modulus since can be complex

$$h_{\mu\nu} = Re\left(H_{\mu\nu} e^{ik^{\mu}x_{\mu}}\right) = |H_+| \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos\omega(t - \xi)$$

$-t_0$ since can be complex

ξ is phase that comes from it being complex

Along λ $\xi = 0$ equations are

(since $h_{0\mu} = h_{3\mu} = 0$)

$$\frac{d^2 Y^0}{dt^2} = \frac{d^2 Y^3}{dt^2} = 0$$

(no relative acceleration in direction of the wave)

(turn. and no correction to time coordinate)

cannot solve explicitly linear eqn with oscillating coefficient ↓ but can solve perturbatively

$$\frac{d^2 Y^1}{dt^2} = -\tfrac{1}{2}\omega^2 |H_+| \cos\omega(t - t_0)\, Y^1$$

$$\frac{d^2 Y^2}{dt^2} = \tfrac{1}{2}\omega^2 |H_+| \cos\omega(t - t_0)\, Y^2$$

H_+ is small, so solve perturbatively with $\frac{dY^1}{dt} = \frac{dY^2}{dt} = 0$ $\Rightarrow Y^1 = Y^1_0 + O(\varepsilon^2)$, $Y^2 = Y^2_0 + O(\varepsilon^2)$

put back into

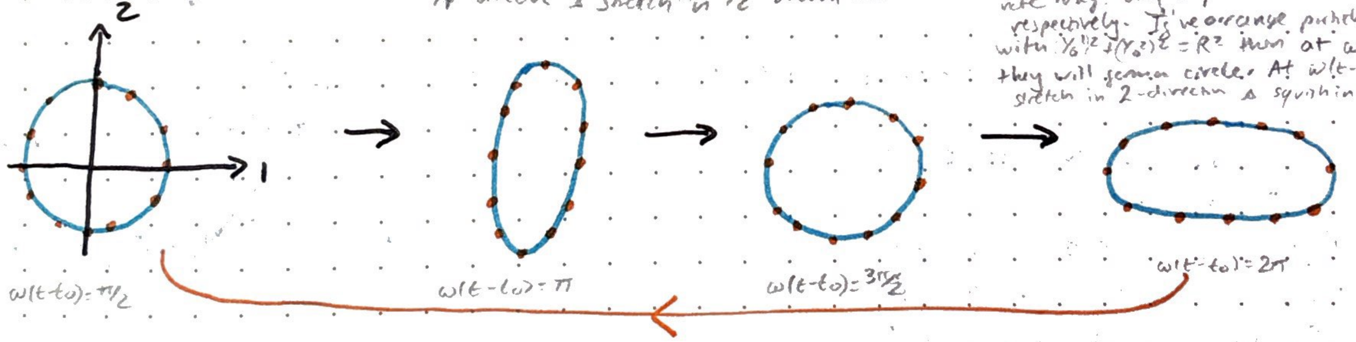
$Y^1 = Y_0{}^1\left(1 + \tfrac{1}{2}|H_+|\cos\omega\left(t - t_0\right)\right)$

$Y^2 = Y_0{}^2\left(1 - \tfrac{1}{2}|H_+|\cos\omega\left(t - t_0\right)\right)$

initially i am sat at origin & around me are collection of test masses on circle. as time passes this is going to squash in Y_1 direction & stretch in Y_2 direction then

stretch other way since oscillatory.

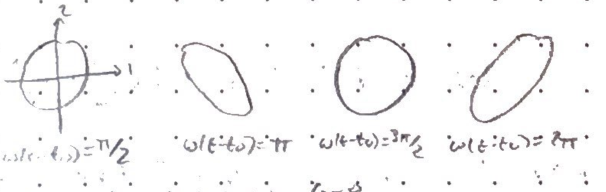
note to self: avg displacement is Y_0^1; Y_0^2 respectively. If we arrange particles in 12 plane with $(Y_0^1)^2 + (Y_0^2)^2 = R^2$ then at $\omega(t-t_0) = \pi/2$ they will form a circle. At $\omega(t-t_0) = \pi$ they stretch in 2-direction & squish in 1-direction etc.



Exercise: Find solution for $\times$ polarisation.

note to self: solve perturbatively
attempt:
zeroth order solution (no wave):
since $\frac{dY^1}{dt} = 0 \Rightarrow Y^1 = Y^1_0 + O(\varepsilon^2)$
first order correction:
plug in $Y^1 = Y^1_0 + \delta Y$ into $\frac{d^2 Y}{dt^2}$
$\Rightarrow \delta Y^1 = \tfrac{1}{2}|H_+|\cos\omega(t - t_0) Y^1_0$
so put together $Y = Y_0 + \delta Y$ gives:
$Y^1 = Y_0^1(1 + \tfrac{1}{2}|H_+|\cos\omega(t - t_0)) Y_0$

attempt: solve as for + polarisation
$Y^2 = Y^2_0 + \tfrac{1}{2}|H_\times|\cos\omega(t - t_0) Y_0^1$
$Y^1 = Y^1_0 + \tfrac{1}{2}|H_\times|\cos\omega(t - t_0) Y_0^2$

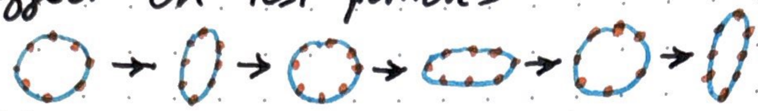


same sol. rotated by 45°

Lecture 18 — 20.11.24

Last lecture: • monochromatic plane gravitational waves
• effect on test particles

circle of particles freely falling, wave travels into the page



(+ POLARISATION)

Found using geodesic deviation eqn.

(X POLARISATION  (??))

THE FIELD FAR FROM A SOURCE

Return to linearised Einstein equations with matter

thinking this like E+M equations in Lorentz gauge
→ potential satisfies wave eqn
→ 4-divergence free → Leonard Fischer potential analog.

$$(*)\ \partial_\rho \partial^\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \qquad \partial_\mu \bar{h}^\mu{}_\nu = 0$$

wave gauge condition

As is the case for electromagnetism, we can solve $(*)$ explicitly using a retarded Greens function:

as is standard in Minkowski use t as x^0 component

modulus computed in euclidean metric → exactly as we normally think of it

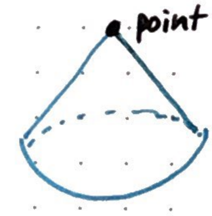
$$\bar{h}_{\mu\nu}(t, \underline{x}) = 4\int d^3x' \frac{T_{\mu\nu}(t - |\underline{x} - \underline{x}'|, \underline{x}')}{|\underline{x} - \underline{x}'|} \qquad (\dagger)$$

Where $|\underline{x} - \underline{x}'|$ is computed in the Euclidean metric

(from pov far away)

If matter is concentrated within a distance d of the origin (so $T_{\mu\nu}(t,\underline{x}) = 0$ for $|\underline{x}'| > d$), we can expand in the far field region where $r = |\underline{x}| >> |\underline{x}'| \sim d$.) Then

interested in look our past lightcone of point (the effect of RHS) integrate over past lightcone with some factor telling me how much decay I see as I go from one side to another)



point $(t, \underline{x})$

past lightcone

$$|\underline{x} - \underline{x}'|^2 = r^2 - 2\underline{x}\cdot\underline{x}' + |\underline{x}'|^2 = r^2\left(1 - \tfrac{2}{r}\hat{\underline{x}}\cdot\underline{x}' + O\left(\tfrac{d^2}{r^2}\right)\right)$$

with $\hat{x} = x/r$. Hence, (use binomial expansion)

unit vector

correction is inner product of vector pointing from origin & position vector (')

$$|\underline{x} - \underline{x}'| = r\left(1 - \tfrac{1}{r}\hat{\underline{x}}\cdot\underline{x}' + O\left(\tfrac{d^2}{r^2}\right)\right) = r - \hat{\underline{x}}\cdot\underline{x}' + O\left(\tfrac{d^2}{r}\right)$$

so fun will not is you most

and

$$T_{\mu\nu}(t - |\underline{x} - \underline{x}'|, \underline{x}') = T_{\mu\nu}(t', \underline{x}') + \hat{\underline{x}}\cdot\underline{x}'\,\partial_0 T_{\mu\nu}(t', \underline{x}') + \dots$$

$\approx T_{\mu\nu}(t - r + \hat{x}\cdot x', x') =$

where $t' = t - r$.

what are some corrections will be ...

note to self small distance over large time so neglect?

what do I mean by this

note to self remember typical timescale of change $\tau \sim \frac{L}{v}$?

If $T_{\mu\nu}$ varies on a timescale of τ so that $\partial_0 T_{\mu\nu} \sim \frac{1}{\tau} T_{\mu\nu}$, then second term above is $O(d/\tau)$ which we can neglect if matter moves non-relativistically. We thus have,

I want to know $\bar{h}_{ij}$ at point in time, I integrate over T_{ij} at the retarded time and add up contributions to get ...

$$\bar{h}_{ij} = \frac{4}{r}\int d^3x'\, T_{ij}(t', \underline{x}') \qquad t' = t - r \qquad (**)$$

This gives the spatial components of $\bar{h}_{ij}$. To find remaining components we use gauge condition: $\partial_0 \bar{h}_{0i} = \partial_j \bar{h}_{ji}$, $\partial_0 \bar{h}_{00} = \partial_i \bar{h}_{0i}$.

note to self from $\partial^\mu \bar{h}_{\mu\nu} = 0$ ν = i component and $\nu = 0$

First solve for $\bar{h}_{0i}$ then $\bar{h}_{00}$.

notes. $\partial_\mu T^{\mu\nu}=0$.

We can simplify the integral in (**) by recalling that $\partial_\mu T^{\mu}{}_{\nu}=0$ and that $T_{\mu\nu}(t',\underline{x}')$ vanishes for $|\underline{x}|>d$.
(EM conservation)

(gonna drop primes in next bit so don't have to write it out & dummy index so doesn't matter)

Dropping primes in the integral

subst $T=T'$ & use dummy index do integration by parts using fact divergence vanishes → integral of t — integ deriv of T^{ij}

think $T^{ij}=T^{ik}\delta^j_k$
$=T^{ik}\partial_k x^j$
$=\partial_k(T^{ik}x^j)-(\partial_k T^{ik})x^j$

$$\int d^3x\, T^{ij}(t,\underline{x}) = \int d^3x\, \partial_k (T^{ik}x^j) - (\partial_k T^{ik})x^j$$

subtract what happens when deriv hits the x

divergence theorem. write as int over a surface

$$= \int_{|\underline{x}|=2d} \underbrace{T^{ik}x^j n^i dS}_{0 \text{ as } T_{ij}=0 \text{ for } |\underline{x}|>d} - \int d^3x\,(\partial_k T^{ik})x^j$$

n^k?

drop surface term

convert into surface int over suff large sphere both vanish by assumption $T_{ij}=0$, $|\underline{x}|>d$

relate spatial divergence of T to a time derivative

$(\partial_0 T^{0i}+\partial_j T^{ji}=0)$

(LHS symmetric in i,j so replace RHS w symmetric part.)

(recall T_{ij} is symmetric)

$$= \int d^3x\, \partial_0 T^{0i}x^j = \partial_0 \int d^3x\, T^{0i}x^j$$

(since $\partial_0 x^j = 0$?)

Symmetrising on i,j

$$\int d^3x\, T^{ij}(t,\underline{x}) = \partial_0 \int d^3x \left(\tfrac{1}{2}T^{0i}x^j + \tfrac{1}{2}T^{0j}x^i\right)$$

realise again this looks like divergence of something + a correction

$$= \partial_0 \int d^3x \left\{ \tfrac{1}{2}\partial_k (T^{0k}x^i x^j) - \tfrac{1}{2}\underbrace{\partial_k T^{0k}}_{=-\partial_0 T^{00}} x^i x^j \right\}$$

as above by divergence theorem

again notice spatial divergence of energy-momentum which I know as above I can replace w/ time divergence of T^{00} since

$(\partial_0 T^{00}+\partial_k T^{0k}=0)$

again pull out time deriv since this is integral over spatial derivs.

$$= \partial_0 \int d^3x\, \partial_0 T^{00}x^i x^j$$

$$= \tfrac{1}{2}\partial_0\partial_0 \int d^3x\, T^{00}x^i x^j = \tfrac{1}{2}\ddot{I}_{ij}(t)$$

2nd time deriv of I_{ij} at time t

Where $I^{ij}(t) = \int d^3x\, T^{00}(t,\underline{x})x^i x^j$

(in these coords)

2nd moment of 00 component of EM tensor (mass energy) evaluated at time t.

Noting $T_{00}=T^{00}$ and $T_{ij}=T^{ij}$ we deduce,

$$\bar{h}_{ij} = \tfrac{2}{r}\ddot{I}_{ij}(t-r). \qquad r\gg d,\ \tau\gg d$$

2nd derivative of I time light ray would have had to leave dist to arrive at me now. then dilute by $1/r$ to reflect fact we are on a sphere (?).

Now reconstruct remaining components using gauge condition

RHS: two time derivs that commute w anything

(these are euclidean indices so don't worry too much about upstairs vs. downstairs)

$$\partial_0 \bar{h}_{0i} = \partial_j \bar{h}_{ji} = \partial_j\left(\tfrac{2}{r}\ddot{I}_{ij}(t-r)\right)$$

evaluated at $t-r$

$$\Rightarrow \bar{h}_{0i} = \partial_j\left(\tfrac{2}{r}\dot{I}_{ij}(t-r)\right) + k_i(\underline{x})$$

const of integration depend only on $\underline{x}$

using chain rule and this fact

$$\bar{h}_{0j} = -\frac{2\hat{x}_j}{r^2}\dot{I}_{ij}(t-r) - 2\frac{\hat{x}_j}{r}\ddot{I}_{ij}(t-r) + k_i(\underline{x})$$

$\partial_i r = \frac{x_i}{r} = \hat{x}_i$

hopefully familiar from vector calculus

so far assumed $r\gg d$, $\tau\gg d$ but not imposed relation between r and τ until now.

We now assume $r\gg\tau$ so we are in radiation zone so can drop first term (it's $O(\tau/r)$ relative to second) and get

(now want to find $\bar{h}_{00}$)

$$\bar{h}_{0i} = -\frac{2\hat{x}_j}{r}\ddot{I}_{ij}(t-r) + k_i(\underline{x})$$

Now use,

$$\partial_0 \bar{h}_{00} = \partial_i \bar{h}_{0i} = \partial_i\left(-\frac{2\hat{x}_j}{r}\ddot{I}_{ij}(t-r) + k_i(\underline{x})\right)$$

integrate as before

$\hookrightarrow$ divergence of k

$$\Rightarrow \quad \bar{h}_{00} = -2\partial_i\left(\frac{\hat{x}_j}{r}\dot{I}_{ij}(t-r)\right) + t\,\partial_i k_i(\underline{x}) + f(\underline{x})$$

another integration const, similarly 2

$$= \frac{2\hat{x}_i\hat{x}_j}{r}\ddot{I}_{ij}(t-r) + t\,\partial_i k_i + f(\underline{x}) + \text{TERMS SUBLEADING IN } \tau/r$$

To fix constants of integration, return to

$$\bar{h}_{\mu\nu} = 4\int d^3x' \, \frac{T_{\mu\nu}(t-|\underline{x}-\underline{x}'|, \underline{x}')}{|\underline{x}-\underline{x}'|}$$

and observe that to leading order in d/τ

"trust me on this, or go do the computation yourselves"

$$\bar{h}_{00} \approx \frac{4E}{r} \qquad \bar{h}_{0i} \approx -\frac{4P_i}{r}$$

same expression as we derived before for $\bar{h}_{ij}$ which tells us we can replace w. 4 times integral at retarded time

where $E = \int d^3x' \, T_{00}(t', \underline{x}')$, $P_i = \int d^3x' \, T_{0i}(t', \underline{x}')$

E, P actually const. in time although in principle they depend on retarded time

Observing that $\partial_0 \int d^3x' \, T_{0\mu}(t', x') = \int d^3x' \, \partial_0 T_{0\mu}(t', x')$

$$= \int d^3x' \, \partial_i T_{i\mu}(t', x')$$

So $E, \underline{P}$ constant in time.

Exercise: By a gauge transformation generated by a multiple of

we have chosen Minkowski gauge s.t. centre of mass sits at origin

$$\xi^\mu = (\underline{P}\cdot\underline{x}, -\underline{P}t)$$

we can set $\underline{P} = 0$. This is the centre of momentum frame.

We've shown that:

what do these eq. give us?

$$\boxed{\begin{aligned} * \; \bar{h}_{00}(t,\underline{x}) &= \frac{4M}{r} + \frac{2\hat{x}_i\hat{x}_j}{r}\ddot{I}_{ij}(t-r) \\ * \; \bar{h}_{0i}(t,\underline{x}) &= -\frac{2\hat{x}_j}{r}\ddot{I}_{ij}(t-r) \\ * \; \bar{h}_{ij} &= \frac{2}{r}\ddot{I}_{ij}(t-r) \end{aligned}}$$

tells us perturbation depends only on 2nd moment eq.

formulae to explicitly compute h_{ij} [illegible] regime e.g. binary [illegible] system [illegible] around [illegible] rel & far enough away

energy is mass of [illegible]

(far enough away so in radiation zone) — r / so non rel. sources

where $r \gg \tau \gg d$, in centre of momentum frame where $E = M$

next time: assign energy to these perturbations so I can understand how waves that are sourced by moving bodies actually carry energy away from the system

(energy here leads you almost certainly "not to look over" is lots of computation)

Lecture 19 22.11.24

Last Lecture

* Linearised field far from a non-relativistic source:

$$\bar{h}_{00}(t,\underline{x}) = \frac{4M}{r} + \frac{2\hat{x}_i\hat{x}_j}{r}\ddot{I}_{ij}(t-r)$$

$$\bar{h}_{0i}(t,\underline{x}) = -\frac{2\hat{x}_j}{r}\ddot{I}_{ij}(t-r)$$

$$\bar{h}_{ij}(t,\underline{x}) = \tfrac{2}{r}\ddot{I}_{ij}(t-r)$$

where $I_{ij}(t') = \int d^3x'\, T_{00}(t',\underline{x}')x_i'x_j'$, $r = |\underline{x}|$, $M = \int d^3x'\, T_{00}(t',\underline{x}')$

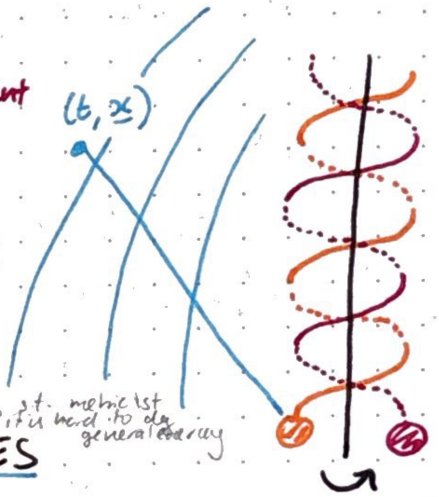
and we are in centre-of-momentum frame $P_i = \int d^3x'\, T_{0i}(t',\underline{x}') = 0$

valid in radiation zone:

$$r \gg T \gg d$$

distance from source (r), timescale of motion of source (T), spatial extent of source (d)

what source was doing when our past light cone intersects it?



today: these waves carry energy away from system which will affect it e.g. rotation → how do waves carry energy away → what do we mean by energy?

hard problem to def energy in GR!

energy hard to assign energy to grav. field because it is geometric quantity; normally define as deriv. but we can always pick coords s.t. metric 1st deriv vanishes so it is hard to def general energy

ENERGY IN GRAVITATIONAL WAVES

Defining the local energy / local energy flux for a gravitational field is hard in general because we can always choose coords s.t.

$$\partial_\mu g_{\sigma\tau}|_p = 0$$

also other reason why def energy is hard: often want to def energy via noether theorem (energy usually quantity conserved due to time symmetry) but in general background we don't have time translation symmetry

There is no hope of an energy density quadratic in first derivatives. – how we are used to define energy

∴ prob of def. energy is hard, but! we can get round this by working in our perturbation theory

In the context of perturbation theory there are various ways to define an energy. To do this we consider how to continue a perturbative solution beyond linear order.

We consider the ungauged vaccum Einstein equations and suppose

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h^{(1)}_{\mu\nu} + \varepsilon^2 h^{(1)}_{\mu\nu}$$

(1) indicates this is linear piece; (correction to $O(\varepsilon^2)$). Work to $O(\varepsilon^2)$

We observe (linearised Ricci tensor as function of the metric around minkowski space) (order 0 term = 0 as minkowski is Ricci flat)

$$R_{\mu\nu}[\eta_{\mu\nu} + \varepsilon h_{\mu\nu}] = \varepsilon R^{(1)}_{\mu\nu}[h] + \varepsilon^2 R^{(1)}_{\mu\nu}[h]$$

where (ungauged ricci tensor to 1st order)

$$R^{(1)}_{\mu\nu}[h] = \partial^\rho\partial_{(\mu}h_{\nu)\rho} - \tfrac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \tfrac{1}{2}\partial_\mu\partial_\nu h$$

are linear terms.

and

$$R^{(2)}_{\mu\nu}[h] = \tfrac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - h^{\rho\sigma}\partial_\rho\partial_{(\mu}h_{\nu)\sigma} + \tfrac{1}{4}\partial_\mu h_{\rho\sigma}\partial_\nu h^{\rho\sigma}$$
$$+ \partial^\sigma h^\rho{}_\nu\partial_{[\sigma}h_{\rho]\mu} + \tfrac{1}{2}\partial_\sigma(h^{\sigma\rho}\partial_\rho h_{\mu\nu}) - \tfrac{1}{4}\partial^\rho h\partial_\rho h_{\mu\nu} - (\partial_\sigma h^{\rho\sigma} - \tfrac{1}{2}\partial^\rho h)\partial_{(\mu}h_{\nu)\rho}$$

are quadratic terms.

This implies (ricci tensor to quadratic order) brackets: evaluated on $h^{(1)}$ — call these terms this

$$R_{\mu\nu}[\eta_{\mu\nu}+\varepsilon h^{(1)}_{\mu\nu}+\varepsilon^2 h^{(2)}_{\mu\nu}] = \varepsilon R^{(1)}_{\mu\nu}[h^{(1)}] + \varepsilon^2\left(R^{(1)}_{\mu\nu}[h^{(2)}] + R^{(2)}_{\mu\nu}[h^{(1)}]\right)$$

Thus (what we really want is Einstein tensor, got this 2) (einstein tensor at linear order)

$$G_{\mu\nu}[\eta+\varepsilon h^{(1)}+\varepsilon^2 h^{(2)}] = \varepsilon G^{(1)}_{\mu\nu}[h^{(1)}] \quad \Big\} \text{ linear order in } \varepsilon$$

(*) quadratic order in ε

$$\underbrace{+\ \varepsilon^2\Big(G^{(1)}_{\mu\nu}[h^{(2)}]}_{\text{from } \varepsilon^2(\dots) \text{ piece}} + \underbrace{R^{(2)}_{\mu\nu}[h^{(1)}] - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\sigma\tau}R^{(2)}_{\sigma\tau}[h^{(1)}]}_{-8\pi t_{\mu\nu}[h^{(1)}]}$$

$$\underbrace{+\ \tfrac{1}{2}\eta_{\mu\nu}h^{(1)\sigma\tau}R^{(1)}_{\sigma\tau}[h^{(1)}] - \tfrac{1}{2}h_{\mu\nu}\eta^{\sigma\tau}R^{(1)}_{\sigma\tau}[h^{(1)}]\Big)}_{\text{from incl. metric } (\varepsilon\times\varepsilon \text{ terms})}$$

remember when I take trace, I also get terms from order ε correction in metric

turns out these terms actually 0 when we impose vacuum eq.

[note: what is meant by $G^{(1)}$?
$G^{(1)}_{\mu\nu}[h] = R^{(1)}_{\mu\nu}[h] - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\sigma\tau}R^{(1)}_{\sigma\tau}[h]$]

We now consider the contracted Bianchi identity,

$$g^{\mu\rho}\nabla_\rho G_{\mu\nu} = 0$$

which holds for any metric. Using (*) and expanding gives (to be explicit about what we are carrying/losing)

$$(\dagger)\quad 0 = \varepsilon\,\eta^{\mu\sigma}\partial_\sigma G^{(1)}_{\mu\nu}[h^{(1)}] + \varepsilon^2\Big(\eta^{\mu\sigma}\partial_\sigma G^{(1)}_{\mu\nu}[h^{(2)}] - 8\pi\partial_\mu t^{\mu}_{\ \sigma\nu}[h^{(1)}]\eta^{\mu\sigma}$$
$$+ \underbrace{h^{(1)}\cdot R^{(1)}[h^{(1)}]}_{\text{SCHEMATIC}}\Big).$$

lots more terms but all prop. to linearised ricci tensor evaluated on linearised metric so of some form: — don't bother writing out all the terms, write this instead

Considering $G_{\mu\nu}[\eta+\varepsilon h^{(1)}+\varepsilon^2 h^{(2)}]=0$, using (*) order by order, we deduce

linearised EEs for $h^{(1)}$: RHS vanishes
$$G^{(1)}_{\mu\nu}[h^{(1)}] = 0 \qquad \left(\Rightarrow R^{(1)}_{\mu\nu}[h^{(1)}] = 0\right)$$

linearised EEs for $h^{(2)}$: (not really an energy momentum tensor) — as we called this earlier
$$G^{(1)}_{\mu\nu}[h^{(2)}] = -R^{(2)}_{\mu\nu}[h^{(1)}] + \tfrac{1}{2}\eta_{\mu\nu}\eta^{\sigma\tau}R^{(2)}_{\sigma\tau}[h^{(1)}] = 8\pi t_{\mu\nu}[h^{(1)}]$$

Thus $h^{(2)}$ solves the linearised Einstein equations sourced by an 'energy momentum tensor'

want it to be divergence free: use bianchi identities

From ($\dagger$) we deduce that

true for any metric: we have not imposed linearised EEs
$$\eta^{\mu\sigma}\partial_\mu G^{(1)}_{\sigma\nu}[h] = 0,$$ which holds for ANY pertubation h.

And $\eta^{\mu\sigma}\partial_\mu t_{\sigma\nu}[h^{(1)}] = 0$ when $h^{(1)}$ satisfies linearised E.E.s. (total integral over space) (div free but not gauge invariant!)

We can identify $t_{\mu\nu}$ with energy-momentum of the gravitational field, however, it is not gauge invariant. If $h^{(1)}$ decays sufficiently at ∞, then $\int t_{00}\,d^3x$ is invariant, (so gives total energy of field) but no gauge invariant local conservation.

We can get approximate gauge invariance by averaging.

Let W be smooth, vanish for $|\underline{x}|^2+t^2>a$, and satisfy

$$\int_{\mathbb{R}^4} W(\underline{x},t)\,d^3x\,dt = 1.$$

(integral over spacetime) use this to get a "local average" on regions of scale a

We define the average of a tensor in almost inertial coordinates by

W supported on ball of size a so non-zero outside ball of size a so deriv of order W/a

$$\langle X_{\mu\nu}(x)\rangle = \int_{\mathbb{R}^4} W(y-x)\,X_{\mu\nu}(y)\,d^4y.$$

spacetime x, y

idea is we can avg over suitable region to recover gauge inv

Suppose we're in far field regime, with radiation of wavelength λ and we average over a region of size $a \gg \lambda$. Since $\partial_\mu W \sim W/a$, we have

same as age integration by parts

$$\langle \partial_\mu X_{\rho\sigma}\rangle = \int_{\mathbb{R}^4} \partial_\mu W(y-x)\,X_{\rho\sigma}(y)\,d^4y$$

roughly of order

$$\sim \frac{\langle X_{\rho\sigma}\rangle}{a}$$

$$\sim \frac{\lambda}{a}\langle \partial_\mu X_{\rho\sigma}\rangle$$

We can ignore total derivatives inside averages, and thus

$$\langle A\partial B\rangle = \langle \partial(AB)\rangle - \langle(\partial A)B\rangle \simeq -\langle(\partial A)B\rangle$$

kind of 'integration by parts formula'

With this we can show:

$\boxed{\text{EXERCISE:}}$ (SHEET 3)

1) If h solves vaccum linearised E.E.

$$\langle \eta^{\mu\nu}R^{(2)}{}_{\mu\nu}[h]\rangle = 0.$$

(raise + lower with η)

this is kind of the formula we were looking for all along

2) $\langle t_{\mu\nu}\rangle = \frac{1}{32\pi}\langle \partial_\mu \bar{h}_{\rho\sigma}\partial_\nu \bar{h}^{\rho\sigma} - \frac{1}{2}\partial_\mu \bar{h}\,\partial_\nu \bar{h} - 2\partial_\sigma \bar{h}^{\rho\sigma}\partial_{(\mu}\bar{h}_{\nu)\rho}\rangle.$

3) $\langle t_{\mu\nu}\rangle$ is gauge invariant.

Using this formula and last lecture's results, we can find energy lost by a system producing gravitational waves.

The averaged spatial energy flux is $S_i = -\langle t_{0i}\rangle$.

We calculate average energy flux across a sphere of radius r centered on source

$$\langle P\rangle = -\int_{S_r} r^2\,d\Omega\,\langle t_{0i}\rangle\hat{x}_i$$

In wave gauge

$$\langle t_{0i}\rangle = \frac{1}{32\pi}\langle \partial_0 \bar{h}_{\rho\sigma}\partial_i \bar{h}^{\rho\sigma} - \frac{1}{2}\partial_0 \bar{h}\,\partial_i \bar{h}\rangle$$

expand out

$$= \frac{1}{32\pi}\langle \partial_0 \bar{h}_{jk}\partial_i \bar{h}_{jk} - 2\partial_0 \bar{h}_{0j}\partial_i \bar{h}_{0j} + \partial_0 \bar{h}_{00}\partial_i \bar{h}_{00} - \frac{1}{2}\partial_0 \bar{h}\,\partial_i \bar{h}\rangle$$

Using $\bar{h}_{ij} = \frac{2}{r}\ddot{I}_{ij}(t-r)$

0 IN RADIATION GAUGE

$$\partial_0 \bar{h}_{jk} = \frac{2}{r}\dddot{I}_{jk}(t-r) \qquad \partial_i \bar{h}_{jk} = \left(-\frac{2}{r}\dddot{I}_{jk}(t-r) - \cancel{\frac{2}{r^2}\ddot{I}_{jk}(t-r)}\right)\hat{x}_i$$

$$\therefore -\frac{1}{32\pi}\int r^2\,d\Omega\,\langle \partial_0 \bar{h}_{jk}\partial_i \bar{h}_{jk}\rangle\hat{x}_i = \frac{1}{2}\langle \dddot{I}_{ij}\dddot{I}_{ij}\rangle_{t-r}.$$

AVERAGE OVER WINDOW CENTERED AT $t-r$

of $\langle t_{0i}\rangle$

(ran out of time so will put any notes for the other terms on moodle but in principle compute all terms to work out power)

Lecture 20 25.11.24

Last lecture:

* EM tensor for linearised gravitational perturbations
  * $t_{\mu\nu} = -\frac{1}{8\pi}\left(R^{(2)}_{\mu\nu}[h^{(1)}] - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}[h^{(1)}]\right)$

  where $R_{\mu\nu}[\eta_{\mu\nu}+\varepsilon h_{\mu\nu}] = \varepsilon R^{(1)}_{\mu\nu}[h] + \varepsilon^2 R^{(2)}_{\mu\nu}[h] + O(\varepsilon^2)$

* After averaging over a window of size $a \gg \lambda$ (λ typical wavelength of grav. radiation) have

$$\langle t_{\mu\nu}\rangle = \frac{1}{32\pi}\langle \partial_\mu \bar{h}_{\rho\sigma}\partial_\nu \bar{h}^{\rho\sigma} - \frac{1}{2}\partial_\mu \bar{h}\,\partial_\nu \bar{h} \underbrace{- 2\partial_\sigma \bar{h}^{\rho\sigma}\partial_{(\mu}\bar{h}_{\nu)\rho}}_{=0 \text{ IN WAVE GAUGE}}\rangle$$

NEW CONTENT ↓

* Insert expressions from radiation zone solution derived in previous lectures to compute flux through large sphere

$$\langle P\rangle_t = -\int_{\{|x|=r\}} r^2 d\Omega \,\langle t_{0i}\rangle \hat{x}_i = \frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle_{t-r}$$

retarded time (rare triple deriv in physics long working on moodle)

where

$$Q_{ij} = I_{ij} - \frac{1}{3}I_{kk}\delta_{ij} \text{ is the } \underline{\text{quadratic tensor}}$$

DIFFERENTIAL FORMS

We want to derive Einstein's equations from an action principle. For this we need to discuss integration on manifolds, to do so we first discuss differential forms.

$\boxed{\text{DEF:}}$ A p-form is a totally antisymmetric $(0,p)$-tensor field on M. The space of p-forms is $\Omega^p M$.

Note: If $p > M$ a p-form must vanish. A 1-form is a covector field.

We have a natural product on forms. If X is a p-form and Y a q-form then $(X\wedge Y)$ is the $(p+q)$-form

$$(X\wedge Y)_{a_1\ldots a_p b_1 \ldots b_q} = \frac{(p+q)!}{p!\,q!} X_{[a_1\ldots a_p}Y_{b_1\ldots b_q]}$$

It has the properties (CHECK!):

* $X\wedge Y = (-1)^{pq} Y\wedge X$ ($\Rightarrow X\wedge X = 0$ if p odd)
* $(X\wedge Y)\wedge Z = X\wedge(Y\wedge Z)$

If $\{\delta^{\mu}\}_{\mu_1=1}^{n}$ is a dual basis (i.e. basis of covectors), then
$\{\delta^{\mu_1}\wedge\ldots\wedge\delta^{\mu_p}\}_{\mu_1<\mu_2<\ldots<\mu_p}$
and we can write

$$X = \frac{1}{p!} X_{\mu_1\ldots\mu_p}\delta^{\mu_1}\wedge\ldots\wedge\delta^{\mu_p}.$$

Another important feature of forms is that we can define a derivative $d: \Omega^p M \to \Omega^{p+1} M$ by

EXTERIOR DERIVATIVE

DEF: If X is a p-form, then in a coordinate basis

$$(dX)_{\mu_1\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1} X_{\mu_2\ldots\mu_{p+1}]} \quad (*)$$

SYMMETRIC IN μ_{p+1}, μ_1

Suppose ∇ is any symmetric connection, then

$$\nabla_{\mu_1} X_{\mu_2\ldots\mu_{p+1}} = \partial_{\mu_1} X_{\mu_2\ldots\mu_{p+1}} - \Gamma^{\sigma}_{\mu_2\mu_1} X_{\sigma\mu_3\ldots\mu_{p+1}} - \ldots - \Gamma^{\sigma}_{\mu_{p+1}\mu_1} X_{\mu_2\ldots\mu_p\sigma}$$

$$\Rightarrow \nabla_{[\mu_1} X_{\mu_2\ldots\mu_{p+1}]} = \partial_{[\mu_1} X_{\mu_2\ldots\mu_{p+1}]}$$

$$\therefore (dX)_{\mu_1\ldots\mu_{p+1}} = (p+1)\nabla_{[\mu_1} X_{\mu_2\ldots\mu_{p+1}]}$$

tensor eq $\Rightarrow$ propagate to AIN

Thus $(dX)_{a_1\ldots a_{p+1}} = (p+1)\nabla_{[a_1} X_{a_2\ldots a_{p+1}]}$ is well-defined independently of coordinates. However, $(*)$ shows it does not depend on a metric or connection.

EXERCISE (sheet 4) show:

(X p-form, Y q-form)

- $d(dX) = 0$
- $d(X\wedge Y) = dX\wedge Y + (-1)^p X\wedge dY$
- $\phi^* dX = d(\phi^* X)$ if $\phi: N\to M$

The last property implies that d commutes with lie derivatives i.e. $\mathcal{L}_v(dX) = d(\mathcal{L}_v X)$.

d is called the EXTERIOR DERIVATIVE.

We say X is CLOSED if $dX=0$, and X is EXACT if $X=dY$ for some Y. EXACT $\Rightarrow$ CLOSED, but the converse is only true locally.

POINCARÉ LEMMA: If X is a closed p-form $(p\geq 1)$, then for any $r\in M$ there is an open neighbourhood $N\subset M$ with $r\in N$ and a $(p-1)$-form Y defined on N such that $X = dY$

The extent to which CLOSED $\neq$ EXACT captures topological properties of M.

E.g. On S^1 the form $d\theta$ (see ex. 1.3) is closed, but <u>not</u> exact (despite confusing notation).

THE TETRAD FORMALISM

called something else in more dim. this is what we call it in 4 dim (we only care about 4 dim)

In GR it's often useful to work with an orthonormal basis of vector fields (<u>TETRAD</u>) $\{e^a_\mu\}_{\mu=0}^3$, satisfying

$$g_{ab}e^a_\mu e^b_\nu = \eta_{\mu\nu} \qquad (\delta_{\mu\nu} \text{ if Riemannian})$$

Recall that the dual basis $\{f^\mu_a\}_{\mu=0}^3$ is defined by

$$\delta^\mu_\nu = f^\mu(e_\nu) = f^\mu_a e^a_\nu .$$

We <u>claim</u> that

$$f^\mu_a = \eta^{\mu\sigma} g_{ab} e^b_\sigma$$

PROOF: $(\eta^{\mu\sigma} g_{ab} e^b_\sigma) e^a_\nu = \eta^{\mu\sigma}\eta_{\sigma\nu} = \delta^\mu_\nu .$

introduce convention to sep. orthonormal indices from abstract indices but orthonormal basis not unique: can be transf.

Recalling that g_{ab} raises + lowers roman indices, and introducing the CONVENTION that $\eta_{\mu\nu}$ raises + lowers greek indices, we have

$$f^\mu_a = e^\mu_a = \eta^{\mu\sigma} g_{ab} e^b_\sigma ,$$

we will thus denote basis vectors e_μ and dual basis vectors e^μ.

Recall that two orthonormal bases are related by

$$e'^a_\mu = (A^{-1})^\nu_\mu e^a_\nu \qquad \text{where } \eta_{\mu\nu} A^\mu_\rho A^\nu_\sigma = \eta_{\rho\sigma} .$$

Unlike in special relativity, A^μ_ν need not be constant.

'GR arises by gauging the Lorentz symmetry of SR'

take sum over covectors because the 2-tensor which is metric: cf earlier statement [illegible] but say w. diff ways

CLAIM: $\eta_{\mu\nu} e_a^\mu e_b^\nu = g_{ab}$, $e_a^\mu e^b_\mu = \delta_a^b$

PROOF: Contract with e^b_ρ:

$$\eta_{\mu\nu} e_a^\mu e_b^\nu e^b_\rho = \eta_{\mu\nu} e_a^\mu \delta^\nu_\rho = \eta_{\mu\rho} e_a^\mu = (e_a)_\rho = g_{ab} e^b_\rho$$

Since equation holds ~~for~~ contracted with any basis vector, it holds in general.

Second equation follows from first by raising b.

3 per lec + spare

Lecture 21

27.11.24

Last Lecture:

* Differential forms, $\Omega^r M$
  → $X \wedge Y$ 'wedge product'
  → dX 'exterior derivative'
* Tetrad formalism $\{e_\mu\}$ on basis, $\{e^\mu\}$ dual basis $e_a{}^\mu = g_{ab}\eta^{\mu\sigma}e^b{}_\sigma$
  → raise + lower { LATIN indices w/ g_{ab}; GREEK indices w/ $\eta_{\mu\nu}$
  → $g_{ab} = e_a{}^\mu e_b{}^\nu \eta_{\mu\nu}$, $\eta_{\mu\nu} = e^a{}_\mu e^b{}_\nu g_{ab}$, $e^a{}_\mu e_b{}^\mu = \delta^a_b$, $e^a{}_\mu a_a{}^\nu = \delta^\nu_\mu$.

CONNECTION 1-FORMS

Let ∇ be the Levi-Civita connection. The connection 1-forms are defined to be

1-forms have a downstairs abstract index

$$(\omega^\mu{}_\nu)_a := e_b{}^\mu \nabla_a e^b{}_\nu$$

seen before: taking ∇ of basis vecs is related to connection components → this is just diff way of encoding the Γs

(just defining Γ)

Recalling $e^a{}_\nu \nabla_a e^b{}_\mu = \Gamma_\mu{}^\sigma{}_\nu e^b{}_\sigma$

multiply by $\times e_c{}^\nu$

$$\Rightarrow \nabla_c e^b{}_\mu = \Gamma_\mu{}^\sigma{}_\nu e^b{}_\sigma e_c{}^\nu$$

go through algebra yourself using above relations (last lec)

$$\therefore (\omega^\mu{}_\nu)_a = \Gamma_\rho{}^\mu{}_\sigma e^\rho{}_a .$$

so $(\omega^\mu{}_\nu)_a$ encodes the connection components.

so what? ↳ we'll see later but first some lemmas,

LEMMA: $(\omega_{\mu\nu})_a = -(\omega_{\nu\mu})_a$

this just set of constants so if differentiate get 0

PROOF: $(\omega_{\mu\nu})_a = (e_b)_\mu \nabla_a e^b{}_\nu \overset{\text{chain rule}}{=} \nabla_a (\underbrace{(e_b)_\mu e^b{}_\nu}_{\eta_{\mu\nu}}) - e^b{}_\nu \nabla_a (e_b)_\mu$

same as line above w/ indices swapped + minus sign

$$= -(e_b)_\nu \nabla_a e^b{}_\mu = -(\omega_{\nu\mu})_a$$

Now consider the exterior derivative of a basis 1-form.

LEMMA: The 1-form e^μ satisfies Cartan's first structure eqn.

(remember: μ,ν not tensor indices, de^μ means think of it as one-form and find exterior deriv. for each index)

$$de^\mu + \omega^\mu{}_\nu \wedge e^\nu = 0.$$

check: consistent? de^μ (deriv of 1 form) is a 2 form and 2nd term also 2-form?

PROOF: Note $(\omega^\nu{}_\mu)_a e^b{}_\nu = (e_c{}^\nu \nabla_a e^c{}_\mu) e^b{}_\nu = \nabla_a e^b{}_\mu$

Thus

$$\nabla_a e_b{}^\mu = (\omega_\nu{}^\mu)_a e_b{}^\nu = -(\omega^\mu{}_\nu)_a e_b{}^\nu$$

by antisymmetry property

$$\Rightarrow (de^\mu)_{ab} = 2\nabla_{[a} e^\mu{}_{b]} = -2(\omega^\mu{}_\nu)_{[a} e_{b]}{}^\nu = -(\omega^\mu{}_\nu \wedge e^\nu)_{ab}.$$

from defining of wedge product (last lec)

what does this mean?
↳ I can find components of ω by computing exterior derivative of e.
↳ efficient way of finding connection components!

Note that in the orthonormal basis,

so if i compute this I know antisymmetrised components of ω

$$(de^\mu)_{\nu\sigma} = 2(\omega^\mu{}_{[\nu})_{\sigma]},$$

so computing de^μ leads to $(\omega^\mu{}_{[\nu})_{\sigma]}$ since $(\omega_{\mu\nu})_\rho = -(\omega_{\nu\mu})_\rho$.

("we're never gonna use this formula, just write to show our Cartan's eq. after we know de^μ they uniquely determines the ωs → show example of how to use this later)

We can check $(\omega_{\mu\nu})_\rho = (\omega_{\mu[\nu})_{\rho]} + (\omega_{\nu[\rho})_{\mu]} - (\omega_{\rho[\mu})_{\nu]}$.

So we can determine $\omega^\mu{}_\nu$ and hence Γ by computing de^μ.

Example: The Schwarzchild Metric

(reminder: metric $-e^{0\,2}+e^{1\,2}+e^{2\,2}+e^{3\,2}$)

has an obvious tetrad:

$$e^0 = \gamma dt \qquad e^1 = \tfrac{1}{\gamma} dr \qquad e^2 = r d\theta \qquad e^3 = r\sin\theta\, d\phi$$

(computing exterior derivs is efficient: can see where they vanish easily)

where $\gamma = \sqrt{1 - 2\frac{M}{r}}$. fixes the metric

read wedge

Then
$$de^0 = d\gamma \wedge dt + \gamma\, d(dt) = \gamma' dr \wedge dt = \gamma' e^1 \wedge e^0$$
$$de^1 = d(\tfrac{1}{\gamma}) \wedge dr + \tfrac{1}{\gamma} d(dr) = -\tfrac{\gamma'}{\gamma^2} dr \wedge dr = 0$$
$$de^2 = dr \wedge d\theta$$
$$de^3 = \sin\theta\, dr \wedge d\phi + r\cos\theta\, d\theta \wedge d\phi = \tfrac{\gamma}{r} e^1 \wedge e^3 + \tfrac{\cot\theta}{r} e^2 \wedge e^3$$

(compare w expressions above)

$$de^0 = -\omega^0{}_\mu \wedge e^\mu \Rightarrow \omega^0{}_1 = \gamma' e^0,\ \omega^0{}_2 \propto e^2,\ \omega^0{}_3 \propto e^3$$
$$de^1 = -\omega^1{}_\mu \wedge e^\mu \Rightarrow \omega^1{}_0 \propto e^0,\ \omega^1{}_2 \propto e^2,\ \omega^1{}_3 \propto e^3$$
$$de^2 = -\omega^2{}_\mu \wedge e^\mu \Rightarrow \omega^2{}_1 = \tfrac{\gamma}{r} e^2,\ \omega^2{}_0 \propto e^0,\ \omega^2{}_3 \propto e^3$$
$$de^3 = -\omega^3{}_\mu \wedge e^\mu \Rightarrow \omega^3{}_0 \propto e^0,\ \omega^3{}_1 = \tfrac{\gamma}{r} e^3,\ \omega^3{}_2 = \tfrac{\cot\theta}{r} e^3$$

(can check all these statements are consistent w/ eachother)

We have:

$$\omega_{01} = -\omega_{10} = -\gamma' e^0$$
$$\omega_{21} = -\omega_{12} = \tfrac{\gamma}{r} e^2$$
$$\omega_{31} = -\omega_{13} = \tfrac{\gamma}{r} e^3$$
$$\omega_{32} = -\omega_{23} = \tfrac{\cot\theta}{r} e^3$$

all other components vanish

(you'll have to believe me this is an efficient way to compute 'connection components' → more valuably we can compute curvature components from these curvature 2-forms)

CURVATURE 2-FORMS

We compute $d\omega^\mu{}_\nu$

$$(d\omega^\mu{}_\nu)_{ab} = \nabla_a(\omega^\mu{}_\nu)_b - \nabla_b(\omega^\mu{}_\nu)_a$$
$$= \nabla_a(e_c{}^\mu \nabla_b e^c{}_\nu) - \nabla_b(e_c{}^\mu \nabla_a e^c{}_\nu)$$

(expand out using Leibniz)

$$= e_c{}^\mu(\nabla_a\nabla_b e^c{}_\nu - \nabla_b\nabla_a e^c{}_\nu) + \nabla_a e_c{}^\mu \nabla_b e^c{}_\nu - \nabla_b e_c{}^\mu \nabla_a e^c{}_\nu$$
$$= e_c{}^\mu (R^c{}_{dab}) e^d{}_\nu + e^d{}_\sigma(\nabla_a e_d{}^\mu) e_\gamma^\sigma(\nabla_b e^\gamma{}_\nu) - e^d{}_\sigma(\nabla_b e_d{}^\mu) \cdot e_\gamma^\sigma(\nabla_a e^\gamma{}_\nu)$$

$$= (\Theta^{\mu}{}_{\nu})_{ab} + (\omega_{\sigma}{}^{\mu} \wedge \omega^{\sigma}{}_{\nu})_{ab}$$

where $\Theta^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\sigma\tau} e^{\sigma} \wedge e^{\tau}$ are the curvature 2-forms. We've shown Cartan's second structure equation,

$$d\omega^{\mu}{}_{\nu} + \omega^{\mu}{}_{\sigma} \wedge \omega^{\sigma}{}_{\nu} = \Theta^{\mu}{}_{\nu},$$

gives an efficient way to compute $R^{\mu}{}_{\nu\sigma\tau}$ in an o/n basis.

Returning to our example:

$$d\omega^{0}{}_{1} = d(f' e^{0}) = d(f' f \, dt) = (f'' f + f'^{2}) \underbrace{dr \wedge dt}_{e^1 \wedge e^0} = (f'' f + f'^{2}) e^{1} \wedge e^{0}$$

only non zero when $\mu = 1$ ($\mu = 2$)

$$\omega^{0}{}_{\mu} \wedge \omega^{\mu}{}_{1} = \omega^{0}{}_{1} \wedge \omega^{1}{}_{1} + \omega^{0}{}_{2} \wedge \omega^{2}{}_{1} + \omega^{0}{}_{3} \wedge \omega^{3}{}_{1} + \omega^{0}{}_{0} \wedge \omega^{0}{}_{1} = 0$$ (all vanish)

$$\therefore \Theta^{0}{}_{1} = -(f'' f + f'^{2}) e^{0} \wedge e^{1}$$

$$\text{hence } R^{0}{}_{101} = -(f'' f + f'^{2}) = -\tfrac{1}{2}(f^{2})'' = \frac{2M}{r^{3}}$$

we found wrt o/n basis: at least 1 component of R tensor but actually we found all components since 0 otherwise → that is why this is efficient, we don't have to calc lots of things and find what is zero; zero components just follow from the argument

$$\text{and } R^{\mu}{}_{\nu 01} = 0 \text{ otherwise.}$$

Exercise: Find other $\Theta^{\mu}{}_{\nu}$ and show R_{ab}.

tetrad stuff was a bit of a side track → remember original goal for introducing differential forms was to do integration on manifold so we can derive Einstein's equations from an action principle: now lets return to this original goal (to understand integration)

← first step is to look at volume form

VOLUME FORM AND HODGE DUAL

We say a manifold is ORIENTABLE if it admits a nowhere vanishing n-form ($n = \dim M$) $\varepsilon_{a_1 \dots a_n}$, an orientation form.

Two such forms are equivalent if $\varepsilon' = f\varepsilon$ for some smooth, everywhere positive f. $[\varepsilon]_N$ is an orientation.

A basis of vectors $\{e_\mu\}_{\mu=1}^{n}$ is right-handed if

$$\varepsilon(e_1, \dots, e_n) > 0.$$

(canonical choice of normalisation is to set this $= 1$)

(corresponding basis / coord vectors)

A coordinate system is right-handed if $\left\{\frac{\partial}{\partial x^\mu}\right\}_{\mu=1}^{n}$ are right-handed.

why care about handedness?
→ if we use e.g. divergence / stokes theorem we need to know direction / normal to surface etc.
intrinsically handedness/orientation is bound up in integrating!

Lecture 22

29.11.24

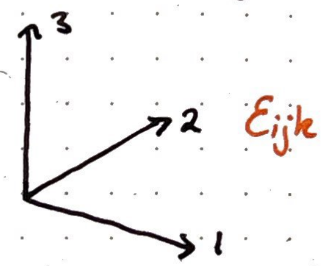
Last lecture

* Tetrad formalism
  * Cartan's 1st + 2nd structure equations
    $$de^{\mu} + \omega^{\mu}{}_{\nu} \wedge e^{\nu} = 0 \qquad \omega_{\mu\nu} = -\omega_{\nu\mu} \qquad \text{connection 1-forms}$$
    $$d\omega^{\mu}{}_{\nu} + \omega^{\mu}{}_{\sigma} \wedge \omega^{\sigma}{}_{\nu} = \Theta^{\mu}{}_{\nu} \qquad \Theta^{\mu}{}_{\nu} = \tfrac{1}{2} R^{\mu}{}_{\nu\sigma\tau} e^{\sigma} \wedge e^{\tau} \qquad \text{curvature 2-forms}$$
* Manifold is orientable if it admits a nowhere vanishing n-form, ε.
  * Two such forms equivalent if $\varepsilon' = f\varepsilon$ ($f > 0$ everywhere)
  * $[\varepsilon]_{\sim}$ is an orientation
  * $\{e_{\mu}\}_{\mu=1}^{n}$ is right-handed if $\varepsilon(e_1, \dots, e_n) > 0$. coord. system is RH if $\left\{\frac{\partial}{\partial x^{\mu}}\right\}_{\mu=1}^{n}$ is RH.

means orientable & we chosen orientation

no preferred normalisation e.g. $\frac{1}{2}\varepsilon_{ijk}$ is as good as $5\varepsilon_{ijk}$ until we introduce a metric

in $\mathbb{R}^3$ this is standard orientation ε_{ijk} (right-handed basis)



An oriented manifold with metric has a preferred normalisation for ε. For a right-handed orthonormal basis, we define the volume form ε by

$$\varepsilon(e_1, \dots, e_n) = 1 \qquad \text{(indep. of choice of RH o/n basis)}$$

If we work in a RH coord. system $\{x^{\alpha}\}_{\alpha=1}^{n}$, then

$$\frac{\partial}{\partial x^{\alpha}} = e_{\alpha}{}^{\mu} e^{\beta}{}_{\mu} \frac{\partial}{\partial x^{\beta}} = e_{\alpha}{}^{\mu} e_{\mu}$$

Then

plug into volume form w/ this choice of normalisation

$$\varepsilon\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) = \varepsilon\left(e_1{}^{\mu_1} e_{\mu_1}, \dots, e_n{}^{\mu_n} e_{\mu_n}\right)$$

can pull out $e_n{}^{\mu_n}$ as just numbers? then sign (± 1) depends on permutation

sum over permutations on n indices

$$= \sum_{\pi \in \mathrm{Sym}(n)} \sigma(\pi) e_1{}^{\pi(1)} \times \cdots \times e_n{}^{\pi(n)} = \det(e_{\alpha}{}^{\mu})$$

sign of permutation

but $e_{\alpha}{}^{\mu} e_{\beta}{}^{\nu} \eta_{\mu\nu} = g_{\alpha\beta}$ $\Rightarrow$ $\det(e_{\alpha}{}^{\mu}) = \sqrt{|g|}$ where $g = \det(g_{\alpha\beta})$

thinking these as matrices

deduce after reminding ourselves about combinatorial factors:

$$\therefore \ \varepsilon = \sqrt{|g|}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

Equivalently, $\varepsilon_{123\dots n} = \sqrt{|g|}$ in coord. basis.

EXERCISE: In the same coord. basis

$$\varepsilon^{123\dots n} = \pm \frac{1}{\sqrt{|g|}} \qquad \begin{cases} + & \text{RIEMANNIAN} \\ - & \text{LORENTZIAN} \end{cases}$$

LEMMA: $\nabla \varepsilon = 0$

PROOF: In normal coords. at p; $\partial_{\mu} g_{\nu\sigma}|_p = 0$, $\Gamma^{\sigma}_{\mu\nu}|_p = 0$

$\Rightarrow \nabla_{\mu_1} \varepsilon_{\mu_2 \cdots \mu_{n+1}} = \partial_{\mu_1} \varepsilon_{\mu_2 \cdots \mu_{n+1}} + (\Gamma \cdot \varepsilon) = 0$ at p

tensor equation so holds everywhere $\square$

LEMMA: $\varepsilon^{a_1 \cdots a_p c_{p+1} \cdots c_n} \varepsilon_{b_1 \cdots b_p c_{p+1} \cdots c_n} = \pm p!(n-p)!\, \delta^{a_1}_{[b_1} \delta^{a_2}_{b_2} \cdots \delta^{a_p}_{b_p]}$ $\begin{cases} + & \text{RIEMANNIAN} \\ - & \text{LORENTZIAN} \end{cases}$

PROOF: (exercise)

p-forms on manifold — (n-p)-forms on manifold

We can use ε to relate $\Omega^p M$ to $\Omega^{n-p} M$:

DEF: On an oriented manifold with metric, the HODGE DUAL of a p-form X is

$$(*X)_{a_1 \cdots a_{n-p}} = \frac{1}{p!} \varepsilon_{a_1 \cdots a_{n-p} b_1 \cdots b_p} X^{b_1 \cdots b_p}$$

From previous results we can show

LEMMA: $*(*X) = \pm(-1)^{p(n-p)} X$ $\begin{cases} + & \text{RIEMANNIAN} \\ - & \text{LORENTZIAN} \end{cases}$

obvious thing we would guess to take divergence of X is actually correct (check this)

$(*d*X)_{a_1 \cdots a_{p-1}} = \pm(-1)^{p(n-p)} \nabla^b X_{a_1 \cdots a_{p-1} b}$. $\begin{cases} + & \text{RIEMANNIAN} \\ - & \text{LORENTZIAN} \end{cases}$

This seems esoteric but we are actually quite familiar w this do it without even realising in vector calc when we take divs, grads & curls

Examples

(3D in euclidean often dont distinguish between vector fields & 1-forms because we can & lower w/ δ, we often dont bother to keep track)

① In Euclidean space, identifying a vector field X^a with the one form X_a, the usual operations of vector calculus become

$$\nabla f = df, \quad \operatorname{div} X = *d*X, \quad \operatorname{curl} X = *dX$$

why need hodge dual? so hodge dual operation needed to write down some of these things (for curl need ε) → turns out in higher dim. this is how we generalise vector calc.

remembering that acting twice with d gives 0, * gives ± what I started with

note $d^2 = 0 \Rightarrow \operatorname{curl} \nabla f = 0$ and $\operatorname{div} \operatorname{curl} X = 0$.

no longer makes sense to think of curl as a vector in >3D.

② Maxwell's Equations

w/ source $\nabla^a F_{ab} = -4\pi j_b$ and $\nabla_{[a} F_{bc]} = 0$ source-free

can be written

$$d*F = -4\pi *j, \quad dF = 0.$$

Poincaré's lemma implies we can write $F = dA$ for some one-form A, locally. cant necessarily do this globally

recover usual statement that we can write a maxwell field in terms of potential (locally)

we have introduced this formulation w/ ds & *s, turns out this makes integration on manifolds & stoke's thrm quite elegant.

INTEGRATION ON MANIFOLDS

challenge: can always pick a coord patch to integrate over w/ manifolds but changing coords in general changes jacobian factor so I won't get same ans. → want to figure out what i can integrate to get same ans under coord change...

Suppose on a manifold M we have a RH coordinate chart $\phi: \mathcal{O} \to \mathcal{U}$ with coordinates x^μ. If X is an n-form which vanishes outside $\mathcal{O}$, we can write

$$X = X_{1\dots n}\, dx^1 \wedge \cdots \wedge dx^n.$$

If $\psi : O \to \mathcal{U}$ is another RH coordinate chart with coords $\{y^\mu\}$ then

$$X = \hat{X}_{1\dots n}\, dy^1 \wedge \cdots \wedge dy^n = \hat{X}_{1\dots n} \frac{\partial y^1}{\partial x^{\mu_1}} \cdots \frac{\partial y^n}{\partial x^{\mu_n}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$$

$$= \hat{X}_{1\dots n} \det\left(\frac{\partial y^\mu}{\partial x^\nu}\right) dx^1 \wedge \cdots \wedge dx^n$$

$$\therefore\ X_{1\dots n} = \hat{X}_{1\dots n} \det\left(\frac{\partial y^\mu}{\partial x^\nu}\right)$$

As a result

$$\int_{\mathcal{U}} X_{1\dots n}\, dx^1 \dots dx^n = \int_{M} \hat{X}_{1\dots n}\, dy^1 \dots dy^n.$$

we have jacobian factor that exactly fixes things: we have found something we can integrate that doesn't care about coord patch we integrate over.

We can define

$$\int_M X = \int_O X := \int_{\mathcal{U}} X_{1\dots n}\, dx^1 \dots dx^n$$

(if you don't know what that is ignore it) (any manifold relevant is in GR)

On any (2nd countable) manifold, we can find a countable atlas of charts (ϕ_i, O_i), and smooth functions $\chi_i : M \to [0,1]$ such that χ_i vanishes outside O_i and

$$\sum_{i=1}^{\infty} \chi_i(p) = 1$$

$\forall p \in M$, and the sum is locally finite

only at most finitely many of these terms are non-zero

(don't really need to worry about this in practice, just writing so you have the definitions)

Then for any n-form X we define

$$\int_M X = \sum_{i=1}^{\infty} \int_M \chi_i X = \sum_{i=1}^{\infty} \int_{O_i} \chi_i X$$

don't worry about this too much, in practice in GR we can cover w/ 1 chart or at least mostly w/ 1 chart (sufficiently small uncovered bit of M can ignore)

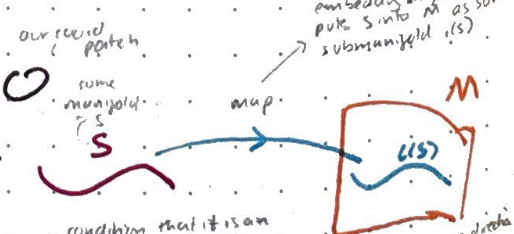
This doesn't depend on a choice of χ_i's.

<u>Remarks</u>

- Computation showing coord. invariance implies that for a diffeomorphism $\phi : M \to M$
$$\int_M X = \int_M \phi^* X$$
- If M is a manifold with metric and vol. form ε, then if $f : M \to \mathbb{R}$ is a scalar, $f\varepsilon$ is an n-form, and we can define $\int_M f = \int_M f\varepsilon$.

In local coordinates, if f vanishes outside O

$$\int_M f = \int_{\mathcal{U}} f(x) \sqrt{|g|}\, dx^1 \dots dx^n = \int_M f\, d\mathrm{vol}_g.$$



SUBMANIFOLDS AND STOKES THEOREM

$\boxed{\text{DEF:}}$ Suppose S, M are manifolds and $\dim S = m < n = \dim M$. A smooth map $\iota : S \to M$ is an embedding if it is an injection (i.e. $\iota_* : T_pS \to T_{\iota(p)}M$ is injective) and if ι is injective i.e. $\iota(p) = \iota(q) \Rightarrow p = q$.

If ι is an embedding, then $\iota(S)$ is an embedded submanifold. (If $m = n-1$ we call it a hypersurface) (mostly drop ι when obvious from context and write $\iota(S) = S$).

If S, M are orientable, and $\iota(S)$ is an embedded submanifold of M. We define the integral of an n-form X over $\iota(S)$ by

$$\int_{\iota(S)} X = \int_S \iota^*(X).$$

Note that if $X = dY$ then

$$\int_{\iota(S)} dY = \int_S d(\iota^* Y).$$

$(d\phi^* y = \phi^* dy)$

Lecture 23

2.12.24

Last lecture

* If M is an oriented n-dim manifold, X is an n-form defined $\int_M X$.
* If M carries a metric (or otherwise has a preferred choice of volume form ε) defined

$$\int_M f := \int_M f\varepsilon = \int_M f \, dvol_g .$$

if ε is volume form of g

* If S is an m-dimensional manifold $(m<n)$, then $\iota: S \to M$ is an embedding if it is an IMMERSION and is INJECTIVE.

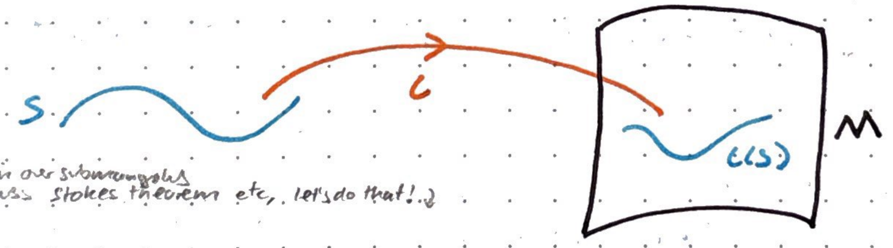
$\iota(p) = \iota(q) \Rightarrow p = q$

$\iota_*: T_pS \to T_{\iota(p)}M$ injective

If so, $\iota(S)$ is an m-dim SUBMANIFOLD of M.

* If $\iota(S)$ is an m-dim submanifold and Y is an m-form on M, then

$$\int_{\iota(S)} Y := \int_S \iota^*(Y)$$



we talked about integration over submanifolds because we want to discuss Stokes theorem etc, let's do that!

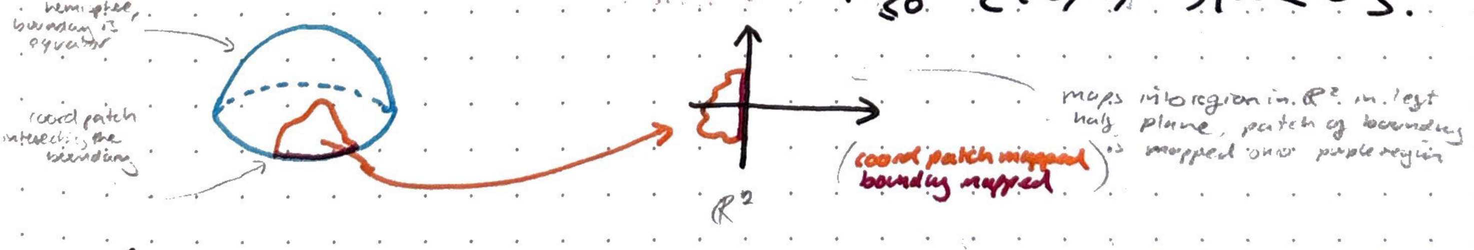
DEF: A manifold with boundary, M, is defined just as for a manifold, (as previously) but charts are maps

$$\phi_\alpha : \mathcal{O}_\alpha \to \mathcal{U}_\alpha ,$$ where $\mathcal{U}_\alpha$ is an open subset of

$\hat{M}$

not $\mathbb{R}^n$, instead halfspace

$$\mathbb{R}^n_{\leq 0} = \{(x^1, \dots, x^n) \mid x^1 \leq 0\}.$$



The boundary of M, ∂M is the set of points mapped in any chart to $\{x^1 = 0\}$. It is naturally an $n-1$ dim. manifold with an embedding $\iota: \partial M \to M$.

If M is oriented ∂M inherits an orientation by requiring $(x^2, \dots, x^n)$ is a RH chart on ∂M when $(x^1, \dots, x^n)$ is RH on M.

"in practice we are not going to use this definition but it's good to have it" now we have defn. consider stokes

STOKES THEOREM

If N is an oriented n-diml manifold with boundary, and X is an $(n-1)$-form, then

$$\int_N dX = \int_{\partial N} X.$$

elegant form doesn't require structure on manifold. – except boundary stuff?
(basis of all integrating by parts arguments on manifolds)

Stokes theorem is the basis of all 'integration by parts' arguments. If N carries a metric, we can reformulate stokes theorem as the divergence theorem. slightly more convenient form?

If V is a vector field on N, then we define

V "hook" epsilon

$$(V \lrcorner \varepsilon)_{a_2 \dots a_n} = V^a \varepsilon_{a a_2 \dots a_n}$$

natural n-1 form

We can check that $d(V \lrcorner \varepsilon) = (\nabla_a V^a)\varepsilon$. check yourself

If we define the flux of V through an embedded hypersurface S by

think of this as contraction between vector measure of surface

$$\int_S V \cdot dS := \int_S V \lrcorner \varepsilon$$

a little bit of a cheat, really defn RHS to be what we want it to look like for div theorem so check this

then Stokes theorem implies $\int_N \nabla_a V^a \, dvol_g = \int_{\partial N} V \cdot dS$.

Recall that a hypersurface $\iota(S)$ is

$$\begin{cases} \text{SPACELIKE} & \text{if } h = \iota^* g \text{ is RIEMANNIAN} \\ \text{TIMELIKE} & \text{if } h = \iota^* g \text{ is LORENTZIAN} \end{cases}.$$

In this case we can relate $\iota^*(V \lrcorner \varepsilon_g)$ to the volume form on (S, h).
Pick $b_2, \dots, b_n$ a RH o/n basis on S (w.r.t. h). Then $\iota_* b_2, \dots, \iota_* b_n$ are o/n in N. The unit normal to S is the unique unit vector $\vec{n}$ orthogonal to $\iota_* b_2, \dots, \iota_* b_n$ with

$$\varepsilon(\vec{n}, \iota_* b_2, \dots, \iota_* b_n) = g(\vec{n}, \vec{n}) \ (= \pm 1).$$

this is unit vector, because we are in Lor. space, this equals ±1

iff

$$\text{If } \iota(S) \begin{cases} \text{SPACELIKE} \longleftrightarrow \vec{n} \text{ TIMELIKE} \\ \text{TIMELIKE} \longleftrightarrow \vec{n} \text{ SPACELIKE} \end{cases}$$

with this definition,

$$V \lrcorner \varepsilon(\iota_* b_2, \dots, \iota_* b_n) = V^a \vec{n}_a$$

thus $\iota^*(V \lrcorner \varepsilon_g) = \iota^*(V^a \vec{n}_a)\varepsilon_h$

volume form on induced metric of h.

$$\int_S V \cdot dS = \int_S V^a \vec{n}_a \, dvol_h.$$

what we normally call flux of vector field through a submanifold (bit fiddly w/ direction of normal)

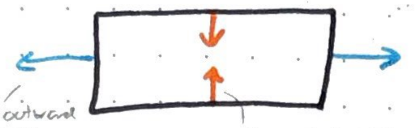
We've shown

$$\int_{\partial N} V^a \vec{n}_a \, dvol_h = \int_N \nabla_a V^a \, dvol_g.$$

(boundary)

Checking the definitions, $\vec{n}^a$ points 'out' of N for ∂N TIMELIKE (or g Riemannian) and 'into' N for ∂N SPACELIKE.

choosing normals to apply div theorem:



choose outward normals for timelike

inward directed for spacelike

(normal spacetime diagram time up & space across)

(div theorem: only difference to what we are used to in vector calc. is we have to insert signs into directions of unit vectors)

[what about null boundaries? issue is no well-defined unit normal b/c normal to null surface is null vector & we can't normalise null vector so flux makes sense for null surface (n^a dvol_h well defined) but can't separately normalise V^a & $dvol_h$? (don't worry about this too much)]

THE EINSTEIN HILBERT ACTION

We want to derive Einstein's equations from an action principle, we expect an action of the form

action depends on g & possibly some matter fields

$$S[g, \text{MATTER}] = \int_M L(g, \text{MATTER})\, dvol_g$$

need a scalar that depends on g to insert in here to integrate, an 'obvious' guess is scalar curvature (work through to see this works)

where L is a scalar Lagrangian. If we ignore matter for now, an obvious guess for L is $L=R$, scalar curvature. This gives the Einstein-Hilbert action.

$$S_{EH}[g] = \frac{1}{16\pi}\int_M R_g\, dvol_g.$$

(in order to derive eom (or unknown) from this, consider variation of g that vanishes outside bounded region. We want that the action does not change to 1st order when we make this variation (principle of least action!))

In order to derive e.o.m. from an action, we consider $g+\delta g$ where δg vanishes outside a compact (=bounded) set in M, and expand to first order in δg (considered small). We require variation of S_{EH} to vanish at this order. We want to compute

(g appears in both R_g & $dvol_g$ so need to figure out how both of these change when I vary g)

$$\delta S_{EH} = S_{EH}[g+\delta g] - S_{EH}[g] \qquad \text{(ignore } O((\delta g)^2)\text{)}$$

First we consider $dvol_g = \sqrt{|g|}\, dx^1 \dots dx^n$ (assume δg non-zero in a single coord patch)

To compute $\delta|g|$, recall we can write the determinant as

$$g = g_{\bar{\mu}\nu}\Delta^{\bar{\mu}\nu}$$

$\bar{\mu}$ means fixed value, no sum over $\bar{\mu}$.

where $\Delta^{\mu\nu}$ is cofactor matrix $\Delta^{\mu\nu} = (-1)^{\mu+\nu}\left|\begin{pmatrix} g_{11} & \cdots & g_{1\nu} & \cdots & g_{1n} \\ \vdots & & \vdots & & \vdots \\ g_{\mu 1} & \cdots & g_{\mu\nu} & \cdots & g_{\mu n} \\ \vdots & & \vdots & & \vdots \\ g_{n1} & \cdots & g_{n\nu} & \cdots & g_{nn}\end{pmatrix}\right|$

delete ν'th column & μ'th row

① delete this column & row, find det of resulting matrix & multiply by factor ± 1 gives component of cofactor matrix

② $\mu\nu$th component of matrix is indep of $g_{\mu\nu}$ because we deleted this from matrix when computing det

$\Delta^{\mu\nu}$ is independent of $g_{\mu\nu}$, and satisfies $\Delta^{\mu\nu} = g g^{\mu\nu}$ ("Jacobi's formula or something") $\therefore$ consider under $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, $\delta g = \frac{\partial g}{\partial g_{\mu\nu}}\delta g_{\mu\nu} = \Delta^{\mu\nu}\delta g_{\mu\nu} = g g^{\mu\nu}\delta g_{\mu\nu}$

$$\therefore \delta\sqrt{-g} = \delta\sqrt{|g|} = \frac{1}{2}\frac{1}{\sqrt{-g}}(-\delta g) = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu}$$

$$\therefore \delta(dvol_g) = \frac{1}{2} g^{ab}\delta g_{ab}\, dvol_g.$$

we have found variation in volume form $dvol_g$ after variation of metric, now work out change in R_g

To compute δR_g, we first consider $\delta\Gamma_{\nu}{}^{\mu}{}_{\rho}$. The difference of two connections is a tensor, so this is a tensor $\delta\Gamma_{a}{}^{b}{}_{c}$. To compute this, we can consider normal coordinates for g_{ab} at some point p. Then since $\partial_\mu g_{\nu\sigma}|_p = 0$

only non zero contrib. comes from varying derivative piece

$$\delta\Gamma_{\nu}{}^{\mu}{}_{\rho}|_p = \tfrac{1}{2} g^{\mu\sigma}(\delta g_{\sigma\nu,\rho} + \delta g_{\rho\sigma,\nu} - \delta g_{\nu\rho,\sigma})|_p$$

$$= \tfrac{1}{2} g^{\mu\sigma}(\delta g_{\sigma\nu;\rho} + \delta g_{\rho\sigma;\nu} - \delta g_{\nu\rho;\sigma})|_p$$

tensor eq. both sides, true for all p so promote to AIN (since p arbitrary)

expr for variation in christoffel symbols ; -> (use this to give variation in R)

$$\therefore \delta\Gamma_{bc}^{a} = \tfrac{1}{2} g^{ad}(\delta g_{db;c} + \delta g_{cd;b} - \delta g_{bc;d}).$$

Next we consider $\delta R^{\mu}{}_{\nu\rho\sigma}$. Again work in normal coords at p,

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho(\Gamma_{\nu}{}^{\mu}{}_{\sigma}) - \partial_\sigma(\Gamma_{\nu}{}^{\mu}{}_{\rho}) + \Gamma\circ\Gamma$$

$$\Rightarrow \delta R^{\mu}{}_{\nu\rho\sigma}|_p = [\partial_\rho(\delta\Gamma_{\nu}{}^{\mu}{}_{\sigma}) - \partial_\sigma(\delta\Gamma_{\nu}{}^{\mu}{}_{\rho})]|_p$$

$$= [\nabla_\rho\delta\Gamma_{\nu}{}^{\mu}{}_{\sigma} - \nabla_\sigma\delta\Gamma_{\nu}{}^{\mu}{}_{\rho}]|_p$$

same as before, promote to AIN

$$\therefore \delta R^{a}{}_{bcd} = \nabla_c\delta\Gamma_{bd}^{a} - \nabla_d\delta\Gamma_{bc}^{a}$$

$$\Rightarrow \delta R_{ab} = \nabla_c\delta\Gamma_{ab}^{c} - \nabla_b\delta\Gamma_{ac}^{c}$$

observing that $\delta(g^{ab}g_{bc}) = 0 \Rightarrow (\delta g^{ab}) = -g^{ac}g^{bd}\delta g_{cd}$. We finally have

$$\delta R_g = \delta(g^{ab}R_{ab}) = (\delta g^{ab})R_{ab} + g^{ab}\delta R_{ab}$$

$$= -R^{ab}\delta g_{ab} + g^{ab}(\nabla_c\delta\Gamma_{ab}^{c} - \nabla_b\delta\Gamma_{ac}^{c})$$

$$= -R^{ab}\delta g_{ab} + \nabla_c X^c$$

where $X^c = g^{ab}\delta\Gamma_{ab}^{c} - g^{cb}\delta\Gamma_{b}{}^{a}{}_{a}$.

Lecture 24 — 4.12.24

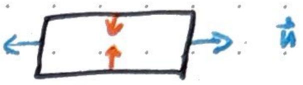
Last lecture

proposed action for grav field where we integrate scalar curvature of the metric over the manifold

action principle: action should be stationary under small perturbation of field

* Stokes / divergence theorem:

$$\int_M \nabla_a X^a \, dvol_g = \int_{\partial M} X^a n_a \, dvol_h$$



* Einstein–Hilbert action

$$S_{EH}[g] = \frac{1}{16\pi} \int_M R_g \, dvol_g$$

issue if manifold infinite then may not get finite action S_{EH} (but generally we ignore this) and anyway we only really care about the variation in the action which makes sense even if it is infinite

* Under a variation $g \to g + \delta g$ (δg small, vanish outside coord chart)

$$\delta(dvol_g) = \tfrac{1}{2} g^{ab} \delta g_{ab} \, dvol_g \, ; \qquad \delta(R_g) = -R^{ab}\delta g_{ab} + \nabla_a X^a$$

ricci tensor contracted into variation in g — some divergence or term involving variation of Γ's symbols

$$X^a = g^{cd} \delta \Gamma^a_{cd} - g^{ac} \delta \Gamma^d_{cd}.$$

We deduce from the formulae for δR_g, $\delta dvol_g$ that

$$\delta S_{EH} = \frac{1}{16\pi} \int_M \{ (\tfrac{1}{2} g^{ab} R - R^{ab}) \delta g_{ab} + \nabla_c X^c \} \, dvol_g .$$

[usual story when we are deriving E-L equation: (something nice) × variation + piece we integrate away by parts]

$$= \frac{1}{16\pi} \int_M -G^{ab} \delta g_{ab} \, dvol_g$$

where we've used the fact that δg and hence X vanish on ∂M to drop the last term using the divergence theorem.
We immediately see that $\delta S_{EH} = 0$ for all variations δg_{ab} if and only if g_{ab} solves the vaccum Einstein equations.

Suppose we also have a contribution from matter fields

$$S_{tot.} = S_{EH} + S_{matter}, \qquad S_{matter} = \int_M L[\Phi, g] \, dvol_g .$$

Under a variation $g \to g + \delta g$ we must have

after doing integration by parts we're gonna have something contracted into δg

$$\delta S_{matter} = \tfrac{1}{2} \int_M T^{ab} \delta g_{ab} \, dvol_g$$

T is just whatever we need to put in to make this statement true

For some symmetric 2-tensor T^{ab}.

[note: this is not the only poss definition for the energy-M tensor but this defn is nicely consistent w/ relativity.]

Varying g in S_{tot} gives $\boxed{G^{ab} = 8\pi T^{ab}}$ i.e. Einstein's equations.

E.g. If ψ is a scalar and $L_{matter} = -\tfrac{1}{2} g^{ab} \nabla_a \psi \nabla_b \psi$, then under $g \to g + \delta g$

variation of matter part of action

$$\delta S_{matter} = -\int_M [\underbrace{\tfrac{1}{2} \delta(g^{ab}) \nabla_a \psi \nabla_b \psi \, dvol_g}_{\text{piece from varying } g \text{ (in } L_{matter})} + \underbrace{\tfrac{1}{2} g^{ab} \nabla_a \psi \nabla_b \psi \, \delta(dvol_g)}_{\text{piece from varying volume form (which we integrate } L \text{ over)}}]$$

using stuff from last lecture

$$= \tfrac{1}{2} \int_M (g^{ac} g^{bd} \nabla_c \psi \nabla_d \psi \, \delta g_{ab} - \tfrac{1}{2} (g^{cd} \nabla_c \psi \nabla_d \psi) g^{ab} \delta g_{ab}) \, dvol_g$$

$$= \tfrac{1}{2} \int_M T^{ab} \delta g_{ab} \, dvol_g$$

where $T^{ab} = \nabla^a \psi \nabla^b \psi - \tfrac{1}{2} g^{ab} \nabla_c \psi \nabla^c \psi .$

Exercise: Show that varying $\psi \to \psi + \delta\psi$ gives the wave equation
$$\nabla_c \nabla^c \psi = 0.$$

It can be shown that diffeomorphism invariance of the matter action implies $\nabla T^{ab} = 0$. (T^{ab} is divergence free)

[CONCLUDE: So we have nice way of formulating einstein's eqns of GR in terms of an action principle which is nice because this is how we think about most modern physical theories (i.e. starting from an action principle)]

"c-owe-chy"

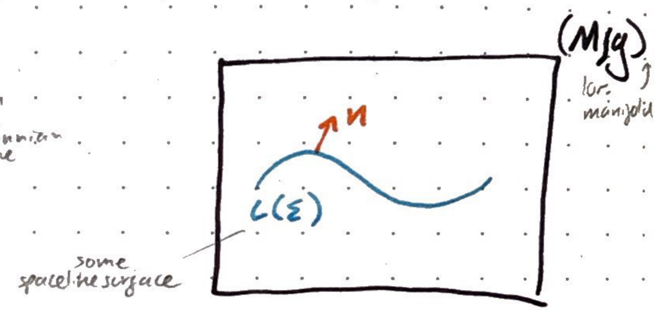
THE CAUCHY PROBLEM FOR EINSTEIN'S EQUATIONS

Saw in wave gauge earlier: EEs are wave eqns: given function & 1st deriv of function at initial time we can solve wave eqns to get future
↳ but a bit unsatisfactory because we had fixed particular gauge & we had fixed notion of time → now our goal is to formulate these ideas in a more geometric way

We expect Einstein's equations can be solved given data on a spacelike hypersurface Σ. What is the right data?

generalise wave eqn approach: on a manifold → don't have an obvious notion of what $t=0$ means but we do have a notion of a hypersurface

Suppose $\iota : \Sigma \to M$ is an embedding, such that $\iota(\Sigma)$ is spacelike. Then $h = \iota^*(g)$ is Riemannian.



metric in big spacetime induces a Riemannian metric on the surface

Let n be a choice of unit normal to $\iota(\Sigma)$. We define for X, Y vector fields on Σ
$$k(X,Y) = \iota^*\left(g(n, \nabla_{\hat{X}} \hat{Y})\right)$$
(strictly speaking need to extend these off the surface but that gives same ans so ignore this subtlety)

where $\iota_* X = \hat{X}$, $\iota_* Y = \hat{Y}$ on $\iota(\Sigma)$.

We pick local coordinates $\{y^i\}$ on Σ and $\{x^\mu\}$ on M such that $\iota : (y^1, y^2, y^3) \longmapsto (0, y^1, y^2, y^3)$, then $n_a \propto dx^0$ say $n_\mu = \alpha \delta^0_\mu$.

If X, Y vector fields on Σ, say $X = X^i \frac{\partial}{\partial y^i}$, $Y = Y^i \frac{\partial}{\partial y^i}$, take
$$\hat{X} = X^i \frac{\partial}{\partial x^i}, \quad \hat{Y} = Y^i \frac{\partial}{\partial x^i}.$$

Then
$$\begin{aligned} k(X,Y) &= g_{\mu\nu} n^\mu \hat{X}^\sigma \nabla_\sigma \hat{Y}^\nu \\ &= \alpha \delta^0_\nu \hat{X}^\sigma (\partial_\sigma \hat{Y}^\nu + \Gamma^\nu_{\sigma\tau} \hat{Y}^\tau) = \alpha \Gamma^0_{ij} X^i Y^j \end{aligned}$$
(1st term vanishes since $Y^0 = 0$?)

$\therefore$ k is a symmetric 2-tensor (called 2nd fundamental form) on Σ. We can show (example sheet 4) that if (M,g) solves the vaccum Einstein equations, then Einstein constraint equations hold:

what are these? → eqns relating k & h, they hold in manifold Σ, not on larger manifold.

(Lev. Civita w.r.t. h) all raising & lowering done with h
$$\left.\begin{aligned} {}^{(h)}\nabla_i k^i{}_j - {}^{(h)}\nabla_j k^i{}_i &= 0 \\ R_h - k^i{}_j k^j{}_i + k^i{}_i k^j{}_j &= 0 \end{aligned}\right\} \quad (\dagger).$$

(what do these say? → if i want to characterise initial conditions for EEs in terms of h & k we can think of k as some kind of normal deriv. to g)

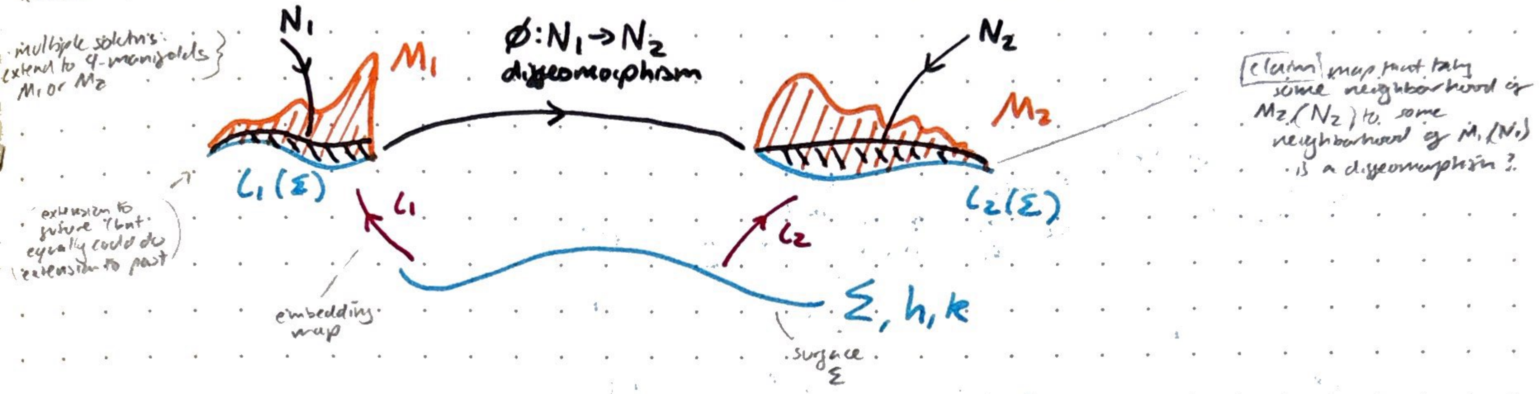
(k, h kind of initial conditions for EES, but they are not completely free as they have to satisfy these constraints?)

Conversely, if we are given

Σ A 3-manifold
h A Riemannian metric on Σ
k A symmetric 2-tensor on Σ

such that (†) hold. Then there exists a solution (M, g) of the vaccum Einstein equations, and an embedding $\iota : \Sigma \to M$ such that h is the induced metric on Σ, and k is the 2nd fundamental form of $\iota(\Sigma)$. ← the embedded manifold

idea now: forget the 4D manifold & think about 3D manifold Σ on which we are given data which solves these constraints
claim is: from this data I can construct a solutn in which the data embeds as an initial hypersurface. Moreover
(nontrivial). it is unique up to extension → what does this mean? ↴



(This result is due to Choquet-Bruhat (existance) and Choquet-Bruhat + Geroch (geometric uniqueness).)

gives us way of looking at EES as evolution problem but w/o having to begin by fixing a gauge (finding ICs that satisfy the constraint equations is difficult.)

well done ☺