note: Even if ∇ is torsion free, $\nabla_a \nabla_b X^c \neq \nabla_b \nabla_a X^c$ in general.

THE LEVI-CIVITA CONNECTION

For a manifold with metric, there is a preferred connection

THM (fundamental theorem of Riemannian geometry)

If (M, g) is a manifold with a metric, there is a unique torsion free connection ∇ satisfying $\nabla g = 0$. This is called the levi-civita connection

(i.e. $\nabla_X g = 0$)

PROOF: Suppose such a connection exists. By Leibniz rule, if X, Y, Z are smooth vector fields

$$* \quad X(g(Y,Z)) = \nabla_X(g(Y,Z)) = \underbrace{(\nabla_X g)}_{=0}(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \qquad a)$$

$$Y(g(Z,X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \qquad b)$$

$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \qquad c)$$

* a) + b) − c):

since metric is symmetric

$$X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) = g(\nabla_X Y + \nabla_Y X, Z) + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_Y Z - \nabla_Z Y, X)$$

* Use $\nabla_X Y - \nabla_Y X = [X,Y]$

this step helps b.c. no dependence on ∇ on RHS

$$X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) = 2g(\nabla_X Y, Z) - g([X,Y],Z) - g([Z,X],Y) + g([Y,Z],X)$$

determines uniquely, Z is arbitrary & g is non degenerate

$$* \quad \Rightarrow g(\nabla_X Y, Z) = \tfrac{1}{2}\{X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X)\} \qquad (†)$$

non-degenerate means if $g(v,Z) = g(w,Z)$ $\forall Z$ then $v = w$

then since everything on RHS is a known quantity → we know it completely, there is no room for freedom i.e. $\nabla_X Y$ is unique. (?)

This determines $\nabla_X Y$ uniquely since g is non-degenerate. Conversely we can use (†) to define $\nabla_X Y$. Then need to check properties of a symmetric connection hold.

$$* \quad \text{E.g. } g(\nabla_{fX} Y, Z) = \tfrac{1}{2}\{fX(g(Y,Z)) + Y(fg(Z,X)) - Z(fg(X,Y)) + g([fX,Y],Z) + g([Z,fX],Y) - g([Y,Z],fX)\}$$

show $\nabla_{fX} Y = f\nabla_X Y$

expand $Y(fg) = fY(g) + Y(f)g$ and $[fX,Y] = f[X,Y] - XY(f)$

$$= \tfrac{1}{2}\{fX(g(Y,Z)) + fY(g(Z,X)) - fZ(g(X,Y)) \cancel{+ Y(f)g(Z,X)} \cancel{- Z(f)g(X,Y)} + g(f[X,Y] \cancel{- Y(f)X}, Z) + g(f[Z,X] \cancel{+ Z(f)X}, Y) - fg([Y,Z],X)\}$$

$$* \quad \Rightarrow g(\nabla_{fX} Y) = g(f\nabla_X Y, Z) \Rightarrow g(\nabla_{fX} Y - f\nabla_X Y, Z) = 0 \quad \forall Z$$

* so $\nabla_{fX} Y = f\nabla_X Y$ as g non-degenerate.

∃ some $X \neq 0$ s.t. $g(X,Y) = 0$ $\forall Y$

Exercise: check other properties

Use the expression (†) we found

apparently the others are easier ?

In a coord. basis we can compute

$$g(\nabla_{e_\mu} e_\nu, e_\sigma) = \tfrac{1}{2}\{e_\mu(g(e_\nu,e_\sigma)) + e_\nu(g(e_\sigma,e_\mu)) - e_\sigma(g(e_\mu,e_\nu))\}$$

$$g(\Gamma^\tau_{\nu\mu} e_\tau, e_\sigma) = \Gamma^\tau_{\nu\mu} g_{\tau\sigma} = \tfrac{1}{2}(g_{\sigma\nu,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma})$$

multiplying inv. metric

$$\Rightarrow \Gamma^\tau_{\nu\mu} = \underbrace{\tfrac{1}{2} g^{\sigma\tau}(g_{\sigma\nu,\mu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma})}_{\text{CHRISTOFFEL SYMBOLS}}$$

so we now can take ∇ to be levi-civita connection - makes sense since the difference b/w two connections is a tensor field, we can write any connection as the levi-civita connection and a (1,2) tensor field.

note: very important term?

notes, we have shown the christoffel symbols are the components of the levi-civita connection wrt a coordinate basis

If ∇ is levi-civita can raise/lower indices and this commutes with covariant differentiation.

i.e. if ∇ is levi-civita then $g_{ab}\nabla_c X^a = \nabla_c(g_{ab}X^a) = \nabla_c X_b$

now think more about curves that extremize proper time & ∇

GEODESICS

We found that a curve extremizing proper time satisfies

the geodesic eqn.

$$(\ddagger)\quad \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x(\tau))\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

τ proper time along curve

note to self: before parametrising by arbitrary param. before had $V^\mu = \frac{dx^\mu}{du}$, now use proper time parametrisation so $X^\mu = \frac{dx^\mu}{d\tau} = V^\mu$

The tangent vector X^a to the curve has components $X^\mu = \frac{dx^\mu}{d\tau}$. Extending this off the curve we get a vector field, of which the geodesics is an integral curve. We note

$$\frac{d^2x^\mu}{d\tau^2} = \frac{d}{d\tau}\Big(\overset{X^\mu(x(\tau))}{\frac{dx^\mu}{d\tau}}\Big) = \frac{\partial X^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau} = X^\mu{}_{,\nu}X^\nu \qquad \text{(chain rule)}$$

I don't know how this works (you'll just have to believe)

$(\ddagger)$ becomes $X^\mu{}_{,\nu}X^\nu + \Gamma^\mu_{\nu\rho}X^\nu X^\rho = 0 \Leftrightarrow X^\nu X^\mu{}_{;\nu} = 0 \Leftrightarrow \nabla_X X = 0.$

where we are using the levi civita connection so $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$ (?) - so it is just symmetric b/c it is multiplied by a symmetric thing?

Extend to any connection ← now extend

any connection

DEF: Let M be a manifold with connection ∇. An AFFINELY PARAMETRIZED GEODESIC satisfies

$$\nabla_X X = 0$$

note: an integral curve of a vector field X satisfying $\nabla_X X = 0$

tangent vector to curve defined only on curve itself but

where X is the tangent vector.

(lecture 10) <u>note</u>: if we reparametrise $t \to t(u)$ then

consider a curve satisfying this def with parameter t & tangent X: ...

$$\underbrace{\frac{dx^\mu}{du}}_{Y^\mu} = \underbrace{\frac{dx^\mu}{dt}}_{X^\mu}\underbrace{\frac{dt}{du}}_{h}$$

conclude (?) the space of affine params is parametrised by two real numbers (i.e. there is a 2 parameter family of affine parameters for any geodesic) $t = \alpha u + \beta$

what is meant by affinely p: if we reparametrise: ∇_Y describes the same curve $\nabla_Y Y = X(h)Y \neq 0$, but in general not affinely parametrised. but it is always possible to find a parameter s.t. it is. in this case $\nabla_Y Y = 0 \Leftrightarrow u = at + b$. wlog we can always restrict to APGs

so $X \to Y = hX$ with $h > 0$

$$\nabla_Y Y = \nabla_{hX}(hX) = h\nabla_X(hX) = h^2 \cancel{\nabla_X X}^{\,0} + hX \cdot X(h) = \delta Y$$

affine reparametrisation

with $\delta = X(h) = \frac{d}{dt}(h) = \frac{1}{h}\frac{dh}{du} = \frac{1}{h}\frac{d^2t}{du^2}$, so $\nabla_Y Y = 0 \Leftrightarrow t = \alpha u + \beta,\ \alpha,\beta \in \mathbb{R},\ \alpha > 0$

nb $X(\delta) := \frac{d}{dt}\delta = \frac{dx^\mu}{dt}\frac{d}{dx^\mu}\delta$?

otherwise $Y = hX$ doesn't ...

note to self: $\nabla_Y Y = 0 \Leftrightarrow \frac{dt}{du}\frac{d^2t}{du^2} = 0$, $\frac{du}{dt} \neq 0$ so take $\frac{d^2t}{du^2} = 0$

line complete

Exercise in notes: let X be tangent to an APG of the Levi-Civita connection. Show that $\nabla_X(g(X,X))=0$:
attempt: $\nabla_X(g(X,X)) = \nabla_X g(X,X) + g(\nabla_X X, X) + g(X, \nabla_X X) = 2g(0,X) = 0$
if $\nabla g = 0$ ie one of them $=0$
so therefore the tangent vector cannot change from eg timelike to null along the geodesic. a geodesic is either timelike, spacelike or null

Lecture 10

1.11.24

Theorem: given $p \in M$, $X_p \in T_pM$, there exists a unique A.P.G. $\lambda: I \to M$ satisfying

$$\lambda(0) = p \qquad \dot{\lambda}(0) = X_p$$

$\lambda(t)$.

PROOF: Choose coordinates with $\phi(p) = 0$ satisfies $\nabla_X X = 0$ with $X = X^\mu \frac{\partial}{\partial x^\mu}$ this becomes

$$x^\mu(t) = \phi(\lambda(t)) \qquad X^\mu = \frac{dx^\mu}{dt}.$$

2nd order diff. unique solution if we specify 2 init conditions

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{dt}\frac{dx^\sigma}{dt} = 0 \qquad \text{(GE)}$$

and $x^\mu(0) = 0 \quad \frac{dx^\mu}{dt}(0) = X_p{}^\mu$

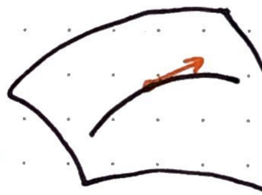
2nd order ODE w/ two BCs → uniqueness follows from standard ODE theory

This has a unique solution $x^\mu: (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ for ε sufficiently small by standard ODE theory. □

GEODESIC POSTULATE

not acted on by any force except gravity

In general relativity, free particles move along geodesics of the Levi-Civita connection



These are TIMELIKE for massive particles and NULL/LIGHTLIKE for massless particles.

Normal Coordinates

locally specify points on manifold by direction & distance to get there e.g. give directions to waitrose. but in canada there are many paths — breaks down long distances — uniqueness problem

the "exponential map"

If we fix $p \in M$, we can map T_pM into M by setting

ψ sends X_p to the point unit affine parameter distance along the geodesic through p w/ tangent X_p at point p

$$\psi(X_p) = \lambda_{X_p}(1)$$ where λ_{X_p} is the unique affinely parametrised geodesic with

$$\lambda_{X_p}(0) = p \quad , \dot{\lambda}_{X_p}(0) = X_p.$$

Notice that

rescale vector equiv to rescaling argument

$$\lambda_{\alpha X_p}(t) = \lambda_{X_p}(\alpha t) \quad \text{for } \alpha \in \mathbb{R}$$

since if t is affine param any $u = at + b$ is also affine param for $a, b \in \mathbb{R}$

since if $\hat{\lambda}(t) = \lambda_{X_p}(at)$ affine reparametrisation so still geodesic, and $\dot{\hat{\lambda}}(0) = \alpha \dot{\lambda}_{X_p}(0) = \alpha X_p$, $\hat{\lambda}(0) = p$.

i.e. the map that sends X_p to 1 also sends αX_p to α

Moreover, $\alpha \longmapsto \psi(\alpha X_p)$ is an affinely parametrised geodesic $= \lambda_{X_p}(\alpha)$.

define map from tangent space to M. claim: as long as sufficiently small neighbourhood e.g. origin then, it is bijective map

heuristically:
- like throwing a ball and seeing where it lands
- if i throw it w/ initial velocity 2x faster it goes twice the distance.

X_p

$2X_p$

i.e. the exponential map sends X_p to the point unit distance along the geodesic through p w tangent X_p at p. but it sends tX_p distance t along that same geodesic.

CLAIM: If $U \subset T_pM$ is a sufficiently small neighbourhood of the origin, then $\psi: T_pM \to M$ is one-to-one and onto.

(don't prove here) — can prove this easily, check jacobian is invertible (not part of this course)

DEF: Construct <u>normal coordinates</u> at p, (supposing $\{e_\mu\}$ is a basis for T_pM), as follows. For

$q \in \psi(U) \subset M$, we define $\phi(q) = (X^1, \dots, X^n)$ where X^μ are components of the unique $X_p \in U$ with $\psi(X_p) = q$ (w.r.t. $\{e_\mu\}$)

(at each point q there is a unique X_p in T_pM ... ; $X_p = X^\mu e_\mu$ defines a point q, we assign q coordinates X^μ)

By our previous observation, the curve given in normal coordinates by $X^\mu(t) = tY^\mu$ for Y^μ constant is an affinely parametrised geodesic so from the geodesic eqn. **(GE)**

(we know its a geodesic so can just plug it in)

$$\Gamma^\mu_{\nu\sigma}(tY) Y^\nu Y^\sigma = 0$$

(reminder: antisymmetric part is 0 since we contract w/ symmetric thing)

Set $t=0$ deduce (since Y arbitrary) that $\Gamma^\mu_{(\nu\sigma)}\big|_p = 0$

(symmetric part of Γ at p vanishes)

So if ∇ is torsion free, $\Gamma^\mu_{\nu\sigma}\big|_p = 0$ in normal coordinates. (components vanish (at p))

(reminder: connection components not a tensor otherwise this would be a stronger result)

If ∇ is the levi-civita connection of a metric, then

$$g_{\mu\nu,\rho}\big|_p = 0$$

(use $\Gamma^\tau_{\nu\mu} g_{\tau\sigma} = \frac{1}{2}(g_{\sigma\nu,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma})$)

(rmk: normal coords are coords where the spacetime looks like minkowski space (at a point) to 1st order)

Since $g_{\mu\nu,\rho} = \frac{1}{2}(g_{\mu\nu,\rho} + g_{\rho\nu,\mu} - g_{\mu\rho,\nu}) + \frac{1}{2}(g_{\mu\nu,\rho} + g_{\mu\rho,\nu} - g_{\rho\nu,\mu})$

$$= \Gamma^\sigma_{\mu\rho} g_{\sigma\nu} + \Gamma^\sigma_{\rho\nu} g_{\sigma\mu} = 0 \quad \text{at } p.$$

(basically if Γ vanishes then the deriv of the metric vanishes)

(can arrange nice form of metric if we choose basis?) (at any point we can introduce normal coords s.t. the first partial deriv of the metric vanishes) (if in normal coords at $p=0$ and ∇ is levi civita connection then Γ vanishes)

We can always choose the basis $\{e_\mu\}$ for T_pM on which base the normal coordinates to be orthonormal. We have

LEMMA: On a Riemannian/Lorentzian manifold (& levi-civita ∇?) we can choose normal coordinates at p s.t. $g_{\mu\nu,\rho}\big|_p = 0$ and

$$g_{\mu\nu}\big|_p = \begin{cases} \delta_{\mu\nu} & \text{RIEMANNIAN} \\ \eta_{\mu\nu} & \text{LORENTZIAN} \end{cases}$$

(1st derivative vanish at p) (at p: $\frac{\partial}{\partial x^\mu}\big|_p = e_\mu$) (i.e. locally looks flat: Euclidean if flat riemannian, Minkowski if flat lorentzian)

PROOF: The curve given in normal coordinates by $t \mapsto (t, 0, \dots, 0)$ is the APG with $\lambda(0) = p$, $\dot\lambda(0) = e_1$ by previous argument. But by defn. of coord basis this vector is $\left(\frac{\partial}{\partial x_1}\right)_p$. So if $\{e_\mu\}$ is ON at p $\left(\frac{\partial}{\partial x^\mu}\right)_p$ form an ON basis. $\square$

(orthonormal?)

(tany: if we pick the initial basis $\{e_\mu\}$ to be orthonormal then the geodesics will point in orthogonal directions which ensures the metric takes the form $g_{\mu\nu}(p) = \delta_{\mu\nu}$)

CURVATURE

look for defn intrinsic to manifold ~~that the~~ that tells us its not flat (solution) by considering parallel transport

sphere

transporting a vector around changes the angle, tells us it is curved

Parallel transport

[note to self: if λ is a curve with tangent vector X^a, then a tensor field T is parallelly transported along λ if $\nabla_X T = 0$]

Suppose $\lambda : I \to M$ is a curve with tangent vector $\dot{\lambda}(t)$ we say a tensor field T is parallely transported/propagated along λ if

$$\nabla_{\dot{\lambda}} T = 0 \quad \text{on } \lambda \qquad (PP)$$

"looks a bit like geodesic eqn (GE)"

affinely parametrised

* If λ is an APG then $\dot{\lambda}$ is parallely propagated along λ.

a geodesic is a curve whose tangent vector is parallelly transported along the curve

* A parallely propagated tensor is determined everywhere on λ by its value at one point:

E.g. If T is a (1,1) tensor then in coordinates (PP) becomes

use $\nabla_\rho T^\mu{}_\nu = T^\mu{}_{\nu;\rho} = T^\mu{}_{\nu,\rho} + \Gamma^\mu_{\sigma\rho} T^\sigma{}_\nu - \Gamma^\sigma_{\nu\rho} T^\mu{}_\sigma$

then $\nabla_X T^\mu{}_\nu = X^\rho \nabla_\rho T^\mu{}_\nu$

since $X^\rho = \frac{dx^\rho}{dt}$

components of tangent vector

$$0 = \frac{dx^\mu}{dt} T^\nu{}_{\sigma;\mu} = \frac{dx^\mu}{dt}\Big(\underbrace{T^\nu{}_{\sigma,\mu}}_{\text{1st partial}} + \Gamma^\nu_{\rho\mu} T^\rho{}_\sigma - \Gamma^\rho_{\sigma\mu} T^\nu{}_\rho\Big)$$

but $T^\nu{}_{\sigma,\mu} \frac{dx^\mu}{dt} = \frac{d}{dt}(T^\nu{}_\sigma)$ so

deriv of T in direction is $\frac{d}{dt}(T^\nu{}_\sigma)$

linear in T

$$0 = \frac{d}{dt} T^\nu{}_\sigma + \left(\Gamma^\nu_{\rho\mu} T^\rho{}_\sigma - \Gamma^\rho_{\sigma\mu} T^\nu{}_\rho\right)\frac{dx^\mu}{dt}$$

1st order eqn → solution determined entirely by values at a point

This is a 1st order linear ODE for $T^\nu{}_\sigma(x(t))$ so ODE theory gives a unique solution once $T^\nu{}_\sigma(x(0))$ specified.

T uniquely determined by knowing its value at one point?

* Parallel transport along a curve from p to q gives an isomorphism between tensors at p and q. This depends on the choice of curve in general. ← important

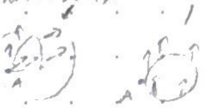
1st order i.e.

$\frac{d}{dt} T^\nu{}_\sigma = -$(linear combination of the components of T)

rem: the isomorphism depends on the choice of path. On a curved manifold, parallel transporting around a loop may not return you to the same tensor?

isomorphism means the map is invertible & preserves the tensor structure i.e. maps tensors to tensors

wiki: isomorphism - a structure preserving map that can be reversed by an inverse mapping



Lecture 11 4.11.24

THE RIEMANN TENSOR

The Riemann tensor captures the extent to which parallel transport depends on the curve.

LEMMA: Given X, Y, Z vector fields, ∇ a connection, define

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Then $(R(X,Y)Z)^a = R^a{}_{bcd} X^c Y^d Z^b$ for a $(1,3)$-tensor $R^a{}_{bcd}$, the Riemann tensor.

PROOF: Suppose f is smooth function, then

prove it is a tensor by showing it's linear in X, Y, Z

$$\begin{aligned} R(fX,Y)Z &= \nabla_{fX} \nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z \\ &= f\nabla_X \nabla_Y Z - \nabla_Y (f\nabla_X Z) - \nabla_{f[X,Y] - Y(f)X} Z \\ &= f\nabla_X\nabla_Y Z - f\nabla_Y \nabla_X Z - \cancel{Y(f)\nabla_X Z} - f\nabla_{[X,Y]} Z + \cancel{Y(f)\nabla_X Z} \\ &= f R(X,Y)Z. \ ^{(1)} \text{ linear in X} \end{aligned}$$

linear in Y $^{(2)}$

Since $R(X,Y)Z = -R(Y,X)Z$, we have $R(X, fY)Z = fR(X,Y)Z$

(antisymmetric in X, Y means we don't need to separately check linearity in Y)

linear in Z

Exercise check $R(X,Y)(fZ) = fR(X,Y)Z^{(3)}$ similar computation to above, we have 3 results now

this is fine (remember $Y(f)$, $X(f)$, f etc. are scalar so $\nabla_Y X(f) = Y(X(f))$ etc. think.)

Now suppose we pick a basis $\{e_\mu\}$ with dual basis $\{f^\nu\}$

$$\begin{aligned} R(X,Y)Z &= R(X^\rho e_\rho, Y^\sigma e_\sigma)(Z^\nu e_\nu) = X^\rho Y^\sigma Z^\nu R(e_\rho, e_\sigma)e_\nu \\ &= (R^\mu{}_{\nu\rho\sigma} X^\rho Y^\sigma Z^\nu) e_\mu \end{aligned}$$

holds in one basis so holds in any basis

note to self: Summary: 1. show that it's linear in X, Y, Z. 2. then also show its components are what we want (think)

where $R^\mu{}_{\nu\rho\sigma} = f^\mu(R(e_\mu, e_\sigma)e_\nu)$ are components of $R^a{}_{bcd}$ in this basis. Since result holds in one basis, it holds in all bases. $\square$

note to self: b/c partial derivatives commute!

In a coordinate basis $e_\mu = \frac{\partial}{\partial x^\mu}$ and $[e_\mu, e_\nu] = 0$ so

$$\begin{aligned} R(e_\rho, e_\sigma)e_\nu &= \nabla_{e_\rho}(\nabla_{e_\sigma} e_\nu) - \nabla_{e_\sigma}(\nabla_{e_\rho} e_\nu) = \nabla_{e_\rho}(\Gamma^\tau_{\nu\sigma} e_\tau) - \nabla_{e_\sigma}(\Gamma^\tau_{\nu\rho} e_\tau) \\ &= \partial_\rho(\Gamma^\tau_{\nu\sigma}) e_\tau + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} e_\mu - \partial_\sigma(\Gamma^\tau_{\nu\rho}) e_\tau - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma} e_\mu \end{aligned}$$

use $R^\mu{}_{\nu\rho\sigma} = f^\mu(R(e_\mu, e_\sigma)e_\nu)$

Hence $$\boxed{R^\mu{}_{\nu\rho\sigma} = \partial_\rho(\Gamma^\mu_{\nu\sigma}) - \partial_\sigma(\Gamma^\mu_{\nu\rho}) + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma}}$$

notation is useful (especially last 2 terms)

note to self: $\nabla_\rho(\Gamma) = e_\rho(\Gamma) = \partial_\rho(\Gamma)$ since Γ are not tensor components, they are scalars so you treat them as scalars (right?)

In normal coordinates we can drop the last two terms.

(if we also have the Levi-Civita connection?)

(flatness at p so $\Gamma = 0$ but in general still curved so $\partial\Gamma \neq 0$?)

(Lorentzian & normal coords means Γ vanishes at a point but if also have Minkowski space then Γ vanishes everywhere)

Example: For the Levi-Civita connection of Minkowski space in an inertial frame, $\Gamma^{\nu}_{\mu\sigma} \equiv 0$, so $R^{\mu}{}_{\nu\sigma\tau} = 0$

(can't inertial coords exist globally in flat spacetime)

(inertial coords are normal coords but normal coords are not inertial coords)

(normal coords valid for not flat spacetime but are only valid locally)

i.e. choose inertial coords

true in 1 basis so true in all

hence $R^{a}{}_{bcd} = 0$. Such a connection with $R^{a}{}_{bcd} = 0$ everywhere is called *flat*. Conversely, for a Lorentzian spacetime with flat L-C connection, we can locally find coordinates such that $g_{\mu\nu} = \mathrm{diag}(-1,1,\dots,1)$.

$\equiv \eta_{\mu\nu}$

this spacetime is "locally isometric" to Minkowski spacetime

<u>A note of caution</u>:

$$(\nabla_X \nabla_Y Z)^c = X^a \nabla_a (Y^b \nabla_b Z^c) \neq X^a Y^b \nabla_a \nabla_b Z^c$$

<u>hence</u>
$$(R(X,Y)Z)^c = X^a \nabla_a (Y^b \nabla_b Z^c) - Y^a \nabla_a (X^b \nabla_b Z^c) - [X,Y]^b \nabla_b Z^c$$
$$= X^a Y^b \nabla_a \nabla_b Z^c - Y^a X^b \nabla_a \nabla_b Z^c + (\nabla_X Y - \nabla_Y X - [X,Y])^b \nabla_b Z^c$$

So if ∇ is torsion free,

$$\boxed{\nabla_a \nabla_b Z^c - \nabla_b \nabla_a Z^c = R^{c}{}_{dab} Z^d}$$ RICCI IDENTITY

on ex. sheet 2 there's a question to generalise for an expression for

$$\nabla_{[a} \nabla_{b]} T^{c_1 \dots c_r}{}_{d_1 \dots d_s}.$$

We can construct a new tensor from $R^{a}{}_{bcd}$ by contraction:

(side note: $R^{a}{}_{bcd} = -R^{a}{}_{bdc} \Rightarrow R^{a}{}_{b(cd)} = 0$)

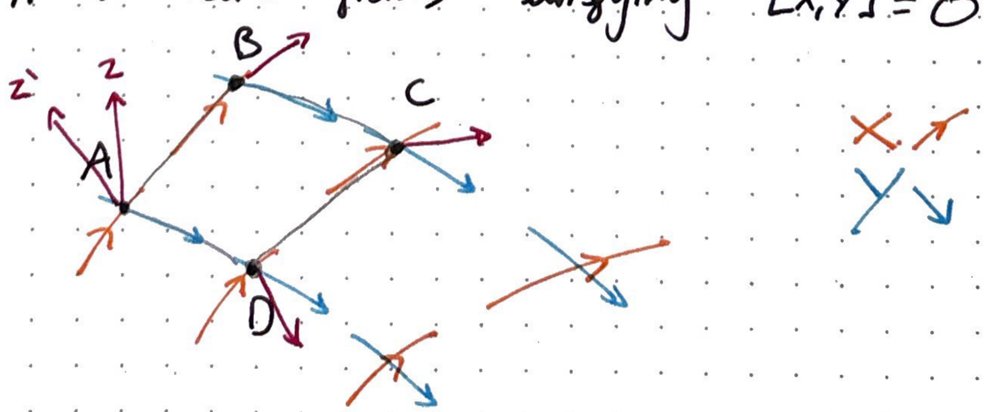
Definition: The RICCI TENSOR is the $(0,2)$-tensor

$$R_{ab} = R^{c}{}_{acb}$$

(note to self: contract middle?)

why not contract diff 2 indices: $R^{a}{}_{acd} = 0$, others are the same up to a minus sign so this is the only one of interest

Suppose X, Y are vector fields satisfying $[X,Y] = 0$



(once I've parallel transport it around the curves it's not necessarily the same vector)

flow distance ε along integral curve of X through a point then ε along integral curve of X through a point then back ε then back ε i.e. get to the same point

Go from A to B by flowing parameter distance ε along int. curve of X
B to C " " ε " Y
C to D " " $-\varepsilon$ " X
D to A " " $-\varepsilon$ " Y

(property that commutators vanish)

Since $[X,Y]=0$, we indeed return to start.

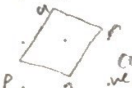
remark
$R^{\mu}{}_{\nu\rho\sigma}$ measures path dependence of parallel transport i.e. difference pqr vs psr but can equally be interpreted as the effect of parallel transporting a vector around the closed loop rqpsr

CLAIM: (See H. Reall notes.)

assuming
(w. levi-civita connection?)

If Z is parallely transported around ABCD to a vector Z', then

$$(Z-Z')^{\mu}=\varepsilon^2 R^{\mu}{}_{\nu\rho\sigma}Z^{\nu}X^{\rho}Y^{\sigma}+O(\varepsilon^3)$$



consider what we get if we parallel transport Z_p around pqr $(Z^{\mu}{}_r)$ vs psr $(Z'^{\mu}{}_r)$ $\rightarrow$ $\Delta Z^{\mu}{}_r \equiv Z'^{\mu}{}_r - Z^{\mu}{}_r = (R^{\mu}{}_{\nu\rho\sigma}Z^{\nu}X^{\rho}Y^{\sigma})_p\,\delta s\,\delta t + O(\delta s)$

$\Rightarrow (R^{a}{}_{bcd}Z^{b}X^{c}Y^{d})_r = \lim_{\delta\to 0}\frac{\Delta Z^{a}{}_r}{\delta s\,\delta t}$

this is one way of visualising curvature, let's consider another way: geodesic deviation

i.e. riemann tensor measures path dependence of parallel transport //

Geodesic Deviation

Let ∇ be a symmetric connection. Suppose $\lambda: I \to M$ is an APG through p. We can pick normal coordinates centred at p such that λ is given by $t \mapsto (t,0,\dots,0)$.

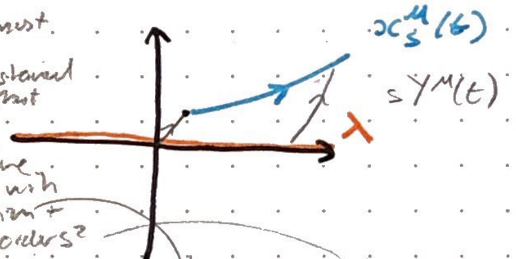
Suppose we start a geodesic with

start close to λ & almost in the same direction

$$x^{\mu}_s(0)=s x_0{}^{\mu} \qquad |s|\ll 1.$$

$$\dot{x}^{\mu}_s(0)=s x_1{}^{\mu}+(1,0,\dots,0)$$

find from standard properties of ODE that curve $x^{\mu}_s(t)$ looks like curve for λ with correction + terms order s^2



Then we find $x^{\mu}_s(t)=x^{\mu}(s,t)=(t,0,0,\dots,0)+sY^{\mu}(t)+O(s^2)$

$x^{\mu}(s,t)=x^{\mu}(0,t)+s\left.\frac{dx^{\mu}(s,t)}{ds}\right|_{s=0}+O(s^2)$

$Y^{\mu}(t)=\left.\frac{\partial x^{\mu}}{\partial s}\right|_{s=0}$ are components of a vector field along λ.

Measuring the (infinitesimal) deviation of the geodesics, we have

$$\frac{\partial^2 x^{\mu}}{\partial t^2}+\Gamma^{\mu}_{\nu\sigma}(x^{\mu}(s,t))\frac{\partial x^{\nu}}{\partial t}\frac{\partial x^{\sigma}}{\partial t}=0 \qquad \text{take } \left.\frac{\partial}{\partial s}\right|_{s=0}$$

$$\Rightarrow \frac{\partial^2 Y^{\mu}}{\partial t^2}+\partial_{\rho}(\Gamma^{\mu}_{\nu\sigma})_{s=0}T^{\nu}T^{\sigma}Y^{\rho}+2\Gamma^{\mu}_{\rho\sigma}\frac{\partial}{\partial t}Y^{\rho}T^{\sigma}=0 \qquad T^{\mu}=\left.\frac{\partial x^{\mu}}{\partial t}\right|_{s=0}$$

$$\Rightarrow T^{\nu}(T^{\sigma}Y^{\mu}{}_{,\sigma})_{,\nu}+\partial_{\rho}(\Gamma^{\mu}_{\nu\sigma})_{s=0}T^{\nu}T^{\sigma}Y^{\rho}+2\Gamma^{\mu}_{\rho\sigma}\frac{\partial Y^{\rho}}{\partial t}T^{\sigma}=0$$

At $p=0$, $\Gamma=0$, so $T^{\nu}(T^{\sigma}Y^{\mu}{}_{;\sigma}-\Gamma^{\mu}_{\rho\sigma}T^{\sigma}Y^{\rho})_{;\nu}+(\partial_{\rho}\Gamma^{\mu}_{\nu\sigma})_p T^{\nu}T^{\sigma}Y^{\rho}=0.$

only thing left over is when partial deriv. hits Γ

t.b.c.
(lect. 12) $\Rightarrow T^{\nu}(T^{\sigma}Y^{\mu}{}_{;\sigma})_{;\nu}+(\partial_{\rho}\Gamma^{\mu}_{\nu\sigma}-\partial_{\nu}\Gamma^{\mu}_{\rho\sigma})T^{\nu}T^{\sigma}Y^{\rho}=0$ (reminder at $p=0$)

$$\Rightarrow (\nabla_T\nabla_T Y)^{\mu}+R^{\mu}{}_{\sigma\rho\nu}T^{\nu}T^{\sigma}Y^{\rho}=0$$

$$\Rightarrow \boxed{\nabla_T\nabla_T Y+R(Y,T)=0}$$

GEODESIC ~~EQN~~ DEVIATION JACOBI EQN.

//

note: $\nabla_T\nabla_T S=R(T,S)T$

main $T^{a}\nabla_a(T^{b}\nabla_b S^{d})=T^{a}S^{b}T^{c}R^{d}{}_{cab}$

result $\Rightarrow$ curvature results in relative acceleration of geodesics

(c.f. in euclidean & minkowski space, in which parallel geodesics remain parallel forever

SUMMARY
- $R^a{}_{b(cd)} = 0$;
- $R^a{}_{[bcd]} = 0$; $R^a{}_{b[cd;e]} = 0$ (∇ TORSION FREE)
- $R_{abcd} = R_{cdab}$ (∇ LEVI CIVITA)

Lecture 12

6.11.24

SYMMETRIES OF THE RIEMANN TENSOR

From the definition it's clear that $R^a{}_{bcd} = -R^a{}_{bdc} \Leftrightarrow R^a{}_{b(cd)} = 0$

PROPOSITION: If ∇ is torsion free then $R^a{}_{[bcd]} = 0$

antisymmetric over 3 downstairs indices

PROOF: Fix $p \in M$, choose normal coordinates at p and work in coordinate basis, then

(use normal coords to simplify)

$$\Gamma^\sigma_{\mu\nu}|_p = 0 \quad \text{and} \quad \Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu} \quad \text{everywhere}$$

torsion free

then

antisymmetrise over downstairs indices then bottom these terms vanish

$$R^\mu{}_{\nu\rho\sigma}|_p = \partial_\rho(\Gamma^\mu_{\nu\sigma})|_p - \partial_\sigma(\Gamma^\mu_{\nu\rho})|_p$$

do the long way or note role uses: 1st term symmetric in $\nu\sigma$, 2nd term symmetric in $\nu\rho$ so antisymmetrising both vanish (since we keep only antisymmetric parts!)

useful trick we will use again & again to prove stuff: establish it holds at arbitrary point p so holds everywhere (normal coords)

$$\Rightarrow R^\mu{}_{[\nu\rho\sigma]}|_p = 0 \quad \text{as} \quad \partial_\rho \Gamma^\mu_{\nu\sigma}|_p = \partial_\rho \Gamma^\mu_{\sigma\nu}|_p$$

p arbitrary and so $R^\mu{}_{[\nu\rho\sigma]} = 0$ everywhere □.

(basis component of tensor vanish in one basis then they vanish in all bases)

PROPOSITION: If ∇ is torsion free then the Bianchi Identity holds:

BIANCHI IDENTITY

$$\boxed{R^a{}_{b[cd;e]} = 0}$$

(there is a 1st, 2nd bianchi identity, if we say just bianchi identity we usually mean this one)

PROOF: Choose coordinates as above then $R^\mu{}_{\nu\rho\sigma;\tau}|_p = R^\mu{}_{\nu\rho\sigma,\tau}|_p$ (since $\Gamma|_p = 0$) schematically,

(Γ contracted into R & Γ vanish at p)

$$R \sim \partial\Gamma + \Gamma\Gamma \quad \text{so} \quad \partial R \sim \partial\partial\Gamma + \partial\Gamma \cdot \Gamma$$

(only terms that are going to contribute are linear 2nd order derivatives of Γ)

is what purpose this step is? just use same expression for R as above? see if I symmetrise

and since $\Gamma|_p = 0$ we deduce

$$R^\mu{}_{\nu\rho\sigma,\tau}|_p = \partial_\tau\partial_\rho\Gamma^\mu_{\nu\sigma}|_p - \partial_\tau\partial_\sigma(\Gamma^\mu_{\nu\rho})|_p.$$

By symmetry of the mixed partial derivatives, we see

(same logic as above: 1st term symmetric in $\tau\rho$, second in $\tau\sigma$ so $[\rho\sigma\tau]$ makes it all vanish)

$$R^\mu{}_{\nu[\rho\sigma;\tau]}|_p = 0$$

(these identities hold for **any** torsion free connection)

since p arbitrary result follows □.

gentle game can unlock possibilities

* Suppose ∇ is the Levi-Civita connection of a manifold with metric g we can lower an index with g_{ab} and consider R_{abcd}.

(reminder: since $g_{ab}\nabla_c X^a = \nabla_c(g_{ab}X^a) = \nabla_c X_b$)

claim R_{abcd} has additional symmetries, lots of symmetries

PROPOSITION: R_{abcd} satisfies $R_{abcd} = R_{cdab}$ $(\Rightarrow R_{(ab)cd} = 0)$

PROOF: Pick normal coordinates at p so that $\partial_\mu g_{\nu\rho} = 0$. we notice that

connection component vanish at p but also partial deriv. vanishes

$$0 = \partial_\mu \delta^\nu_\sigma|_p = \partial_\mu(g^{\nu\tau}g_{\tau\sigma})|_p = (\partial_\mu g^{\nu\tau})g_{\tau\sigma}|_p$$

because g non degenerate

$$\Rightarrow \partial_\mu g^{\nu\tau}|_p = 0$$

hence $\partial_\rho(\Gamma^{\mu}_{\nu\sigma})|_p = \partial_\rho(\tfrac{1}{2}g^{\mu\tau}(g_{\tau\sigma,\nu} + g_{\nu\tau,\sigma} - g_{\nu\sigma,\tau}))|_p$
$= \tfrac{1}{2}g^{\mu\tau}(g_{\tau\sigma,\nu\rho} + g_{\nu\tau,\sigma\rho} - g_{\nu\sigma,\tau\rho})|_p$

we have $R_{\mu\nu\rho\sigma}|_p = g_{\mu\kappa}(\partial_\rho \Gamma^{\kappa}_{\nu\sigma} - \partial_\sigma \Gamma^{\kappa}_{\nu\rho})|_p$ ("I have a look at this not using normal coords if you don't want to sleep tonight")

(If I swap 1st 2 indices it goes into itself (?) so)
$= \tfrac{1}{2}(g_{\mu\sigma,\nu\rho} + g_{\nu\rho,\mu\sigma} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma})|_p$

note to self. (use equality of mixed partial derivatives & symmetry of the metric)

This satisfies $R_{\mu\nu\rho\sigma}|_p = R_{\rho\sigma\mu\nu}|_p$ hence true everywhere $\square$.

note: "tensor eqns hold so valid in any basis"

<u>COROLLARY</u>: The Ricci tensor is symmetric $R_{ab} = R_{ba}$ $\rightarrow$ $R_{ab} = g^{dc}R_{dacb} = g^{dc}R_{cbda} = R^{d}{}_{bda} = R_{ba}$ //

(use $R_{ab} = R^{c}{}_{acb}$ and $R^{e}{}_{bcd} = g^{ea}R_{abcd}$)

(w/ metric we can go further, one more contraction)

DEFINITION: * The <u>Ricci scalar</u> (scalar curvature) is $R = R_a{}^a = g^{ab}R_{ab}$.
* The <u>Einstein tensor</u> is $G_{ab} = R_{ab} - \tfrac{1}{2}g_{ab}R$

(construct from ricci scalar)

<u>Exercise</u>: The Bianchi identity implies $\nabla_a G^a{}_b = 0$ (contracted Bianchi identity)

(important for when we construct einstein equations)

($\leftarrow$ the gauge group of einstein eqns.)

DIFFEOMORPHISMS AND THE LIE DERIVATIVE

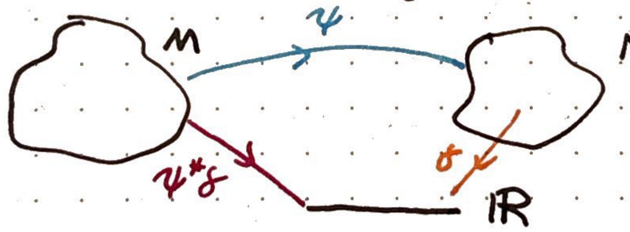
Suppose $\psi: M \to N$ is a smooth map, then ψ induces maps between corresponding vector/covector bundles.

DEFINITION: Given $f: N \to \mathbb{R}$, the PULL BACK of f by ψ is the map $\psi^* f: M \to \mathbb{R}$ given by $\psi^* f(p) = f(\psi(p))$

"f acting on image of p"

(notes: $\psi: M \to N$ is smooth iff $\phi_\alpha \circ \psi \circ \phi_\alpha^{-1}$ is smooth for all charts of M and N (ϕ_α & ϕ_α resp.))

(notes: 'pull back' a fn. on N to define a fn. on M)



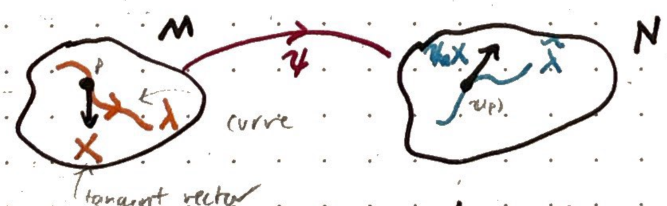
[notes: def $\psi^*(f) = f \circ \psi$]

DEFINITION: Given $X \in T_pM$, we define the PUSH FORWARD of X by ψ, $\psi_* X \in T_{\psi(p)}N$ as follows,

Let $\lambda: I \to M$ be a curve with $\lambda(0) = p$, $\dot\lambda(0) = X$. Then set $\hat\lambda = \psi \circ \lambda$, $\hat\lambda: I \to N$ gives a curve in N with $\hat\lambda(0) = \psi(p)$. We set $\psi_* X = \dot{\hat\lambda}(0)$.

(notes: we can "push forward" a curve λ in M to a curve $\psi \circ \lambda$ in N, hence we can push forward vectors from M to N)

(notes: if λ is a curve in M with $\lambda(0) = p$, $\dot\lambda(0) = X$, then $\psi_* X$ is defined as $\dot{\hat\lambda}(0)$ where $\hat\lambda = \psi \circ \lambda$ is a curve in N with $\hat\lambda(0) = \psi(p)$)



<u>Note</u>: If $f: N \to \mathbb{R}$ then $\psi_* X(f) = \frac{d}{dt}(f \circ \hat\lambda(t))|_{t=0} = \frac{d}{dt}(f \circ \psi \circ \lambda(t))|_{t=0}$
$= \frac{d}{dt}(\psi^* f \circ \lambda(t))|_{t=0} = X(\psi^* f)$

(reminder: $X_p(f) := \frac{d}{dt} f(\lambda(t))|_{t=0}$)

$\psi_* X(f) = X(\psi^* f)$

$((\psi^* \omega_f)_p(v) = X(f)?)$

Exercise: If x^μ are coords on M near p, y^α are coords on N near $\psi(p)$ then ψ gives a map $y^\alpha(x^\mu)$. Show that in a coordinate basis

important

$$(\psi_* X)^\alpha = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)_p X^\mu \quad \left(\text{or } \psi_*\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)_p \left(\frac{\partial}{\partial y^\alpha}\right)\right)$$

(attempt: like this?? $f \circ \psi \circ \lambda(t) = f \circ \phi_\alpha^{-1} \circ \phi_\alpha \circ \lambda(t)$ so $\frac{d}{dt}(f(\lambda(t))) = \frac{\partial}{\partial y^\alpha}(f \circ \phi_\alpha^{-1}) \frac{dy^\alpha}{dx^\mu}\frac{dx^\mu}{dt} \Rightarrow (\psi_* X)^\alpha = \frac{dy^\alpha}{dx^\mu}\frac{dx^\mu}{dt}$?? ; or in short $f \circ \psi \circ \lambda = f \circ y(x(t))$)

On cotangent bundle we go backwards

DEFINITION: If $\eta \in T_{\psi(p)}^* N$, then the pullback of η, $\psi^*\eta \in T_p^* M$, is defined by

$$\psi^*\eta(X) = \eta(\psi_* X) \quad \forall\, X \in T_pM$$

(note to self: e.g. $(df)_p(X) = X(f)$?)

(worth checking where these identities come from; look back)

Note: If $f : N \to \mathbb{R}$, $\psi^*(df)[X] = df[\psi_* X] = \psi_* X(f) = X(\psi^* f)$
$= d(\psi^* f)[X]$

(use: $\psi^*\eta(X) = \eta(\psi_* X)$; $df(X) = X(f)$; $\psi_* X(f) = X(\psi^* f)$.)

(since X arbitrary)

$$\Rightarrow \psi^* df = d(\psi^* f)$$

(notes "i.e. gradient commutes with pull back")

Exercise: With notation as before, show that

important

$$(\psi^*\eta)_\mu = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)_p \eta_\alpha \quad \left(\text{or } \psi^*(dy^\alpha)_p = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)_p (dx^\mu)_p\right).$$

(attempt (like it?): • consider $(\psi^*\eta)(X) = \eta(\psi_* X)$ • plug in $\psi_* X = X^\mu \frac{dy^\alpha}{dx^\mu}\frac{\partial}{\partial y^\alpha}$ • remember $\eta(e_\alpha) = \eta_\alpha$ • so get $(\psi^*\eta)(X) = X^\mu \frac{dy^\alpha}{dx^\mu}\eta_\alpha$ → extract components $(\psi^*\eta)_\mu = \frac{dy^\alpha}{dx^\mu}\eta_\alpha$?)

(likely consider $\psi^*(dy^\alpha)(X) = X^\mu \frac{dy^\alpha}{dx^\mu}$; compare to what we want $\frac{dy^\alpha}{dx^\mu}dx^\mu(X)$)

* We can extend the pullback to map a $(0,s)$-tensor at $\psi(p) \in N$ to a $(0,s)$-tensor $\psi^* T$ at $p \in M$ by requiring

$$\psi^* T(X_1, \dots, X_s) = T(\psi_* X_1, \dots, \psi_* X_s) \quad \forall X_i \in T_pM.$$

* Similarly we can push forward a $(s,0)$-tensor S at $p \in M$ to a $(s,0)$-tensor $\psi_* S$ at $\psi(p) \in N$ by

$$\psi_* S(\eta_1, \dots, \eta_s) = S(\psi^*\eta_1, \dots, \psi^*\eta_s) \quad \forall\, \eta_i \in T_{\psi(p)}^* N.$$

(note to self: in conventions pullback ~ normal things to downstairs indices $\frac{dy}{dx}$; pushforward ~ normal things to upstairs indices $\frac{dy}{dx}$)

* If $\psi : M \to N$ has the property that $\psi_* : T_pM \to T_{\psi(p)}N$ is injective (one-to-one), we say ψ is an **immersion** ($\dim N \geq \dim M$).

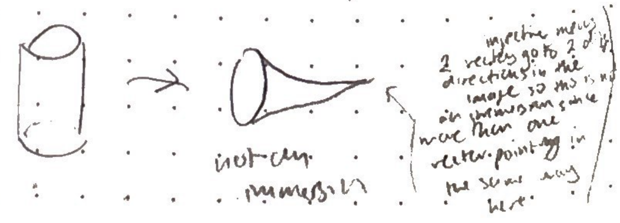
(see lecture notes for examples)

* If N is a manifold with metric g, and $\psi : M \to N$ is an immersion, we can consider $\psi^* g$ (pull back of g)

* If g is Riemannian, then $\psi^* g$ is non-degenerate and positive definite, so defines a metric on M, the **induced metric**.

Note: If we pull back a Lorentzian metric, there is no guarantee it ends up on the manifold.
→ will talk about Lorentzian metric later.

SUMMARY
- maps between manifolds induce maps between cotangent bundles
- $\psi^* f = f \circ \psi$; $\psi_* X(f) = X(\psi^* f)$; $\psi^*\eta(X) = \eta(\psi_* X)$
  ($\psi : M \to N$, $X \in T_pM$, $f : N \to \mathbb{R}$, $\eta \in T^*_{\psi(p)}N$.)
- extend to general tensor
- If $\psi_* : T_pM \to T_{\psi(p)}N$ injective, ψ is an IMMERSION
- If ψ is an immersion, N riemannian manifold, then $\psi^* g$ is a metric on M



(r,s) tensors (notes)

$\psi^* S_{\mu_1 \dots \mu_s} = \left(\frac{dy^{\alpha_1}}{dx^{\mu_1}}\right)_p \cdots \left(\frac{dy^{\alpha_s}}{dx^{\mu_s}}\right)_p S_{\alpha_1 \dots \alpha_s}$

$\psi_* T^{\alpha_1 \dots \alpha_r} = \left(\frac{dy^{\alpha_1}}{dx^{\mu_1}}\right)_p \cdots \left(\frac{dy^{\alpha_r}}{dx^{\mu_r}}\right)_p T^{\mu_1 \dots \mu_r}$

$(\psi^* g)_{\mu\nu} = \left(\frac{dy^\alpha}{dx^\mu}\right)\left(\frac{dy^\beta}{dx^\nu}\right) g_{\alpha\beta}$

Lecture 13

8.11.24

Exercise: Let $(N,g) = (\mathbb{R}^3, \delta)$, $M = S^2$. Let ψ be the map taking a point on S^2 w/ spherical coordinates (θ, ϕ) to $(x^1, x^2, x^3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ then

$$\psi^*\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right) = d\theta^2 + \sin^2\theta\, d\phi^2 ?$$

→ Find $(\psi^* g)_{\theta\theta} = 1$, $(\psi^* g)_{\theta\phi} = 0$ and $(\psi^* g)_{\phi\phi} = \sin^2\theta$ → $(\psi^* g) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$ ✓

[eg this: we took the standard metric δ on $\mathbb{R}^3$, pulled it back via the embedding to the sphere & recovered the familiar spherical metric $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$]

If you can do this you understand last lecture

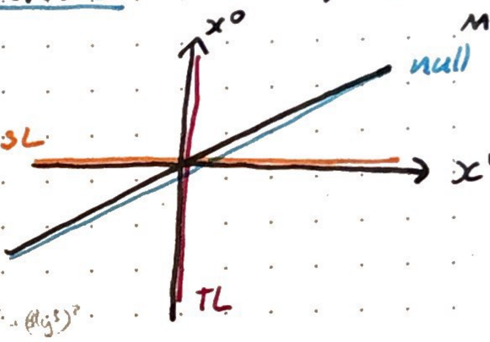
If ψ immersion, (N,g) is Lorentzian, then $\psi^* g$ is not in general a metric on M. There are 3 important cases:

* $\psi^* g$ is a Riemannian metric $\Rightarrow \psi(M)$ is spacelike
* $\psi^* g$ is a Lorentzian metric $\Rightarrow \psi(M)$ is timelike
* $\psi^* g$ is everywhere degenerate $\Rightarrow \psi(M)$ is null

MINKOWSKI

$\psi: (y^1, y^2, y^3) \mapsto (0, y^1, y^2, y^3)$

$\psi(\mathbb{R}^3)$ is spacelike.



$(\psi^* g) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \Rightarrow ds^2 = (dy^1)^2 + (dy^2)^2 + (dy^3)^2$ riemannian ⇒ spacelike

(⇒ $\dim M = \dim N$)?

Recall that $\psi: M \to N$ is a diffeomorphism if it is bijective with a smooth inverse (& is smooth). If we have a diffeomorphism we can push forward a general (r,s)-tensor at p to an (r,s)-tensor at $\psi(p)$ by

notes: if have a diffeomorphism we can extend our definitions of push forward & pull back to apply to a general tensor

$$\psi_* T(\eta^1, \dots, \eta^r, X_1, \dots, X_s) = T(\psi^*\eta^1, \dots, \psi^*\eta^r, \psi_*^{-1} X_1, \dots, \psi_*^{-1} X_s)$$

$\forall\, \eta_1, \dots, \eta_r \in T^*_{\psi(p)} N$

$\forall\, X_1, \dots, X_s \in T_{\psi(p)} N$

Define a pull-back by $\psi_*^{-1} = \psi^*$

notes: push forward of (1,1) tensor components $(\psi_* T)^\mu{}_\nu = \left(\frac{\partial y^\mu}{\partial x^\rho}\right)\left(\frac{\partial x^\sigma}{\partial y^\nu}\right) T^\rho{}_\sigma$

If M, N are diffeomorphisms, we often don't distinguish between them we can think of $\psi: M \to M$

(ψ an isometry means $\psi_* g$ is a symmetry ($\psi_* g = g$))

We say a diffeomorphism $\psi: M \to M$ is a symmetry of T if $\psi_* T = T$. If T is the metric, we say ψ is an Isometry. e.g in Minkowski space w/ an inertial frame

$$\psi(x^0, x^1, \dots, x^n) = (x^0 + 1, x^1, \dots, x^n) \text{ is a symmetry of } g.$$

An important class of diffeomorphisms are those generated by a vector field.

If X is a smooth vector field, we associate to each point $p \in M$ the point $\psi_t^X(p) \in M$ given by flowing a parameter distance t along reminder to self: integral curve of a vector field is curve where the tangent to the curve is equal to the vector field at that point

note to self: summary
$\Psi_t^X : M \to M$
* flows distance t along the integral curve of X
* from $p \in M$ to $\Psi_t^X(p) \in M$ (defined for small enough t)
* Theorem shows Ψ_t^X is a diffeomorphism

the integral curve of X starting at p.

Suppose $\Psi_t^X(p)$ is well defined for all $t \in I \subset \mathbb{R}$ for each $p \in M$ then $\Psi_t^X : M \to M$ is ~~well defined~~ a diffeomorphism for all $t \in I$.

reminder: bijective, smooth, smooth inverse

Further

* If $t, s, t+s \in I$ then $\boxed{\Psi_t^X \circ \Psi_s^X = \Psi_{t+s}^X}$ and $\Psi_0^X = id$ $(*)$
* If $I = \mathbb{R}$ this gives $\{\Psi_t^X\}_{t \in \mathbb{R}}$ the structure of a one parameter abelian group. (i.e. Ψ_t^X well defined for all $t \in \mathbb{R}$) 'don't worry what this is if you don't know'
* If Ψ_t is any smooth family of diffeomorphisms satisfying $(*)$ we can define a vector field by
$$X(p) = \frac{d}{dt}(\Psi_t(p))\Big|_{t=0} \quad \text{then } \Psi_t = \Psi_t^{X}$$

(note to self: if we satisfy $(*)$ then $\Psi_t^X(p)$ is a curve through p parametrised by t so we define a vector field to be the tangent to this curve at p by taking the d/dt of it)

We can use Ψ_t^X to compare tensors at different points as $t \to 0$ this gives a new notion of derivative.

(note to self (cont): integral curve X – only defined one curve at a time (defined for one point p at a time); not a diffeomorphism; can't pullback tensors. Flow Ψ_t^X – defined for all points p so can recover entire vector field from the flow; is a diffeomorphism; can)

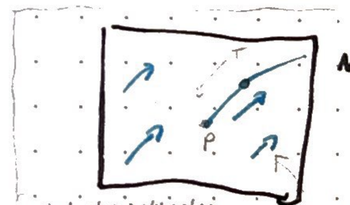
THE LIE DERIVATIVE

Suppose $\Psi_t^X : M \to M$ is the smooth one-parameter family of diffeomorphisms generated by a vector field X.

DEFINITION: For a tensor field T, the lie derivative of T with respect to X is

$$(\mathcal{L}_X T)_p = \lim_{t \to 0} \frac{((\Psi_t^X)^* T)_p - T_p}{t}$$

(note to self: can also phrase in terms of pushforward $(\Psi_{-t})_* = (\Psi_t)^*$)



* start at p & flow along integral curve to point distance t away.
* take tensor field at new point & pull it back to p
* then can compare it with T_p since both tensors are at the same point so live in the same tensor space
* take lim as $t \to 0$

vector field defined on any manifold

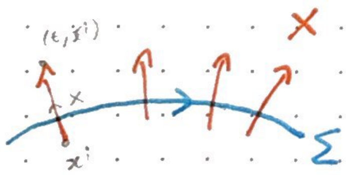
It's easy to see that the constant α, β and (r,s) tensors S, T

$$\mathcal{L}_X(\alpha S + \beta T) = \alpha \mathcal{L}_X S + \beta \mathcal{L}_X T$$

(plug into def)

To see how $\mathcal{L}_X$ acts in components, it's helpful to construct coordinates adapted to X.

X nowhere tangent to Σ



$i = 1, \dots, n-1$

Near p we can construct an $(n-1)$-surface Σ which is transverse to X (i.e. nowhere tangent). Pick coords x^i on Σ and assign the coordinate (t, x^i) to the point a parameter distance t along the integral curve of X ~~satisfying~~ starting at x^i on Σ.

In these coords, $X = \frac{\partial}{\partial t}$ and $\Psi_t^X(\tau, x^i) = (\tau + t, x^i)$.

So if $y^\mu = (\Psi_t^X)^*(x^\mu)$ then $\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu$ and

(if Ψ_t^X takes the point w/ coords $x^\mu = (\tau, x^i)$ to point w/ coords $y^\mu = (\tau+t, x^i)$ then $\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu$)

$$\left[(\Psi_t^X)^* T\right]^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\Big|_{(\tau, x^i)} = T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\Big|_{(\tau+t, x^i)}$$

general: $\Psi_t^* T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}|_p = \left(\frac{\partial x^{\mu_1}}{\partial y^{\rho_1}}\right) \cdots \left(\frac{\partial y^{\sigma_s}}{\partial x^{\nu_s}}\right) T^{\rho_1 \dots \rho_r}{}_{\sigma_1 \dots \sigma_s}|_{\Psi_t(p)}$ → with $\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu$ → $\Psi_t^* T^{\mu_1 \dots}{}_{\nu_1 \dots}|_p = T^{\mu_1 \dots}{}_{\nu_1 \dots}|_{\Psi_t(p)}$

$\left(\lim_{t\to 0} \frac{T(t+t, x_i) - T(t, x_i)}{t} \quad \frac{\partial T}{\partial t}\right)$

thus $(\mathcal{L}_X T)^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\big|_p = \frac{\partial T^{\mu_1 \dots \mu_r}}{\partial t}{}_{\nu_1 \dots \nu_s}\big|_p$

So in these coords, $\mathcal{L}_X$ acts on components by $\frac{\partial}{\partial t}$.
In particular we immediately see

$\mathcal{L}_X$ obeys leibniz:

① $\cdot$ $\mathcal{L}_X(S \otimes T) = (\mathcal{L}_X S) \otimes T + S \otimes \mathcal{L}_X T$ ② $\cdot$ $\mathcal{L}_X$ commutes w/ contraction

e.g. $\left(\mathcal{L}_X(T(X,Y)) = (\mathcal{L}_X T)(X,Y) + T(\mathcal{L}_X X, Y) + T(X, \mathcal{L}_X Y)\right)$ ← i think?

To write $\mathcal{L}_X$ in a coordinate-free fashion. We can simply seek a basis independent expression that agrees with $\mathcal{L}_X$ in these coords.

$\boxed{\text{E.g.}}$ * For a function $\mathcal{L}_X f = \frac{\partial f}{\partial t} = X(f)$ in these coords

(note: $\mathcal{L}_X f = X(f)$, both sides are scalars so this must be verified in any basis)

* For a vector field Y we observe that

$$(\mathcal{L}_X Y)^\mu = \frac{\partial Y^\mu}{\partial t} = X^\sigma \frac{\partial}{\partial x^\sigma}(Y^\mu) - Y^\sigma \frac{\partial}{\partial x^\sigma} X^\mu = [X,Y]^\mu$$

(note this only follows since in these coords $X = \frac{\partial}{\partial t}$ so $X^\mu = \delta^\mu_t$, components are const so this vanishes)

point basis to compute ← $\therefore \mathcal{L}_X Y = [X,Y]$

note: basis indep result

(note: if 2 vectors have the same components in one basis then they are equal in all bases so)

Exercise: • In any coord basis show if ω_a is a covector field,

$$(\mathcal{L}_X \omega)_\mu = X^\sigma \partial_\sigma \omega_\mu + \omega_\sigma \partial_\mu X^\sigma$$

if ∇ is torsion free

$$(\mathcal{L}_X \omega)_a = X^b \nabla_b \omega_a + \omega_b \nabla_a X^b$$

($(\mathcal{L}_X \omega)_\mu = X^\sigma \partial_\sigma \omega_\mu + \omega_\sigma \partial_\mu X^\sigma$; work in normal coords & ∇ torsion free $\Rightarrow \Gamma = 0$ so $= X^\sigma \nabla_\sigma \omega_\mu + \omega_\sigma \nabla_\mu X^\sigma$ tensor eq so promote to ATN//)

(think like we did for ∇: leibniz: compute $\mathcal{L}_X(\omega(Y)) = (\mathcal{L}_X \omega)(Y) + \omega(\mathcal{L}_X Y)$; LHS is $\mathcal{L}_X g$ ugh; 1st term is what we want; 2nd term is $\mathcal{L}_X$ of vector $= [X,Y]$ compute in coords)

(attempt: $(\mathcal{L}_X \omega)_\mu = X(\omega_\mu) = \frac{\partial \omega_\mu}{\partial t} = \frac{\partial x^\sigma}{\partial t}\frac{\partial}{\partial x^\sigma}\omega_\mu = X^\sigma \partial_\sigma \omega_\mu$; 2nd term vanishes in coords; if they have same components in one basis they have same components in all bases//)

• If g_{ab} is a metric tensor, ∇ levi-civita, then

$$(\mathcal{L}_X g)_{ab} = \nabla_a X_b + \nabla_b X_a$$

(similarly consider $\mathcal{L}_X(g(Y,Z))$; find $(\mathcal{L}_X g)_{\mu\nu} = X^\delta \partial_\delta g_{\mu\nu} + g_{\mu\delta}\partial_\nu X^\delta + g_{\delta\nu}\partial_\mu X^\delta$; work in normal coords: $\partial_\delta g_{\mu\nu}$ vanishes; promote to ∇ (since $\Gamma = 0$ again) and use the fact ∇ commutes w/ metric $\Rightarrow (\mathcal{L}_X g)_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu$ ⇒ tensor eq so promote to ain//)

If ψ^X_t is a one-parameter family of isometries for a manifold with metric g, then $\mathcal{L}_X g = 0$.

Conversley, if $\mathcal{L}_X g = 0$ then X generates a one-parameter family of isometries.

(consider: isometry $\psi^*_t g = g$ (ψ_t is a symmetry of the metric))

$\boxed{\text{DEFINITION:}}$ A vector field K satisfying $\mathcal{L}_K g = 0$ is called a KILLING VECTOR. It satisfies Killing's equation

$$\nabla_a K_b + \nabla_b K_a = 0 \qquad (\nabla \text{ levi-Civita})$$

(use above $\partial_t g_{ab} = \nabla_a X_b + \nabla_b X_a$ →)

$\boxed{\text{LEMMA:}}$ Suppose K is killing and $\lambda: I \to M$ is a geodesic of the Levi-Civita connection. Then $g_{ab}\dot\lambda^a K^b$ is const. along λ.

(tangent to geodesic satisfies: $\nabla_X X$ assuming affinely parametrised?)

$\boxed{\text{PROOF:}}$ $\frac{d}{dt}(K_b \dot\lambda^b) = \dot\lambda^a \nabla_a (K_b \dot\lambda^b) = \underbrace{(\nabla_a K_b)}_{0 \text{ (K killing)}} \dot\lambda^a \dot\lambda^b + K_b \underbrace{\dot\lambda^a \nabla_a \dot\lambda^b}_{\text{vanishes for geodesic}}$

$\square$.

($\frac{d}{dt} = \dot\lambda$ since ... $X(f) = \nabla_X f$ so $\dot\lambda(f) = \nabla_{\dot\lambda} f = \dot\lambda^a \nabla_a$)

(this is symmetric in a,b so only $\nabla_a K_b$ symmetric part left which vanishes since killing?)

(basically this says geometry ... new einstein eq.)

note to self: we assume ∇ is Levi-Civita here? not sure

note to self:
(pseudo-Riemannian (M,g): for any $p \in M$ we can introduce normal coordinates s.t. in a neighbourhood of p, $g_{\mu\nu} = \eta_{\mu\nu}$ + higher order terms, so locally looks like Minkowski (to 1st order))
?

Lecture 14

11.11.24

Physics in Curved SpaceTime

Minkowski space (special relativity)

notes: begin w/ special relativity. In an inertial frame, Levi-Civita ∇_μ is the same as ∂_μ, so to promote our spec. rel. equations so they hold for an arbitrary basis, we replace ∂_μ with ∇_μ and replace Greek indices w/ Latin ones.

We review physical theorems in Minkowski $\mathbb{R}^{1+3}$ equipped with $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,)$ we set $c=1$.

(enlarge range of validity of equations: (1) $\partial_\mu \to \nabla_\mu$ (2) $\mu \to a$)

(1) $\boxed{\text{Klein-Gordon equation}}$

$$\partial_\mu \partial^\mu \phi - m^2 \phi = 0 \quad (a)$$

Note that in inertial cords $\partial_\mu = \nabla_\mu$ so we can write this in a covariant manner as

promote partial to nabla

$$\nabla_a \nabla^a \phi - m^2 \phi = 0$$

((3) if also replace η_{ab} with arbitrary metric g_{ab} (taking ∇ to be Levi-Civita connecting the metric) then we get eq. valid in an arbitrary spacetime)

Associate to (a) is the energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - \tfrac{1}{2}\eta_{\mu\nu}(\partial_\sigma \phi \, \partial^\sigma \phi + m^2\phi^2)$$

or covariantly

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \tfrac{1}{2}\eta_{ab}(\nabla_c \phi \nabla^c \phi + m^2 \phi^2)$$

This satisfies $T_{ab} = T_{ba}$, (check) $\nabla_a T^a{}_b = 0$

(2) $\boxed{\text{Maxwell's eq}^{\underline{n}}}$ equations

The Maxwell field is an anti-symmetric $(0,2)$-tensor

$F_{\mu\nu} = -F_{\nu\mu}$ where $F_{0i} = E_i$, $F_{ij} = \varepsilon_{ijk} B_k$ $(i,j = 1,2,3)$

If j_μ is the charge current density, then Maxwell's eq$^{\underline{n}}$ are

$$\left.\begin{aligned} \partial_\mu F^\mu{}_\nu &= 4\pi j_\nu & \qquad \nabla_a F^a_c &= 4\pi j_c \\ \partial_{[\mu} F_{\nu\sigma]} &= 0 & \qquad \nabla_{[a} F_{bc]} &= 0 \end{aligned}\right\} (b)$$

Associated to (b) is the energy-momentum tensor

$$T_{\mu\nu} = F_\mu{}^\sigma F_{\nu\sigma} - \tfrac{1}{4}\eta_{\mu\nu} F_{\sigma\tau}F^{\sigma\tau}$$

$$\to T_{ab} = F_a^c F_{bc} - \tfrac{1}{4}\eta_{ab} F_{cd} F^{cd}$$

Check $T_{ab} = T_{ba}$, $\nabla_a T^a{}_b = 0$ for source-free Maxwell

'use $\nabla_{[a} F_{bc]}$' and $F_{ab} = -F_{ba}$ (did this on paper 2024?)

(3) $\boxed{\text{Perfect Fluid}}$

A perfect fluid is described by a local velocity field U^μ

normalize (idealised fluid (no viscosity), perfectly described by P, ρ and a 4-velocity)

satisfying $U^\mu U_\mu = -1$, together with a pressure P and density ρ. They satisfy the first law of thermodynamics

$$U^\mu \partial_\mu \rho + (\rho + P)\partial_\mu U^\mu = 0$$

covariant eq

$$\rightarrow U^a \nabla_a \rho + (\rho + P)\nabla_a U^a = 0$$

and Euler's equation (read as conserving momentum law)

$$(\rho + P)U^\nu \partial_\nu U^\mu + \partial^\mu P + U^\mu U^\nu \partial_\nu P = 0$$

see cosmology course we deal w/ this

fluid velocity: moves along w/ fluid

$$\rightarrow (P+\rho)U^b \nabla_b U^a + \nabla^a P + U^a U^b \nabla_b P = 0$$

because if I set a=0 I get nothing; essentially 2 eq. need eq of state) to find unknowns (3 deg eq. ...)

Associated is the energy-momentum tensor

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + P\eta_{\mu\nu}$$

(check)

$$\rightarrow T_{ab} = (\rho + P)U_a U_b + P\eta_{ab} \qquad \text{again } T_{ab} = T_{ba},\ \nabla_a T^a{}_b = 0$$

(g_{ab} in lecture notes? mistake?)

Notice that in all these cases, we can, if we wish, promote the Minkowski η to a general Lorentzian metric g, and take ∇ to be the Levi-Civita connection. Consider normal coordinates, we see that coords exist near any point $q \in M$, such that the physics described is approximately Minkowskian, with corrections of the order of curvature.

notes say do this ex. sheet 2 (estimate correction terms)

minimal coupling approach: we could have eg. extra term to T_{ab} that just happen to vanish in minkowski: we don't know if these exist! but simplest approach is to say they don't.

GENERAL RELATIVITY

In Einstein's theory of general relativity we postulate that spacetime is a 4-dimensional Lorentzian manifold (M, g). We also require any matter model to consist of

equations

* some matter fields Φ°
* eq$^{\underline{n}}$ of motion for Φ° which are expressed geometrically in terms of g ($+\nabla, R \cdots$)
* an energy-momentum tensor T_{ab} depending on Φ° satisfies $T_{ab} = T_{ba}$, $\nabla_a T^a{}_b = 0$

The matter should reduce to a non-grav theory when (M, g) is fixed to be Minkowski.

The metric g should satisfy the Einstein equations

LHS geometric

case of vacuum RHS = 0

Einstein knew RHS he wanted divergence free; needed to add term. Λg_{ab} to get there

$$R_{ab} - \frac{R}{2}g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}$$

(view: RHS energy matter, ... energy of matter depends on geometry ... gravitation)

Λ cosmological constant ($\Lambda > 0$ but small), G Newton's constant.

(view: for our purposes, ignore Λ from now onwards)

The Einstein eq$^{\underline{n}}$ together with the EoMs for Φ°, constitute a coupled system which must be solved simultaneously.

note: at point p on the trajectory of a test body (free particle), we can introduce a local inertial frame. equivalence principle states in this frame, any experiments should be indistinguishable from if idid the same thing in an inertial frame in minkowski so. ⇒ the acceleration of the curve at p: $\nabla_X X$ must vanish as it does in minkowski

(nothing so the curve must be a geodesic → motivates the geodesic postulate!)

(notes: geodesics of the levi civita connection)

GEODESIC POSTULATE

Free test particles move along timelike/null geodesics if they have non zero/zero rest mass.

view einstein eq as evolution equation

Gauge Freedom

notes: we have a gauge freedom/gauge symmetry if there exist distinct configurations that are physically equivalent. a gauge transformation maps a field configuration to a physically equivalent configuration.

diffeomorphisms are gauge transformations in GR

Consider Maxwell with no sources

$$\partial_\mu F^\mu{}_\nu = 0 \quad \text{— m1}$$

$$\partial_{[\mu} F_{\nu\sigma]} = 0 \quad \text{— m2}$$

dont worry about proofs of 2, claude doesnt write them

a standard approach to solve m2 is to introduce the gauge potential A_μ s.t. $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$, then m1 becomes

$$\partial_\mu \partial^\mu A_\nu - \partial_\mu \partial_\nu A^\mu = 0 \quad (*)$$

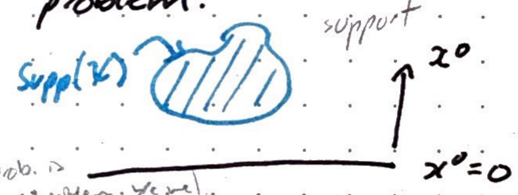
We'd like to solve $(*)$ given data at $\{x^0 = 0\} = \Sigma$. However this eq$^=$ doesnt give a good evolution problem.

(want to specify initial data then solve for A_μ uniquely at a time. but cant b/c of gauge)

(initial data at surface Σ)

(i.e. a smooth fn on $\mathbb{R}^{1+3}$ that vanishes near Σ)

If $\chi \in C^\infty(\mathbb{R}^{1+3})$ which vanishes near Σ, then $\tilde{A}_\mu = A_\mu + \partial_\mu \chi$ will also solve $(*)$ and $\partial_{[\mu} \tilde{A}_{\nu]} = \partial_{[\mu} A_{\nu]}$.



(initial value prob. is not a well-posed problem. we don't have uniqueness of solutions!)

(gauge transformations does not change the observable quantity $F_{\mu\nu}$)

To resolve this, we can fix a gauge e.g. if we assume $\partial_\mu A^\mu = 0$ then $(*)$ becomes

(choose gauge condition)

$$\partial_\mu \partial^\mu A_\nu = 0 \quad (*)$$

a wave eq$^=$ for each cpt of A_ν.

component

(i.e.: wave eq for each component of A_ν with unique solutions given by specifying A_μ and its first time derivative on Σ (unique sols. to initial value problem))

L15:

$\partial_\mu g_{\nu\rho} \neq 0$ in general since metric may change in space

	what g commutes with?
∇_μ	yes (for levi civita)
∂_μ	no (unless g is const.)
Γ	yes
tensor contraction	yes

A

Prove: $R_{\rho\sigma\mu\nu} = \frac{1}{2}(g_{\rho\nu,\mu\sigma} + g_{\sigma\mu,\nu\rho} - g_{\rho\mu,\nu\sigma} - g_{\sigma\nu,\mu\rho}) - \Gamma_{\mu\lambda\rho}\Gamma^\lambda_{\nu\sigma} + \Gamma_{\nu\rho}\Gamma^\lambda \ldots$

① start w expression for $R^\mu{}_{\nu\rho\sigma}$ in terms of connection components.

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma}$$

② lower μ, remember g commutes w/ Γ but not ∂_ρ! so last 2 terms look like $\Gamma^\tau_{\nu\sigma}\Gamma_{\kappa\tau\rho}$

③ but 1st and 2nd term look like $g_{\mu\kappa}\partial_\rho \Gamma^\mu_{\nu\sigma}$ → need to find this

$$\rightarrow g_{\mu\kappa}\partial_\rho \Gamma^\mu_{\nu\sigma} = \underbrace{\partial_\rho(g_{\mu\kappa}\Gamma^\mu_{\nu\sigma})}_{\partial_\rho \Gamma_{\kappa\nu\sigma}} - g_{\mu\kappa,\rho}\Gamma^\mu_{\nu\sigma}$$

find this in terms of Γ: use expr for Γ in terms of g, and lower index:

$$\rightarrow \Gamma_{\nu\kappa\mu} = \frac{1}{2}(g_{\kappa\nu,\mu} + g_{\mu\kappa,\nu} - g_{\mu\nu,\kappa})$$

→ to isolate $g_{\mu\kappa,\nu}$ symmetrise over 2nd two indices ($\kappa\mu$) (would equally work for 1st two)

this gives $g_{\mu\kappa,\nu} = \Gamma_{\nu\kappa\mu} + \Gamma_{\nu\mu\kappa}$

④ plug in $\partial_\rho \Gamma$ terms and simplify, gives

$$R_{\kappa\nu\rho\sigma} = \partial_\rho \Gamma_{\nu\kappa\sigma} - \partial_\sigma \Gamma_{\nu\kappa\rho} - \Gamma_{\rho\mu\kappa}\Gamma^\mu_{\nu\sigma} + \Gamma_{\sigma\mu\kappa}\Gamma^\mu_{\nu\rho}$$

⑤ find $\partial_\rho \Gamma_{\nu\kappa\sigma}$ from our expression for $\Gamma_{\nu\kappa\mu}$ above & plug that in //

B Prove $R_{\sigma\nu} = -\frac{1}{2} g^{\mu\rho} g_{\sigma\nu,\mu\rho} + \frac{1}{2}\partial_\sigma \Gamma^\mu_{\mu\nu} + \frac{1}{2}\partial_\nu \Gamma^\mu_{\mu\sigma} - \ldots$ etc.

①. $R_{\sigma\nu} = g^{\tau\nu} R_{\tau\sigma\nu\rho}$, plug in our expression in [A]

the tricky bit is the $\frac{1}{2}(g_{\rho\nu,\mu\sigma} + g_{\sigma\mu,\nu\rho} - g_{\rho\mu,\nu\sigma})$ term

② we found before → $g_{\rho\nu,\mu\sigma} = \partial_\sigma(\Gamma_{\mu\rho\nu} + \Gamma_{\mu\nu\rho})$

so? $g_{\sigma\mu,\nu\rho} = \partial_\nu(\Gamma_{\rho\sigma\mu} + \Gamma_{\rho\mu\sigma})$

$\rightarrow g_{\rho\mu,\nu\sigma} = \frac{1}{2}\partial_\nu(\Gamma_{\sigma\rho\mu} + \Gamma_{\sigma\mu\rho}) + \frac{1}{2}\partial_\sigma(\Gamma_{\nu\rho\mu} + \Gamma_{\nu\mu\rho})$

(fiddly: i want to cancel terms so recognise symmetries in $\nu\sigma$ without ∂_ν, ∂_σ and rearrange two terms)

③ combine & simplify then note: if i plug this into our main expr for $R_{\sigma\nu}$ it is contracted w/ $g^{\tau\nu}$ so all the γ factor half terms vanish since they are antisymmetric in τ,ν

④ ... $g^{\mu\rho}(\partial_\sigma \Gamma_{\mu\nu\rho})$, notice this $= \partial_\sigma \Gamma_{\mu\nu}{}^\mu - \Gamma_{\mu\nu\rho}\partial_\sigma g^{\mu\rho}$

⑤ final piece is to figure out $\partial_\sigma g^{\mu\rho}$: i know from [A] $g_{\mu\kappa,\nu}$ ... with that $\partial_\sigma \delta^\mu_\nu = 0 \Rightarrow \partial_\sigma g_{\nu\tau} g^{\mu\tau} = 0$ to find $\partial_\sigma g^{\mu\rho} = -g^{\mu\kappa}\partial_\sigma g_{\kappa\tau} g^{\tau\rho}$

 ⑥ put everything together! //

Lecture 15 13.11.24

Recap: Maxwell $\partial_\mu F^\mu{}_\nu = 0$, $\partial_{[\mu} F_{\nu\sigma]} = 0$. Let $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$ then M1 $\Rightarrow$ $\boxed{\Box A_\nu - \partial_\mu \partial_\nu A^\mu = 0}$ $(*)$ ($\Box = \partial_\mu \partial^\mu$). But cannot solve this as good evolution problem. For example, $\tilde{A}_\mu = A_\mu + \partial_\mu \chi$ also solves $(*)$ with some F and some initial data $A_\nu|_{t=0}$, $\partial_0 A_\nu|_{t=0}$ (provided $\mathrm{supp}(\lambda) \cap \{t=0\} = \emptyset$). To resolve, we fix a gauge via $\partial_\mu A^\mu = 0$. Then $(*) \Rightarrow \Box A_\nu = 0$ $(\dagger)$ which has unique solution given $A_\nu|_{t=0}, \partial_0 A_\nu|_{t=0}$.

CLAIM: There is no non-trivial χ s.t. $\tilde{A}_\nu$ also satisfies $\partial_\mu \tilde{A}^\mu = 0$ and the same initial condition. If we solve $(\dagger)$ for data s.t.

$$\partial_\mu A^\mu|_{t=0}, \quad \partial_0(\partial_\mu A^\mu)|_{t=0}$$

then we find $\partial_\mu A^\mu \equiv 0$, and so our solution solves $(*)$ and hence Maxwell's equations

Gauge Freedom for Einstein

If (M,g) solves Einstein equations with E/M tensor T (energy-momentum tensor) and $\psi: M \to M$ a diffeomorphism, then $\psi^* g$ solves the Einstein equations with E/M tensor $\psi^* T$ (energy-momentum). At a local level, this arises (manifests) as the coordinate ~~indices~~ independence/invariance of the Einstein equations.

In order to solve EEs, we need to find a way to fix coordinates. There are several approaches, we consider wave/harmonic gauge.

(note to self: fixing harmonic coords turns vacuum EE into a well-posed evolution problem, show this ↓)

LEMMA: In any local coordinate system

(Riemann tensor)

$$R_{\rho\sigma\mu\nu} = \tfrac{1}{2}(g_{\rho\nu,\mu\sigma} + g_{\sigma\mu,\rho\nu} - g_{\rho\mu,\nu\sigma} - g_{\sigma\nu,\mu\rho}) - \Gamma_{\mu\lambda\rho}\Gamma^\lambda{}_{\nu\sigma} + \Gamma_{\nu\lambda\rho}\Gamma^\lambda{}_{\mu\sigma}$$

(if manifold, like this is the wave operator, (curved wave operator))

$$\text{and } R_{\sigma\nu} = \underbrace{-\tfrac{1}{2} g^{\mu\rho} g_{\sigma\nu,\mu\rho}}_{g^{\mu\rho}\partial_\mu\partial_\rho g_{\sigma\nu}} + \tfrac{1}{2}\partial_\sigma \Gamma^\mu{}_{\mu\nu} + \tfrac{1}{2}\partial_\nu \Gamma^\mu{}_{\mu\sigma} - \Gamma^\mu{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} + \Gamma_{\lambda\tau\nu}\Gamma^{\lambda\tau}{}_\sigma + \Gamma_{\lambda\tau\nu}\Gamma^{\tau\lambda}{}_\sigma + \Gamma_{\lambda\tau\sigma}\Gamma^{\tau\lambda}{}_\nu$$

PROOF: on moodle

This form of Ricci is well-adapted to wave/harmonic coordinates.

(note: choosing coords fixes our gauge (choosing coords is gauge fixing in this context))

$\underline{\nu}$ means don't treat as vector component

Suppose we choose a system of coords $\{x^{\underline{\nu}}\}$ $\underline{\nu} = 0,1,2,3$ which satisfy the wave equation $\nabla_\mu \nabla^\mu x^{\underline{\nu}} = 0$

(idea: choose coords s.t. each coordinate satisfies the wave equation)

$$\Rightarrow 0 = \nabla^\mu(\partial_\mu x^{\underline{\nu}}) = \underbrace{\partial^\mu \partial_\mu x^{\underline{\nu}}}_{=0} + \Gamma^{\mu\sigma}{}_\mu \underbrace{\partial_\sigma x^{\underline{\nu}}}_{\delta_\sigma^{\underline{\nu}}}$$

(note to self: $x^{\underline{\nu}}$ is scalar not the components of a vector!) so $\nabla_\mu x^{\underline{\nu}} = \partial_\mu x^{\underline{\nu}}$ then $\partial_\mu x^{\underline{\nu}}$ is covector so we can take ∇^μ as we would a vector)

(since 2nd deriv of scalar is zero)

$\nabla^\mu \omega_\nu = \omega_\nu \ldots + \Gamma^{\mu\sigma}{}_\nu \omega_\sigma$

order $\Gamma^{\mu\nu}{}_{\mu}$

the three terms in blue vanish!

$$\Rightarrow \Gamma^{\mu\nu}{}_{\mu} = 0 \iff \tfrac{1}{2} g^{\mu\sigma}\left(g_{\mu\kappa,\sigma} - \tfrac{1}{2} g_{\mu\sigma,\kappa}\right) = 0$$

For such coordinates,

$$R_{\sigma\nu} = -\tfrac{1}{2} g^{\mu\rho} g_{\sigma\nu,\mu\rho} + \underbrace{\left(\Gamma_{\lambda\tau\nu}\Gamma^{\lambda\tau}{}_{\sigma} + \Gamma_{\lambda\tau\nu}\Gamma^{\tau\lambda}{}_{\sigma} + \Gamma_{\lambda\tau\sigma}\Gamma^{\tau\lambda}{}_{\nu}\right)}_{\equiv P_{\sigma\nu}(g,\partial g)}$$

(g)et $P_{\sigma\nu}$ is a nonlinear function in g and 1st derivative

(g)ot vacuum einstein equation (no matter/energy so $T_{ab}=0$ so EE reduces to $R_{ab}=0$ → solutions ascribe spacetime curvature generated purely by gravity affecting itself)

locally means in time by locality

analogous to what we did w/ Maxwell eq

So $R_{\sigma\nu} = 0$ reduces to a system of nonlinear wave equations in this gauge. We can solve this (locally) given initial data. Further, we can show that if the gauge condition is initially satisfied, it will remain time $\forall (t,x)$ (Choquet-Bruhat '54) prop in this gauge (wave coords)

(i.e. if gauge condition holds initially it will continue to hold under time evolution)

(g)ot the point is, $R_{\sigma\nu}$ reduces to a form where the 2nd derivs of the metric appear linearly so vacuum EE becomes $\Box_g g_{\sigma\nu} = P_{\sigma\nu}$ F non-linear terms i.e. non linear wave eq

Linearised Theory

Suppose we are in a situation where the gravitational field is weak. We expect to be able to describe the nature as a pertubation of Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}, \quad \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$$

$\varepsilon \ll 1$ is a small parameter and we work to $O(\varepsilon)$ $\left(\begin{matrix}\text{i.e. neglect}\\ O(\varepsilon^2)\end{matrix}\right)$.

If the metric has this form we say we're in "almost inertial" coordinates. One can check

$$g^{\mu\nu} = \eta^{\mu\nu} - \varepsilon h^{\mu\nu}, \quad \text{where } h^{\mu\nu} = \eta^{\mu\sigma} h_{\sigma\tau}\eta^{\tau\nu}.$$

(minus sign!)

note to self: show that $g_{\mu\lambda} g^{\lambda\nu} = \delta^{\nu}_{\mu}$ holds to 1st order: make an ansatz: $g^{\mu\nu} = \eta^{\mu\nu} + \varepsilon k^{\mu\nu}$ ⇒ want $k^{\mu\nu} = -\eta^{\mu\rho} h_{\rho\lambda}\eta^{\lambda\nu}$ for identity to hold to first order in perturbations

i.e. for $O(\varepsilon)$ quantities can raise / lower with η.

Suppose our metric is also in wave gauge, then

I think this is just the gauge condition?

$$0 = g^{\mu\sigma}\left(g_{\mu\kappa,\sigma} - \tfrac{1}{2} g_{\mu\sigma,\kappa}\right) = \varepsilon\,\partial^{\mu}\left(h_{\mu\kappa} - \tfrac{1}{2}\eta_{\mu\kappa} h\right), \quad h = \eta^{\mu\nu} h_{\mu\nu}$$

Using our expression for the Ricci tensor

need also stress energy tensor to order ε

$$R_{\mu\nu} = -\tfrac{\varepsilon}{2}\eta^{\sigma\tau}\partial_{\sigma}\partial_{\tau} h_{\mu\nu}$$

note to self: $\Gamma\Gamma$ terms are second order in perturbation so vanish; then any term w/ $\partial_\rho \eta_{\mu\nu}$ vanishes since minkowski constant

In order for Einstein equations to hold, we ~~would~~ must have

relabel to avoid confusion with E-mom tensor — (will remain)

$$T_{\mu\nu} = \varepsilon T_{\mu\nu}$$

note to self: EEs $R_{ab} - \tfrac{R}{2} g_{ab} = 8\pi G T_{ab}$ → $R = R^{\lambda}{}_{\lambda} = g^{\kappa\lambda} R_{\kappa\lambda}$ just plug in our expressions for $R_{\mu\nu}$ (remember η commutes w/ ∂_ρ)

then to order $O(\varepsilon)$ in the Einstein equations

$$-\tfrac{1}{2}\Box h_{\mu\nu} + \tfrac{1}{4}\eta_{\mu\nu}\eta^{\sigma\tau}\partial_{\sigma}\partial_{\tau} h = 8\pi G T_{\mu\nu}$$

$(h_{\mu\nu}\eta^{\mu\nu})$

will usually set $G=1$ by choice of coords

note to self: get $R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = \varepsilon\left(-\tfrac{1}{2}\eta^{\sigma\tau}\partial_\sigma\partial_\tau h_{\mu\nu} + \tfrac{1}{4}\eta^{\sigma\tau}\eta_{\mu\nu}\partial_\sigma\partial_\tau h\right) = 8\pi G T_{\mu\nu}$

since LHS is 1st order $T_{\mu\nu}$ must be 1st order in order for EE to hold so we rename it as $T_{\mu\nu} = \varepsilon T_{\mu\nu}$

so to 1st order in ε, EE becomes: $-\tfrac{1}{2}\eta^{\sigma\tau}\partial_\sigma\partial_\tau h_{\mu\nu} + \tfrac{1}{4}\eta^{\sigma\tau}\eta_{\mu\nu}\partial_\sigma\partial_\tau h = 8\pi G T_{\mu\nu}$ //

usually set $G=1$

$\Box = \eta^{\sigma\tau}\partial_\sigma\partial_\tau$

trace reversed metric perturbation

$$\Rightarrow \boxed{\begin{aligned} \Box \bar{h}_{\mu\nu} &= -16\pi G T_{\mu\nu} \\ \partial_\mu \bar{h}^{\mu}{}_{\nu} &= 0 \end{aligned}}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$

$\Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}$

gauge condition

initially say we impose this but it looks like it follows from gauge fix our coords. not. I can't figure it out?

These are the linearised Einstein equations in a wave (aka harmonic, Lorenz, de Donder) gauge. We can solve these given initial data at $\{t=0\} = \Sigma$.

If the (initial) data satisfies $\partial_\mu \bar{h}^{\mu}{}_{\nu}|_\Sigma = \partial_0(\partial_\mu \bar{h}^{\mu}{}_{\nu})|_\Sigma = 0$, then since $\partial_\mu T^{\mu}{}_{\nu} = 0$, we have

$$\Box(\partial_\mu \bar{h}^{\mu}{}_{\nu}) = 0 \quad \text{so gauge holds for all times.}$$

gauge condition holds for all time.

note: this is a condition on the EM tensor apparently → energy is conserved?

Linearised gauges

(remember we can treat 2 spacetimes (M,g) and (M,g') as equiv if there is a diffeomorphism?)

Suppose we have a physically equivalent solution not necessarily in wave gauge. We've seen that 2 such equivalent solutions are related by a diffeomorphism.

In order that the diffeomorphism reduces smoothly to the identity as $\varepsilon \to 0$, it should take the form Φ^{ξ}_{ε} for some vector field ξ. some vector field def. on Minkowski

ψ^{ξ}_{ε} ← flows the param distance ε along integral curve ξ

If S is any tensor field, then from the definition of Lie derivative

$\psi^{\xi *}_{\varepsilon} S = S + \varepsilon \mathcal{L}_\xi S$

(from $\mathcal{L}_X T|_p = \lim_{t\to 0} \frac{\psi^{X*}_t T|_p - T_p}{t}$)

$$(\Phi^{\xi}_{\varepsilon})^* S = S + \varepsilon \mathcal{L}_\xi S + O(\varepsilon^2).$$

change in object is lie derivative of ξ times the object

(covariant deriv is same as partial deriv at order ε since Christoffel symbols already at $O(\varepsilon)$.)

In particular

$$* \quad ((\Phi^{\xi}_{\varepsilon})^* \eta)_{\mu\nu} = \eta_{\mu\nu} + \underbrace{\varepsilon(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)}_{h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu} + O(\varepsilon^2).$$

$((\psi^{\xi}_{\varepsilon})^*\eta)_{\mu\nu} = \eta_{\mu\nu} + \varepsilon(\mathcal{L}_\xi \eta)_{\mu\nu}$ use $(\mathcal{L}_X g)_{ab} = \nabla_a X_b + \nabla_b X_a$

(then $\nabla \to \partial$ since $\varepsilon\Gamma$ term are $O(\varepsilon)$?) (note: must Γ provide $\varepsilon \xi \nu \varepsilon \Gamma$ ... $O(\varepsilon)$)

(i.e. in the unperturbed spacetime)

(L16)

So if $S = O(\varepsilon)$ it will be invariant under gauge transformations to $O(\varepsilon)$. Tensors vanishing on Minkowski are gauge invariant in linear pertubation theory. In particular $T_{\mu\nu}$ is gauge invariant.

(so e.g. EM tensor is gauge inv. as it's $O(\varepsilon)$ as we saw from Einstein eqs)

However, *, this $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ represents a linear gauge transformation (c.f. $A_\mu \to A_\mu + \partial_\mu t$).

summary (added later for L16 intro): If we have a physically equivalent solution, not necessarily in wave gauge, then it should be related to any other such solution by a diffeomorphism Φ^{ξ}_{ε} for some vector field ξ. From def of lie derivative for any tensor field S: $(\Phi^{\xi}_{\varepsilon})^* S = S + \varepsilon \mathcal{L}_\xi S + O(\varepsilon^2)$ (change to order ε in our tensor is $\varepsilon \mathcal{L}_\xi S$). So, if $S = O(\varepsilon)$, it will be invariant under gauge transformations to $O(\varepsilon)$. Tensors vanishing on Minkowski are gauge invariant in linear perturbation.

(most things we are interested in will vanish in Minkowski e.g. $T_{\mu\nu}$ but one important thing that doesn't vanish is the metric itself!)

(reminder $G=c=1$)

Lecture 16

15.11.24

note: motivation in course structure: gauge fix then linearize, why not linearize then gauge fix? → we know doing it the first way we have our gauge comes from a gauge condition that makes sense in the full non-linear problem

(recall back the three terms we sacked off (but one is ?? so only 2 extra terms)?

From formulae in last lecture (or on Moodle), if we linearise $R_{\mu\nu}$ without fixing a gauge, we find

$$R_{\mu\nu} = \varepsilon\left(\partial^\rho \partial_{(\mu} h_{\nu)\rho} - \tfrac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \tfrac{1}{2}\partial_\mu\partial_\nu h\right)$$

Substituting in $h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ we can check $R_{\mu\nu}$ vanishes, so $h_{\mu\nu}$ solves the vaccum Einstein equations. We call such a solution a pure gauge solution

note to self (right) → gauge fixing is choosing a rule to eliminate redundancy → pure gauge sol. is one that can be transformed away by gauge fixing, it is an unphysical mode

EXERCISE: Show that if $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$ then

(total perturbation, "actual" physical perturbation, unphysical gauge perturbation; see p.48)

$$\partial_\mu \bar{h}'^{\mu}{}_{\nu} = \partial_\mu \bar{h}^{\mu}{}_{\nu} + \partial_\mu \partial^\mu \xi_\nu$$

(reminder $\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2} h \eta_{\mu\nu}$)

Deduce that:

* Any linearised perturbation can be put into wave gauge by a suitable gauge transformation
* Any pure gauge solution of the wave gauge fixed equations with $\xi_\mu|_{t=0} = 0$, $\partial_0 \xi_\mu|_{t=0} = 0$ vanishes everywhere

attempt (right): to ensure wave gauge we want $\partial_\mu \bar{h}'^\mu{}_\nu = 0$, this happens if $\Box\xi_\nu = -\partial_\mu \bar{h}^\mu{}_\nu$ which always has a solution for linear pert. so we can always find a vector ξ that transforms us to wave gauge

pure gauge sol. $h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$ today becomes $\partial_\mu \bar{h}^\mu{}_\nu = \Box\xi_\nu$, so for it in wave gauge requires $\Box\xi_\nu = 0$: with these ICs the solution is $\xi_\nu = 0$ everywhere ⇒ $h_{\mu\nu} = 0$. pure gauge sol that satisfies wave gauge & has trivial ICs vanishes everywhere!

(i.e. only the residual freedom in the wave gauge comes down to the gauge ... initial time ...)

(e.g. tests of whether this is a good theory)

don't cover some stuff like light bending etc. here, because more suited to be a 2nd course in GR

THE NEWTONIAN LIMIT

We'd expect that if GR is a good theory of gravity, we should be able to recover Newton's theory of gravitation in the limit where fields are weak and matter is slowly moving in comparison with the speed of light ($c=1$ in our units).

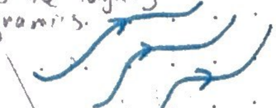
(need source to plug into LHS of eq: choose fluid)

Let us suppose that matter is modelled by a perfect fluid with velocity field U^a, density ρ and pressure p.

In all but the most extreme situations, $\rho \gg p$ (in standard units $P/\rho \approx V_s^2 \ll c^2$).

(integral curves $\frac{\partial}{\partial t}$ correspond to flow lines thinking of the light Lagrangian coords in fluid dynamics)

We choose coordinates such that $\boxed{U^a = \frac{\partial}{\partial t}}$. [Think of as Lagrangian coordinates for fluid c.f. Mach's principle.]



Note that this does not imply the fluid is at rest: distances are measured with the METRIC.

(we are moving with the perturbed metric not Minkowski metric so effectively time dependence of fluid is put in the metric rather than the fluid)

The condition that the fluid moves non-relativistically becomes the assumption that we are in the weak field limit

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad (h_{\mu\nu} = O(\varepsilon))$$

(drop ε for ease of notation, just let $h_{\mu\nu}$ mean $\varepsilon h_{\mu\nu}$)

also want to require that the motions of the particles are not too dramatic: small relative velocities & accelerations

many ways for reasons we will see later on; not a single way to do this; choose powers of ε more 'natural'

time derivatives are small ⟹ velocities & accelerations are small

$$\partial_0 h_{\mu\nu} \sim \varepsilon^{1/2} h_{\mu\nu}$$
$$\partial_0\partial_0 h_{\mu\nu} \sim \varepsilon h_{\mu\nu}$$

(reminder to self: $h_{\mu\nu}$ is $O(\varepsilon)$ so $\partial_0\partial_0 h_{\mu\nu} \sim O(\varepsilon^2)$)

For consistency require $\rho = O(\varepsilon)$ and $p = O(\varepsilon^2)$.

The linearised Einstein equations become

(mixing various scales here; don't worry about this too much)

$$\partial^\mu \partial_\mu \bar{h}_{\sigma\tau} = -16\pi T_{\sigma\tau}$$

$$T_{00} = \rho, \quad T_{0i} = T_{ij} = 0 \quad (\text{to } O(\varepsilon^2)).$$

Deduce that

(since condition on time derivs $\partial_0, \partial_0\partial_0$; 2nd time deriv is $O(\varepsilon^2)$, so $\partial^\mu\partial_\mu \bar{h}_{00} = -16\pi\rho$ ⟹ $\nabla^2 \bar{h}_{00} = -16\pi\rho$)

3D Laplacian ∇^2

$$\Delta \bar{h}_{00} = -16\pi\rho$$

$$\bar{h}_{0i}, \bar{h}_{ij} \text{ vanish at } O(\varepsilon)$$

Since

$\bar{h} = \bar{h}_{\mu\nu}\eta^{\mu\nu}$ (trace)

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\bar{h} \qquad \bar{h} = -\bar{h}_{00}$$

$$\Rightarrow h_{00} = \tfrac{1}{2}\bar{h}_{00}$$

(second the first non-trivial order?)

$$\therefore \quad \Delta\left(-\frac{h_{00}}{2}\right) = 4\pi\rho \qquad (\text{c.f. } \Delta\phi = 4\pi G\rho)$$

($\Delta = \nabla^2$)

N's theory doesn't just tell us potential from a mass, it also tells us how particles move under potentials so we need to go further

Suggests we identify $-\frac{h_{00}}{2}$ with the Newtonian potential ϕ.

(its own field is small enough that we can ignore)

Consider a test particle in this background. By geodesic postulate its motion is determined by the Lagrangian.

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$$
$$= \eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$
$$= -\dot{t}^2 + |\dot{\underline{x}}|^2 + h_{00}\dot{t}^2 + O(\varepsilon^2)$$

$$\cdot \equiv \frac{\partial}{\partial\tau}$$

$$x^\mu = (t, \underline{x})$$

Suppose motion is non-relativistic, so $|\dot{\underline{x}}|^2 = O(\varepsilon)$. Conservation of $\mathcal{L}$ gives,

(timelike geodesic, Minkowski metric)

$$-\dot{t}^2 = -1 + O(\varepsilon) \Rightarrow \dot{t} = 1 + O(\varepsilon)$$

note to self:
by def of proper time:
$\frac{d\tau}{d\tau} = 1 = -g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\mathcal{L}$
$\Rightarrow -1 = -\dot{t}^2 + |\dot{\underline{x}}|^2 + h_{00}\dot{t}^2 + O(\varepsilon^2)$
since $|\dot{\underline{x}}|^2$, h_{00} are $O(\varepsilon)$
$\Rightarrow \dot{t}^2 = 1 + O(\varepsilon) \Rightarrow \dot{t} = 1 + O(\varepsilon)$
(think $\dot{t} = (1 + O(\varepsilon))^{1/2}$)
$\Rightarrow \mathcal{L} = -\dot{t}^2 + |\dot{\underline{x}}|^2 + h_{00}\dot{t}^2 + O(\varepsilon^2)$
re: EL eq. since $\dot{t}$ = const + $O(\varepsilon)$, treat it as a constant and remember $h_{00}(x)$

x^i Euler-Lagrange equation:

note to self: this is $\dot{t}$ not $\dot{x}$? for some reason?

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i}$$

$$2\frac{d}{d\tau}(\dot{x}_i) = 2\ddot{x}_i = h_{00,i}\dot{t}^2$$

note to self: can see this since by def of τ → $\frac{d\tau}{d\tau} = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$, $\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt}$

Since $\dot{t} = 1 + O(\varepsilon)$, $\frac{d}{dt} = \frac{d}{d\tau} + O(\varepsilon)$, we have

$$\frac{d^2x^i}{dt^2} = \tfrac{1}{2}h_{00,i} + O(\varepsilon^2) = \partial_i\phi + O(\varepsilon^2)$$

WE'VE RECOVERED NEWTON'S LAWS OF GRAVITATION!

note to self: $\ddot{x} = -\nabla\phi$?

GRAVITATIONAL WAVES

One of the most spectacular recent results in gravitational physics was the measurement in 2015 of gravitational waves sourced by two colliding black holes.

Near the source, the field is <u>not</u> weak, but by the time we detect the waves, the weak field approximation is relevant.

* from p 46

we know that under diffeomorphism i.e. gauge transformation metric perturbation transforms as

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \quad \text{(from defn of lie deriv.)}$$

so we want to find out how $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$ transforms, so plug in transformed $h_{\mu\nu}$

$$\bar{h}_{\mu\nu} \to h'_{\mu\nu} - \tfrac{1}{2} h'_{\rho\sigma} \eta^{\rho\sigma} \eta_{\mu\nu}$$

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial^\rho \xi_\rho \leftarrow \text{(correction term)?}$$

so now want to find $\partial^\mu \bar{h}'_{\mu\nu}$: just plug in $\bar{h}'_{\mu\nu}$

$$\partial^\mu \bar{h}'_{\mu\nu} = \partial^\mu \bar{h}_{\mu\nu} + \partial^\mu \partial_\mu \xi_\nu + \cancel{\partial^\mu \partial_\nu \xi_\mu} - \cancel{\partial_\nu \partial^\mu \xi_\mu}$$

terms cancel! $\Rightarrow \quad \partial^\mu \bar{h}'_{\mu\nu} = \partial^\mu \bar{h}_{\mu\nu} + \partial^\mu \partial_\mu \xi_\nu$ //

**

If ξ_ν obeys wave gauge then plug in $\partial_\mu \xi_\nu$, $\partial_\nu \xi_\mu$, $\partial^\rho \xi_\rho$ to our expression for $\bar{h}'_{\mu\nu}$:

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + (X_\nu k_\mu + X_\mu k_\nu) e^{ik^\sigma x_\sigma} - \eta_{\mu\nu} X^\rho k_\rho e^{ik^\sigma x_\sigma}$$

so since $\bar{h}_{\mu\nu} = H_{\mu\nu} e^{ik^\sigma x_\sigma}$, we can see gauge freedom acts on $H_{\mu\nu}$ as:

$$H_{\mu\nu} \to H'_{\mu\nu} = H_{\mu\nu} + X_\nu k_\mu + X_\mu k_\nu - \eta_{\mu\nu} X^\rho k_\rho$$

Lecture 17

GW propagate in vaccum: we want to solve $\Box \bar{h}_{\mu\nu} = 0$

18.11.24

convenient gauge for grav. waves discussion

Seek a propagating wave solution to the vaccum linearised Einstein equations in wave gauge

$$\Box \bar{h}_{\mu\nu} = 0 \qquad \partial_\mu \bar{h}^\mu{}_\nu = 0$$

Making the Ansatz

$$\bar{h}_{\mu\nu} = Re\left(H_{\mu\nu} e^{ik^\mu x_\mu}\right)$$

plane wave propagating in direction of k^μ

(note to self: sloppy notation? should be σ ok?)

(reminder: in minkowski vector is null if $\eta(x,x)=0$)

(note to self: $k^\mu k_\mu = 0 \Rightarrow \omega^2 = k^2$ so $v = \frac{\omega}{k} = 1$?)

(note: suppress Re from now on)

symmetric $H_{\mu\nu} = + H_{\nu\mu}$ const.

k^μ const.

plug its in

We see that this is a solution if

(say k is in z direction wlog)

call direction wave travel in z direction

(i got $-H_{\mu\nu} k^\sigma k_\sigma e^{ik\cdot x}$ from $\Box \bar{h}_{\mu\nu}=0$, from this $k^\sigma k_\sigma = 0$) $\rightarrow$ $k^\mu k_\mu = 0 \Rightarrow k^\mu$ is <u>null</u>: waves travel at speed of light

(note to self: from null condition, if k in z dir. $\to -k_0 k_0 + k_3 k_3 = 0 \to k_0 = k_3$ so $k^\mu = \omega(1,0,0,1)$)

from gauge condition $\rightarrow$ $k_\mu H^\mu{}_\nu = 0$ take $k^\mu = \omega(1,0,0,1)$ wlog

(note to self: $\Rightarrow$ waves are transverse, i.e. $H_{\mu\nu}$ has no component along k^μ so no perturbation in the direction of propagation)

There is still some residual gauge freedom: recalling that the diffeomorphism generated by ξ acts on $\partial^\mu \bar{h}_{\mu\nu}$ by

$$* \quad \partial^\mu \bar{h}_{\mu\nu} \rightarrow \partial^\mu \bar{h}_{\mu\nu} + \partial_\mu \partial^\mu \xi_\nu .$$

notes: the condition $\partial_\mu \bar{h}^\mu{}_\nu = 0$ does not eliminate all gauge freedom: we still have residual $\partial_\mu\partial^\mu \xi_\nu$

If $\partial_\mu \partial^\mu \xi_\nu = 0$, then gauge condition is preserved. Let

(if ξ_ν obeys wave eq then we preserve the gauge condition $\partial_\mu \bar{h}^\mu{}_\nu = 0$)

(since k_μ null)

$$\xi_\nu = Re\left(-iX_\nu e^{ik^\mu x_\mu}\right) \quad \text{then} \quad \partial_\mu \partial^\mu \xi_\nu = 0.$$

(complex no. represents phase shift)

corrects for the fact H is $\bar{h}$ thing not h thing

And (check)

(how residual gauge freedom acts on H)

$$** \quad H_{\mu\nu} \rightarrow H_{\mu\nu} + k_\mu X_\nu + k_\nu X_\mu - \eta_{\mu\nu} k_\sigma X^\sigma$$

use up remaining freedom to further set $H_{0\mu}=0$ & $H_\mu{}^\mu = 0$

<u>Exercise</u> * show that if we take $X_0 = 0$, $X_i = -\frac{H_{0i}}{k_0}$

then we can set $\boxed{H_{0\mu} = 0}$

forces / can do gauge trans. that forces $H_{0\mu}$ to vanish

(note to self: idk? how to do this? show that $H_{0\mu} \to 0$, use wlog $k^\mu = \omega(1,0,0,1)$ and $H_{\mu\nu}$ not $h_{\mu\nu}$ = 0?)

* Making a further transformation of the form

does not mess up previous fixes: $H_{0\mu}$ still vanishes, and also H is traceless

$\rightarrow H_\mu{}^\mu \to H_\mu{}^\mu - 2k_\mu X^\mu$ (minkowski $\eta^{\mu\nu}\eta_{\mu\nu}=4$!) $= H_\mu{}^\mu - 2\alpha(k_0 k_0 - k^i k_i)$

$$X_0 = \alpha k_0 \; , \; X_i = -\alpha k_i$$

(got $\alpha = \frac{1}{\ldots H_\mu{}^\mu}$??) ??? (*)

we're effectively 'used up' dof to get constraints

show we can additionally impose $\boxed{H_\mu{}^\mu = 0}$ $\Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu}$

"traceless" condition

i think $\bar{h} = 0 \Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu}$ then show this implies $\bar{h} = 0$ so $h_{\mu\nu} = \bar{h}_{\mu\nu}$

Since $H_{0\mu} = 0$ and $H_{\nu\mu} k^\nu = \omega(H_{0\mu} + H_{3\mu}) = 0$, we deduce $H_{3\mu} = 0$. With symmetry + tracelessness of $H_{\mu\nu}$ we have

one term equal to minus other term (check) on diag?

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$H_+, H_\times$ are constants corresponding to two independent polarisations of the wave.

$H_{0\mu} = 0$, $H_{3\mu} = 0$ rows are 0 (columns?) then fill in the rest by symmetry & tracelessness

what does this solution mean physically?

traceless means sum of stuff on main diagonal is zero

can't just consider single particle in free fall -> will always be freely falling
instead need to consider collection of test particles all moving in a similar direction

To understand the consequences of such a gravitational wave, we recall the geodesic deviation equation. If λ is a geodesic with tangent T, then a vector Y joining λ to a nearby geodesic satisfies

(geodesic deviation eq. $\nabla_T \nabla_T Y = R(T,Y)T$)

haven't specified coords yet: imagine I am freely falling particle & set up my coords as follows

$$T^a \nabla_a (T^b \nabla_b Y^c) = R^c{}_{dab} T^d T^a Y^b.$$

Suppose a freely falling observer sets up a frame consisting of $e_0 = T$ (tangent to observer's worldline), together with three spacelike vectors e_1, e_2, e_3 which are parallelly transported along the worldline and initially satisfy $g(e_\mu, e_\nu) = \eta_{\mu\nu}$. Since parallel transport preserves inner products, we can assume $\{e_\mu\}$ is O/N for all time. Since $T^a \nabla_a (e_\alpha) = 0$.

"meant to write α, β here but it doesn't really matter."

Geodesic equation implies

(parallel transport $\nabla_T (e_\alpha) = 0$)

basis vectors e_α

$= e^c{}_\alpha R_{cdab} T^d T^a Y^b = e^c{}_\alpha R_{cdab} e^d_0 e^a_0 Y^b$ (using $e_0 = T$)

$$T^a \nabla_a (T^b \nabla_b (Y_c e^c_\alpha)) = R_{abcd} e^a_\alpha e^b_0 e^c_0 Y^d.$$

$\rightarrow$ ($\nabla_T (Y_\alpha) = T(Y_\alpha) = \frac{\partial Y_\alpha}{\partial \tau}$ then apparently $\frac{dY_\alpha}{d\tau}$ (right things) is also scalar so $\nabla_T (\frac{dY_\alpha}{d\tau}) = T(\frac{dY_\alpha}{d\tau}) = \frac{d^2 Y_\alpha}{d\tau^2}$?)

Now $Y_c e^c_\alpha =: Y_\alpha$ is a scalar, so equation becomes

take components along curve wrt parallely transported local coords: effectively observer's local frame: evolution is governed by this eq

of stuff see frame?

$$\frac{d^2 Y_\alpha}{d\tau^2} = R_{abcd} e^a_\alpha e^b_0 e^c_0 Y^\beta e^d_\beta.$$

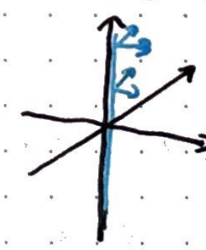
minkowski observer falls/curve where sits at origin for all time bring with me set of vectors that i parallel transp. which is local inertial frame

here: raise & lower α, β indices w/ $\eta_{\alpha\beta}$.

For our problem of a gravitational wave spacetime, the Riemann curvature is $O(\varepsilon)$ so we only need e_α to leading order. We can assume $e_0 = \partial_t$, $e_i = \partial_i$ and λ is $t \mapsto (t,0,0,0)$

τ

(notationally the α is to show the $e^a{}_\alpha$ etc are vectors while α labels the basis vectors - so the components are $e^a_\alpha = \frac{dx^a}{dx^\alpha} = \delta^a_\alpha$, $e^b_0 = \frac{dx^b}{dx^0} = \delta^b_0$ etc.?)



both the same - doesn't imply but sort to be consistent

Then

$$\frac{d^2 Y_\alpha}{d\tau^2} = R_{\alpha 0 0 \beta} Y^\beta + O(\varepsilon^2).$$

note (to order ε t & τ use interchangeably)

To $O(\varepsilon)$ we have

$$R_{\rho\sigma\mu\nu} = \tfrac{1}{2}(h_{\rho\nu,\mu\sigma} + h_{\sigma\mu,\nu\rho} - h_{\rho\mu,\nu\sigma} - h_{\sigma\nu,\mu\rho})$$

remember $H_{0\mu} = 0 \Rightarrow \bar{h}_{0\nu} = 0$, and $H_\mu{}^\mu = 0 \Rightarrow \bar{h}_{\mu\nu} = h_{\mu\nu}$ so we have $h_{0\mu} = 0$ (21))

$\rightarrow$ but since $h_{0\mu} = 0$, we find

$$R_{\alpha 0 0 \beta} = \tfrac{1}{2} h_{\alpha\beta,00}$$

so

by defining wrt this moving observer? Interpret that depends on perturbation?

$$\frac{d^2 Y_\alpha}{dt^2} = \frac{1}{2} \frac{\partial^2 h_{\alpha\beta}}{\partial t^2} Y^\beta$$ (used $t = \tau$ to $O(\varepsilon)$).

Let's consider the + POLARISATION: