GENERAL RELATIVITY

office hours 8.40 (20 mins before lecture)

Prof Claude Warnick (Printed notes by Prof Harvey Reall)
↳ also look at Tony notes (although slightly diff.)

General relativity is our best theory of gravitation on the largest scales.
It is:

- CLASSICAL: No quantum effects
- GEOMETRICAL: space + time are combined in a curved spacetime
- DYNAMICAL: In contrast to Newton's theory of gravity, Einstein's gravitational field has its own non-trivial dynamics

Differentiable Manifolds

The basic object of study in differential geometry is the (differentiable) manifold. This is an object which locally looks like $\mathbb{R}^n$ and has enough structure to let us do calculus.

DEF: A differentiable manifold of dimension n is a set M, together with a collection of coordinate charts (O_α, ϕ_α) where

- $O_\alpha \subset M$ are subsets of M such that $\bigcup_\alpha O_\alpha = M$
- ϕ_α is a bijective map (one-to-one and onto) from O_α to U_α, an open subset of $\mathbb{R}^n$
- If $O_\alpha \cap O_\beta \neq \emptyset$ (their intersection is not empty / emptyset), then $\phi_\beta \circ \phi_\alpha^{-1}$ is a smooth (infinitely differentiable) map from $\phi_\alpha(O_\alpha \cap O_\beta) \subset U_\alpha$ to $\phi_\beta(O_\alpha \cap O_\beta) \subset U_\beta$

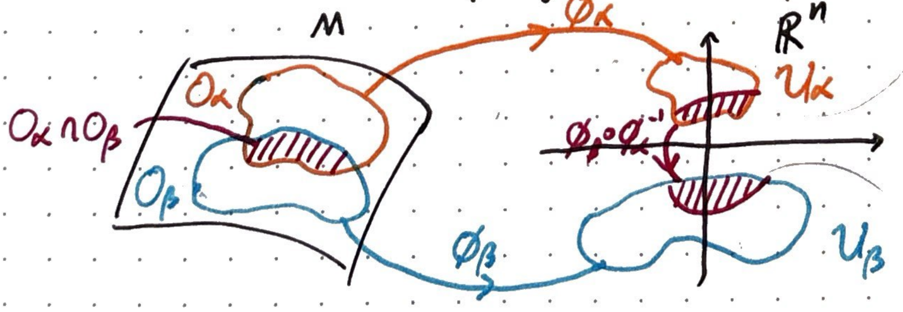


image of $O_\alpha \cap O_\beta$ when mixed with ϕ_α

image of $O_\alpha \cap O_\beta$ when I apply ϕ_β

condition: map between these regions ϕ_α inverse then ϕ_β I want to be smooth (shortcut the loop)

one-to-one and onto: each point from LHS maps to exactly one point on RHS

REMARKS:

- Could replace smooth with finite differentiability (e.g. k-times differentiable)
- The charts define a topology on the original manifold M: $\mathcal{R} \subset M$ is open iff $\phi_\alpha(\mathcal{R} \cap O_\alpha)$ is open in $\mathbb{R}^n$ for all α (curly R) (don't worry too much about this) (think this is maybe just point of interest for those who have done topology)
- Every open subset of M is itself a manifold (Restrict charts to $\mathcal{R}$)

The collection $\{(\mathcal{O}_\alpha, \phi_\alpha)\}$ is called an <u>atlas</u>. Two atlases are <u>compatible</u> if their union is an atlas.

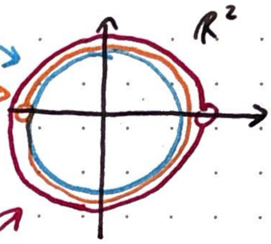
An atlas A is <u>maximal</u> if there exists no atlas B with $A \subsetneq B$ (subset but not equal to). Every atlas is contained to a maximal atlas (consider the union of all compatible atlases). We can assume without loss of generality that we work with a maximal atlas.

<u>EXAMPLES</u>

1. If $\mathcal{U} \subset \mathbb{R}^n$ is open, we can take $\mathcal{O} = \mathcal{U}$, $\phi(x^1, \dots, x^n) = (x^1, \dots, x^n)$ $\quad \phi : \mathcal{O} \to \mathcal{U}$ $\quad \{(\mathcal{U}, \phi)\}$ is an atlas

(2D sphere?)

2. $S^1 = \{ p \in \mathbb{R}^2 \mid |p| = 1 \}$



(the circles are on top of each other)

diff. coordinate charts using angles around the circle

if $p \in S^1 \setminus \{(-1,0)\} = \mathcal{O}_1$, there is a unique $\theta_1 \in (-\pi, \pi)$ s.t. $p = (\cos\theta_1, \sin\theta_1)$ — open interval (excluding $\pi, -\pi$)

open interval

If $p \in S^1 \setminus \{(1,0)\} = \mathcal{O}_2$, then there is a unique $\theta_2 \in (0, 2\pi)$ s.t. $p = (\cos\theta_2, \sin\theta_2)$ (minus/not including that point)

$\phi_1 : p \to \theta_1 \qquad p \in \mathcal{O}_1 \qquad \mathcal{U}_1 = (-\pi, \pi)$

$\phi_2 : p \to \theta_2 \qquad p \in \mathcal{O}_2 \qquad \mathcal{U}_2 = (0, 2\pi)$

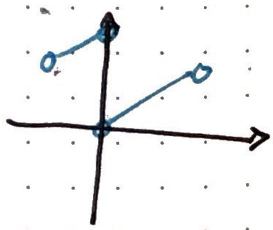
exercise

$\phi_1(\mathcal{O}_1 \cap \mathcal{O}_2) = (-\pi, 0) \cup (0, \pi)$ (open set)

transition function defined on this set

$$\phi_2 \circ \phi_1^{-1}(\theta) = \begin{cases} \theta & \theta \in (0, \pi) \\ \theta + 2\pi & \theta \in (-\pi, 0) \end{cases}$$

(composite function? 1 then 2?)

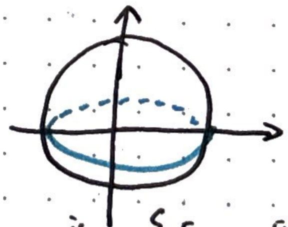


smooth where defined similarly for $\phi_1 \circ \phi_2^{-1}$

S^1 is a manifold.

higher dimensional sphere

3. $S^n = \{ \underline{p} \in \mathbb{R}^{n+1} \mid |\underline{p}| = 1 \}$



Define charts by stereographic projection, if $\{\underline{E}_1, \dots, \underline{E}_{n+1}\}$ is a standard basis for $\mathbb{R}^{n+1}$ and $\{\underline{e}_1, \dots, \underline{e}_n\}$ the basis for $\mathbb{R}^n$, write

$\underline{p} = p^1 \underline{E}_1 + \dots + p^{n+1} \underline{E}_{n+1}$

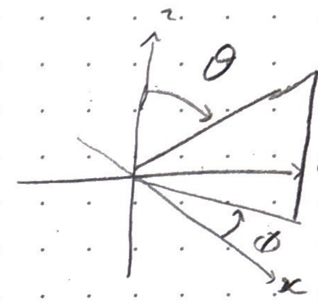
point on surface of sphere

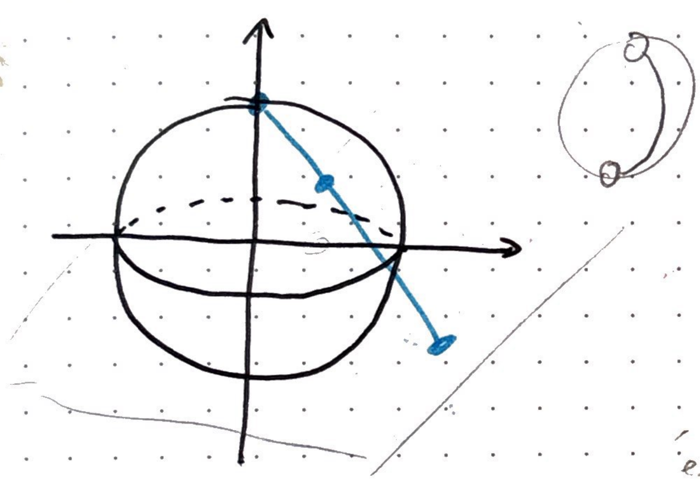
set $O_1 = S^n \setminus \{\underline{E}_{n+1}\}$ (sphere minus north pole)

$\varphi_1(\underline{p}) = \frac{1}{1-p^{n+1}}(p^1\underline{e}_1 + \dots + p^n\underline{e}_n)$

$O_2 = S^n \setminus \{-\underline{E}_{n+1}\}$

$\varphi_2(\underline{p}) = \frac{1}{1+p^{n+1}}(p^1\underline{e}_1 + \dots + p^n\underline{e}_n)$





1st map takes point & projects onto plane transverse to $n+1$ direction through the north pole

Other map does the same through the south pole

'excluding the origin'

Claim $\phi_1(O_1 \cap O_2) = \mathbb{R}^n \setminus \{\underline{0}\}$ and $\phi_2 \circ \phi_1^{-1}(\underline{x}) = \frac{\underline{x}}{|\underline{x}|^2}$

smooth on $\mathbb{R}^n \setminus \{\underline{0}\}$

similar for $\phi_1 \circ \phi_2^{-1}$, S^n is an n-manifold

well $\Phi_1(\underline{p}) = \underline{x}$, then we see $\Phi_2 \circ \Phi_1^{-1}(\underline{x}) = \frac{1-p^{n+1}}{1+p^{n+1}}\underline{x}$

then since point p on n-sphere has unit norm $\rightarrow (p^1)^2 + \dots + (p^n)^2 = 1-(p^{n+1})^2$.

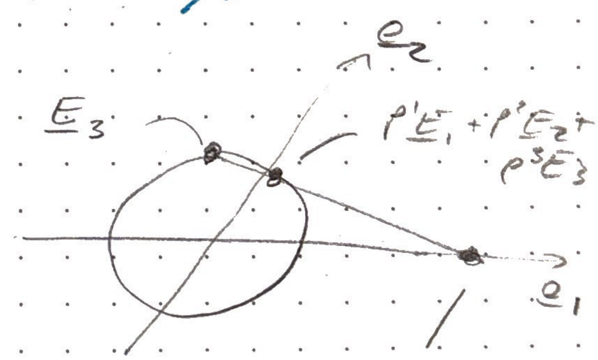
and so $|\underline{x}| = \frac{(p^1)^2 + \dots + (p^n)^2}{(1-p^{n+1})^2} = \frac{1-(p^{n+1})^2}{(1-p^{n+1})^2} = \frac{1+p^{n+1}}{1-p^{n+1}}$

so $\Phi_2 \circ \Phi_1^{-1}(\underline{x}) = \frac{\underline{x}}{|\underline{x}|^2}$

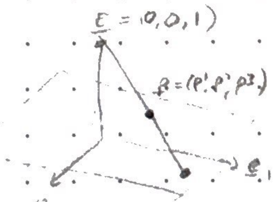
which is smooth

$\underline{p} = p^1\underline{E}_1 + p^2\underline{E}_2 + p^3\underline{E}_3$

$\psi_1(\underline{p}) = \frac{1}{1-p^3}(p^1\underline{e}_1 + p^2\underline{e}_2)$



$\frac{1}{1-p^3}(p^1\underline{e}_1 + p^2\underline{e}$

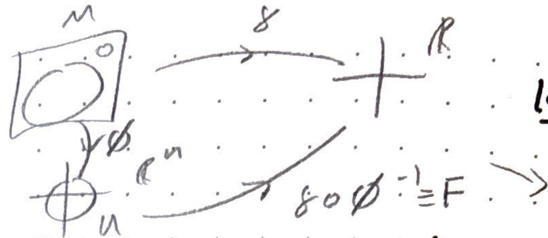


$(x,y,z) = (0,0,1) + \lambda(p^1, p^2, p^3-1)$

$z = 0$

$\Rightarrow x = \frac{1}{1-p^3}p^1, \; y = \frac{1}{1-p^3}p^2$

so $\psi(p) = \frac{1}{1-p^3}(p^1\underline{e}_1 + p^2\underline{e}_2)$



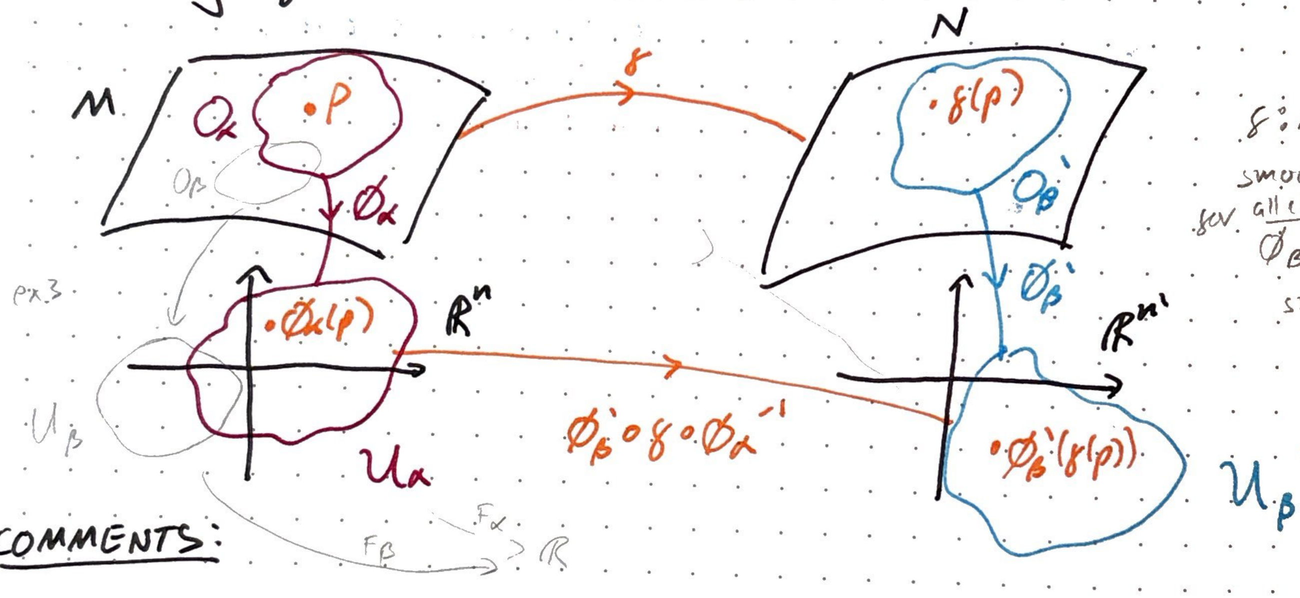
$f: M \to \mathbb{R}$ smooth iff for all ~~[illegible]~~ charts ϕ says $F \equiv f \circ \phi^{-1} : U \to \mathbb{R}$ is smooth
all in [illegible]

Lecture 2

14.10.24

Smooth functions on manifolds

Suppose M, N are manifolds of dim n, n' respectively
let $f: M \to N$
Then let $p \in M$ and pick charts (O_α, ϕ_α) for M and (O'_β, ϕ'_β) for N with $p \in O_\alpha$, $f(p) \in O'_\beta$. Then $\phi'_\beta \circ f \circ \phi_\alpha^{-1}$ maps an open neighbourhood of $\phi_\alpha(p)$ in $U_\alpha \subset \mathbb{R}^n$ to $U'_\beta \subset \mathbb{R}^{n'}$
If this function is smooth for all possible choices of chart, and p
we say $f: M \to N$ is smooth



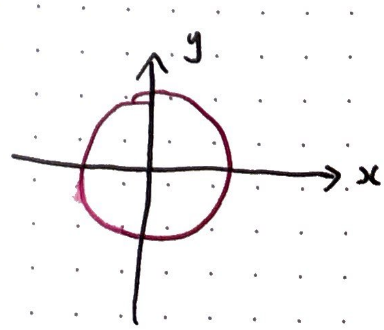
$f: M \to N$ smooth if for all choices of chart $\phi'_\beta \circ f \circ \phi_\alpha$ smooth

COMMENTS:

- A smooth map $\psi: M \to N$ which has a smooth inverse is called a <u>diffeomorphism</u> $(n = n')$
- If $N = \mathbb{R} / \mathbb{C}$ we sometimes call f a scalar field
- If $M = I \subset \mathbb{R}$ an open interval (open interval on the reals) then $f: I \to N$ is a smooth curve in N
- If f is smooth in one atlas, it is smooth in all compatible atlases

EXAMPLES:

1. Recall $S^1 = \{ \underline{x} \in \mathbb{R}^2 \ | \ |\underline{x}| = 1 \}$
let $f(x,y) = x$ $\qquad f: S^1 \to \mathbb{R}$



Using previous charts

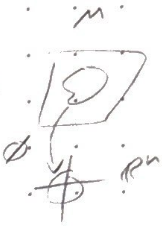
$$f \circ \phi_1^{-1} : (-\pi, \pi) \to \mathbb{R}$$

similarly $f \circ \phi_1^{-1}(\theta_1) = \cos\theta_1$

$f \circ \phi_2^{-1} : (0, 2\pi) \to \mathbb{R}$

$f \circ \phi_2^{-1}(\theta_2) = \cos\theta_2 \quad \therefore f$ is smooth

notes: if ϕ is any other chart, then $f \circ \phi^{-1} = (f \circ \phi_i^{-1}) \circ (\phi_i \circ \phi^{-1})$ which is smooth since we have shown $f \circ \phi_i^{-1}$ is smooth, & $\phi_i \circ \phi^{-1}$ is smooth by defining of a manifold



2. If (O, ϕ) is a coordinate chart on M, write

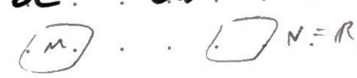
$$\phi(p) = (x^1(p), x^2(p), \dots, x^n(p)) \qquad p \in O$$

Then $x^i(p)$ defines a map from O to $\mathbb{R}$. This is a smooth for each $i = 1, \dots, n$. If (O', ϕ') is another overlapping coordinate chart then $x^i \circ \phi'^{-1}$ is the ith component of $\phi \circ (\phi')^{-1}$, hence smooth

(s $f \circ \phi'^{-1}$ is smooth because it's the first component of $\phi \circ \phi'^{-1}$ which is smooth by definition of a manifold)

often convenient to def. a fn. by specifying F instead of f:

3. We can define a smooth function chart-by-chart. For simplicity $N = \mathbb{R}$, let $\{(O_\alpha, \phi_\alpha)\}$ be an atlas on M. Define smooth functions



coordinate chart representations of f: $F_\alpha : U_\alpha \to \mathbb{R}$

and suppose $F_\alpha \circ \phi_\alpha = F_\beta \circ \phi_\beta$ on $O_\alpha \cap O_\beta$ for all α, β.

Then for some point $p \in M$ we can define $f(p) = F_\alpha \circ \phi_\alpha(p)$ where (O_α, ϕ_α) is any chart with $p \in O_\alpha$. f is smooth as

$$f \circ \phi_\beta^{-1} = F_\alpha \circ \underbrace{\phi_\alpha \circ \phi_\beta^{-1}}_{\text{smooth}}$$

smooth since we defined it to be smooth

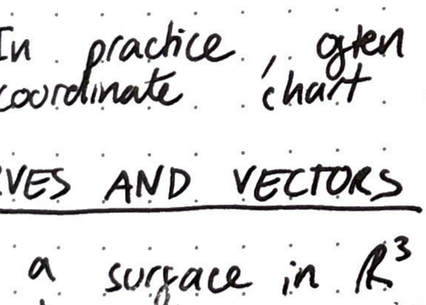
i.e. if $O_\alpha \cap O_\beta \neq \emptyset$, $\phi_\alpha \circ \phi_\beta^{-1}$ smooth

m: $f(p) = F_\alpha \circ \phi_\alpha(p)$ since we have specified the region for the chart overlaps, this definition is chart independent in those regions

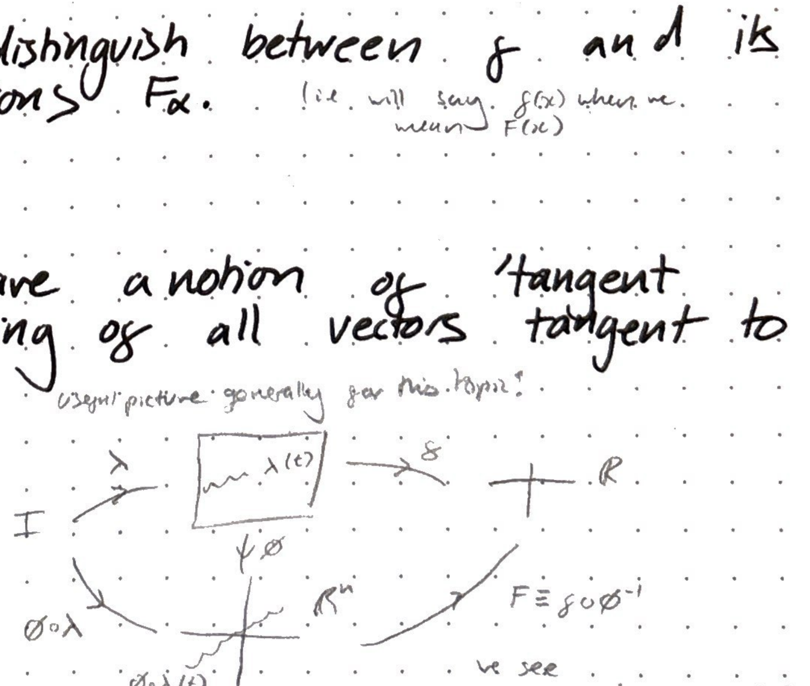
In practice, often don't distinguish between f and its coordinate chart representations F_α. (i.e. will say $f(x)$ when we mean $F(x)$)

CURVES AND VECTORS

For a surface in $\mathbb{R}^3$, we have a notion of 'tangent space' at a point consisting of all vectors tangent to the surface



useful picture generally for this topic!



we see

$$f \circ \lambda(t) = \underbrace{f \circ \phi^{-1}}_{F} \circ \underbrace{\phi \circ \lambda(t)}_{\text{curve in } \mathbb{R}^n}$$

think?
- λ is a map $\lambda : I \to M$
- $\lambda(t)$ is a curve in M
- ϕ is a map $\phi : M \to \mathbb{R}$
- $\phi(p)$ is a point in $\mathbb{R}^n$

so we move onto the manifold to $\mathbb{R}^n$ where we can do differentiation and eval. $\frac{d}{dt}(f(\lambda(t)))$

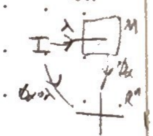
The tangent spaces are vector spaces (copies of $\mathbb{R}^2$).
Different points have different tangent spaces

In order to define the tangent space for a manifold, we first consider tangent vectors of a curve.

<u>Recall</u> $\lambda : \underset{\mathbb{R}\ \text{interval}}{\underset{\cap}{I}} \to M$ a smooth map is a smooth curve in M

I is open interval in $\mathbb{R}$ e.g. $(0,1)$ or $(-1,\infty)$

(a smooth curve in a manifold M is a smooth function $\lambda : I \to M$) → by smooth function we mean $\phi \circ \lambda$ is a smooth map from I to $\mathbb{R}^n$ for all charts ϕ



If $\lambda(t)$ is a smooth curve in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function. The chain rule gives

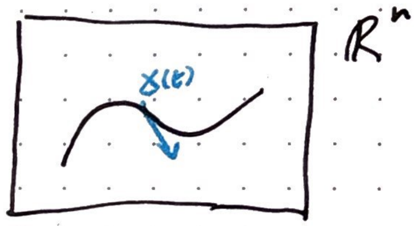
$$\frac{d}{dt}\left[f(\underline{\lambda}(t))\right] = \underline{X}(t) \cdot \underline{\nabla} f(\underline{\lambda}(t))$$

where $\underline{X}(t) = \dfrac{d\underline{\lambda}(t)}{dt}$ is the tangent vector to $\underline{\lambda}$ at t.

<u>IDEA</u>: treat as though equivalent?

this map takes $f \to \underline{X} \cdot \underline{\nabla} f$

identify $\underline{X}(t)$ with $\underline{X}(t) \cdot \underline{\nabla}$



<u>DEF</u> Let $\lambda : I \to M$ be a smooth curve with (wlog) $\lambda(0) = p$.
The tangent vector to λ at p is the linear map X_p from the space of smooth functions $f : M \to \mathbb{R}$ given by

$$X_p(f) := \frac{d}{dt} f(\lambda(t))\Big|_{t=0}$$

X_p is the vector, $X_p(f)$ specifies how the vector acts on a function, it a map

map that takes f to $\frac{d}{dt} f = \underline{X} \cdot \underline{\nabla} f$

<u>We observe</u>:

i) X_p is linear: $X_p(f + ag) = X_p f + a X_p g$ $\quad f, g$ smooth, $a \in \mathbb{R}$

ii) X_p satisfies Leibniz rule $X_p(fg) = (X_p f) g(p) + f(p) X_p g$

If (O, ϕ) is a chart $p \in O$, write $\phi(p) = (x^1(p), \dots, x^n(p))$
Let $F = f \circ \phi^{-1}$ and $x^i(t) = x^i(\lambda(t))$

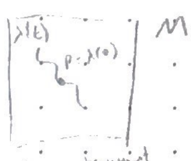
Then $\quad f \circ \lambda(t) = f \circ \phi^{-1} \circ \phi \circ \lambda(t) = F \circ x(t)$

$f \circ \phi$ f to itself

and $\quad \dfrac{d}{dt}\left(f(\lambda(t))\right)\Big|_{t=0} = \underbrace{\dfrac{\partial F}{\partial x^\mu}(x)\Big|_{\phi(p)}}_{\text{depends on } f, \phi} \ \underbrace{\dfrac{dx^\mu}{dt}\Big|_{t=0}}_{\text{depends on } \lambda, \phi}$

(Einstein summation convention: sum over repeated $\mu = 1, \dots, n$)

(see preceeding page)



X_p is tangent vector to λ at p.

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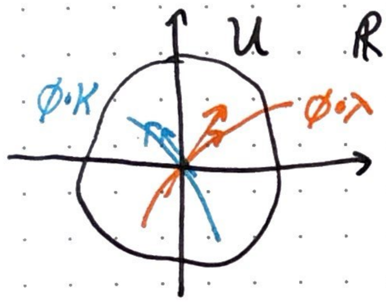
$(*)\ X_p(f) = \frac{\partial F}{\partial x^\mu}(\phi(p)) \cdot \frac{dx^\mu(t)}{dt}\Big|_{t=0}$

Prop the set of tangent vectors to curves at p forms a vector space T_pM of dimension $n = \dim M$. We call T_pM the tangent space to M at p

dim of manifold

Pf/ Given X_p, Y_p tangent vectors, we need to show $\alpha X_p + \beta Y_p$ is a tangent vector for $\alpha, \beta \in \mathbb{R}$.
Let λ, κ be smooth curves with $\lambda(0) = \kappa(0) = p$ and whose tangent vectors at p are X_p, Y_p resp.
Let (O, ϕ) be a chart with $p \in O$, $\phi(p) = 0$ (chart centred at p)



Let $\nu(t) = \phi^{-1}[\alpha \phi(\lambda(t)) + \beta \phi(\kappa(t))]$

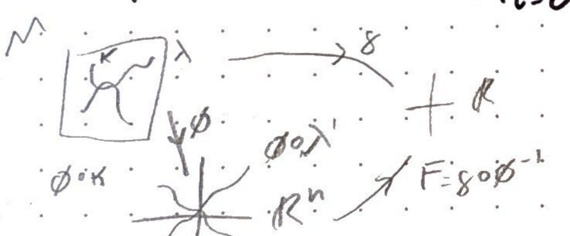
$\nu(0) = \phi^{-1}(0) = p$

From $(*)$ we have that if Z_p is the tangent to ν at p:

$$Z_p(f) = \frac{d}{dt}(f(\nu(t)))\Big|_{t=0} = \frac{\partial F}{\partial x^\mu}\Big|_0 \frac{d}{dt}\left[\alpha x^\mu(\lambda(t)) + \beta x^\mu(\kappa(t))\right]\Big|_{t=0}$$

$$= \alpha \frac{\partial F}{\partial x^\mu}\Big|_0 \frac{d}{dt} x^\mu(\lambda(t)) + \beta \frac{\partial F}{\partial x^\mu}\Big|_0 \frac{d}{dt} x^\mu(\kappa(t))\Big|_{t=0}$$

$$= \alpha X_p(f) + \beta Y_p(f)$$



Thus T_pM is a vector space.

To see T_pM is n-dimensional consider the curves

$$\lambda_\mu(t) = \phi^{-1}(0, \dots, 0, t, 0, \dots, 0)$$

μ component

think of it like "basis" vector in $\mathbb{R}^n$ getting moved to M (see pic below)

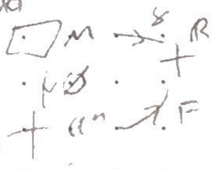
We denote the tangent vector to λ_μ at p by $\left(\frac{\partial}{\partial x^\mu}\right)_p$
To see why, note that $(*)$

$F(\phi(p)) = f \circ \phi^{-1}(\phi(p)) = f \circ p = f(p)$

$$\left(\frac{\partial}{\partial x^\mu}\right)_p f = \frac{\partial F}{\partial x^\mu}\Big|_{\phi(p)=0}$$

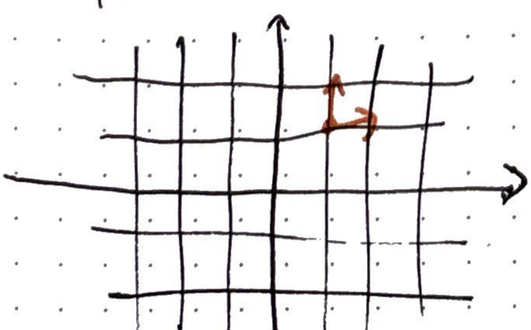
(approach from other end)

tong:



don't know how to diff f on M but do know how to diff $f \circ \phi^{-1}$ on $\mathbb{R}^n$

$\mathbb{R}^n$



since λ_μ

$X_p(f) = \frac{\partial F}{\partial x^\mu} \cdot 1$

call it $= \frac{\partial}{\partial x^\mu}\Big|_p$?

tong says: $\frac{\partial f}{\partial x^\mu}\Big|_p := \frac{\partial (f \circ \phi^{-1})}{\partial x^\mu}\Big|_{\phi(p)}$

The vectors $\left(\frac{\partial}{\partial x^\mu}\right)_p$ are linearly independent. Otherwise $\exists\, \alpha^\mu \in \mathbb{R}$ s.t. not all zero s.t.

$$\alpha^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p = 0$$

$\Rightarrow \alpha^\mu \frac{\partial F}{\partial x^\mu}\Big|_0 = 0 \quad \forall F$. setting $F = x^\nu$ gives $\alpha^\nu = 0$

Further, $\left(\frac{\partial}{\partial x^\mu}\right)_p$ form a basis for T_pM, since if λ is any curve with tangent X_p at p (*) gives

$$X_p(f) = \frac{\partial F}{\partial x^\mu}\Big|_0 \frac{d}{dt} x^\mu(\lambda(t))\Big|_{t=0} = X^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p f$$

where $X^\mu = \frac{d}{dt} x^\mu(\lambda(t))\Big|_{t=0}$ are the components of X_p w.r.t. the basis $\left\{\left(\frac{\partial}{\partial x^\mu}\right)_p\right\}_{\mu=1}^n$ for T_pM // complete proof

true for any f so $X_p = X^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p$ ← can express any vector $X_p \in T_pM$ as a linear combination of basis vectors $\left(\frac{\partial}{\partial x^\mu}\right)_p$

Notice that $\left\{\left(\frac{\partial}{\partial x^\mu}\right)_p\right\}_{\mu=1}^n$ depends on the coordinate chart ϕ.

⇒ Tong $\frac{\partial f}{\partial x^\mu}\Big|_p := \frac{\partial (f \circ \phi^{-1})}{\partial x^\mu}\Big|_{\phi(p)}$

Suppose we choose another chart (O', ϕ'), again centered at p. Write $\phi' = (x'^1, \dots, x'^n)$. Then if $F' = f \circ \phi'^{-1}$ we have

/ p mapped to origin by ϕ'

$$F(x) = f \circ \phi^{-1}(x) = f \circ \phi'^{-1} \circ \phi' \circ \phi^{-1}(x)$$
$$= F'(x'(x))$$

summation convention sum over ν

$$\text{so } \left(\frac{\partial}{\partial x^\mu}\right)_p f = \frac{\partial F}{\partial x^\mu}\Big|_{\phi(p)} = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial F'}{\partial x'^\nu}\right)_{\phi'(p)} = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\nu}\right)_p f$$

We deduce that

jacobian? (linearity rem' dims

$$\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\nu}\right)_p$$

Let X^μ be components of X_p w.r.t. $\left\{\left(\frac{\partial}{\partial x^\mu}\right)_p\right\}_{\mu=1}^n$ and X'^μ be components of X_p w.r.t. $\left\{\left(\frac{\partial}{\partial x'^\mu}\right)_p\right\}_{\mu=1}^n$

i.e. $X_p = X^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p = X'^\mu \left(\frac{\partial}{\partial x'^\mu}\right)_p$

$$= X^\mu \left(\frac{\partial x'^\sigma}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\sigma}\right)_p$$

so $X'^{\mu} = \left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)_{\phi(p)} X^{\nu}$

We do not have to choose a coordinate basis such as $\left\{\left(\frac{\partial}{\partial x^{\mu}}\right)_p\right\}_{\mu=1}^{n}$ with respect to a general basis $\{e_\mu\}_{\mu=1}^{n}$ for T_pM we write $X_p = X^{\mu} e_{\mu}$ for $X^{\mu} \in \mathbb{R}$ are components w.r.t. $\{e_\mu\}_{\mu=1}^{n}$

We always use summation convention: we always contract one upstairs and one downstairs index. The index on $\frac{\partial}{\partial x^{\mu}}$ counts as downstairs.

side note how does this relate to X? funny: a general vector in V can be written as $X = X^{\mu} e_{\mu}$

COVECTORS

Recall that if V is a vector space over $\mathbb{R}$, the dual space V^* is the space of linear maps from V to $\mathbb{R}$. If V is n-dimensional, so is V^*. Given a basis $\{e_\mu\}_{\mu=1}^{n}$ for V, we define the dual basis $\{f^{\mu}\}_{\mu=1}^{n}$ for V^* by requiring $f^{\mu}(e_\nu) = \delta^{\mu}_{\nu} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}$

think of matrix multiplication row & column vector gives real number, for usual basis vectors ~~other~~ dual is the transpose

this is actually what the note in the lecture? I'm lost

theorem?

If V is finite dimensional then $V^{**} = (V^*)^*$ is isomorphic to V: to an element X of V we the linear map

$$\Lambda_X : V^* \to \mathbb{R} \qquad \Lambda_X(\omega) = \omega(X) \qquad \omega \in V^*$$

= there is a bijection between them?

Def the dual space of T_pM is denoted T_p^*M and called the cotangent space to M at p. An element of T_p^*M is a covector at p. If $\{e_\mu\}_{\mu=1}^{n}$ is a basis for T_pM and $\{f^{\mu}\}_{\mu=1}^{n}$ the dual basis for T_p^*M, we can ~~extend~~ expand a covector η as

$$\eta = \eta_\mu f^{\mu} \quad \text{for } \eta_\mu \in \mathbb{R} \text{ the components of } \eta.$$

notes: (isomorphic means there is a bijection between V^{**} and V i think)
the bijection (aka the isomorphism) is $\phi : V \to V^{**}$, where $\phi(X)(\omega) = \omega(X)$ for all $\omega \in V^*$.

Lecture 4

18.10.24

Recap:
- Defined T_pM: space of tangent vectors at p. basis $\{e_\mu\}_{\mu=1}^n$
- Coord basis $\left\{\left(\frac{\partial}{\partial x^\mu}\right)_p\right\}_{\mu=1}^n$
- change of basis $\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)}\left(\frac{\partial}{\partial x'^\nu}\right)_p$, $x'(x) = \phi'\circ\phi^{-1}(x)$
- Dual space T_p^*M: space of covectors
  - linear maps η from T_pM to $\mathbb{R}$
  - dual basis $\{f^\mu\}_{\mu=1}^n$ satisfying $f^\mu(e_\nu) = \delta^\mu_\nu$; $\eta = \eta_\mu f^\mu$ ($\eta_\mu \in \mathbb{R}$) general components

NOTE (linear map that eats a vector)

$*$ $\eta(e_\nu) = \eta_\mu f^\mu(e_\nu) = \eta_\mu \delta^\mu_\nu = \eta_\nu$ zero except η_ν component

general action on X $*$ $\eta(X) = \eta(X^\mu e_\mu) = X^\mu \eta(e_\mu) = X^\mu \eta_\mu$ contraction between components

DEF differentials of smooth fn: natural set of covectors

If $f : M \to \mathbb{R}$ is a smooth function, define $(df)_p \in T_p^*M$, the differential (aka gradient in real notes) of f at p by
$$(df)_p(X) = X(f) \quad \text{for any } X \in T_pM$$

$(df)_p$ is a covector that acts on vectors X to give $X(f)$ (X is a column of measured?)

$(df)_p$ is sometimes also called the gradient of f at p.

$*$ If f is constant $X(f) = 0 \Rightarrow (df)_p = 0$

$*$ If (O, ϕ) is a coord chart with $p \in O$ and $\phi = (x^1, \dots, x^n)$ then we can set $f = x^\mu$ to find $(dx^\mu)_p$ now
wrong

same as basis vector acting on fn x^μ

$$(dx^\mu)_p\left(\frac{\partial x^\mu}{\partial x^\nu}\right)_{\phi(p)} = \delta^\mu_\nu$$

use $(df)_p(X) = X(f)$

$\rightarrow$ (condition for basis of cotangent space to be dual to basis of tangent space

- Hence $\{(dx^\mu)_p\}_{\mu=1}^n$ is the dual basis to $\left\{\left(\frac{\partial}{\partial x^\mu}\right)_p\right\}_{\mu=1}^n$
- In this basis we can compute using $\eta(e_\nu) = \eta_\nu$!

components $[(df)_p]_\mu = (df)_p\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial}{\partial x^\mu}\right)_p f = \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)}$ $(F = f\circ\phi^{-1})$

Justifying the language 'GRADIENT' why we call it gradient of f at p

Exercise: show that if (O', ϕ') is another chart with $p \in O'$, then

evaluated at p.
$$(dx^\mu)_p = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)_{\phi'(p)}(dx'^\nu)_p \quad , \quad x(x') = \phi\circ\phi'^{-1}$$

and hence if η_μ, η'_μ are components w.r.t. these bases

$$\eta'_\mu = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right)_{\phi'(p)}\eta_\nu$$

(simple; just plug into $(dx^\mu)_p\eta_\mu = (dx'^\nu)_p\eta'_\nu$)

previously show that each side gives the same for general vector X then remove X (like we did with f before):

$(dx^\mu)_p(X) = (dx^\mu)_p\left(X^\sigma\left(\frac{\partial}{\partial x^\sigma}\right)_p\right)$

(use transf. law for the basis $\frac{\partial}{\partial x'^\nu}$!) $\rightarrow$ $= (dx^\mu)_p\left(X^\sigma\frac{dx'^\nu}{dx^\sigma}\left(\frac{\partial}{\partial x'^\nu}\right)_p\right)$

$= \frac{dx^\mu}{dx^\nu}(dx'^\nu)_p\underbrace{\left(X^\sigma\frac{\partial}{\partial x^\sigma}\right)}_{X}$ //

tangent space: vectors tangent to manifold

cotangent space: less clear, vectors annihilated by cotangent vector

glue the objects together as the point varies: cotangent bundle

The (co)tangent bundle

TM is 2n dimensional if M is dim. n because element of TM is specified by a point on the manifold along with a vector in the corresponding tangent space T_pM } loca

We can glue together the tangent spaces T_pM as p varies to get a new $2n$ dimensional manifold TM, the **tangent bundle**. We get

TM is the collection of all the tangent spaces for all points in a manifold

(union over all points in the manifold) $$TM = \bigcup_{p \in M} \{p\} \times T_pM$$

The set of ordered pairs (p, X), with $p \in M$, $X \in T_pM$. If $\{(O_\alpha, \phi_\alpha)\}$ is an atlas on M, we obtain an atlas for TM by setting

patch of original atlas and at each point corresponding tangent space $$\tilde{O}_\alpha = \bigcup_{p \in O_\alpha} \{p\} \times T_pM$$

and

$$\tilde{\phi}_\alpha(p, X) = (\phi_\alpha(p), X^\mu) \in U_\alpha \times \mathbb{R}^n = \tilde{U}_\alpha$$

where

X^μ are the components of X w.r.t. the coord bases of ϕ_α

work out transition fns for this atlas

<u>Exercise</u>: If (O, ϕ) and (O', ϕ') are two charts on M, show that on $\tilde{U} \cap \tilde{U}'$, if we write $\phi' \circ \phi^{-1}(x) = x'(t)$ then

"don't really need this for most of course but may as well state it."

$$\tilde{\phi}' \circ \tilde{\phi}^{-1}(x, X^\mu) = \left(x'(x), \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)_x X^\nu\right)$$

deduce TM is a manifold

A similar construction permits us to define the cotangent bundle

$$T^*M = \bigcup_{p \in M} \{p\} \times T_p^*M$$

<u>Exercise</u>: Show that the map $\pi\ TM \to M$ which takes $(p, X) \mapsto p$ is smooth

"for the keener geometers"

["mostly forget the last 10 mins of what I was saying → bundle construction is not going to play much of a role in the rest of the course"]

ABSTRACT INDEX NOTATION

– (capturing the value of index notation without having to assume we've chosen a coord chart or basis.)

We've used greek letters μ, ν etc. to label components of vectors (or covectors) w.r.t. the basis $\{e_\mu\}_{\mu=1}^n$ (resp $\{f^\mu\}_{\mu=1}^n$). Equations involving these quantities refer to the specific basis. E.g. if we write $X^\mu = \delta^\mu_1$ ← (no longer true if change to diff basis.) This says X only has one non-zero component in <u>current basis</u>. This won't be true in other bases.

We know some equations hold in <u>all bases</u>, e.g.

$$\eta(X) = X^\mu \eta_\mu$$

(abstract index promotes this kind of statement to latin indices)

To capture this, we can use abstract index notation (AIN). We denote a vector by X^a where the latin index a does not denote a component, rather it tells us X^a is a vector.

(if eqn. true in all basis we're allowed to write it w/ latin indices replacing greek ones)

downstairs

Similarly we denote a covector η by η_a. If an equation is true in all bases we can replace greek indices by latin indices.

can do this because switch out a for μ in any basis this is true

i.e. $\eta(X) = X^a \eta_a = \eta_a X^a$

or $X(f) = X^a (df)_a$

An equation in AIN can always be turned into an equation for components by picking a basis and changing $a \to \mu$, $b \to \nu$ etc.

TENSORS

some quantities not described by either a scalar or a vector, even in Newtonian physics e.g. momentum inertia – need higher rank object

In Newtonian physics, we know some quantities are described by higher rank objects (e.g. inertia tensor of a body).

DEF: A tensor of type (r,s) is a multilinear map

$$T: \underbrace{T_p^*(M) \times \dots \times T_p^*(M)}_{r \text{ factors}} \times \underbrace{T_pM \times \dots \times T_pM}_{s \text{ factors}} \to \mathbb{R}$$

Multilinear means linear in each argument

Examples

1. A tensor of type $(0,1)$ is a linear map $T_pM \to \mathbb{R}$ i.e. a covector.

dual to the cotangent space at p — 'isomorphic to' ?

2. A tensor of type $(1,0)$ is a linear map $T_p^*M \to \mathbb{R}$ i.e. an element $(T_p^*M)^* \simeq T_pM$ a vector.

define it to be the map where the covector eats the vector

3. We can define a $(1,1)$ tensor, δ, by $\delta(\omega, X) = \omega(X)$, $\omega \in T_p^*M$, $X \in T_pM$

(components are what we get if we feed basis vectors into every one of the slots of T)

If $\{e_\mu\}$ is a basis for T_pM and $\{f^\mu\}$ the dual basis, the components of an (r,s) tensor T are

$$\boxed{T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} := T(f^{\mu_1}, \dots, f^{\mu_r}, e_{\nu_1}, \dots, e_{\nu_s})}$$

(by definition)

In AIN we denote T by $T^{a_1 \dots a_r}{}_{b_1 \dots b_s}$. Tensors at p form a vector space over $\mathbb{R}$ of DIM n^{r+s}. (not going to prove this here but can check)

EXAMPLES

1. consider δ above

$$\delta^\mu{}_\nu := \delta(f^\mu, e_\nu) = f^\mu(e_\nu) = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

we can write δ as $\delta^a{}_b$ in AIN

kronecker delta defines a (1,1) tensor

2. consider a $(2,1)$ tensor T, let $\omega, \eta \in T_p^*M$, $X \in T_pM$

$$T(\omega, \eta, X) = T(\omega_\mu f^\mu, \eta_\nu f^\nu, X^\sigma e_\sigma)$$

follows because multilinear

$$= \omega_\mu \eta_\nu X^\sigma T(f^\mu, f^\nu, e_\sigma)$$

defining components

$$= \omega_\mu \eta_\nu X^\sigma T^{\mu\nu}{}_\sigma$$

in AIN $T(\omega, \eta, X) = \omega_a \eta_b X^c T^{ab}{}_c$

generalisation to higher ranks

basically given r covectors & s vectors, a tensor type (r,s) will give me a real number

Lecture 5 — 21.10.24

CHANGE OF BASES

We've seen how components of X or η w.r.t. a coordinate basis (X^μ, η_ν resp.) change under a change of coordinates. We don't have to only consider coordinate bases. — i.e. can also examine how they transform under an arbitrary change in basis

Suppose $\{e_\mu\}_{\mu=1}^n$ and $\{e'_\mu\}_{\mu=1}^n$ are two bases for T_pM with dual bases $\{f^\mu\}_{\mu=1}^n$ and $\{f'^\mu\}_{\mu=1}^n$.

We can expand $f'^\mu = A^\mu{}_\nu f^\nu$ and $e'_\mu = B^\nu{}_\mu e_\nu$ for some $A^\mu{}_\nu, B^\nu{}_\mu \in \mathbb{R}$

but
$$\delta^\mu_\nu = f'^\mu(e'_\nu) = A^\mu{}_\tau f^\tau(B^\sigma{}_\nu e_\sigma)$$
f linear map
$$= A^\mu{}_\tau B^\sigma{}_\nu f^\tau(e_\sigma) = A^\mu{}_\tau B^\sigma{}_\nu \delta^\tau_\sigma$$
$$= A^\mu{}_\sigma B^\sigma{}_\nu = \delta^\mu_\nu \text{ identity matrix}$$

answering the question: how do general basis vectors transform?

Thus $B^\mu{}_\nu = (A^{-1})^\mu{}_\nu$.

we can see our previous results for coord. basis transforms indeed satisfy this (?)

If $e_\mu = \left(\frac{\partial}{\partial x^\mu}\right)_p$ and $e'_\mu = \left(\frac{\partial}{\partial x'^\mu}\right)_p$

We've already seen $A^\mu{}_\nu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)_{\phi(p)}$, $B^\mu{}_\nu = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)_{\phi(p)}$

Which indeed satisfy $A^\mu{}_\sigma B^\sigma{}_\nu = \delta^\mu{}_\nu$ by the chain rule.

A change of bases induces a transformation of tensor components.

note also easy to show how components wrt general coord bases transform (e.g. plug into $X^\mu e_\mu = X'^\nu e'_\nu$ as before) we find
$X'^\mu = A^\mu{}_\nu X^\nu$
$\eta'_\mu = (A^{-1})^\nu{}_\mu \eta_\nu$
↑ how components of (1,0)-vector transform under an arbitrary change of basis

E.g. if T is a (1,1)-tensor
$$T^\mu{}_\nu = T(f^\mu, e_\nu)$$
$$T'^\mu{}_\nu = T(f'^\mu, e'_\nu) = T(A^\mu{}_\sigma f^\sigma, (A^{-1})^\tau{}_\nu e_\tau)$$
$$= A^\mu{}_\sigma (A^{-1})^\tau{}_\nu T(f^\sigma, e_\tau) = A^\mu{}_\sigma (A^{-1})^\tau{}_\nu T^\sigma{}_\tau$$

(is also easy to show components of (2,1) tensor transform as $T'^{\mu\nu}{}_\rho = A^\mu{}_\sigma A^\nu{}_\tau (A^{-1})^\lambda{}_\rho T^{\sigma\tau}{}_\lambda$)

TENSOR OPERATIONS

Given an (r,s)-tensor, we can form an $(r-1,s-1)$-tensor by contraction.

i.e. show how it acts on vector & covector

For simplicity assume T is a (2,2)-tensor. Define a (1,1)-tensor S by
$$S(\omega, X) = T(\omega, f^\mu, X, e_\mu) \quad (*)$$

To see this is independent of the choice of basis:
$$T(\omega, f'^\mu, X, e'_\mu) = T(\omega, A^\mu{}_\sigma f^\sigma, X, (A^{-1})^\tau{}_\mu e_\tau)$$
$$= A^\mu{}_\sigma (A^{-1})^\tau{}_\mu T(\omega, f^\sigma, X, e_\tau)$$
$$= \delta^\tau_\sigma T(\omega, f^\sigma, X, e_\tau) = T(\omega, f^\tau, X, e_\tau) = S(\omega, X)$$

So $(*)$ does not depend on the choice of basis. S and T have components related by

$$S^{\mu}{}_{\nu} = T^{\mu\sigma}{}_{\nu\sigma}$$

note to self: def. S (a (1,1) tensor) as a (2,2) tensor w/ sth upstairs & downstairs index contracted. This def holds in any basis we can write $S^a{}_b = T^{ac}{}_{bc}$

In any basis in AIN we write $S^a{}_b = T^{ac}{}_{bc}$

note to self: b/c upstairs gives factor A which cancels w/ factor A' from downstairs)

Generalise to contract any upstairs index with any downstairs index in a general (r,s)-tensor.

(note e.g. $S^{bc}{}_e = T^{dbc}{}_{de}$, $S^{ab}{}_c = T^{abd}{}_{cd}$ — there are diff ways to contract $T^{abc}{}_{de}$)

Another way to make new tensors from old is to form the $\boxed{\text{tensor product.}}$

If S is a (p,q) tensor and T is an (r,s)-tensor then $S \otimes T$ is a $(p+r, q+s)$ tensor:

outer product

$\eta^i, \omega^i \in T_p^*M$
$X_i, Y_i \in T_pM$

$$S\otimes T(\omega^1,\dots,\omega^p,\eta^1,\dots,\eta^r,X_1,\dots,X_q,Y_1,\dots,Y_s)$$

equivalent definitions

$$= S(\omega^1,\dots,\omega^p,X_1,\dots,X_q)\,T(\eta^1,\dots,\eta^r,Y_1,\dots,Y_s)$$

In ~~AIN~~ $(S\otimes T)^{a_1,\dots,a_p,b_1,\dots,b_r}{}_{c_1,\dots,c_q,d_1,\dots,d_s} = S^{a_1,\dots,a_p}{}_{c_1,\dots,c_q}\,T^{b_1,\dots,b_r}{}_{d_1,\dots,d_s}$

show that they're equivalent?: need to show that this def does not depend on choice of basis i think

Exercise: for any (1,1)-tensor, T, in a basis we have $T = T^{\mu}{}_{\nu}\, e_\mu \otimes f^\nu$

(→ (show both sides acting on (η, X) give the same thing?)

The final tensor operations we require are $\boxed{\text{(anti-)symmetrisation}}$

If T is a (0,2)-tensor, we can define two new tensors

$$S(X,Y) := \tfrac{1}{2}\left(T(X,Y) + T(Y,X)\right)$$

$$A(X,Y) := \tfrac{1}{2}\left(T(X,Y) - T(Y,X)\right)$$

In AIN $S_{ab} = \tfrac{1}{2}(T_{ab} + T_{ba})$ we write $S_{ab} = T_{(ab)}$

$A_{ab} = \tfrac{1}{2}(T_{ab} - T_{ba})$ $A_{ab} = T_{[a,b]}$

These operations can be applied to any pair of matching indices in a more general tensor

e.g. $T^{a(bc)}{}_{de} := \tfrac{1}{2}(T^{abc}{}_{de} + T^{acb}{}_{de})$ etc.

We can (anti-)symmetrise over more than two indices

- * To symmetrise over n indices, sum over all permutations of the indices and divide by $n!$
- * To anti-symmetrise over n indices sum over all permutations weighted by sign $\left(\begin{array}{l}\text{even} = +\\ \text{odd} = -\end{array}\right)$ and divide by $n!$

e.g. $T^{(abc)} := \frac{1}{3!}\left(T^{abc} + T^{bca} + T^{cab} + T^{acb} + T^{cba} + T^{bac}\right)$

$T^{[abc]} := \frac{1}{3!}\left(T^{abc} + T^{bca} + T^{cab} - T^{acb} - T^{cba} - T^{bac}\right)$

To exclude indices from (anti-)symmetrisation, use vertical lines

e.g. $T^{(a|b|c)} = \frac{1}{2}\left(T^{abc} + T^{cba}\right)$

→ notes: exercise: show that $T^{(ab)}X_{[a|cd|b]} = 0$
↳ attempt $T^{(ab)}X_{[a|cd|b]} = \frac{1}{2}T^{(ab)}X_{acdb} - \frac{1}{2}T^{(ab)}X_{bcda}$
since $T^{(ab)}$ symmetric $= \frac{1}{2}T^{(ab)}X_{acdb} - \frac{1}{2}T^{(ba)}X_{bcda}$
then relabel dummy indices in 2nd term $= \frac{1}{2}T^{(ab)}X_{acdb} - \frac{1}{2}T^{(ab)}X_{acdb}$
$= 0$ //

TENSOR BUNDLES

(not super relevant)

The space of (r,s)-tensors at a point p is the vector space $(T^r_s)_pM$, these can be glued together to form the bundle of (r,s)-tensors

$$T^r_sM = \bigcup_{p\in M} \{p\} \times (T^r_s)_pM$$

If (O,ϕ) is a coordinate chart on M, set

$$\tilde{O} = \bigcup_{p\in O} \{p\} \times (T^r_s)_pM \subset T^r_sM$$

$$\tilde{\phi}(p, S_p) = (\phi(p), \underbrace{S^{\mu_1,\dots,\mu_r}{}_{\nu_1,\dots,\nu_s}})$$

components of S w.r.t. coordinate basis

(this stuff isn't in notes)

T^r_sM is a manifold, with a natural smooth map $\pi : T^r_sM \to M$ such that $\pi(p, S_p) = p$.

so far looked at stuff defined at a point but in physics we want to consider how stuff varies in spacetime → introduce concept of a field

An (r,s)-TENSOR FIELD is a smooth map $T : M \to T^r_sM$ such that $\pi \circ T = \mathrm{id}$.

If (O,ϕ) is a coordinate chart on M then

$$\tilde{\phi} \circ T \circ \phi^{-1}(x) = (x, T^{\mu_1,\dots,\mu_r}{}_{\nu_1,\dots,\nu_s}(x))$$

which is smooth provided the components $T^{\mu_1,\dots,\mu_r}{}_{\nu_1,\dots,\nu_s}(x)$ are smooth functions of x.

SPECIAL CASE If $T^r_sM = T^1_0M \cong TM$ ← the tangent bundle → the tangent spaces T_pM glued together as p varies

↳ vector field is map from M to TM!

The tensor field is called a vector field in a local coord. patch, if X is a vector field, we can write

$$X(p) = (p, X_p) \quad \text{with} \quad X_p = \underbrace{X^\mu(x)}_{\text{smooth}} \left(\frac{\partial}{\partial x^\nu}\right)_p$$

point on M! vector at point p ⇒ (p, X_p) is an element of TM!

← (vector field requires X_p to be smooth, so for vector field we require $X^\mu(x)$ smooth ($\frac{\partial}{\partial x^\nu}$ always smooth)) slight abuse of notation

In particular $\frac{\partial}{\partial x^\mu}$ are always smooth (but only defined locally)

(tang) a vector field is a smooth assignment of a tangent vector X_p to each point $p \in M$
$(X(f))(p) = X_p(f)$
so if you give it a fn, it spits back another fn.
(notation) maps $p \mapsto X_p(f)$
is smooth iff $X(f)$ smooth for any f

tensor bundle is gluing together the spaces of tensors at each point p as p varies
a tensor field is constructed by picking a member of

aka tangent vector space T_pM is the set of all tangent vectors to curves at p
vector bundle (tangent bundle) is gluing together all these spaces T_pM as p varies
vector field picks out a member of T_pM for each point in a smooth manner

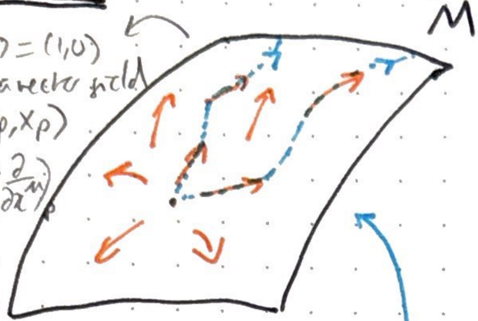
TM is a set of ordered pairs (p, X_p) with $p \in M$, $X_p \in T_pM$
so vector field $X(p) = (p, X_p)$ maps $M \to TM$ (provided $X^\mu(x)$ is smooth)

Lecture 6

23.10.24

assign to every point a 'velocity' like a fluid velocity field

RECAP
$I_f(r,s) = (1,0)$ we get a vector field $X(p) = (p, X_p)$
$X = X^\mu(x)\left(\frac{\partial}{\partial x^\mu}\right)_p$
w/ this smooth



A vector field can act on a function $f: M \to \mathbb{R}$ to give a new function Xf by
$$Xf(p) = X_p(f)$$

In coordinates
$$Xf(p) = X^\mu(\phi(p)) \left.\frac{\partial F}{\partial x^\mu}\right|_{\phi(p)}$$

INTEGRAL CURVES

Given a vector field X on M, we say a curve $\lambda: I \to M$ (interval) is an integral curve of X if its tangent at every point is X. i.e. denote the tangent vector to λ at t by $\frac{d\lambda}{dt}(t)$ then,
$$(\dagger) \quad \frac{d\lambda}{dt}(t) = X_{\lambda(t)} \quad \forall t \in I$$

note to self: remember tangent vector at p was $X_p(f) = \frac{dx^\mu}{dt}\Big|_{t=0}\left(\frac{\partial}{\partial x^\mu}\right)_p f$ where $\phi(p) = (x^1(p), \ldots, x^n(p))$ and $x^\mu(t) = x^\mu(\lambda(t))$.

Through each point p, an integral curve passes, unique up to extension / shift of the parameter.

justify:
To see this, pick a chart ϕ with $\phi = (x^1, \ldots, x^n)$ and $\phi(p) = 0$. In this chart $(\dagger)$ becomes

note to self: our vector field has $X_p = X^\mu(x)\left(\frac{\partial}{\partial x^\mu}\right)_p$ so it's just showing the components

$$(*) \quad \frac{dx^\mu}{dt}(t) = X^\mu(x(t)) \qquad x^\mu(t) = x^\mu(\lambda(t))$$

So arrive at system of diff eqns with initial condition ⇒ solvable

the components of the vector field in a given coord system X^μ are given by this eq.

RHS fn. sensible fn of unknowns then solvable

Assuming wlog that $\lambda(0) = p$ we ~~see that~~ get an initial condition $(**)$ $x^\mu(0) = 0$

don't do details here but basically

Standard ODE theory gives that $(*)$ with $(**)$ has a solution unique up to extension.

therefore there is a unique integral curve of X through any point p (up to extension)

COMMUTATORS

(see tony notes for more examples of integral curves)

Suppose X and Y are two vector fields and $f: M \to \mathbb{R}$ is smooth then $X(Y(f))$ is a smooth fn. Is it of the form $K(f)$ for some vector field K? No, because
$$X(Y(fg)) = X(fY(g) + gY(f)) = X(fY(g)) + X(gY(f))$$
$$= fX(Y(g)) + gX(Y(f)) + X(f)Y(g) + X(g)Y(f)$$

(leibniz requires $X(Y(fg)) = fX(Y(g)) + X(Y(f))g$)

So leibniz doesn't hold. But if we consider
$$[X,Y](f) := X(Y(f)) - Y(X(f))$$
then Leibniz does hold.

cancel by this expression

In fact $[X,Y]$ defines a vector field

we can build a new vector by taking the commutator $[X,Y]$ which acts on functions as

the commutator aka the lie bracket

To see this use coordinates

vector fields act on $C^\infty(p)$ as

since $X(f)(p) = X_p(f) = X^\mu \frac{\partial F}{\partial x^\mu}$; components

$$[X,Y](f) = X\left(Y^\nu \frac{\partial F}{\partial x^\nu}\right) - Y\left(X^\mu \frac{\partial F}{\partial x^\mu}\right)$$

$$= X^\mu \frac{\partial}{\partial x^\mu}\left(Y^\nu \frac{\partial F}{\partial x^\nu}\right) - Y^\nu \frac{\partial}{\partial x^\nu}\left(X^\mu \frac{\partial F}{\partial x^\mu}\right)$$

(use leibniz rule / linearity of X, Y)

terms cancel

$$= \cancel{X^\mu Y^\nu \frac{\partial^2 F}{\partial x^\mu \partial x^\nu}} - \cancel{Y^\nu X^\mu \frac{\partial^2 F}{\partial x^\nu \partial x^\mu}} + X^\mu \frac{\partial Y^\nu}{\partial x^\mu}\frac{\partial F}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\frac{\partial F}{\partial x^\mu}$$

equals?

$$= \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\right)\frac{\partial F}{\partial x^\nu} = [X,Y]^\nu \frac{\partial F}{\partial x^\nu}$$

Where $[X,Y]^\nu = X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}$ are the components of the commutator. (in this basis)

(notes: the RHS is a vector field hence $[X,Y]$ is a vector field) [see recall notes for example]

we can drop it & write:

Since f arbitrary: $[X,Y] = [X,Y]^\nu \frac{\partial}{\partial x^\nu}$, valid only in a coordinate basis. (i.e. can't write this in a...) (i.e. have found a way of generating a vector field from two other vector fields)

METRIC TENSOR

- talked about manifolds so far but not geometry stuff e.g. notions of length & angle → do this now

We're familiar from Euclidean geometry (and special relativity) with the fact that the fundamental object when talking about distance and angles (time intervals / rapidity) is an inner product between vectors.

E.g. * $\underline{x} \cdot \underline{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$ $\qquad \mathbb{R}^3$ w/ Euclidean geometry

(familiar from special rel.)

* $X \cdot Y = -X^0 Y^0 + X^1 Y^1 + X^2 Y^2 + X^3 Y^3$ $\qquad \mathbb{R}^{3+1}$ w/ Minkowski geometry $(c=1)$

scalar product! maps a pair of vectors to a scalar $T_pM \times T_pM \to \mathbb{R}$

→ want to generalise this to curved spacetime:

multilinear map that eats 2 vectors

DEF: A metric tensor at $p \in M$ is a $(0,2)$-tensor g satisfying

i) g is symmetric $g(X,Y) = g(Y,X)$ $\forall\, X, Y \in T_pM$ $(g_{ab} = g_{ba})$

ii) g is non-degenerate $g(X,Y) = 0$ for all $Y \in T_pM$ iff $X = 0$

NOTATION: sometimes write $g(X,Y) = \langle X, Y \rangle = \langle X, Y \rangle_g = X \cdot Y$

"if you don't know what that is you can just believe that this is true"

By adapting the Gram-Schmidt algorithm we can always find a basis $\{e_\mu\}_{\mu=1}^n$ for T_pM such that

$$g(e_\mu, e_\nu) = \begin{cases} 0 & \mu \neq \nu \\ +1 \text{ or } -1 & \mu = \nu \end{cases}$$

← orthonormal basis

i.e. $g_{\mu\nu} = \begin{pmatrix} -1 & & & & 0 \\ & -1 & & & \\ & & \ddots & & \\ & & & +1 & \\ 0 & & & & +1 \end{pmatrix}$ The number of -1's and $+1$'s appearing does not depend on choice of basis (Sylvestre's law of inertia) and is called the signature

- If g has signature $++\dots+$ we say it is **RIEMANNIAN**
- If g has signature $-++\dots+$ we say it is **LORENTZIAN**

DEF: A Riemannian (resp. Lorentzian) manifold is a pair (M,g) where M is a manifold and g is a Riemannian (resp. Lorentzian) metric tensor field

(on riemannian use metric to def lengths & angles)

notes: Lorentzian manifold sometimes also called a spacetime

remarks:

* On a Riemannian manifold the norm of a vector $X \in T_pM$ is

$$|X| = \sqrt{g(X,X)}$$

* The angle between $X, Y \in T_pM$ is given by

$$\cos\theta = \frac{g(X,Y)}{|X||Y|}$$

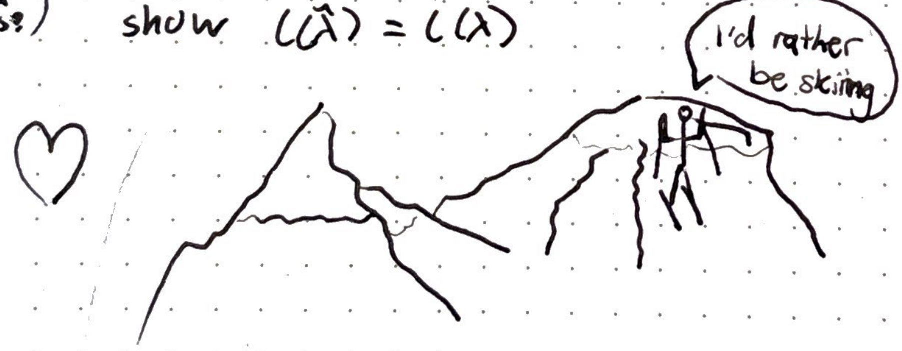
* The length of a curve $\lambda : (a,b) \to M$ is given by

$$L(\lambda) = \int_a^b \left| \frac{d\lambda}{dt}(t) \right| dt$$

EXERCISE! If $\tau : (c,d) \to (a,b)$ with $\frac{d\tau}{du} > 0$ $\tau(c)=a$ $\tau(d)=b$ then $\hat{\lambda} = \lambda \circ \tau : (c,d) \to M$ is a representation of λ

show $L(\hat{\lambda}) = L(\lambda)$

(if reparametrise curve do i get the same ans?)



is our def of length is independent of parametrisation

attempt: $L(\lambda) = \int_a^b \left|\frac{d\lambda}{dt}\right| dt$

$L(\hat{\lambda}) = \int_c^d \frac{d\hat{\lambda}}{dt} dt$

use $\hat{\lambda} = \lambda(\tau(t)) \Rightarrow L(\hat{\lambda}) = \int_c^d \left|\frac{d\lambda}{d\tau}\frac{d\tau}{dt}\right| dt$

$= \int_a^b \left|\frac{d\lambda}{d\tau}\right| d\tau = L(\lambda)$?

so our def of length is indep. of parametrisation

→ notes: can also show that $Y^a = \frac{dt}{d\tau} X^a$ if Y is tangent vec to $\hat{\lambda}$ and X to λ.

attempt: components $Y^\mu = \frac{dx^\mu}{d\tau}$, $X^\mu = \frac{dx^\mu}{dt}$

$\Rightarrow Y^\mu = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \frac{dt}{d\tau} X^\mu$?

true for all bases so $\mu \to a$?

Lecture 7

(reminder: can write (0,2) tensor as $T = T_{\mu\nu} f^\mu \otimes f^\nu$ (show this as exercise) In curvilinear view basis $f^\mu = (dx^\mu)$)

25.10.24

In a coordinate basis, $g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$. We often write $dx^\mu dx^\nu := \frac{1}{2}(dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu)$ and by convention write $g = ds^2$ so that

$$g = ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

(note to self: we alegbra & to we symmetric anyway so what's the point of this?)

(notes just say $g_{\mu\nu} dx^\mu d^\nu$ is an abbreviating $g_{\mu\nu} dx^\mu \otimes dx^\nu$)

Examples:

flat (Riemannian example)

i) $\mathbb{R}^n$ $= \{(x^1, \dots, x^n)\}$ with

$$g = ds^2 = (dx^1)^2 + (dx^2)^2 + \dots + (dx^n)^2 = \delta_{\mu\nu}\, dx^\mu dx^\nu$$

$(\mathbb{R}^n, g)$ is called Euclidean space. Any chart covering $\mathbb{R}^n$ in which the metric takes this form (diag(1, 1, ..., 1)) is called CARTESIAN.

flat (Lorentz example)

ii) $\mathbb{R}^{1+3} = \{(x^0, x^1, x^2, x^3)\}$ with

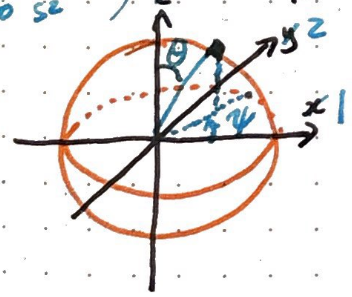
$$g = ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$$= \eta_{\mu\nu}\, dx^\mu dx^\nu \qquad \left(\eta_{\mu\nu} = \begin{cases} -1 & \mu = \nu = 0 \\ 1 & \mu = \nu \neq 0 \\ 0 & \text{otherwise} \end{cases} \right)$$

$(\mathbb{R}^{3+1}, \eta)$ Is Minkowski space. A coordinate chart covering $\mathbb{R}^{1+3}$ in which the metric takes this form (i.e. diag(-1, 1, 1, 1)) is an inertial frame such that

iii) On $S^2 = \{\underline{x} \in \mathbb{R}^3 \mid |\underline{x}| = 1\}$ define a chart by (curved example)

(map an open square in $\mathbb{R}^2$ into S^2)



$$\phi^{-1} : (0, \pi) \times (-\pi, \pi) \rightarrow S^2$$

(no? don't think so, other way)

$$(\theta, \psi) \longmapsto (\sin\theta\cos\psi,\ \sin\theta\sin\psi,\ \cos\theta)$$

In this chart the round metric is

$$g = ds^2 = d\theta^2 + \sin^2\theta\, d\psi^2$$

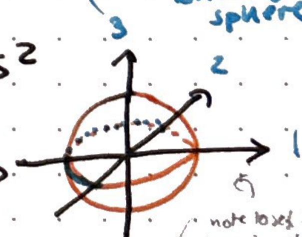
set of points where all satisfied (:)

This covers $S^2 \setminus \{\underbrace{|\underline{x}| = 1}_{\text{surface}},\ \underbrace{x^2 = 0}_{\text{plane } x^2=0},\ \underbrace{x^1 \leq 0}_{x^1 \text{ -ve}}\}$

(miss poles & curt out of back of sphere)

To cover the rest let $\phi'^{-1} : (0, \pi) \times (-\pi, \pi) \rightarrow S^2$

$$(\theta', \psi') \mapsto (-\sin\theta'\cos\psi',\ \cos\theta',\ \sin\theta'\sin\psi')$$



which covers $S^2 \setminus \{|\underline{x}| = 1 \text{ and } x^3 = 0 \text{ and } x^1 \geq 0\}$

(note to self: imagine picture before but relabel axes $x \to -x$ and $y \leftrightarrow z$)

setting $g = d\theta'^2 + \sin^2\theta'\, d\psi'^2$ defines a metric on all of S^2. (check this by finding coordinate transform from unprimed to primed then on region of overlap the metric defined in each set of coords is the same tensor)

("painful" to do this)

(need to do this?)

(b/c upstairs indices transform w/ inv. transformation basis to down stairs)

Since g_{ab} is non-degenerate, it is invertible as a matrix in any basis. We can check the inverse defines a symmetric (2,0)-tensor $\boxed{g^{ab}}$ satisfying (tong: $\hat{g} = g^{\mu\nu}\partial_\mu \otimes \partial_\nu$)

$$g^{ab}g_{bc} = \delta^a_c$$

(tong: physicists say that we use the metric to lower the index given X^μ to X_μ ... in their heart, they mean "a metric determines a natural isomorphism between vectors & covectors")

Example: In the $\theta\phi$ coordinates of the S^2 example

$$g^{\mu\nu} = \left(1, \frac{1}{\sin^2\theta}\right)$$

An important property of the metric is that it induces a canonical identification of T_pM and T_p^*M.

(i.e. given X^a we can get X_a & vice versa)

(covector i get by lowering w/ metric)

* Given $X^a \in T_pM$ (vector) we define a covector $g_{ab}X^b = X_a$ and
* given $\eta_a \in T_p^*M$ (covector) we define a vector $g^{ab}\eta_b = \eta^a$

In $(\mathbb{R}^3, \delta)$ ("standard inner product") Euclidean space we often do this without realising. (metric & its inverse are the identity in these coordinates: don't really distinguish covectors & vectors)

More generally, this allows us to raise tensor indices with g^{ab} and lower with g_{ab}.

Example: If $T^{ab}{}_c$ is a (2,1)-tensor then $T_a{}^{bc}$ is the (1,2)-tensor given by

$$T_a{}^{bc} = g_{ad}\, g^{ce}\, T^{db}{}_e \quad \text{etc.}$$

LORENTZIAN SIGNATURE

→ "relevant b/c this is what we assume spacetime to have"

(notes: on Lorentzian manifold, we take basis indices μ,ν to run from 0 to $n-1$)

In Lorentzian signature, indices $0,1,\dots,n$ (manifold $n+1$ dim.), at any point p in a Lorentzian manifold we can find a basis (orthonormal)

(tong: at any point on general Lorentzian manifold we can find an orthonormal basis of T_pM s.t. locally the metric looks like the Minkowski metric)

$$\{e_\mu\}_{\mu=0}^{n} \quad \text{s.t.} \quad g(e_\mu, e_\nu) = \eta_{\mu\nu} \equiv \mathrm{diag}(-1, 1, \dots, 1)$$

(can always diagonalise at a point s.t. looks like Minkowski space)

This basis is not unique, if $e_\mu' = (A^{-1})^\nu{}_\mu e_\nu$ is another such basis then (at point p)

$$\eta_{\mu\nu} = g(e_\mu', e_\nu') = (A^{-1})^\sigma{}_\mu (A^{-1})^\tau{}_\nu\, g(e_\sigma, e_\tau) = (A^{-1})^\sigma{}_\mu (A^{-1})^\tau{}_\nu\, \eta_{\sigma\tau}$$

$$\Rightarrow A^\mu{}_\kappa A^\nu{}_\rho\, \eta_{\mu\nu} = \eta_{\kappa\rho}$$

(defining eq. of a Lorentz transform)

which is the condition that $A^\mu{}_\nu$ is a **LORENTZ TRANSFORMATION** (cf. special relativity)

(notes: different orthonormal bases at p are related by Lorentz transformations)

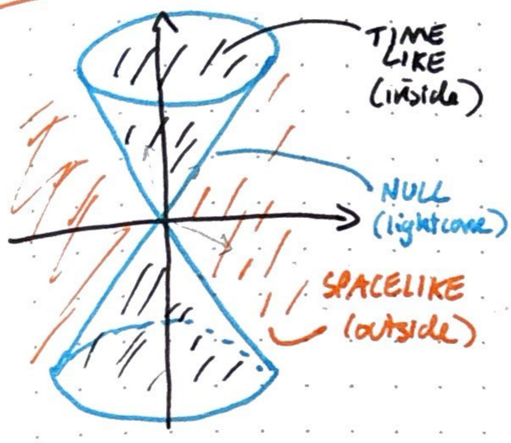
(also reminder we saw vectors transform as $X^\mu = A^\mu{}_\nu X^\nu$ which is the same as the transformation law of components in special relativity)

The tangent space at p has $\eta_{\mu\nu}$ as metric tensor (in this basis)

(i.e. viewed locally we recover structure of special relativity) so has the structure of Minkowski space in particular

on a Lorentzian manifold (M, g)

$\boxed{\text{DEF:}}$ $X \in T_pM$ is SPACELIKE, NULL/LIGHTLIKE, TIMELIKE IF
$$\begin{cases} g(X,X) > 0 \\ g(X,X) = 0 \\ g(X,X) < 0 \end{cases}$$



(every vector has to be one of these whilst a curve can be none → start off one become another)

notes: a causal vector is timelike or null

note the causal vectors split into two disconnected sets

A curve $\lambda : I \to M$ in a Lorentzian manifold is spacelike or timelike or null if the tangent vector is everywhere spacelike or timelike or null resp.

A spacelike curve has a well-defined LENGTH (given by the same formula as in Riemannian case.

notes: $\left(\int_a^b dt \sqrt{g(X,X)}\big|_{\lambda(t)} \right)$

For a timelike curve $\lambda : (a,b) \to M$, the relevent quantity is the PROPER TIME

(u is parameter along curve)

$$\tau(\lambda) = \int_a^b \sqrt{-g_{ab} \frac{d\lambda^a}{du} \frac{d\lambda^b}{du}}\, du$$

so in coordinates: $\tau = \int_a^b du \sqrt{-\left(g_{\mu\nu} \frac{dx^\mu}{du} \frac{dx^\nu}{du}\right)}\Big|_{\lambda(u)}$

If $g_{ab} \frac{d\lambda^a}{du} \frac{d\lambda^b}{du} = -1$ for all u, then λ is parametrised by proper time.

In this case we call the tangent vector $u^a := \frac{d\lambda^a}{du}$ the 4-VELOCITY of λ.

notes: def: if proper time is used to parametrise a timelike curve, then the tangent vec to the curve is called the 4-velocity of the curve. in coordinate basis: $u^\mu = \frac{dx^\mu}{d\tau}$

from def $\Rightarrow d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu$ which implies 4-velocity is a unit timelike vector: $g_{ab} u^a u^b = -1$.

CURVES OF EXTREMAL PROPER TIME

Suppose $\lambda : (0,1) \to M$ is timelike, satisfies $\lambda(0) = p$, $\lambda(1) = q$, and extremises proper time among all such curves. This is a 'variational problem associated to (in a coordinate chart)

$$\tau[\lambda] = \int_0^1 G(x^\mu(u), \dot{x}^\mu(u))\, du \qquad \left(\dot{} \equiv \frac{d}{du} \text{ here} \right)$$

with $G(x^\mu(u), \dot{x}^\mu(u)) = \sqrt{-g_{\mu\nu}(x(u))\, \dot{x}^\mu(u)\, \dot{x}^\nu(u)}$

EULER LAGRANGE EQN is (*) $\left[\frac{d}{du}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) = \frac{\partial G}{\partial x^\mu} \right]$, we can compute:

$$\frac{\partial G}{\partial \dot{x}^\mu} = -\frac{1}{G} g_{\mu\nu} \dot{x}^\nu \qquad \frac{\partial G}{\partial x^\mu} = -\frac{1}{2G} \underbrace{\frac{\partial}{\partial x^\mu}(g_{\sigma\tau})}_{= g_{\sigma\tau,\mu}} \dot{x}^\sigma \dot{x}^\tau = -\frac{1}{2G} g_{\sigma\tau,\mu} \dot{x}^\sigma \dot{x}^\tau$$

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CURVES OF EXTREMAL PROPER TIME cont.

(before had arbitrary parametrisation u)

* Now fix parametrisation so curve is parametrised by proper time τ. Doing this

$$\frac{dx^\mu}{d\tau} = \dot{x}^\mu \frac{du}{d\tau} \quad \text{and} \quad -1 = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

chain rule — condition τ is proper time along curve

(alternatively: from definition in notes $\frac{d\tau}{du} = \sqrt{-(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)}$, write this in coordinates $\left(\frac{d\tau}{du}\right)^2 = -g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} = G^2 \Rightarrow \frac{d\tau}{du} = G$ so $\frac{d}{du} = G\frac{d}{d\tau}$)

* Deduce $-1 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\left(\frac{du}{d\tau}\right)^2 \Rightarrow \frac{du}{d\tau} = 1/G \Rightarrow \frac{1}{G}\frac{d}{du} = \frac{d}{d\tau}$

* Returning to (†) (EL) we find

(use $\frac{d}{d\tau}g_{\mu\nu} = \frac{dx^\rho}{d\tau}\frac{d}{dx^\rho}g_{\mu\nu}$)

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) = \frac{1}{2}g_{\nu\rho,\mu}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$$

which we can solve

$$\Rightarrow g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + g_{\mu(\nu,\rho)}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} - \frac{1}{2}g_{\sigma\rho,\mu}\frac{dx^\sigma}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

multiplied by $\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$ symmetric in ν,ρ

(note to reminder: when we contract a symmetric term with a non-antisymmetric in the same indices we get zero since the antisymmetric part of $g_{\mu\nu,\rho}$ is zero so can replace it with just the symmetrical part $g_{\mu(\nu,\rho)}$)

(symmetrise & contract w/ inverse metric)

* Thus
$$\boxed{\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\mu\rho}\frac{dx^\mu}{d\tau}\frac{dx^\rho}{d\tau} = 0} \qquad (*)$$

the geodesic equation

* where $\Gamma^\nu_{\mu\sigma} := \frac{1}{2}g^{\nu\sigma}\left(g_{\mu\sigma,\rho} + g_{\sigma\rho,\mu} - g_{\nu\rho,\sigma}\right)$ are the

μ?

CHRISTOFFEL SYMBOLS of g

comments

NOTE:
- $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$ (symmetric in downstairs indices)
- $\Gamma^\mu_{\nu\rho}$ are not tensor components (cf exercise 2.1)
- We can solve (*) with standard ODE theory, solutions are called GEODESICS
- The same equation governs curves of extremal length in a Riemannian manifold (or spacelike curves in a Lorentzian manifold) parametrised by arc length

i.e. extremise $L(\lambda) = \int du\sqrt{g_{ab}\dot{x}^a\dot{x}^b}$ vs $\tau(\lambda) = \int du\sqrt{-g_{ab}\dot{x}^a\dot{x}^b}$ (mink)

EXERCISE! Show that (*) can be obtained as the Euler-Lagrange equation for the Lagrangian

$$L = -g_{\mu\nu}(x(\tau))\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)$$

(strongly recommend doing this exercise!!)

(reminder? for finding geodesic eq / christoffel symbols?)

$$\frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial\dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0$$

an easier way to derive the geodesic eqn or christoffel symbols

(c.f. it said we want to extremise τ for this one, but why? I think both lead to the same eqn)

EXAMPLES:

(i.e. the components of the metric are constant)

1) In Minkowski space in an inertial frame $g_{\mu\nu} = \eta_{\mu\nu}$ so $\Gamma^{\nu}_{\mu\rho} = 0$ and geodesic equation is

$$\frac{d^2 x^\mu}{d\tau^2} = 0$$

easy to solve → (solutions are straight lines)

geodesics are straight lines

notes: in Minkowski spacetime, timelike curves of extremal proper time are straight lines. It can be shown that these lines maximise the proper time between two points (in a general spacetime this is only true locally) only in the neighbourhood of p

2) The Schwarzschild metric in Schwarzschild coords. (t, r, θ, ϕ) is a metric on $M = \mathbb{R}_t \times (2m, \infty)_r \times S^2_{\theta,\phi}$ given by

(i.e. this means it depends on the reals parametrised by t & on the interval (2m,∞) parametrised by r & on the 2-sphere parametrised by θ, ϕ)

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

$f = 1 - \frac{2m}{r}$ then

metric does not depend on t explicitly

(emphasize t is just a (timelike) coordinate in spacetime, τ is a parameter along curve)

$$L = f\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{f}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2$$

(note to self: 4 eqns, one for each $x^\mu = t, r, \theta, \phi$. consider t:)

E-L equation for $t(\tau)$ is $\frac{d}{d\tau}\left(\frac{\partial L}{\partial t'}\right) = \frac{\partial L}{\partial t}$ $\quad t' = \frac{dt}{d\tau}$

$$\Rightarrow 2\frac{d}{d\tau}\left(f\frac{dt}{d\tau}\right) = 0$$

$$\Rightarrow f\frac{d^2t}{d\tau^2} + \frac{df}{dr}\left(\frac{dr}{d\tau}\right)\left(\frac{dt}{d\tau}\right) = 0$$

$x^\mu = (t, r, \theta, \phi)$ (indices 0, 1, 2, 3)

remember symmetry! $(\Gamma^0_{10} + \Gamma^0_{01}) = \frac{1}{f}\frac{df}{dr}$

Compare to (*) to see $\Gamma^0_{\ 1 0} = \Gamma^0_{\ 0 1} = \frac{1}{2} \cdot \frac{1}{f}\frac{df}{dr}$

$$\Gamma^0_{\mu\nu} = 0 \quad \text{otherwise}$$

note doing this is often more efficient than using formula using metric & doing explicit computation

Rest of symbols can be found (done on examples sheet) from other EL equations

→ just consider each of the 4 EL eq. and read off the Γ symbols. remember symmetry $\Gamma^0_{10} = \Gamma^0_{01}$ gives factor ½ and all symbols not seen are zero. (f remember is g_{00} is f)

COVARIANT DERIVATIVE

For a function $f: M \to \mathbb{R}$, we know that $\frac{df}{dx^\mu}$ are components of a covector $(df)_\mu$

For a vector field, we can't just differentiate components

b/c partial deriv of tensor field does not give another tensor field

EXERCISE: show that if V is a vector field then

$$T^\mu_{\ \nu} := \frac{dV^\mu}{d\tau^\nu} \text{ are not components of a (1,1)-tensor}$$

(notes have $\frac{dV^\mu}{dx^\mu}$?) → components of tensor transform as $T'^\mu_{\ \nu} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}T^\rho_{\ \sigma}$

→ thinking $T'^\mu_{\ \nu} = T(f'^\mu, e'_\nu)$ & how f'^μ, e'_ν transform

components of vector transform as $V'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu}V^\nu(x)$

so find $T'^\mu_{\ \nu} = \frac{\partial}{\partial x'^\nu}V'^\mu$

$$\left[T'^\mu_{\ \nu} \stackrel{?}{=} \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}T^\rho_{\ \sigma} + \left(\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x'^\nu}\right)V^\rho\right]$$

(I think)

(extra term! does not transf. as tensor!)

doing this you will find if change to new coord chart I also get a term involving derivative of jacobian

DEF: A covariant derivative ∇ on a manifold M is a map sending X, Y smooth vector fields to a vector field $\nabla_X Y$ satisfying (X, Y, Z smooth V fields, f, g functions)

want it to satisfy

i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$

note to self, i.e. for scalars covariant deriv is just partial deriv $\nabla_\mu g = e_\mu(g) = \frac{\partial g}{\partial x^\mu}$

ii) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$

note to self: (1,1) Tensor is a multilinear map $T: T_pM^* \times T_pM \to \mathbb{R}$; equivalently we can think of it as a linear map $T: T_pM \to T_pM$, i.e. $V^\mu \mapsto T^\mu{}_\nu V^\nu$ (?) eqv.

leibniz type rule.

iii) $\nabla_X(fY) = f\nabla_X Y + (\nabla_X f)Y$ where $\nabla_X f := X(f)$

any acting ∇ on functions by

since ∇Y is linear in X^μ so $\nabla_X Y$ is a vector $\in T_pM$ so map $T_pM \to T_pM \Rightarrow \nabla Y$ is (1,1) tensor. (?)

Note i) implies that $\nabla Y: X \longmapsto \nabla_X Y$ is a linear map of T_pM to itself, so defines a (1,1)-tensor, the COVARIANT DERIVATIVE of Y $\equiv$ AFFINE CONNECTION

In AIN $(\nabla Y)^a{}_b = \nabla_b Y^a$ or $Y^a{}_{;b}$

notation: $\nabla_\mu Y$ shorthand for $\nabla_{e_\mu} Y$ and $\nabla_\mu Y^\nu$ shorthand for $(\nabla_\mu Y)^\nu$

tonny says the semicolon notation is stupid (I agree.)

DEF: In a basis $\{e_\mu\}$ the CONNECTION COMPONENTS $\Gamma^\mu{}_{\nu\rho}$ are defined by

assuming basis is basis of smooth vector fields

$$\left(\nabla_\rho e_\nu = \right) \quad \nabla_{e_\rho} e_\nu = \Gamma^\mu{}_{\nu\rho} e_\mu$$

(note to self remember swaps $\rho\nu \leftrightarrow \nu\rho$)

These determine ∇

$$\nabla_X Y = \nabla_{X^\mu e_\mu}(Y^\nu e_\nu) \overset{\text{property (i)}}{=} X^\mu \nabla_{e_\mu}(Y^\nu e_\nu) \overset{(ii),(iii)}{=} X^\mu (e_\mu(Y^\nu) e_\nu + Y^\sigma \nabla_{e_\mu} e_\sigma)$$

("skipped a couple of lines")

$$= (X^\mu e_\mu(Y^\nu) + \Gamma^\nu{}_{\sigma\mu} Y^\sigma X^\mu) e_\nu$$

Hence

$$(\nabla_X Y)^\nu = X^\mu (e_\mu(Y^\nu) + \Gamma^\nu{}_{\sigma\mu} Y^\sigma)$$

the νth component of $\nabla_X Y$

components not tensors but they have transformation law that compensates for fact this is not a tensor

and $\left((\nabla_\mu Y)^\nu = \right) \; Y^\nu{}_{;\mu} = e_\mu(Y^\nu) + \Gamma^\nu{}_{\sigma\mu} Y^\sigma$

In a coord basis $e_\mu = \frac{\partial}{\partial x^\mu}$ then $Y^\nu{}_{;\mu} = Y^\nu{}_{,\mu} + \Gamma^\nu{}_{\sigma\mu} Y^\sigma$

notes? $Y^\nu{}_{,\mu} = \frac{dY^\nu}{dx^\mu}$

but notice $\Gamma - \Gamma$ transforms as a tensor

$\Gamma^\nu{}_{\mu\sigma}$ are not components of a tensor

i.e. Γ not a tensor (1,2) tensor but nothing to stop it being one! not! we find $\Gamma'^\mu{}_{\nu\rho} = (A^{-1})^\mu{}_\tau A^\sigma{}_\rho A^\lambda{}_\nu \Gamma^\tau{}_{\lambda\sigma} + (A^{-1})^\mu{}_\tau A^\sigma{}_\rho e_\sigma(A^\tau{}_\nu)$ ← take $\Gamma'^\mu{}_{\nu\rho} e'_\mu = \nabla_{e'_\rho} e'_\nu$ and plug in e'_ν, e'_ρ in eqs

e.g. e_ν? $\Gamma^\mu{}_{\nu\rho} e_\mu$, (1,2) tensor transforms as $T'^\mu{}_{\nu\rho} = (A^{-1})^\mu{}_\tau A^\lambda{}_\nu A^\sigma{}_\rho T^\tau{}_{\lambda\sigma}$

E.g. for η a tensor field (covector field?) we define $(\nabla_X \eta)(Y) := \nabla_X(\eta(Y)) - \eta(\nabla_X Y)$

extend defs to act on more general tensor fields?

since $\eta = \eta_\sigma dx^\sigma$?

In components $(\nabla_X \eta) Y = X^\mu e_\mu(\eta_\sigma Y^\sigma) - \eta_\sigma (\nabla_X Y)^\sigma$

(since $\nabla_X Y$ is a vector we just get exactly just components of ?)

$\nabla_X \eta) Y = X^\mu (e_\mu(\eta_\nu) - \eta_\sigma \Gamma^\sigma{}_{\nu\mu}) Y^\nu$

$$= X^\mu e_\mu(\eta_\sigma) Y^\sigma + X^\mu \cancel{\eta_\sigma e_\mu(Y^\sigma)} - \eta_\sigma(\cancel{X^\nu e_\nu(Y^\sigma)} + X^\tau \Gamma^\sigma{}_{\tau\nu} Y^\tau)$$

linear in $Y^\nu \Rightarrow \nabla_X \eta$ is a covector w components $(\nabla_X \eta)_\nu$; $(\nabla_X \eta)_\nu$ linear in $X^\mu \Rightarrow \nabla_\mu \eta$ is a (0,2) tensor field w components $(\nabla_\mu \eta)_\nu = \eta_{\nu;\mu}$ $= e_\mu(\eta_\nu) - \eta_\sigma \Gamma^\sigma{}_{\nu\mu}$

$$= \underbrace{(e_\mu(\eta_\sigma) - \Gamma^\nu{}_{\sigma\mu} \eta_\nu)}_{\text{linear in } X, Y} X^\mu Y^\sigma \quad \therefore \nabla \eta \text{ is a tensor - (0,2)}$$

$$\nabla_\mu \eta_\sigma = e_\mu(\eta_\sigma) - \Gamma^\nu{}_{\sigma\mu} \eta_\nu =: \eta_{\sigma;\mu}$$

in coord basis $\eta_{\sigma;\mu} = \eta_{\sigma,\mu} - \Gamma^\nu{}_{\sigma\mu} \eta_\nu$

in coord basis $\eta_{\nu;\mu} = \eta_{\nu,\mu} - \eta_\sigma \Gamma^\sigma{}_{\nu\mu}$ //

can use leibniz rule to find the basis components of ∇T for a general tensor

Lecture 9

30.10.24

(writing X^μ, ω_μ are not scalars since they transform under a change of basis vectors (scalars don't) so $X^\mu{}_{;\nu} \neq X^\mu{}_{,\nu}$)

useful expressions (we have shown in coords):

$$\nabla_\nu X^\mu = X^\mu{}_{;\nu} = X^\mu{}_{,\nu} + \Gamma^\mu{}_{\sigma\nu} X^\sigma$$

$$\nabla_\nu \omega_\mu = \omega_{\mu;\nu} = \omega_{\mu,\nu} - \Gamma^\sigma{}_{\mu\nu} \omega_\sigma$$

$\left(X^\mu{}_{;\nu} = e_\nu(X^\mu) + \Gamma^\mu{}_{\sigma\nu} X^\sigma \right.$
$\left. \omega_{\mu;\nu} = e_\nu(\omega_\mu) - \Gamma^\sigma{}_{\mu\nu}\omega_\sigma \right)$ not in coords

(easy not rigorous check is scalars have no free indices or basis vectors)

Exercise: in a coordinate basis

$$T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s ; \rho} = T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s , \rho} + \Gamma^{\mu_1}{}_{\sigma\rho} T^{\sigma \mu_2 \dots \mu_r}{}_{\nu_1 \dots \nu_s} + \dots + \Gamma^{\mu_r}{}_{\sigma\rho} T^{\mu_1 \dots \mu_{r-1}\sigma}{}_{\nu_1 \dots \nu_s} - \Gamma^\sigma{}_{\nu_1 \rho} T^{\mu_1 \dots \mu_r}{}_{\sigma \nu_2 \dots \nu_s} - \dots - \Gamma^\sigma{}_{\nu_s \rho} T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_{s-1}\sigma}$$

(for every lower index: $-\Gamma$ term, upper index: $+\Gamma$ term)

show for (1,1): write (1,1) tensor as $T(\omega, X) = \omega_\mu X^\nu T^\mu{}_\nu$
→ we want to find $T^\mu{}_{\nu;\rho} = \nabla_\rho T^\mu{}_\nu$

① consider Leibniz
$\nabla_\rho T = \nabla_\rho(\omega_\mu X^\nu T^\mu{}_\nu) = (\nabla_\rho \omega_\mu) X^\nu T^\mu{}_\nu + \omega_\mu (\nabla_\rho X^\nu) T^\mu{}_\nu + \omega_\mu X^\nu (\nabla_\rho T^\mu{}_\nu)$
→ plug in our expressions for $\nabla_\rho \omega_\mu$ and $\nabla_\rho X^\nu$

② consider scalar deriv. since $\nabla_\rho T = \nabla_\rho(\omega_\mu X^\nu T^\mu{}_\nu)$
$= e_\rho(\ ) \overset{\text{coords}}{=} \partial_\rho(\omega_\mu X^\nu T^\mu{}_\nu) = (\partial_\rho \omega_\mu) X^\nu T^\mu{}_\nu + \omega_\mu(\partial_\rho X^\nu) T^\mu{}_\nu + \omega_\mu X^\nu (\partial_\rho T^\mu{}_\nu)$ (in coords)

[$T(\omega, X)$ is scalar: remember contracting (1,1) tensor w vector & covector gives scalar (or less rigorously we have no free indices or basis vectors in the brackets) so this is just covariant deriv of a scalar = normal deriv]

③ compare both expressions, cancel terms & simplify ⇒
$\nabla_\rho T^\mu{}_\nu = T^\mu{}_{\nu;\rho} = T^\mu{}_{\nu,\rho} + \Gamma^\mu{}_{\sigma\rho} T^\sigma{}_\nu - \Gamma^\sigma{}_{\nu\rho} T^\mu{}_\sigma$ //

- **Remark** If $T^a{}_b$ is a (1,1) tensor, then $T^a{}_{b;c}$ is a (1,2) tensor and we can take further covariant derivatives

$$(T^a{}_{b;c})_{;d} =: T^a{}_{b;cd} = \nabla_d \nabla_c T^a{}_b$$

(i.e. two things after semicolon just means take covariant derivative ∇ twice)

(nearest index to ; is first deriv. to be taken)

In general $T^a{}_{b;cd} \neq T^a{}_{b;dc}$

- If f is a function $f_{;a} \overset{?}{=} (df)_a$ is a covector. In a coordinate basis

($= f_{,a}$ also true)

(use this & plug in expression for $\omega_{\mu;\nu}$ with $\omega = df$) (components of (df))

(reminder def $(df)_p(X) = X(f)$ and so $\nabla_X f := X(f) = (df)_p(X)$? $\nabla_X f = X(f) = X^\mu e_\mu(f) = X^\mu(\partial_\mu f)$ in coords → but saw before $\partial_\mu f = [df]_\mu$ so $X^\mu \nabla_\mu f = X^\mu [df]_\mu$ ⇒ $\nabla_\mu f = [df]_\mu$ also $= \partial_\mu f$!)

$$f_{;\mu} = f_{,\mu} \Rightarrow f_{;\mu\nu} = f_{,\mu\nu} - \Gamma^\sigma{}_{\mu\nu} f_{,\sigma}$$

(i.e. matters which order we take derivatives)

$$\Rightarrow f_{;[\mu\nu]} = -\Gamma^\sigma{}_{[\mu\nu]} f_{,\sigma}$$

($f_{,\mu\nu} = f_{,\nu\mu}$, $\partial_\mu\partial_\nu f = \partial_\nu\partial_\mu f$)

DEF: A connection (= covariant derivative) is torsion free or symmetric if

$$\nabla_a \nabla_b f - \nabla_b \nabla_a f = 0$$

For any function f in a coordinate basis this is equivalent to (not f ... only for functions!)

$$\Gamma^\rho{}_{[\mu\nu]} = 0 \iff \Gamma^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu}$$

(most of the time we will work w/ symmetric ones so don't need to be as worried abt ordering bottom indices)

LEMMA: If ∇ is torsion free, then for X, Y vector fields

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

PROOF: In a coordinate basis

(use results from earlier)

$$\begin{aligned}
(\nabla_X Y - \nabla_Y X)^\mu &= X^\sigma Y^\mu{}_{;\sigma} - Y^\sigma X^\mu{}_{;\sigma} \\
&= X^\sigma (Y^\mu{}_{,\sigma} + \Gamma^\mu{}_{\rho\sigma} Y^\rho) - Y^\sigma (X^\mu{}_{,\sigma} + \Gamma^\mu{}_{\rho\sigma} X^\rho) \\
&= [X,Y]^\mu + 2X^\sigma Y^\rho \Gamma^\mu{}_{[\rho\sigma]} = [X,Y]^\mu
\end{aligned}$$

This is a tensor equation so if true in one basis, true in all. □

note: Even if ∇ is torsion free, $\nabla_a \nabla_b X^c \neq \nabla_b \nabla_a X^c$ in general.

THE LEVI-CIVITA CONNECTION

For a manifold with metric, there is a preferred connection

TFRG (fundamental theorem of Riemannian geometry)

If (M,g) is a manifold with a metric, there is a unique torsion free connection ∇ satisfying $\nabla g = 0$. This is called the levi-civita connection

(i.e. $\nabla_X g = 0$)

PROOF: Suppose such a connection exists. By Leibniz rule, if X, Y, Z are smooth vector fields

$$* \quad X(g(Y,Z)) = \nabla_X(g(Y,Z)) = \underbrace{(\nabla_X g)}_{0}(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \qquad a)$$

Similarly:

$$Y(g(Z,X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \qquad b)$$

$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \qquad c)$$

$*$ a) + b) − c):

since metric is symmetric

$$X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) = g(\nabla_X Y + \nabla_Y X, Z) + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_Y Z - \nabla_Z Y, X)$$

$*$ Use $\nabla_X Y - \nabla_Y X = [X,Y]$

this step helps b.c. no dependence on ∇ on RHS

$$X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) = 2g(\nabla_X Y, Z) - g([X,Y],Z) - g([Z,X],Y) + g([Y,Z],X)$$

determines uniquely, Z is arbitrary & g is non degenerate

$$* \quad \Rightarrow g(\nabla_X Y, Z) = \tfrac{1}{2}\{X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X)\} \quad (\dagger)$$

non degenerate means: if $g(V,Z) = g(W,Z)$ $\forall Z$ then $V = W$

then since everything on RHS is a known quantity → we know it completely, there is no room for freedom i.e. $\nabla_X Y$ is unique. (?)

This determines $\nabla_X Y$ uniquely since g is non-degenerate. Conversely we can use $(\dagger)$ to define $\nabla_X Y$. Then need to check properties of a symmetric connection hold.

we've shown uniqueness, but now need to show that this satisfies the properties of a connection

$$* \text{ E.g. } g(\nabla_{fX} Y, Z) = \tfrac{1}{2}\{fX(g(Y,Z)) + Y(fg(Z,X)) - Z(fg(X,Y)) + g([fX,Y],Z) + g([Z,fX],Y) - g([Y,Z],fX)\}$$

show $\nabla_{fX} Y = f\nabla_X Y$

expand $Y(fg) = fY(g) + Y(f)g$ and $[fX,Y] = f[X,Y] - Y(f)X$

$$= \tfrac{1}{2}\{fX(g(Y,Z)) + fY(g(Z,X)) - fZ(g(X,Y)) + \cancel{Y(f)g(Z,X)} - \cancel{Z(f)g(X,Y)} + g(f[X,Y] - \cancel{Y(f)X}, Z) + g(f[Z,X] + \cancel{Z(f)X}, Y) - fg([Y,Z],X)\}$$

$$* \quad \Rightarrow g(\nabla_{fX} Y, Z) = g(f\nabla_X Y, Z) \Rightarrow g(\nabla_{fX} Y - f\nabla_X Y, Z) = 0 \quad \forall Z$$

$*$ so $\nabla_{fX} Y = f\nabla_X Y$ as g non-degenerate.

reminder: degenerate means: $\exists$ some $X \neq 0$ s.t. $g(X,Y) = 0$ $\forall Y$

Exercise: check other properties

use the expression $(\dagger)$ we found

apparently the others are easier ?

In a coord. basis we can compute

$$g(\nabla_{e_\mu} e_\nu, e_\sigma) = \tfrac{1}{2}\{e_\mu(g(e_\nu,e_\sigma)) + e_\nu(g(e_\sigma,e_\mu)) - e_\sigma(g(e_\mu,e_\nu))\}$$

$$g(\Gamma^\tau_{\nu\mu} e_\tau, e_\sigma) = \Gamma^\tau_{\nu\mu} g_{\tau\sigma} = \tfrac{1}{2}(g_{\sigma\nu,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma})$$

multiplying into make target index

$$\Rightarrow \underbrace{\Gamma^\tau_{\nu\mu} = \tfrac{1}{2} g^{\sigma\tau}(g_{\sigma\nu,\mu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma})}_{\text{CHRISTOFFEL SYMBOLS}}$$

we can now take ∇ to be the levi-civita connection since the difference between two connections is a tensor field, we can write any connection as the levi-civita connection and a (1,2) tensor field.

note to self: important term?

note, we have shown the christoffel symbols are the components of the levi-civita connection in a coordinate basis

If ∇ is levi-civita can raise/lower indices and this commutes with covariant differentiation.

i.e. if ∇ is levi-civita then $g_{ab}\nabla_c X^a = \nabla_c (g_{ab}X^a) = \nabla_c X_b$

now think more about curves that extremise proper time & ∇

GEODESICS

We found that a curve extremizing proper time satisfies

the geodesic eqn.

$$(\ddagger)\quad \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x(\tau))\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

τ proper time along curve

note to self: (?) before parametrised by arbitrary param t before had $X^\mu = \frac{dx^\mu}{du}$ now $V^\mu = \frac{dx^\mu}{d\tau}$, now use proper time parametrisation so $X^\mu = \frac{dx^\mu}{d\tau} = V^\mu$

The tangent vector X^a to the curve has components $X^\mu = \frac{dx^\mu}{d\tau}$. Extending this off the curve we get a vector field, of which the geodesic is an integral curve. We note

I don't think this works (?) is it true to be here

$$\frac{d^2x^\mu}{d\tau^2} = \frac{d}{d\tau}\Big(\overset{X^\mu(x(\tau))}{\frac{dx^\mu}{d\tau}}\Big) = \frac{\partial X^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau} = X^\mu{}_{,\nu}X^\nu \qquad \text{(chain rule)}$$

$(\ddagger)$ becomes $\quad X^\mu{}_{,\nu}X^\nu + \Gamma^\mu_{\nu\rho}X^\nu X^\rho = 0 \Leftrightarrow X^\nu X^\mu{}_{;\nu} = 0 \Leftrightarrow \nabla_X X = 0.$

where we are using the levi civita connection so $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$ (?) – so its just symmetric b/c its multiplied by something like this?

Extend to any connection ← now extend

any connection

DEF: Let M be a manifold with connection ∇. An AFFINELY PARAMETRIZED GEODESIC satisfies

$$\nabla_X X = 0$$

where X is the tangent vector.

note: an integral curve of a vector field X satisfying $\nabla_X X = 0$

tangent vector to curve defined only on curve itself but

(lecture 10) <u>note</u>: if we reparametrise $t \to t(u)$ then

consider a curve satisfying this def with parameter t & tangent X: 2

$$\underbrace{\frac{dx^\mu}{du}}_{Y^\mu} = \underbrace{\frac{dx^\mu}{dt}}_{X^\mu}\underbrace{\frac{dt}{du}}_{h}$$

conclude (last): the space of affine params is parametrised by two real numbers (i.e. there is a 2 parameter family of affine parameters for any geodesic) $t = \alpha u + \beta$

what is meant by 'affinely param' if we reparametrise: ∇_Y describes the same curve $\nabla_Y Y = \chi(h) Y \neq 0$ but in general not affinely parametrised but it is always possible to find a parameter s.t. it is. in this case $\nabla_Y Y = 0 \Leftrightarrow u = at + b$ wlog we can always restrict to APGs

so $\quad X \to Y = hX \quad$ with $\quad h > 0$

$$\nabla_Y Y = \nabla_{hX}(hX) = h\nabla_X(hX) = h^2 \cancel{\nabla_X X}^{\,0} + hX\cdot X(h) = \chi Y$$

with $\quad \chi = X(h) = \frac{d}{dt}(h) = \frac{1}{h}\frac{dh}{du} = \frac{1}{h}\frac{d^2t}{du^2}$, so $\nabla_Y Y = 0 \Leftrightarrow t = \alpha u + \beta, \; \alpha, \beta \in \mathbb{R}, \; \alpha > 0$

affine reparametrisation, line comprises

nb $X(f) := \frac{d}{dt}f = \frac{dx^\mu}{dt}\frac{d}{dx^\mu}f$?

otherwise, $Y = hX$ describes the same ... not a reparametrisation

note to self: $\nabla_Y Y = 0 \Leftrightarrow \frac{d}{dt}\frac{dt}{du} = 0$, $\frac{du}{dt} \neq 0$ so we have $\frac{d^2t}{du^2} = 0$

Exercise in notes: let X be tangent to an APG of the Levi-Civita connection, show that $\nabla_X(g(X,X))=0$:

attempt: $\nabla_X(g(X,X)) = \nabla_X g(X,X) + g(\nabla_X X, X) + g(X, \nabla_X X) = 2g(0,X) = 0$

if $= 0$ if one arg $= 0$

so therefore the tangent vector cannot change from e.g. timelike to null along the geodesic. a geodesic is either timelike, spacelike or null

Lecture 10

1.11.24

Theorem: given $p \in M$, $X_p \in T_pM$, there exists a unique A.P.G. $\lambda : I \to M$ satisfying

$$\lambda(0) = p \qquad \dot{\lambda}(0) = X_p$$

$\lambda(t)$.

PROOF: Choose coordinates with $\phi(p)=0$ satisfies $\nabla_X X = 0$ with $X = X^\mu \frac{\partial}{\partial x^\mu}$ this becomes

$x^\mu(t) = \phi(\lambda(t))$ $\quad X^\mu = \frac{dx^\mu}{dt}$.

2nd order odg. unique solution if we specify 2 init conditions

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{dt}\frac{dx^\sigma}{dt} = 0 \qquad (GE)$$

and $x^\mu(0) = 0 \quad \frac{dx^\mu}{dt}(0) = X_p{}^\mu$

2nd order ODE w/ two BCs → uniqueness follows from standard ODE theory

This has a unique solution $x^\mu : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ for ε sufficiently small by standard ODE theory. $\square$

GEODESIC POSTULATE

not acted on by any force except gravity

In general relativity, free particles move along geodesics of the Levi-Civita connection

These are TIMELIKE for massive particles and NULL/LIGHTLIKE for massless particles.

Normal Coordinates

locally specify points on manifold by directions & distance to get there e.g. give directions

kind of like going to waitrose, but in canada there are many paths

breaks down long distances

uniqueness problem

"the exponential map"

If we fix $p \in M$, we can map T_pM into M by setting

Ψ sends X_p to the point unit affine parameter distance along the geodesic through p w/ tangent X_p at point p

$\Psi(X_p) = \lambda_{X_p}(1)$ where λ_{X_p} is the unique affinely parametrised geodesic with

$$\lambda_{X_p}(0) = p \quad , \dot{\lambda}_{X_p}(0) = X_p.$$

Notice that

rescale ~~rescle~~ vector equiv to rescaling argument

$$\lambda_{\alpha X_p}(t) = \lambda_{X_p}(\alpha t) \quad \text{for } \alpha \in \mathbb{R}$$

since if t is affine param then $u = at + b$ is also affine param for $a, b \in \mathbb{R}$ then

since if $\hat{\lambda}(t) = \lambda_{X_p}(\alpha t)$ affine reparametrisation so still geodesic, and $\dot{\hat{\lambda}}(0) = \alpha \dot{\lambda}_{X_p}(0) = \alpha X_p$, $\hat{\lambda}(0) = p$.

locally, the map that sends X_p to 1 also sends αX_p to α

Moreover, $\alpha \longmapsto \Psi(\alpha X_p)$ is an affinely parametrised geodesic $= \lambda_{X_p}(\alpha)$.

define map from tangent space to M. claim is long as sufficiently small neighbourhood e.g. origin then it is bijective map

intuitively:

- like throwing a ball and seeing where it lands
- if i throw it w/ initial velocity 2x faster it goes twice the distance.

X_p

$2X_p$

i.e. the exponential map sends X_p to the point unit distance along the geodesic through p w/ tangent X_p at p.
but it sends tX_p distance t along that same geodesic.

CLAIM: If $U \subset T_pM$ is a sufficiently small neighbourhood of the origin, then $\psi : T_pM \to M$ is one-to-one and onto.

(don't prove here) (can prove this easily, check jacobian is invertible (not part of this course))

DEF: Construct <u>normal coordinates</u> at p, (supposing $\{e_\mu\}$ is a basis for T_pM) as follows. For

$q \in \psi(U) \subset M$, we define $\phi(q) = (X^1, \dots, X^n)$ where X^μ are components of the unique $X_p \in U$ with $\psi(X_p) = q$ (w.r.t. $\{e_\mu\}$)

(at least how I'm thinking: so where it lands (basically the point the map lands), is given by the initial velocity needed to reach that point in say 1 second. $X_p = X^\mu e_\mu$ defines a point q, we assign q coordinates X^μ)

By our previous observation, the curve given in normal coordinates by $X^\mu(t) = tY^\mu$ for Y^μ constant is an affinely parametrised geodesic so from the geodesic eqn. (GE)

(we know its a geodesic so can just plug it in?)

$$\Gamma^\mu_{\nu\sigma}(tY)Y^\nu Y^\sigma = 0$$

(reminder: antisymmetric part is 0 since we contract w symmetric thing; so left with $\Gamma^\mu_{(\nu\sigma)}Y^\nu Y^\sigma = 0$ then since Y arbitrary)

Set $t=0$ deduce (since Y arbitrary) that $\Gamma^\mu_{(\nu\sigma)}|_p = 0$

So if ∇ is torsion free, $\Gamma^\mu_{\nu\sigma}|_p = 0$ (components vanish (at p)) in normal coordinates.

(reminder: connection components not a tensor otherwise this would be a stronger result)

If ∇ is the levi-civita connection of a metric, then

$$g_{\mu\nu,\rho}|_p = 0$$

(use $\Gamma^\tau_{\nu\mu} g_{\tau\sigma} = \frac{1}{2}(g_{\sigma\nu,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma})$)

(rmk: normal coords are coords where the spacetime looks like minkowski space (at a point) to 1st order)

Since $g_{\mu\nu,\rho} = \frac{1}{2}(g_{\mu\nu,\rho} + g_{\rho\nu,\mu} - g_{\mu\rho,\nu}) + \frac{1}{2}(g_{\mu\nu,\rho} + g_{\mu\rho,\nu} - g_{\rho\nu,\mu})$

$$= \Gamma^\sigma_{\mu\rho} g_{\sigma\nu} + \Gamma^\sigma_{\rho\nu} g_{\sigma\mu} = 0 \quad \text{at } p.$$

(can arrange nice form of metric if we choose basis?) (at any point we can introduce normal coords s.t. the first partial deriv of the metric vanishes) (if in normal coords at $p = 0$ and ∇ is levi civita connection then Γ vanishes) (basically if Γ vanishes then the deriv of the metric vanishes)

We can always choose the basis $\{e_\mu\}$ for T_pM on which base the normal coordinates to be orthonormal. We have

LEMMA: On a Riemannian/Lorentzian manifold (levi-civita ∇?) we can choose normal coordinates at p s.t. $g_{\mu\nu,\rho}|_p = 0$ (1st derivative vanish at p) and

$$g_{\mu\nu}|_p = \begin{cases} \delta_{\mu\nu} & \text{RIEMANNIAN} \\ \eta_{\mu\nu} & \text{LORENTZIAN} \end{cases}$$

(i.e. locally looks flat (euclidean if flat riemannian, minkowski if flat lorentzian))

PROOF: The curve given in normal coordinates by $t \mapsto (t, 0, \dots, 0)$ is the APG with $\lambda(0) = p$, $\dot\lambda(0) = e_1$ by previous argument. But by defn. of coord basis this vector is $\left(\frac{\partial}{\partial x_1}\right)_p$. So if $\{e_\mu\}$ is ON at p $\left(\frac{\partial}{\partial x^\mu}\right)_p$ form an O/N basis. $\square$

(orthonormal?)

(tony: if we pick the initial basis $\{e_\mu\}$ to be orthonormal then the geodesics will point in orthogonal directions which ensures the metric takes the form $g_{\mu\nu}(p) = \delta_{\mu\nu}$)

CURVATURE

look for defn intrinsic to manifold ~~fat the~~ that tells us its not flat (do this by considering parallel transport

sphere: transporting a vector around changes the angle, tells us it is curved

Parallel transport

[alternatively: if λ is a curve with tangent vector X^a, then a tensor field T is parallelly transported along λ if $\nabla_X T = 0$]

Suppose $\lambda : I \to M$ is a curve with tangent vector $\dot\lambda(t)$ ~~if~~ we say a tensor field T is parallely transported/propagated along λ if

$$\nabla_{\dot\lambda} T = 0 \quad \text{on } \lambda \qquad (PP)$$

affinely parametrised

"looks a bit like geodesic eqn (GE)"

* If λ is an APG then $\dot\lambda$ is parallely propagated along λ.

a geodesic is a curve whose tangent vector is parallelly transported along the curve

* A parallely propagated tensor is determined everywhere on λ by its value at one point:

E.g. If T is a $(1,1)$ tensor then in coordinates (PP) becomes

use $\nabla_\rho T^\mu{}_\nu = T^\mu{}_{\nu;\rho} = T^\mu{}_{\nu,\rho} + \Gamma^\mu_{\sigma\rho} T^\sigma{}_\nu - \Gamma^\sigma_{\nu\rho} T^\mu{}_\sigma$, then $\nabla_X T^\mu{}_\nu = X^\rho \nabla_\rho \ldots$, where $X^\rho = \frac{dx^\rho}{dt}$ components of tangent vector

$$0 = \frac{dx^\mu}{dt} T^\nu{}_{\sigma;\mu} = \frac{dx^\mu}{dt}\Big(\underbrace{T^\nu{}_{\sigma,\mu}}_{\text{1st partial}} + \Gamma^\nu_{\rho\mu} T^\rho{}_\sigma - \Gamma^\rho_{\sigma\mu} T^\nu{}_\rho\Big)$$

but $T^\nu{}_{\sigma,\mu} \dfrac{dx^\mu}{dt} = \dfrac{d}{dt}\left(T^\nu{}_\sigma\right)$ so

deriv of T in a direction is $\frac{d}{dt}(T^\nu{}_\sigma)$

linear in T

$$0 = \frac{d}{dt} T^\nu{}_\sigma + \left(\Gamma^\nu_{\rho\mu} T^\rho{}_\sigma - \Gamma^\rho_{\sigma\mu} T^\nu{}_\rho\right)\frac{dx^\mu}{dt}$$

1st order eqn → solution determined entirely by values at a point

This is a 1st order linear ODE for $T^\nu{}_\sigma(x(t))$ so ODE theory gives a unique solution once $T^\nu{}_\sigma(x(0))$ specified.

T uniquely determined by knowing its value at one point ?

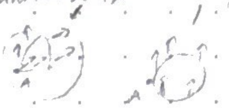
* Parallel transport along a curve from p to q gives an isomorphism between tensors at p and q. This depends on the choice of curve in general. ← important

1st order i.e. $\frac{d}{dt} T^\nu{}_\sigma = $ (linear combination of the components of T)

expt: the isomorphism depends on the choice of path. On a curved manifold, parallel transporting around a loop may not return you to the same tensor ?

isomorphism means the map is invertible & preserves the tensor structure i.e. maps tensors to tensors

wiki: isomorphism = structure preserving map that can be reversed by an inverse mapping



Lecture 11 — 4.11.24

THE RIEMANN TENSOR

The Riemann tensor captures the extent to which parallel transport depends on the curve.

LEMMA: Given X, Y, Z vector fields, ∇ a connection, define

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Then $(R(X,Y)Z)^a = R^a{}_{bcd} X^c Y^d Z^b$ for a $(1,3)$-tensor $R^a{}_{bcd}$, the Riemann tensor.

PROOF: Suppose f is smooth function, then

prove it is a tensor by showing it's linear in X, Y, Z

$$\begin{aligned}
R(fX,Y)Z &= \nabla_{fX} \nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z \\
&= f\nabla_X \nabla_Y Z - \nabla_Y (f \nabla_X Z) - \nabla_{f[X,Y] - Y(f)X} Z \\
&= f\nabla_X \nabla_Y Z - f\nabla_Y \nabla_X Z - \cancel{Y(f)\nabla_X Z} - f\nabla_{[X,Y]} Z + \cancel{Y(f)\nabla_X Z} \\
&= f R(X,Y)Z. \quad (1) \text{ linear in } X
\end{aligned}$$

linear in Y (2)

Since $R(X,Y)Z = -R(Y,X)Z$, we have $R(X,fY)Z = fR(X,Y)Z$

(antisymmetric in X, Y means we don't need to separately check linearity in Y)

linear in Z

Exercise check $R(X,Y)(fZ) = fR(X,Y)Z$ (3)

similar computation to above, we have 3 results now

this is fine (remember $Y(f), X(f)$, etc. are scalar so $\nabla_Y X(f) = Y(X(f))$ etc. think.)

Now suppose we pick a basis $\{e_\mu\}$ with dual basis $\{f^\nu\}$

$$\begin{aligned}
R(X,Y)Z = R(X^\rho e_\rho, Y^\sigma e_\sigma)(Z^\nu e_\nu) &= X^\rho Y^\sigma Z^\nu R(e_\rho, e_\sigma) e_\nu \\
&= (R^\mu{}_{\nu\rho\sigma} X^\rho Y^\sigma Z^\nu) e_\mu
\end{aligned}$$

holds in one basis so holds in any basis

note to self: summary 1. show that it's linear in X, Y, Z. 2. then also show its components are what we want (think)

where $R^\mu{}_{\nu\rho\sigma} = f^\mu (R(e_\rho, e_\sigma) e_\nu)$ are components of $R^a{}_{bcd}$ in this basis. Since result holds in one basis, it holds in all bases. $\square$

note to self: b/c partial derivatives commute!

In a coordinate basis $e_\mu = \frac{\partial}{\partial x^\mu}$ and $[e_\mu, e_\nu] = 0$ so

$$\begin{aligned}
R(e_\rho, e_\sigma) e_\nu &= \nabla_{e_\rho}(\nabla_{e_\sigma} e_\nu) - \nabla_{e_\sigma}(\nabla_{e_\rho} e_\nu) = \nabla_{e_\rho}(\Gamma^\tau_{\nu\sigma} e_\tau) - \nabla_{e_\sigma}(\Gamma^\tau_{\nu\rho} e_\tau) \\
&= \partial_\rho(\Gamma^\tau_{\nu\sigma}) e_\tau + \Gamma^\tau_{\nu\sigma} \Gamma^\mu_{\tau\rho} e_\mu - \partial_\sigma(\Gamma^\tau_{\nu\rho}) e_\tau - \Gamma^\tau_{\nu\rho} \Gamma^\mu_{\tau\sigma} e_\mu
\end{aligned}$$

use $R^\mu{}_{\nu\rho\sigma} = f^\mu(R(e_\rho, e_\sigma) e_\nu)$

Hence $$\boxed{R^\mu{}_{\nu\rho\sigma} = \partial_\rho(\Gamma^\mu_{\nu\sigma}) - \partial_\sigma(\Gamma^\mu_{\nu\rho}) + \Gamma^\tau_{\nu\sigma} \Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho} \Gamma^\mu_{\tau\sigma}}$$

note to self: $\nabla_\rho(\Gamma) = e_\rho(\Gamma) = \partial_\rho(\Gamma)$ since Γ are not tensor components, they're scalar fixed components so treat them as scalars (right?)

note structure (especially last 2 terms)

In normal coordinates we can drop the last two terms.

(if we also have the Levi-Civita connection?)

(flatness at p so $\Gamma = 0$ but in Minkowski still curved so $\partial\Gamma \neq 0$?)

(note if Γ vanishes the derivative of Γ do not necessarily vanish)

(Lorentzian & normal coords means Γ vanishes at a point but if also have Minkowski space then derivative is zero so Γ vanishes everywhere)

Example: For the Levi-Civita connection of Minkowski space in an inertial frame, (i.e. choose inertial coords) $\Gamma^{\nu}_{\mu\sigma} \equiv 0$, so $R^{\mu}{}_{\nu\sigma\tau} = 0$

(can't inertial coords exist globally in flat spacetime)

(inertial coords are normal coords but normal coords are not inertial coords)

(normal coords valid for not flat spacetime but are only valid locally)

true in 1 basis so true in all

hence $R^{a}{}_{bcd} = 0$. Such a connection with $R^{a}{}_{bcd} = 0$ everywhere is called *flat*. Conversely, for a Lorentzian spacetime with flat L-C connection, we can locally find coordinates such that $g_{\mu\nu} = \mathrm{diag}(-1, 1, \dots, 1)$. ($\equiv \eta_{\mu\nu}$)

this spacetime is "locally isometric" to Minkowski spacetime

<u>A note of caution</u>:

$$(\nabla_X \nabla_Y Z)^c = X^a \nabla_a (Y^b \nabla_b Z^c) \neq X^a Y^b \nabla_a \nabla_b Z^c$$

<u>hence</u>

$$(R(X,Y)Z)^c = X^a \nabla_a (Y^b \nabla_b Z^c) - Y^a \nabla_a (X^b \nabla_b Z^c) - [X,Y]^b \nabla_b Z^c$$

$$= X^a Y^b \nabla_a \nabla_b Z^c - Y^a X^b \nabla_a \nabla_b Z^c + (\nabla_X Y - \nabla_Y X - [X,Y])^b \nabla_b Z^c$$

So if ∇ is torsion free,

$$\boxed{\nabla_a \nabla_b Z^c - \nabla_b \nabla_a Z^c = R^{c}{}_{dab} Z^d}$$ RICCI IDENTITY

on ex. sheet 2 there's a question to generalise for an expression for

$$\nabla_{[a} \nabla_{b]} T^{c_1 \dots c_r}{}_{d_1 \dots d_s}.$$

We can construct a new tensor from $R^{a}{}_{bcd}$ by contraction: (side note to self: $R^{a}{}_{bcd} = -R^{a}{}_{bdc} \Rightarrow R^{a}{}_{b(cd)} = 0$)

Definition: The RICCI TENSOR is the (0,2)-tensor

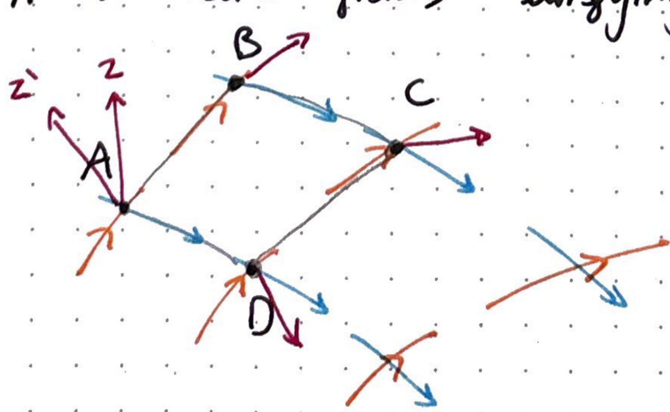
$$R_{ab} = R^{c}{}_{acb}$$

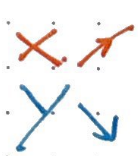
(note to self: contract middle?)

why not contract diff 2 indices $R^{a}{}_{acd} = 0$, others are the same up to a minus sign so this is the only one of interest

Suppose X, Y are vector fields satisfying $[X,Y] = 0$

(once I use parallel transport it around the curves it's not necessarily the same vector)





flow distance ε along integral curve of X through a point then ε along integral curve of Y through a point then back ε then back ε i don't get to the same point

Go from A to B by flowing parameter distance ε along int. curve of X
B to C " " ε " Y
C to D " " $-\varepsilon$ " X
D to A " " $-\varepsilon$ " Y

(property that commutators vanish)

Since $[X,Y]=0$, we indeed return to start.

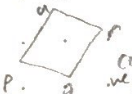
remark: $R^{\mu}{}_{\nu\rho\sigma}$ measures path dependence of parallel transport i.e. difference pqr vs psr but can equally be interpreted as the effect of parallel transporting a vector around the closed loop rqpsr

$\boxed{\text{CLAIM:}}$ (See H. Reall notes.)

assuming (w. Levi-Civita connection?)

If Z is parallely transported around ABCD to a vector Z', then

$$(Z-Z')^{\mu} = \varepsilon^2 R^{\mu}{}_{\nu\rho\sigma} Z^{\nu} X^{\rho} Y^{\sigma} + O(\varepsilon^3)$$



consider what we get if we parallel transport Z_p around pqr $(Z^{\mu}{}_r)$ vs psr $(Z'^{\mu}{}_r)$ $\rightarrow$ $\Delta Z^{\mu}{}_r \equiv Z'^{\mu}{}_r - Z^{\mu}{}_r = (R^{\mu}{}_{\nu\rho\sigma} Z^{\nu} X^{\rho} Y^{\sigma})_p \delta s \delta t + O(\delta s)$

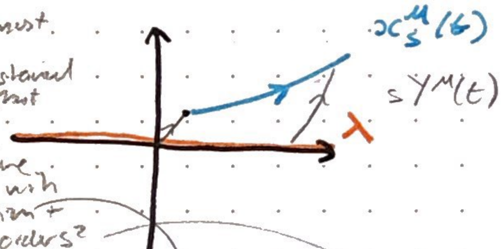
$\Rightarrow (R^{a}{}_{bcd} Z^b X^c Y^d)_r = \lim_{\delta\to 0} \frac{\Delta Z^a{}_r}{\delta s \delta t}$

i.e. Riemann tensor measures path dependence of parallel transport

this is one way of visualising curvature, let's consider another way: geodesic deviation

Geodesic Deviation

Let ∇ be a symmetric connection. Suppose $\lambda: I \to M$ is an APG through p. We can pick normal coordinates centred at p such that λ is given by $t \mapsto (t,0,\dots,0)$.



start close to λ & almost in the same direction

Suppose we start a geodesic with

$$x^{\mu}_s(0) = s x_0{}^{\mu} \qquad |s| \ll 1.$$

$$\dot{x}^{\mu}_s(0) = s x_1{}^{\mu} + (1,0,\dots,0)$$

find from standard properties of ODE that curve $x^{\mu}_s(t)$ looks like curve for λ with correction + terms orders s^2

Then we find $x^{\mu}_s(t) = x^{\mu}(s,t) = (t,0,0,\dots,0) + sY^{\mu}(t) + O(s^2)$

$x^{\mu}(s,t) = x^{\mu}(0,t) + s\left.\frac{dx^{\mu}(s,t)}{ds}\right|_{s=0} + O(s^2)$

$Y^{\mu}(t) = \left.\frac{\partial x^{\mu}}{\partial s}\right|_{s=0}$ are components of a vector field along λ.

Measuring the (infinitesimal) deviation of the geodesics, we have

$$\frac{\partial^2 x^{\mu}}{\partial t^2} + \Gamma^{\mu}_{\nu\sigma}(x^{\mu}(s,t)) \frac{\partial x^{\nu}}{\partial t}\frac{\partial x^{\mu}}{\partial t} = 0 \qquad \text{take } \left.\frac{\partial}{\partial s}\right|_{s=0}$$

$$\Rightarrow \frac{\partial^2 Y^{\mu}}{\partial t^2} + \partial_{\rho}(\Gamma^{\mu}_{\nu\sigma})_{s=0} T^{\nu} T^{\sigma} Y^{\rho} + 2\Gamma^{\mu}_{\rho\sigma}\frac{\partial}{\partial t} Y^{\rho} T^{\sigma} = 0 \qquad T^{\mu} = \left.\frac{\partial x^{\mu}}{\partial t}\right|_{s=0}$$

$$\Rightarrow T^{\nu}(T^{\sigma} Y^{\mu}{}_{,\sigma})_{,\nu} + \partial_{\mu}(\Gamma^{\mu}_{\nu\sigma})_{s=0} T^{\nu} T^{\sigma} Y^{\rho} + 2\Gamma^{\mu}_{\rho\sigma}\frac{\partial Y^{\rho}}{\partial t} T^{\sigma} = 0$$

At $p=0$, $\Gamma=0$, so $T^{\nu}(T^{\sigma} Y^{\mu}{}_{;\sigma} - \Gamma^{\mu}_{\rho\sigma} T^{\sigma} Y^{\rho})_{;\nu} + (\partial_{\rho}\Gamma^{\mu}_{\nu\sigma})_p T^{\nu} T^{\sigma} Y^{\rho} = 0.$

only thing left over is when partial deriv. hits Γ

t.b.c. (lect. 12) $\Rightarrow T^{\nu}(T^{\sigma} Y^{\mu}{}_{;\sigma})_{;\nu} + (\partial_{\rho}\Gamma^{\mu}_{\nu\sigma} - \partial_{\nu}\Gamma^{\mu}_{\rho\sigma}) T^{\nu} T^{\sigma} Y^{\rho} = 0$ (reminder at $p=0$)

$$\Rightarrow (\nabla_T \nabla_T Y)^{\mu} + R^{\mu}{}_{\sigma\rho\nu} T^{\nu} T^{\sigma} Y^{\rho} = 0$$

$$\Rightarrow \boxed{\nabla_T \nabla_T Y + R(Y,T) = 0}$$

GEODESIC ~~EQN~~ DEVIATION JACOBI EQN.

//

note: $\nabla_T \nabla_T S = R(T,S)T$

main $T^a \nabla_a (T^b \nabla_b S^d) = T^a S^b T^c R^{d}{}_{cab}$

result $\Rightarrow$ curvature results in relative acceleration of geodesics

(c.f. in euclidean & minkowski space, in which parallel geodesics remain parallel forever)

SUMMARY
- $R^a{}_{b(cd)} = 0$;
- $R^a{}_{[bcd]} = 0$; $R^a{}_{b[cd;e]} = 0$ (∇ TORSION FREE)
- $R_{abcd} = R_{cdab}$ (∇ LEVI CIVITA)

Lecture 12

6.11.24

SYMMETRIES OF THE RIEMANN TENSOR

From the definition it's clear that $R^a{}_{bcd} = -R^a{}_{bdc} \Leftrightarrow R^a{}_{b(cd)} = 0$

PROPOSITION: If ∇ is torsion free then $R^a{}_{[bcd]} = 0$

antisymmetric over 3 downstairs indices

PROOF: Fix $p \in M$, choose normal coordinates at p and work in coordinate basis, then

use normal coords to simplify

$$\Gamma^\sigma_{\mu\nu}|_p = 0 \quad \text{and} \quad \Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu} \text{ everywhere}$$

torsion free

then

antisymmetrise over downstairs indices then both of these terms vanish

$$R^\mu{}_{\nu\rho\sigma}|_p = \partial_\rho(\Gamma^\mu_{\nu\sigma})|_p - \partial_\sigma(\Gamma^\mu_{\nu\rho})|_p$$

do the long way or not? note use: 1st term symmetric in $\nu\sigma$, 2nd term symmetric in $\nu\rho$ so antisymmetrising both vanish (since we keep only antisymmetric part :))

useful trick we will use again & again to prove stuff: establish it holds at arbitrary point p so holds everywhere (normal coords)

$$\Rightarrow R^\mu{}_{[\nu\rho\sigma]}|_p = 0 \quad \text{as} \quad \partial_\rho\Gamma^\mu_{\nu\sigma}|_p = \partial_\rho\Gamma^\mu_{\sigma\nu}|_p$$

p arbitrary and so $R^\mu{}_{[\nu\rho\sigma]} = 0$ everywhere $\square$.

(i.e. if components go to vanish in one basis then they vanish in all bases)

PROPOSITION: If ∇ is torsion free then the Bianchi Identity holds:

BIANCHI IDENTITY

$$\boxed{R^a{}_{b[cd;e]} = 0}$$

(there is a 1st, 2nd bianchi identity, if we say just bianchi identity we usually mean this one)

PROOF: Choose coordinates as above then $R^\mu{}_{\nu\rho\sigma;\tau}|_p = R^\mu{}_{\nu\rho\sigma,\tau}|_p$ schematically,

since $\Gamma|_p = 0$

(Γ contracted into R & Γ vanish at p)

$$R \sim \partial\Gamma + \Gamma\Gamma \quad \text{so} \quad \partial R \sim \partial\partial\Gamma + \partial\Gamma\cdot\Gamma$$

(only terms that are going to contribute are linear 2nd order derivatives of Γ)

what purpose of this step is? just use same expression for R as above? see if I symmetrise

and since $\Gamma|_p = 0$ we deduce

$$R^\mu{}_{\nu\rho\sigma,\tau}|_p = \partial_\tau\partial_\rho\Gamma^\mu_{\nu\sigma}|_p - \partial_\tau\partial_\sigma(\Gamma^\mu_{\nu\rho})|_p .$$

By symmetry of the mixed partial derivatives, we see

$$R^\mu{}_{\nu[\rho\sigma;\tau]}|_p = 0$$

(same logic as above: 1st term symmetric in $\tau\rho$, second in $\tau\sigma$ so $[\rho\sigma\tau]$ indices it all vanish)

(these identities hold for any torsion free connection)

since p arbitrary result follows $\square$.

just this going to unlock possibilities

---

* Suppose ∇ is the Levi-Civita connection of a manifold with metric g we can lower an index with g_{ab} and consider R_{abcd}.

(reminder: since $g_{ab}\nabla_c X^a = \nabla_c(g_{ab}X^a) = \nabla_c X_b$)

claim R_{abcd} has additional symmetries, lots of symmetries

PROPOSITION: R_{abcd} satisfies $R_{abcd} = R_{cdab}$ $\quad (\Rightarrow R_{(ab)cd} = 0)$

PROOF: Pick normal coordinates at p so that $\partial_\mu g_{\nu\rho} = 0$. we notice that

connection component vanish at p but also partial deriv. vanishes

$$0 = \partial_\mu \delta^\nu_\sigma|_p = \partial_\mu(g^{\nu\tau}g_{\tau\sigma})|_p = (\partial_\mu g^{\nu\tau})g_{\tau\sigma}|_p$$

because g non degenerate

$$\Rightarrow \partial_\mu g^{\nu\tau}|_p = 0$$

hence $\partial_\rho(\Gamma^\mu_{\nu\sigma})|_p = \partial_\rho(\tfrac{1}{2}g^{\mu\tau}(g_{\tau\sigma,\nu} + g_{\nu\tau,\sigma} - g_{\nu\sigma,\tau}))|_p$
$= \tfrac{1}{2}g^{\mu\tau}(g_{\tau\sigma,\nu\rho} + g_{\nu\tau,\sigma\rho} - g_{\nu\sigma,\tau\rho})|_p$

we have $R_{\mu\nu\rho\sigma}|_p = g_{\mu\kappa}(\partial_\rho\Gamma^\kappa_{\nu\sigma} - \partial_\sigma\Gamma^\kappa_{\nu\rho})|_p$
$= \tfrac{1}{2}(g_{\mu\sigma,\nu\rho} + g_{\nu\rho,\mu\sigma} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma})|_p$

(I have to look at this not using normal coords if you don't want to sleep tonight!!)

(note to self: use equality of mixed partial derivatives & symmetry of the metric)

(If I swap 1st 2 indices it goes into itself (?) so)

This satisfies $R_{\mu\nu\rho\sigma}|_p = R_{\rho\sigma\mu\nu}|_p$ hence true everywhere $\square$.

(note: "tensor eqn holds so valid in any basis")

<u>COROLLARY:</u> The Ricci tensor is symmetric $R_{ab} = R_{ba}$ $\rightarrow$ $R_{ab} = g^{dc}R_{dacb} = g^{dc}R_{cbda} = R^d{}_{bda} = R_{ba}$ //

(we $R_{ab} = R^c{}_{acb}$ and $R^a{}_{bcd} = g^{ae}R_{ebcd}$)

(w/ metric we can go further, one more contraction)

DEFINITION: * The Ricci scalar (scalar curvature) is $R = R_a{}^a = g^{ab}R_{ab}$.
* The Einstein tensor is $G_{ab} = R_{ab} - \tfrac{1}{2}g_{ab}R$

(construct from Ricci scalar)

<u>Exercise:</u> The Bianchi identity implies $\nabla_a G^a{}_b = 0$ (contracted Bianchi identity)

(important for when we construct einstein equations)

($\leftarrow$ the gauge group of einstein eqns.)

DIFFEOMORPHISMS AND THE LIE DERIVATIVE

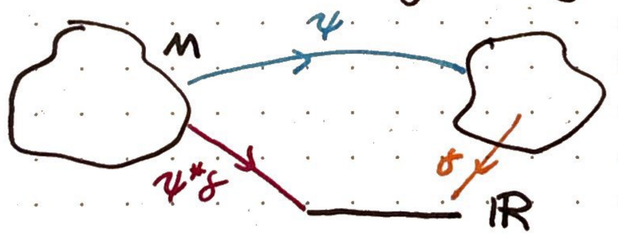
Suppose $\psi: M \to N$ is a smooth map, then ψ induces maps between corresponding vector/covector bundles.

DEFINITION: Given $f: N \to \mathbb{R}$, the PULL BACK of f by ψ is the map $\psi^* f: M \to \mathbb{R}$ given by $\psi^* f(p) = f(\psi(p))$

("f acting on image of p"...)

(notes: $\psi: M \to N$ is smooth iff $\phi_\alpha \circ \psi \circ \phi_\alpha^{-1}$ is smooth for all charts of M and N (ϕ_α & ϕ_α resp.))

(notes: 'pull back' a fn. on N to define a fn. on M)



[notes: def $\psi^*(f) = f \circ \psi$]

DEFINITION: Given $X \in T_pM$, we define the PUSH FORWARD of X by ψ, $\psi_* X \in T_{\psi(p)}N$ as follows,

Let $\lambda: I \to M$ be a curve with $\lambda(0) = p$, $\dot\lambda(0) = X$. Then set $\tilde\lambda = \psi \circ \lambda$, $\tilde\lambda: I \to N$ gives a curve in N with $\tilde\lambda(0) = \psi(p)$. We set $\psi_* X = \dot{\tilde\lambda}(0)$.

(notes: we can "push forward" a curve λ in M to a curve $\psi \circ \lambda$ in N, hence we can push forward vectors from M to N)

(notes: if λ is a curve in M with $\lambda(0) = p$, $\dot\lambda(0) = X$, then $\psi_* X$ is defined as $\dot{\tilde\lambda}(0)$ where $\tilde\lambda = \psi \circ \lambda$ is a curve in N with $\tilde\lambda(0) = \psi(p)$)

<u>Note:</u> If $f: N \to \mathbb{R}$ then $\psi_* X(f) = \frac{d}{dt}(f \circ \tilde\lambda(t))|_{t=0} = \frac{d}{dt}(f \circ \psi \circ \lambda(t))|_{t=0}$
$= \frac{d}{dt}(\psi^* f \circ \lambda(t))|_{t=0} = X(\psi^* f)$

(reminder: $X_p(f) := \frac{d}{dt} f(\lambda(t))|_{t=0}$)

$\psi_* X(f) = X(\psi^* f)$

$((\psi^* \omega)_p(v) = \ldots ?)$

Exercise: If x^μ are coords on M near p, y^α are coords on N near $\psi(p)$ then ψ gives a map $y^\alpha(x^\mu)$. Show that in a coordinate basis

important

$$(\psi_* X)^\alpha = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)_p X^\mu \qquad \left(\text{or } \psi_*\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)_p \left(\frac{\partial}{\partial y^\alpha}\right)\right)$$

(attempt: like this??) $f \circ \psi \circ \lambda(t) = f \circ \phi_\alpha^{-1} \circ \phi_\alpha \circ \lambda(t)$ so $\frac{d}{dt}(f(\lambda(t))) = \frac{\partial}{\partial y}(f \circ \phi_\alpha^{-1}) \frac{dy^\alpha}{dx^\mu}\frac{dx^\mu}{dt} \Rightarrow (\psi_* X)^\alpha = \frac{dy^\alpha}{dx^\mu}\frac{dx^\mu}{dt}$?? (or in short $f \circ \psi \circ \lambda = f \circ y(x(t))$)

$\Rightarrow \psi_*\left(\frac{\partial}{\partial x^\mu}\right)_p(f) = \left(\frac{\partial}{\partial x^\mu}\right)_p(\psi^* f) = \left(\frac{\partial}{\partial x^\mu}\right)_p(f \circ y(x^\mu))$ ...

On cotangent bundle we go backwards

DEFINITION: If $\eta \in T_{\psi(p)}^*N$, then the pullback of η, $\psi^*\eta \in T_p^*M$, is defined by

(note this is eg: $df(X) = X(f)$?)

$$\psi^*\eta(X) = \eta(\psi_* X) \qquad \forall X \in T_pM$$

(worth checking where these identities come from, look back)

Note: If $f: N \to \mathbb{R}$, $\psi^*(df)[X] = df[\psi_* X] = \psi_* X(f) = X(\psi^* f)$
$$= d(\psi^* f)[X]$$

(use $\psi^*\eta(X) = \eta(\psi_* X)$, $df(X) = X(f)$, $\psi_* X(f) = X(\psi^* f)$.)

(since X arbitrary)
$$\Rightarrow \psi^* df = d(\psi^* f)$$

notes "i.e. gradient commutes with pull back"

Exercise: With notation as before, show that

important

$$(\psi^*\eta)_\mu = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)_p \eta_\alpha \qquad \left(\text{or } \psi^*(dy^\alpha)_p = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)_p (dx^\mu)_p\right).$$

attempt (i.e. ...):
- consider $(\psi^*\eta)(X) = \eta(\psi_* X)$
- plug in $\psi_* X = X^\mu \frac{dy^\alpha}{dx^\mu}\frac{\partial}{\partial y^\alpha}$
- remember $\eta(e_\alpha) = \eta_\alpha$
- so get $(\psi^*\eta)(X) = X^\mu \frac{dy^\alpha}{dx^\mu}\eta_\alpha \to$ extract components $(\psi^*\eta)_\mu = \frac{dy^\alpha}{dx^\mu}\eta_\alpha$?

(likewise consider $\psi^*(dy^\alpha)(X) = X^\mu \frac{dy^\alpha}{dx^\mu}$, compare this w/ what we want $\frac{dy^\alpha}{dx^\mu} dx^\mu$)

* We can extend the pullback to map a $(0,s)$-tensor at $\psi(p) \in N$ to a $(0,s)$-tensor ψ^*T at $p \in M$ by requiring

$$\psi^*T(X_1, \dots, X_s) = T(\psi_* X_1, \dots, \psi_* X_s) \qquad \forall X_i \in T_pM.$$

* Similarly we can push forward a $(s,0)$-tensor S at $p \in M$ to a $(s,0)$-tensor $\psi_* S$ at $\psi(p) \in N$ by

(note here: in components pullback $\Rightarrow$ normal things to downstairs indices $\frac{dy}{dx}$; pushforward $\Rightarrow$ normal things to upstairs indices $\frac{dy}{dx}$)

$$\psi_* S(\eta_1, \dots, \eta_s) = S(\psi^*\eta_1, \dots, \psi^*\eta_s) \qquad \forall \eta_i \in T_{\psi(p)}^*N.$$

* If $\psi: M \to N$ has the property that $\psi_*: T_pM \to T_{\psi(p)}N$ is injective (one-to one), we say ψ is an immersion $(\dim N \geq \dim M)$.

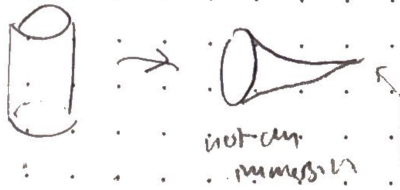
see lecture notes for examples

* If N is a manifold with metric g, and $\psi: M \to N$ is an immersion, we can consider $\psi^* g$ pull back of g

* If g is Riemannian, then $\psi^* g$ is non-degenerate and positive definite, so defines a metric on M, the induced metric.

Note: If we pull back a Lorentzian metric, there is no guarantee it ends up on the manifold. will talk about Lorentzian metric later.

SUMMARY
- maps between manifolds induce maps between cotangent bundles
- $\psi^* f = f \circ \psi$; $\psi_* X(f) = X(\psi^* f)$; $\psi^*\eta(X) = \eta(\psi_* X)$
  ($\psi: M \to N$, $X \in T_pM$, $f: N \to \mathbb{R}$, $\eta \in T^*_{\psi(p)}N$)
- extend to general tensor
- if $\psi_*: T_pM \to T_{\psi(p)}N$ injective, ψ is an IMMERSION
- if ψ is an immersion, N riemannian manifold, then $\psi^* g$ is a metric on M



injective means 2 vectors go to 2 diff directions in the image, so this is not an immersion since more than one vector pointing in the same way here.

(or tensors, notes) $\psi^* S_{\mu_1 \dots \mu_s} = \left(\frac{dy^{\alpha_1}}{dx^{\mu_1}}\right)_p \cdots \left(\frac{dy^{\alpha_s}}{dx^{\mu_s}}\right)_p S_{\alpha_1 \dots \alpha_s}$ $\quad (\psi^* g)_{\mu\nu} = \left(\frac{dy^\alpha}{dx^\mu}\right)\left(\frac{dy^\beta}{dx^\nu}\right) g_{\alpha\beta}$

$\psi_* T^{\alpha_1 \dots \alpha_r} = \left(\frac{dy^{\alpha_1}}{dx^{\mu_1}}\right)_p \cdots \left(\frac{dy^{\alpha_r}}{dx^{\mu_r}}\right)_p T^{\mu_1 \dots \mu_r}$ $\quad = \left(\frac{dy^\alpha}{dx^\mu}\right)\left(\frac{dy^\alpha}{dx^\nu}\right)$

Lecture 13

8.11.24

δ_2 → find $(\psi^* g)_{\theta\theta} = 1$, $(\psi^* g)_{\theta\phi} = (\psi^* g)_{\phi\theta} = 0$ and $(\psi^* g)_{\phi\phi} = \sin^2\theta$ → $\psi^* g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$:)

Exercise: Let $(N, g) = (\mathbb{R}^3, \delta)$, $M = S^2$. Let ψ be the map taking a point on S^2 w/ spherical coordinates (θ, ϕ) to $(x^1, x^2, x^3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ then

(if you can do this you understood last lecture)

$$\psi^*((dx^1)^2 + (dx^2)^2 + (dx^3)^2) = d\theta^2 + \sin^2\theta\, d\phi^2 ?$$

[get: we took the standard metric δ on $\mathbb{R}^3$, pulled it back via the embedding of the sphere & recovered the familiar spherical metric $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$]

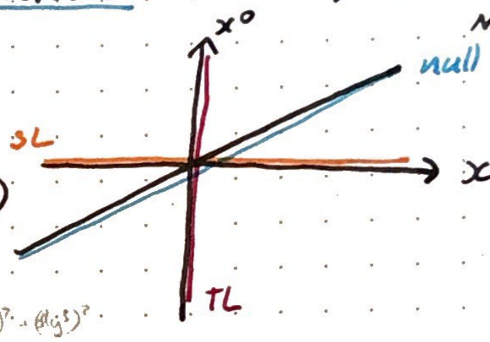
If ψ immersion, (N, g) is Lorentzian, then $\psi^* g$ is not in general a metric on M. There are 3 important cases:

* $\psi^* g$ is a Riemannian metric $\Rightarrow \psi(M)$ is spacelike
* $\psi^* g$ is a Lorentzian metric $\Rightarrow \psi(M)$ is timelike
* $\psi^* g$ is everywhere degenerate $\Rightarrow \psi(M)$ is null

MINKOWSKI

$\psi : (y^1, y^2, y^3) \mapsto (0, y^1, y^2, y^3)$

$\psi(\mathbb{R}^3)$ is spacelike.



$(\psi^* g) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \Rightarrow ds^2 = (dy^1)^2 + (dy^2)^2 + (dy^3)^2$ riemannian ⇒ spacelike

$(\dim M = \dim N)$?

Recall that $\psi : M \to N$ is a diffeomorphism if it is bijective with a smooth inverse (& is smooth). If we have a diffeomorphism we can push forward a general (r,s)-tensor at p to an (r,s)-tensor at $\psi(p)$ by

note: if have a diffeomorphism we can extend our definitions of push forward & ~~pull back~~ to apply to a general tensor

$$\psi_* T(\eta^1, \dots, \eta^r, X_1, \dots, X_s) = T(\psi^*\eta^1, \dots, \psi^*\eta^r, \psi_*^{-1} X_1, \dots, \psi_*^{-1} X_s)$$

note: $\psi^{-1}/*$

push forward of (1,1) tensor components

$\forall\, \eta_1, \dots, \eta_r \in T^*_{\psi(p)} N$

notes: $(\psi_* T)^\mu{}_\nu = \left(\frac{\partial y^\mu}{\partial x^\rho}\right)\left(\frac{\partial x^\sigma}{\partial y^\nu}\right) T^\rho{}_\sigma$

$\forall\, X_1, \dots, X_s \in T_{\psi(p)} N$

Define a pull-back by $\psi_*^{-1} = \psi^*$

general r-s tensor (?): $(\psi_* S)^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = \left(\frac{\partial y^{\mu_1}}{\partial x^{\rho_1}}\right) \cdots \left(\frac{\partial y^{\mu_r}}{\partial x^{\rho_r}}\right)\left(\frac{\partial x^{\sigma_1}}{\partial y^{\nu_1}}\right) \cdots \left(\frac{\partial x^{\sigma_s}}{\partial y^{\nu_s}}\right) S^{\rho_1 \dots \rho_r}{}_{\sigma_1 \dots \sigma_s}$

$(\psi^* T)^{\rho_1 \dots \rho_r}{}_{\sigma_1 \dots \sigma_s} = \left(\frac{dx^{\rho_1}}{dy^{\mu_1}}\right) \cdots \left(\frac{dx^{\rho_r}}{dy^{\mu_r}}\right)\left(\frac{dy^{\nu_1}}{dx^{\sigma_1}}\right) \cdots \left(\frac{dy^{\nu_s}}{dx^{\sigma_s}}\right) T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}$

If M, N are diffeomorphisms, we often don't distinguish between them we can think of $\psi : M \to M$

(ψ an isometry means $\psi_* g$ is a symmetry ($\psi_* g = g$))

We say a diffeomorphism $\psi : M \to M$ is a symmetry of T if $\psi_* T = T$. If T is the metric, we say ψ is an Isometry. e.g in Minkowski space w/ an inertial frame

$$\psi(x^0, x^1, \dots, x^n) = (x^0 + 1, x^1, \dots, x^n) \text{ is a symmetry of } g.$$

An important class of diffeomorphisms are those generated by a vector field.

If X is a smooth vector field, we associate to each point $p \in M$ the point ~~ψ~~ $\psi_t^X(p) \in M$ given by flowing a parameter distance t along reminder to self: integral curve of a vector field is curve where the tangent to the curve is equal to the vector field at that point

the integral curve of X starting at p.

Suppose $\psi_t^X(p)$ is well defined for all $t \in I \subset \mathbb{R}$ for each $p \in M$ then $\psi_t^X : M \to M$ is ~~well defined~~ a diffeomorphism for all $t \in I$.

reminder: bijective, smooth, smooth inverse

Further

* If $t, s, t+s \in I$ then $\boxed{\psi_t^X \circ \psi_s^X = \psi_{t+s}^X \text{ and } \psi_0^X = id}$ $(*)$

  If $I = \mathbb{R}$ this gives $\{\psi_t^X\}_{t \in \mathbb{R}}$ the structure of a one parameter abelian group. 'dont worry what this is if you dont know'

  (i.e. ψ_t^X well defined for all $t \in \mathbb{R}$)

* If ψ_t is any smooth family of diffeomorphisms satisfying $(*)$ we can define a vector field by

$$X(p) = \frac{d}{dt}(\psi_t(p))\Big|_{t=0} \quad \text{then } \psi_t = \psi_t^X$$

note to self: if we satisfy $*$ then $\psi_t^X(p)$ is a curve through p parametrised by t so we can define a vector field to be the tangent to this curve at p by doing this for all p, e.g. $X_{\lambda(t)} = \frac{d\lambda}{dt}(t)$

We can use ψ_t^X to compare tensors at different points as $t \to 0$ this gives a new notion of derivative.

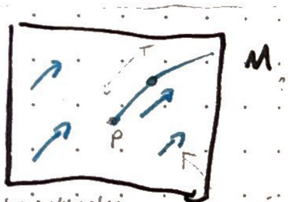
note to self: summary: $\psi_t^X : M \to M$ flows distance t along the integral curve of X from $p \in M$ to $\psi_t^X(p) \in M$ (defined for small enough t). * theorem shows ψ_t^X is a diffeomorphism

THE LIE DERIVATIVE

Suppose $\psi_t^X : M \to M$ is the smooth one-parameter family of diffeomorphisms generated by a vector field X.

DEFINITION: For a tensor field T, the Lie derivative of T with respect to X is

$$(\mathcal{L}_X T)_p = \lim_{t \to 0} \frac{((\psi_t^X)^* T)_p - T_p}{t}$$

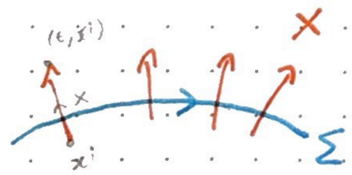


- start at p & flow along integral curve to point distance t away
- take tensor field at new point & pull it back to p
- then can compare it with T_p since both tensors are at the same point so live in the same tensor space
- take lim as $t \to 0$

It's easy to see that the constant α, β and (r,s) tensors S, T

$$\mathcal{L}_X(\alpha S + \beta T) = \alpha \mathcal{L}_X S + \beta \mathcal{L}_X T$$

To see how $\mathcal{L}_X$ acts in components, it's helpful to construct coordinates adapted to X.



Near p we can construct an $(n-1)$-surface Σ which is transverse to X (i.e. nowhere tangent). Pick coords x^i on Σ and assign the coordinate (t, x^i) to the point a parameter distance t along the integral curve of X ~~satisfying~~ starting at x^i on Σ.

In these coords, $X = \frac{\partial}{\partial t}$ and $\psi_t^X(\tau, x^i) = (\tau + t, x^i)$.

So if $y^\mu = (\psi_t^X)^*(x^\mu)$ then $\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu$ and

$$\left[(\psi_t^X)^* T\right]^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\Big|_{(\tau, x^i)} = T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\Big|_{(\tau + t, x^i)}$$

$\left(\lim_{t \to 0} \frac{T(t+t, x^i) - T(t, x^i)}{t} \quad \frac{\partial T}{\partial t} \right)$

thus $\quad (\mathcal{L}_X T)^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\big|_p = \frac{\partial T}{\partial t}^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\big|_p$

So in these coords, $\mathcal{L}_X$ acts on components by $\frac{\partial}{\partial t}$.
In particular we immediately see

$\mathcal{L}_X$ obeys leibniz:

① $\mathcal{L}_X (S \otimes T) = (\mathcal{L}_X S) \otimes T + S \otimes \mathcal{L}_X T$ ② $\mathcal{L}_X$ commutes w/ contraction

e.g. $\left(\mathcal{L}_X (T(X,Y)) = (\mathcal{L}_X T)(X,Y) + T(\mathcal{L}_X X, Y) + T(X, \mathcal{L}_X Y) \right)$ ← i think?

To write $\mathcal{L}_X$ in a coordinate-free fashion, we can simply seek a basis independent expression that agrees with $\mathcal{L}_X$ in these coords.

E.g. * For a function $\mathcal{L}_X f = \frac{\partial f}{\partial t} = X(f)$ in these coords

(note: $\mathcal{L}_X f = X(f)$. both sides are scalars so this must be vertical in any basis)

* For a vector field Y we observe that

$$(\mathcal{L}_X Y)^\mu = \frac{\partial Y^\mu}{\partial t} = X^\sigma \frac{\partial}{\partial x^\sigma}(Y^\mu) - Y^\sigma \frac{\partial}{\partial x^\sigma} X^\mu = [X, Y]^\mu$$

(note this follows since in these coords $X = \frac{\partial}{\partial t}$ so $X^\mu = \delta^\mu_t$; components are const so this vanishes)

point basis to compute ← $\therefore \mathcal{L}_X Y = [X, Y]$

note: if 2 vectors have the same components in one basis then they are equal in all bases so

note: basis indep result

Exercise: • In any coord basis show if ω_a is a covector field,

$$(\mathcal{L}_X \omega)_\mu = X^\sigma \partial_\sigma \omega_\mu + \omega_\sigma \partial_\mu X^\sigma$$

(attempt: $(\mathcal{L}_X \omega)_\mu = X(\omega_\mu) = \frac{\partial \omega_\mu}{\partial t} = \frac{\partial x^\sigma}{\partial t} \frac{\partial}{\partial x^\sigma} \omega_\mu = X^\sigma \partial_\sigma \omega_\mu$ ← 2nd term vanishes in coords; if they have same components in one basis they have same components in all bases//)

think like we did for ∇: leibniz: consider $\mathcal{L}_X(\omega(Y)) = (\mathcal{L}_X \omega)(Y) + \omega(\mathcal{L}_X Y)$ · LHS is $\mathcal{L}_X$ of fn. · 1st term is what we want · 2nd term is $\mathcal{L}_X$ of vector $= [X, Y]$ · now write in coords

($(\mathcal{L}_X \omega)_\mu = X^\sigma \partial_\sigma \omega_\mu + \omega_\sigma \partial_\mu X^\sigma$ work in normal coords & ∇ torsion free $\Rightarrow \Gamma = 0$ so $= X^\sigma \nabla_\sigma \omega_\mu + \omega_\sigma \nabla_\mu X^\sigma$ tensor eq so promote to ATN//)

if ∇ is torsion free

$$(\mathcal{L}_X \omega)_a = X^b \nabla_b \omega_a + \omega_b \nabla_a X^b$$

• If g_{ab} is a metric tensor, ∇ levi-civita, then

similarly consider $\mathcal{L}_X(g(Y,Z))$. find $(\mathcal{L}_X g)_{\mu\nu} = X^\delta \partial_\delta g_{\mu\nu} + g_{\delta\nu} \partial_\mu X^\delta + g_{\mu\delta} \partial_\nu X^\delta$ work in normal coords: $\partial_\delta g_{\mu\nu}$ vanishes; promote to ∇ (since $\Gamma = 0$ again) and use the fact ∇ commutes w/ metric $\Rightarrow (\mathcal{L}_X g)_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu$ > tensor eq so promote to ain//

$$(\mathcal{L}_X g)_{ab} = \nabla_a X_b + \nabla_b X_a$$

If ψ_t^X is a one-parameter family of isometries for a manifold with metric g, then $\mathcal{L}_X g = 0$.

Conversley, if $\mathcal{L}_X g = 0$ then X generates a one-parameter family of isometries.

(consider: isometry $\psi_t^* g = g$ (ψ_t is a symmetry of the metric))

DEFINITION: A vector field K satisfying $\mathcal{L}_K g = 0$ is called a KILLING VECTOR. It satisfies Killing's equation

use above $\partial_\mu g_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu \rightarrow$

$$\nabla_a K_b + \nabla_b K_a = 0 \qquad (\nabla \text{ levi-Civita})$$

LEMMA: Suppose K is killing and $\lambda : I \to M$ is a geodesic of the Levi-Civita connection. Then $g_{ab} \dot\lambda^a K^b$ is const. along λ.

(tangent to geodesic satisfies: $\nabla_X X$ assuming affinely parametrised?)

PROOF: $\frac{d}{dt}(K_b \dot\lambda^b) = \dot\lambda^a \nabla_a (K_b \dot\lambda^b) = \underbrace{(\nabla_a K_b)}_{0 \text{ (K killing)}} \dot\lambda^a \dot\lambda^b + K_b \underbrace{\dot\lambda^a \nabla_a \dot\lambda^b}_{\text{vanishes for geodesic}}$ □

($\frac{d}{dt} = \dot\lambda$ since ... $X(f) = \nabla_X f$ so $\dot\lambda(f) = \nabla_{\dot\lambda} f = \dot\lambda^a \nabla_a$)

basically gravity is geometry ... the ... new einstein eq. ?

this is symmetric in a,b so when $\nabla_a K_b$ symmetric part left when ... since killing?

note to self: we assume ∇ is Levi-Civita here? not sure

note to self: (spacetime (M,g) for any $p \in M$ we can introduce normal coordinates s.t. in a neighbourhood of p $g_{\mu\nu} = \eta_{\mu\nu}$ + higher order terms so locally looks like Minkowski (to 1st order) ?)

Lecture 14

11.11.24

Physics in Curved SpaceTime

Minkowski space (special relativity)

notes: begin w/ special relativity in an inertial frame, Levi-Civita ∇_μ is the same as ∂_μ so to promote our spec. rel. equations so they valid for an arbitrary basis, we replace ∂_μ with ∇_μ and replace Greek indices w/ Latin ones

We review physical theorems in Minkowski $\mathbb{R}^{1+3}$ equipped with $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,)$ we set $c=1$.

(enlarge range of validity of equations: ① $\partial_\mu \to \nabla_\mu$ ② $\mu \to a$)

① $\boxed{\text{Klein-Gordon equation}}$

$$\partial_\mu \partial^\mu \phi - m^2 \phi = 0 \quad (a)$$

Note that in inertial cords $\partial_\mu = \nabla_\mu$ so we can write this in a covariant manner as

promote partial to nabla

$$\nabla_a \nabla^a \phi - m^2 \phi = 0$$

(③. if also replace η_{ab} with arbitrary metric g_{ab} (taking ∇ to be Levi-Civita connecting the metric) then we get eq. valid in an arbitrary spacetime)

Associate to (a) is the energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \tfrac{1}{2} \eta_{\mu\nu} (\partial_\sigma \phi \partial^\sigma \phi + m^2 \phi^2)$$

or covariantly

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \tfrac{1}{2} \eta_{ab} (\nabla_c \phi \nabla^c \phi + m^2 \phi^2)$$

This satisfies $T_{ab} = T_{ba}$, (check) $\nabla_a T^a{}_b = 0$

② $\boxed{\text{Maxwell's eq}^{\underline{n}}}$ (equations)

The Maxwell field is an anti-symmetric $(0,2)$-tensor

$F_{\mu\nu} = -F_{\nu\mu}$ where $F_{0i} = E_i$, $F_{ij} = \varepsilon_{ijk} B_k$ $\quad (i,j = 1,2,3)$

If j_μ is the charge current density, then Maxwell's eq$^{\underline{n}}$ are

$$\left.\begin{aligned} \partial_\mu F^\mu{}_\nu &= 4\pi j_\nu & \qquad \nabla_a F^a{}_c &= 4\pi j_c \\ \partial_{[\mu} F_{\nu\sigma]} &= 0 & \qquad \nabla_{[a} F_{bc]} &= 0 \end{aligned}\right\} \quad (b)$$

Associated to (b) is the energy-momentum tensor

$$T_{\mu\nu} = F_\mu{}^\sigma F_{\nu\sigma} - \tfrac{1}{4} \eta_{\mu\nu} F_{\sigma\tau} F^{\sigma\tau}$$

$$\to T_{ab} = F_a{}^c F_{bc} - \tfrac{1}{4} \eta_{ab} F_{cd} F^{cd}$$

Check $T_{ab} = T_{ba}$, $\nabla_a T^a{}_b = 0$ for source-free Maxwell

(use $\nabla_{[a} F_{bc]}$ and $F_{ab} = -F_{ba}$ (did this on paper 2024?))

③ $\boxed{\text{Perfect Fluid}}$

A perfect fluid is described by a local velocity field U^μ

normalize (idealised fluid (no viscosity), perfectly described by P, ρ and 4-velocity)

satisfying $U^\mu U_\mu = -1$, together with a pressure P and density ρ. They satisfy the first law of thermodynamics

$$U^\mu \partial_\mu \rho + (\rho + P)\partial_\mu U^\mu = 0$$

covariant eq

$$\rightarrow U^a \nabla_a \rho + (\rho + P)\nabla_a U^a = 0$$

and Euler's equation (read as conservation momentum law)

$$(\rho + P)U^\nu \partial_\nu U^\mu + \partial^\mu P + U^\mu U^\nu \partial_\nu P = 0$$

see cosmology course, we deal w/ this

fluid velocity: moves along w/ fluid

$$\rightarrow (P + \rho)U^b \nabla_b U^a + \nabla^a P + U^a U^b \nabla_b P = 0$$

because if i set $a=0$ i get nothing. essentially 2 eq. need eq. ?? (eq of state) to find unknowns (3 deg of freedom)

Associated is the energy-momentum tensor

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + P\eta_{\mu\nu}$$

(check)

$$\rightarrow T_{ab} = (\rho + P)U_a U_b + P\eta_{ab} \qquad \text{again } T_{ab} = T_{ba},\ \nabla_a T^a{}_b = 0$$

(g_{ab} in lecture notes? or what?)

Notice that in all these cases, we can, if we wish, promote the Minkowski η to a general Lorentzian metric g, and take ∇ to be the Levi-Civita connection. Consider normal coordinates, we see that coords exist near any point $q \in M$, such that the physics described is approximately Minkowskian, with correction of the order of curvature.

notes: do this ex. sheet 2 (estimate correction terms)

minimal coupling approach: we could have e.g. extra term to T_{ab} that just happens to vanish in minkowski: we don't know if these exist! but simplest approach is to say they don't.

GENERAL RELATIVITY

In Einstein's theory of general relativity we postulate that spacetime is a 4-dimensional Lorentzian manifold (M, g). We also require any matter model to consist of

equations

* some matter fields ϕ^o
* eq$^{\underline{n}}$ of motion for ϕ^o which are expressed geometrically in terms of g ($+\nabla, R \cdots$)
* an energy-momentum tensor T_{ab} depending on ϕ^o satisfies $T_{ab} = T_{ba}$, $\nabla_a T^a{}_b = 0$

The matter should reduce to a non-grav theory when (M, g) is fixed to be Minkowski.

The metric g should satisfy the Einstein equations

case of vacuum RHS $= 0$

einstein tensor RHS: he wanted divergence free, needed to add term Λg_{ab} to get there

LHS geometric

$$R_{ab} - \frac{R}{2}g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}$$

(view: RHS energy matter, i.e. energy of matter responsible for geometry is you choose i.e. gravitation)

Λ cosmological constant ($\Lambda > 0$ but small), G Newton's constant.

(tiny: for our purposes ignore Λ from now onwards)

The Einstein eq$^{\underline{n}}$ together with the EoMs for ϕ^o, constitute a coupled system which must be solved simultaneously.

note: at point p on the trajectory of a test body (free particle), we can introduce a local inertial frame. equivalence principle states in this frame, any experiments should be indistinguishable from if it did the same thing in an inertial frame in minkowski. so, ⇒ the acceleration of the curve at p: $\nabla_{\dot{x}}\dot{x}$ must vanish as it does in minkowski

(holding so the curve must be a geodesic → motivates the geodesic postulate!)

(notes: geodesics of the levi civita connection)

GEODESIC POSTULATE

Free test particles move along timelike/null geodesics if they have non zero/zero rest mass.

Gauge Freedom

view einstein eq as evolution equations

notes: we have a gauge freedom/gauge symmetry if there exist distinct configurations that are physically equivalent. a gauge transformation maps a field configuration to a physically equivalent configuration

diffeomorphisms are gauge transformations in GR

Consider Maxwell with no sources

$$\partial_\mu F^\mu{}_\nu = 0 \quad \text{— m1}$$

$$\partial_{[\mu} F_{\nu\sigma]} = 0 \quad \text{— m2}$$

don't worry about factors of 2, claude doesn't include them

a standard approach to solve m2 is to introduce the gauge potential A_μ s.t. $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$, then m1 becomes

so A_μ

$$\partial_\mu \partial^\mu A_\nu - \partial_\mu \partial_\nu A^\mu = 0 \quad (*)$$

we want to specify initial data then solve for A_μ uniquely at a time. but can't bc of gauge

We'd like to solve $(*)$ given data at $\{x^0 = 0\} = \Sigma$. However this eq$^{\underline{n}}$ doesn't give a good evolution problem.

(initial data at surface Σ)

(a smooth fn on $\mathbb{R}^{1+3}$ that vanishes near Σ)

If $\chi \in C^\infty(\mathbb{R}^{1+3})$ which vanishes near Σ, then $\tilde{A}_\mu = A_\mu + \partial_\mu \chi$ will also solve $(*)$ and $\partial_{[\mu} \tilde{A}_{\nu]} = \partial_{[\mu} A_{\nu]}$.

(initial value prob. is not a well-posed problem. we don't have uniqueness of solutions?)

(gauge transformation since does not change the observable quantity $F_{\mu\nu}$)

To resolve this, we can fix a gauge e.g. if we assume $\partial_\mu A^\mu = 0$ then $(*)$ becomes

choose gauge condition

$$\partial_\mu \partial^\mu A_\nu = 0 \quad (*)$$

a wave eq$^{\underline{n}}$ for each cpt of A_ν.

component

(tra: wave eq for each component of with unique solutions given by specifying A_μ and its first time derivative on Σ (unique sols. to initial value problem))

L15:

$\partial_\mu g_{\nu\rho} \neq 0$ in general since metric may change in space

	which g commutes with?
∇_μ	yes (for levi civita).
∂_μ	no (unless g is const.)
Γ	yes
tensor contraction	yes

[A] Prove: $R_{\rho\sigma\mu\nu} = \frac{1}{2}(g_{\rho\nu,\mu\sigma} + g_{\sigma\mu,\nu\rho} - g_{\rho\mu,\nu\sigma} - g_{\sigma\nu,\mu\rho}) - \Gamma_{\mu\lambda\rho}\Gamma^\lambda{}_{\nu\sigma} + \Gamma_{\nu\rho}\Gamma^\mu$

① start w expression for $R^\mu{}_{\nu\rho\sigma}$ in terms of connection components.

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma}$$

② lower μ, remember g commutes w/ Γ but not ∂_ρ! so last 2 terms look like $\Gamma^\tau_{\nu\sigma}\Gamma_{\kappa\tau\rho}$

③ but 1st and 2nd term look like $g_{\mu\kappa}\partial_\rho \Gamma^\mu_{\nu\sigma}$ → need to find this

$$\to g_{\mu\kappa}\partial_\rho \Gamma^\mu_{\nu\sigma} = \underbrace{\partial_\rho (g_{\mu\kappa}\Gamma^\mu_{\nu\sigma})}_{\partial_\rho \Gamma_{\nu\mu\sigma}} - g_{\mu\kappa,\rho}\Gamma^\mu_{\nu\sigma}$$

find this in terms of Γ: use expr for Γ in terms of g, and lower index:

$$\to \Gamma_{\nu\kappa\mu} = \tfrac{1}{2}(g_{\kappa\nu,\mu} + g_{\mu\kappa,\nu} - g_{\mu\nu,\kappa})$$

→ to isolate $g_{\mu\kappa,\nu}$ symmetrize over 2nd two indices ($\kappa\mu$) (would equally work for 1st two)

this gives $g_{\mu\kappa,\nu} = \Gamma_{\nu\kappa\mu} + \Gamma_{\nu\mu\kappa}$

④ plug in $\partial_\rho \Gamma$ terms and simplify, gives

$$R_{\kappa\nu\rho\sigma} = \partial_\rho \Gamma_{\nu\kappa\sigma} - \partial_\sigma \Gamma_{\nu\kappa\rho} - \Gamma_{\rho\mu\kappa}\Gamma_\nu{}^\mu{}_\sigma + \Gamma_{\sigma\mu\kappa}\Gamma_\nu{}^\mu{}_\rho$$

⑤ find $\partial_\rho \Gamma_{\nu\kappa\sigma}$ from our expression for $\Gamma_{\nu\kappa\mu}$ above & plug that in //

[B] Prove $R_{\sigma\nu} = -\frac{1}{2}g^{\mu\rho}g_{\sigma\nu,\mu\rho} + \frac{1}{2}\partial_\sigma \Gamma^\mu_{\mu\nu} + \frac{1}{2}\partial_\nu \Gamma^\mu_{\mu\sigma} - \ldots$ etc.

①. $R_{\sigma\nu} = g^{\tau\nu} R_{\tau\sigma\nu\rho}$, plug in our expression in [A]

the tricky bit is the $\frac{1}{2}(g_{\rho\nu,\mu\sigma} + g_{\sigma\mu,\nu\rho} - g_{\rho\mu,\nu\sigma})$ term

② we found before → $g_{\rho\nu,\mu\sigma} = \partial_\sigma(\Gamma_{\mu\rho\nu} + \Gamma_{\mu\nu\rho})$

so → $g_{\sigma\mu,\nu\rho} = \partial_\nu(\Gamma_{\rho\sigma\mu} + \Gamma_{\rho\mu\sigma})$

$\to g_{\rho\mu,\nu\sigma} = \frac{1}{2}\partial_\nu(\Gamma_{\sigma\rho\mu} + \Gamma_{\sigma\mu\rho}) + \frac{1}{2}\partial_\sigma(\Gamma_{\nu\rho\mu} + \Gamma_{\nu\mu\rho})$

oddly: i want to cancel so, terms to cancel... recognise symmetries in $\nu\sigma$ with other two terms ∂_σ, ∂_ν

③ combine & simplify, then note: if i plug this into our main expr for $R_{\sigma\nu}$ it is contracted w/ $g^{\tau\nu}$ so all the go-for-half terms vanish since they are antisymmetric in τ,ν

④ integrate the $\partial\sigma$ terms, like $g^{\mu\rho}(\partial_\sigma \Gamma_{\mu\nu\rho})$, notice this $= \partial_\sigma \Gamma_{\mu\nu}{}^\mu - \Gamma_{\mu\nu\rho}\partial_\sigma g^{\mu\rho}$

⑤ special piece is to figure out $\partial_\sigma g^{\mu\rho}$: i know from [A] $g_{\mu\kappa,\nu}$, so write that $\partial_\sigma \delta^\mu_\nu = 0 \Rightarrow \partial_\sigma g_{\nu\tau}g^{\mu\tau} = 0$ to find $\partial_\sigma g^{\mu\rho} = -g^{\mu\kappa}\partial_\sigma g_{\kappa\tau}g^{\tau\rho}$

⑥ put everything together! //

Lecture 15 13.11.24

Recap: Maxwell $\partial_\mu F^\mu{}_\nu = 0$, $\partial_{[\mu}F_{\nu\sigma]} = 0$. Let $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ then M1 $\Rightarrow$ $\boxed{\Box A_\nu - \partial_\mu \partial_\nu A^\mu = 0}$ (*) But cannot solve this as good evolution problem. For example, $\tilde{A}_\mu = A_\mu + \partial_\mu \chi$ also solves (*) with some F and some initial data $A_\nu|_{t=0}$, $\partial_0 A_\nu|_{t=0}$ (provided $\mathrm{supp}(\lambda) \cap \{t=0\} = \emptyset$). To resolve, we fix a gauge via $\partial_\mu A^\mu = 0$. Then (*) $\Rightarrow \Box A_\nu = 0$ (†) which has unique solution given $A_\nu|_{t=0}$, $\partial_0 A_\nu|_{t=0}$.

$\partial_\mu \partial^\mu$

CLAIM: There is no non-trivial χ s.t. $\tilde{A}_\nu$ also satisfies $\partial_\mu \tilde{A}^\mu = 0$ and the same initial condition. If we solve (†) for data s.t.

$$\partial_\mu A^\mu|_{t=0},\ \partial_0(\partial_\mu A^\mu)|_{t=0}$$

then we find $\partial_\mu A^\mu \equiv 0$, and so our solution solves (*) and hence Maxwell's equations

Gauge Freedom for Einstein

If (M,g) solves Einstein equations with E/M tensor T and $\psi: M \to M$ a diffeomorphism, then $\psi^* g$ solves the Einstein equations with E/M tensor $\psi^* T$. At a local level, this arises as the coordinate ~~indices~~ of the Einstein equations.

energy-momentum tensor

energy-momentum

independence/ invariance

manifests

In order to solve EEs, we need to find a way to fix coordinates. There are several approaches, we consider wave harmonic gauge.

(note to self: fixing harmonic coords turns vacuum EE into a well-posed evolution problem show this ↓)

LEMMA: In any local coordinate system

Riemann tensor

$$R_{\rho\sigma\mu\nu} = \tfrac{1}{2}(g_{\rho\nu,\mu\sigma} + g_{\sigma\mu,\rho\nu} - g_{\rho\mu,\nu\sigma} - g_{\sigma\nu,\mu\rho}) - \Gamma_{\mu\lambda\rho}\Gamma^\lambda{}_{\nu\sigma} + \Gamma_{\nu\lambda\rho}\Gamma^\lambda{}_{\mu\sigma}$$

if Minkowski metric this is the wave operator (curved wave operator)

and
$$R_{\sigma\nu} = \underbrace{-\tfrac{1}{2} g^{\mu\rho} g_{\sigma\nu,\mu\rho}}_{g^{\mu\rho}\partial_\mu\partial_\rho g_{\sigma\nu}} + \tfrac{1}{2}\partial_\sigma \Gamma^\mu{}_{\mu\nu} + \tfrac{1}{2}\partial_\nu \Gamma^\mu{}_{\mu\sigma} - \Gamma^\mu{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} + \Gamma_{\lambda\tau\nu}\Gamma^{\lambda\tau}{}_\sigma + \Gamma_{\lambda\tau\nu}\Gamma^{\tau\lambda}{}_\sigma + \Gamma_{\lambda\tau\sigma}\Gamma^{\tau\lambda}{}_\nu$$

PROOF: on moodle

note to self: ordering indices

This form of Ricci is well-adapted to wave/harmonic coordinates.

(choosing coords fixes our gauge choosing coords is gauge fixing in this context)

$\underline{\nu}$ means don't treat as vector component

Suppose we choose a system of coords $\{x^{\underline{\nu}}\}$ $\underline{\nu} = 0,1,2,3$ which satisfy the wave equation $\nabla_\mu \nabla^\mu x^{\underline{\nu}} = 0$

(idea: choose coords s.t. each coordinate satisfies the wave equation)

$$\Rightarrow 0 = \nabla^\mu(\partial_\mu x^{\underline{\nu}}) = \underbrace{\partial^\mu \partial_\mu x^{\underline{\nu}}}_{=0} + \Gamma^{\mu\sigma}{}_\mu \underbrace{\partial_\sigma x^{\underline{\nu}}}_{\delta_\sigma^{\underline{\nu}}}$$

(note to self: $x^{\underline{\nu}}$ is scalar, not the components of a vector! so $\nabla_\mu x^{\underline{\nu}} = \partial_\mu x^{\underline{\nu}}$ then $\partial_\mu x^{\underline{\nu}}$ is covector so we can take ∇ as we would covector)

since 2nd deriv of scalar is zero

$\nabla^\mu \omega_\nu = \omega_\nu{}^{,\mu} + \Gamma^{\mu\sigma}{}_\nu \omega_\sigma$

ordo $\Gamma^{\mu\nu}{}_{\mu}$

$$\Rightarrow \Gamma^{\mu\nu}{}_{\mu} = 0 \iff \tfrac{1}{2} g^{\mu\sigma}(g_{\mu\kappa,\sigma} - \tfrac{1}{2} g_{\mu\sigma,\kappa}) = 0$$

the three terms in blue vanish!

For such coordinates,

$$R_{\sigma\nu} = -\tfrac{1}{2} g^{\mu\rho} g_{\sigma\nu,\mu\rho} + \underbrace{\left(\Gamma_{\lambda\tau\nu}\Gamma^{\lambda\tau}{}_{\sigma} + \Gamma_{\lambda\tau\nu}\Gamma^{\tau\lambda}{}_{\sigma} + \Gamma_{\lambda\tau\sigma}\Gamma^{\tau\lambda}{}_{\nu}\right)}_{\equiv P_{\sigma\nu}(g,\partial g)}$$

cgpt $P_{\sigma\nu}$ is a nonlinear function in g and 1st derivative

cgpt: vacuum einstein equation (no matter/energy so $T_{ab}=0$ so EE reduces to $R_{ab}=0$ → solutions describe spacetime curvature generated purely by gravity affecting itself

locally means in locality in time by locally

analogise to what we did w/ Maxwell eq

So $R_{\sigma\nu} = 0$ reduces to a system of nonlinear wave equations in this gauge. We can solve this (locally) given initial data. Further, we can show that if the gauge condition is initially satisfied, it will remain time $\forall (t,x)$ (Choquet-Bruhat '54) ^prop in this gauge (wave coords)

→ (i.e. if gauge condition holds initially it will continue to hold under time evolution)

cgpt: the point is, $R_{\sigma\nu}$ reduces to a form where the 2nd derivs of the metric appear linearly so vacuum EE becomes $\Box g_{\sigma\nu} = P_{\sigma\nu}$ - nonlinear terms ... non linear wave eq

Linearised Theory

Suppose we are in a situation where the gravitational field is weak. We expect to be able to describe the nature as a pertubation of Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu} \quad , \quad \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$$

$\varepsilon \ll 1$ is a small parameter and we work to $O(\varepsilon)$ $\left(\begin{matrix}\text{i.e. neglect} \\ O(\varepsilon^2)\end{matrix}\right)$.

If the metric has this form we say we're in "almost inertial" coordinates. One can check

$$g^{\mu\nu} = \eta^{\mu\nu} - \varepsilon h^{\mu\nu} \quad , \quad \text{where } h^{\mu\nu} = \eta^{\mu\sigma} h_{\sigma\tau} \eta^{\tau\nu}.$$

(minus sign!)

→ note to self: show that $g_{\mu\lambda} g^{\lambda\nu} = \delta^\nu_\mu$ holds to 1st order: make an ansatz. $g^{\mu\nu} = \eta^{\mu\nu} + \varepsilon k^{\mu\nu}$ ⇒ want $k^{\mu\nu} = -\eta^{\mu\rho} h_{\rho\lambda} \eta^{\lambda\nu}$ for identity to hold to 1st order in perturbation

i.e. for $O(\varepsilon)$ quantities can raise / lower with η.

Suppose our metric is also in wave gauge, then

I think this is the gauge condition? → I think this is?

$$0 = g^{\mu\sigma}(g_{\mu\kappa,\sigma} - \tfrac{1}{2} g_{\mu\sigma,\kappa}) = \varepsilon \partial^\mu (h_{\mu\kappa} - \tfrac{1}{2}\eta_{\mu\kappa} h), \quad h = \eta^{\mu\nu} h_{\mu\nu}$$

Using our expression for the Ricci tensor

need also stress/energy tensor to order ε

$$R_{\mu\nu} = -\tfrac{\varepsilon}{2} \eta^{\sigma\tau} \partial_\sigma \partial_\tau h_{\mu\nu}$$

← note to self: $\Gamma\Gamma$ terms are second order in perturbation so vanish; then only term w/ $\partial \rho \eta_{\mu\nu}$ vanishes since minkowski constant

In order for Einstein equations to hold, we ~~would~~ must have

relabel to avoid confusion with E-mom tensor — (will remain)

$$T_{\mu\nu} = \varepsilon T_{\mu\nu}$$

note to self: EEs $R_{ab} - \tfrac{R}{2} g_{ab} = 8\pi G T_{ab}$ → $R = R^\lambda{}_\lambda = g^{\alpha\lambda} R_{\alpha\lambda}$ just plug in our expression for $R_{\mu\nu}$. (remember η commutes w/ ∂)

then to order $O(\varepsilon)$ in the Einstein equations

(using $\eta^{\mu\nu}$)

$$-\tfrac{1}{2}\Box h_{\mu\nu} + \tfrac{1}{4}\eta_{\mu\nu}\eta^{\sigma\tau}\partial_\sigma\partial_\tau h = 8\pi G T_{\mu\nu}$$

note to self: $\Rightarrow R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = \varepsilon(-\tfrac{1}{2}\eta^{\sigma\tau}\partial_\sigma\partial_\tau h_{\mu\nu} + \tfrac{1}{4}\eta^{\sigma\tau}\eta_{\mu\nu}\partial_\sigma\partial_\tau h) = 8\pi G T_{\mu\nu}$

will usually set $G=1$ by choice of coords

since LHS is 1st order, $T_{\mu\nu}$ must be 1st order in order for EE to hold so we rename it as $T_{\mu\nu} = \varepsilon T_{\mu\nu}$

so to 1st order in ε, EE becomes: $-\tfrac{1}{2}\eta^{\sigma\tau}\partial_\sigma\partial_\tau h_{\mu\nu} + \tfrac{1}{4}\eta^{\sigma\tau}\eta_{\mu\nu}\partial_\sigma\partial_\tau h = 8\pi G T_{\mu\nu}$ //

usually set $G=1$

$\Box = \eta^{\sigma\tau}\partial_\sigma\partial_\tau$

trace reversed metric perturbation

$$\Rightarrow \boxed{\begin{aligned} \Box \bar{h}_{\mu\nu} &= -16\pi G T_{\mu\nu} \\ \partial_\mu \bar{h}^\mu{}_\nu &= 0 \end{aligned}}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$

$\Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}$

gauge condition

they say we impose this but it looks like it follows from gauge fix our coords. nat. I can't figure it out?

These are the linearised Einstein equations in a wave (aka harmonic, Lorenz, de Donder) gauge. We can solve these given initial data at $\{t=0\} = \Sigma$.

If the (initial) data satisfies $\partial_\mu \bar{h}^\mu{}_\nu|_\Sigma = \partial_0(\partial_\mu \bar{h}^\mu{}_\nu)|_\Sigma = 0$, then since $\partial_\mu T^\mu{}_\nu = 0$, we have

$$\Box(\partial_\mu \bar{h}^\mu{}_\nu) = 0 \quad \text{so gauge holds for all times.}$$

gauge condition holds for all time.

this is a condition on EM tensor apparently → energy is conserved?

Linearised gauges

(remember we can treat 2 spacetimes (M,g) and (M,g') as equiv if there is a diffeomorphism)

Suppose we have a physically equivalent solution not necessarily in wave gauge. We've seen that 2 such equivalent solutions are related by a diffeomorphism.

In order that the diffeomorphism reduces smoothly to the identity as $\varepsilon \to 0$, it should take the form Φ^ξ_ε for some vector field ξ. (some vector field def. on Minkowski)

ψ^ξ_ε ← flows the param distance ε along integral curve ξ

If S is any tensor field, then from the definition of Lie derivative

$\psi^{\xi*}_\varepsilon S = S + \varepsilon \mathcal{L}_\xi S$

(from $\mathcal{L}_X T|_p = \lim_{t\to 0} \frac{(\phi^{X*}_t T)|_p - T_p}{t}$)

$$(\Phi^\xi_\varepsilon)^* S = S + \varepsilon \mathcal{L}_\xi S + O(\varepsilon^2).$$

change in object is lie derivative of ξ times the object

In particular

(covariant deriv is same as partial deriv at order ε since Christoffel symbols always at $O(\varepsilon)$)

$$* \quad ((\Phi^\xi_\varepsilon)^*\eta)_{\mu\nu} = \eta_{\mu\nu} + \underbrace{\varepsilon(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)}_{h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu} + O(\varepsilon^2).$$

$((\psi^\xi_\varepsilon)^*\eta)_{\mu\nu} = \eta_{\mu\nu} + \varepsilon(\mathcal{L}_\xi \eta)_{\mu\nu}$, use $(\mathcal{L}_X g)_{ab} = \nabla_a X_b + \nabla_b X_a$

(then $\nabla \to \partial$ since Γ term are $O(\varepsilon)$?, i.e. (note: must Γ include ε so $\varepsilon\Gamma \sim O(\varepsilon^2)$))

(wrt the unperturbed spacetime)

(L16)

So if $S = O(\varepsilon)$ it will be invariant under gauge transformations to $O(\varepsilon)$. Tensors vanishing on Minkowski are gauge invariant in linear pertubation theory. In particular $T_{\mu\nu}$ is gauge invariant.

(so e.g. EM tensor is gauge inv. as it's $O(\varepsilon)$ as we saw from Einstein eqs)

However, $*$, this $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ represents a linear gauge transformation (c.f. $A_\mu \to A_\mu + \partial_\mu t$).

(added later for L16 intro)

summary: If we have a physically equivalent solution, not necessarily in wave gauge, then it should be related to any other such solution by a diffeomorphism Φ^ξ_ε for some vector field ξ. From def of lie derivative for any tensor field S: $(\Phi^\xi_\varepsilon)^* S = S + \varepsilon\mathcal{L}_\xi S + O(\varepsilon^2)$ (change to order ε in our tensor is $\varepsilon\mathcal{L}_\xi S$). So, if $S = O(\varepsilon)$, it will be invariant under gauge transformations to $O(\varepsilon)$. Tensors vanishing on Minkowski are gauge invariant in linear perturbation.

(most things we are interested in will vanish in Minkowski e.g. $T_{\mu\nu}$ but one important thing that doesn't vanish is the metric itself!)

(reminder $G=c=1$)

Lecture 16

15.11.24

note: motivation in course structure: gauge fix then linearize, why not linearize then gauge fix? → we know doing it the first way we know our gauge comes from a gauge condition that makes sense in the full non-linear problem

From formulae in last lecture (or on Moodle), if we linearise $R_{\mu\nu}$ without fixing a gauge, we find

$$R_{\mu\nu} = \varepsilon\left(\partial^\rho \partial_{(\mu} h_{\nu)\rho} - \tfrac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \tfrac{1}{2}\partial_\mu \partial_\nu h\right)$$

Substituting in $h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ we can check $R_{\mu\nu}$ vanishes, so $h_{\mu\nu}$ solves the vaccum Einstein equations. We call such a solution a pure gauge solution

note to self (right?) → gauge fixing is choosing a rule to eliminate redundancy → pure gauge sol. is one that can be transformed away by gauge fixing → it's an unphysical mode

EXERCISE: Show that if $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu \xi_\mu$ then

$$\partial_\mu \bar{h}'^{\mu}{}_{\nu} = \partial_\mu \bar{h}^{\mu}{}_{\nu} + \partial_\mu \partial^\mu \xi_\nu$$

see p.48

reminder $\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2} h \eta_{\mu\nu}$

Deduce that:

* Any linearised perturbation can be put into wave gauge by a suitable gauge transformation

* Any pure gauge solution of the wave gauge fixed equations with $\xi_\mu|_{t=0} = 0$ $\partial_0 \xi_\mu|_{t=0} = 0$ vanishes everywhere

attempt (right?) to ensure wave gauge we want $\partial_\mu \bar{h}'^\mu{}_\nu = 0$, this happens if $\Box \xi_\nu = -\partial_\mu \bar{h}^\mu{}_\nu$ which always has a solution for linear perturb. → we can always find a vector ξ that transforms us to wave gauge

pure gauge sol $h'_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ 2nd eq becomes $\partial_\mu \bar{h}^\mu{}_\nu = \Box \xi_\nu$, so fixing wave gauge requires $\Box \xi_\nu = 0$: with these BCs the solution is $\xi_\nu = 0$ everywhere $\Rightarrow h'_{\mu\nu} = 0$ pure gauge sol that satisfies wave gauge & has trivial BCs vanishes everywhere!

THE NEWTONIAN LIMIT

We'd expect that if GR is a good theory of gravity, we should be able to recover Newton's theory of gravitation in the limit where fields are weak and matter is slowly moving in comparison with the speed of light ($c=1$ in our units).

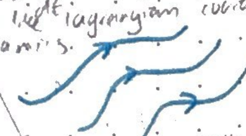
need sources to plug into EFE eq: choose fluid

Let us suppose that matter is modelled by a perfect fluid with velocity field U^a, density ρ and pressure p.

In all but the most extreme situations, $\rho >> p$ (in standard units $P/\rho \approx V_s^2 << c^2$).

We choose coordinates such that $\left[U^a = \frac{\partial}{\partial t}\right]$. [Think of as Lagrangian coordinates for fluid e.g. Mach's principle.]

integral curves $\frac{\partial}{\partial t}$ correspond to flow lines think of the Lagrangian coords in fluid dynamics



Note that this does not imply the fluid is at rest: distances are measured with the METRIC.

we are measuring w time perturbed metric not minkowski metric so effectively time dependence of the fluid is put in the metric rather than the fluid

The condition that the fluid moves non-relativistically becomes the assumption that we are in the weak field limit

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad (h_{\mu\nu} = O(\varepsilon))$$

(drop ε for ease of notation just let $h_{\mu\nu}$ mean $\varepsilon h_{\mu\nu}$)

also want to require that the motions of the particles are not too dramatic: small relative velocities & accelerations

[illegible] for reasons we will see later on
not a single way to do this; choose powers of ε more 'natural'

time derivatives are small
velocities & accelerations are small

$$\partial_0 h_{\mu\nu} \sim \varepsilon^{1/2} h_{\mu\nu}$$
$$\partial_0\partial_0 h_{\mu\nu} \sim \varepsilon h_{\mu\nu}$$

(reminder to self: $h_{\mu\nu}$ is $O(\varepsilon)$ so $\partial_0\partial_0 h_{\mu\nu} \sim O(\varepsilon^2)$)

For consistency require $\rho = O(\varepsilon)$ and $p = O(\varepsilon^2)$.

The linearised Einstein equations become

$$\partial^\mu \partial_\mu \bar{h}_{\sigma\tau} = -16\pi T_{\sigma\tau}$$

(mixing various scales here, don't worry about this too much)

$$T_{00} = \rho, \quad T_{0i} = T_{ij} = 0 \quad (\text{to } O(\varepsilon^2)).$$

Deduce that

3D laplacian ∇^2

$$\Delta \bar{h}_{00} = -16\pi\rho$$
$$\bar{h}_{0i}, \bar{h}_{ij} \text{ vanish at } O(\varepsilon)$$

since condition on time derivs $\partial_0, \partial_0\partial_0$: 2nd time deriv is $O(\varepsilon^2)$
so $\partial^\mu\partial_\mu \bar{h}_{00} = -16\pi\rho$
$\Rightarrow \nabla^2 \bar{h}_{00} = -16\pi\rho$

Since

trace
$\bar{h} = \bar{h}_{\mu\nu}\eta^{\mu\nu}$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\bar{h} \qquad \bar{h} = -\bar{h}_{00}$$

$$\Rightarrow h_{00} = \tfrac{1}{2}\bar{h}_{00}$$

(second the first non-trivial order?)

$$\therefore \quad \overset{\nabla^2}{\Delta}\left(-\frac{h_{00}}{2}\right) = 4\pi\rho \qquad (\text{c.f. } \Delta\phi = 4\pi G\rho)$$

N's theory doesn't just tell us potential from a mass, it also tells us how particles move under potential, so we need to go further

Suggests we identify $-\frac{h_{00}}{2}$ with the Newtonian potential ϕ.

its own field is small enough that we can ignore

Consider a test particle in this background. By geodesic postulate its motion is determined by the Lagrangian.

$$\begin{aligned} \mathcal{L} &= g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \\ &= \eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \\ &= -\dot{t}^2 + |\dot{\underline{x}}|^2 + h_{00}\dot{t}^2 + O(\varepsilon^2) \end{aligned}$$

$$\dot{} \equiv \frac{\partial}{\partial\tau}$$

$$x^\mu = (t, \underline{x})$$

Suppose motion is non-relativistic, so $|\dot{\underline{x}}|^2 = O(\varepsilon)$. Conservation of $\mathcal{L}$ gives,

(near Minkowski metric)

$$-\dot{t}^2 = -1 + O(\varepsilon) \Rightarrow \dot{t} = 1 + O(\varepsilon)$$

note to self:
by def of proper time:
$\frac{d\tau}{d\tau} = 1 = -g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\mathcal{L}$
$\Rightarrow -1 = -\dot{t}^2 + |\dot{\underline{x}}|^2 + h_{00}\dot{t}^2 + O(\varepsilon^2)$
since $|\dot{\underline{x}}|^2$, h_{00} are $O(\varepsilon)$
$\Rightarrow \dot{t}^2 = 1 + O(\varepsilon) \Rightarrow \dot{t} = 1 + O(\varepsilon)$
(think $\dot{t} = (1 + O(\varepsilon))^{1/2}$?)
$\Rightarrow \mathcal{L} = -\dot{t}^2 + |\dot{\underline{x}}|^2 + h_{00}\dot{t}^2 + O(\varepsilon^2)$
reach EL eq. since $\dot{t}$ [illegible] treat it as a constant and remember $h_{00}(x)$

x^i Euler-Lagrange equation:

note to self: this isn't $\mathcal{L}$ for some reason?

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i}$$

$$2\frac{d}{d\tau}(\dot{x}_i) = 2\ddot{x}_i = h_{00,i}\,\dot{t}^2$$

note to self: can see this from def of $\tau \rightarrow \frac{d\tau}{d\tau} = -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$
$\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt}$

Since $\dot{t} = 1 + O(\varepsilon)$, $\frac{d}{dt} = \frac{d}{d\tau} + O(\varepsilon)$, we have

$$\frac{d^2x^i}{dt^2} = \tfrac{1}{2}h_{00,i} + O(\varepsilon^2) = \partial_i\phi + O(\varepsilon^2)$$

WE'VE RECOVERED NEWTON'S LAWS OF GRAVITATION!

note to self:
$\ddot{x} = -\nabla\phi$?

GRAVITATIONAL WAVES

One of the most spectacular recent results in gravitational physics was the measurement in 2015 of gravitational waves sourced by two colliding black holes.

Near the source, the field is not weak, but by the time we detect the waves, the weak field approximation is relevant.

* from p 46
we know that under diffeomorphism i.e. gauge transformation metric perturbation transforms as

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \quad \text{(from defn of lie deriv.)}$$

so we want to find out how $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$ transforms, so plug in transformed $h_{\mu\nu}$

$$\bar{h}_{\mu\nu} \to h'_{\mu\nu} - \frac{1}{2} h'_{\rho\sigma} \eta^{\rho\sigma} \eta_{\mu\nu}$$

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial^\rho \xi_\rho \leftarrow \text{(correction term)?}$$

so now want to find $\partial^\mu \bar{h}'_{\mu\nu}$: just plug in $\bar{h}'_{\mu\nu}$

$$\partial^\mu \bar{h}'_{\mu\nu} = \partial^\mu \bar{h}_{\mu\nu} + \partial^\mu \partial_\mu \xi_\nu + \cancel{\partial^\mu \partial_\nu \xi_\mu} - \cancel{\partial_\nu \partial^\mu \xi_\mu}$$

terms cancel! $\Rightarrow$ $\partial^\mu \bar{h}'_{\mu\nu} = \partial^\mu \bar{h}_{\mu\nu} + \partial^\mu \partial_\mu \xi_\nu$ //

**
If ξ_μ obeys wave gauge then plug in $\partial_\mu \xi_\nu$, $\partial_\nu \xi_\mu$, $\partial^\rho \xi_\rho$ to our expression for $\bar{h}'_{\mu\nu}$:

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + (X_\nu k_\mu + X_\mu k_\nu) e^{ik^\sigma x_\sigma} - \eta_{\mu\nu} X^\rho k_\rho e^{ik^\sigma x_\sigma}$$

so since $\bar{h}_{\mu\nu} = H_{\mu\nu} e^{ik^\sigma x_\sigma}$, we can see gauge freedom acts on $H_{\mu\nu}$ as:

$$H_{\mu\nu} \to H'_{\mu\nu} = H_{\mu\nu} + X_\nu k_\mu + X_\mu k_\nu - \eta_{\mu\nu} X^\rho k_\rho$$

GW propagate in vaccum: we want to solve $\Box \bar{h}_{\mu\nu}=0$

convenient gauge for grav. waves discussion

Seek a propagating wave solution to the vaccum linearised Einstein equations in wave gauge

$$\Box \bar{h}_{\mu\nu} = 0 \qquad \partial_\mu \bar{h}^\mu{}_\nu = 0$$

Making the Ansatz

(note to self: sloppy notation? should be σ ok?)

symmetric

$$\bar{h}_{\mu\nu} = Re\,(H_{\mu\nu} e^{ik^\mu x_\mu}) \qquad H_{\mu\nu} = + H_{\nu\mu} \text{ const.} \quad k^\mu \text{ const.}$$

plane wave propagating in direction of k^μ

(reminder: in minkowski vector is null if $\eta(x,x)=0$)

(note to self: suppress Re from now on)

(note to self: $k^\mu k_\mu = 0 \Rightarrow \omega^2 = k^2$ so $v = \frac{\omega}{k} = 1$?)

plug it in

We see that this is a solution if

(say k is in z direction wlog) call direction wave travel in z direction

(i got $-H_{\mu\nu} k^\sigma k_\sigma e^{\dots}$ from this so $k^\sigma k_\sigma = 0$) from $\Box \bar{h}_{\mu\nu} = 0$

$$k^\mu k_\mu = 0 \quad \Rightarrow k^\mu \text{ is } \underline{\text{null}}\text{: waves travel at speed of light}$$

(note to self: from null condition, if k in z dir $\Rightarrow -k_0^2 + k_3^2 = 0 \Rightarrow k_0 = k_3$ so $k^\mu = \omega(1,0,0,1)$)

from gauge condition

$$k_\mu H^\mu{}_\nu = 0 \qquad \text{take } k^\mu = \omega(1,0,0,1) \text{ wlog}$$

(note to self: $\Rightarrow$ waves are transverse, i.e. $H_{\mu\nu}$ has no component along k^μ so no perturbation in the direction of propagation)

There is still some residual gauge freedom: recalling that the diffeomorphism generated by ξ acts on $\partial^\mu \bar{h}_{\mu\nu}$ by

$$* \quad \partial^\mu \bar{h}_{\mu\nu} \to \partial^\mu \bar{h}_{\mu\nu} + \partial_\mu \partial^\mu \xi_\nu .$$

(notes: the condition $\partial_\mu \bar{h}^\mu{}_\nu = 0$ does not eliminate all gauge freedom: we still have residual $\partial_\mu \partial^\mu \xi_\nu$)

If $\partial_\mu \partial^\mu \xi_\nu = 0$, then gauge condition is preserved. Let

(if ξ_ν obeys wave eq then we preserve the gauge condition $\partial_\mu \bar{h}^\mu{}_\nu = 0$)

(since k_μ null)

$$\xi_\nu = Re\,(-iX_\nu e^{ik^\mu x_\mu}) \qquad \text{then } \partial_\mu \partial^\mu \xi_\nu = 0.$$

(complex no represents phase shift)

correcting for the fact H is $\bar{h}$ thing not h thing

And (check)

(how residual gauge freedom acts on H)

$$** \quad H_{\mu\nu} \to H_{\mu\nu} + k_\mu X_\nu + k_\nu X_\mu - \eta_{\mu\nu} k_\sigma X^\sigma$$

use up remaining freedom to further set $H_{0\mu}=0$ & $H_\mu{}^\mu = 0$

<u>Exercise</u> * show that if we take $X_0 = 0$, $X_i = -\frac{H_{0i}}{k_0}$

(note to self: idk? how to do this? show that $H_{0\mu} \to 0$ use wlog $k^\mu = \omega(1,0,0,1)$ and $H_{\mu\nu}$ not $\bar{h}_{\mu\nu} = 0$?)

forces

can do gauge transf that forces $H_{0\mu}$ to vanish

then we can set $\boxed{H_{0\mu} = 0}$

does not mess up previous fixes: $H_{0\mu}$ still vanishes, and also H is traceless

$\to H_\mu{}^\mu \to H_\mu{}^\mu - 2k_\mu X^\mu$ (minkowski $\eta^{\mu\nu}\eta_{\mu\nu} = 4$!) $= H_\mu{}^\mu - 2\alpha(k^0 k_0 - k^i k_i)$

* Making a further transformation of the form

(got $\alpha = \frac{1}{\dots H^\mu{}_\mu}$??) ??? (*)

$$X_0 = \alpha k_0 , \quad X_i = -\alpha k_i$$

we're effectively 'used up' dofs to get constraints

show we can additionally impose $\boxed{H_\mu{}^\mu = 0} \Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu}$

"traceless" condition

(i think $\bar{h} = 0 \Rightarrow \bar{h}_{\mu\nu} = h_{\mu\nu}$, then show this requires $\bar{h} = 0$ so $\bar{h}_{\mu\nu} = h_{\mu\nu}$)

Since $H_{0\mu} = 0$ and $H_{\nu\mu} k^\nu = \omega(H_{0\mu} + H_{3\mu}) = 0$, we deduce $H_{3\mu} = 0$. With symmetry + tracelessness of $H_{\mu\nu}$ we have

one term equal to minus other term (check) on diag?

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$H_+, H_\times$ are constants corresponding to two independent polarisations of the wave.

$H_{0\mu} = 0$, $H_{3\mu} = 0$ rows are 0 (columns?) then fill in the rest by symmetry & tracelessness

what does this solution mean physically? ↓

traceless means sum of stuff on main diagonal is zero

can't just consider single particle in free fall -> will always be freely falling
instead need to consider collection of test particles all moving in a similar direction

To understand the consequences of such a gravitational wave, we recall the geodesic deviation equation. If λ is a geodesic with tangent T, then a vector Y joining λ to a nearby geodesic satisfies

(geodesic deviation eq. $\nabla_T \nabla_T Y = R(T,Y)T$) geodesics w/ tangent T

haven't specified coords yet: imagine I am freely falling particle & set up my coords as follows

$$T^a \nabla_a (T^b \nabla_b Y^c) = R^c{}_{dab} T^d T^a Y^b.$$

Suppose a freely falling observer sets up a frame consisting of $e_0 = T$ (tangent to observer's worldline), together with three spacelike vectors e_1, e_2, e_3 which are parallelly transported along the worldline and initially satisfy $g(e_\mu, e_\nu) = \eta_{\mu\nu}$. Since parallel transport preserves inner products, we can assume $\{e_\mu\}$ is O/N for all time. Since $T^a \nabla_a (e_\alpha) = 0$. Geodesic equation implies

"meant to write α,β here but it doesn't really matter."

(parallel transport. $\nabla_T (e_\alpha) = 0$ basis vectors e_α)

$= e^c_\alpha R_{cdab} T^d T^a Y^b = e^c_\alpha R_{cdab} e^d_0 e^a_0 Y^b$ (using $e_0 = T$)

$$T^a \nabla_a (T^b \nabla_b (Y_c e^c_\alpha)) = R_{abcd} e^a_\alpha e^b_0 e^c_0 Y^d.$$

$\to$ ($\nabla_T (Y_\alpha) = T(Y_\alpha) = \frac{\partial Y_\alpha}{d\tau}$ then apparently $\frac{dY_\alpha}{d\tau}$ (right fuzzy) is also scalar so $\nabla_T (\frac{dY_\alpha}{d\tau}) = T(\frac{dY_\alpha}{d\tau}) = \frac{d^2 Y_\alpha}{d\tau^2}$?)

Now $Y_c e^c_\alpha =: Y_\alpha$ is a scalar, so equation becomes

take components along curve wrt parallely transported orth. coords. effectively observer's local frame: evolution is governed by this eq. / of stuff seen from my frame?

$$\frac{d^2 Y_\alpha}{d\tau^2} = R_{abcd} e^a_\alpha e^b_0 e^c_0 Y^\beta e^d_\beta .$$

Minkowski observer sits somewhere sits at origin for all time bring with me set of vectors that is parallel transp. which is local inertial frame

note: raise & lower α,β indices w/ $\eta_{\alpha\beta}$.

For our problem of a gravitational wave spacetime, the Riemann curvature is $O(\varepsilon)$ so we only need e_α to leading order. We can assume $e_0 = \partial_t$, $e_i = \partial_i$ and λ is $t \mapsto (t,0,0,0)$

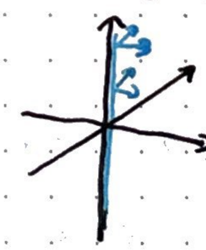
τ

both the same: doesn't imply but seems to be consistent

(note: lovely the α is to show the e^a_α etc are vectors while α labels the basis vectors. So the components are $e^a_\alpha = \frac{dx^a}{dx^\alpha} = \delta^a_\alpha$, $e^b_0 = \frac{dx^b}{dx^0} = \delta^b_0$ etc.?)

Then

$$\frac{d^2 Y_\alpha}{d\tau^2} = R_{\alpha 0 0 \beta} Y^\beta + O(\varepsilon^2).$$



note (to order ε t & τ use interchangeably)

To $O(\varepsilon)$ we have

$$R_{\rho\sigma\mu\nu} = \tfrac{1}{2}(h_{\rho\nu,\mu\sigma} + h_{\sigma\mu,\nu\rho} - h_{\rho\mu,\nu\sigma} - h_{\sigma\nu,\mu\rho})$$

remember $H_{0\mu} = 0 \Rightarrow \bar{h}_{0\mu} = 0$, and $H_\mu{}^\mu = 0 \Rightarrow \bar{h}_{\mu\nu} = h_{\mu\nu}$ so we have $h_{0\mu} = 0$ (21)

$\to$ but since $h_{0\mu} = 0$, we find

$$R_{\alpha 0 0 \beta} = \tfrac{1}{2} h_{\alpha\beta,00}$$

so

if we drop τ, where stuff moving around me? determine what depends on perturbation?

$$\to \frac{d^2 Y_\alpha}{dt^2} = \tfrac{1}{2} \frac{\partial^2 h_{\alpha\beta}}{\partial t^2} Y^\beta \qquad \text{(used } t = \tau \text{ to } O(\varepsilon)\text{)}.$$

Let's consider the + POLARISATION:

modulus since can be complex

$$h_{\mu\nu} = Re(H_{\mu\nu} e^{ik^{\mu}x_{\mu}}) = |H_+| \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos\omega(t-\xi)$$

$-t_0$ since can be complex

ξ is phase that comes from it being complex

Along λ $\xi = 0$ equations are

(since $h_{0\mu} = h_{3\mu} = 0$) $$\frac{d^2Y^0}{dt^2} = \frac{d^2Y^3}{dt^2} = 0$$ (no relative acceleration in direction of the wave)

(turn. and no correction to time coordinate)

cannot solve explicitly linear eqn with oscillating coefficient

↓ but can solve perturbatively

$$\frac{d^2Y^1}{dt^2} = -\tfrac{1}{2}\omega^2 |H_+| \cos\omega(t-t_0) Y^1$$

$$\frac{d^2Y^2}{dt^2} = \tfrac{1}{2}\omega^2 |H_+| \cos\omega(t-t_0) Y^2$$

H_+ is small, so solve perturbatively with $\dfrac{dY^1}{dt} = \dfrac{dY^2}{dt} = 0$ $\Rightarrow$ $Y^1 = Y^1_0 + O(\varepsilon^2)$, $Y^2 = Y^2_0 + O(\varepsilon^2)$ put back into

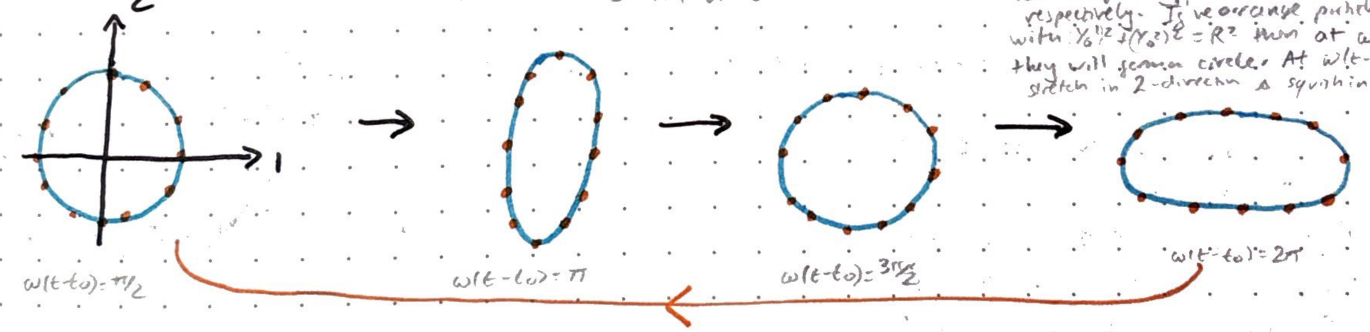
$Y^1 = Y_0^1(1 + \tfrac{1}{2}|H_+| \cos\omega($ $(t-t_0))$

$Y^2 = Y_0^2(1 - \tfrac{1}{2}|H_+| \cos\omega(t-$

initially i am sat at origin & around me are collection of test masses on circle as time passes this is going to squash & in Y_1 direction & stretch in Y_2 direction then

stretch other way since oscillating.

note key: avg displacement is Y_0^1; Y_0^2 respectively. If we arrange particles in 12 plane with $(Y_0^1)^2 + (Y_0^2)^2 = R^2$ then at $\omega(t-t_0) = \pi/2$ they will form a circle. At $\omega(t-t_0) = \pi$ they stretch in 2-direction & squish in 1-direction etc.



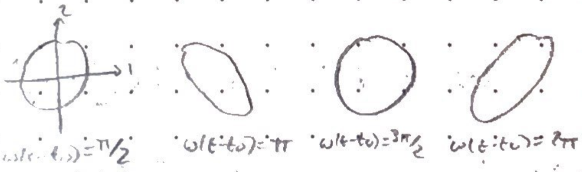
$\omega(t-t_0) = \pi/2$ $\omega(t-t_0) = \pi$ $\omega(t-t_0) = 3\pi/2$ $\omega(t-t_0) = 2\pi$

Exercise: Find solution for $\times$ polarisation.

attempt: solve as for + polarisation

$Y^2 = Y^2_0 + \tfrac{1}{2}|H_\times| \cos\omega(t-t_0) Y_0^1$

$Y^1 = Y_0^1 + \tfrac{1}{2}|H_\times| \cos\omega(t-t_0) Y_0^2$



$\omega(t-t_0) = \pi/2$ $\omega(t-t_0) = \pi$ $\omega(t-t_0) = 3\pi/2$ $\omega(t-t_0) = 2\pi$

same sol. rotated by 45°

note to self: solve perturbatively

attempt:

zeroth order solution (no wave):

since $\frac{dY^1}{dt} = 0 \Rightarrow Y^1 = Y_0^1 + O(\varepsilon^2)$

first order correction:

plug in $Y^1 = Y^1_0 + \delta Y$ into $\frac{d^2Y^1}{dt^2}$

$\Rightarrow \delta Y^1 = \tfrac{1}{2}|H_+| \cos\omega(t-t_0) Y_0^1$

so put together $Y = Y_0 + \delta Y$ gives:

$Y^1 = Y_0^1(1 + \tfrac{1}{2}|H_+| \cos\omega(t-t_0)) Y_0$

Lecture 18   20.11.24

Last lecture:
- monochromatic plane gravitational waves
- effect on test particles

circle of particles freely falling, wave travels into the page

(+ POLARISATION)

Found using geodesic deviation eqn.

(X POLARISATION

(??)

THE FIELD FAR FROM A SOURCE

Return to linearised Einstein equations with matter

thinking this like E+M equations in Lorenz gauge
→ potential satisfies wave eqn
→ 4-divergence free → Leonard Fischer potential analog.

$$(*)\quad \partial_\rho \partial^\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \qquad \partial_\mu \bar{h}^{\mu}{}_{\nu} = 0$$

wave gauge condition

As is the case for electromagnetism, we can solve $(*)$ explicitly using a retarded Greens function:

as is standard in Minkowski use t as x^0 component

modulus computed in euclidean metric → exactly as we normally think of it

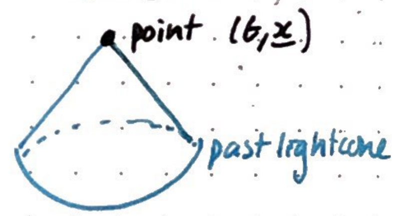
$$\bar{h}_{\mu\nu}(t,\underline{x}) = 4\int d^3x' \, \frac{T_{\mu\nu}(t-|\underline{x}-\underline{x}'|, \underline{x}')}{|\underline{x}-\underline{x}'|} \qquad (\dagger)$$

Where $|\underline{x}-\underline{x}'|$ is computed in the Euclidean metric

(from pov far away)

If matter is concentrated within a distance d of the origin (so $T_{\mu\nu}(t,\underline{x}) = 0$ for $|\underline{x}'| > d$), we can expand in the far field region where $r = |\underline{x}| >> |\underline{x}'| \sim d$.) Then

interested in look our past lightcone of point (the effect of RHS) integrate over past lightcone with some factor telling me how much decay I see as I go from one side to another.



$$|\underline{x}-\underline{x}'|^2 = r^2 - 2\underline{x}\cdot\underline{x}' + |\underline{x}'|^2 = r^2\left(1 - \tfrac{2}{r}\hat{\underline{x}}\cdot\underline{x}' + O\left(\tfrac{d^2}{r^2}\right)\right)$$

with $\hat{\underline{x}} = x/r$ (unit vector). Hence, (use binomial expansion)

correction is inner product of vector pointing from origin & position vector $(\underline{x}')$

$$|\underline{x}-\underline{x}'| = r\left(1 - \tfrac{1}{r}\hat{\underline{x}}\cdot\underline{x}' + O\left(\tfrac{d^2}{r^2}\right)\right) = r - \hat{\underline{x}}\cdot\underline{x}' + O\left(\tfrac{d^2}{r}\right)$$

so then will look as you want

and

$$T_{\mu\nu}(t - |\underline{x}-\underline{x}'|, \underline{x}') = T_{\mu\nu}(t', \underline{x}') + \hat{\underline{x}}\cdot\underline{x}' \, \partial_0 T_{\mu\nu}(t', \underline{x}') + \ldots$$

$\approx T_{\mu\nu}(t - r + \hat{\underline{x}}\cdot\underline{x}', \underline{x}') =$

where $t' = t - r$.

note to self: small distance over large time so neglect?

what do I mean by this

note to self: fundamental physics timescale of change $\tau \sim \frac{d}{c}$?

If $T_{\mu\nu}$ varies on a timescale of τ so that $\partial_0 T_{\mu\nu} \sim \frac{1}{\tau} T_{\mu\nu}$, then second term above is $O(d/\tau)$ which we can neglect if matter moves non-relativistically. We thus have,

I want to know h_{ij} at point in time, I integrate over T_{ij} at the retarded time and add up contributions to get

$$\bar{h}_{ij} = \frac{4}{r}\int d^3x' \, T_{ij}(t', \underline{x}') \qquad t' = t - r \qquad (**)$$

This gives the spatial components of $\bar{h}_{ij}$. To find remaining components we use gauge condition: $\partial_0 \bar{h}_{0i} = \partial_j \bar{h}_{ji}$, $\partial_0 \bar{h}_{00} = \partial_i \bar{h}_{0i}$.

note to self: from $\partial^\mu \bar{h}_{\mu\nu} = 0$ 'v' component and $v = 0$

First solve for $\bar{h}_{0i}$ then $\bar{h}_{00}$.

notes. $\partial_\mu T^{\mu\nu}=0$.

We can simplify the integral in (**) by recalling that $\partial_\mu T^\mu{}_\nu = 0$ (EM conservation) and that $T_{\mu\nu}(t', \underline{x}')$ vanishes for $|\underline{x}| > d$.

(gonna drop primes, invert bit so don't have to write it out & dummy index so doesn't matter)

Dropping primes in the integral

subst $T = T'$ use dummy index, do integration by parts using fact divergence vanishes → integral of t — int of deriv of T^{ij}

think $T^{ij} = T^{ik}\delta^j_k = T^{ik}\partial_k x^j = \partial_k(T^{ik}x^j) - (\partial_k T^{ik})x^j$

$$\int d^3x\, T^{ij}(t,\underline{x}) = \int d^3x\, \partial_k (T^{ik}x^j) - (\partial_k T^{ik})x^j$$

subtract what happens when deriv hits the x

divergence theorem. write as int over a surface

$$= \underbrace{\int_{|\underline{x}|=2d} T^{ik}x^j n^i\, dS}_{0 \text{ as } T_{ij}=0 \text{ for } |\underline{x}|>d} - \int d^3x\, (\partial_k T^{ik})x^j$$

drop surface term

convert into surface int over suff. large sphere but vanish by assumption $T_{ij}=0$, $|x|>d$

relate spatial divergence of T to a time derivative $(\partial_0 T^{0i} + \partial_j T^{ji} = 0)$

LHS symmetric in i,j so replace RHS w symmetric part. (recall T_{ij} is symmetric)

$$= \int d^3x\, \partial_0 T^{0i} x^j = \partial_0 \int d^3x\, T^{0i}x^j$$

(since $\partial_0 x^j = 0$)

Symmetrising on i,j

$$\int d^3x\, T^{ij}(t,\underline{x}) = \partial_0 \int d^3x\, \left(\tfrac{1}{2}T^{0i}x^j + \tfrac{1}{2}T^{0j}x^i\right)$$

realise again this looks like divergence of something + a correction

$$= \partial_0 \int d^3x \left\{ \underbrace{\tfrac{1}{2}\partial_k (T^{0k}x^i x^j)}_{\text{as above by divergence theorem}} - \tfrac{1}{2}\overbrace{\partial_k T^{0k}}^{=-\partial_0 T^{00}} x^i x^j \right\}$$

again notice spatial divergence of energy-momentum which I know as above I can replace w/ time divergence of T^{00} since $(\partial_0 T^{00} + \partial_k T^{0k} = 0)$

again pull out time deriv since this is integral over spatial derivs.

$$= \partial_0 \int d^3x\, \partial_0 T^{00} x^i x^j$$

$$= \tfrac{1}{2}\partial_0\partial_0 \int d^3x\, T^{00}x^i x^j = \tfrac{1}{2}\ddot{I}_{ij}(t)$$

2nd time deriv of I_{ij} at time t

Where $I^{ij}(t) = \int d^3x\, T^{00}(t,\underline{x})x^i x^j$ (in these coords)

2nd moment of 00 component of EM tensor (mass energy) evaluated at time t.

Noting $T_{00} = T^{00}$ and $T_{ij} = T^{ij}$ we deduce,

$$\bar{h}_{ij} = \tfrac{2}{r}\ddot{I}_{ij}(t-r). \qquad r \gg d, \quad \tau \gg d$$

2nd derivative of I at time light ray would have had to leave dist to arrive at me now. then dilute by $1/r$ to reflect fact we are on a sphere (?).

Now reconstruct remaining components using gauge condition

RHS: two time derivs that commute w anything

(these are euclidean indices so don't worry too much about upstairs vs. downstairs)

$$\partial_0 \bar{h}_{0i} = \partial_j \bar{h}_{ji} = \partial_j \left(\tfrac{2}{r}\ddot{I}_{ij}(t-r)\right)$$

evaluated at $t-r$

$$\Rightarrow \bar{h}_{0i} = \partial_j \left(\tfrac{2}{r}\dot{I}_{ij}(t-r)\right) + k_i(\underline{x})$$

const of integration depend only on $\underline{x}$

using chain rule and this fact

$$\bar{h}_{0j} = -\frac{2\hat{x}_j}{r^2}\dot{I}_{ij}(t-r) - 2\frac{\hat{x}_j}{r}\ddot{I}_{ij}(t-r) + k_i(\underline{x})$$

$\partial_i r = \frac{x_i}{r} = \hat{x}_i$ hopefully familiar from vector calculus

so far assumed $r > d$, $\tau > d$ but not imposed relation between r and τ until now:

We now assume $r \gg \tau$ so we are in radiation zone so can drop first term (it's $O(\tau/r)$ relative to second) and get

(now want to find $\bar{h}_{00}$)

$$\bar{h}_{0i} = -\frac{2\hat{x}_j}{r}\ddot{I}_{ij}(t-r) + k_i(\underline{x})$$

Now use,

$$\partial_0 \bar{h}_{00} = \partial_i \bar{h}_{0i} = \partial_i\left(-\frac{2\hat{x}_j}{r}\ddot{I}_{ij}(t-r) + k_i(\underline{x})\right)$$

integrate as before

($\times$ divergence of k)

$$\Rightarrow \quad \bar{h}_{00} = -2\partial_i\left(\frac{\hat{x}_j}{r}\dot{I}_{ij}(t-r)\right) + t\,\partial_i k_i(\underline{x}) + f(\underline{x})$$

(another integration const, similarly 2)

$$= \frac{2\hat{x}_i\hat{x}_j}{r}\ddot{I}_{ij}(t-r) + t\,\partial_i k_i + f(\underline{x}) + \text{TERMS SUBLEADING IN } \tau/r$$

To fix constants of integration, return to

$$\bar{h}_{\mu\nu} = 4\int d^3x' \, \frac{T_{\mu\nu}(t-|\underline{x}-\underline{x}'|, \underline{x}')}{|\underline{x}-\underline{x}'|}$$

and observe that to leading order in d/τ

"trust me on this or go do the computation yourselves"

$$\bar{h}_{00} \approx \frac{4E}{r} \qquad \bar{h}_{0i} \approx -\frac{4P_i}{r}$$

same expansion as we did before for $\bar{h}_{ij}$ which tells us we can replace with 4 times integral at retarded time

where $E = \int d^3x' \, T_{00}(t', \underline{x}')$, $P_i = \int d^3x' \, T_{0i}(t', \underline{x}')$

E, P actually const. in time although in principle they depend on retarded time

Observing that $\partial_0 \int d^3x' \, T_{0\mu}(t', x') = \int d^3x' \, \partial_0 T_{0\mu}(t', x')$

$$= \int d^3x' \, \partial_i T_{i\mu}(t', x')$$

So $E, \underline{P}$ constant in time.

Exercise: By a gauge transformation generated by a multiple of

we've chosen Minkowski gauge s.t. centre of mass sits at origin

$$\xi^\mu = (\underline{P}\cdot\underline{x}, -\underline{P}t)$$

we can set $\underline{P} = 0$. This is the centre of momentum frame.

We've shown that:

what do these eq. give us?

$$\boxed{\begin{aligned} * \ \bar{h}_{00}(t,\underline{x}) &= \frac{4M}{r} + \frac{2\hat{x}_i\hat{x}_j}{r}\ddot{I}_{ij}(t-r) \\ * \ \bar{h}_{0i}(t,\underline{x}) &= -\frac{2\hat{x}_j}{r}\ddot{I}_{ij}(t-r) \\ * \ \bar{h}_{ij} &= \frac{2}{r}\ddot{I}_{ij}(t-r) \end{aligned}}$$

tells us perturbation depends only on 2nd moment eq.

formulae to explicitly compute h_{ij} out in regime e.g. binary pulsar system emitting gravitational waves, far enough away

(energy is mass of configuration)

(far enough away so in radiation zone) $\gg r$ / so non rel. sources

where $r \gg \tau \gg d$, in centre of momentum frame where $E = M$

next time: assign energy to these perturbations so I can understand how waves that are sourced by moving bodies actually carry energy away from the system ("energy there tells you almost certainly how to look over" is lots of computation)

Lecture 19

22.11.24

Last Lecture

* Linearised field far from a non-relativistic source:

$$\bar{h}_{00}(t,\underline{x}) = \frac{4M}{r} + \frac{2\hat{x}_i\hat{x}_j}{r}\ddot{I}_{ij}(t-r)$$

$$\bar{h}_{0i}(t,\underline{x}) = -\frac{2\hat{x}_j}{r}\ddot{I}_{ij}(t-r)$$

$$\bar{h}_{ij}(t,\underline{x}) = \frac{2}{r}\ddot{I}_{ij}(t-r)$$

where $I_{ij}(t') = \int d^3x'\, T_{00}(t',\underline{x}')x_i'x_j'$

$r = |\underline{x}|$

$M = \int d^3x'\, T_{00}(t',\underline{x}')$

and we are in centre-of-momentum frame

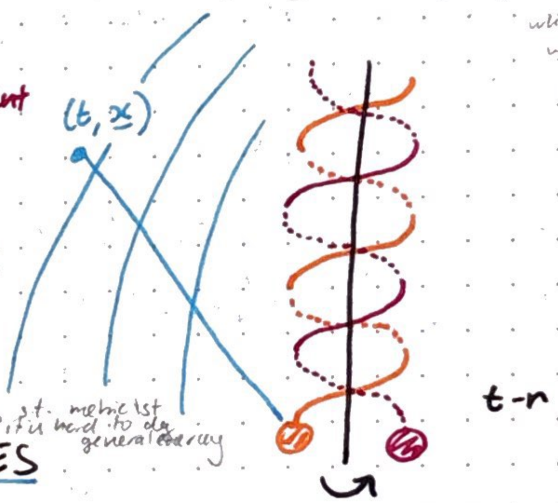
$$P_i = \int d^3x'\, T_{0i}(t',\underline{x}') = 0$$

valid in radiation zone:

$$r \gg T \gg d$$

distance from source → r; T ← timescale of motion of source; d ← spatial extent of source

what source was doing when our past light cone intersects w/ it?



today: these waves carry energy away from system which will affect it e.g. rotation
→ how do waves carry energy away
→ what do we mean by energy?

hard problem to def energy in GR!

hard to assign energy to grav. field because it is geometric quantity
energy normally define as deriv but we can always pick coords s.t. metric 1st deriv vanishes so it is hard to def general energy

ENERGY IN GRAVITATIONAL WAVES

Defining the local energy / local energy flux for a gravitational field is hard in general because we can always choose coords s.t.

$$\partial_\mu g_{\sigma\tau}|_p = 0$$

also other reasons why def energy is hard: also often want to def energy via noether theorem (energy usually quantity conserved due to time symmetry) but in general background we don't have time translation symmetry

There is no hope of an energy density quadratic in first derivatives — how we are used to def energy

∴ prob of def energy is hard, but! we can get round this by working in our perturbation theory

In the context of perturbation theory there are various ways to define an energy. To do this we consider how to continue a perturbative solution beyond linear order.

We consider the ungauged vaccum Einstein equations and suppose

(1) indicates this is linear piece; (correction to $O(\varepsilon^2)$)

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h^{(1)}_{\mu\nu} + \varepsilon^2 h^{(1)}_{\mu\nu}$$

Work to $O(\varepsilon^2)$

We observe (linearise Ricci tensor as function of the metric around minkowski space) (order 0 term = 0 as minkowski is Ricci flat)

$$R_{\mu\nu}[\eta_{\mu\nu} + \varepsilon h_{\mu\nu}] = \varepsilon R^{(1)}_{\mu\nu}[h] + \varepsilon^2 R^{(1)}_{\mu\nu}[h]$$

where (ungauged ricci tensor to 1st order)

$$R^{(1)}_{\mu\nu}[h] = \partial^\rho\partial_{(\mu}h_{\nu)\rho} - \tfrac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \tfrac{1}{2}\partial_\mu\partial_\nu h$$

are linear terms.

and

$$R^{(2)}_{\mu\nu}[h] = \tfrac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - h^{\rho\sigma}\partial_\rho\partial_{(\mu}h_{\nu)\sigma} + \tfrac{1}{4}\partial_\mu h_{\rho\sigma}\partial_\nu h^{\rho\sigma}$$
$$+ \partial^\sigma h^\rho{}_\nu\partial_{[\sigma}h_{\rho]\mu} + \tfrac{1}{2}\partial_\sigma(h^{\sigma\rho}\partial_\rho h_{\mu\nu}) - \tfrac{1}{4}\partial^\rho h\,\partial_\rho h_{\mu\nu} - (\partial_\sigma h^{\rho\sigma} - \tfrac{1}{2}\partial^\rho h)\partial_{(\mu}h_{\nu)\rho}$$

are quadratic terms.

This implies (ricci tensor to quadratic order)

brackets: evaluated on $h^{(1)}$ — call these terms this

$$R_{\mu\nu}[\eta_{\mu\nu}+\epsilon h^{(1)}_{\mu\nu}+\epsilon^2 h^{(2)}_{\mu\nu}] = \epsilon R^{(1)}_{\mu\nu}[h^{(1)}] + \epsilon^2\left(R^{(1)}_{\mu\nu}[h^{(2)}] + R^{(2)}_{\mu\nu}[h^{(1)}]\right)$$

Thus (what we really want is Einstein tensor, got this 2) (einstein tensor at linear order)

$$G_{\mu\nu}[\eta+\epsilon h^{(1)}+\epsilon^2 h^{(2)}] = \epsilon G^{(1)}_{\mu\nu}[h^{(1)}] \quad \Big\}\ \text{linear order in } \epsilon$$

remember when I take trace, I also get terms from order ϵ correction in metric

(*) quadratic order in ϵ:

$$\text{from } \epsilon^2(\dots) \text{ piece} \quad + \epsilon^2\Big(G^{(1)}_{\mu\nu}[h^{(2)}] + \underbrace{R^{(2)}_{\mu\nu}[h^{(1)}] - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\sigma\tau}R^{(2)}_{\sigma\tau}[h^{(1)}]}_{-8\pi t_{\mu\nu}[h^{(1)}]}$$

$$\text{from inv. metric } (\epsilon\times\epsilon \text{ terms}) \quad + \tfrac{1}{2}\eta_{\mu\nu}h^{(1)\sigma\tau}R^{(1)}_{\sigma\tau}[h^{(1)}] - \tfrac{1}{2}h_{\mu\nu}\eta^{\sigma\tau}R^{(1)}_{\sigma\tau}[h^{(1)}]\Big)$$

turns out these terms actually 0 when we impose vacuum eq.

[note: what is meant by $G^{(1)}$?
$$G^{(1)}_{\mu\nu}[h] = R^{(1)}_{\mu\nu}[h] - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\sigma\tau}R^{(1)}_{\sigma\tau}[h]$$]

We now consider the contracted Bianchi identity,

$$g^{\mu\rho}\nabla_\rho G_{\mu\nu} = 0$$

which holds for any metric. Using (*) and expanding gives (to be explicit about what we are carrying/losing)

$$(\dagger)\quad 0 = \epsilon\,\eta^{\mu\sigma}\partial_\sigma G^{(1)}_{\mu\nu}[h^{(1)}] + \epsilon^2\Big(\eta^{\mu\sigma}\partial_\sigma G^{(1)}_{\mu\nu}[h^{(2)}] - 8\pi\,\partial_\mu t_{\sigma\nu}[h^{(1)}]\,\eta^{\mu\sigma}$$
$$+ \underbrace{h^{(1)}\cdot R^{(1)}[h^{(1)}]}_{\text{SCHEMATIC}}\Big).$$

lots more terms but all prop. to linearised ricci tensor evaluated on linearised metric so of some form

don't bother writing out all the terms, write this instead

Considering $G_{\mu\nu}[\eta+\epsilon h^{(1)}+\epsilon^2 h^{(2)}]=0$, using (*) order by order, we deduce

linearised EEs for $h^{(1)}$: RHS vanishes

$$G^{(1)}_{\mu\nu}[h^{(1)}] = 0 \qquad \left(\Rightarrow R^{(1)}_{\mu\nu}[h^{(1)}]=0\right)$$

linearised EEs for $h^{(2)}$ — as we called this earlier

$$G^{(1)}_{\mu\nu}[h^{(2)}] = -R^{(2)}_{\mu\nu}[h^{(1)}] + \tfrac{1}{2}\eta_{\mu\nu}\eta^{\sigma\tau}R^{(2)}_{\sigma\tau}[h^{(1)}] = 8\pi t_{\mu\nu}[h^{(1)}]$$

not really an energy momentum tensor

Thus $h^{(2)}$ solves the linearised Einstein equations sourced by an 'energy momentum tensor'

want it to be divergence free: use bianchi identities

From $(\dagger)$ we deduce that

true for any metric: we have not imposed linearised EEs

$$\eta^{\mu\sigma}\partial_\mu G^{(1)}_{\sigma\nu}[h] = 0,$$ which holds for ANY pertubation h.

And $\eta^{\mu\sigma}\partial_\mu t_{\sigma\nu}[h^{(1)}] = 0$ when $h^{(1)}$ satisfies linearised E.E.s.

total integral over space — div free but not gauge invariant!

We can identify $t_{\mu\nu}$ with Energy-momentum of the gravitational field, however, it is not gauge invariant. If $h^{(1)}$ decays sufficiently at ∞, then $\int t_{00}\, d^3x$ is invariant, (so gives total energy of field) but no gauge invariant local conservation.

We can get approximate gauge invariance by averaging.

Let W be smooth, vanish for $|\underline{x}|^2+t^2>a$, and satisfy

integral over spacetime — use this to get a 'local average' on regions of scale a

$$\int_{\mathbb{R}^4} W(\underline{x},t)\,d^3x\,dt = 1.$$

We define the average of a tensor in almost inertial coordinates by

ball of
W supported on size a so non-zero outside ball of size a so deriv of order W/a

spacetime of y

$$\langle X_{\mu\nu}(x) \rangle = \int_{\mathbb{R}^4} W(y-x) X_{\mu\nu}(y) \, d^4y$$

idea is we can avg over suitable region to recover gauge inv

Suppose we're in far field regime, with radiation of wavelength λ and we average over a region of size $a \gg \lambda$. Since $\partial_\mu W \sim W/a$, we have

same as age integration by parts

$$\langle \partial_\mu X_{\rho\sigma} \rangle = \int_{\mathbb{R}^4} \partial_\mu W(y-x) X_{\rho\sigma}(y) \, d^4y$$

roughly of order

$$\sim \frac{\langle X_{\rho\sigma} \rangle}{a}$$

$$\sim \frac{\lambda}{a} \langle \partial_\mu X_{\rho\sigma} \rangle$$

We can ignore total derivatives inside averages, and thus

$$\langle A \partial B \rangle = \langle \partial(AB) \rangle - \langle (\partial A) B \rangle \simeq -\langle (\partial A) B \rangle$$

← similarly 'integration by parts formula'

With this we can show:

$\boxed{\text{EXERCISE:}}$ (SHEET 3)

1) If h solves vaccum linearised E.E.

$$\langle \eta^{\mu\nu} R^{(2)}{}_{\mu\nu}[h] \rangle = 0.$$

(raise + lower with η)

this is kind of the formula we were looking for all along →

2) $$\langle t_{\mu\nu} \rangle = \frac{1}{32\pi} \langle \partial_\mu \bar{h}_{\rho\sigma} \partial_\nu \bar{h}^{\rho\sigma} - \tfrac{1}{2} \partial_\mu \bar{h} \partial_\nu \bar{h} - 2 \partial_\sigma \bar{h}^{\rho\sigma} \partial_{(\mu} \bar{h}_{\nu)\rho} \rangle.$$

3) $\langle t_{\mu\nu} \rangle$ is gauge invariant.

Using this formula and last lecture's results, we can find energy lost by a system producing gravitational waves.

The averaged spatial energy flux is $S_i = -\langle t_{0i} \rangle$.

We calculate average energy flux across a sphere of radius r centered on source

$$\langle P \rangle = -\int_{S_r} r^2 d\Omega \, \langle t_{0i} \rangle \hat{x}_i$$

In wave gauge

$$\langle t_{0i} \rangle = \frac{1}{32\pi} \langle \partial_0 \bar{h}_{\rho\sigma} \partial_i \bar{h}^{\rho\sigma} - \tfrac{1}{2} \partial_0 \bar{h} \partial_i \bar{h} \rangle$$

expand out

$$= \frac{1}{32\pi} \langle \partial_0 \bar{h}_{jk} \partial_i \bar{h}_{jk} - 2 \partial_0 \bar{h}_{0j} \partial_i \bar{h}_{0j} + \partial_0 \bar{h}_{00} \partial_i \bar{h}_{00} - \tfrac{1}{2} \partial_0 \bar{h} \partial_i \bar{h} \rangle$$

Using $\bar{h}_{ij} = \frac{2}{r} \ddot{I}_{ij}(t-r)$

0 IN RADIATION GAUGE

$$\partial_0 \bar{h}_{jk} = \frac{2}{r} \dddot{I}_{jk}(t-r) \qquad \partial_i \bar{h}_{jk} = \left(-\frac{2}{r} \dddot{I}_{jk}(t-r) - \cancel{\frac{2}{r^2} \ddot{I}_{jk}(t-r)} \right) \hat{x}_i$$

$$\therefore -\frac{1}{32\pi} \int r^2 d\Omega \, \langle \partial_0 \bar{h}_{jk} \partial_i \bar{h}_{jk} \rangle \hat{x}_i = \frac{1}{2} \langle \dddot{I}_{ij} \dddot{I}_{ij} \rangle_{t-r} .$$

of $\langle t_{0i} \rangle$

← AVERAGE OVER WINDOW CENTERED AT $t-r$

Ran out of time so will put one I note for the other terms on moodle but in principle compute all terms to work out power

Lecture 20 — 25.11.24

Last lecture:

* EM tensor for linearised gravitational perturbations
  * $t_{\mu\nu} = -\frac{1}{8\pi}\left(R^{(2)}_{\mu\nu}[h^{(1)}] - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}[h^{(1)}]\right)$

  where $R_{\mu\nu}[\eta_{\mu\nu} + \varepsilon h_{\mu\nu}] = \varepsilon R^{(1)}_{\mu\nu}[h] + \varepsilon^2 R^{(2)}_{\mu\nu}[h] + O(\varepsilon^2)$

* After averaging over a window of size $a \gg \lambda$ (λ typical wavelength of grav. radiation) have

$$\langle t_{\mu\nu}\rangle = \frac{1}{32\pi}\langle \partial_\mu \bar{h}_{\rho\sigma}\partial_\nu \bar{h}^{\rho\sigma} - \frac{1}{2}\partial_\mu \bar{h}\,\partial_\nu \bar{h} \underbrace{- 2\partial_\sigma \bar{h}^{\rho\sigma}\partial_{(\mu}\bar{h}_{\nu)\rho}}_{=0 \text{ IN WAVE GAUGE}}\rangle$$

NEW CONTENT

* Insert expressions from radiation zone solution derived in previous lectures to compute flux through large sphere

$$\langle P\rangle_t = -\int_{\{|x|=r\}} r^2 d\Omega \,\langle t_{0i}\rangle \hat{x}_i = \frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle_{t-r}$$

derivation moodle

(see triple deriv in physics long working on moodle

where

$$Q_{ij} = I_{ij} - \frac{1}{3}I_{kk}\delta_{ij} \quad \text{is the } \underline{\text{quadratic tensor}}$$

DIFFERENTIAL FORMS

We want to derive Einstein's equations from an action principle. For this we need to discuss integration on manifolds, to do so we first discuss differential forms.

DEF: A p-form is a totally antisymmetric $(0,p)$-tensor field on M. The space of p-forms is $\Omega^p M$.

Note: If $p > M$ a p-form must vanish. A 1-form is a covector field.

We have a natural product on forms. If X is a p-form and Y a q-form then $(X\wedge Y)$ is the $(p+q)$-form

$$(X\wedge Y)_{a_1\dots a_p b_1\dots b_q} = \frac{(p+q)!}{p!\,q!} X_{[a_1\dots a_p}Y_{b_1\dots b_q]}$$

It has the properties (CHECK!):

* $X\wedge Y = (-1)^{pq} Y\wedge X$ $\quad$ ($\Rightarrow X\wedge X = 0$ if p odd)
* $(X\wedge Y)\wedge Z = X\wedge (Y\wedge Z)$

If $\{\delta^{\mu}\}_{\mu_1=1}^{n}$ is a dual basis (i.e. basis of covectors), then
$\{\delta^{\mu_1}\wedge\dots\wedge\delta^{\mu_p}\}_{\mu_1<\mu_2<\dots<\mu_p}$
and we can write

$$X = \frac{1}{p!} X_{\mu_1\dots\mu_p}\delta^{\mu_1}\wedge\dots\wedge\delta^{\mu_p}.$$

Another important feature of forms is that we can define a derivative $d: \Omega^p M \to \Omega^{p-1} M$ by

EXTERIOR DERIVATIVE

$\boxed{\text{DEF:}}$ If X is a p-form, then in a coordinate basis

$$(dX)_{\mu_1\dots\mu_{p+1}} = (p+1)\,\partial_{[\mu_1} X_{\mu_2\dots\mu_{p+1}]} \quad (*)$$

Suppose ∇ is any symmetric connection, then

$$\nabla_{\mu_1} X_{\mu_2\dots\mu_{p+1}} = \partial_{\mu_1} X_{\mu_2\dots\mu_{p+1}} - \Gamma^{\sigma}_{\mu_2\mu_1} X_{\sigma\mu_3\dots\mu_{p+1}} - \dots - \overbrace{\Gamma^{\sigma}_{\mu_{p+1}\mu_1}}^{\text{SYMMETRIC IN } \mu_{p+1},\mu_1} X_{\mu_2\dots\mu_p\sigma}$$

$$\Rightarrow \nabla_{[\mu_1} X_{\mu_2\dots\mu_{p+1}]} = \partial_{[\mu_1} X_{\mu_2\dots\mu_{p+1}]}$$

$$\therefore\ (dX)_{\mu_1\dots\mu_{p+1}} = (p+1)\nabla_{[\mu_1} X_{\mu_2\dots\mu_{p+1}]}$$

tensor eq $\Rightarrow$ promote to AIN

Thus $(dX)_{a_1\dots a_{p+1}} = (p+1)\nabla_{[a_1} X_{a_2\dots a_{p+1}]}$ is well-defined independently of coordinates. However, $(*)$ shows it does not depend on a metric or connection.

$\boxed{\text{EXERCISE}}$ (sheet 4) show:

$\left(\begin{array}{l} X \ p\text{-form} \\ Y \ q\text{-form} \end{array}\right)$

* $d(dX) = 0$
* $d(X\wedge Y) = dX\wedge Y + (-1)^p X\wedge dY$
* $\phi^* dX = d(\phi^* X)$ if $\phi: N \to M$

The last property implies that d commutes with Lie derivatives
i.e. $\mathcal{L}_v(dX) = d(\mathcal{L}_v X)$.

d is called the EXTERIOR DERIVATIVE.

We say X is CLOSED if $dX=0$, and X is EXACT if $X = dY$ for some Y. EXACT $\Rightarrow$ CLOSED, but the converse is only true locally.

$\boxed{\text{POINCARÉ LEMMA:}}$ If X is a closed p-form $(p \geq 1)$, then for any $r \in M$ there is an open neighbourhood $N \subset M$ with $r \in N$ and a $(p-1)$-form Y defined on N such that $X = dY$

The extent to which CLOSED $\neq$ EXACT captures topological properties of M.

E.g. On S^1 the form $d\theta$ (see ex. 1.3) is closed, but <u>not</u> exact (despite confusing notation).

THE TETRAD FORMALISM

called something else in more dim. this is what we call it in 4 dim (we only care about 4 dim)

In GR it's often useful to work with an orthonormal basis of vector fields (<u>TETRAD</u>) $\{e^a_\mu\}_{\mu=0}^3$, satisfying

$$g_{ab} e^a_\mu e^b_\nu = \eta_{\mu\nu} \qquad (\delta_{\mu\nu} \text{ if Riemannian})$$

Recall that the dual basis $\{\theta^\mu_a\}_{\mu=0}^3$ is defined by

$$\delta^\mu_\nu = \theta^\mu(e_\nu) = \theta^\mu_a e^a_\nu .$$

We <u>claim</u> that

$$\theta^\mu_a = \eta^{\mu\sigma} g_{ab} e^b_\sigma$$

PROOF: $(\eta^{\mu\sigma} g_{ab} e^b_\sigma) e^a_\nu = \eta^{\mu\sigma}\eta_{\sigma\nu} = \delta^\mu_\nu$.

introduce convention to sep. orthonormal indices from abstract indices but orthonormal basis not unique: can be transf.

Recalling that g_{ab} raises + lowers roman indices, and introducing the CONVENTION that $\eta_{\mu\nu}$ raises + lowers greek indices, we have

$$\theta^\mu_a = e^\mu_a = \eta^{\mu\sigma} g_{ab} e^b_\sigma ,$$

we will thus denote basis vectors e_μ and dual basis vectors e^μ.

Recall that two orthonormal bases are related by

$$e'^a_\mu = (A^{-1})^\nu{}_\mu e^a_\nu \quad \text{where} \quad \eta_{\mu\nu} A^\mu_\rho A^\nu_\sigma = \eta_{\rho\sigma} .$$

Unlike in special relativity, A^μ_ν need not be constant.

'GR arises by gauging the Lorentz symmetry of SR'

take sum over covectors recover the 2-tensor which is metric. Cf earlier statement they look similar but say diff things

CLAIM: $\eta_{\mu\nu} e_a{}^\mu e_b{}^\nu = g_{ab}$, $e_a{}^\mu e^b_\mu = \delta_a{}^b$

PROOF: Contract with e^b_ρ:

$$\eta_{\mu\nu} e_a{}^\mu e_b{}^\nu e^b_\rho = \eta_{\mu\nu} e_a{}^\mu \delta^\nu_\rho = \eta_{\mu\rho} e_a{}^\mu = (e_a)_\rho = g_{ab} e^b_\rho$$

Since equation holds ~~for~~ contracted with any basis vector, it holds in general.

Second equation follows from first by raising b.

3 pr lec + spare

Lecture 21 — 27.11.24

Last Lecture:

* Differential forms, $\Omega^p M$
  - $\to X \wedge Y$ 'wedge product'
  - $\to dX$ 'exterior derivative'
* Tetrad formalism $\{e_\mu\}$ oh basis, $\{e^\mu\}$ dual basis $e_a{}^\mu = g_{ab}\eta^{\mu\sigma}e^b{}_\sigma$
  - $\to$ raise + lower $\begin{cases} \text{LATIN indices w/ } g_{ab} \\ \text{GREEK indices w/ } \eta_{\mu\nu} \end{cases}$
  - $\to g_{ab} = e_a{}^\mu e_b{}^\nu \eta_{\mu\nu}$, $\eta_{\mu\nu} = e^a{}_\mu e^b{}_\nu g_{ab}$, $e^a{}_\mu e_b{}^\mu = \delta^a_b$, $e^a{}_\mu e_a{}^\nu = \delta^\nu_\mu$.

CONNECTION 1-FORMS

Let ∇ be the Levi-Civita connection. The connection 1-forms are defined to be

1-forms have a downstairs abstract index

seen before: taking ∇ of basis vecs is related to connection components
$\Rightarrow$ this is just diff way of encoding the Γs

$$(\omega^\mu{}_\nu)_a := e_b{}^\mu \nabla_a e^b{}_\nu$$

(just defining Γ)

Recalling
$$e^a{}_\nu \nabla_a e^b{}_\mu = \Gamma_\mu{}^\sigma{}_\nu e^b{}_\sigma$$

multiply by $\times e_c{}^\nu$

$$\Rightarrow \quad \nabla_c e^b{}_\mu = \Gamma_\mu{}^\sigma{}_\nu e^b{}_\sigma e_c{}^\nu$$

go through algebra yourself using above relations (last lec)

$$\therefore \quad (\omega^\mu{}_\nu)_a = \Gamma_\rho{}^\mu{}_\sigma e^\rho{}_a .$$

so $(\omega^\mu{}_\nu)_a$ encodes the connection components.

so what? $\to$ we'll see later. but first some lemmas.

LEMMA: $(\omega_{\mu\nu})_a = -(\omega_{\nu\mu})_a$

this just set of constants so if i differentiate I get 0

PROOF:
$$(\omega_{\mu\nu})_a = (e_b)_\mu \nabla_a e^b{}_\nu \overset{\text{chain rule}}{=} \nabla_a(\underbrace{(e_b)_\mu e^b{}_\nu}_{\eta_{\mu\nu}}) - e^b{}_\nu \nabla_a (e_b)_\mu$$

same as line above w/ indices swapped + minus sign

$$= -(e_b)_\nu \nabla_a e^b{}_\mu = -(\omega_{\nu\mu})_a$$

Now consider the exterior derivative of a basis 1-form.

LEMMA: The 1-form e^μ satisfies Cartan's first structure eqn.

(remember μ,ν not tensor indices, de^μ means thinking of it as one-form and find exterior deriv. for each index)

$$de^\mu + \omega^\mu{}_\nu \wedge e^\nu = 0.$$

check: consistent? de^μ (deriv of 1 form) is a 2 form and 2nd term also 2-form?

PROOF: Note $(\omega^\nu{}_\mu)_a e^b{}_\nu = (e_c{}^\nu \nabla_a e^c{}_\mu) e^b{}_\nu = \nabla_a e^b{}_\mu$

Thus
$$\nabla_a e_b{}^\mu = (\omega_\nu{}^\mu)_a e_b{}^\nu = -(\omega^\mu{}_\nu)_a e_b{}^\nu$$

by antisymmetry property

$$\Rightarrow (de^\mu)_{ab} = 2\nabla_{[a} e^\mu{}_{b]} = -2(\omega^\mu{}_\nu)_{[a} e_{b]}{}^\nu = -(\omega^\mu{}_\nu \wedge e^\nu)_{ab}.$$

from defn of wedge product (last lec)

what does this mean?
$\hookrightarrow$ I can find components of ω by computing exterior derivative of e.
$\hookrightarrow$ efficient way of finding connection components.

Note that in the orthonormal basis,

$$(de^\mu)_{\nu\sigma} = 2(\omega^\mu{}_{[\nu})_{\sigma]},$$

(so if i compute this I know antisymmetrised components of ω)

so computing de^μ leads to $(\omega^\mu{}_{[\nu})_{\sigma]}$ since $(\omega_{\mu\nu})_\rho = -(\omega_{\nu\mu})_\rho$.

("we're never gonna use this formula" just write to show our Cartan's eq. → show example of how to use this later)

(once we know de^μ they uniquely determines the ωs)

We can check $(\omega_{\mu\nu})_\rho = (\omega_{\mu[\nu})_{\rho]} + (\omega_{\nu[\rho})_{\mu]} - (\omega_{\rho[\mu})_{\nu]}$.

So we can determine $\omega^\mu{}_\nu$ and hence Γ by computing de^μ.

Example: The Schwarzchild Metric

(reminder: metric $-e^{0\,2}+e^{1\,2}+e^{2\,2}+e^{3\,2}$)

has an obvious tetrad:

$$e^0 = \gamma dt \qquad e^1 = \tfrac{1}{\gamma} dr \qquad e^2 = r d\theta \qquad e^3 = r\sin\theta\, d\phi$$

where $\gamma = \sqrt{1 - \frac{2M}{r}}$. fixes the metric

(computing exterior derivs is efficient: can see where they vanish easily.)

read "wedge"

Then
$$de^0 = d\gamma \wedge dt + \gamma d(dt) = \gamma' dr \wedge dt = \gamma' e^1 \wedge e^0$$
$$de^1 = d(\tfrac{1}{\gamma}) \wedge dr + \tfrac{1}{\gamma} d(dr) = -\tfrac{\gamma'}{\gamma^2} dr \wedge dr = 0$$
$$de^2 = dr \wedge d\theta$$
$$de^3 = \sin\theta\, dr \wedge d\phi + r\cos\theta\, d\theta \wedge d\phi = \tfrac{\gamma}{r} e^1 \wedge e^3 + \tfrac{\cot\theta}{r} e^2 \wedge e^3$$

(compare w expressions above)

$$de^0 = -\omega^0{}_\mu \wedge e^\mu \Rightarrow \omega^0{}_1 = \gamma' e^0,\ \omega^0{}_2 \propto e^2,\ \omega^0{}_3 \propto e^3$$
$$de^1 = -\omega^1{}_\mu \wedge e^\mu \Rightarrow \omega^1{}_0 \propto e^0,\ \omega^1{}_2 \propto e^2,\ \omega^1{}_3 \propto e^3$$
$$de^2 = -\omega^2{}_\mu \wedge e^\mu \Rightarrow \omega^2{}_1 = \tfrac{\gamma}{r} e^2,\ \omega^2{}_0 \propto e^0,\ \omega^2{}_3 \propto e^3$$
$$de^3 = -\omega^3{}_\mu \wedge e^\mu \Rightarrow \omega^3{}_0 \propto e^0,\ \omega^3{}_1 = \tfrac{\gamma}{r} e^3,\ \omega^3{}_2 = \tfrac{\cot\theta}{r} e^3$$

(can check all these statements are consistent w/ eachother)

We have:

$$\omega_{01} = -\omega_{10} = -\gamma' e^0$$
$$\omega_{21} = -\omega_{12} = \tfrac{\gamma}{r} e^2$$
$$\omega_{31} = -\omega_{13} = \tfrac{\gamma}{r} e^3$$
$$\omega_{32} = -\omega_{23} = \tfrac{\cot\theta}{r} e^3$$

all other components vanish

(you'll have to believe me this is an efficient way to compute connection components)

(more valuably we can compute curvature components from these curvature 2-forms)

CURVATURE 2-FORMS

We compute $d\omega^\mu{}_\nu$

$$(d\omega^\mu{}_\nu)_{ab} = \nabla_a (\omega^\mu{}_\nu)_b - \nabla_b (\omega^\mu{}_\nu)_a$$
$$= \nabla_a (e_c{}^\mu \nabla_b e^c{}_\nu) - \nabla_b (e_c{}^\mu \nabla_a e^c{}_\nu)$$

(expand out using Leibniz)

$$= e_c{}^\mu (\nabla_a \nabla_b e^c{}_\nu - \nabla_b \nabla_a e^c{}_\nu) + \nabla_a e_c{}^\mu \nabla_b e^c{}_\nu - \nabla_b e_c{}^\mu \nabla_a e^c{}_\nu$$
$$= e_c{}^\mu (R^c{}_{dab}) e^d{}_\nu + e^d{}_\sigma (\nabla_a e_d{}^\mu) e_\gamma^\sigma (\nabla_b e^\gamma{}_\nu) - e^d{}_\sigma (\nabla_b e_d{}^\mu) \cdot e_\gamma^\sigma (\nabla_a e^\gamma{}_\nu)$$

$$= (\Theta^\mu{}_\nu)_{ab} + (\omega_\sigma{}^\mu \wedge \omega^\sigma{}_\nu)_{ab}$$

where $\Theta^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\sigma\tau} e^\sigma \wedge e^\tau$ are the curvature 2-forms. We've shown Cartan's second structure equation,

$$d\omega^\mu{}_\nu + \omega^\mu{}_\sigma \wedge \omega^\sigma{}_\nu = \Theta^\mu{}_\nu,$$

gives an efficient way to compute $R^\mu{}_{\nu\sigma\tau}$ in an o/n basis.

Returning to our example:

$$d\omega^0{}_1 = d(f'e^0) = d(f'f\,dt) = (f''f + f'^2)\,\overbrace{dr \wedge dt}^{e^1 \wedge e^0} = (f''f + f'^2)e^1 \wedge e^0$$

only non zero when $\mu = 1$ ($\mu = 2$)

$$\omega^0{}_\mu \wedge \omega^\mu{}_1 = \omega^0{}_1 \wedge \omega^1{}_1 + \omega^0{}_2 \wedge \omega^2{}_1 + \omega^0{}_3 \wedge \omega^3{}_1 + \omega^0{}_0 \wedge \omega^0{}_1 = 0 \quad \text{(all vanish)}$$

$$\therefore \Theta^0{}_1 = -(f''f + f'^2)e^0 \wedge e^1$$

hence $R^0{}_{101} = -(f''f + f'^2) = -\frac{1}{2}(f^2)'' = \frac{2M}{r^3}$

and $R^\mu{}_{\nu 01} = 0$ otherwise.

we found w.r.t. o/n basis: at least 1 component of R tensor but actually we found none (components) since 0 otherwise → that is why this is efficient, we don't have to calc lots of things and find what is zero; zero components just follow from the argument

Exercise: Find other $\Theta^\mu{}_\nu$ and show R_{ab}.

tetrad stuff was a bit of a side track → remember original goal for introducing differential forms was to do integration on manifold so we can derive Einstein's equations from an action principle: now let's return to this original goal (to understand integration)

← first step is to look at volume form

VOLUME FORM AND HODGE DUAL

We say a manifold is ORIENTABLE if it admits a nowhere vanishing n-form ($n = \dim M$) $\varepsilon_{a_1 \dots a_n}$, an orientation form.

Two such forms are equivalent if $\varepsilon' = f\varepsilon$ for some smooth, everywhere positive f. $[\varepsilon]_\sim$ is an orientation.

A basis of vectors $\{e_\mu\}_{\mu=1}^n$ is right-handed if

$$\varepsilon(e_1, \dots, e_n) > 0.$$

(canonical choice of normalisation is to set this $= 1$)

← (corresponding basis of covectors)

A coordinate system is right-handed if $\left\{\frac{\partial}{\partial x^\mu}\right\}_{\mu=1}^n$ are right-handed.

why care about handedness?
→ if we use e.g. divergence / Stokes theorem we need to know direction / normal to surface etc.
intrinsically handedness/orientation is bound up in integrating!

Lecture 22

29.11.24

Last lecture

- ***** Tetrad formalism
  - Cartan's 1st + 2nd structure equations

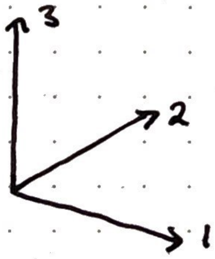
$$de^\mu + \omega^\mu{}_\nu \wedge e^\nu = 0 \qquad \omega_{\mu\nu} = -\omega_{\nu\mu} \quad \text{connection 1-forms}$$
$$d\omega^\mu{}_\nu + \omega^\mu{}_\sigma \wedge \omega^\sigma{}_\nu = \Theta^\mu{}_\nu \qquad \Theta^\mu{}_\nu = \tfrac{1}{2} R^\mu{}_{\nu\sigma\tau} e^\sigma \wedge e^\tau \quad \text{curvature 2-forms}$$

- ***** Manifold is <u>orientable</u> if it admits a nowhere vanishing n-form, ε.
  - Two such forms equivalent if $\varepsilon' = f\varepsilon$ ($f>0$ everywhere)
  - $[\varepsilon]_N$ is an orientation
  - $\{e_\mu\}_{\mu=1}^n$ is right-handed if $\varepsilon(e_1,\dots,e_n) > 0$. coord. system is RH if $\left\{\frac{\partial}{\partial x^\mu}\right\}_{\mu=1}^n$ is RH.

means orientable & we chosen orientation

no preferred normalisation e.g. $\frac{1}{2}\varepsilon_{ijk}$ is as good as $5\varepsilon_{ijk}$ until we introduce a metric

in $\mathbb{R}^3$ this is standard orientation ε_{ijk} (right-handed basis)



ε_{ijk}

An oriented manifold with metric has a preferred normalisation for ε. For a right-handed orthonormal basis, we define the <u>volume form</u> ε by

$$\varepsilon(e_1,\dots,e_n) = 1 \qquad \text{(indep. of choice of RH o/n basis)}$$

If we work in a RH coord. system $\{x^\alpha\}_{\alpha=1}^n$, then

$$\frac{\partial}{\partial x^\alpha} = e_\alpha{}^\mu e^\beta{}_\mu \frac{\partial}{\partial x^\beta} = e_\alpha{}^\mu e_\mu$$

Then

plug into volume form w/ this choice of normalisation

$$\varepsilon\left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\right) = \varepsilon\left(e_1{}^{\mu_1} e_{\mu_1},\dots,e_n{}^{\mu_n} e_{\mu_n}\right)$$

can pull out $e_n{}^{\mu_n}$ as just numbers? then sign (± 1) depends on permutation

$$= \sum_{\pi \in \mathrm{Sym}(n)} \sigma(\pi)\, e_1{}^{\pi(1)} \times \dots \times e_n{}^{\pi(n)} = \det(e_\alpha{}^\mu)$$

sum over permutations on n indices; sign of permutation

but $e_\alpha{}^\mu e_\beta{}^\nu \eta_{\mu\nu} = g_{\alpha\beta}$ $\Rightarrow$ $\det(e_\alpha{}^\mu) = \sqrt{|g|}$ where $g = \det(g_{\alpha\beta})$

thinking of these as matrices

deduce after reminding ourselves about combinatorial factors:

$$\therefore \varepsilon = \sqrt{|g|}\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^n.$$

Equivalently, $\varepsilon_{123\dots n} = \sqrt{|g|}$ in coord. basis.

<u>EXERCISE</u>: In the same coord. basis

$$\varepsilon^{123\dots n} = \pm \frac{1}{\sqrt{|g|}} \quad \begin{cases} + & \text{RIEMANNIAN} \\ - & \text{LORENTZIAN} \end{cases}$$

LEMMA: $\nabla \varepsilon = 0$

PROOF: In normal coords. at p; $\partial_\mu g_{\nu\sigma}|_p = 0$, $\Gamma^\sigma_{\mu\nu}|_p = 0$

$$\Rightarrow \nabla_{\mu_1} \varepsilon_{\mu_2 \cdots \mu_{n+1}} = \partial_{\mu_1} \varepsilon_{\mu_2 \cdots \mu_{n+1}} + (\Gamma \cdot \varepsilon) = 0 \quad \text{at } p$$

tensor equation so holds everywhere □

LEMMA: $\varepsilon^{a_1 \cdots a_p c_{p+1} \cdots c_n} \varepsilon_{b_1 \cdots b_p c_{p+1} \cdots c_n} = \pm p!(n-p)! \, \delta^{a_1}_{[b_1} \delta^{a_2}_{b_2} \cdots \delta^{a_p}_{b_p]}$ $\begin{cases} + & \text{RIEMANNIAN} \\ - & \text{LORENTZIAN} \end{cases}$

PROOF: (exercise)

p-forms on manifold — (n-p)-forms on manifold

We can use ε to relate $\Omega^p M$ to $\Omega^{n-p} M$:

DEF: On an oriented manifold with metric, the HODGE DUAL of a p-form X is

$$(\ast X)_{a_1 \cdots a_{n-p}} = \frac{1}{p!} \varepsilon_{a_1 \cdots a_{n-p} b_1 \cdots b_p} X^{b_1 \cdots b_p}$$

From previous results we can show

LEMMA: $\ast(\ast X) = \pm(-1)^{p(n-p)} X$ $\begin{cases} + & \text{RIEMANNIAN} \\ - & \text{LORENTZIAN} \end{cases}$

obvious thing we would guess to take divergence of X is actually correct (check this)

$(\ast d \ast X)_{a_1 \cdots a_{p-1}} = \pm (-1)^{p(n-p)} \nabla^b X_{a_1 \cdots a_{p-1} b}$ $\begin{cases} + & \text{RIEMANNIAN} \\ - & \text{LORENTZIAN} \end{cases}$

This seems esoteric but we are actually quite familiar w this do it without even realising in vector calc when we take divs, grads & curls

Examples

(3D in euclidean often dont distinguish between vector fields & 1-forms because we raise & lower w/ δ, we often dont bother to keep track)

① In Euclidean space, identifying a vector field X^a with the one form X_a, the usual operations of vector calculus become

$$\nabla f = df, \quad \operatorname{div} X = \ast d \ast X, \quad \operatorname{curl} X = \ast dX$$

why need hodge dual? so hodge dual operation needed to write down some of these things (for curl need ε) → turns out in higher dim. this is how we generalise vector calc.

remembering that acting twice with d gives 0, ∗ gives ± what I started with

note $d^2 = 0 \Rightarrow \operatorname{curl} \nabla f = 0$ and $\operatorname{div} \operatorname{curl} X = 0$.

no longer makes sense to think of curl as a vector in >3D

② Maxwell's Equations

w/ source $\nabla^a F_{ab} = -4\pi j_b$ and $\nabla_{[a} F_{bc]} = 0$ source-free

can be written

$$d \ast F = -4\pi \ast j, \quad dF = 0.$$

recover usual statement that we can write a maxwell field in terms of potential (locally)

Poincaré's lemma implies we can write $F = dA$ for some one-form A, locally. cant necessarily do this globally

we have introduced this formulation w/ ds & ∗s, turns out this makes integration on manifolds & stoke's thm. quite elegant.

INTEGRATION ON MANIFOLDS

challenge: can always pick a coord patch to integrate over w/ manifolds but changing coords in general changes jacobian factor so I won't get same ans.
→ want to figure out what i can integrate to get same ans under coord change...

Suppose on a manifold M we have a RH coordinate chart $\phi: \mathcal{O} \to \mathcal{U}$ with coordinates x^μ. If X is an n-form which vanishes outside $\mathcal{O}$, we can write

$$X = X_{1\ldots n}\, dx^1 \wedge \cdots \wedge dx^n .$$

If $\psi : \mathcal{O} \to \mathcal{U}$ is another RH coordinate chart with coords $\{y^\mu\}$ then

$$X = \hat{X}_{1\ldots n}\, dy^1 \wedge \cdots \wedge dy^n = \hat{X}_{1\ldots n} \frac{\partial y^1}{\partial x^{\mu_1}} \cdots \frac{\partial y^n}{\partial x^{\mu_n}}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$$

$$= \hat{X}_{1\ldots n} \det\left(\frac{\partial y^\mu}{\partial x^\nu}\right) dx^1 \wedge \cdots \wedge dx^n$$

$$\therefore\ X_{1\ldots n} = \hat{X}_{1\ldots n} \det\left(\frac{\partial y^\mu}{\partial x^\nu}\right)$$

As a result

$$\int_{\mathcal{U}} X_{1\ldots n}\, dx^1 \ldots dx^n = \int_{\mathcal{M}} \hat{X}_{1\ldots n}\, dy^1 \ldots dy^n .$$

we have jacobian factor that exactly fixes things: we have found something we can integrate that doesn't care about coord patch we integrate over.

We can define

$$\int_M X = \int_{\mathcal{O}} X := \int_{\mathcal{U}} X_{1\ldots n}\, dx^1 \ldots dx^n$$

(if you don't know what that is, ignore it) (any manifold relevant in GR)

On any (2nd countable) manifold, we can find a countable atlas of charts $(\phi_i, \mathcal{O}_i)$, and smooth functions $\chi_i : M \to [0,1]$ such that χ_i vanishes outside $\mathcal{O}_i$ and

$$\sum_{i=1}^{\infty} \chi_i(p) = 1$$

$\forall p \in M$, and the sum is locally finite

only at most finitely many of these terms are non-zero

(don't really need to worry about this in practice, just writing so you have the definitions)

Then for any n-form X we define

$$\int_M X = \sum_{i=1}^{\infty} \int_M \chi_i X = \sum_{i=1}^{\infty} \int_{\mathcal{O}_i} \chi_i X$$

(don't worry about this too much, in practice in GR we can cover w/ 1 chart or at least mostly w/ 1 chart (sufficiently small uncovered bit of M can ignore))

This doesn't depend on a choice of χ_i's.

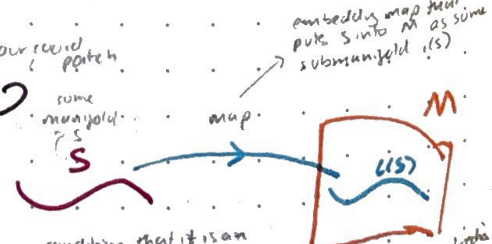
<u>Remarks</u>

- Computation showing coord. invariance implies that for a diffeomorphism $\phi : M \to M$
$$\int_M X = \int_M \phi^* X$$
- If M is a manifold with metric and vol. form ε, then if $f : M \to \mathbb{R}$ is a scalar, $f\varepsilon$ is an n-form, and we can define $\int_M f = \int_M f\varepsilon$.

our coord patch

In local coordinates, if f vanishes outside $\mathcal{O}$

$$\int_M f = \int_{\mathcal{U}} f(x)\sqrt{|g|}\, dx^1 \ldots dx^n = \int_M f\, d\mathrm{vol}_g .$$



condition that it is an <u>embedding</u>: no self intersecting in blue curve

condition that <u>injective</u>: ι(S) has as many tangent directions along point as S does

<u>SUBMANIFOLDS AND STOKES THEOREM</u>

$\boxed{\text{DEF:}}$ Suppose S, M are manifolds and $\dim S = m < n = \dim M$. A smooth map $\iota : S \to M$ is an embedding if it is an injection (i.e. $\iota_* : T_pS \to T_{\iota(p)}M$ is injective) and if ι is injective i.e. $\iota(p) = \iota(q) \Rightarrow p = q$.

If ι is an embedding, then $\iota(S)$ is an embedded submanifold. (If $m = n-1$ we call it a hypersurface) (mostly drop ι when obvious from context and write $\iota(S) = S$).

If S, M are orientable, and $\iota(S)$ is an embedded submanifold of M. We define the integral of an n-form X over $\iota(S)$ by

$$\int_{\iota(S)} X = \int_S \iota^*(X).$$

Note that if $X = dY$ then

$$\int_{\iota(S)} dY = \int_S d(\iota^* Y).$$

$(d\phi^* y = \phi^* dy)$

Lecture 23

2.12.24

<u>Last lecture</u>

* If M is an oriented n-dim manifold, X is an n-form defined $\int_M X$.
* If M carries a metric (or otherwise has a preferred choice of volume form ε) defined

$$\int_M f := \int_M f\varepsilon = \int_M f \, d\mathrm{vol}_g.$$

if ε is volume form of g

* If S is an m-dimensional manifold $(m<n)$, then $\iota : S \to M$ is an embedding if it is an IMMERSION and is INJECTIVE.

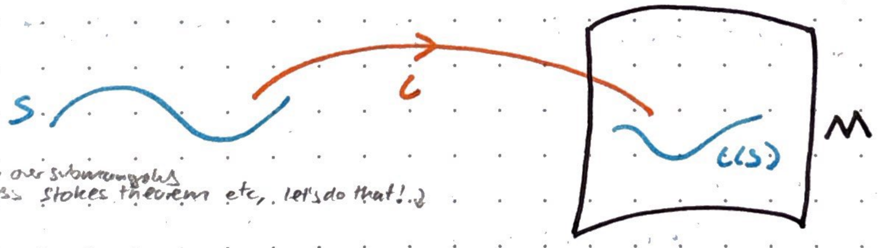
$\iota(p) = \iota(q) \Rightarrow p = q$

$\iota_* : T_pS \to T_{\iota(p)}M$ injective

If so, $\iota(S)$ is an m-dim SUBMANIFOLD of M.

* If $\iota(S)$ is an m-dim submanifold and Y is an m-form on M, then

$$\int_{\iota(S)} Y := \int_S \iota^*(Y)$$



we talked about integrating over submanifolds because we want to discuss Stokes theorem etc, let's do that!

DEF: A manifold with boundary, M, is defined just as for a manifold, (as previously) but charts are maps

$$\phi_\alpha : \mathcal{O}_\alpha \to \mathcal{U}_\alpha, \quad \text{where } \mathcal{U}_\alpha \text{ is an open subset}$$

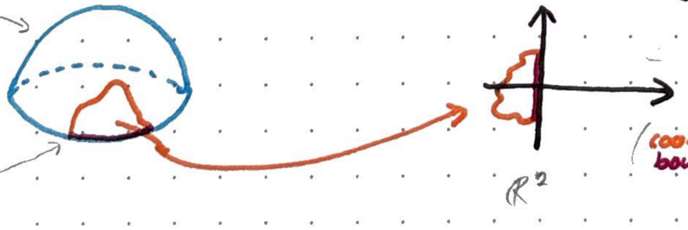
$\hat{M}$

of $\mathbb{R}^n_{\leq 0} = \{(x^1, \dots, x^n) \mid x^1 \leq 0\}$.

not $\mathbb{R}^n$, instead half space

hemisphere, boundary is equator

coord patch intersecting the boundary



maps into region in $\mathbb{R}^2$ in left half plane, patch of boundary is mapped onto purple region

(coord patch mapped
boundary mapped)

The boundary of M, ∂M is the set of points mapped in any chart to $\{x^1 = 0\}$. It is naturally an $n-1$ dim'l manifold with an embedding $\iota : \partial M \to M$.

If M is oriented, ∂M inherits an orientation by requiring $(x^2, \dots, x^n)$ is a RH chart on ∂M when $(x^1, \dots, x^n)$ is RH on M.

"in practice we are not going to use this definition but it's good to have it" now we have defn. consider stokes

STOKES THEOREM

If N is an oriented n-diml manifold with boundary, and X is an $(n-1)$-form, then

$$\int_N dX = \int_{\partial N} X .$$

elegant form doesnt reqire structure on manifold. — except boundary stuff?
(basis of all integrating by parts arguments on manifolds)

Stokes theorem is the basis of all 'integration by parts' arguments. If N carries a metric, we can reformulate stokes theorem as the divergence theorem. slightly more convenient form?

If V is a vector field on N, then we define

V "hook" epsilon

$$(V \lrcorner \varepsilon)_{a_2 \dots a_n} = V^a \varepsilon_{a a_2 \dots a_n}$$

natural $n-1$ form

We can check that $d(V \lrcorner \varepsilon) = (\nabla_a V^a)\varepsilon$. check yourself

If we define the flux of V through an embedded hypersurface S by

think of this as contraction between vector measure of surface

$$\int_S V \cdot dS := \int_S V \lrcorner \varepsilon$$

a little bit of a cheat, reverse defn RHS to be what we want it to look like for div theorem so check this

then Stokes theorem implies $\int_N \nabla_a V^a \, dvol_g = \int_{\partial N} V \cdot dS$.

Recall that a hypersurface $\iota(S)$ is

$$\begin{cases} \text{SPACELIKE} & \text{if } h = \iota^* g \text{ is RIEMANNIAN} \\ \text{TIMELIKE} & \text{if } h = \iota^* g \text{ is LORENTZIAN} \end{cases}.$$

In this case we can relate $\iota^*(V \lrcorner \varepsilon_g)$ to the volume form on (S, h).

Pick $b_2, \dots, b_n$ a RH o/n basis on S (w.r.t. h). Then $\iota_* b_2, \dots, \iota_* b_n$ are o/n in N. The unit normal to S is the unique unit vector $\vec{n}$ orthogonal to $\iota_* b_2, \dots, \iota_* b_n$ with

this is unit vector, becuse we are in Lor. space, this equals ± 1

$$\varepsilon(\vec{n}, \iota_* b_2, \dots, \iota_* b_n) = g(\vec{n}, \vec{n}) \quad (= \pm 1).$$

iff

If $\iota(S)$ $\begin{cases} \text{SPACELIKE} \longleftrightarrow \vec{n} \text{ TIMELIKE} \\ \text{TIMELIKE} \longleftrightarrow \vec{n} \text{ SPACELIKE} \end{cases}$

with this definition,

$$V \lrcorner \varepsilon(\iota_* b_2, \dots, \iota_* b_n) = V^a \vec{n}_a$$

thus $\iota^*(V \lrcorner \varepsilon_g) = \iota^*(V^a \vec{n}_a)\, \varepsilon_h$

volume form on induced metric of h.

$$\int_S V \cdot dS = \int_S V^a \vec{n}_a \, dvol_h .$$

what we normally call flux of vector field through an submanifold (bit fiddly w/ direction of normal)

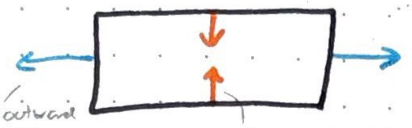
We've shown

$$\int_{\partial N} V^a \vec{n}_a \, dvol_h = \int_N \nabla_a V^a \, dvol_g .$$

(boundary)

Checking the definitions, $\vec{n}^a$ points 'out' of N for ∂N TIMELIKE (or g Riemannian) and 'into' N for ∂N SPACELIKE.

choosing normals to apply div theorem:



choose outward normals for timelike

inward directed for spacelike

(normal spacetime diagram time up & space across)

div theorem: (only difference to what we are used to in vector calc. is we have to insert signs into directions of unit vectors)

[what about null boundaries? → issue is no well-defined unit normal b/c normal to null surface is null vector & we can't normalise null vector so flux makes sense for null surface (n·dvolh well defined) but can't separately normalise V^a & dvolh? (don't worry about this too much)]

THE EINSTEIN HILBERT ACTION

We want to derive Einstein's equations from an action principle, we expect an action of the form

action depends on g & possibly some matter fields

$$S[g, \text{MATTER}] = \int_M L(g, \text{MATTER})\, d\mathrm{vol}_g$$

(need a scalar that depends on g to insert in here to integrate, an 'obvious' guess is scalar curvature (work through to see this works))

where L is a scalar Lagrangian. If we ignore matter for now, an obvious guess for L is $L=R$, scalar curvature. This gives the Einstein-Hilbert action.

$$S_{EH}[g] = \frac{1}{16\pi}\int_M R_g\, d\mathrm{vol}_g.$$

(in order to derive eom from this, consider variation of g (our unknown) that varies outside bounded region. We want that the action does not change to 1st order when we make this variation (principle of least action))

In order to derive e.o.m. from an action, we consider $g+\delta g$ where δg vanishes outside a compact (=bounded) set in M, and expand to first order in δg (considered small). We require variation of S_{EH} to vanish at this order. We want to compute

(g appears in both R_g & dvol$_g$ so need to figure out how both of these change when I vary g)

$$\delta S_{EH} = S_{EH}[g+\delta g] - S_{EH}[g] \qquad \text{(ignore } O((\delta g)^2)\text{)}$$

First we consider $d\mathrm{vol}_g = \sqrt{|g|}\, dx^1 \ldots dx^n$ (assume δg non-zero in a single coord patch)

To compute $\delta|g|$, recall we can write the determinant as

$$g = g_{\bar{\mu}\nu}\Delta^{\bar{\mu}\nu}$$

$\bar{\mu}$ means fixed value, no sum over $\bar{\mu}$.

where $\Delta^{\mu\nu}$ is cofactor matrix $\Delta^{\mu\nu} = (-1)^{\mu+\nu}\begin{vmatrix} g_{11} & \cdots & g_{1\nu} & \cdots & g_{1n} \\ \vdots & & \vdots & & \vdots \\ g_{\mu 1} & \cdots & g_{\mu\nu} & \cdots & g_{\mu n} \\ \vdots & & \vdots & & \vdots \\ g_{n1} & \cdots & g_{n\nu} & \cdots & g_{nn}\end{vmatrix}$

delete ν'th column & μth row

① → delete this column & row, find det of resulting matrix & multiply by factor ±1 → δg gives component of cofactor matrix

② note $\mu\nu$th component of matrix is indep of $g_{\mu\nu}$ because we deleted this from matrix when computing det

"Jacobi's formula or something"

$\Delta^{\mu\nu}$ is independent of $g_{\mu\nu}$, and satisfies $\Delta^{\mu\nu} = g g^{\mu\nu}$ ∴ consider under $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, $\delta g = \frac{\partial g}{\partial g_{\mu\nu}}\delta g_{\mu\nu} = \Delta^{\mu\nu}\delta g_{\mu\nu} = g g^{\mu\nu}\delta g_{\mu\nu}$

$$\therefore \delta\sqrt{-g} = \delta\sqrt{|g|} = \tfrac{1}{2}\frac{1}{\sqrt{-g}}(-\delta g) = \tfrac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu}$$

$$\therefore \delta(d\mathrm{vol}_g) = \tfrac{1}{2} g^{ab}\delta g_{ab}\, d\mathrm{vol}_g.$$

we have found variation in volume form dvol$_g$ after variation of metric, now work out change in R_g ↓

To compute δR_g, we first consider $\delta\Gamma_{\nu}{}^{\mu}{}_{\rho}$. The difference of two connections is a tensor, so this is a tensor $\delta\Gamma_a{}^b{}_c$. To compute this, we can consider normal coordinates for g_{ab} at some point p. Then since $\partial_\mu g_{\nu\sigma}|_p = 0$

only non zero contrib. comes from varying derivative piece

$$\delta\Gamma_{\nu}{}^{\mu}{}_{\rho}\big|_p = \tfrac{1}{2} g^{\mu\sigma}\left(\delta g_{\sigma\nu,\rho} + \delta g_{\rho\sigma,\nu} - \delta g_{\nu\rho,\sigma}\right)\big|_p$$

$$= \tfrac{1}{2} g^{\mu\sigma}\left(\delta g_{\sigma\nu;\rho} + \delta g_{\rho\sigma;\nu} - \delta g_{\nu\rho;\sigma}\right)\big|_p$$

tensor eq. both sides, true for all p so promote to a.i.n. (since p arbitrary)

expr for variation in christoffel symbols ; -> (use this to give variation in R)

$$\therefore\ \delta\Gamma^a_{bc} = \tfrac{1}{2} g^{ad}\left(\delta g_{db;c} + \delta g_{cd;b} - \delta g_{bc;d}\right).$$

Next we consider $\delta R^{\mu}{}_{\nu\rho\sigma}$. Again work in normal coords at p,

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho\left(\Gamma_{\nu}{}^{\mu}{}_{\sigma}\right) - \partial_\sigma\left(\Gamma_{\nu}{}^{\mu}{}_{\rho}\right) + \Gamma\circ\Gamma$$

$$\Rightarrow\ \delta R^{\mu}{}_{\nu\rho\sigma}\big|_p = \left[\partial_\rho\left(\delta\Gamma_{\nu}{}^{\mu}{}_{\sigma}\right) - \partial_\sigma\left(\delta\Gamma_{\nu}{}^{\mu}{}_{\rho}\right)\right]\big|_p$$

$$= \left[\nabla_\rho\delta\Gamma_{\nu}{}^{\mu}{}_{\sigma} - \nabla_\sigma\delta\Gamma_{\nu}{}^{\mu}{}_{\rho}\right]\big|_p$$

same as before, promote to AIN

$$\therefore\ \delta R^a{}_{bcd} = \nabla_c\delta\Gamma^a_{bd} - \nabla_d\delta\Gamma^a_{bc}$$

$$\Rightarrow \delta R_{ab} = \nabla_c\delta\Gamma^c_{ab} - \nabla_b\delta\Gamma^c_{ac}$$

observing that $\delta(g^{ab}g_{bc}) = 0 \Rightarrow (\delta g^{ab}) = -g^{ac}g^{bd}\delta g_{cd}$. We finally have

$$\delta R_g = \delta(g^{ab}R_{ab}) = (\delta g^{ab})R_{ab} + g^{ab}\delta R_{ab}$$

$$= -R^{ab}\delta g_{ab} + g^{ab}\left(\nabla_c\delta\Gamma^c_{ab} - \nabla_b\delta\Gamma^c_{ac}\right)$$

$$= -R^{ab}\delta g_{ab} + \nabla_c X^c$$

where $X^c = g^{ab}\delta\Gamma^c_{ab} - g^{cb}\delta\Gamma^a_{ba}$.

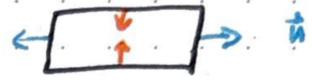
Lecture 24

4.12.24

Last lecture

* Stokes / divergence theorem:

$$\int_M \nabla_a X^a \, dvol_g = \int_{\partial M} X^a n_a \, dvol_h$$



proposed action for grav field where we integrate scalar curvature of the metric over the manifold

* Einstein-Hilbert action

$$S_{EH}[g] = \frac{1}{16\pi}\int_M R_g \, dvol_g$$

issue if manifold infinite then may not get finite action SEH (but generally we ignore this) and anyway we only really care about the variation in the action which makes sense even if it is infinite

action principle: action should be stationary under small perturbations of field

* Under a variation $g \to g + \delta g$ **(δg small, vanish outside coord chart)**

$$\delta(dvol_g) = \tfrac{1}{2} g^{ab}\delta g_{ab} \, dvol_g \; ;$$

ricci tensor contracted into variation in g — some divergence of term involving variation of Γ symbols

$$\delta(R_g) = -R^{ab}\delta g_{ab} + \nabla_a X^a$$

$$X^a = g^{cd}\delta\Gamma^a_{cd} - g^{ac}\delta\Gamma^d_{cd}.$$

We deduce from the formulae for δR_g, $\delta dvol_g$ that

$$\delta S_{EH} = \frac{1}{16\pi}\int_M \{ (\tfrac{1}{2}g^{ab}R - R^{ab})\delta g_{ab} + \nabla_c X^c \} \, dvol_g .$$

[usual story when we are deriving E-L equation: (something nice) × variation + piece we integrate away by parts]

$$= \frac{1}{16\pi}\int_M -G^{ab}\delta g_{ab} \, dvol_g$$

where we've used the fact that δg and hence X vanish on ∂M to drop the last term using the divergence theorem.
We immediately see that $\delta S_{EH} = 0$ for all variations δg_{ab} if and only if g_{ab} solves the vaccum Einstein equations.

Suppose we also have a contribution from matter fields

$$S_{tot.} = S_{EH} + S_{matter}, \qquad S_{matter} = \int_M L[\Phi, g] \, dvol_g .$$

Under a variation $g \to g + \delta g$ we must have

after doing integration by parts we're gonna have something contracted into δg

$$\delta S_{matter} = \tfrac{1}{2}\int_M T^{ab}\delta g_{ab} \, dvol_g$$

(T is just whatever we need to put in to make this statement true)

For some symmetric 2-tensor T^{ab}.

[note: this is not the only poss definition for the energy-M tensor but this defn is nicely consistent w/ relativity]

Varying g in S_{tot} gives $\boxed{G^{ab} = 8\pi T^{ab}}$ i.e. **Einstein's equations.**

E.g. If ψ is a scalar and $L_{matter} = -\tfrac{1}{2}g^{ab}\nabla_a\psi\nabla_b\psi$, then under $g \to g + \delta g$

variation of matter part of action

$$\delta S_{matter} = -\int_M [\underbrace{\tfrac{1}{2}\delta(g^{ab})\nabla_a\psi\nabla_b\psi \, dvol_g}_{\text{piece from varying } g \text{ (in } L_{matter})} + \underbrace{\tfrac{1}{2}g^{ab}\nabla_a\psi\nabla_b\psi\,\delta(dvol_g)}_{\text{piece from varying volume form (which we integrate L over)}}]$$

using stuff from last lecture

$$= \tfrac{1}{2}\int_M (g^{ac}g^{bd}\nabla_c\psi\nabla_d\psi\,\delta g_{ab} - \tfrac{1}{2}(g^{cd}\nabla_c\psi\nabla_d\psi)g^{ab}\delta g_{ab}) \, dvol_g$$

$$= \tfrac{1}{2}\int_M T^{ab}\delta g_{ab} \, dvol_g$$

where $T^{ab} = \nabla^a\psi\nabla^b\psi - \tfrac{1}{2}g^{ab}\nabla_c\psi\nabla^c\psi.$

Exercise: Show that varying $\psi \to \psi + \delta\psi$ gives the wave equation
$$\nabla_c \nabla^c \psi = 0.$$

It can be shown that diffeomorphism invariance of the matter action implies $\nabla T^{ab} = 0$. (T^{ab} is divergence free)

[CONCLUDE: So we have nice way of formulating einstein's eqns of GR in terms of an action principle which is nice because this is how we think about most modern physical theories (i.e. starting from an action principle)]

"c-owe-chy"

THE CAUCHY PROBLEM FOR EINSTEIN'S EQUATIONS

Saw in wave gauge earlier: EEs are wave eqns: given function & 1st deriv of function at initial time we can solve wave eqns to get future
↳ but a bit unsatisfactory because we had fixed particular gauge & we had fixed notion of time. So now our goal is to formulate these ideas in a more geometric way.

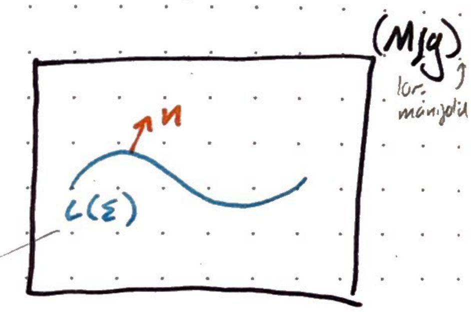
We expect Einstein's equations can be solved given data on a spacelike hypersurface Σ. What is the right data?

generalise wave eqn approach: on a manifold → don't have an obvious notion of what $t=0$ means but we do have a notion of a hypersurface

Suppose $\iota: \Sigma \to M$ is an embedding, such that $\iota(\Sigma)$ is spacelike. Then $h = \iota^*(g)$ is Riemannian.

Let n be a choice of unit normal to $\iota(\Sigma)$. We define for X, Y vector fields on Σ

metric in big spacetime induces a Riemannian metric on the surface



$$k(X,Y) = \iota^*(g(n, \nabla_{\hat{X}} \hat{Y}))$$

(strictly speaking need to extend these off the surface but that gives same ans so ignore this subtlety)

where $\iota_* X = \hat{X}$, $\iota_* Y = \hat{Y}$ on $\iota(\Sigma)$.

We pick local coordinates $\{y^i\}$ on Σ and $\{x^\mu\}$ on M such that $\iota: (y^1, y^2, y^3) \longmapsto (0, y^1, y^2, y^3)$, then $n_a \propto dx^0$ say $n_\mu = \alpha \delta^0_\mu$.

If X, Y vector fields on Σ, say $X = X^i \frac{\partial}{\partial y^i}$, $Y = Y^i \frac{\partial}{\partial y^i}$, take
$$\hat{X} = X^i \frac{\partial}{\partial x^i}, \quad \hat{Y} = Y^i \frac{\partial}{\partial x^i}.$$

Then
$$k(X,Y) = g_{\mu\nu} n^\mu \hat{X}^\sigma \nabla_\sigma \hat{Y}^\nu$$
$$= \alpha \delta^0_\nu \hat{X}^\sigma (\partial_\sigma \hat{Y}^\nu + \Gamma^\nu_{\sigma\tau} \hat{Y}^\tau) = \alpha \Gamma^0_{ij} X^i Y^j$$

(1st term vanishes since $Y^0 = 0$?)

$\therefore$ k is a symmetric 2-tensor (called 2nd fundamental form) on Σ. We can show (example sheet 4) that if (M,g) solves the vaccum Einstein equations, then Einstein constraint equations hold:

what are these? → eqns relating k & h, they hold in manifold Σ, not on larger manifold.

$$\left.\begin{aligned} {}^{(h)}\nabla_i k^i{}_j - {}^{(h)}\nabla_j k^i{}_i &= 0 \\ R_h - k^i{}_j k^j{}_i + k^i{}_i k^j{}_j &= 0 \end{aligned}\right\} \quad (\dagger).$$

(levi civita wrt. h) all raising & lowering done with h

(what do these say: → if i want to characterise initial conditions for EEs in terms of h & k we can think of k as some kind of normal deriv. to g

k, h kind of initial conditions for EEs, but they are not completely free as they have to satisfy these constraints?)

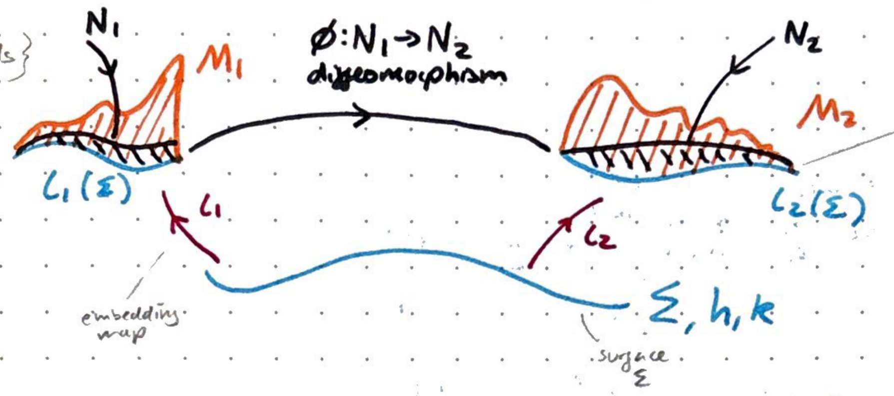
Conversely, if we are given

- Σ — A 3-manifold
- h — A Riemannian metric on Σ
- k — A symmetric 2-tensor on Σ

such that (†) hold. Then there exists a solution (M, g) of the vaccum Einstein equations, and an embedding $\iota : \Sigma \to M$ such that h is the induced metric on Σ, and k is the 2nd fundamental form of $\iota(\Sigma)$. ← the embedded manifold

idea now: forget the 4D manifold & think about 3D manifold Σ on which we are given data which solves these constraints
claim is: from this data I can construct a soln in which the data embeds as an initial hypersurface. Moreover (nontrivial), it is unique up to extension → what does this mean?

multiple solutions: extend to 4-manifolds M_1 or M_2



extension to future that equally could do extension to past

[claim] map that takes some neighborhood of M_2 (N_2) to some neighborhood of M_1 (N_1) is a diffeomorphism?

(This result is due to Choquet-Bruhat (existance) and Choquet-Bruhat + Geroch (geometric uniqueness).)

gives us way of looking at EES as evolution problem but w/o having to begin by fixing a gauge (finding ICs that satisfy the constraint equations is difficult.)

well done :)