϕ, $\delta\phi$
allow to vary by $\delta\phi$ & extremize its variation

see by: Taylor expanding ϕ about background value

obtain $\delta S = \int d^4x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\delta\phi\,\partial_\nu\phi - \delta\phi V'(\phi)\right]$

like previous derivation but now without assuming ϕ const.

* We can now integrate by parts
writing $\sqrt{-g}g^{\mu\nu}\partial_\mu\delta\phi\,\partial_\nu\phi = \partial_\mu(\sqrt{-g}g^{\mu\nu}\delta\phi\,\partial_\nu\phi) - \delta\phi\,\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)$.

$\to 0$ (integral over total deriv: can assume $\delta\phi$ is zero on boundary so $=0$)

variation in action

$$\Rightarrow \delta S = -\int d^4x\sqrt{-g}\left[\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) + V'(\phi)\right]\delta\phi = 0$$

extremize: $\delta S = 0$ for all $\delta\phi$

follows from extremizing action
$\hookrightarrow$ **e.o.m.**

$$\Rightarrow \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) + V'\phi = 0$$

* Since $g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1/a^2 & & \\ & & -1/a^2 & \\ 0 & & & -1/a^2 \end{pmatrix}$ (inv. of $g_{\mu\nu}$) and $\sqrt{-g} \equiv \sqrt{-\det g_{\mu\nu}} = a^3$

$\downarrow$ just plug these in & simplify this diagonal so becomes!

$$\frac{1}{a^3}\partial_\mu(a^3 g^{\mu\nu}\partial_\nu\phi) + \partial_\phi V = 0$$

$$\Rightarrow \quad \ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + \partial_\phi V = 0$$

we have recovered klein gordon eq. but now it includes spatial derivatives

now write in conformal time for convenience

* In conformal time $d\tau = \frac{dt}{a}$

$$\dot{\phi} = \frac{1}{a}\phi' \,;\quad \ddot{\phi} = \frac{1}{a^2}\phi'' - \mathcal{H}\phi'\frac{1}{a^2}$$

$$\Rightarrow \quad \boxed{\phi'' + 2\mathcal{H}\phi' - \nabla^2\phi + a^2\partial_\phi V = 0} \qquad (**)$$

KLEIN GORDON W/ SPATIAL DERIVATIVES IN CONFORMAL TIME

Classical Perturbation Evaluation

* Want eq. for perturbations $\delta\phi$; $\cancel{\delta g_{\mu\nu}}$ around classical background

spatially flat gauge $\delta g_{\mu\nu}$ suppressed

(proof in daniel bowman book)

* Can just take $\phi(\underline{x},\tau) = \bar{\phi}(\tau) + \delta\phi(\underline{x},\tau)$ in $(**)$:

in spatially flat gauge dont have to worry about what happens to metric

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi + a^2\partial^2_{\phi\phi}V\delta\phi = 0.$$

(only non trivial term: take taylor expansion to get 2nd deriv.?)

* Now define $\gamma(\underline{x},\tau) \equiv a(\tau)\,\delta\phi(\underline{x},\tau)$ (for convenience)

Plug in $\quad \gamma'' - \nabla^2\gamma + \left(a^2 V_{,\phi\phi} - \frac{a''}{a}\right)\gamma = 0.$

Slow roll $\quad \frac{V_{,\phi\phi}}{H^2} = 3M_{Pl}^2\frac{V_{,\phi\phi}}{V} = 3\eta_V \ll 1.$

(recognise defining 2nd hubble slow roll param which has to be small.)

$$a' = a^2 H \,,\quad a'' = 2aa'H \rightarrow \frac{a''}{a} = 2a'H = 2a^2H^2 \gg a^2 V_{,\phi\phi}$$

so get rid of $\rightarrow a^2 V_{\phi\phi}$ term!

$$\boxed{\gamma''(\underline{x},\tau) - \nabla^2\gamma(\underline{x},\tau) - \frac{a''}{a}\gamma(\underline{x},\tau) = 0}$$

MUKHANOV-SASAKI EQUATION

classical eom for perturbation to inflation field, how it grows

* For fourier mode $\underline{k}$; $\nabla^2 \rightarrow -k^2$

$$\boxed{\gamma''(\underline{k},\tau) + \left(k^2 - \frac{a''}{a}\right)\gamma(\underline{k},\tau) = 0}$$

M-S EQUATION (in Fourier space)

starting to look more like harmonic oscillator

* Take subhorizon limit $k^2 \gg \frac{a''}{a} = 2(aH)^2$

$$\gamma''_{\underline{k}} + k^2\gamma_{\underline{k}} = 0$$

a simple harmonic oscillator!

$\hookrightarrow$ quantise this to understand our (quantum fluctuations?)

Lecture 22: Quantising the Inflation

4.12.24

* **Last time**: showed that for $\delta\phi = f/a$

$$f''(\underline{k},\tau) + \left(k^2 - \frac{a''}{a}\right) f(\underline{k},\tau) = 0 \qquad \text{M-S}$$

* subhorizon limit $k^2 >> a''/a = 2(aH)^2$

$$f'' + k^2 f = 0 \rightarrow \underline{\text{SHO!}}$$

we showed that fluctuations in inflaton field f can be described by M-S equation, which becomes a SHO at subhorizon scales

$\longrightarrow$ today we quantise SHO, i.e. inflaton fluctuations

HO has zero-point fluctuations (recall x^2 on average in ground state, or energy these are not zero!), these fluctuations will also appear here: zero-point fluctuations of this field lead to the origin of all structure!

Canonical Quantisation of SHO

$\rightarrow$ go through this then proceed by analogy to consider inflation

* Consider classical harmonic oscillator, coordinate q, mass $m=1$ in a potential $V(q) = \frac{\omega^2 q^2}{2}$

* Lagrangian is $L = \frac{1}{2}\dot{q}^2 - \frac{\omega^2 q^2}{2}$

$\Rightarrow$ e.o.m. is $\underline{\ddot{q} + \omega^2 q = 0}$ solutions $q = Ae^{\pm i\omega t}$

conjugate momentum $p = \frac{\delta L}{\delta \dot{q}} = \dot{q}$; Hamiltonian $H = \frac{1}{2}p^2 + \frac{\omega^2 q^2}{2}$

* In canonical quantisation, we promote $q(t), p(t) \Rightarrow \hat{q}(t), \hat{p}(t)$ Heisenberg picture operators, then:

1) Impose canonical commutation relations $[\hat{q}(t), \hat{p}(t)] = i$ $(\hbar = 1)$.

2) Expand in modes $\hat{q}(t) = q(t)\hat{a} + q(t)\hat{a}^+$, where $\hat{a}, \hat{a}^+$ are time-independent operators.
$\hat{p}(t) = \dot{q}(t)\hat{a} + \ddot{q}(t)\hat{a}^+$.

now ask what do commutation relations imply for mode expansion?

Inserting commutation relations 1) mode expansion 2) :

$$[\hat{q}, \hat{p}] = \underbrace{(q\dot{q}^* - \dot{q}q^*)}[\hat{a}, \hat{a}^+] = i$$

$\leftarrow$ (after a couple of lines of work)

this function $q\dot{q}^* - \dot{q}q^* \equiv iW(q, \dot{q})$ defines Wronskian W.

3) We choose the normalisation such that $W = 1$

(standard result of QM.) $\leftarrow$

4) This implies the commutation relations $[\hat{a}, \hat{a}^+] = 1$

The Hamiltonian is then

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2 \hat{q}^2}{2} = \frac{1}{2}(\dot{q}\hat{a} + \dot{q}^*\hat{a}^+)^2 + \frac{\omega^2}{2}(q\hat{a} + q^*\hat{a}^+)^2$$

$$= \frac{1}{2}\left((\dot{q}^*)^2 + \omega^2 (q^*)^2\right)^2 + \frac{1}{2}(\dot{q}\dot{q}^* + \omega^2 q q^*)\hat{a}\hat{a}^+ + \dots$$

terms involving $\hat{a}^+\hat{a}$, $\hat{a}^2$

not quite done: need to find our vacuum/ground state $\Downarrow$ how do we define this?

need to know $q(t)$ s.t. $\hat{a}$ is unambiguously defined ← since $\hat{a}$ is defined in combination w/ $q(t)$ so need to know $q(t)$ to know exactly what $\hat{a}$ is

Define vaccum state $\hat{a}|0\rangle = 0$. But, this condition isn't enough, we still haven't fixed $|0\rangle$ as we have not yet fixed $q(t)$.

5) To define vaccum fully, fix $q(t)$ by imposing extra condition that vaccum must be an eigenstate of $\hat{H}$.

look at $\hat{H}|0\rangle$

1st term: problem raising vaccum twice not $\propto|0\rangle$!

$$\hat{H}|0\rangle = \tfrac{1}{2}\underbrace{((\dot{q}^*)^2 + \omega^2(q^*)^2)^2\hat{a}^{\dagger 2}|0\rangle}_{\text{not } \propto|0\rangle \text{ unless require } \dot{q}^2+\omega^2q^2=0 \Rightarrow \dot{q}=\pm i\omega q} + \tfrac{1}{2}\underbrace{(\dot{q}\dot{q}^* + \omega^2 q q^*)\hat{a}\hat{a}^\dagger|0\rangle}_{\propto|0\rangle}$$

term ok: good prop. to vacuum

Require $W=1$, positive
Since $W = \mp 2\omega|q|^2$ only true for +ve frequency solution

$\Rightarrow q(t) = Ae^{-i\omega t}$

Choose A so that $W=1 \Rightarrow q(t) = \frac{1}{\sqrt{2\omega}}e^{-i\omega t}$

by fixing mode fn. everything is fully defined (operators, vaccum etc.)

q obeys classical equation of motion

* With mode function fixed we can now calculate zero-pt. fluctuations

non-zero! oscillator does not sit at zero, its fluctuates around zero

$$\langle 0|\hat{q}|0\rangle = 0 \quad ; \quad \langle 0|\hat{q}^2|0\rangle = \langle 0|\ldots + q(t)q^*(t)\hat{a}\hat{a}^\dagger|0\rangle$$
$$= \tfrac{1}{2\omega}.$$

mean square non-zero ⇒ for inflaton, mean field fluctuations in inflaton value

Quantising the Inflaton Field

* Will follow analagous procedure, modulo small changes to ordering
* Wish to quantise $f_{\underline{k}}$ with

$$f''_{\underline{k}} + (k^2 - \tfrac{a''}{a})f_{\underline{k}} = 0 \qquad \text{(obeys M-S eq.)}$$

As during inflation, $a(\tau) = -\frac{1}{H\tau}$, $f''_{\underline{k}} + (k^2 - 2/\tau^2)f_{\underline{k}} = 0$.

The conjugate momentum to f is $\pi = f'$. (time deriv of f) (see notes)

now lets go through same steps to quantise this, we have correctly momentum! f

To canonically quantise f:

1) Impose canonical commutation relations

promote f, π to heisenberg picture operators w/ following commutation relations

$$[\hat{f}(\tau,\underline{x}), \hat{\pi}(\tau,\underline{\tilde{x}})] = i\delta^{(3)}(\underline{x}-\underline{\tilde{x}})$$

2) Expand f, π operators in mode functions

subtlety: want to write this in real space so write via FT

$$\hat{f}(\tau,\underline{x}) = \int\frac{d^3k}{(2\pi)^3}\left(f^*_{\underline{k}}(\tau)\hat{a}^\dagger_{\underline{k}}e^{-i\underline{k}\cdot\underline{x}} + f_{\underline{k}}(\tau)\hat{a}_{\underline{k}}e^{i\underline{k}\cdot\underline{x}}\right)$$

can be written as before in terms of time evolving raising & lowering operators

numbered to match previous method / switched order for convenience here.)

$$\pi(\tau, \underline{x}) = \delta' = \int \frac{d^3 \underline{k}}{(2\pi)^3} \left(\delta_{\underline{k}}^{\prime *}(\tau) \hat{a}_{\underline{k}}^{+} e^{-i\underline{k}\cdot\underline{x}} + \delta_{\underline{k}}^{\prime}(\tau) \hat{a}_{\underline{k}} e^{i\underline{k}\cdot\underline{x}} \right)$$

4) For creation and annihilation operators, take as before

$$[\hat{a}_{\underline{k}}, \hat{a}_{\underline{\tilde{k}}}] = [\hat{a}_{\underline{k}}^{+}, \hat{a}_{\underline{\tilde{k}}}^{+}] = 0 \; ; \; [\hat{a}_{\underline{k}}, \hat{a}_{\underline{\tilde{k}}}^{+}] = (2\pi)^3 \delta^{(3)}(\underline{k} - \underline{\tilde{k}})$$

assume δ fn. as analogue to before then check canonical commutator

Inserting mode expansion in commutation relations

$$[\hat{\delta}(\tau, \underline{x}), \hat{\pi}(\tau, \underline{x})] = \int \frac{d^3k\, d^3\tilde{k}}{(2\pi)^6} [(\delta_{\underline{k}} e^{i\underline{k}\cdot\underline{x}} \hat{a}_{\underline{k}} + \delta_{\underline{k}}^{*} e^{-i\underline{k}\cdot\underline{x}} \hat{a}_{\underline{k}}^{+}), (\delta_{\underline{\tilde{k}}}' e^{i\underline{\tilde{k}}\cdot\underline{\tilde{x}}} \hat{a}_{\underline{\tilde{k}}} + \delta_{\underline{\tilde{k}}}^{\prime *} e^{-i\underline{\tilde{k}}\cdot\underline{\tilde{x}}} \hat{a}_{\underline{\tilde{k}}}^{+})]$$

written as:

$$= \frac{1}{(2\pi)^3} \int d^3k \, i W e^{i\underline{k}(\underline{x} - \underline{\tilde{x}})} \qquad \text{with } iW \equiv \delta_{\underline{k}} \delta_{\underline{k}}^{\prime *} - \delta_{-\underline{k}}^{*} \delta_{-\underline{k}}'$$

δ fn. for result W.

3) This implies we get canonical commutation relations from $[\hat{a}_{\underline{k}}, \hat{a}_{\underline{\tilde{k}}}^{+}] = (2\pi)^3 \delta^{(3)}(\underline{k} - \underline{\tilde{k}})$ if $W = 1$. (then get δ fn.)

ambiguity, what exactly is δ_k?

5) But mode function not fully determined, so vaccum not fixed

Require that, as for SHO, it obeys equation of motion (δ_k)

$$\delta_{\underline{k}}'' + (k^2 - 2/\tau^2)\delta_{\underline{k}} = 0 \qquad (M\text{-}S)$$

Solution (general M-S solution)

$$\delta_k(\tau) = a \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) + b \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right)$$

but how do I choose a, b?

In horizon → SHO

inside hor. this becomes SHO eq.

early times, everything is subhor. of mode fn. needs to go back to harmonic oscillator solution

Require as $\tau \to -\infty$, limit of $\delta k \to$ initial Minkowski / SHO solution.

Matching onto Minkowski initial conditions, we obtain $a=1$, $b=0$, so our solution is

$$\boxed{\delta_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right)}$$

BUNCH-DAVIES MODE FUNCTION

This also fully fixes vaccum state.

Now know full field operator and can do calculations!

$$\hat{\delta}(\tau, \underline{x}) = \int \frac{d^3k}{(2\pi)^3} [\delta_{\underline{k}}^{*}(\tau) \hat{a}_{\underline{k}}^{+} e^{-i\underline{k}\cdot\underline{x}} + \delta_{\underline{k}}(\tau) \hat{a}_{\underline{k}} e^{i\underline{k}\cdot\underline{x}}]$$

e.g. →

Lecture 23: Quantum fluctuations + observables 6.12.24

mean square of δ given by bracketing operator in |0⟩

* Can now calculate fluctuations $\langle \delta^2 \rangle = \langle 0|\hat{\delta}|0\rangle$ → just like HO → mean is 0 but mean² not zero ⇒ random quantum fluctuations

* Main quantity of interest is power spectrum → (real space version of PS is correlation fn) related to correlation function via

$$\xi^{\phi}(r) = \langle \delta\phi(\underline{x},\tau)\,\delta\phi(\underline{x}+r,\tau)\rangle = \int \frac{d^3k}{(2\pi)^3} P^{\phi}(k) e^{-i\underline{k}\cdot r}$$

our strategy: → want to convert PS in φ to PS in R ↓

$$= \int \frac{d^3k}{(2\pi)^3}\left[\frac{2\pi^2}{k^3}\Delta_{\phi}^2(k)\right] e^{-i\underline{k}\cdot\underline{r}}$$ dimless

* Calculate

$$\xi^{\phi} = \langle 0|\delta\hat{\phi}(\underline{x},\tau)\,\delta\hat{\phi}(\underline{x}+\underline{r},\tau)|0\rangle$$
$$= \frac{1}{a^2}\langle 0|\hat{f}(\underline{x},\tau)\hat{f}(\underline{x}+\underline{r},\tau)|0\rangle$$
$$= \frac{1}{a^2}\iint \frac{d^3k\, d^3\tilde{k}}{(2\pi)^6}\langle 0|\left(\cancel{f_{\underline{k}}^{*}(\tau)\hat{a}_{\underline{k}}^{+}e^{-i\underline{k}\cdot\underline{x}}}^{\,0} + f_{\underline{k}}(\tau)e^{i\underline{k}\cdot\underline{x}}\right)\times\left(f_{\tilde{k}}^{*}(\tau)\hat{a}_{\tilde{k}}^{+}e^{-i\tilde{k}\cdot(\underline{x}+\underline{r})} + \cancel{f_{\tilde{k}}(\tau)\hat{a}_{\tilde{k}}e^{i\tilde{k}\cdot(\underline{x}+\underline{r})}}^{\,0}\right)|0\rangle$$
$$= \frac{1}{a^2}\iint\frac{d^3k\,d^3\tilde{k}}{(2\pi)^6}\langle 0|\hat{a}_{\underline{k}}\hat{a}_{\tilde{k}}^{+}|0\rangle f_{\underline{k}}f_{\tilde{k}}^{*}\, e^{i\underline{k}\cdot\underline{x}}e^{-i\tilde{k}\cdot(\underline{x}+\underline{r})}$$

this immediately disappears — $\cancel{\hat{a}_{\tilde{k}}^{+}\hat{a}_{\underline{k}}} + (2\pi)^3\delta^{(3)}(\underline{k}-\tilde{k})$

RESULT correlation fn predicted by random fluctuations of inflaton is →

$$\left[= \frac{1}{a^2}\int\frac{d^3k}{(2\pi)^3}|f_k(\tau)|^2 e^{-i\underline{k}\cdot\underline{r}}\right]$$ non zero!

* Obtain (dimless PS) $\Delta_{\phi}^2(k) = \dfrac{k^3}{2\pi^2}\dfrac{|f_k(\tau)|^2}{a^2}$ — PS of inflaton fluctuations predicted by quantum mechanical treatment → even this

* Use B-D mode function f_k:

$\mathrm{Re}^2 + \mathrm{Im}^2$

$$\Delta_{\phi}^2(k) = \frac{1}{4\pi^2}\frac{k^2}{a^2}\left(1 + \frac{1}{k^2\tau^2}\right)$$ as $a = -\dfrac{1}{H\tau}$ (as seen many times before)

$$= \frac{1}{4\pi^2}\frac{k^2}{a^2}\left(1+\frac{a^2H^2}{k^2}\right) = \frac{1}{4\pi^2}H^2\left(1+\frac{k^2}{a^2H^2}\right)$$

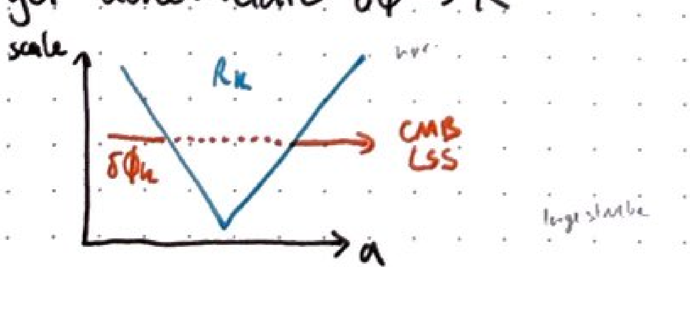
(recall would have ... when $k \ll aH$ horizon exit ↓)

* Soon after horizon exit $k \lesssim aH = \mathcal{H}$ → * slowly varying but varying nonetheless → extent to which varies depends

immediately after ↓ now relate δφ to R ↓

$$(1)\quad \boxed{\Delta_{\phi}^2(k) = \left(\frac{H}{2\pi}\right)^2\Big|_{k\simeq aH} \text{ (horizon cross)}}$$ $\simeq$ const. ~ scale invariant spectrum as $H \simeq$ const.

Not yet done: relate $\delta\phi \to R$



large scale

scale-time plot
quantum fluctuations in φ (δφ)
want to turn into classical
perturbations so can
compute initial conditions CMB

* Recall $\mathcal{R} \equiv \cancel{C} - \cancel{\tfrac{1}{3}\nabla^2 E} - \mathcal{H}(B+v)$ definition
in spatially flat gauge $C = E = 0$
look back at notes (...)

* We showed, perturbing matter
$$\delta T^0{}_j = -(\bar{\rho} + \bar{P})\partial_j (B+v) \quad (*)$$
Evaluating perturbation to ϕ's $T_{\mu\nu}$ directly
$T_{\mu\nu}$ for scalar field

$$\left[T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}\left(\tfrac{1}{2} g^{\alpha\beta}\partial_\alpha \phi \partial_\beta \phi - V(\phi)\right)\right]$$
perturbing $^0{}_j$ element
phi bar prime

$$\delta T^0_j = \bar{g}^{0\mu}\partial_\mu \phi \partial_j \delta\phi = \bar{g}^{00}\partial_0 \bar{\Phi}\partial_j \delta\phi = \frac{\bar{\Phi}'}{a^2}\partial_j \delta\phi$$
(...) $\delta\phi$ and $\delta T^0{}_j$ (related to spatial deriv of $\delta\phi$)

Equating, using $\bar{\rho} + \bar{P} = \dot{\bar{\phi}}^2 = \tfrac{1}{a}\bar{\Phi}'^2$
relate inflaton fluctuations to curvature (rigorously)
$$\rightarrow B + V = -\frac{\delta\phi}{\bar{\Phi}'} \Rightarrow \boxed{\mathcal{R}(\underline{x}) = -\mathcal{H}\frac{\delta\phi(x)}{\bar{\Phi}'}} \quad (II)$$

* Combining (I) and (II) we obtain
notice! handy looks like slow-roll parameter
$$\Delta^2_{\mathcal{R}}(k) = \underbrace{\left(\frac{\mathcal{H}}{\bar{\Phi}'}\right)^2}_{= \frac{1}{2\varepsilon M_{pl}^2}} \Delta^2_{\delta\phi}(k)$$
Now using definition $\varepsilon = \dfrac{\tfrac{1}{2}\dot{\bar{\phi}}^2}{H^2 M_{pl}^2} = \dfrac{\bar{\Phi}'^2}{2\mathcal{H}^2 M_{pl}^2}$
why? H slowly varying so even though diff ks cross horizon at diff times this doesn't matter since H is slowly varying

* $$\boxed{\Delta^2_{\mathcal{R}}(k) = \frac{1}{2\varepsilon M_{pl}^2}\left(\frac{H}{2\pi}\right)^2\Bigg|_{k \simeq aH}}$$
$\simeq$ const. since $H + \varepsilon$ slowly varying
core result
everyone rises from their seats & erupts in applause

* i.e. $\underline{P_{\mathcal{R}}(k) =} \dfrac{2\pi^2}{k^3} \underbrace{\dfrac{1}{2\varepsilon M_{pl}^2}\left(\dfrac{H}{2\pi}\right)^2}_{\text{const.}} \underline{\Bigg|_{k = aH}}$

Near-scale invariant power spectrum of $\mathcal{R}$ arises naturally from considering quantum fluctuations of $\delta\phi$!

$\Rightarrow$ origin of all structure! (probably)

* Can we be more precise?
$$\Delta_{\mathcal{R}}^2(k) \propto k^{n_s - 1} \; ; \; n_s - 1 \equiv \frac{d\ln \Delta^2_{\mathcal{R}}}{d\ln k}$$

$$n_s - 1 = \frac{d\ln \Delta_{\mathcal{R}}^2}{d\ln k} = \underbrace{\frac{d\ln \Delta^2_{\mathcal{R}}}{dN}}_{-2\varepsilon - \eta} \times \underbrace{\frac{dN}{d\ln k}}_{1 + O(\varepsilon)} = -2\varepsilon - \eta \; .$$
notes:

Predicts near scale invariant spectrum but not exactly.

* Inflation tests

- near scale-inv. spectrum ✓ $n_s = 0.967 \pm 0.004$
- superhorizon CMB ✓
- Gaussianity ✓
- Adiabatic
- flat universe

ans to question asked in lecture (?):

non-rel Φ vs Ψ rel

$$ds^2 = \underline{(1+2\Phi)}dt^2 - \underline{(1-2\Psi)}dx^2$$

growth vs expansion

$$\nabla^2\Phi = g(k)\delta$$

Lecture 24: ("Bonus" lecture) Observing and testing inflationary physics

* Last time: derived

$$\boxed{\Delta^2_R(k) = \frac{1}{2\varepsilon M_{Pl}^2}\left(\frac{H}{2\pi}\right)^2\Bigg|_{k=aH}}$$

$\simeq$ const. since H, ε slowly varying with t and hence k.
→ $\simeq$ scale invariant

* Can we be more precise? Parametrise
$\Delta_R^2(k) \propto k^{n_s-1}$; → $n_s - 1 \equiv \frac{d\ln\Delta^2_R}{d\ln k}$

* Calculate inflation prediction:

$$n_s - 1 = \frac{d\ln\Delta_R^2(k)}{d\ln k} = \frac{d\ln\Delta^2_R(k)}{dN} \times \frac{dN}{d\ln k}$$

N is no. of e-folds $= \ln a$.

1st factor: $-2\varepsilon-\eta$

? 2nd factor

* The first factor is (from *):

$$\frac{d\ln\Delta^2_R}{dN} = 2\frac{d\ln H}{dN} - \frac{d\ln\varepsilon}{dN} = -2\varepsilon - \eta$$

* For the second, use $k = aH$

→ $\ln k = N + \ln H$

→ $\frac{dN}{d\ln k} = \left[\frac{d\ln k}{dN}\right]^{-1} = \left[1 + \frac{d\ln H}{dN}\right]^{-1} \simeq 1+\varepsilon$

put together ⇒ $\boxed{n_s - 1 = 2\varepsilon - \eta}$

Inflation predicts near-scale invariant spectrum, but not exactly.

* Inflation tests, already observed:

1) Fluctuations that are
   - near-scale-invariant ✓
     $n_s = 0.967 \pm 0.004$
   - Gaussian ✓
   - Adiabatic ✓

2) Flat universe ✓

$$\Omega_k = 0.0023 \pm 0.004$$

But: how to definitively confirm inflation and find out more about its exact physics?

<u>Bonus Material</u> : <u>The search for Inflationary Gravitational Waves with the CMB B-mode Polarisation</u>

(non-examinable)

Inflationary predictions: gravitational waves

* Not just fluctuations in inflation, fluctuations in metric

$$ds^2 = a^2(\eta)\left[d\eta^2 - (\delta_{ij} + 2\hat{E}_{ij})\, dx^i dx^j\right]$$

* Can write E as two gravitational wave polarisations

$$\frac{M_{PL}}{2} a \hat{E}_{ij} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} f_+ & f_\times & 0 \\ f_\times & -f_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

* Insert in action

$$S = \frac{M_{PL}^2}{2} \int d^4x \sqrt{-g}\, R$$

* Get exact same equation of motion for f_+ and $f_\times$ as for inflaton fluctuation f! So power also a near scale invariant power spectrum of gravitational waves:

$$\frac{k^3}{2\pi^2} P_t \equiv 2\, \frac{k^3}{2\pi^2} P_{\hat{E}} \approx \text{const.}$$

$$\sim (H/2\pi)^2$$