QUANTUM FIELD THEORY

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Office hours Fri (info on moodle)
every Weds post summary of lectures over the week (on moodle)

Notes: David Tong's notes mainly & book Matt Schwarz QFT, sometimes additional notes posted

QFT encapsulates a lot about how we think about physics in the modern era
↳ it is a 'mindset' over a theory: bringing together concepts in theoretical physics

GOAL: To combine quantum mechanics with special relativity (L1)

OUTPUT: number of particles is not conserved

$\boxed{\text{HIGHLIGHT:}}$ Robust & systematic => governed by few principles

→ LOCALITY
→ SYMMETRIES
→ RENORMALIZATION

} → almost unique theory

(not in this course but in AQFT)

david tong: QFT is "literally the language in which the laws of nature are written"

Units & conventions

L, T, M
length, time, mass

speed of light c $[c] = [LT^{-1}]$
Planck's constant $\hbar$ $[\hbar] = [L^2 M T^{-1}]$
Newton's constant G $[G] = [L^3 M^{-1} T^{-2}]$

(G doesn't appear in this course)

Natural units: $c = 1 = \hbar$ will be used $\Rightarrow L = T = M^{-1}$

$[G] = [M^{-2}] = -2$

just keep power of M, M is basically a unit of energy

M = energy $(E = Mc^2)$

'mass dimension' 'energy dimension' equivalently
(note: say dimension to mean mass dimension i.e. G has dimension −2)

Relativistic notation

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

note: opposite convention to GR!!

$x^\mu = (t, x, y, z)$

$F(t, \vec{x}) = F(x^\mu) \equiv F(x)$

CLASSICAL FIELD THEORY

Classical mechanics a natural object is the action

$$S(t_1, t_2) = \int_{t_1}^{t_2} dt \left(m \underbrace{\sum_{i=1}^{3} \left(\frac{dx^i}{dt}\right)^2}_{\text{kinetic term}} - \underbrace{V(x^i)}_{\text{potential}} \right)$$

unlike field theory fundamental variable is position of the particle

Basic facts:

1) equation of motion are given by extremising S
2) boundary conditions are supplied externally
3) S is built on symmetries

Field theory

$\Longrightarrow$

Declare in F.T. that the fundamental object is a field

(unlike for classical mech where it is position)

we do QFT in 4 dimensions

$$\phi_a(t, \vec{x}) : \mathbb{R}^{3,1} \to \mathbb{R} \text{ or } \mathbb{C} \text{ or } \mathbb{R}^n$$

field depending on time & space

$\hookrightarrow$ a indicates the type of field which dictates which space we land in

(notes: notice position is fundamental variable in classical mechanics but relegated to a mere label in field theory)

First consequence is that we are dealing with an infinite no. of degrees of freedom (dof) $\to$ (c.f. classical mech only 3 dof)

Example: Electricity & magnetism $\to$ (example of a field theory)

$$A^\mu(t, \vec{x}) = (\phi(x), \vec{A}(x)) \quad : \text{gauge field}$$

"a"

(note: using shorthand)

From here, construct E & M fields $\longrightarrow$

electric field

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

magnetic field

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$\xRightarrow{\text{equations of motion}}$

$$\vec{\nabla} \cdot \vec{E} = \rho$$

$$\vec{\nabla} \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}$$

(immediately we see two of Maxwell's equations hold as identities)

Two identities

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\frac{d\vec{B}}{dt} = -\vec{\nabla} \times \vec{E}$$

note to self (google):
symmetry transformation → transformation that leaves the system unchanged in some important way

LAGRANGIANS

$$L = T - V$$ (lagrangian) definition

notes: lagrangian is fn. of $\phi(\underline{x},t)$, $\dot{\phi}(\underline{x},t)$ and $\nabla\phi(\underline{x},t)$

$$S = \int dt\, L$$ (action)

$L(t)$

$$L = \int d^3x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a)$$

↳ functional: LAGRANGIAN DENSITY

jargon: refer to lagrangian density as 'lagrangian' (the actual lagrangian won't pop up anymore.)

$$S = \int dt\, L = \int d^4x\, \mathcal{L}$$

want to extremize this wrt fields?

Equations of motion (eom):

* Eom are determined by extremising w.r.t. fields

notes: also restrict to ϕ & $\dot{\phi}$, not $\ddot{\phi}$

Assumption: $\mathcal{L}(\phi_a, \partial_\mu \phi_a)$ and not $\mathcal{L}(\phi_a, \partial\phi_a, \partial^2\phi_a, \dots)$

we only consider $\mathcal{L}$ depending on $\nabla\phi$ & not $\nabla^2\phi$ etc. with an eye to later Lr. invariance

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta(\partial_\mu \phi_a)\right]$$

about to integrate by parts

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right)\delta\phi_a + \underbrace{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a\right)}_{\text{total derivative} \rightarrow 0}\right]$$

Assumption: fields decay at infinity → $\delta\phi_a(\underline{x}, t_1) = \delta\phi_a(\underline{x}, t_2) = 0$

Requiring $\delta S = 0$ yields eom.

$$\boxed{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0}$$ EULER-LAGRANGE

Example: Free massive scalar field

$$\mathcal{L} = \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \tfrac{1}{2}m^2\phi^2$$

(simplest lagrangian we can write, version of harmonic oscillator for QFT)

$$= \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\vec{\nabla}\phi)^2 - \tfrac{1}{2}m^2\phi^2$$

notes: this term is 'gradient energy'

notes: usually the phrase 'potential' or 'potential energy' is reserved for just this term

Traditional classical mechanics picture $\begin{cases} \tfrac{1}{2}\dot{\phi}^2 & \text{: kinetic energy} \\ \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2 & \text{: potential energy} \end{cases}$

In QFT jargon
kinetic terms: any bilinear of fields

$\begin{cases} \eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi & \text{: kinetic} \\ m^2\phi^2 & \text{: mass term} \end{cases}$

EoM $$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0$$

$$\Rightarrow \boxed{\partial_\mu\partial^\mu\phi + m^2\phi = 0}$$

(RELATIVISTIC) KLEIN-GORDON EQ.

$$\Box\phi + m^2\phi = 0$$

laplacian in minkowski space denoted $\Box$

Lecture 2 14.10.24

Hamiltonian

In this setup one starts by defining the canonical momentum

$$\Pi^a(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_t \Phi_a)} = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_a}$$

the hamiltonian density is

canonical momentum

(memorize? important later to find hamiltonian)

$$\mathcal{H} = \Pi^a \partial_t \Phi_a - \mathcal{L} \qquad (\text{"}p\dot{x} - L\text{"})$$

the hamiltonian is

$$H = \int d^3x\, \mathcal{H}$$

(note always refer to hamiltonian vs hamiltonian correctly unlike L)

Example: scalar field with a potential $V(\phi)$

$$\mathcal{L} = \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)$$

canonical momentum

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$$

hamiltonian

$$H = \int d^3x\,(\Pi\partial_t\phi - \mathcal{L}) = \int d^3x\left(\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi)\right)$$

notes: elim. $\dot{\phi}$ in favour of Π^a as in classical mech

$\hookrightarrow H = \int d^3x\left[\tfrac{1}{2}\Pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi)\right]$

EOM arise from $\dot{\phi} = \frac{\partial H}{\partial \Pi}$, $\dot{\Pi} = -\frac{\partial H}{\partial \phi}$

(note: here $\mathcal{H}$ agrees with def. of total energy E (p155) we get from applying noether's theorem for time translation invariance)

$$\boxed{\text{SYMMETRIES}}$$

→ dictate the actions we write
→ dictate the class of fields (operators) used
→ control the observables we will compute

Ex. time translations, rotation, U(1), time reversal, spatial translation, boosts, gauge sym, parity, conformal symmetry, SUSY, SU(N) ...

Lorentz invariance

Lorentz group is defined $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

which preserve the interval $s^2 = x^\mu x^\nu \eta_{\mu\nu} = t^2 - \vec{x}^2$

$s^2 \to s'^2 = s^2$

This condition implies $\boxed{\eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}}$ ① $(\Lambda^T \eta \Lambda = \eta)$

Examples:

1) Rotation $t' = t$ $\quad \Lambda^i{}_j = R^i{}_j$ (orthogonal matrix)

(about z axis)

rot x-y plane

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2) Boost : mix time & space

boost (t, x) plane

by $v(1)$

$$\equiv \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad w/ \ \gamma = \frac{1}{\sqrt{1-v^2}}$$

$$\Lambda = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

η rapidity

$$\cosh\eta = \frac{1}{\sqrt{1-v^2}}$$

$$\sinh\eta = \frac{v}{\sqrt{1-v^2}}$$

more generally, from (1) : take determinant

$$\boxed{\det(\Lambda)^2 = 1} \rightarrow \det\Lambda = \pm 1$$

*If $\det\Lambda = 1$, proper lorentz transformation — focus on proper ones
$\Rightarrow$ continuously connect to identity

*If $\det\Lambda = -1$, improper lorentz transformation
$\Rightarrow$ parity, time reversal

Focus of $\det\Lambda = 1$, we can then write

Kronecker delta — small parameter

$$\boxed{\Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\nu} + \varepsilon^{\mu}_{\ \nu} + O(\varepsilon^2)} \qquad (2)$$

what are the properties of $\varepsilon^{\mu}_{\ \nu}$?

Insert (2) in (1):

$$\cancel{\eta_{\rho\sigma}} = \eta_{\mu\nu}\left(\delta^{\mu}_{\rho} + \varepsilon^{\mu}_{\ \rho} + \cdots\right)\left(\delta^{\nu}_{\sigma} + \varepsilon^{\nu}_{\ \sigma} + \cdots\right)$$

(expand, keep only linear terms in ε)

$$= \eta_{\mu\nu}\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} + \eta_{\mu\nu}\varepsilon^{\mu}_{\ \rho}\delta^{\nu}_{\sigma} + \eta_{\mu\nu}\delta^{\mu}_{\rho}\varepsilon^{\nu}_{\ \sigma} + O(\varepsilon^2)$$

$$= \cancel{\eta_{\rho\sigma}} + \varepsilon_{\sigma\rho} + \varepsilon_{\rho\sigma} + \cdots$$

$\Rightarrow \varepsilon_{\sigma\rho} = -\varepsilon_{\rho\sigma}$ $\Rightarrow$ in (3+1) dim we have (# components ε) $= \frac{4\cdot 3}{2} = 6$

anti-symmetric tensor

(total # of how many unique ways to select 2 $= \binom{4}{2} = \frac{4!}{2!2!}$)

Therefore we have 6 generators for the lorentz group
→ 3 rotations
→ 3 boosts

(since elements of diag of antisymmetric matrix are zero $A_{ii} = -A_{ii} \Rightarrow A_{ii} = 0$)

(note: active transform in which the field is truly shifted $\phi'(x) = \phi(\Lambda^{-1}x)$ vs passive transform where we relabel our choice of coordinates we would instead have $\phi'(x) = \phi(\Lambda x)$)

Fields revisited

have symmetry → what are objects that transform under that symmetry

A field is an object that depends on coordinates and has a definite transformation under Lorentz.

$$x \to x' = \Lambda x \,, \qquad \phi_a(x) \to \phi'_a(x) = \underbrace{D[\Lambda]_a^{\ b}}_{\text{form a representation of the Lorentz group}} \phi_b(\Lambda^{-1}x)$$

identities

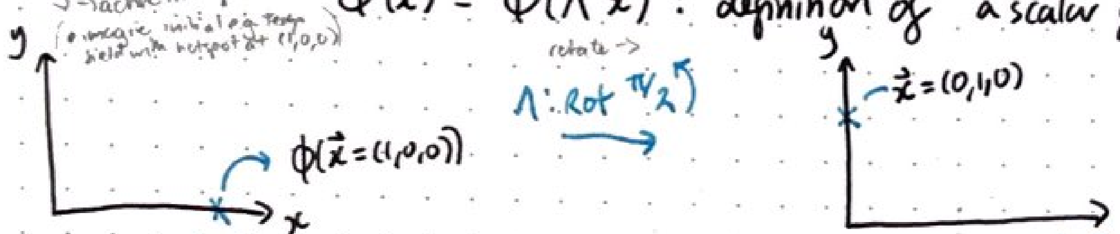
$$D[\Lambda_1]D[\Lambda_2] = D[\Lambda_1\Lambda_2] \qquad D[\Lambda^{-1}] = D[\Lambda]^{-1} \qquad D[\mathbb{I}] = 1$$

(what/why? active transformation) Consider trivial rep $D[\Lambda] = 1$

why inverse Λ? - active transformation (imagine initial a.g. temp field with hotspot at (1,0,0))

$$\phi'(x) = \phi(\Lambda^{-1}x) : \text{definition of a scalar field}$$

(transforms as $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$) (?)



• hotspot now at (0,1,0) $x \to x' = \Lambda x$

$$\phi'(x) = \phi(\Lambda^{-1}x)$$

← • to write ϕ' in terms of the old field ϕ, we need to place ourselves at (0,1,0) and ask what the old field looked like where we've come from at (1,0,0) ⇒ $\phi'(x) = \phi(\Lambda^{-1}x)$

(notes: def. of lorentz inv. theory is that if $\phi(x)$ solves EOM, then $\phi(\Lambda^{-1}x)$ also solves EOM)

Ex 1.: trivial representation → scalar field

↳ $D[\Lambda] = 1$ (?)

Ex 2.: vector representation

$$D[\Lambda]^\mu_{\ \nu} = \Lambda^\mu_{\ \nu}$$

(cf.: passive transform we would relabel coordinates to instead have $\phi'(x) = \phi(\Lambda x)$)

note inv. lowered vs raised indices ?

$$A^\mu(x) \to A'^\mu(x) = \Lambda^\mu_{\ \nu} A^\nu(\Lambda^{-1}x)$$

$$\partial_\mu \phi \to \partial_\mu \phi'(x) = (\Lambda^{-1})^\nu_{\ \mu}\, \partial_\nu \phi(\Lambda^{-1}x)$$

derivative of the scalar field transforms as a vector

Actions revisited: how it constrains actions

$$\text{Ex.} \quad \mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial_\nu\phi\,\eta^{\mu\nu} - \tfrac{1}{2}m^2\phi^2$$

— reminder: lagrangian of free massive scalar field. (?)

$$S = \int d^4x\, \mathcal{L} \quad , \text{ verify that it is invariant under Lorentz}$$

derivative scalar field transforms as a vector: $\partial_\mu\phi(x) \to \partial_\mu\phi'(x) = \Lambda^{-1\rho}_{\ \ \mu}\partial_\rho\phi(y)$, $(y = \Lambda^{-1}x)$

so, lagrangian of free massive scalar field $\mathcal{L}(x) = \tfrac{1}{2}\partial_\mu\phi(x)\partial_\nu\phi(x)\eta^{\mu\nu} - \tfrac{1}{2}m^2\phi^2(x)$

transforms as: $\mathcal{L}(x) \to \mathcal{L}'(x) = \tfrac{1}{2}\Lambda^{-1\rho}_{\ \ \mu}\partial_\rho\phi(y)\Lambda^{-1\sigma}_{\ \ \nu}\partial_\sigma\phi(y)\eta^{\mu\nu} - \tfrac{1}{2}m^2\phi^2(y)$ $\quad (y = \Lambda^{-1}x)$

$$= \tfrac{1}{2}\partial_\rho\phi(y)\partial_\sigma\phi(y)\underbrace{\Lambda^{-1\rho}_{\ \ \mu}\Lambda^{-1\sigma}_{\ \ \nu}\eta^{\mu\nu}}_{} - \tfrac{1}{2}m^2\phi^2(y)$$

since $\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu_{\ \rho}\Lambda^\nu_{\ \sigma}$

$\to \eta^{\rho\sigma} = \eta^{\mu\nu}\Lambda^{-1\rho}_{\ \ \mu}\Lambda^{-1\sigma}_{\ \ \nu}$

(taking inv. of this expression? using $\eta_{\rho\sigma} = (\eta^{\rho\sigma})^{-1}$)

$$= \tfrac{1}{2}\partial_\rho\phi(y)\partial_\sigma\phi(y)\eta^{\rho\sigma} - \tfrac{1}{2}m^2\phi^2(y)$$

$$= \mathcal{L}(y)$$

so, the action $S = \int d^4x\, \mathcal{L}(x) \to S' = \int d^4x\, \mathcal{L}'(x) = \int d^4x\, \mathcal{L}(y) = \int d^4y\, \mathcal{L}(y) = S$

↑ follows since we don't pick up a jacobian factor from $\int d^4x$ to $\int d^4y$ since $\det\Lambda = 1$ ($\hookrightarrow d^4y = d^4x \det\Lambda = d^4x$) ?

∴ action is invariant under lorentz! (for free massive scalar field)

Lecture 3 16.10.24

Classical field theory cont. - Lorentz inv. cont.

Under Lorentz, we know $x \to x' = \Lambda x$

Fields transform as

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x) = \phi(y) \qquad y = \Lambda^{-1}x$$

$$\partial_\mu \phi \to (\Lambda^{-1})_\mu^{\ \nu} \partial_\nu \phi(y)$$

Replace in the Lagrangian

$$\mathcal{L} \to \tfrac{1}{2}\eta^{\mu\nu}(\Lambda^{-1})_\mu^{\ \rho}\partial_\rho\phi(y)(\Lambda^{-1})_\nu^{\ \sigma}\partial_\sigma\phi(y) - \tfrac{1}{2}m^2\phi^2(y)$$

$$= \tfrac{1}{2}\eta^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi - \tfrac{1}{2}m^2\phi^2$$

Therefore: $\mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(y)$

and then $S \to \int d^4x\, \mathcal{L}(y) = \int d^4y\, \mathcal{L}(y)$

by definition $\det(\Lambda) = 1$ $\Rightarrow$ action is Lorentz invariant

notes: can show similarly action is lorentz invariant for maxwell lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ - in general to see if action is lor. invariant, make sure all indices $\mu = 0,1,2,3$ are connected w/ lorentz invariant objects e.g. the metric $\eta_{\mu\nu}$

NOETHER THEOREM

1) Every continuous symmetry of the Lagrangian gives rise to a conserved current j^μ

The e.o.m. imply $\partial_\mu j^\mu = 0$: conservation $\left(\frac{\partial j^0}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0\right)$

equivalent ($\partial_0 j_0 - \partial_i j_i = 0$)

2) Provided suitable boundary conditions, a conserved current will give rise to a conserved charge Q, where

$$Q = \int d^3x\, j^0$$

notes: can see this immediately: $\frac{dQ}{dt} = \int_{\mathbb{R}^3} d^3x \frac{\partial j^0}{\partial t} = -\int_V d^3x\, \nabla\cdot\vec{J} = -\int_{\delta V} d\vec{S}\cdot\vec{j}$

states divergence theorem relation -> integral over enclosing surface

V is all of space ($\mathbb{R}^3$) so if have BCs that $\vec{j} \to 0$ fast enough as $|\vec{x}| \to \infty$ then $\frac{dQ}{dt} = 0 \Rightarrow$ conserved charge

this local conservation of charge holds in any special FT.

notes: however, existence of a current is a much stronger condition than existence of a conserved charge b/c it implies charge is conserved locally, take charge in finite volume V:

$Q_V = \int_V d^3x\, j^0 \to \frac{dQ_V}{dt} = -\int_V d^3x\, \vec{\nabla}\cdot\vec{j} = -\int_A \vec{j}\cdot d\vec{S}$

(local conservation of charge: any charge leaving V must be accounted for by flow of current 3-vector $\vec{j}$ out of V)

PROOF:

1) Definition: A transformation is continuous if there is an inf. parameter in it

- Internal transformations: do not act on coords, act on fields
- Local transformations: act on coords + fields

In both cases, the defn. of continuous transf.

$$\delta\phi_a \equiv \phi_a'(x) - \phi_a(x)$$

notes: transformation is a symmetry if $\delta\mathcal{L} = \partial_\mu F^\mu$

When is such a transf. a symmetry of the system?

$\longrightarrow$ Action is invariant under transf.

$$S[\Phi] \rightarrow S[\Phi'] = \int d^4x\, \mathcal{L}(x)$$

$$\delta S = S[\Phi'] - S[\Phi] = 0$$

↳ symmetry

This implies for the Lagrangian

$$\delta\mathcal{L} = \mathcal{L}'(x) - \mathcal{L}(x) = \underbrace{\partial_\mu F^\mu}_{\text{total derivative}}$$

Let's quantify the change in $\mathcal{L}$:

using def. of symmetry for symmetry transform $\delta\Phi_a = X_a(\Phi)$, $\delta\mathcal{L} = \partial_\mu F^\mu$ ↓ symm.

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Phi_a}\delta\Phi_a + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_a}\delta\partial_\mu\Phi_a$$

$$= \left(\frac{\partial\mathcal{L}}{\partial\Phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_a}\right)\right)\delta\Phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_a}\delta\Phi_a\right) = \partial_\mu F^\mu$$

$$\Rightarrow -\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\Phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_a}\right)\right)}_{\text{e.o.m.}}\delta\Phi_a = \partial_\mu\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_a}\delta\Phi_a - F^\mu\right)}_{j^\mu}$$

If e.o.m are imposed $\Rightarrow \partial_\mu j^\mu = 0$ //

where $$\boxed{j^\mu \equiv \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_a}\delta\Phi_a - F^\mu}$$ q.e.d.

(conserved current gives rise to conserved charge) ↓

2) We have $Q = \int d^3x\, j^0$

what does it mean to be conserved? its time derivative is zero ~ conservation of glory charge across surface

$$\frac{dQ}{dt} = \int_{V=\mathbb{R}^3} d^3x\, \frac{\partial j^0}{\partial t} \underset{\text{conserved } j^\mu}{=} -\int_{V=\mathbb{R}^3} d^3x\, \vec{\nabla}\cdot\vec{J} = -\int_{\partial V \to |\vec{x}|\to\infty} d\vec{A}\cdot\vec{j} = 0$$

↳ provided fields decay as $|\vec{x}| \to \infty$

(examples)

ENERGY-MOMENTUM TENSOR

– undeleted: example of local symmetry: transf. acts on coords & fields?

Local transformation: translations

$$x^\mu \rightarrow x'^\mu = x^\mu - \varepsilon^\mu$$

ε^μ = constant vector → continuous parameter

Under translations, the fields transform as

→ + not-sense active transformation

taylor expand, treat as infinitesimal ↓

$$\Phi_a(x) \rightarrow \Phi_a'(x) = \Phi_a(x+\varepsilon)$$

↓ inf.

$$= \Phi_a(x) + \varepsilon^\mu\partial_\mu\Phi_a + O(\varepsilon^2)$$

$$\boxed{\begin{aligned}\delta\Phi_a &= \Phi_a'(x) - \Phi_a(x)\\ &= \varepsilon^\mu\partial_\mu\Phi_a\end{aligned}}$$

Similarly lagrangian transforms as
$\mathcal{L}(x) \Rightarrow \mathcal{L}'(x) = \mathcal{L}(x+\varepsilon) = \mathcal{L}(x) + \varepsilon^\mu \partial_\mu \mathcal{L}(x)$
$\Rightarrow \delta\mathcal{L} = \varepsilon^\mu \partial_\mu \mathcal{L}$
(recall Noether: symmetry transform. $\delta\mathcal{L} = \partial_\mu F^\mu \Rightarrow$ conserved current j^μ)
(gives four conserved currents, one for each of $\nu = 0,1,2,3$)

The Lagrangian

$$\delta\mathcal{L} = \varepsilon^\mu \partial_\mu \mathcal{L} = \partial_\mu(\underbrace{\varepsilon^\mu \mathcal{L}}_{F^\mu})$$

Conserved current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \underbrace{\varepsilon^\nu \partial_\nu \phi_a}_{\delta\phi_a} - \underbrace{\varepsilon^\mu \mathcal{L}}_{F^\mu}$$

$(j^\mu)_\nu$

$$= \varepsilon^\nu \underbrace{\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \partial_\nu\phi_a - \delta^\mu_\nu \mathcal{L} \right)}$$

$$\equiv \varepsilon^\nu T^\mu_\nu$$

T^μ_ν : energy-momentum tensor

where, if e.o.m. $=0$, $\partial_\mu j^\mu = 0 \Rightarrow \partial_\mu T^\mu_\nu = 0$

We can construct 4 conserved charges

total momentum of the field configuration

Momentum: $p^i = \int d^3x \, T^{0i}$ $\qquad \varepsilon = (0,1,0,0), (0,0,1,0), (0,0,0,1)$

total energy of the field configuration

Energy: $E = \int d^3x \, T^{00}$ $\qquad \varepsilon^\mu = (1,0,0,0)$

- $\partial_\mu T^\mu_0 = 0$ ↳ $\int d^3x T^{00} = E$ conserved
- $\partial_\mu T^\mu_i = 0$ ↳ $\int d^3x T^{0i} = p^i$ conserved

(note to self: conclude invariance under local translations gives me energy/momentum conservation?)

Ex. Free massive scalar field

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \partial_\nu\phi - \partial^\mu_\nu \mathcal{L}$$ ← just plug in $\mathcal{L}$ for scalar field

$$T^{\mu\nu} = \partial^\mu\phi \, \partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$$ ← raise w η

$$T^{00} = \tfrac{1}{2}\dot\phi^2 + \tfrac{1}{2}(\vec\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2$$

$T^{0i} = \dot\phi \, \partial^i\phi$ (since, remember $\eta^{0i} = 0$)

$$E = \int d^3x \, T^{00} = H \quad , \quad p^i = \int d^3x \, T^{0i} = \int d^3x \, \dot\phi \, \partial^i \phi$$

comment: $T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \partial_\nu\phi_a - \delta^\mu_\nu \mathcal{L}$

from here we will not always have a symmetric tensor. we like $T_{\mu\nu} = T_{\nu\mu}$ because of GR

how to make it symmetric? 1) $\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho T^{\rho\mu\nu}$
where $T^{\rho\mu\nu} = -T^{\mu\rho\nu}$ s.t. $\partial_\mu \Theta^{\mu\nu} = 0$

↓ $\partial_\mu(T^{\mu\nu} + \partial_\rho T^{\rho\mu\nu}) = 0$
so require: $\partial_\mu\partial_\rho T^{\rho\mu\nu} = 0$
which is true if $T^{\rho\mu\nu} = -T^{\mu\rho\nu}$
(antisymmetric in 1st two indices)

(same way as in GR)

2) Couple fields to $g_{\mu\nu}$ $(\eta^{\mu\nu} \to g^{\mu\nu})$

$$\Theta^{\mu\nu} = \left(-\frac{2}{\sqrt{-g}} \frac{\partial}{\partial g_{\mu\nu}} (\sqrt{-g}\,\mathcal{L}) \right) \Big|_{g=\eta}$$

definition for EM tensor
$T^\mu_\nu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \partial_\nu\phi_a - \delta^\mu_\nu \mathcal{L}$
comes from Noether's theorem
however this is generally not symmetric
we can make it symmetric by adding an extra term that satisfies $T^{\rho\mu\nu} = -T^{\mu\rho\nu}$
then $\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho T^{\rho\mu\nu}$ is symmetric
why would you want to do this?
is potentially to make contact w/ GR
another trick to make a symmetric EM tensor is to construct it with GR in mind

(*) idk how you arrive at this

Summary
NOETHER'S THEOREM (notes)
every continuous symmetry of the lagrangian gives rise to a conserved current $j^\mu(x)$ s.t. the eq. of motion imply
$\partial_\mu j^\mu = 0$

* continuous transf. is a symmetry of the system if the action is invariant under that transform / equivalently if $\delta\mathcal{L} = \partial_\mu F^\mu$

see notes: (example 1.5)
lorentz transform is (another local symmetry?)
- consider infinitesimal form of lor transforms
$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$
- plug into def. of lor transf.
$\Rightarrow \omega^{\mu\nu} = -\omega^{\nu\mu}$ antisymmetric
- find $\delta\phi$, $\delta\mathcal{L}$
$\Rightarrow$ conserved current $j^\mu = -\omega^\rho{}_\nu T^\mu{}_\rho x^\nu$ (6 unique choice of $\omega^\rho_\nu \Rightarrow$ 6 conserved charges) $\left\{\begin{matrix} 0 i \times 3 \\ ij \times 3 \end{matrix}\right.$
- $\rho, \sigma = 1,2,3$ lor transform is a rotation & ⇒ conservation of angular mom.
- $0i$ lor transf is a boost & get field theoretic version of newton's I (of field)

note to self? to find Q write j^μ st all indices upper

recap NOETHER
1) for each cont. symm $\exists\, j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a = F^\mu$ s.t. $\partial_\mu j^\mu|_{eom} = 0$
2) $\exists\, Q = \int d^3x\, j^0$, $\dot{Q} = 0$

Lecture 4

18.10.24

Noether theorem (cont. 2nd example)

Internal Symmetry → does not act on coordinates

(x, only acts on fields ϕ)

Ex. complex scalar field $\psi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$ — ϕ_i real scalar fields

where $|\psi|^2 = \psi^*\psi$

a lagrangian for this field $\mathcal{L} = \partial_\mu\psi\,\partial^\mu\psi^* - V(|\psi|^2)$

notes: could expand ψ in terms of ϕ_1, ϕ_2 & work as before but easier & equivalent to work w/ ψ, ψ^* as indep variables instead → leads to these EOM if we eval E-L with ψ, ψ^*

E.o.m. $\partial_\mu\partial^\mu\psi + \frac{\partial V(\psi^*\psi)}{\partial\psi^*} = 0$, $\partial_\mu\partial^\mu\psi^* + \frac{\partial V(\psi^*\psi)}{\partial\psi} = 0$

(note to self: to find this eom recall $\partial_\mu\psi\,\partial^\mu\psi = \eta^{\mu\nu}\partial_\mu\psi^*\eta_{\mu\nu}\partial^\nu\psi = \partial^\nu\psi\,\partial_\nu\psi$ since $\eta^{\mu\nu} = \eta_{\mu\nu}$)

The internal symm. of this system

$$\psi(x) \to \psi'(x) = e^{i\alpha}\psi(x)$$
$$\psi^* \to \psi'^* = e^{-i\alpha}\psi^* \quad (*)$$
$$\mathcal{L} \to \mathcal{L}' = \mathcal{L}$$
$$S \to S' = S$$

(choose example st it has symmetry eg under lorentz?)

under this set of transf. Lagrang and S invariant

(easy to see that this transf. is symmetry by plugging into $\mathcal{L}$ (does not change))

α is the continuous param. of the transformation

$$\delta\psi = \psi'(x) - \psi(x) = i\alpha\psi$$
$$\delta\psi^* = -i\alpha\psi^*$$

(if α is infinitesimal $\psi \to \psi' = e^{i\alpha}\psi \Rightarrow \delta\psi = i\alpha\psi$ (see from taylor exp. of $e^{i\alpha}$))

Construct current [remind: need to vary for ψ and ψ^*]

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi^*}\delta\psi^* \qquad (F^\mu = 0)$$

(internal symmetries gives conserved charge?)

$$= \partial^\mu\psi^*\,\delta\psi + \partial^\mu\psi\,\delta\psi^*$$

going to learn this is charge; is precursor of how we quantify charge

$$= i\alpha(\psi\partial^\mu\psi^* - \psi^*\partial^\mu\psi) \to Q = \int d^3x\, j^0$$

arbitrary param & usually set = 1

later learn conserved charges arising from currents of this type have interpretation of electric charge or particle no.

also possible to just new transf. — symmetry trans. that rotates around scalar fields

same real scalar fields ϕ_1, ϕ_2

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix} = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \quad (*)$$

$(*) = (*)$ say the same thing, equivalent

(now done with classical field theory!)

FREE THEORY
- lagrangian is quadratic in fields
- linear EOM
- no interactions (no scattering / forces / decays)

QUANTUM FIELDS: FREE THEORY

canonical quantization: move from hamiltonian formalism to quantum theory by promoting fields to "operator valued fn.s" i.e. fns that output an operator eg. classical mech → get quantum theory by promoting coords q_a and conjugate momenta p^a to operators eg $[q_a, p^b] = i\delta_a^b$

Take a Hamiltonian approach, and (reminder that) we will follow rules of QM in QM:

$$[x^i, p^j] = i\hbar\delta^{ij}$$

(follow same prescription & see how far we get) reminder: variables in FT are fields not p, x

In QFT now $\phi_a(x)$, $\pi^a(x)$ $= \frac{\partial\mathcal{L}}{\partial\dot{\phi}_a}$

'quantum field' is an operator valued fn. of (fn. of space only in schrod. picture) (think 2) space obeying the commutation relations

the rule $[\phi_a(\vec{x}, t), \pi^b(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y})\delta_a^b$

not going to prove this, just accept it?

also notes: $[\phi_a(\vec{x}), \phi_b(\vec{y})] = [\pi^a(\vec{x}), \pi^b(\vec{y})] = 0$

(where in schrodinger picture fn of space only?)

(notes) note: state $|\psi\rangle$ in QFT is now a function of every possible configuration of the field ϕ

big goal, will take a couple of weeks

GOAL: implement for free massive real scalar field

PLAN: 1) Canonical Quantization 2) Hamiltonian 3) Fock space 4) Causality 5) Propagators

note to self: i think in all of these discussions we focus on a free massive scalar field. i think when they say scalar field they often are actually referring to free massive scalar field? — i.e. the simplest/easiest example

1) Canonical Quantisation

difference here to written notes

david tong?

note: DT CQ is done in Schrodinger picture ($t=0$), here we will use Heisenberg pict.

return to scalar field aka free massive scalar field ?(?)

Our theory $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

(choose this since a theory for which we can characterise all solutions)

e.o.m. $\partial_\mu\partial^\mu\phi + m^2\phi = 0$ minus?

solutions: $\phi \sim \exp(i\vec{k}\cdot\vec{x} + i\omega t)$
where $-\omega^2 + \vec{k}^2 + m^2 = 0 \rightarrow \omega = \pm\sqrt{\vec{k}^2+m^2}$ — constraint

however k, ω not free i.e. they are constrained via mass term

Adopt the notation $\omega_{(\vec{k})} \equiv \sqrt{\vec{k}^2+m^2}$

most general solution arbitrary superposition

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\left[a(\vec{k})e^{i\vec{k}\cdot\vec{x}-i\omega t} + b(\vec{x})e^{i\vec{k}\cdot\vec{x}+i\omega t}\right]$$

but needs to be real! imposes restrictions on a, b

note: write this as a linear sum, or an infinite sum of creation & annihilation operators indexed by $\vec{k}$

note to self: have to use $a(\vec{k})e^{i\vec{k}\cdot\vec{x}-i\omega t} = a(-\vec{k})e^{-i\vec{k}\cdot\vec{x}-i\omega t}$ etc. ??

1) note: ϕ is real!
→ take $\phi^* = \phi \Rightarrow$ $a^*(-\vec{k}) = b(\vec{k})$, $b^*(-\vec{k}) = a(\vec{k})$

$x = (\vec{x}, t)$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\left[a(\vec{k})e^{i\vec{k}\cdot\vec{x}-i\omega t} + a^*(\vec{k})e^{-i\vec{k}\cdot\vec{x}+i\omega t}\right]$$

most general solution that is compatible with reality of the field

$$= \int \frac{d^3k}{(2\pi)^3}\left[a(\vec{k})e^{-ikx} + a^*(\vec{k})e^{ikx}\right]$$

more relativistic notation

where $kx = k^\mu x_\mu = \omega_{\vec{k}} t - \vec{k}\cdot\vec{x}$, $k^2 = \omega^2 - \vec{k}^2 = m^2$

note to self: $k = (\omega, -\vec{k})$

2) note: normalisation

we will choose to normalise $a(k)$ & $a^*(k)$ s.t.

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left[a(\vec{k})e^{-ikx} + a^*(\vec{k})e^{ikx}\right]$$

(arbitrary choice of normalisation) → no real physics here

~~Next~~ $\pi(x) = \dot{\phi} = \int \frac{d^3k}{(2\pi)^3}\frac{1}{i}\sqrt{\frac{\omega_{\vec{k}}}{2}}\left[a(\vec{k})e^{-ikx} - a^*(\vec{k})e^{ikx}\right]$

Next, we quantize, i.e. we declare (introduce commutation relations)

note: equal time commutation relations

$$(1)\quad \begin{cases} [\phi(\vec{x},t), \phi(\vec{x}',t)] = 0 \\ [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0 \\ [\phi(\vec{x},t), \pi(\vec{x}',t)] = i\delta^3(\vec{x}-\vec{x}') \end{cases}$$

same t: equal time commutation relations in heisenberg picture are same as commutation relations in schrodinger picture (?)

note: claim commutation relations for ϕ & π are equivalent to the below commutation relations for $a(\vec{k})$ and $a(\vec{k})^\dagger$

Claim: These commutation relations promote a to an operator
$a^* \to a^\dagger$ also operator

(2)
$$\text{and} \quad [a(\vec{k}), a(\vec{k}')] = 0, \quad [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0$$
$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

(the $a(k)$ were kind of like Fourier coefficients, by imposing this now they have become operators (beauty in this))

→ note $(2\pi)^3$ comes from notation (choice of normalisation factor $\frac{1}{\sqrt{2\omega}}$ → choose to pick one here

PROOF: can show (1) ⇒ (2) or (2) ⇒ (1)

(note to self to show reverse (1)⇒(2) find exp. of $a^\dagger$ in terms of ϕ (by inversely ϕ = ... then integrating $\int d^3x$ etc.) then plug into $[a, a^\dagger]$ and show similarly as from (2)⇒(1))

show (2) → (1) & pick one (rest is exercise)

$$[\phi(\vec{x},t), \pi(\vec{y},t)] = \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2i} \sqrt{\frac{\omega_q}{\omega_p}} \Big(\big[a(p) e^{i\vec{p}\cdot\vec{x} - i\omega t} + a^\dagger(p) e^{-i\vec{p}\cdot\vec{x} + i\omega t}, \; a(q) e^{i\vec{q}\cdot\vec{y} - i\omega t} - a^\dagger(q) e^{-i\vec{q}\cdot\vec{y} + i\omega t} \big] \Big)$$

↓(2)

time dependence drops out

$$= \int \Big(-[a(p), a^\dagger(q)] e^{i\vec{p}\cdot\vec{x}} e^{-i\vec{q}\cdot\vec{y}} + [a^\dagger(p), a(q)] e^{-i\vec{p}\cdot\vec{x}} e^{i\vec{q}\cdot\vec{y}} \Big)$$

note to self
→ after plug in commutators to get $\delta(\vec{q} - \vec{p})$ terms,
→ split exponent up
e.g. $e^{ipx - iqy} = e^{i\omega_p t - i\vec{p}\cdot\vec{x} - i\omega_q t + i\vec{q}\cdot\vec{y}}$
→ then integrate over d^3q
→ $\omega_p t, \omega_q t$ terms disappear
→ gives something like
$\int \frac{d^3p}{(2\pi)^3} \frac{i}{2} \left[e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \right]$
→ then use
$\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = \delta^3(\vec{x}-\vec{y})$
(and $\delta(x) = \delta(-x)$)
→ $[\phi, \pi] = i\delta^3(\vec{x}-\vec{y})$ //

Summary (start of L5)

$$[\phi(\vec{x},t), \pi(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y})$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} \left(a(\vec{k}) e^{-ikx} + a^\dagger(\vec{k}) e^{ikx} \right)$$

$$\pi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{i} \sqrt{\frac{\omega}{2}} \left(a(\vec{k}) e^{-ikx} - a^\dagger(\vec{k}) e^{ikx} \right)$$

$$[a(\vec{k}), a(\vec{k}')] = 0$$

$$[a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 \delta(\vec{k} - \vec{k}')$$

'only' non-trivial for a with $a^\dagger$

Lecture 5 (Quantum fields: free theory cont.) 21.10.24

Proof:

Only non-trivial commutator a w/ $a^\dagger$ $\rightarrow \omega_q = \omega_p$

$$[\phi(\vec{x},t), \pi(\vec{y},t)] = \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2i}\sqrt{\frac{\omega_q}{\omega_p}}\Big(-\underbrace{[a(p), a^\dagger(q)]}_{(2\pi)^3\delta(\vec{p}-\vec{q})} e^{i\vec{p}\cdot\vec{x}-i\vec{q}\cdot\vec{y}} e^{it(\omega_q-\omega_p)} + \underbrace{[a^\dagger(p), a(q)]}_{-(2\pi)^3\delta(\vec{p}-\vec{q})} e^{-i\vec{p}\cdot\vec{x}} e^{i\vec{q}\cdot\vec{y}} e^{it(\omega_p-\omega_q)}\Big)$$

Outline
1) canonical quantisation ✓
2) hamiltonian
3) fock space
⋮

$$= i\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = i\delta^{(3)}(\vec{x}-\vec{y})$$

2) HAMILTONIAN

$$H = \int d^3x\, \mathcal{H} = \tfrac{1}{2}\int d^3x\,(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2)$$

(note to self: don't plug in $\pi^2 = \pi\pi^\dagger$ just plug in $\pi\pi$ etc.! ↳ b/c they are real so doesn't matter anyway! (& use p, q dummy for each!))

same w/ exact across out switch p/q

be careful not mixing around positions of operators

write H in terms of $a, a^\dagger$

$$H = \tfrac{1}{2}\int d^3x \int \frac{d^3p\,d^3q}{(2\pi)^6}\Big(-\frac{\sqrt{\omega_p\omega_q}}{2}(ae^{-ipx} - a^\dagger e^{ipx})(p\leftrightarrow q)$$
$$-\tfrac{1}{2}\frac{1}{\sqrt{\omega_p\omega_q}}(ae^{-ipx} - a^\dagger e^{ipx})(p\leftrightarrow q)\,\vec{p}\cdot\vec{q} + \frac{m^2}{2}\frac{1}{\sqrt{\omega_q\omega_p}}(ae^{ipx}+a^\dagger e^{-ipx})(p\leftrightarrow q)\Big)$$

now multiply out

(warnings: 1) recall $a, a^\dagger$ are operators! 2) $\int d^3x\, e^{i(\vec{p}\pm\vec{q})\cdot\vec{x}} \propto \delta(\vec{p}\pm\vec{q})$)

(skipping steps: do your own work through) $\rightarrow \omega_p = \omega_q$ $(\vec{p} = \pm\vec{q})$

↳ - have integral over $d^3x\, d^3p\, d^3q$ then use $\int d^3x\, e^{i\vec{x}\cdot(\vec{p}\pm\vec{q})} = (2\pi)^3\delta(\vec{p}\pm\vec{q})$ to integrate over x ↳ then integrate over q to get rid of all qs using δ fn.s then: something get final ans!

$$= \tfrac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\Big[\underbrace{(-\omega_p^2 + \vec{p}^{\,2} + m^2)}_{=0 \text{ e.o.m.?}}(a_pa_pe^{-2i\omega t} + a_p^\dagger a_p^\dagger e^{2i\omega t})$$
$$+ \underbrace{(\omega_p^2 + \vec{p}^{\,2} + m^2)}_{2\omega_p^2}(a_p^\dagger a_p + a_pa_p^\dagger)\Big]$$

(*) redo this derivation by hand

remember $\omega_p = \sqrt{\vec{p}^{\,2}+m^2}$ & δ fn applies to x, q only not ωt

$$H = \tfrac{1}{2}\int\frac{d^3p}{(2\pi)^3}\omega\,(a_p^\dagger a_p + a_pa_p^\dagger)$$

use $[a, a^\dagger]$ (commutation relations)

$$= \int\frac{d^3p}{(2\pi)^3}\omega a_p^\dagger a_p + \int\frac{d^3p}{(2\pi)^3}\omega\cdot\tfrac{1}{2}(2\pi)^3\delta^3(0)$$

??

should give some hesitation

$= \infty$? delta fn at 0 has infinite spike? & integral over ωp diverges at large p what do we do?

$\boxed{\text{what does this mean?}}$

(annihilated by a_p)

- Introduce a vacuum $|0\rangle$ s.t. $a_p|0\rangle = 0$

attempt to find energy of this ground state →

- $H|0\rangle = \int\frac{d^3p}{(2\pi)^3}\tfrac{1}{2}\omega(2\pi)^3\delta(0)|0\rangle = \infty|0\rangle$ $= E_0|0\rangle$

don't like this don't like infinity in physics
↳ this infinity has a meaning
↳ try to understand how this came about

- To understand the nature of this, we need to see the origin of the divergence

what I was expecting

=> we actually have two infinities → one from delta fn. in this expression $H|0\rangle$

in physics infrared stands for long distance

infrared divergence → infinity that arises because space is infinitely large

- to extract out this infinity consider $(2\pi)^3\delta(0)$ & put it in a box w/ side length $2L$ & take limit $L\to\infty$
- find the $\delta(0)$ divergence arises because system is infinite in size → solve by considering energy density instead

1) Infrared divergence : $(2\pi)^3\delta(0)$

$$(2\pi)^3\delta(0) = \lim_{L\to\infty}\int_{-L}^{L} d^3x\, e^{i\vec{x}\cdot\vec{p}}\Big|_{\vec{p}=0} = \lim_{L\to\infty}\int_{-L} d^3x = V$$

$\left(\delta^{(n)}(x) = \int \frac{d^nk}{(2\pi)^n} e^{ikx}\right)$ using dirac fourier representation

but still divergent!

no surprise (of course if you integrate over everything you get infinite volume)

- Infinite size system: $\boxed{\text{cure}}$ discuss energy density

$$\mathcal{E}_0 = \frac{E_0}{V} = \int \frac{d^3p}{(2\pi)^3}\tfrac{1}{2}\omega_p \sim \int d^3p\, \vec{p}^2 \to \infty$$

spoiler: not convergent integral

divergent! still infinite!

2) UV divergence

(should we some max frequency/cut off that we expect the theory to hold up to)

$$\int_0^{p_{max}} d^3p\sqrt{\vec{p}^2+m^2} \xrightarrow[p_{max}\to\infty]{} \infty$$

(high frequency divergence (hence 'UV') (arbitrarily)

- high freq./short distance divergence known as 'UV' divergence
- arises because we have assumed our theory is valid to arbitrarily high freq. corresponding to arbitrarily high energies
- solve by imposing a cut off at high p

or we are practically, since we only care about energy differences in physics just redefine H by subtracting this ∞

- Absurd to think that the theory is valid for arb. high energies
- Solution (practical): declare $H = \int \frac{d^3p}{(2\pi)^3}\omega_p a_p^+ a_p$ (we've just dropped 1st term)
- with this $\boxed{H|0\rangle = 0}$ (there is a state with zero energy (great!)
- The origin is due to an ambiguity in multiplying fields. The cure is NORMAL ODERING

(basically we are trying to fix an ambiguity (ambiguity with what hamiltonian means when moving to quantum fields)

what we've done here is called normal ordering (e.g. prev. expression $H = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\omega_p(a_p^+a_p + a_pa_p^+)$ apply normal ordering $:H: = \int\frac{d^3p}{(2\pi)^3}\omega_p a_p^+ a_p$ as expected!)

$\boxed{\text{Definition:}}$ if you have a list of fields, we define N.O. as

$$:\phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n):$$

divergences appear from moving from classical field to a quantum one

where this is the usual product with all $a(p)$ operators placed to the right of $a^+(p)$

perspective: quantum theory is the good one then take limit to find classical theory

⟹ first destroy, then create ← (way to remember this)

3) FOCK SPACE

have dealt with the vacuum $|0\rangle$, now look at excitations of the field!

We have $|0\rangle$, we want to construct excited states

$$[H, a_p^+] = \omega_p a_p^+ \qquad [H, a_p] = -\omega_p a_p \qquad a_p \equiv a(\vec{p})$$

check these are correct yourself → note to self: see by plugging in for $H = \int\frac{d^3q}{(2\pi)^3}\omega_q a_q^+ a_q$ and use $[AB,C] = A[B,C] + [A,C]B$

⟹ Construct energy eigenstates by e.g.

$$|\vec{p}\rangle = a^+(p)|0\rangle : \boxed{\text{single particle state}}$$ NOMENCLATURE

then $H|\vec{p}\rangle = \omega_p|\vec{p}\rangle \qquad \omega_p^2 = \vec{p}^2 + m^2$

is an eigenstate of H with e-value ω_p and also e-state of momentum operator!

see by $H|\vec{p}\rangle = Ha_p^+|0\rangle = ([H,a_p^+] - a_p^+H)|0\rangle$
$= \omega_p a_p^+|0\rangle - \cancel{a_p^+H|0\rangle}$ 0.
$= \omega_p|\vec{p}\rangle$

(⟹ single particle state is eigenstate of H with e-value ω_p)

recognise this as relativistic dispersion relation $E_{\vec{p}}^2 = \vec{p}^2 + m^2$

from now on write $E_{\vec{p}}$ instead of ω_p

Lecture 6 23.10.24

Recap: $\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} [a(\vec{k})e^{ikx} + a^\dagger(\vec{k})e^{ikx}]$

$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$ $a(\vec{k}) \equiv a_k$ — notation

$H = :\int d^3x (\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\vec{\nabla}\phi)^2 + \tfrac{1}{2}m^2\phi^2): = \int \frac{d^3k}{(2\pi)^3} \omega_k a_k^\dagger a_k$ $H|0\rangle = 0$

$H|\psi\rangle = E_\psi|\psi\rangle$

3) FOCK SPACE – cont.

p155: classical total (for scalar field?) momentum: $\vec{P}' = \int d^3x \, \dot\phi \, \partial\phi$ now promote this to operator!

- we can also consider (another conserved quantity in the system) $\vec{P} = -:\int d^3x \, \pi \vec{\nabla}\phi: = \int \frac{d^3k}{(2\pi)^3} \vec{k} a_k^\dagger a_k$

(similar process to promoting H to operator, gives something like $\int \frac{d^3k}{(2\pi)^3} \frac{\vec{k}}{2}(a_p a_{-p} + a_p^\dagger a_p + a_p a_p^\dagger + a_p^\dagger a_{-p}^\dagger)$ → but only mixed terms survive, because of odd integral !? → then normal order (see sheet 2))

single particle state: $\vec{P}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle$ so state $|\vec{p}\rangle$ has momentum $\vec{p}$

- $|\vec{p}\rangle$ is a momentum + energy eigenstate with $E^2 = \omega_p^2 = \vec{p}^2 + m^2$

(act on 1 particle state w/ zero momentum and it has no internal angular mom. i.e. spin 0 particle)

- Note: angular momentum J^2, $J^2|\vec{p}=0\rangle = 0$ → spin 0

(notes $J^i = \epsilon^{ijk}\int d^3x (J^0)^{jk}$ take classical angular mom. of field & promote to operator)

(promoting to operators: remember to sets: k i.e. p & q for each π, $\nabla\phi$ but same x (exactly as we did for H))

- With this we can create more states

multiparticle state: $|\vec{p}_1 \cdots \vec{p}_n\rangle \equiv a^\dagger(p_1) \cdots a^\dagger(p_n)|0\rangle$: $\boxed{\text{n-particle state (multi)}}$

(n particle state is state in which n $a^\dagger$s act on the vacuum (notes))

- since $a^\dagger$ commute $|\vec{p}_1, \vec{p}_2\rangle = |\vec{p}_2, \vec{p}_1\rangle$

state is symmetric under exchange of 2 particles: particles are bosons!

<u>Fock space</u>: all possible combinations of $a^\dagger$ acting on $|0\rangle$

full hilbert space spanned by acting on $|0\rangle$ w/ all combinations of $a^\dagger$: $|0\rangle, a_1^\dagger|0\rangle, a_1^\dagger a_2^\dagger|0\rangle, a_1^\dagger a_2^\dagger a_n^\dagger|0\rangle$ Known as Fock space!

Interesting to introduce

(counts no. of particles in a given state in the Fock space)

$N = \int \frac{d^3p}{(2\pi)^3} a_p^\dagger a_p$: $\boxed{\text{number operator}}$

satisfies $N|\vec{p}_1 \cdots \vec{p}_n\rangle = n|\vec{p}_1 \cdots \vec{p}_n\rangle$

- For a <u>free theory</u> $[N, H] = 0$

commute w. hamiltonian ⇒ conserved quantity → momentum tied to this ("free theory") ⇒ particles conserved (reminder from RQM)

⇒ particles conserved in free theory (but not true for interactions: create & destroy particles)

Fock space $= \bigoplus_n \mathcal{H}_n$

(Fock space is just the sum of the n-particle hilbert spaces for $n \geq 0$)

(Noticeably: angular momentum: n-the current associated w/ Lorentz transforms (rot. and boosts) is: $(J^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}$ • gives 6 conserved charges (3 corresponding to rotation) (ρ,σ) is 23 to invert 01 • conservation of angular momentum comes from the transforms that are rotations i.e. $\rho,\sigma = 1,2,3$: rotational charges $J^{ij} = \int d^3x (x^i T^{0j} - x^j T^{0i})$ (take 0 component & integrate). get: des the generator of rotations important easy as $J^i = \frac{1}{2}\epsilon^{ijk} J^{jk}$ ⇒ $J^i = \epsilon^{ijk}\int d^3x x^j T^{0k}$ (makes sense because I only have 3 rot. so ϵ^{ijk} means I only have label i which makes sense?) compute for scalar field this looks like $J^i = \epsilon^{ijk}\int d^3x x^j \pi \partial^k\phi$)

Relativistic Normalisation

How do we normalise states? pick $\langle 0|0\rangle = 1$ (could pick anything e.g. unicorn)

For 1-particle state $|\vec{p}\rangle = a_p^\dagger|0\rangle \Rightarrow \langle\vec{p}|\vec{q}\rangle = (2\pi)^3\delta^3(\vec{p}-\vec{q})$ we don't like this

⇒ NOT Lorentzian invariant

Dream/hope $|\vec{p}\rangle \underset{\text{let }\Lambda \text{ (translat.) ?}}{=} |\vec{p}\rangle = U(\Lambda)|\vec{p}\rangle$ — unitary transformation

To figure out a proper definition of $|\vec{p}\rangle$ use relativity

$$|\vec{q}\rangle = \int \frac{d^3p}{(2\pi)^3} |\vec{p}\rangle\langle\vec{p}|\vec{q}\rangle$$

$$1 = \underbrace{\int \frac{d^3p}{(2\pi)^3}}_{\text{not Lorentz inv. integral}} |\vec{p}\rangle\langle\vec{p}|$$

see how we can compensate p such that this is equal to 1

not Lorentz inv. integral → GOAL: make it Lorentz inv.

$$\int \frac{d^3p}{(2\pi)^3} \rightarrow \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\text{Lor. inv.}} \underbrace{\delta^4(p^2-m^2)}_{\text{Lor. inv.}} \underbrace{\Theta(p^2)}_{\text{Lor. inv.}}$$

step fn.

get rid of higher dim

$$\xrightarrow{\text{use delta } \Theta} \int d^3p \frac{1}{2\sqrt{\vec{p}^2+m^2}}$$

using deltafn. identity ?

$$\left(\int dx\, \delta(f(x)) = \sum_a \frac{1}{|f'(x_a)|},\ f(x_a)=0 \right)$$

↳ has solution ω_+, ω_- then Θ gets rid of one solution?

$$= \underbrace{\int d^3p \frac{1}{2\omega_p}}_{\text{Lor. inv.}}$$

$$1 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} |\vec{p}\rangle\langle\vec{p}|$$

$$|\vec{p}\rangle = \sqrt{2\omega_p}\, a_p^\dagger |0\rangle \quad : \boxed{\text{Relativistic Normalisation}}$$

(Concludes Fock space)

4) CAUSALITY

... 3) Fock space 4) causality 5) propagators

Why equal time commutators are compatible with SR (causality)?

"we can't influence our grandparents"

→ Measurements influencing each other in a timelike fashion

(key word: what is meant by 'influence')

↳ commutators vanishing or not

want our theory to be causal: i.e. a measurement at x cannot affect a measurement of y when x & y are not causally connected → i.e. require spacelike operators commute: $[O_1(x), O_2(y)] = 0 \ \forall\ (x-y)^2 < 0$

Define $\Delta(x-y) \equiv [\phi(x), \phi(y)]$

i.e. operators have spacelike separation.

not talked about measurement yet but introduce as an example e.g. if commutator zero then one measurement does not influence the other

Evaluate for free theory $\Delta = [\phi(x), \phi(y)] =$ → (*)

$$= \int \frac{d^3k}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_k \omega_p}} \left([a_k, a_p^\dagger] e^{-ikx} e^{ipy} + [a_k^\dagger, a_p] e^{ikx} e^{-ipy} \right)$$

↓ use (*) (?)

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right) = \Delta(x-y)$$

Comments / Properties

1) $\Delta(x-y)$ is Lorentz invariant, also for free theory a commutator

(due to appearance of Lor. measure $\int \frac{d^3p}{2\omega_p}$)

doesn't vanish for time-like separations:

2) Time-like separated $(x-y)_T = (t,0,0,0)$

if events time like separated they can influence each other

$$\Delta(x-y)_T = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(e^{-i\omega_p t} - e^{i\omega_p t}\right) \sim e^{-imt} - e^{imt} \neq 0$$

gpt says this is long time/large t limit

(using in this case $\omega_p^2 = \vec{p}^2 + m^2 \approx m^2$ says gpt)

vanishes for space-like separations:

3) space-like separated $(x-y)_S = (0, \vec{x}-\vec{y})$

$$\Delta(x-y)_S = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right) = 0$$

$\vec{p} \to -\vec{p}$

can flip sign of $\vec{p}$ since it is integration variable. (vera says so $d^3p \to -d^3p$ but integration limits also flip so essentially $d^3p \to d^3p$. gpt says technically $d^3p \to -d^3p$ but we're integrating over all space so this doesn't matter - flipping sign doesn't change the value so you get $\vec{p} \to -\vec{p}$ invariance under $\vec{p} \to -\vec{p}$)

& note, since $\Delta(x-y)$ is lor inv., it can only depend on $(x-y)^2$ and must therefore vanish for all $(x-y)^2 < 0$

(we have shown for $t=0$ but since $\Delta(x-y)$ is lor inv., this must also be true for all spacelike separations)

$\Rightarrow$ **any spacelike event has zero commutator**

compatible with causality

we will see things compatible w lorentz symmetry are therefore causal

our theory is causal $\to$ commutators vanish outside the lightcone

last topic of this chapter

5) PROPAGATORS

prepare a particle at point y in spacetime, compute the amplitude of finding it at point x by

(*)

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p} e^{-ip(x-y)}$$

$$\equiv D(x-y)$$

define this as D, the propagator

(or show from def of $\phi(x), \phi(y)$:)

$\langle 0|\phi(x)\phi(y)|0\rangle$

$= \int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{2\sqrt{\omega_p\omega_q}} \langle 0|a_p a_p^\dagger|0\rangle e^{-ip(x-y)}$

$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p} e^{-ip(x-y)}$

[reminder: use $a_p|0\rangle = 0$ & $\langle 0|a_p^\dagger = 0$]

For spacelike events

the propagator decays exponentially outside the lightcone

(quantum field 'leaks' outside the lightcone)

(gpt says this is from using known integral results → don't need to derive myself?)

$$D(x-y) \sim e^{-m|\vec{x}-\vec{y}|} \neq 0$$

but is nevertheless non vanishing! → how do we reconcile this with knowledge that spacelike operators commute?

But $[\phi(x), \phi(y)] = D(x-y) - D(y-x) = 0$ $\Rightarrow$ spacelike

(late note: for spacelike separated points in free theory $[\phi(x), \phi(y)] = 0$ we like this! we have shown our theory is causal)

interpret this!

[when $(x-y)^2 < 0$ there is no lor. inv. way to order events: if particle can travel in a spacelike direction from $x \to y$ it can just as easily travel from $y \to x$, in any measurement, the amplitudes for these two events cancel]

(unlike the commutator $\Delta(x-y) \equiv [\phi(x), \phi(y)]$, the propagator $D(x-y)$ (& is also the feynman propagator $D_F(x-y)$?) is non zero outside the lightcone)

Lecture 7: Quantum Fields: Free theory cont. 25.10.24

recap: **PROPAGATORS**

$$D(x-y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle \quad (\star)$$

propagator

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p} e^{-ip(x-y)}$$

$D(x-y)$ does not vanish for spacelike separated points

v. important in interacting field theory

FEYNMAN PROPAGATOR

this is what "F" does

Defn: $\Delta_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle \equiv \begin{cases} D(x-y) & x^0>y^0 \\ D(y-x) & y^0>x^0 \end{cases}$

TIME ORDERING – place all operators evaluated at later times to the left (→ i.e. if $y^0>x^0$, $T\phi(x)\phi(y) = \phi(y)\phi(x)$)

Why?

final state. initial state

$\langle f|i\rangle$ → past (initial state)

future (final state)

initial state should happen before final...

(note to self: $\Delta_F(x-y)$ is the amplitude of going from $y\to x$ if x happens after y ($x^0>y^0$), and the amplitude of going from $x\to y$ if y happens after x ($y^0>x^0$) → time ordering makes sense b/c future states should happen after initial states?)

we want to find a more useful expression for $\Delta_F(x-y)$!

CLAIM: $$\Delta_F(x-y) = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-y)}$$

[we want to find an expression for this object, and to verify the following claim:]

note to self: without the $i\epsilon$ this integral is ill-defined b/c the denominator $= (p^0)^2 - \vec{p}^2 - m^2$ produces a pole when $(p^0)^2 = \omega_p^2 = \vec{p}^2 - m^2$ (note we integrate over 4 momentum unlike only 3 mom before)

we'll get to what this is → $i\epsilon$ tells us how to deform the contour to avoid poles

PROOF: (long)

$*$ $\langle 0|T\phi(x)\phi(y)|0\rangle = \langle 0|\phi(x)\phi(y)|0\rangle\,\Theta(x^0-y^0) + \langle 0|\phi(y)\phi(x)|0\rangle\,\Theta(y^0-x^0)$

step fn: this term $=1$ when $x^0>y^0$; this term $y^0>x^0$

$*$ $$\langle 0|T\phi(x)\phi(y)|0\rangle \underset{(\star)}{=} \int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-i\omega_k(x^0-y^0)}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\,\Theta(x^0-y^0) + \int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-i\omega_k(y^0-x^0)}e^{i\vec{k}\cdot(\vec{y}-\vec{x})}\,\Theta(y^0-x^0)$$

valid since ω_p depends only on $\vec{p}^2$ (again when integrate over $\vec{p}$, d^3p is invariant under $\vec{p}\to-\vec{p}$ reversal(?))

flip in 2nd integral $k\to -k$

$$= \int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left(e^{-i\omega_k\tau}\theta(\tau) + e^{i\omega_k\tau}\theta(-\tau)\right)$$

defining $\tau = x^0-y^0$

$*$ Focus on time-dependence and show that

intermediate step (this will take the most time)

$$e^{-i\omega_k\tau}\theta(\tau) + e^{i\omega_k\tau}\theta(-\tau) = \lim_{\epsilon\to 0}\frac{(-2\omega_k)}{2\pi i}\int_{-\infty}^{\infty}d\omega\,\frac{e^{-i\omega\tau}}{\omega^2-\omega_k^2+i\epsilon}$$

↑ the identity we want to prove

$*$ Start from RHS of the eqn.

$$\frac{1}{\omega^2-\omega_k^2+i\epsilon} = \frac{1}{(\omega-(\omega_k-i\tilde{\epsilon}))(\omega-(-\omega_k+i\tilde{\epsilon}))} \quad \text{with } \epsilon = 2\omega_k\tilde{\epsilon}+\dots$$

note to self: (think $(\omega-a)(\omega-(-a))$, k!

rescale epsilon (going to carry on writing ϵ not $\tilde{\epsilon}$?)

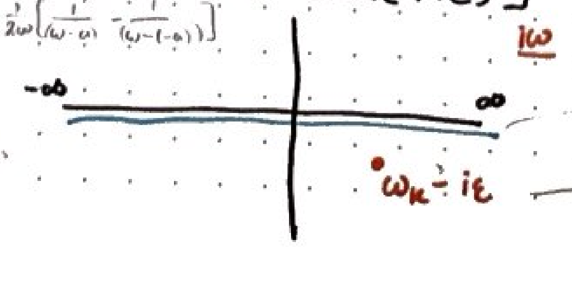
redefine for simplicity → $\tilde{\epsilon}\to\epsilon$

$$= \frac{1}{2\omega_k}\left[\frac{1}{\omega-(\omega_k-i\epsilon)} - \frac{1}{\omega-(-\omega_k+i\epsilon)}\right] + O(\epsilon^2)$$

(→ think $\frac{1}{2\omega}\left[\frac{1}{(\omega-a)} - \frac{1}{(\omega-(-a))}\right]$)

$*$ Consider

$$I_1 := \int_{-\infty}^{\infty}d\omega\,\frac{e^{-i\omega\tau}}{\omega-(\omega_k-i\epsilon)}$$



• complex ω plane

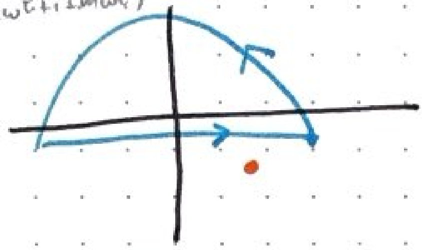
integrate in the integral for ... by "closing the contour"

pole in the complex plane;

from real analysis → analyse how we close this contour ↻
be pedantic: τ has real & im parts ↻

* $e^{-i\omega\tau} = e^{\mathrm{Im}\,\omega\tau}\, e^{-i\mathrm{Re}\,\omega\tau}$, if $\tau < 0$ close contour above

$(\omega\tau = \mathrm{Re}\,\omega\tau + i\,\mathrm{Im}\,\omega\tau)$



$\Rightarrow I_1 = 0$

if $\tau < 0$, as $\mathrm{im}\,\omega\tau \to \infty$, $e^{-i\omega\tau} \to 0$ so close contour in upper half-plane (?)

note to self: jordan's lemma & residue theorem

$\oint_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$

? if $f(z)$ (usually exponential) goes to zero fast enough as $|z| \to \infty$ in the appropriate half-plane as we take lim to ∞:

$\int_{-\infty}^{\infty} f(z)dz = \oint_C f(z)dz$

now residue theorem says:

$\oint_C f(z)dz = 2\pi i \sum(\text{residues inside the contour})$

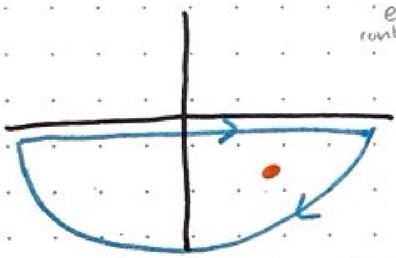
where residue is the "coefficient of the $\frac{1}{z-z_0}$ term"

i.e. use jordan's lemma to show we're ok turning our integral into contour integral → then use residue theorem to deal w/ singularities inside contour

* If $\tau > 0$, close contour below

if $\tau > 0$, as $\mathrm{Im}\,\omega\tau \to -\infty$, $e^{-i\omega\tau} \to 0$, so close contour in lower half-plane (?)

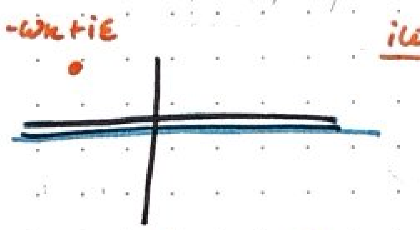
minus because integral clockwise



$\Rightarrow I_1 = -2\pi i e^{-i\omega_k\tau}\theta(\tau) + O(\epsilon)$

(residue i.e. coefficient of $\frac{1}{z-z_0}$ (?))

* $I_2 := \int_{-\infty}^{\infty} d\omega \dfrac{e^{-i\omega\tau}}{\omega - (-\omega_k + i\epsilon)}$

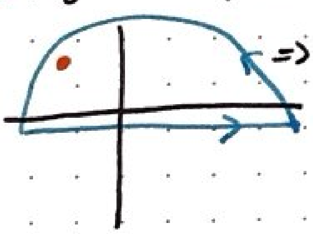


similar problem
new pole

(shift down)

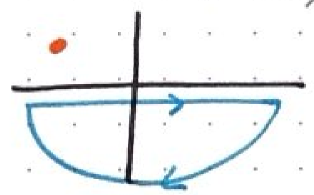
same again:

* If $\tau < 0$, close above



$\Rightarrow I_2 = 2\pi i e^{i\omega_k\tau}\theta(-\tau) + O(\epsilon)$

* If $\tau > 0$, close below



$\Rightarrow I_2 = 0$

* Collect I_1 & I_2 in $(\star\star)$

split integral into 2 poles & eval. these integrals separately
I_1 only contributes for +ve step fn
I_2 only contributes for -ve step fn

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega^2 - \omega_k^2 + i\epsilon} = \lim_{\epsilon \to 0}\frac{1}{2\omega_k}(I_1 - I_2)$$

$$= \frac{1}{2\omega_k}\left(-2\pi i e^{-i\omega_k\tau}\theta(\tau) - 2\pi i e^{i\omega_k\tau}\theta(-\tau)\right)$$

// q.e.d.

* Putting $(\star\star)$ into $\langle 0|T\phi(x)\phi(y)|0\rangle$

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{i}{2\pi} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega^2 - \omega_k^2 + i\epsilon}$$

k^0 ; $\omega_k^2 = \vec{k}^2 + m^2$

→ convention/nomenclature is to drop lim
because we are lazy → this is understood

$$=: \Delta_F(x-y) \quad = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)}$$

// q.e.d.

$\left((k^0)^2 - \vec{k}^2 - m^2 = k^2 - m^2\right)$

as $k = (\omega_k, -\vec{k})$
$\omega_k^2 = \vec{k}^2 + m^2$

→ $i\epsilon$ prescription

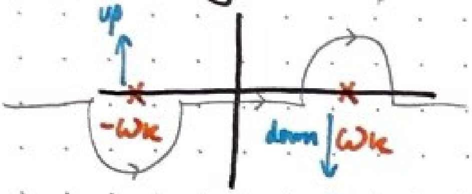
↳ this way of writing the contour is called the $i\epsilon$ prescription
↳ taking $i\epsilon$ infinitesimal shifts the poles off the real axis so we can equate integral to the contours discussed

Properties / Comments

the contour for the feynman propagator

remember our poles were at $-\omega_k+i\varepsilon$ & $\omega_k-i\varepsilon$ which decides the ways the contours go (?) i think

1) Time ordering $\longleftrightarrow$ contour prescription (i.e. how to introduce ε)



• without epsilon just points ×
• the ordering tells us move up or down
(• could have moved both poles up or both down → would give diff expression)

2). $\Delta_F(x-y)$ is Lorentz invariant

Δ_F will be one of our building blocks → amazing that it is Lorentz inv.

3). $$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} e^{-ik(x-y)}$$: off-shell (i.e. no e.o.m. for k°)

$k^2 \neq m^2$

off: integral includes all possible momenta including those where $\omega^2 \neq \vec{p}^2+m^2$

on shell: $\omega^2=\vec{p}^2+m^2$ particle satisfies this eq. → real particles
off shell → does not satisfy this eq. → not physical particles → virtual particles.

off shell virtual particles that don't satisfy classical propagation laws are allowed to temporarily exist but only for short times (?)

4) Don't forget i

5). $\Delta_F(x-y)$ is a Green's function

just ignore the $i\varepsilon$??

dropping ε here because not important for this argument

e.g. $$(\partial_\mu\partial^\mu + m^2)\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}(-k^2+m^2)e^{-ik(x-y)}$$
$$= -\int \frac{d^4k}{(2\pi)^4} ie^{-ik\cdot(x-y)}$$
$$= -i\delta^4(x-y)$$

if fields satisfy this then the propagator will satisfy the green's fn eq

propagator satisfies an e.o.m with a delta fn on the side

reminder: Greens fn
$LG(x,s) = \delta(s-x)$
linear operator $L(x)$; greens fn; delta fn.

$\boxed{(\Box_x+m^2)\Delta_F(x-y) = -i\delta^4(x-y)}$ ← "basically an identity" ← IMPORTANT LATER (or course?)

Δ_F is the Green's func. associated to K.G. opt. (klein gordon?)

Lecture 8

28.10.24

time ordering fields = normal ordering + feynman propagator

6) CLAIM: $$T\phi(x)\phi(y) = :\phi(x)\phi(y): + \Delta_F(x-y)$$
$$\equiv :\phi(x)\phi(y): + \overline{\phi\phi}$$ shorthand notation

(lecture notes: define "contraction" as $\overline{\phi_1\phi_2} = \Delta_F(x_1-x_2)$)

PROOF: decompose field

$$\phi = \phi^+ + \phi^-$$
$$\phi^+ = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}} a_p e^{-ipx}$$
$$\phi^- = \quad '' \quad a_p^\dagger e^{ipx}$$

Choose $x^0 > y^0$

$$T(\phi(x)\phi(y)) = \phi(x)\phi(y) = \phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x) + [\phi^+(x),\phi^-(y)] + \phi^-(x)\phi^-(y)$$

choose $y^0 > x^0$

(note in step 2 can immediately drop $[\phi^+(y), \phi^+(x)]$ and $[\phi^-(y), \phi^-(x)]$ terms b/c $[a_p, a_q] = 0$ and $[a_p^\dagger, a_q^\dagger] = 0$)

$$= :\phi(x)\phi(y): + D(x-y)$$

(note to self: can show this yourself)

$$D(x-y) = [\phi^+(x), \phi^-(y)]$$

$$T(\phi(x)\phi(y)) = :\phi(x)\phi(y): + D(y-x) \quad // \text{ q.e.d.}$$

SUMMARY
1. show for $x^0 > y^0$, $T\phi(x)\phi(y) = :\phi(x)\phi(y): + D(x-y)$
2. show for $y^0 > x^0$, $T\phi(x)\phi(y) = :\phi(x)\phi(y): + D(y-x)$
3. combine these using def. for Feynman propagator $\Delta_F(x-y) = \begin{cases} D(x-y), & x^0 > y^0 \\ D(y-x), & y^0 > x^0 \end{cases}$
4. gives $T\phi(x)\phi(y) = :\phi(x)\phi(y): + \Delta_F(x-y)$

generalise the above to give Wick's theorem:

Wick's Theorem: for free theory

(prove it yourself by induction → works for n, works for n+1 etc. (see p. 58 David Tong)) (base case is the one we already proved: $T\phi_1\phi_2 = :\phi_1\phi_2: + \overline{\phi_1\phi_2}$!)

$$T(\phi(x_1)\cdots\phi(x_n)) = :\phi(x_1)\cdots\phi(x_n): + \text{possible contractions}$$

↳ propagators

Example

(note to self: wondered?! i.e. only have term $\overline{\phi_1\phi_2}:\phi_3\phi_4:$ and not $\overline{\phi_2\phi_1}:\phi_3\phi_4:$, $\overline{\phi_1\phi_2}:\phi_4\phi_3:$ etc. thus $4C2 = 6$ terms w/ one bracket and $\frac{4C2}{2!} = 3$ terms w/ two brackets)

$$T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) = :\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4):$$

$$+ \overline{\phi_1\phi_2}:\phi_3\phi_4: + \overline{\phi_1\phi_3}:\phi_2\phi_4: + \overline{\phi_1\phi_4}:\phi_2\phi_3:$$

$$+ \overline{\phi_2\phi_3}:\phi_1\phi_4: + \overline{\phi_2\phi_4}:\phi_1\phi_3: + \overline{\phi_3\phi_4}:\phi_1\phi_2:$$

$$+ \overline{\phi_1\phi_2}\,\overline{\phi_3\phi_4} + \overline{\phi_1\phi_3}\,\overline{\phi_2\phi_4} + \overline{\phi_1\phi_4}\,\overline{\phi_2\phi_3}$$

shorthand: $\phi_i = \phi(x_i)$

$$\overline{\phi_i\phi_j} := \Delta_F(x_i - x_j)$$

new topic!

QUANTUM FIELDS: INTERACTIONS

(deviate slightly from lecture notes)

↳ note to self: previously looked at free theory, now introduce interactions

1) COUPLINGS

Free theory are "simple" because we can construct explicitly the Fock space. We want to consider more general Lagrangians

Obstruction: we cannot solve e.o.m. => don't have access to Hilbert space

Approach: perturbative QFT

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$

$\mathcal{L}_0$: ↓ known free theory

$\mathcal{L}_{int}$: ↳ unknown, treat as perturbation

(split up lagrangian into bit we know (free theory) and what we don't know → treat this as perturbation)

note: say $n \geq 3$

Say: $\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$, $\quad \mathcal{L}_{int} = -\sum_{n=3} \frac{\lambda_n}{n!}\phi^n$ ← for simplicity $\lambda_n \in \mathbb{R}$

Naive: perturbation means $\lambda_n \ll 1$ WRONG!

→ what does small mean, what does it mean to be perturbative

To quantify "smallness" (perturbative) recall units

our convention

choose this · means this

$c = 1 = \hbar \qquad [L] = [T] = [M^{-1}] \quad , \quad [M] = 1 \rightarrow$ dimension one

Apply to action: $S = \int d^4x\,\mathcal{L} \quad , \quad [S] = [\hbar] = 0$ dimensionless

(note to self: $S = \int dt\, L$ where $L = T - V$ units energy => S has units energy × time or equivalently the same units as $\hbar$)

($[T][L]^3 = [M]^{-4}$)

what does this imply for the lagrangian? : $[\int d^4x] = -4 \quad => [\mathcal{L}] = 4$

what does this imply for the field? : apply to $\mathcal{L}_0$

$[m] = 1$

$[\partial_\mu] = 1$ if $[\mathcal{L}_0] = 4 \quad => [\phi] = 1 \quad \therefore [\mathcal{L}_{int}] = [\lambda_n\phi^n] => [\lambda_n] = 4 - n$

(field has dimension)

($[T]^{-1}$ or $[L]^{-1}$ ↔ $[M]^1$)

(note to self: remember these are powers $[M]^1$ so add)

$4 = [\lambda_n] + n[\phi]$
$[\lambda_n] + n$

notes (have seen: $[\lambda_n] = 4-n$ → so now we see why we can't just say $\lambda_n \ll 1$ as this only makes sense for dimensionless quantities!)

Cases:

1) $n=3$, $[\lambda_n]=1$ (more generally in any d-dim $[\lambda_n]>0$)

when $n=3$, dimless parameter is $\frac{\lambda_3}{E}$ where E has dims mass

Dimensionless quantity $\frac{\lambda_n}{E}$ where E is some energy scale (of the process of interest)

since dimless, meaningful to say when this is small

- If $\lambda_n \ll E$ (high energies), small perturbation
- If $\lambda_n \gg E$ (low energies), big perturbation

→ λ_n is a $\boxed{\text{RELEVANT}}$ coupling

(called this b/c terms w/ this behaviour are most relevant at low energies (where mostly the physics we see lies))

In a relativistic theory $E > m$, so we can treat perturbatively

(relativistic: $E = \sqrt{p^2+m^2} > m$?)

$\lambda_n \ll m$ ← (so we can always make this perturbation small by taking $\lambda_n \ll m$)

2) $n=4$, $[\lambda_n] = 0$

meaningful to write $\lambda_n \gg 1$

(n=4: this term is small if $\lambda_n \ll 1$ → call these perturbations "marginal")

$\lambda_n \ll 1$ → perturbative

λ_n is a $\boxed{\text{MARGINAL}}$ coupling

3) $n>4$, $[\lambda_n]<0$

dimensionless combination $\lambda_n (E)^{n-4}$ → not important at low energies; → big at high energies

(note lastly: $[\lambda_n] = 4-n$ so this is dimless)

λ_n is an $\boxed{\text{IRRELEVANT}}$ coupling

QFT is simple because at a start we only need Relevant & Marginal couplings → note: basically bc irrelevant couplings are even irrelevant at low energies

("to haunt you in advanced QFT")

(notes: this discussion based on dimensional analysis tells us we only need to focus on the first few terms in the interaction (those that are relevant & marginal))

Note: classification/identity? is classical, in Adv. QFT you will see how quantum effects change this

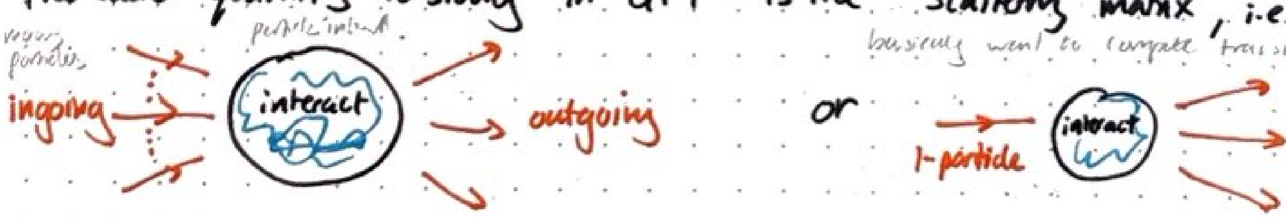
annoying people

2) LSZ REDUCTION FORMULA

discussion of basic observable in QFT → how do we deal with interactions

The basic quantity to study in QFT is the scattering matrix, i.e. S-matrix

basically want to compute transition amplitude



scattering

decay

How to quantify this?:

1) Define states (asymptotic states)
2) How to relate in & out: S-matrix

} → LSZ

3) How to evaluate S] → Schwinger-Dyson ⇒ Feynman Rules

Asymptotic States

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$
$$H = H_0 + H_{int}$$
$\rightarrow$ $|\Omega\rangle$ vaccum state for whole system
($|0\rangle$ vaccum for free theory)

Lecture 9 (LSZ reduction formula cont.) 30.10.24

Asymptotic states

our task: want to find inner product

$$\underbrace{\langle \text{final}, t_f | \text{initial}, t_i\rangle}_{\text{schrodinger}} = \underbrace{\langle f|S|i\rangle}_{\text{heisenberg}}$$

schrodinger: states evolve

heisenberg (operators evolve w/ time)

$\langle f, t_f | i, t_i\rangle = \langle f|S|i\rangle$
↑ schrodinger picture
↑ heisenberg picture is more useful for QFT: all the time evolution is encoded in an operator, the S-matrix

Assumptions/input/declarations:

free part, interacting part

$$\rightarrow H = H_0 + H_{int}, \quad \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$

Define $|\Omega\rangle$ as the vaccum state of H

$\boxed{\text{Assumption:}}$ Hamiltonian does time evolution

i.e. if we take time derivative of the field,

$$i\partial_t \phi = [\phi, H]$$

can generalise

$$i\partial_t \theta = [\theta, H] \quad \text{with } \theta \text{ any operator}$$

note where this is from:
$O_H(t) = e^{iHt} O_s e^{-iHt}$
$\Rightarrow i\dfrac{dO_H(t)}{dt} = [O_H(t), H]$
$\left(+ e^{iHt} \dfrac{dO_s}{dt} e^{-iHt}\right)$
↑ ignore

$\boxed{\text{Declare: (assume)}}$ At some time $t = t_0$, we can match the Hilbert space H_0 to that of H

subscript 0 to indicate free

p denotes $a^\dagger$? destruction

(1) in interacting theory get destruction operator dependent on time

(2) time evolve our operator

$$a_p(t) = e^{iH(t-t_0)} a_p^0 e^{-iH(t-t_0)}$$

(1) free destruction operator

$a_p(t) = e^{iH(t-t_0)} a_p(t_0) e^{-iH(t-t_0)}$
coincide w/ free thing at $t = t_0$ or $t = \pm\infty$? (confused?) $\Rightarrow$ I think we define the time at which they coincide as $t = \pm\infty$, it is not just some arbitrary time
↳ this is just general assumption that at some time the Hilbert spaces are matching
actual working statement is the asymptotic time statement def. below

(consequence of saying the field at some time is the free field then time evolve it
↳ looks similar to free field but w/ time dependence)

$$\underset{\text{int.}}{\phi(x)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p(t) e^{-ipx} + a_p^\dagger(t) e^{ipx}\right)$$

$(\omega_p = \sqrt{\vec{p}^{\,2} + m^2})$

from this pov defining of states...

$p_1, p_2 \rightarrow (\,) \rightarrow p_3, p_4$

With this, we would write states as, e.g.

write lorentz? elaborating normalised states? multi particle states?

reminder: normalised single particle state $|\vec{p}\rangle = \sqrt{2\omega_p}\, a_p^\dagger |0\rangle$

* $|\text{initial}, t_i\rangle = \sqrt{2\omega_1}\sqrt{2\omega_2}\, a_{p_1}^\dagger(t_i) a_{p_2}^\dagger(t_i) |\Omega\rangle$

* $|\text{final}, t_f\rangle = \sqrt{2\omega_3}\sqrt{2\omega_4}\, a_{p_3}^\dagger(t_f) a_{p_4}^\dagger(t_f) |\Omega\rangle$

Summary: initially prepare system to be free then at some time turn on interactions then do measurement at $t = \infty$ after interactions have been turned off

Definition of asymptotic state, we will want interactions to be turned off when $t_i \rightarrow -\infty$, $t_f \rightarrow +\infty$

at limit of infinite times we have the free theory

$$\lim_{t \to \pm\infty} a_p^\dagger(t) = \underset{\text{free theory}}{a_p^{0\dagger}}$$

i.e. asymptotic states are those of the free theory

this addresses 1) defining what we mean by in and out states, asymptotic states have defined asymptotic states, now ask

How to relate states @ $\pm\infty$?

prepare system to be free at $t=-\infty$ & label states in some way, turn of some time turn on interactions, then measure final state at $t=\infty$ after interactions have been turned off again

$$\langle f|S|i\rangle = \langle \text{final}, t_f | \text{initial}, t_i\rangle$$

to compute this, need to know relationship between a_p at $\pm$ infinity

$$(*) \quad = \left(\prod_{i=1}^{4}\sqrt{2\omega_i}\right)\langle\Omega|T\, a_{p_3}(+\infty)a_{p_4}(+\infty)a_{p_1}^\dagger(-\infty)a_{p_2}^\dagger(-\infty)|\Omega\rangle$$

T → time ordered

in order to prove, show identity:

CLAIM:

aim: to relate state at initial vs final times

(i.e. difference between $a_p^\dagger$ at $+\infty$ & at $-\infty$ is equal to this)

will use this to relate states at initial & final times

$$\sqrt{2\omega_p}\left(a_p^\dagger(\infty) - a_p^\dagger(-\infty)\right) = -i\int d^4x\, e^{-ipx}(\Box + m^2)\phi(x)$$

this is a claim made

In the interacting theory, and have $\omega_p = \sqrt{\vec{p}^{\,2}+m^2}$

PROOF:

→ diff should be 0 for free theory: is obvious (a_p since the a_p are time indep!) but we want to show for interacting theory:

Start from RHS

time deriv. spatial deriv.

note to self $\int u\, dv\, dx = uv - \int v\, du\, dx$ just gives this?

$$-i\int d^4x\, e^{-ipx}(\Box+m^2)\phi(x) = -i\int d^4x\, e^{-ipx}(\partial_t^2 - \vec{\nabla}_x^2 + m^2)\phi(x)$$

acts on backwards ← now acts backwards

int by parts 2x (ignore boundary terms!) — the spatial deriv.

$$= -i\int d^4x\, e^{-ipx}(\partial_t^2 - \overleftarrow{\nabla}_x^2 + m^2)\phi(x)$$

just 2x bring down $\vec{p}$ 2x

$$\left(\begin{aligned} px &= p^0x^0 - \vec{p}\cdot\vec{x} \\ &= \omega_p x^0 - \vec{p}\cdot\vec{x}\end{aligned}\right)$$

$$= -i\int d^4x\, e^{-ipx}(\partial_t^2 + \underbrace{\vec{p}^{\,2} + m^2}_{\omega_p^2})\phi(x)$$

→ next interpret this as coming from time deriv of exponential

$$= -i\int d^4x\, \partial_t\left(e^{-ipx}\partial_t\phi - (\partial_t e^{-ipx})\phi\right)$$

→ note cross terms cancel due to − sign

total time deriv → this only depends on $t \to \pm\infty$ ↳ only depends on asymptotic times!

total time derivative so only depends on asymptotic times: eval. at $\pm\infty$

Recall from the free theory

identity: can write a_p & $a_p^\dagger$ in terms of the fields

$$\sqrt{2\omega_p}\, a_p^0 = i\int d^3x\, e^{ipx}\overleftrightarrow{\partial_t}\phi$$

$$\sqrt{2\omega_p}\, a_p^{0\dagger} = -i\int d^3x\, e^{-ipx}\overleftrightarrow{\partial_t}\phi$$

where $f\overleftrightarrow{\partial_t}g = f\partial_t g - (\partial_t f)g$

$$= \sqrt{2\omega_p}\int_{-\infty}^{\infty} dt\, \partial_t(a_p^\dagger(t)) = \sqrt{2\omega_p}\left(a_p^\dagger(\infty) - a_p^\dagger(-\infty)\right)$$

// q.e.d.

Analogously we have:

$$\sqrt{2\omega_p}\left(a_p(\infty) - a_p(-\infty)\right) = i\int d^4x\, e^{ipx}(\Box + m^2)\phi(x)$$

With this, we now write $(*)$ as

$$\langle f|S|i\rangle = \prod_{i=1}^{4}\sqrt{2\omega_i}\,\langle\Omega|T\Big((a_{p_3}(+\infty) - a_{p_3}(-\infty))(a_{p_4}(+\infty) - a_{p_4}(-\infty))$$

we expand ingoing, we out going

$$\cdot\,(a_{p_1}^\dagger(-\infty) - a_{p_1}^\dagger(+\infty))(a_{p_2}^\dagger(-\infty) - a_{p_2}^\dagger(+\infty))\Big)|\Omega\rangle$$

$$= \boxed{\prod_{j=1}^{4}\left(i\int d^4x_j\right)\underbrace{e^{-ip_1x_1}(\Box_1+m^2)e^{-ip_2x_2}(\Box_2+m^2)}_{\text{ingoing}}\underbrace{e^{+ip_3x_3}(\Box_3+m^2)e^{+ip_4x_4}(\Box_4+m^2)}_{\text{outgoing}} \times \underbrace{\langle\Omega|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle}_{\text{4 p.t. correlation function}}}$$

LSZ FOR 2-2 SCATTERING

this construction
why better than
the leiche notes?

Pros of LSZ:

every bit is the box is lorentz inv.

1) Lorentz invariant S-matrix

2) It makes manifest the relation between

nice properties of this construction

$$\underbrace{\langle\Omega|T\phi(x_1)\cdots\phi(x_n)|\Omega\rangle}_{\text{n-point correlation fn.}} \leftrightarrow \langle f|S|i\rangle$$

Note: $\langle\Omega|T(\Box_x+m^2)\phi(x)\ldots|\Omega\rangle$

$$\neq \quad (\Box_x+m^2)\langle\Omega|T\phi(x)\ldots|\Omega\rangle$$

The difference are contact terms

what we care about is this

$$\langle f|S|i\rangle \Rightarrow S = \underbrace{\mathbb{1}}_{\substack{\text{boring bit}\\ \text{(not interesting)}}} + i\underbrace{T}_{\substack{\text{transfer matrix}\\ \text{(interesting bits)}}}$$

"got" the 'uv' terms in integration by parts disappear b/c it is standard assumption in QFT that fields fall off just enough at ∞ that any boundary/surface terms vanish

note to self:
show this (just plug stuff in)

$\rightarrow \sqrt{2\omega_p}\,a_p = i\int d^3x\, e^{ipx}(\dot\phi - i\omega_p\phi)$

$\rightarrow$ plug in $\dot\phi$ & ϕ in terms of q's $\leftarrow$ important!

get something like $= i\int d^3x\, e^{ipx}\left(\int\frac{d^3q}{(2\pi)^3}\frac{1}{i\sqrt2}\left(\sqrt{\omega_q}a_q e^{-iqx} - \sqrt{\omega_q}a_q^\dagger e^{iqx} + \frac{\omega_p}{\sqrt{\omega_q}}a_q e^{-iqx} + \frac{\omega_p}{\sqrt{\omega_q}}a_p^\dagger e^{iqx}\right)\right.$

$\rightarrow$ then separate $e^{ipx} = e^{i\omega_p t - i\vec p\cdot\vec x}$ and similarly for e^{iqx} then use $\int d^3x\, e^{i\vec x\cdot(\vec p-\vec q)} = (2\pi)^3\delta^3(\vec p-\vec q)$

$\rightarrow$ gives $= ie^{i\omega_p t}\frac{1}{i\sqrt2}\left(\sqrt{\omega_p}a_p e^{-i\omega_p t} - \cancel{\sqrt{\omega_p}a_{-p}^\dagger e^{i\omega t}} + \sqrt{\omega_p}a_p e^{-i\omega_p t} + \cancel{\sqrt{\omega_p}a_{-p}^\dagger e^{i\omega_p t}}\right)$

$= \sqrt{2\omega_p}\,a_p$ ✓

take conjugate $\Rightarrow \sqrt{2\omega_p}\,a_p^\dagger = -i\int d^3x\, e^{-ipx}(\dot\phi + i\omega_p\phi)$ ✓

$\rightarrow$ lucas notes: anyhow consider simpler example of $\langle\Omega|T a_{p_1}(\infty)a_{p_1}^\dagger(-\infty)|\Omega\rangle$
then show this is equivalent to $\langle\Omega|T(a_{p_1}(\infty)-a_{p_1}(-\infty))(a_{p_1}^\dagger(-\infty)-a_{p_1}^\dagger(\infty))|\Omega\rangle$
using time ordering & fact commutation relations of $a_p, a_p^\dagger$ hold as usual at equal times. then all extra terms disappear b/c they annihilate the vacuum

?

lucas says can generalise to N ingoing & M outgoing particles to give the LSZ formula

$$\langle f|S|i\rangle = i^{N+M}\left[\prod_{j=1}^{M}\int d^4x_j\, e^{ip_j\cdot x_j}(\Box_j+m^2)\right]\times\left[\prod_{i=1}^{N}\int d^4y_i\, e^{-ip_i\cdot y_i}(\Box_i+m^2)\right]$$
$$\times\underbrace{\langle\Omega|T\phi(x_1)\ldots\phi(x_M)\phi(y_1)\ldots\phi(y_N)|\Omega\rangle}_{(M+N)\text{-point correlation fn.}}$$

Lecture 10 - (Quantum fields interactions cont.) 1.11.24

3) SHCWINGER-DYSON FORMULA

diff ways to do this

We need a way to evaluate $\langle \Omega | T\phi(x_1) \dots \phi(x_n) | \Omega \rangle$

Strategy is to present a Lagrangian approach to this

Assumption 1 at any given time the Hilbert space of interacting theory = Hilbert space of free theory — has implication →

fields at equal t commute

$$[\phi(\vec{x},t), \phi(\vec{x}',t)] = 0 \quad \rightarrow \text{causality}$$

i.e. all our commutation relations for the fields & conjugate momenta are the same as before

in some way restating assumption we made for LSF but just to that in a way to make sure of correlation fn.

$$[\phi(\vec{x},t), \partial_t \phi(\vec{x}',t)] = i\delta^3(\vec{x}-\vec{x}') \quad \rightarrow \text{principle of QM}$$

Assumption 2 Our fields still comply with Euler-Lagrange equations

field obeys eom

Free theory : $(\Box + m^2)\phi^0(x) = 0$

in int theory eom is this.

Int. theory : $(\Box + m^2)\phi(x) - \mathcal{L}'_{int}(\phi) = 0$ $\quad \left(\mathcal{L}'_{int} = \frac{\partial \mathcal{L}_{int}}{\partial \phi}\right)$

↳ assume $\mathcal{L}_{int}(\phi)$ but no $\partial\phi$ for simplicity

reminder to self: EL: $\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0$. for free thry $\mathcal{L}$ (free massive scalar) have: $\partial_\mu\partial^\mu\phi + m^2\phi = 0$. For $\mathcal{L} + \mathcal{L}_{int}$: $\frac{\partial \mathcal{L}}{\partial \phi} + \frac{\partial \mathcal{L}_{int}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \partial_\mu\left(\frac{\partial \mathcal{L}_{int}}{\partial(\partial_\mu\phi)}\right)$; $-m^2\phi$; $-\Box\phi$; given?

Note: in Ham. derivation you would assume

$$\partial_t \phi = i[H, \phi]$$

Notation:

$$\langle \Omega | T\phi(x_1) \dots \phi(x_n) | \Omega \rangle \equiv \langle \phi_1 \dots \phi_n \rangle$$

this bracket is short for { int. vacuum + time ordering } and $\phi_1 \equiv \phi(x_1)$

CLAIM: act w 2 pt.

same as moving inside

$$(\Box_x + m^2)\langle \phi_x \phi_y \rangle = \langle (\Box_x + m^2)\phi_x \phi_y \rangle - i\delta^4(x-y)$$

PROOF: Warm up for free theory (superscript 0)

$$(\Box_x + m^2)\langle \underbrace{\phi^0_x \phi^0_y}_{D_F(x-y)} \rangle = 0 - i\delta^4(x-y) \quad // \text{ q.e.d. for free theory}$$

feynman propagator — this tells us it is Green's fn → we already knew that

↑ use assumption 2 $(\Box_x + m^2)\phi_x = 0$

For interacting theory — take time deriv of object which is time ordered

NB: just consider time derivative we using that integral w/ the time ordering

$$\partial_{x^0}\langle \phi_x \phi_y \rangle = \partial_{x^0}\left(\langle \Omega | \phi_x \phi_y | \Omega \rangle \Theta(x^0 - y^0) + \langle \Omega | \phi_y \phi_x | \Omega \rangle \Theta(y^0 - x^0)\right)$$

take into account time derivative on step fn

$$= \langle \partial_{x^0}\Phi_x\Phi_y\rangle + \langle\Omega|\Phi_x\Phi_y|\Omega\rangle\partial_{x^0}\,\theta(x^0-y^0) + \langle\Omega|\Phi_y\Phi_x|\Omega\rangle\partial_{x^0}\,\theta(y^0-x^0)$$

delta deriv. of step fn.

$$= \langle \partial_{x^0}\Phi_x\Phi_y\rangle + \underbrace{\delta(x^0-y^0)\langle\Omega|[\Phi_x,\Phi_y]|\Omega\rangle}_{\text{commutes at equal time } = 0}$$

$= -\langle\Omega|\Phi_y\Phi_x|\Omega\rangle\partial_{x^0}\theta(x^0-y^0)$ (use delta fn is derivative of step fn?)

now same exercise for ∂^2 -> (start from $\partial_{x^0}\langle\partial_{x^0}\Phi_x\Phi_y\rangle$)

use commutation equal time relations

$$\partial^2_{x^0}\langle\Phi_x\Phi_y\rangle = \langle\partial^2_{x^0}\Phi_x\Phi_y\rangle + \underbrace{\delta(x^0-y^0)\langle\Omega|[\partial_{x^0}\Phi_x,\Phi_y]|\Omega\rangle}_{-i\delta^4(x-y)}$$

$= \theta(x) = \theta(-x)$ think?

combine w/ other δ fn. to get δ^4

$$= \langle\partial^2_{x^0}\Phi_x\Phi_y\rangle - i\delta^4(x-y)$$

$-i\delta^3(\vec{x}-\vec{y})\delta(x^0-y^0) = -i\delta^4(x-y)$ (?)

spatial derivative + $(m)^2$ do nothing $\Longrightarrow$ $(\Box_x+m^2)\langle\Phi_x\Phi_y\rangle = \langle(\Box_x+m^2)\Phi_x\Phi_y\rangle - i\delta^4(x-y)$ // q.e.d.

So from claim (now proved) using assumption 2

ex sheet 2 starts generic

$$\Longrightarrow \boxed{(\Box_x+m^2)\langle\Phi_x\Phi_y\rangle = \langle\mathcal{L}'_{int}(\Phi_x)\Phi_y\rangle - i\delta^4(x-y)}$$

SCHWINGER DYSON EQN.

Generalise $\downarrow$

(which is why i think this is $\to \left.\frac{\partial\mathcal{L}}{\partial\Phi}\right|_{\Phi_x}$?)

$$\boxed{\begin{aligned}(\Box_x+m^2)\langle\Phi_x\Phi_1\dots\Phi_n\rangle &= \langle\mathcal{L}'_{int}(\Phi_x)\Phi_1\dots\Phi_n\rangle\\ &\quad - i\sum_{j=1}^{n}\delta^4(x-x_j)\langle\Phi_1\dots\Phi_{j-1}\Phi_{j+1}\dots\Phi_n\rangle\end{aligned}} \quad (*)$$

2nd term is essentially dealing w/ quantum fields(?)

why is this a useful relation to compute correlation fn.s? -> illustrate why it is useful by going through examples

Using Schwinger-Dyson in examples:

$\boxed{\text{Ex 1}}$ 4pt. function in free theory

(reminder: wick's theorem tells us the difference between time ordering & normal ordering)

(superscript 0 emphasise we're working in free theory)

$$\langle\Phi_1^0\Phi_2^0\Phi_3^0\Phi_4^0\rangle \underset{\text{wick's theorem}}{=} \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4) + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)$$

only these 3 terms remain b/c normal ordering quantities vanish

$$= \Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23}$$

introduce shorthand $\Delta_F(x-y) \equiv \Delta_{xy}$

want to use SD to derive same ans we got from wick's theorem!

Derive w/ Schw-Dyson

(going to stop writing superscript 0 but still free!)

$$\langle\Phi_1^0\Phi_2^0\Phi_3^0\Phi_4^0\rangle = \int d^4x\,\delta^4(x-x_1)\langle\Phi_x\Phi_2\Phi_3\Phi_4\rangle \quad (1)$$

remember Δ_F as a green's fn. basically an identity to memorise:

(1) $\delta^4(x-x_1) = i\Box_x\Delta_{x_1} = i(\Box_x+m^2)\Delta_{x_1}$

$$= i\int d^4x\,\underbrace{(\Box_x\Delta_{x_1})}_{((\Box_x+m^2)\Delta_{x_1})}\langle\Phi_x\Phi_2\Phi_3\Phi_4\rangle$$

remember this is Δ_{x_1}, not Δ_{x_1}...

int. by parts 2x

$$= i\int d^4x\,\Delta_{x_1}\big((\Box_x+m^2)\langle\Phi_x\Phi_2\Phi_3\Phi_4\rangle\big)$$

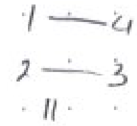
$(*)$ with $\mathcal{L}_{int}=0$

side note: int by parts 2x, boundary terms vanish
$\int u\frac{dv}{dx} \to -\int v\frac{du}{dx}$, do this 2x, 2 -ve signs give +ve
i.e. $\int(\partial^2_\mu\Delta_{x_1})\langle\;\rangle = -\int\partial_\mu\Delta_{x_1}\partial_\mu\langle\;\rangle = \int\Delta_{x_1}\partial^2_\mu\langle\;\rangle$ //

$$= i\int d^4x\,\Delta_{x_1}\big(-i\delta(x-x_2)\langle\Phi_3\Phi_4\rangle - i\delta(x-x_3)\langle\Phi_2\Phi_4\rangle - i\delta(x-x_4)\langle\Phi_2\Phi_3\rangle\big)$$

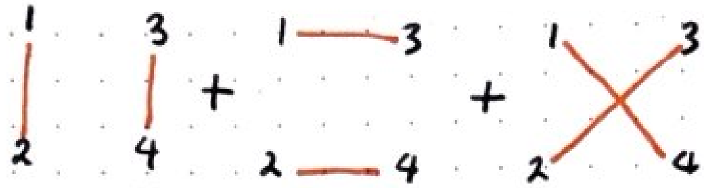
$$= \Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23} \;//$$

Result $= \Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23}$



crossing line doesn't mean anything (proof)(?)

illustrate in pictures

$=$ 

{follow same strategy as before: start by introducing delta fn.}

$\boxed{\text{Ex 2.}}$ Cubic int. $\mathcal{L}_{int} = \frac{g}{3!}\Phi^3$, one-point fn.

← (we will do more complicated in this cubic theory, let's just look at one-pt function)

same trick: replace by δ fn. — normalise coupling by factorials → see in a moment why

$$\langle \Phi_x \rangle = \int d^4y\, \delta(x-y) \langle \Phi_y \rangle$$

(1)

note lightly: makes more sense to be consistent w/ (1) i.e. write $\delta(y-x)$: $\delta(y-x)$ here then integral by parts actually makes sense since everything is w.r.t. y not x

$$= i\int d^4y\, (\Box_x + m^2)\Delta_{xy} \langle \Phi_y \rangle$$

int by parts 2x step (2)

(note: actually use $\delta(y-x)$, $\Box_y$ in this example.

$$= i\int d^3y\, \Delta_{xy} \left((\Box_y + m^2) \langle \Phi_y \rangle \right)$$

(Schw.D) (*) $(\Box_x + m^2)\Delta_{xy} = -i\delta(x-y)$ (*)

$$= i\int d^4y\, \Delta_{xy} \frac{g}{3!} \langle \Phi_y^2 \rangle$$

2pt

approx in free theory — (approx w/ free theory b/c non-zero in free theory)

e.g. in doing perturbation theory & just wants to know what the leading correction i.e. can say this thing to leading order will be = what i got in free theory + higher order corrections

$$= \frac{ig}{2} \int d^4y\, \Delta_{xy} \langle \Phi_y^{0\,2} \rangle + O(g^3)$$

= 0 in free theory

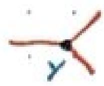
$$= \frac{ig}{2} \int d^4y\, \Delta_{xy}\Delta_{yy} + O(g^3)$$

(this propagator tells me x to y) $x \longrightarrow y$ (this propagator takes y to itself)

$$= \frac{ig}{2}\Big($$  $$\Big) + O(g^3)$$ "loop"

(propagator takes me from point x to point y then propagates y to itself (loop) intersection in picture implicitly means we have an integral)

[**Rule**: $\Delta =$  propagator = line

 $= \int d^4y$ intersection $= \int d^4y$]

the fields are allowed to proliferate (go from one field to two) we have this coupling g (goes from x to y) then something that goes back to itself (loop)

with one power of g so one intersection

particle creates & destroys itself (only allowed to do this b/c the propagator has a δ fn in it which comes from quantisation (classically this will not happen → particle cannot create & destroy itself classically) i.e. delta fn allows particle to create & destroy ?

Lecture 11

4.11.24

(still using $\mathcal{L}_{int} = \frac{g}{3!}\phi^3$)

Ex 3. 3pt. function in ϕ^3 theory

start w/ same steps as usual

$$\langle\phi_1\phi_2\phi_3\rangle = \int d^4x\, \delta(x-x_1)\langle\phi_x\phi_2\phi_3\rangle$$

$$= i\int d^4x\, \Delta_{x1}(\Box_x + m^2)\langle\phi_x\phi_2\phi_3\rangle \quad (1)+(2)$$

$$= \frac{ig}{2}\int d^4x\,\Delta_{x1}\underbrace{\langle\phi_x\phi_x\phi_2\phi_3\rangle}_{\text{approx w/ 4pt. fn. free theory (Ex1)}} + \int d^4x\,\Delta_{x1}\big(\underbrace{\delta(x-x_2)\langle\phi_3\rangle+\delta(x-x_3)\langle\phi_2\rangle}_{\text{(Ex 2)}}\big) \quad \text{(Schw-D)}$$

at this stage, make approx

plug in ans. from ex2

"similar to Born approximation" (?) replace expressions $\langle\rangle$ etc. by free theory or leading order gues. e.g. see $\langle\phi_x\phi_x\phi_2\phi_3\rangle$ 4pt fn. i know the free thy. already gives a non-trivial ans. so replace w/ free theory ans. 1pt fns $\langle\phi\rangle$ are zero in free thy. but we have found ans to order g in ex2. so plug this in

now 4pt fn. (grouping terms that are the same)

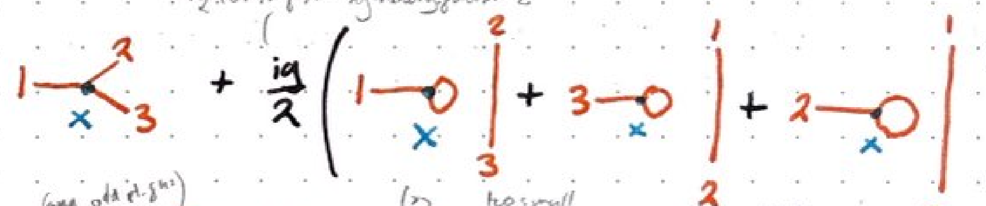
$$= \frac{ig}{2}\int d^4x\,\Delta_{x1}(\Delta_{xx}\Delta_{23} + 2\Delta_{x3}\Delta_{x2})$$
$$+ \frac{ig}{2}\int d^4x\,\Delta_{x1}\delta(x-x_2)\int d^4y\,\Delta_{3y}\Delta_{yy}$$
$$+ \frac{ig}{2}\int d^4x\,\Delta_{x1}\delta(x-x_3)\int d^4y\,\Delta_{2y}\Delta_{yy} + O(g^3)$$

apply δ fn. & relabel $y \to x$ just so it looks nicer & simplify

$$= ig\int d^4x\,\Delta_{x1}\Delta_{x3}\Delta_{x2} + \frac{ig}{2}\int d^4x\,\Delta_{xx}(\Delta_{x1}\Delta_{23}+\Delta_{12}\Delta_{3x}+\Delta_{13}\Delta_{2x}) + O(g^3)$$

½ comes from symmetry factor 2

picture! using rules overleaf

$= ig$  $+ O(g^2)$

(any odd pt. fn. is zero in free theory!)

Ex 4. 2pt. function

Free thy. $:\phi_1\phi_2: + \Delta :\phi_2: + \ldots$

(?) too small

seeing w/ Wick's thm odd no. of fields always leader annihilate vacuum (?)

(have $\mathcal{L}_{int}$ $\frac{g}{2}\int d^4x\,\Delta_{1x}\langle\phi_x^2\phi_2\rangle$ replacing 3pt. fn w/ free thy. just get 0)

can i stop perturbation and just replace by free thy? (no), will just get 0. so stop here, to leading order get one but to order g^2 1st order, first one just get zero. so need to keep applying until i get 1st contribution something interesting i.e. non zero (non-trivial contribution)

$$\langle\phi_1\phi_2\rangle = i\int d^4x\,\Delta_{1x}\left(\tfrac{g}{2}\langle\phi_x^2\phi_2\rangle - i\delta(x-x_2)\right) \quad (1)+(2)+\text{(SchwD)}$$

(keep applying same trick 3 fn etc.)

$$= \Delta_{12} + \frac{ig}{2}\int d^4x\int d^4y\,\delta(y-x_2)\Delta_{1x}\langle\phi_x^2\phi_y\rangle \quad (1)+(2)+\text{(SchwD)}$$

$$= \Delta_{12} + \frac{ig}{2}\int d^4x\,d^4y\; i\Delta_{1x}\Delta_{2y}\Big(\tfrac{g}{2}\underbrace{\langle\phi_x^2\phi_y^2\rangle}_{\text{(Ex1)}} - i2\delta(x-y)\underbrace{\langle\phi_x\rangle}_{\text{(Ex2)}}\Big)$$

4pt. fn. i this is 4pt fn. know from ex1

know this to order g (?)

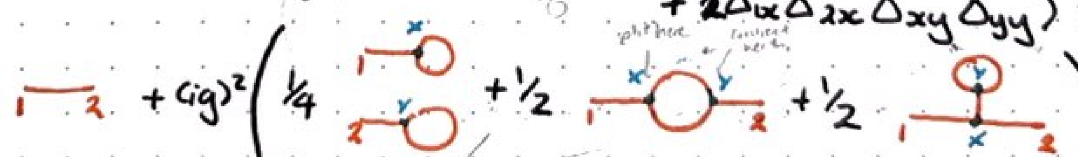
now ask: if i replace w/ free thy. am i going to get an interesting ans?

$$= \Delta_{12} + \tfrac{1}{4}(ig)^2\int d^4x\,d^4y\,(\Delta_{1x}\Delta_{2y}\Delta_{xx}\Delta_{yy} + 2\Delta_{1x}\Delta_{2y}\Delta_{xy}\Delta_{xy}$$
$$+ 2\Delta_{1x}\Delta_{2x}\Delta_{xy}\Delta_{yy}) + O(g^3)$$

symmetry factor 2 × 2 diagrams

note to self: each has 2 vertices, ie $\int d^4x\int d^4y$. middle diagram was slightly less intuitive to draw so check this

split here, connect here

$=$  $+ O(g^3)$

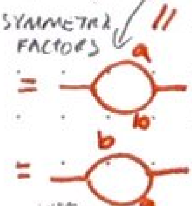
see —O 1x so symm 2

"x or y" doesn't matter, only what it connects to matters!

note: $\Delta_F(x-y) = \Delta_F(y-x)$
$\Delta_{xy} = \Delta_{yx}$
x — y $=$ y — x

feeding 1/2

SYMMETRY FACTORS



from loops between exchange between the two contractions

(? same thing as $\mathcal{L}_{int}$?) shouldn't count them as indep. contractions.

note examples & pics try to illustrate how many ways i can combine/draw according to the rules:

① s.d. $\mathcal{L}_{int}$ term tells me one field can go into 2 so (?)

② can also contract w/ δ

examples demonstrate all possible ways i can split and contract (up to some order)

"combinatorial structure"

c 30

FEYNMAN DIAGRAMS

what we learn from Schw D: every time we have a correlator we will keep on getting terms of we repeat this identity over & over again ⇒ a correlation fn. will always have this property that part of it want to split & part ~~are~~ want to contract

Schw. D

$$\langle \phi_1 \dots \phi_n \rangle = i\int d^4x\, \Delta_{1x} \Big(\langle \mathcal{L}'_{int}[\phi_x]\phi_2 \dots \phi_n \rangle - i \sum_j \delta(x_i - x_j) \langle \phi_2 \dots \phi_{j-1}\phi_{j+1} \dots \phi_n \rangle \Big)$$

Field proliferates

can get this type of connection just from classical thing

class. F.T. ⟶ contraction ⇒ quantum, creates loops

can get more contraction & apply S_D again

but contraction thing allows us to contract & create loops! only get loops in quantum (not classical thing)

$\langle \phi_1 \dots \phi_n \rangle$ via Feynman Diagrams

shortcut instead of applying these rules over and over again. The result is still always check these rules give right ans. like we have done already (will ask to check in exam!)

1) Start with x_i external points $(i = 1, \dots, n)$
   Draw a line from each point x_i ——

2) A line can ~~be~~ either:
   - contract (join) an existing line. This gives $D_F(x_1 - x_1)$

   or

   - split, where the split gives a new vertex. The coefficient will be $i\lambda_n$ for $\mathcal{L}_{int} = \frac{\lambda_n}{n!}\phi^n$, number of lines depends on $\mathcal{L}_{int}$

   (splitting means, you to higher order or the couplings)

   the interacting $\mathcal{L}_{int}$

3) At any given order in $i\lambda_n$, the result is the sum of all diagrams with all lines contracted and integrated over vertices

   i.e. no lines outstanding to worry

   what is going on with factors e.g. ½, little numbers in front of diagrams ⇒ what is going on with those combinatorial problem, issues every way we can split & join is equivalent → overcounting equiv. combinations

   sometimes there are ways we can split & join are equivalent!

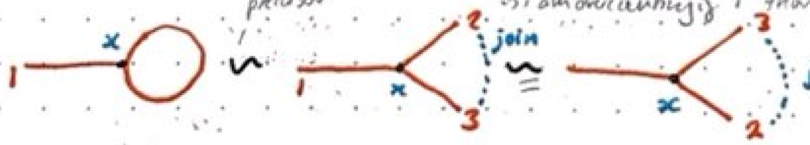
4) Warnings: symmetry factors

Symm factors: why?

these 2 are clearly equivalent ⇒ i am overcounting if i thought there were 2 diff. possibilities

Ex.

precursor



the same!

if i apply these rules i am technically overcounting → i need to be mindful!

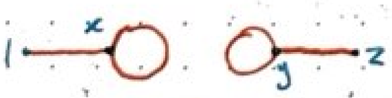
if i encounter this diagram, I know I have

⇒ symmetry factor 2

special to bosonic particles because we can't distinguish them

added from lecture video

Ex.



⇒ symmetry factor $2 \times 2 = 4$

Ex.

these equiv. configs produce the same diagram, so symmetry factor 2 (⇒ getting ½ for diagram)

⇒ symmetry factor 2

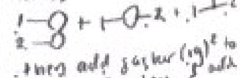
note: basically how many ways can i draw graph if all use ends two labelled by x_i's (each x_i only appears once) for a given no. of vertices

e.g. for 2pt fn.

→ how many of these graphs can i make with zero vertices? just one $1 — 2$

→ how many with one vertex? none

→ how many with 2 vertices? none:



then add factor $(ig)^2$ to all since 2 vertices & add symmetry factors

can think your way similarly though 3pt. example

Lecture 12

6.11.24

SCALAR YUKAWA THEORY

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$

$$\mathcal{L}_0 = \underbrace{\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2}_{\text{real scalar}} + \underbrace{\partial_\mu\psi^\dagger\partial^\mu\psi - M^2\psi^\dagger\psi}_{\text{complex scalar}}$$

$$\mathcal{L}_{int} = \underbrace{-g\psi^\dagger\psi\phi}_{\text{cubic interaction}} \quad \text{with } g \ll M, m$$

Goal: Infer Feynman rules by evaluating S-matrix.

i) Free theory

ii) Interacting — Schw. Dyson — LSZ

i) $\boxed{\text{FREE THEORY}}$ ($\mathcal{L}_0$)

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{-ipx} + a_p^\dagger e^{ipx}\right) \qquad \omega_p \equiv \sqrt{\vec{p}^{\,2}+m^2}$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\tilde{\omega}_p}}\left(b_p e^{-ipx} + c_p^\dagger e^{ipx}\right) \qquad \tilde{\omega}_p \equiv \sqrt{\vec{p}^{\,2}+M^2}$$

$$\phi(x) \longrightarrow [\phi(\vec{x},t), \underbrace{\partial_t\phi(\vec{y},t)}_{\Pi_\phi}] = i\delta(\vec{x}-\vec{y})$$

$$\psi(x) \longrightarrow [\psi(\vec{x},t), \Pi_\psi(\vec{y},t)] = i\delta(\vec{x}-\vec{y}) \rightarrow \Pi_\psi = \partial_t\psi^\dagger$$

$$[\psi^\dagger(\vec{x},t), \Pi_{\psi^\dagger}(\vec{y},t)] = i\delta(\vec{x}-\vec{y}) \rightarrow \Pi_{\psi^\dagger} = \partial_t\psi$$

check $\Rightarrow [a_p, a_{p'}^\dagger] = (2\pi)^3\delta(\vec{p}-\vec{p}')$ $\qquad [c_p, c_{p'}^\dagger] = (2\pi)^3\delta(\vec{p}-\vec{p}')$

$\Rightarrow [b_p, b_{p'}^\dagger] = (2\pi)^3\delta(\vec{p}-\vec{p}')$

this is why I chose to call it $c_p^\dagger$, otherwise I would get minus sign in commutator, so this just looks nice

Hamiltonian

$$H_0 = \int\frac{d^3p}{(2\pi)^3}\left(\omega_p a_p^\dagger a_p + \tilde{\omega}_p b_p^\dagger b_p + \tilde{\omega}_p c_p^\dagger c_p\right)$$

we also have the charge associated to $\psi \to e^{i\alpha}\psi$

$$Q = i\int d^3x : (\dot{\psi}^\dagger\psi - \psi^\dagger\dot{\psi}):$$

$$= \int\frac{d^3p}{(2\pi)^3}\left(c_p^\dagger c_p - b_p^\dagger b_p\right)$$

$$[H_0, Q] = 0$$

commute so can use Q label for states

how study the given space:

Fock Space

a) Vacuum state: $|0\rangle$

$$a_p|0\rangle = b_p|0\rangle = c_p|0\rangle = 0 \quad \Rightarrow \quad H_0|0\rangle = 0$$

define vacuum state as: state annihilated by a_p, b_p, c_p (which implies $H_0|0\rangle = 0$) also

Margin notes:

we want to derive Feynman rules for this theory (basically a toy model)

this theory: has one free scalar & one free complex scalar; it is a toy model for quantum electrodynamics; i.e. all fields within our knowledge

combining 2 theories i.e. 2 types of field

so compatible w/ perturbation theory

cubic g $\phi\psi$ fields

organise our thoughts: always just make sure we understand the free theory

i.e. then get to feynman rules

note: no couplings at this stage? just looking at free theory

real: general soln to KG eqn

complex:

($\psi^\dagger(x)$ is just conjugate of this)

in principle 2 arbitrary complex coefficients

general soln to KG eqn

(equal time commutators)

canonical field

imposing canonical quantisation

(note: also imposing e.g. $[\phi,\phi]=0$ etc.)

just the free theory: we already have eq. of motion, do canonical quantisation then find Hamiltonian. want to know what our state space is: free, assume it is the same as our free (Fock space), derive H like we did for free theory

next step: compute Hamiltonian

free part of H: not deriving again

note to self: to do this, write $a_p, a_p^\dagger$ terms of ϕ by just considering $\dot\phi - i\omega_p\phi$):
$\sqrt{2\omega_p}\,a_p^\dagger = -i\int d^3x\, e^{-ipx}(\dot\phi + i\omega_p\phi)$
$\sqrt{2\omega_p}\,a_p = i\int d^3x\, e^{ipx}(\dot\phi - i\omega_p\phi)$
then plug into $[a_p, a_p^\dagger]$ and use $[\phi,\dot\phi]$ to show

see p136 for derivation, remember we take normal order!

(I think) to derive: $\mathcal{H} = \Pi_i\partial_t\phi_i - \mathcal{L}$; get 3 terms for each of $\phi, \psi, \psi^\dagger$; then $H = \int d^3x\,\mathcal{H}$; plug in expansions of $\phi, \psi, \psi^\dagger$ & use commutation relations to get ans.; REMEMBER: then normal order to get ans!

i.e. charge b/c now I can distinguish between b and c

to show commutation relations of fields (incl. the relations not explicitly shown above but also imposed) imply $[c_p, c_p^\dagger]$ & $[b_p, b_p^\dagger]$ can write down eq. for b_p & $c_p^\dagger$ straight from what we found for $a, a^\dagger$:
$\sqrt{2\tilde\omega_p}\,b_p = i\int d^3x\, e^{ipx}(\dot\psi - i\tilde\omega_p\psi)$
$\sqrt{2\tilde\omega_p}\,c_p^\dagger = -i\int d^3x\, e^{-ipx}(\dot\psi + i\tilde\omega_p\psi)$
then find $b_p^\dagger, c_p$ by taking conjugates & remembering fields are complex now!
$\sqrt{2\tilde\omega_p}\,b_p^\dagger = -i\int d^3x\, e^{-ipx}(\dot\psi^\dagger + i\tilde\omega_p\psi^\dagger)$
$\sqrt{2\tilde\omega_p}\,c_p = i\int d^3x\, e^{ipx}(\dot\psi^\dagger - i\tilde\omega_p\psi^\dagger)$
then plug into $[c_p, c_p^\dagger]$ etc and solve using $[\psi, \Pi_\psi]$ etc.! :)

b) One-particle states

→ "meson" states $|\phi\rangle$: act $a^\dagger$

$$|\phi\rangle = a_p^\dagger|0\rangle \rightarrow H|\phi\rangle = \omega_p|\phi\rangle$$

$$Q|\phi\rangle = 0$$

neutral charge

→ "nucleon" states $|\psi\rangle$

$$|\psi\rangle = c_p^\dagger|0\rangle \longrightarrow H|\psi\rangle = \hat{\omega}_p|\psi\rangle$$

$$Q|\psi\rangle = +|\psi\rangle$$

pos charge

→ "antinucleon" states $|\psi^\dagger\rangle$

$$|\psi^\dagger\rangle = b_p^\dagger|0\rangle \longrightarrow H|\psi^\dagger\rangle = \hat{\omega}_p|\psi^\dagger\rangle$$

$$Q|\psi^\dagger\rangle = -|\psi^\dagger\rangle$$

neg charge

(fock space is defined by all possible ways to act with these creation operators)

c) Multi-particle states (choose any combination you want)

Feynman Propagator

$$\Delta_F(x_1-x_2) = \langle\phi_1\phi_2\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2+m^2+i\epsilon}e^{-ip(x_1-x_2)} \equiv \Delta_{12}$$

shorthand

$$\Delta_F^\psi(x_1-x_2) = \langle\Omega|T\psi(x_1)\psi^\dagger(x_2)|\Omega\rangle$$

$$= \int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2+M^2+i\epsilon}e^{-ip(x_1-x_2)} \equiv \hat{\Delta}_{12}$$

shorthand

same as scalar field except for M instead of m

note:

1) $\langle\psi_1\psi_2\rangle = \langle\psi_1^\dagger\psi_2^\dagger\rangle = 0$

this is why I don't state it as only the propagator

2) to see why $\langle\psi_1\psi_2^\dagger\rangle \neq 0$,

$$(*)\quad \sqrt{2\hat{\omega}_p}\, b_p^\dagger = -i\int d^3x\, e^{-ipx}\overleftrightarrow{\partial_t}\psi^\dagger$$

$$\sqrt{2\hat{\omega}_p}\, b_p = i\int d^3x\, e^{ipx}\overleftrightarrow{\partial_t}\psi$$

$$\sqrt{2\hat{\omega}_p}\, c_p^\dagger = -i\int d^3x\, e^{ipx}\overleftrightarrow{\partial_t}\psi$$

$$\sqrt{2\hat{\omega}_p}\, c_p = i\int d^3x\, e^{-ipx}\overleftrightarrow{\partial_t}\psi^\dagger$$

to create then annihilate: create w/ $\psi^\dagger$, destroy w/ ψ

opposite for c: create w/ ψ, destroy w/ $\psi^\dagger$

ii) INTERACTING THEORY

(EL eqs: $\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} - \frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial\mathcal{L}_{int}}{\partial\phi_a} = 0$ gives 3 EOM, one for each field, includes interaction term)

⇒ Schwinger-Dyson:

$$(\Box+m^2)\phi - (-g\psi^\dagger\psi) = 0$$

$$(\Box+M^2)\psi - (-g\phi\psi) = 0$$

$$(\Box+M^2)\psi^\dagger - (-g\phi\psi^\dagger) = 0$$

$\mathcal{L}_{int}$

By using same assumptions as before

(1) at any given time, interacting fields match Hilbert space of free theory

(2) fields still comply w/ EL eqs

(2 species $\mathcal{g}$ield: 3 SD eq.) → 3 Schwinger Dyson equations: one for ϕ, ψ, ψ^+

$$(\Box_x + m^2)\langle \phi_x \phi_1 \ldots \phi_n \psi_1 \ldots \psi_m \psi_1^+ \ldots \psi_r^+ \rangle$$

use and generalise operator eqns what happens.

$$= -g\langle \psi_x \psi_x^+ \phi_1 \ldots \phi_n \psi_1 \ldots \psi_m \psi_1^+ \ldots \psi_r^+ \rangle$$

get this term & contractions

$$-i \sum_{j=1}^{n} \delta(x-x_j)\langle \phi_1 \ldots \phi_{j-1}\phi_{j+1} \ldots \phi_n \psi_1 \ldots \psi_m \psi_1^+ \ldots \psi_r^+ \rangle$$

!! think !! what are you varying to get the EOM on previous page? → 2nd EOM vary ψ^+ to get $\Box + M$ acting on ψ. watch out for this: see EOM for intuition. ② why do we have $\frac{\delta}{\delta\psi^+}$?) → & note sum upto index. Things are correct in final term. get nontrivial contraction between ψ & ψ^+ $\hat{\Delta}_{12}$?

$$(\Box_x + M^2)\langle \phi_1 \ldots \phi_n \psi_x \psi_1 \ldots \psi_m \psi_1^+ \ldots \psi_r^+ \rangle$$

(you should check)

$$= -g\langle \phi_1 \ldots \phi_n \phi_x \psi_x \psi_1 \ldots \psi_m \psi_1^+ \ldots \psi_r^+ \rangle$$

$$-i \sum_{j=1}^{p \text{ (should be } r \text{ ours)}} \langle \phi_1 \ldots \phi_n \psi_1 \ldots \psi_m \psi_1^+ \ldots \psi_{j-1}^+ \psi_{j+1}^+ \ldots \psi_p^+ \rangle \delta^4(x-x_j)$$

(can analogously do w/ other field)

(now not going to use SD every time because tedious so use feynman rules) (skipping steps ① $\mathcal{g}$ ② SD) to save time

To infer Feynman diagrams, look at 3pt. function

(also see ex sheet 2)

(test: compute this correlator in free theory)

$$\langle \phi_1 \psi_2^+ \psi_3 \rangle = -ig\int d^4x\, \Delta_{1x}\langle \psi_x \psi_x^+ \psi_2^+ \psi_3 \rangle$$

(no approximation made yet. identity. no more contractions I can do)

either use Wick's theorem or S.D free theory without interaction

$$= -ig\int d^4x\, \Delta_{1x}\left(\hat{\Delta}_{xx}\hat{\Delta}_{23} + \hat{\Delta}_{x2}\hat{\Delta}_{3x}\right) + O(g^2)$$

replace in free theory

interacting pair grouping & the rest

only 2 contractions I can do for SD w/ 3 fields

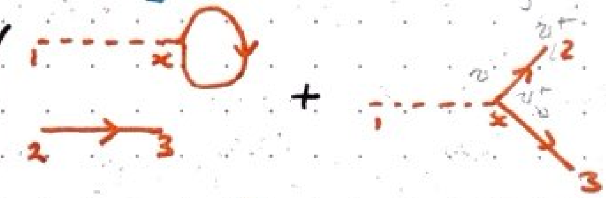
[new symbol: Δ_{12} = - - - - - -
$\hat{\Delta}_{12}$ = ⟶]

(order important for $\hat{\Delta}$?) (you can do ... this order e.g. ... to compute but explain it!)

↳ arrow convention $\hat{\Delta}_{12}\,\psi_1\,\psi_2^+$ whereas $\hat{\Delta}_{21}$ is $\psi_2\,\psi_1^+$. remind you are going between diff things

(derive this on sheet 2)

$$= -ig\Bigg($$



$$\Bigg) + \ldots$$

note to self using Wick's $\langle \psi_x \psi_x^+ \psi_2^+ \psi_3 \rangle = \langle \psi_x \psi_x^+ \rangle \langle \psi_2^+ \psi_3 \rangle + \langle \psi_x \psi_2^+ \rangle \langle \psi_x^+ \psi_3 \rangle + \langle \psi_3 \psi_x \rangle \langle \psi_x^+ \psi_2^+ \rangle$ $= \hat{\Delta}_{xx}\hat{\Delta}_{32} + \hat{\Delta}_{x2}\hat{\Delta}_{x2}$ since $\langle \psi_x^+ \psi_2^+ \rangle = 0$. Or compute theory ($g = 0$) w/ SD get same ans as before (check yourself)

(check arrows have to flow) HANDY

(invisible overlap between initial & final states at $\pm\infty$)

Scattering:

S matrix

We want to evaluate $\langle$final, $+\infty$ | initial, $-\infty\rangle = \langle f|S|i\rangle$.
Our assumptions are as before:
  ⇒ asymptotic states @ $\pm\infty$ match free theory.
Hence to evaluate S, we need, e.g., $\sqrt{2\tilde{\omega}_p}$

$$\sqrt{2\tilde{\omega}_p}\,(b_p^+(\infty) - b_p^+(-\infty)) = -i\int d^4x\, e^{-ipx}(\Box_x + M^2)\psi^+(x)$$

need formulas that tell you e.g. what is relationships at $\pm\infty$

← we see the appropriate thing to put in to create b^+ at $-\infty$ e.g. it will be attached to a ψ^+ with a -ve exponential coming because of (*)

Similar for $c_p, a_p, b_p, a_p^+, c_p^+$

pick an example to see what we mean by this ↳ on Friday!

note to self can prove this (long way) as we did on p.170 using eqn. for a_p, b_p, c_p etc. in terms of ϕ, on prev page. quick way can see from these eq immediately e.g. for b_p^+ from ⊛ we see this eq. comes from ψ^+ attached to a -ve exponential and e.g. c_p^+ will come from ψ attached to a +ve exponential

YES not sure this is true but eqns on prev page are wrong for c ?!

$$\Rightarrow \sqrt{2\tilde{\omega}_p}\,(c_p^+(\infty) - c_p^+(-\infty)) = +i\int d^4x\, e^{ipx}(\Box + M^2)\psi(x)$$ (I think) ?

(quicker than proving each one individually)

SUMMARY OF USEFUL RELATIONSHIPS:

$$\sqrt{2\omega_p}\,(a_p^+(+\infty) - a_p^+(-\infty)) = -i\int d^4x\, e^{-ipx}(\Box + m^2)\phi(x)$$
$$\sqrt{2\tilde{\omega}_p}\,(b_p^+(+\infty) - b_p^+(-\infty)) = -i\int d^4x\, e^{-ipx}(\Box + M^2)\psi^+(x)$$
$$\sqrt{2\tilde{\omega}_p}\,(c_p^+(+\infty) - c_p^+(-\infty)) = -i\int d^4x\, e^{-ipx}(\Box + M^2)\psi(x)$$

(then find others by complex conjugate (i think proof only works for + since then derivs cancel out so have to find a_p, b_p, c_p by taking conjugate ?)

e.g. $\sqrt{2\omega_p}\,(a_p(+\infty) - a_p(-\infty)) = i\int d^4x\, e^{ipx}(\Box + m^2)\phi(x)$

<u>Lecture 13</u> <u>8.11.24</u>

Recap: do this w/ aim of finding out about scattering

SCALAR YUKAWA THEORY cont.

important: propagator for complex scalar is time ordered product of $\psi, \psi^\dagger$

$\boxed{\text{recap:}}$
- $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$, $\mathcal{L}_0 = \partial_\mu \psi^\dagger \partial^\mu \psi - M^2 \psi^\dagger \psi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$

  $\mathcal{L}_{int} = -g\psi^\dagger \psi \phi$
- $\Delta_{ij} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 - i\varepsilon} e^{-ip(x_i - x_j)}$, $\hat{\Delta}_{ij} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - M^2 - i\varepsilon} e^{-ip(x_i - x_j)} = \langle \psi_i \psi_j^\dagger \rangle$

  meson, neutral, $Q=0$ — antinucleon $Q=1$ — nucleon, $Q=-1$
- $a_p^\dagger = |\phi\rangle$, $b_p^\dagger : |\psi^\dagger\rangle \rightsquigarrow i\int d^3x e^{-ipx} \overleftrightarrow{\partial_t} \psi^\dagger$, $c_p^\dagger : |\psi\rangle \rightsquigarrow \int d^3x e^{-ipx} \overleftrightarrow{\partial_t} \psi$

more on choice of notation

species of particle

denotes ? $\rightarrow$ (lesson from free theory: creating a, b introduces $\psi^\dagger$ in LSZ, creating c introduces ψ)

Scattering:

(remember normalisation) $\sqrt{2\hat{\omega}_{p_1}}\sqrt{2\hat{\omega}_{p_2}}$, $\sqrt{2\hat{\omega}_{p_3}}\sqrt{2\hat{\omega}_{p_4}}$

(I) Nucleon scatterings $|\psi\psi\rangle \to |\psi\psi\rangle$

initial state: $\lim_{t\to -\infty} c_{p_1}^\dagger(t) c_{p_2}^\dagger(t)|\Omega\rangle$: initial mom p_1, p_2 ← specifying species (c) and momenta

final state: $\lim_{t\to\infty} c_{p_3}^\dagger(t) c_{p_4}^\dagger(t)|\Omega\rangle$: final mom p_3, p_4

note to self: can see by plugging in $c^\dagger$, c like before

overlap $\langle \text{final}, +\infty | \text{initial}, -\infty \rangle = \langle f|S|i\rangle$

shorthand

$$(*) = i^4 \prod_{j=1}^{4} \int d^4x_j \underbrace{e^{-ip_1x_1}(\Box_1 + M^2) e^{-ip_2x_2}(\Box_2 + M^2)}_{\text{ingoing states}} \overbrace{e^{ip_3x_3}(\Box_3 + M^2) e^{ip_4x_4}(\Box_4 + M^2)}^{\text{outgoing states}}$$

$$\cdot \langle \Omega | T \underbrace{\psi(x_1)\psi(x_2)}_{\text{because of } c^\dagger} \underbrace{\psi^\dagger(x_3)\psi^\dagger(x_4)}_{\text{because of } c} |\Omega\rangle$$

ingoing states: −ve exponent, see eq above

"encourage you to use Sch. D to verify" — is non-trivial repetition.

$\rightarrow$ we need $\langle \psi_1 \psi_2 \psi_3^\dagger \psi_4^\dagger \rangle$, we'll use Feyn. diagrams to leading order g.

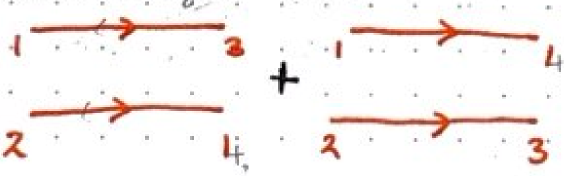
ψ talks to $\psi^\dagger$ but not to itself

(direction of arrow here not important but is important here) — does matter but choose convention? idk

textbooks differ convention for arrows

Order g^0:

(free theory diagrams at order g^0)



Order g: nothing

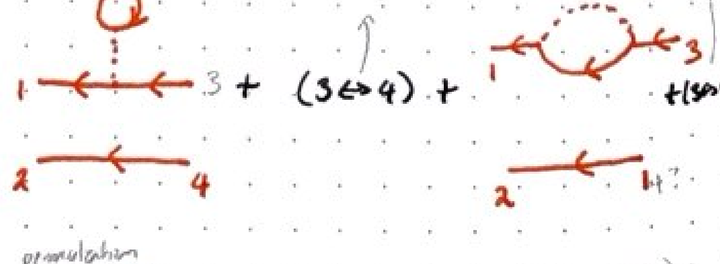
coupling term: need to join & split things.

+ possibility of this one

Order g^2: a) disconnected

(always 4 free points)

permutations exchange 3↔4



b) connected

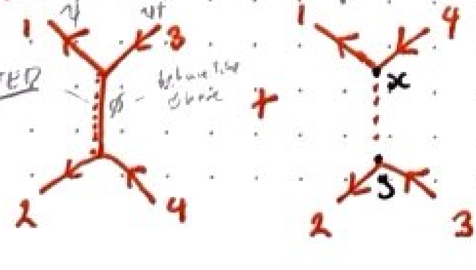
don't have to lift pen to draw

before the ψ here, where the $\psi^\dagger$

(reminder: rule arrows need to flow, they tell us about charge conservation)

DOTTED: ϕ — between the two

permutation



keep increasing power of g: more & more diagrams

let's only focus on connected, particles interacting w/ each other (for scattering purposes) what the pictures tell you in equations:

$$\langle \psi_1 \psi_2 \psi_3^\dagger \psi_4^\dagger \rangle_c = (-ig)^2 \int d^4x\, d^4y \left(\hat{\Delta}_{1x} \hat{\Delta}_{x3} \Delta_{xy} \hat{\Delta}_{2y} \hat{\Delta}_{y4} + \hat{\Delta}_{1x} \hat{\Delta}_{x4} \Delta_{xy} \hat{\Delta}_{2y} \hat{\Delta}_{y3} \right) + O(g^3)$$

only connected part

→ Back to (*) replace $\langle \psi_1 \psi_2 \psi_3^\dagger \psi_4^\dagger \rangle_c$ as last part

have $(\Box + m^2)$ terms acting on the correlator, look for parts that depend on 4

consider term

$$\text{e.g. } (\Box_4 + M^2) \langle \psi_1 \psi_2 \psi_3^\dagger \psi_4^\dagger \rangle \sim \int (\Box_4 + M^2)(\hat{\Delta}_{y4} + \hat{\Delta}_{x4} \cdots)$$

originally external purpose: these boxes make correlators delta functions

after using δ fn. on every term → get rid of $\hat{\Delta}$ terms, just exponentials

note to self:
- replace $(\Box_4 + M^2)\langle \psi_1 \cdots \psi_4^\dagger \rangle$ etc. w/ δ fn identity
- integral over $\int d^4x_1 d^4x_2 d^4x_3$ etc. replace all x_i w/ x or y according to δ fns gives this

$$\langle f|S|i\rangle = i^4(-ig)^2 \int d^4x \int d^4y \Big((-ie^{-ip_1x})(-ie^{-ip_2x}) \Delta_{xy} (-ie^{ip_3x})(-ie^{ip_4x})$$
$$+ (-ie^{-ip_1x})(-ie^{-ip_2y}) \Delta_{xy} (-ie^{ip_3y})(-ie^{ip_4x}) \Big) + O(g^3)$$

→ now use our expression for Δ_{ij} (recap), write down one term

collect terms that depend on x

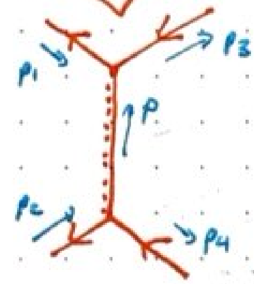
$$\text{e.g.} = \cancel{i^4}(-ig^2) \int d^4x \int d^4y \int \frac{d^4p}{(2\pi)} \frac{i}{p^2 - m^2 - i\epsilon} \left(\left(e^{-ix(p_1 + p - p_3)} e^{-iy(p_2 - p - p_4)} \right) + (3 \leftrightarrow 4) \right)$$

all i's cancel

permutations

$$\langle f|S|i\rangle = \underbrace{(-ig)^2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} (2\pi)^4 \delta(p_1 + p - p_3)(2\pi)^4 \delta(p_2 - p - p_4)} + (3 \leftrightarrow 4)$$

coming from this



note to self: integrate over 1st delta fn replaces p in 2nd delta fn as well as in denominator

(what?)

if arrows of momentum point towards vertex they have + sign.

propagator in the middle

$$= (-ig)^2 (2\pi)^4 \underbrace{\delta(p_1 + p_2 - p_3 - p_4)}_{\text{conservation of 4-mom.}} \left(\frac{i}{(p_3 - p_1)^2 - m^2} + \frac{i}{(p_4 - p_1)^2 \pm m^2} \right) + O(g^3)$$

↓ Energy + momentum

"if you are coming in my path, I'll hit you"

(from symmetries)

↑ we have computed our first scattering process in QFT! ♡

classical: particles moving, we deduce there is a force attracting them; this is the grownup version, what does it mean that 2 particles are attracting? they are exchanging something. in low energy limit this will give $1/r$. Yukawa! amazing

next lecture: feynman rules & more examples

Comments: comment on this result

1) the nucleons are external particles and hence on-shell but the meson is an internal (virtual) particle & hence off-shell.

2) we have computed a 'tree level' answer i.e. diagrams with no loops

note to self: dont worry i think i got confused w/ some integral w/ $\langle \psi_x \psi_x^\dagger \psi_y \psi_y^\dagger \psi_3^\dagger \psi_4^\dagger \rangle$ there are $6C2 \cdot 4C2 \cdot 2C2$ ways of splitting into 3 pairs then divide by 3! we dont care about ordering pairs = 15 ways to split into 3 propagators but connected diagram corresponds to terms w/ $\langle \phi\phi \rangle_A \langle \psi^\dagger \psi \rangle$ of which there are only two! (remember $\langle \psi^\dagger \psi^\dagger \rangle = \langle \psi \psi \rangle = 0$) the ψ_x is integrated over all possibilities so this is not necessarily ψ could be $\psi, \psi^\dagger, \phi$ & it takes this into account; could from the disconnected graphs? not shown this

Lecture 14 — 11.11.24

(in order to see we are removing stuff w/ free, replacing propagators, theory propagator, (can't get to desired correction term))

Note: $\Delta_{xy} = \langle 0 | T \phi^0(x) \phi^0(y) | 0 \rangle$

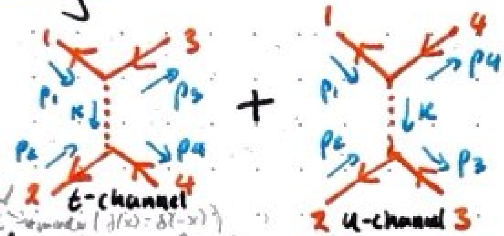
$\hat{\Delta}_{xy} = \langle 0 | T \psi^0(x) \psi^{\dagger 0}(y) | 0 \rangle$ } Feynman Propagators

(stopping w/ notation last time)

(clarify nomenclature: Feynman prop results from free theory)

Recap: Nucleon Scattering $\psi\psi \to \psi\psi$

(add momenta)

$\langle f | S | i \rangle_c = \langle f | S - \mathbb{1} | i \rangle =$ 

(not agreed rules? remember $(f(x) = \delta(-x))$)

(since we integrate out all possibilities: direction of k doesn't matter (internal) but direction of p_i etc. has to be consistent across diagrams)

$$= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \left((2\pi)^4 \delta(k - p_1 + p_3)(2\pi)^4 \delta(k - p_2 + p_4) + (3 \leftrightarrow 4) \right)$$

$$= (-ig)^2 i (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \left(\frac{1}{(p_1 - p_3)^2 - m^2} + \frac{1}{(p_1 - p_4)^2 - m^2} \right)$$

(Important pattern: consistently LSZ external particles removed from correlator/amputated parts, then internal particles, interaction points give δ fn that tells me E conserve momentum at each vertex)

(you can keep writing $+i\epsilon$ if you want but basically take limit $\epsilon \to 0$) (will find out why later)

(internal i.e. Δ_{xy}? gives δ fn w/ conservation of momentum)

FEYNMAN RULES

(write for this particular theory)

(Scalar Yukawa theory) (caveat: compute connected contributions usually denoted c subscript)

→ connected S-matrix

1) Draw the Feynmann diagrams for relevant process, use

(arrow just convention) (diff convention in notes)

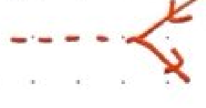
- - - - - - mesons (ϕ) $\phi \to a^\dagger$

⟵ nucleons (ψ) $\psi \to c^\dagger$

⟶ antinucleons ($\psi^\dagger$) $\psi^\dagger \to b^\dagger$

(1) Def. theory 1st step: what are objects we have? note to self: (e.g. ψs, ϕs; 3 →, 1 ← (swaps for outgoing!!!) think))

lines split via



(2nd step: what are the rules for splitting?)

2) For each diagram, the contribution to S-matrix is

i) assign momenta to each line

(blue (momentum) clarity, nothing to do w/ object + 'phase' arrows)

- - - - $\xrightarrow{p_i}$ ⟶ $\xrightarrow{k_i}$ ⟵ $\xrightarrow{q_i}$

(doesn't matter which direction, as long as you are consistent in external lines, internal lines do what you want)

ii) Internal line

(replace internal lines by this)

$$- - - - - \; (\vec{p}) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 - i\epsilon}$$

(internal lines get a factor of this ← for meson)

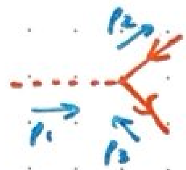
$$\longrightarrow \; (\vec{k}) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - M^2 - i\epsilon}$$

(← for nucleon/anti-nucleon)

iii) External lines, do nothing

iv) To each vertex write the factor

pick convention: arrow towards vertex is +



$$= (-ig)(2\pi)^4\delta(p_1 - p_2 + p_3)$$

3) Sum over all diagrams
Integrate over undetermined momenta.

Comments on Nucleon scattering

will also apply more broadly

1) External particles are on-shell $p_i^2 = M^2$
However, internal particles are not! $k^2 \neq m^2$
↳ virtual particles (call them)

comment on jargon

2) What we computed is a tree-level diagram.
$\equiv$ absence of loops
(no loops)

Tree-level diagrams are the leading connected contributions and emphasis of this course.

ex. 2 Follow rules

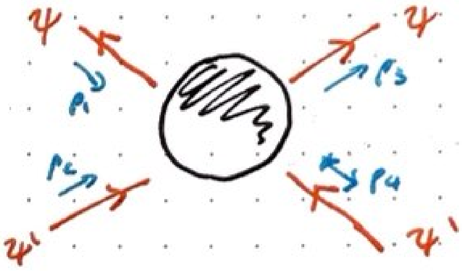
(II) Nucleon-anti-nucleon scattering: $\psi^+\psi \to \psi^+\psi$

(note to self, I think here $\psi_1, \psi_2^+, \psi_3, \psi_4$ are contrary to pic on the left)

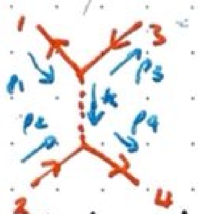
at leading order (+ connected)

what happens in the middle as we compute S-matrix? Follow rules: draw it out (step 1)

order 0 F.T. order 1 nothing as before, so what can I draw order g^2? possibilities

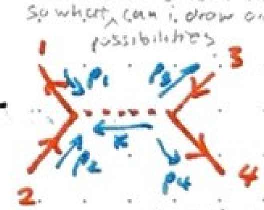


$= \langle f|S|i\rangle_c =$



t-channel

$+$



s-channel

$+\, O(g^3)$

can't exchange 3 & 4 this time!

step 2) label momenta. use same convention for all diagrams

2 iv) write equations:

$$= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2-i\varepsilon}(-ig)(2\pi)^4\delta(p_1-k-p_3)(-ig)(2\pi)^4\delta(k+p_2-p_4)$$

2nd contrib

$$+ \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2-i\varepsilon}(-ig)(2\pi)^4\delta(p_1+p_2+k)(-ig)(2\pi)^4\delta(-k-p_3-p_4)$$

$$= (-ig)^2(2\pi)^4\delta(p_1+p_2-p_3-p_4)\left(\frac{i}{(p_1-p_3)^2-m^2-i\varepsilon} + \frac{i}{(p_1+p_2)^2-m^2-i\varepsilon}\right) + O(g^3)$$

ask yourself, are there values for which this will blow up? this one won't, but this one will for some values! ↳ investigate 2nd diagram

// computed this much faster than last time

↓

The second diagram has an interesting mom. dependence:

center of mass ref.

note to self: ($p_1^2 = E_1^2 - \vec{p}_1^2 = M^2$)
- $(p_1 + p_2)^2 = (E_1+E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$ (w/ cos special rel. somehow)
- assume particles have same mass $\Rightarrow E_1 = E_2$ and work in CoM frame $\Rightarrow \vec{p}_2 = -\vec{p}_1$
- so $(p_1+p_2)^2 = 4E^2 = 4(M^2 + \vec{p}_1^{\,2})$

momenta! p_1 p_2

$$(\vec{p}_1 + \vec{p}_2)^2 = 4(M^2 + \vec{p}_1^{\,2}) \quad \text{in CM ref.}$$

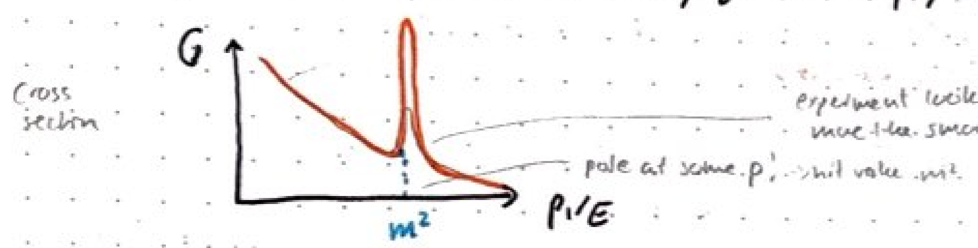
CM

then term in denominator, $(p_1+p_2)^2 - m^2 = 4(M^2 + \vec{p}_1^{\,2}) - m^2$

when will this be zero?

pole in scattering process $\Rightarrow$ discover particles

- * If $m < 2M$, never have $(p_1+p_2)^2 \neq m^2$ $\rightsquigarrow$ remove $i\epsilon$ in these cases
- * If $m > 2M$, it could have, for some $\vec{p}_1$, $(p_1+p_2)^2 = m^2$ (since $\vec{p}^{\,2} \geq 0$!) $\Rightarrow$ have a pole! pole!



(how we discover new particles!)

So sometimes drop $i\epsilon$, sometimes not for these reasons, (but you can keep if you want)

Mandelstam Variables

(introduce more jargon)

assumed in the definition: $p_1 + p_2 = p_3 + p_4$

Useful definition

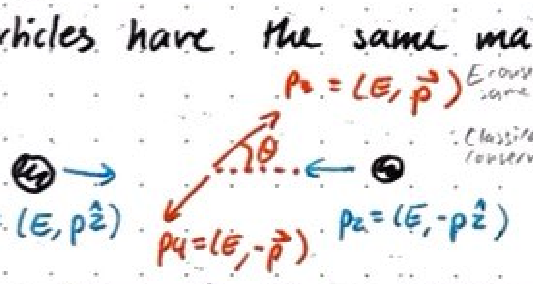
$$\begin{bmatrix} s \equiv (p_1+p_2)^2 = (p_3+p_4)^2 \\ t \equiv (p_1-p_3)^2 = (p_2-p_4)^2 \\ u \equiv (p_1-p_4)^2 = (p_2-p_3)^2 \end{bmatrix}$$

(see diagram labels 2 previous pages e.g. t-channel, s-channel, u-channel)

What do they mean? – kinematic interpretation

Assume all particles have the same mass

centre of mass ref.



E conserved: same E for all

Classical view: conserve E/m

For this process:
- $s = 4E^2$, $t = -2p^2(1-\cos\theta)$, $u = -2p^2(1+\cos\theta)$
- $s + t + u = 4m^2$

Interpretation:

s: total energy in center of mass frame

t, u: momentum exchange

(less intuitive, tells us at which angle things are moving around)

(L15) Note to self

- want to find new, more interesting representations of the group: look at infinitesimal transformations & study the resulting Lie algebra
1. $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$. look at infinitesimal using $\omega^\mu{}_\nu$, this obj is Lor. transf. ($\Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho \eta^{\sigma\rho} = \eta^{\mu\nu}$) $\Rightarrow \omega^{\mu\nu}$ must be antisymmetric!
2. 4x4 antisymmetric Lor. matrix has 6 indep. components which agrees with the 6 transformations of the Lor. group (3 rot 3 boost) $\Rightarrow$ introduce a basis of 6 antisymmetric Lor. matrices: $(M^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}$ where $\mu\nu$ are the components of each matrix & $\rho\sigma$ label each basis element (there are 6: 01, 02, 03, 12, 13, 23)
3. write this more practically by lowering one index: $(M^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu$ & now basis matrices are e.g. $(M^{01})^\mu{}_\nu = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$.
4. now write $\omega^\mu{}_\nu$ as a linear combination of the $M^{\rho\sigma}$: $\omega^\mu{}_\nu = \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu$ where $\Omega_{\rho\sigma}$ are just 6 numbers telling us how much of each transformation to do: think
5. infinitesimal $\omega^\mu{}_\nu$ represented above, so we can write a finite transformation as (e.g. PQM Lie group)

$$\Lambda = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$$

6 numbers telling us how much of each Lor. transform to do

six basis elements, generators for rep

(good to check! not trivial)

The 6 basis matrices $M^{\rho\sigma}$ are called the generators of the Lor. group. These matrices form a representation of the Lorentz group & satisfy the Lie algebra relations: $[M^{\rho\sigma}, M^{\tau\nu}] = \eta^{\sigma\tau}M^{\rho\nu} - \eta^{\rho\tau}M^{\sigma\nu} + \eta^{\rho\nu}M^{\sigma\tau} - \eta^{\sigma\nu}M^{\rho\tau}$

Lecture 15

13.11.24

via symmetry approach

DIRAC EQUATION → introduce the concept of spin-$\frac{1}{2}$

(classical FT for spinors)

contents

1) Representations of Lorentz group
2) Constructing an action
3) Symmetries
4) Plane wave solutions
5) Chiral (weyl) spinors

"butchered & quick discussion"

QM: $|\psi\rangle = |\updownarrow\rangle$ — postulated to us in QM state is spin either up or down

$i\partial_t \psi = (\cdots + \mu_B \vec{B}\cdot\vec{\sigma})\psi \quad \psi = \langle x|\psi\rangle$ — these were just postulated/stated to us: now we derive!

QFT: we will obtain spin for symmetries

1) Lorentz group

(reminder: the matrices $D[\Lambda]$ form a representation of the lorentz group)

(note to self: representation just means $D[\Lambda]$ tells us how lorentz transforms affect fields. (e.g. could have different $D[\Lambda]$ for vector/spinor/scalar fields) $D[\Lambda]$ tells us how to transform our fields as they obey lorentz invariance)

Definition of a field $\quad \Phi^a(x) \to D[\Lambda]^a{}_b \, \Phi_b(\Lambda^{-1}x)$

notes: so far we have only discussed scalar fields s.t. under a lor. transform the field changes as $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$ quantisation of such fields leads to spin 0 particles. now consider fields which transform non-trivially under group → gives rise to particles w/ spin!

Our goal is to identify / consider a more interesting / new representation $D[\Lambda]$.

$$D[\Lambda] = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma} R^{\rho\sigma}\right) \to \text{generators}$$

parameters (6, $\Omega_{\rho\sigma} = -\Omega_{\sigma\rho}$)

expect 6 parameters from transf. of Lorentz group

The generators $R^{\rho\sigma}$, by definition of Lorentz, satisfy

$$[R^{\rho\sigma}, R^{\tau\nu}] = \eta^{\sigma\tau}R^{\rho\nu} - \eta^{\rho\tau}R^{\sigma\nu} + \eta^{\rho\nu}R^{\sigma\tau} - \eta^{\sigma\nu}R^{\rho\tau} \qquad (1)$$

(note to self: if something forms a representation of lorentz algebra, that means it satisfies this eq.)

$\boxed{\text{Ex 1.}}$ Inf-dim. representation

generator $L^{\rho\sigma} = x^\rho\partial^\sigma - x^\sigma\partial^\rho \quad$ check that L's comply with (1)

$\boxed{\text{Ex 2.}}$ a 4×4 representation

(recall: $M^{\rho\sigma}$ form a 4×4 representation of the lorentz group → specifically they tell us how 4-vectors transform under lor transforms)

generator $(M^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu$, check these matrices comply w/ (1)

why some matrix we discussed at beginning of course in exp form

$$\Lambda = D[\Lambda] = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$$

act on vectors $V^\mu \to D[\Lambda]^\mu{}_\nu V^\nu = \Lambda^a{}_b V^b$

rot. Ω_{12}

$$\Lambda = \exp\left(\tfrac{1}{2}\Omega_{12}M^{12}\right) = \mathbb{1}_{4\times4}$$

$\Omega_{12} = 2\pi$

note on rotation overleaf →

let's find another representation of lorentz:

Consider a new representation

i) Spinor representation

first define algebra (this has nothing to do w/ lor group yet)

Consider Clifford algebra, defined by $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1} \quad (2) \quad \mu = 0,1,2,3.$

$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}$

so matrices must satisfy → $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1} \;\to\; \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ if $\mu \neq \nu$

$\to (\gamma^0)^2 = \underline{\mathbb{1}} \quad , \quad (\gamma^i)^2 = \underline{-\mathbb{1}}.$

4x4 matrix is smallest rep you can get that obeys Clifford algebra

A simple representation, chiral representation is

we can construct many representations of the clifford algebra; this one is called the Weyl or chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where σ^i are Pauli matrices $[\sigma^i, \sigma^j] = 2\delta^{ij}$.

notes: $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$

wiki: $\sigma^i\sigma^j = \delta_{ij} + i\varepsilon_{ijk}\sigma^k$

$$S^{\rho\sigma} = \begin{cases} \frac{1}{2}\gamma^\rho\gamma^\sigma, & \rho \neq \sigma \\ 0, & \rho = \sigma \end{cases}$$

What is relation to Lorentz?

CLAIM: $S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma]$ forms a representation of Lorentz algebra (i.e. complies with (1)).

plug in

PROOF: Plug in (1) and only use (2)
First show

$[C, AB] = A[C,B] + [C,A]B$

(if $\rho \neq \sigma$)

$$[S^{\mu\nu}, S^{\sigma\rho}] = \tfrac{1}{2}[S^{\mu\nu}, \gamma^\rho\gamma^\sigma] = \tfrac{1}{2}[S^{\mu\nu}, \gamma^\rho]\gamma^\sigma + \tfrac{1}{2}\gamma^\rho[S^{\mu\nu}, \gamma^\sigma]$$

Second show

use matrix identity (when $\rho \neq \nu$)

$$[S^{\mu\nu}, \gamma^\rho] = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu}$$

$\to$ S 4x4 matrices since γ 4x4 matrices attach indices a, b to remind us we may omit.

$\to$ plug 2nd into first get $= \frac{1}{2}\gamma^\mu\gamma^\sigma\eta^{\nu\rho} - \frac{1}{2}\gamma^\nu\gamma^\sigma\eta^{\rho\mu} + \frac{1}{2}\gamma^\rho\gamma^\mu\eta^{\nu\sigma} - \frac{1}{2}\gamma^\rho\gamma^\nu\eta^{\sigma\mu}$

$\to$ then substitute $S^{\mu\nu}$ back in: $= S^{\mu\sigma}\eta^{\nu\rho} - S^{\nu\sigma}\eta^{\rho\mu} + S^{\rho\mu}\eta^{\nu\sigma} - S^{\rho\nu}\eta^{\sigma\mu}$

Combine and get (1), q.e.d.

Therefore we have $S[\Lambda]^a{}_b = \exp(\frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^a{}_b)$

What transforms under $S[\Lambda]$? => SPINORS $\psi^a(x)$

so we have found new representation of Lorentz

need field for the $S[\Lambda]^a{}_b$ to act on: introduce Dirac spinor $\to$

$$\psi^a(x) \to S[\Lambda]^a{}_b\,\psi^b(\Lambda^{-1}x)$$

$\to$ notes: where $\Lambda = \exp(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma})$, $S[\Lambda] = \exp(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma})$

Properties of this new representation

how can we be sure that spinor rep $S[\Lambda]$ isn't equivalent to our familiar vector rep Λ? $\to$ consider rotation first.

pick i, j means rot. along k

$\leadsto$ Rotations $\quad S[\Lambda] = \exp(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma})$

$\to$ Ω_{ij} rot, Ω_{0i} boost

$\to$ Ω_{ij} is rot. (angle)

=> inspect S^{ij}

reminder: pauli matrices

$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Rotation along z-axis => Ω_{12} $\quad S^{12} = -\frac{i}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$ chiral rep.

recall $\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$ characterises angle

$\leftarrow$ pauli matrices?

Group element $\quad S[\Lambda] = \exp(\frac{1}{2}2\Omega_{12}S^{12}) = \exp\left(-i\frac{\Omega_{12}}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\right)$

(rotation around z axis of angle 2π) $\to$ Pick $\Omega_{12} = 2\pi$ (angle) $\quad S[\Lambda] = \begin{pmatrix} e^{i\pi\sigma_3} & 0 \\ 0 & e^{i\pi\sigma_3} \end{pmatrix} = -\mathbb{1}_{4\times4}$

=> under a 2π rotation, $\psi^a(x) \to -\psi^a(x)$ which would never a vector transform!

=> NOT a vector!!

rot 2π : not in... ?

$\leadsto$ Boost along x-axis $\quad S^{01} = \frac{1}{2}\begin{pmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} \quad \Omega_{i0} = \eta_i$

(Ω_{ij} rot. param about k, $\eta_i = \Omega_{i0}$ boost param along i)

$$S[\Lambda] = \begin{pmatrix} e^{\eta_1\sigma^1/2} & 0 \\ 0 & e^{-\eta_1\sigma^1/2} \end{pmatrix} \overset{\text{for all axes}}{=} \begin{pmatrix} e^{\vec{\eta}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\eta}\cdot\vec{\sigma}/2} \end{pmatrix}$$

very diff. from vector

(we have a field to work w (spinor ψ)
now we want to construct a Lorentz inv. EOM
→ do this by constructing Lor. inv. action)

2) Actions

actions should be Lorentz inv.

We have $\psi(x) \to S[\Lambda]\psi(\Lambda^{-1}x)$

Actions are Lorentz invariant (and moreover hence numbers)
To construct a Lorentz invariant from ψ:
we

$$\text{"..."} \to \text{"...."}\, S[\Lambda]^{-1}$$

our task is to fill in the brackets

then $\text{"..."}\psi \to \text{"..."}S[\Lambda]^{-1}S[\Lambda]\psi(\Lambda^{-1}x) = \text{"..."}\psi(\Lambda^{-1}x)$

per eg.: we can try what $\psi^\dagger$ transf. like

First see what $\psi^\dagger(x) = (\psi^*(x))^T$
we know

$$\psi^\dagger(x) \to \psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger$$

How is $S[\Lambda]^\dagger$ related to $S[\Lambda]^{-1}$?

notes:
- if $S[\Lambda]$ is unitary then $S[\Lambda]S[\Lambda] = 1$,
  $S[\Lambda]^\dagger = \exp(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma\dagger})$ (since $\Omega_{\rho\sigma}$ just real numbers)
- so if $S[\Lambda]$ is unitary, then $S^{\rho\sigma}$ must be antihermitian: $S^{\rho\sigma\dagger} = -S^{\rho\sigma}$
- we have
  $S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma]$, so $S^{\rho\sigma\dagger} = -\frac{1}{4}[\gamma^{\rho\dagger}, \gamma^{\sigma\dagger}]$
  so $S^{\rho\sigma}$ is antihermitian if all of γ^ρ are either hermitian or antihermitian,
  but in the chiral reps (& in general!) γ^0 are hermitian but γ^i are antihermitian so $S[\Lambda]$ is not unitary!
  ⇒ looking at transforming $\psi^\dagger(x)\psi(x)$, we find it is not Lorentz invariant!

using $[A,B]^\dagger = -[A^\dagger, B^\dagger]$

$$S[\Lambda]^\dagger = \exp(\tfrac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\dagger)$$

In the chiral representation

obvious from definition

$$\left.\begin{aligned}(\gamma^0)^\dagger &= \gamma^0 \\ (\gamma^i)^\dagger &= -\gamma^i\end{aligned}\right\} \to \gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$$

from properties of pauli matrices

identity encompasses these 2 properties
0 component: $(\gamma^0)^\dagger = \gamma^0\gamma^0\gamma^0 = \gamma^0$ ✓
i component: $(\gamma^i)^\dagger = \gamma^0\gamma^i\gamma^0 = -\gamma^i\gamma^0\gamma^0 = -\gamma^i$ ✓
↑ since $\{\gamma^0, \gamma^i\} = 0$

$[A,B]^\dagger = [B^\dagger, A^\dagger]$

$$(S^{\rho\sigma})^\dagger = \tfrac{1}{4}[(\gamma^\sigma)^\dagger, (\gamma^\rho)^\dagger] = -\gamma^0 S^{\rho\sigma}\gamma^0$$

$$\Rightarrow S[\Lambda]^\dagger = \exp(-\tfrac{1}{2}\Omega_{\rho\sigma}\gamma^0 S^{\rho\sigma}\gamma^0) = \gamma^0 S[\Lambda]^{-1}\gamma^0$$

$\psi^\dagger$ does not transf. as inverse

then $\psi^\dagger\psi$ is <u>not</u> Lorentz invariant.

note to self:
factor of 2!
$S[\Lambda] = \exp(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma})$
is a weighted sum over ρ,σ.
since $\Omega_{12} = -\Omega_{21}$ and $S^{12} = -S^{21}$
we get two terms in sum
corresponding to rot around
y axis so get factor 2!
when we find group element
(given rot. matrix)

summary/intuition:
$S^{\mu\nu}$: what is basis matrix for a particular transf.
$S[\Lambda]$: how does $\psi(x)$ transform if specify a transformation eg 2π rot around z axis & boost along z-axis by η
(or combination of transforms)

general rot. $S^{ij} = -\frac{i}{2}\epsilon^{ijk}\begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}$ $i \neq j$
wrt ϵ!

write rot parameters $\Omega_{ij} = -\epsilon_{ijk}\varphi^k$
(i.e. $\Omega_{12} = -\varphi^3$)
then rot matrix is general

$$S[\Lambda] = \begin{pmatrix} e^{i\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\varphi}\cdot\vec{\sigma}/2}\end{pmatrix}$$

Similarly general boost along i

$$S^{0i} = \tfrac{1}{2}\begin{pmatrix}-\sigma^i & 0 \\ 0 & \sigma^i\end{pmatrix}$$

write boost parameter $\Omega_{i0} = -\Omega_{0i} = \chi_i$
then general boost matrix is

$$S[\Lambda] = \begin{pmatrix} e^{\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2}\end{pmatrix}$$

Lecture 16 — 15.11.24

Recap:

- Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$
- Chiral rep. $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$; $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ (with $\sigma^\mu = (\mathbb{1}, \sigma^i)$, $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$)

- 2) Action

Spinor: $\psi(x) \to S[\Lambda]\psi(\Lambda^{-1}x)$, $S[\Lambda] = \exp(\tfrac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma})$, $S^{\rho\sigma} = \tfrac{1}{4}[\gamma^\rho, \gamma^\sigma]$

$^{*}\ \psi^\dagger(x) \to \psi^\dagger(\Lambda^{-1}x)\gamma^0 S[\Lambda]^{-1}\gamma^0$

(with the expression for $S[\Lambda]^\dagger$ we found in mind, define the dirac adjoint)

CLAIM: Define $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$ — adjoint spinor

then $\bar{\psi}\psi$ is Lorentz inv. ($\bar{\psi}\psi$ is Lorentz scalar $\Rightarrow$ transforms as a Lorentz scalar)

PROOF: (using $*$)

$$\bar{\psi}\psi = \psi^\dagger(x)\gamma^0\psi(x) \to \psi^\dagger(\Lambda^{-1}x)\gamma^0 S[\Lambda]^{-1}\underbrace{\gamma^0\gamma^0}_{=\mathbb{1}}S[\Lambda]\psi(\Lambda^{-1}x)$$

$$\to \bar{\psi}\psi(\Lambda^{-1}x) \ //$$

stick γ matrix in middle

γ don't transform under Lor., they are just matrices

CLAIM: $\bar{\psi}\gamma^\mu\psi$ transforms as a vector $(*)$

! means what we're supposed to show (i.e. how this should transf. as a vector)

PROOF: $\bar{\psi}\gamma^\mu\psi \to \bar{\psi}S[\Lambda]^{-1}\gamma^\mu S[\Lambda]\psi \overset{!}{=} \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\psi$

$(\bar{\psi}(\Lambda^{-1}x))\,\gamma\,\psi(\Lambda^{-1}(x))$

this is what we have to prove $\Rightarrow S[\Lambda]^{-1}\gamma^\mu S[\Lambda] \overset{!}{=} \Lambda^\mu{}_\nu\gamma^\nu$ (1)

γ^ν both sides of eq

elegant/simple proof is to do this perturbatively (treat Ω small):

$$S[\Lambda] = \mathbb{1} + \tfrac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma} + \dots$$
$$S[\Lambda]^{-1} = \mathbb{1} - \tfrac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma} + \dots$$
$$\Lambda = \exp(\tfrac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}) = \mathbb{1} + \tfrac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma} + \dots$$

plug these in to eq we want to show, see what we get to 1st order in Ω

To first order in Ω (1) implies

$$-[S^{\rho\sigma}, \gamma^\mu] \overset{!}{=} (M^{\rho\sigma})^\mu{}_\nu\gamma^\nu$$
$$= \gamma^\sigma\eta^{\rho\mu} - \eta^{\sigma\mu}\gamma^\rho \ \ /\text{q.e.d.}$$

note to self: the $S^{\rho\sigma}$s are just numbers, so we can take it out of the commutator to arrive at this

in terms of definition

$\to$ we proved on p.186 already $[S^{\rho\sigma}, \gamma^\mu] = \eta^{\sigma\mu}\gamma^\rho - \gamma^\sigma\eta^{\rho\mu}$

$\to$ then plug in def of $M^{\rho\sigma}$ and show these match

More claims:

transforms like

$(**)$
$$\bar{\psi}\gamma^\mu\gamma^\nu\psi \longrightarrow \text{2-tensor}$$
$$\bar{\psi}\partial_\mu\psi \longrightarrow \text{co-vector}$$

Construct action

(now we have $\bar{\psi}\psi$, $\bar{\psi}\gamma^\mu\psi$ ($\bar{\psi}\gamma^\mu\gamma^\nu\psi$) which are Lorentz inv. / transform covariantly under the Lorentz group (same thing), we try to build a Lorentz invariant action!)

1) Lorentz inv.
2) Minimal no. of derivatives (want simplest model)
3) Reality

since we're really ... condition 3)

Combine 1) – 3), the minimal action is
$$\boxed{S = \int d^4x\,(\bar{\psi}i\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi)}$$
DIRAC ACTION

check that this is real

check 3)

$$(\bar\Psi i \gamma^\mu \partial_\mu \Psi)^\dagger = -i\partial_\mu \Psi^\dagger (\gamma^\mu)^\dagger (\gamma^0)^\dagger \Psi$$
$$= -i\partial_\mu \Psi^\dagger \gamma^0 \gamma^\mu \underbrace{\gamma^0 \gamma^0}_{=\mathbb{1}} \Psi$$

(can bring γ^0, γ^μ in and out of derivatives ∂_μ since these are just matrices eg. 1s-1s? OK?)

$$= -i\partial_\mu \bar\Psi \gamma^\mu \Psi$$
$$= -i\partial_\mu (\bar\Psi \gamma^\mu \Psi) + i\bar\Psi \gamma^\mu \partial_\mu \Psi$$

"total deriv. → same as before

same action

this is real 'yay' if we didn't add i it wouldn't be real

we have action principle now consider EOM

Equations of Motion

again remember γ^0, γ^μ are just numbers so can take out of derivs

DT says "it's completely gorgeous"

easy b/c no deriv of $\bar\Psi$

vary w.r.t. $\bar\Psi$: $\boxed{i\gamma^\mu \partial_\mu \Psi - m\Psi = 0}$ DIRAC EQUATION (2) 1st order eqs.

vary w.r.t. Ψ : $\boxed{-i\gamma^\mu \partial_\mu \bar\Psi - m\bar\Psi = 0}$

(this one is called "conjugate eq. dirac eq" in notes)

what is diff compared to scalar: in scalar they no obj γ so have to contract w/ another deriv so minimal actions are second order, in scalar they give want to be Lor. inv. whereas here 1st order

NOTE: FIRST ORDER ACTION (one ∂_μ) → because of γ^μ

Useful to note, a second order equation would be

$$(i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\Psi = 0$$

cross term cancels

$$(\underbrace{-\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu}_{\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu = \eta^{\mu\nu}} - m^2)\Psi = 0$$

gammas don't commute but partial derivs symmetric

b/c of defn of gammas

$$\Rightarrow \boxed{(\Box + m^2)\Psi = 0}$$ (3) satisfies klein-gordon eq.

KLEIN GORDON

(Introduce slash notation:

Notation: $\gamma^\mu \partial_\mu = \not\partial$
$\gamma^\mu A_\mu = \not A$)

slash means the tensor is getting contracted w/ gamma matrices → i.e. Dirac eq. becomes $(i\not\partial - m)\Psi = 0$

3) Symmetries

what do we do for symmetries? apply Noether!

what are symmetries of this action?

i) Lorentz — obviously because we used this to construct it

ii) U(1) Internal

iii) Translations

$\Longrightarrow$ APPLY NOETHER

i) "Angular-Momentum"

ii) Charge Q (how to show this?)

iii) Stress tensor: Energy + Momentum

(iii) Translation: $x^\mu \to x^\mu + \epsilon^\mu$, $\delta\Psi = \epsilon^\mu \partial_\mu \Psi$ → $T^{\mu\nu} = \bar\Psi i\gamma^\mu \partial^\nu \Psi$ eval on-shell

use EOM to get to this eq.

have $T^{\mu\nu} = \bar\Psi i\gamma^\mu \partial^\nu \Psi - \eta^{\mu\nu}\mathcal{L}$ since current is only conserved when EOM are obeyed we can use EOM to set $\mathcal{L} = 0$ so gives expr. above.

this part comes b/c Ψ depends on coords

i) Lorentz $x^\mu \to \Lambda^\mu{}_\nu x^\nu$

$$\delta\Psi_a = -\omega^{\mu\nu}(\underbrace{x_\nu \partial_\mu \Psi_a}_{\text{same as scalar}} \underbrace{- \tfrac{1}{2}(S_{\mu\nu})^b{}_a \Psi_b}_{\text{new, it comes from the rep}})$$

didn't have this for scalar b/c scalar transforms trivially under lorentz

→ $\Psi(x) \to S[\Lambda]\Psi(\delta^\mu_\nu - \omega^\mu{}_\nu x^\nu)$ plug in infinitesimal $S[\Lambda]$ then plug in infinitesimal Λ we get $\Omega_{\rho\sigma} = \omega_{\rho\sigma}$ (analogous to how we got the scalar $L \to L(\Lambda^{-1}x)$?)

current $$(J^\mu)^{\lambda\nu} = \underbrace{-x^\nu T^{\mu\lambda} + x^\lambda T^{\mu\nu}}_{\text{same as before}} + \underbrace{i\bar\Psi \gamma^\mu S^{\lambda\nu}\Psi}_{\text{new → part responsible for spin}}$$

(see notes sheet 1)

ii) internal vector symm.

$\Psi \to e^{i\alpha}\Psi$ $\quad \alpha \in \mathbb{R}$, constant

$\bar\Psi \to e^{-i\alpha}\bar\Psi$

not un-index, V for vector

Associated current $(J^\mu)_V = \bar\Psi \gamma^\mu \Psi$

$Q = \int d^3x\, \bar\Psi \gamma^0 \Psi$ → electric charge

4) Plane Wave Solution

this eq already gives intuition as to what solutions look like ↳ plane waves

Finding solutions to (2) and (3)

From (3) KG:

(U(p), V(p) are both 4 component spinors)

ansatz:

but we also need to solve for Ψ:

$$\Psi(x) \sim U(\vec{p})e^{-ipx} + V(\vec{p})e^{ipx} \quad \text{with } p^2 = m^2$$

(plane waves seem like a reasonable set of solutions)

If we replace this on (2) — both gives 1st eqn

$$(a) \quad (-\not{p} + m)U(\vec{p}) = 0$$

$$(b) \quad (\not{p} + m)V(\vec{p}) = 0$$

$$\not{p} = \gamma^\mu p_\mu$$

these eq just say that the linear algebra problem ... needs eigenvalues that are zero

use chiral rep to visualise this as a 4x4 matrix

In chiral rep, (a) looks like

(use chiral rep to visualise these eq as a 4x4 matrix) — 2x2 identity matrix

$$\begin{pmatrix} m\mathbb{1}_{2\times2} & -p\cdot\sigma \\ -p\cdot\bar{\sigma} & m\mathbb{1}_{2\times2} \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = 0$$

being produced $(-\not{p} + m\mathbb{I})U(\vec{p}) = 0$

check: this gives 2 eq. check that they both give the same condition (drop in identity relations)

this comes from here

$$\Rightarrow \begin{rrcases} mU_1 = p\cdot\sigma U_2 \\ m^2 = (p\cdot\sigma)(p\cdot\bar{\sigma}) \end{rrcases} \Rightarrow \sqrt{p\cdot\sigma}\, U_2 = \sqrt{p\cdot\bar{\sigma}}\, U_1$$

we regarding Γ^μ ... allows to check this

this identity comes from:
$(p\cdot\sigma)(p\cdot\bar{\sigma}) = \mathbb{1}_{2\times2}\, p^\mu \sigma^\mu ...$
$= (p^0 - p^i\sigma^i)(p^0 + p^j\sigma^j)$
(using defs of $\sigma, \bar{\sigma}$ & remember signature)
$= (p^0)^2 - p^i p^j \sigma^i\sigma^j$
$= (p^0)^2 - p^i p^j(\delta^{ij} + i\varepsilon^{ijk}\sigma^k)$
(using expression for $\sigma^i\sigma^j$)
$= (p^0)^2 - |\vec{p}|^2 = m^2$
using on-shell condition
(since $p^i p^j = p^j p^i$ is symmetric (they are just components of the same vector) but ε^{ijk} is antisymmetric so the $p^i p^j \varepsilon^{ijk}\sigma^k$ term $= 0$)

$$U^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^s \\ \sqrt{p\cdot\bar{\sigma}}\, \xi^s \end{pmatrix}$$

$\xi^s \quad s = 1, 2$

$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ — a choice of bases

generally $\sqrt{p\cdot\bar{\sigma}}\, U_1 = \sqrt{p\cdot\sigma}\, U_2 \Rightarrow$ solution is:
$U(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi \\ \sqrt{p\cdot\bar{\sigma}}\, \xi \end{pmatrix}$ where ξ is an arbitrary 2 component spinor
but it is convenient for us to make a choice for ξ which allows us to ...

Similar steps for (b)

(i.e. $\sqrt{p\cdot\bar{\sigma}}\, V_1 = -\sqrt{p\cdot\sigma}\, V_2$)

$$V^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\, \eta^s \end{pmatrix}$$

$\eta^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \eta^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

SIDE NOTE: what do I mean by square root of a matrix?

$$M = U\Lambda U^T = U\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}U^T$$

$$\sqrt{M} = U\begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix}U^T$$

what is the notation we are using?

note to self:
a matrix can be written in the form of an eigendecomposition
$M = U\Lambda U^{-1}$
where Λ is a matrix whose diagonal values are eigenvalues of M and U is matrix whose columns are the eigenvectors of M
then the square root can be written as
$\sqrt{M} = U\Lambda^{\frac{1}{2}}U^{-1}$
$= U\begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix}U^{-1}$
(... orthogonal matrix has $U^{-1} = U^T$)

note: we normalise the basis s.t. $\xi^{r\dagger}\xi^s = \delta^{rs}$ and $\eta^{r\dagger}\eta^s = \delta^{rs}$

note to self:
spinors are orthonormal
$$U^{s\dagger}(\vec{p})U^r(\vec{p}) = 2p_0\delta^{rs}$$
(& other useful relations) find ...
e.g.
$$U^{r\dagger}(\vec{p})U^s(\vec{p}) = \left(\xi^{r\dagger}\sqrt{p\cdot\sigma},\ \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}\right)\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}$$
$$= 2p_0\delta^{rs}$$
(remember trick: $p\cdot\sigma = p_0 + p_i\sigma^i$ and $p\cdot\bar{\sigma} = p_0 - p_i\sigma^i$) etc.

also outer products
$$\sum_s U^s(\vec{p})\bar{U}^s(\vec{p}) = \not{p} + m$$
$$\sum_s V^s(\vec{p})\bar{V}^s(\vec{p}) = \not{p} - m$$

↳ ... with product, in general for all of these proofs just plug in
column vectors $U^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}$, row vectors $U^{s\dagger}(\vec{p}) = \left(\sqrt{p\cdot\sigma}\,\xi^{s\dagger}, \sqrt{p\cdot\bar{\sigma}}\,\xi^{s\dagger}\right)$
and matrices $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
(& similarly for $V^s(\vec{p})$ etc.)

Lecture 17 — 18.11.24

(note (self): block diagonal $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ matrix is reducible since smaller subspaces transform independently (i.e. stuff acted on by A stays separate to stuff acted on by B) so we can split the representation into two smaller (non-reducible) representations that act on separate subspaces)

(stuff in subspace acted on by A stays in that subspace under boosts)

notice: 2×2 blocks ⇒ representation is reducible

57 Chiral (Weyl) Spinors

Recall
$$S[\Lambda_{rot}] = \begin{pmatrix} e^{i\vec{\psi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\psi}\cdot\vec{\sigma}/2} \end{pmatrix} \qquad \Omega_{ij} = -\varepsilon_{ijk}\psi^k$$

chiral rep.

(spinor rotation & boost transformations in the chiral representation)

$$S[\Lambda_{boost}] = \begin{pmatrix} e^{\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix} \qquad \Omega_{0i} = \chi_i$$

(shouldn't $\Omega_{i0} = -\Omega_{0i} = \chi_i$? to get the right − sign?)

(act on disjoint subspaces)

(Dirac spinor representation of the Lorentz group is reducible)

Dirac (4 comp spinor) representation is reducible

→ split into two representations that act only on two-component spinors $U_\pm$ defined in the chiral rep as:

$$\psi(x) = \begin{pmatrix} U_+ \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ U_- \end{pmatrix}$$

(Chiral representation: nothing special about this representation, just 1 way to represent the clifford algebra)

U_+, U_- 2 comp spinors : Weyl (chiral) spinors

Under rotations
$$\left.\begin{aligned} U_+ &\to e^{i\vec{\psi}\cdot\vec{\sigma}/2}U_+ \\ U_- &\to e^{i\vec{\psi}\cdot\vec{\sigma}/2}U_- \end{aligned}\right\} \text{same sign}$$

(Weyl spinors transform in the same way under rotations but oppositely under boosts)

Under boost
$$\left.\begin{aligned} U_+ &\to e^{\vec{\chi}\cdot\vec{\sigma}/2}U_+ \\ U_- &\to e^{-\vec{\chi}\cdot\vec{\sigma}/2}U_- \end{aligned}\right\} \text{opposite sign}$$

(for group matrices $S[\Lambda]$ came out to be block diagonal b/c we chose the representation γ^μ here i.e. the chiral rep.)

Chiral representation $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ it is obvious this split

(want to know: can we define the chiral spinors U_+, U_- for an arbitrary representation?)

A more general way to identify the two Weyl spinors is by introducing
$$\gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 \quad \sim \varepsilon_{\mu\nu\lambda\rho}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho$$

satisfies $\{\gamma^5, \gamma^\mu\} = 0 \qquad (\gamma^5)^2 = \mathbb{1} \quad , \quad [S^{\mu\nu}, \gamma^5] = 0$ → γ^5 is Lor inv.

Define projection
$$P_+ = \tfrac{1}{2}(\mathbb{1} + \gamma^5) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \qquad P_\pm^2 = P_\pm \quad , \quad P_+P_- = 0$$

chiral rep.

$$P_- = \tfrac{1}{2}(\mathbb{1} - \gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \qquad \psi_+ = P_+\psi \overset{\text{chiral rep}}{=} \begin{pmatrix} U_+ \\ 0 \end{pmatrix} \, , \quad \psi_- = P_-\psi \overset{\text{chiral rep}}{=} \begin{pmatrix} 0 \\ U_- \end{pmatrix}$$

(find in chiral rep by plugging in γ^μ)
$$\gamma^5 \overset{\text{chiral rep}}{=} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

(i.e. these projection operators define the chiral spinors U_+, U_- for an arbitrary representation of the clifford algebra)

(in the chiral rep, $P_\pm$ project onto the Weyl (chiral) spinors → but for an arbitrary rep. of the clifford algebra (i.e. not necessarily the chiral rep), we can use γ^5 to define the chiral spinors $\psi_\pm = P_\pm\psi$ via the projection operators $P_\pm = \frac{1}{2}(1 \pm \gamma^5)$!)

Look at Lagrangian (Dirac)

(Dirac $\mathcal{L}$ from Dirac action on p188)

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

chiral rep.

$$= iU_-^+\sigma^\mu\partial_\mu U_- + iU_+^+\bar{\sigma}^\mu\partial_\mu U_+ - m(U_+^+U_- + U_-^+U_+)$$

(i.e. plane wave solutions)

(find in chiral rep from:
$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$
$= \begin{pmatrix} U_-^+ & U_+^+ \end{pmatrix}\begin{pmatrix} -m & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & -m \end{pmatrix}\begin{pmatrix} U_+ \\ U_- \end{pmatrix}$
↑ multiplying out from $\bar{\psi} = \psi^\dagger\gamma^0$)

(first classical properties of this field, can quantise this theory ↓)

QUANTISING DIRAC FIELD

(I) Canonical quantisation free theory
  1) Fock space
  2) Feynman propagator

(II) Interactions Yukawa theory
  1) Feynman rules
  2) Example of scattering

(I) Canonical Quantisation

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi$$

use same principles as we did for the scalar field to quantise this theory

for scalar field we were able to successfully construct a Fock space with states with a positive norm & a hamiltonian bounded from below → this is our first aim!

Follow principles from scalar theory: aim to construct a Fock space
↳ $\langle 1 \rangle > 0$ (positive norm) H bounded below

in order to construct this Fock space → Need to figure out commutation relations

to quantise → promote field & its momentum Π to operators which in our previous examples satisfied canonical commutation relations → but do they now?

(again, canonical commutation relations of fields ⟺ commutation relations of operators)

(canonical momentum of the spinor, usually momentum is a time deriv. → some deg. than we thought)

$$\Pi = \frac{\partial\mathcal{L}}{\partial\partial_t\Psi} = i\bar{\Psi}\gamma^0 = i\Psi^+$$

Route 1 impose equal time relations

route 1: impose that these things commute

$$[\Psi_a(\vec{x},\tau), \Pi_b(\vec{y},\tau)] = i\delta_{ab}\delta(\vec{x}-\vec{y})$$

$$i[\Psi_a(\vec{x},\tau), \Psi_b^+(\vec{y},\tau)] = i\delta_{ab}\delta(\vec{x}-\vec{y})$$

(i think these are t not τ)

$$[\Psi_a(\vec{x},\tau), \Psi_b(\vec{y},\tau)] = 0$$

Route 2 route 2 write anticommutation relations

$$\{\Psi_a(\vec{x},\tau), \Psi_b^+(\vec{x},t)\} = \delta_{ab}\delta(\vec{x}-\vec{y})$$

$$\{\Psi_a(\vec{x},\tau), \Psi_b(\vec{y},t)\} = 0$$

aim: to have reasonable Fock space, both routes will have some shared properties?

SHARED PROPERTIES

For both routes:

(s: 1 or 2 which are the two indep. solutions I have)

sum over S. integrate over all vals of momentum

$$\Psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(b_p^s U^s(\vec{p})e^{-ipx} + c_p^{s+} V^s(\vec{p})e^{+ipx}\right)$$

need to know solution (in momentum) of the field, we have already found plane wave solutions, general solution is arbitrary superposition of these plane wave solutions, & again we include an arbitrary choice of normalisation $\sqrt{2\omega_p}$

$\bar{U} = U^+\gamma^0$ $\bar{V} = V^+\gamma^0$

reverse relationship (how much this gives?)

$$b_p^s = \frac{1}{\sqrt{2\omega_p}}\int d^3x\, e^{ipx}\,\bar{U}^s(\vec{p})\gamma^0\Psi(x) \qquad c_p^s = \frac{1}{\sqrt{2\omega_p}}\int d^3x\, e^{ipx}\,\bar{\Psi}\gamma^0 V^s(\vec{p})$$

$$b_p^{s+} = \frac{1}{\sqrt{2\omega_p}}\int d^3x\, e^{-ipx}\,\bar{\Psi}(x)\gamma^0 U^s(\vec{p}) \qquad c_p^{s+} = \frac{1}{\sqrt{2\omega_p}}\int d^3x\, e^{-ipx}\,\bar{V}^s(\vec{p})\gamma^0\Psi$$

For both routes, Hamiltonian (now find H:)

does not depend on what the commutation / anticommutation!

plug that in get this expr.

$$H = \int d^3x\, \Pi\dot{\Psi} = i\int d^3x\, \Psi^+\dot{\Psi}$$

$H = \int d^3x\,(\Pi\dot{\Psi} - \mathcal{L})$

but we just leave the $\mathcal{L}$ b/c ... (got rid?) that the $\mathcal{L}$ gives spatial derivatives only but we can ignore the spatial terms & focus only on the time dependent part since spatial derivatives are not relevant to computing the canonical commutation relations ... simplifies our canonical quantisation setup? ... ... $\mathcal{L}$? 'a trick'!?

$$= \sum_s \int \frac{d^3p}{(2\pi)^3}\,\omega_p\left(b_p^{s+}b_p^s - c_p^s c_p^{s+}\right)$$

get also a $\delta^3(0)$ term which by dealt with by normal ordering like before?

{ just plug in $\Psi^+ = \sum\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_q}}(b_q^{s+}U^{s+}(\vec{q})e^{iqx} + c_q^s V^{s+}(\vec{q})e^{-iqx})$ and $\dot{\Psi}$ like usual then need to use properties of spinors:
$U^{r+}(\vec{p})U^s(\vec{p}) = V^{r+}(\vec{p})V^s(\vec{p}) = 2\omega_p\delta^{rs}$ and $U^{r+}(\vec{p})V^s(-\vec{p}) = V^{r+}(\vec{p})U^s(-\vec{p}) = 0$

how and one differences between routs?

(can show commutation rels w/ between fields ⇒ these rels b/w operators using color product, ...)

Route 1

$$[b_p^s, b_{p'}^{s'\dagger}] = (2\pi)^3\delta^3(\vec p - \vec p')\delta_{ss'} \Rightarrow [c_p^s, c_{p'}^{s'\dagger}] = -(2\pi)^3\delta^3(\vec p - \vec p')\delta_{ss'}$$ (1)

(all other commutators vanish)

Route 2

$$\{b_p^s, b_{p'}^{s'\dagger}\} = (2\pi)^3\delta(\vec p - \vec p')\delta_{ss'} \Rightarrow \{c_p^s, c_{p'}^{s'\dagger}\} = (2\pi)^3\delta(\vec p - \vec p')\delta_{ss'}$$

i) treat $b^\dagger, c^\dagger$ as creation & annihilation then: complex norm of state I have a problem: norm of C state is neg

ii) (now look at Hamiltonian)

Route 1

consider hamiltonian

$$[H, b_p^{s\dagger}] = \omega_p b_p^{s\dagger} \quad , \quad [H, c_p^{s\dagger}] = \omega_p c_p^{s\dagger}$$

e.g. $H b_p^{s\dagger}|0\rangle = [H, b_p^{s\dagger}]|0\rangle = \omega_p b_p^{s\dagger}|0\rangle$
ii) $c_p^{s\dagger}$ creates then $H c_p^{s\dagger}|0\rangle = \omega_p c_p^{s\dagger}|0\rangle$
but because (1) the states have -ve energy. -ve norm $\langle 0|c_p^s c_p^{s\dagger}|0\rangle$
but if c creates instead then -ve energy

If $b^\dagger, c^\dagger$ create then Energy > 0, but $\langle 1\rangle < 0$ due to (1) the minus sign causes problems!

what is this create in old?

If $b^\dagger, c$ create then $\langle 1\rangle > 0$, $[H, c_p^s] = -\omega_p c_p^s$

we don't like this approach, what about route 2?

∃ neg. energy !!
can have eigen-states but can't have arbitrarily -ve, something to do with can't have states unbounded from above & below?
(if keep acting on w/ c_p^s would get arbitrarily -ve states)

Route 2

→ use $[AB, C] = A\{B,C\} - \{A,C\}B$

$$[H, b_p^{s\dagger}] = \omega_p b_p^{s\dagger} \quad , \quad [H, c_p^{s\dagger}] = \omega_p c_p^{s\dagger}$$

With this we can construct a healthy Fock space ☺

→ construct fock space by acting on vacuum by $b_p^\dagger$ and $c_p^{s\dagger}$ to make tower of energy eigenstates!

Two comments:

choice of commutator not a choice → in QM we are told what has to commute but in QFT Lorentz inv. tells us

1) Spin + statistics theorem ← tells us whether they commute

Rep of Lor. group ⇔ Bosonic (comm.) / Fermionic (anti-comm.)

E.g. $\begin{cases} \text{scalar (spin 0)} \leftrightarrow \text{Bose} \\ \text{spinor (spin 1/2)} \leftrightarrow \text{Fermi} \end{cases}$

3 ingredients:
a) stability : Hamiltonian bounded i.e. Fock space
b) causality : $[\mathcal{O}(x), \mathcal{O}(y)] = 0$ if $(x-y)^2 < 0$
c) Lorentz inv. of S-matrix implies $\langle T\mathcal{O}_1 \dots \mathcal{O}_n\rangle$ needs to be Lorentz invariant (beyond normal ordering)

will see this soon: if want Lor. inv. quantities in scattering process: need to impose correct comm. rel.

now only discuss route 2 → change over to get right creation & annihilation operators

now more around to get the right position of creation & annihilation which gives this extra term

2) for spinors

2 spin states

$$H = \sum_s \int \frac{d^3p}{(2\pi)^3}\omega_p (b_p^{s\dagger}b_p^s + c_p^{s\dagger}c_p^s) - \int \frac{d^3p}{(2\pi)^3}\omega_p\, 2(2\pi)^3\delta^3(\vec 0)$$

said this was important? don't understand ...

parallel
for scalar

$$H = \int \frac{d^3p}{(2\pi)^3}\omega_p a_p^\dagger a_p + \int \frac{d^3p}{(2\pi)^3}\omega_p \tfrac{1}{2}(2\pi)^3\delta(\vec 0)$$

notice sign difference
4 scalars & 1 spinor get zero vacuum energy
feature of supersymmetric theory ... balance to get 0 vac. energy

Balance scalar (bosons) + spinors (fermions) to get 0 vaccum energy.

this was divergent because of the ultraviolet divergence (the inf issue) & also the $\delta(0)$ infrared divergence but we got rid of this using normal ordering

but if we had 4 scalars & one spinor I would get zero vacuum energy!
→ this is a feature (called supersymmetric theory) where we balance bosons & fermions to get 0 vacuum energy (so don't need to normal order?)
(remember in the past we subtracted this infinity term to get rid of it (normal ordering) but here they cancel)

note to self: do this for b_p^s:
• multiply ψ by $u^{s\dagger}(\vec p)$ & integrate over x → (skip some steps)

$\int d^3x\, e^{-ipx}u^{s\dagger}(\vec p)\psi(x) = \sum_r \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_q}}u^{s\dagger}(\vec p)[b_q^r u^r(\vec q)\int d^3x\, e^{i(q-p)x} + c_q^{r\dagger}v^r(\vec q)\int d^3x\, e^{-i(p+q)x}]$

the second term gives $\delta(\vec p + \vec q)$ which apparently vanishes since ... is actually because $u^{s\dagger}(\vec p)v^s(-\vec p) = 0$?
~~but -p since q is positive? not convinced because for hamiltonian ...~~

$= \sum_r \frac{1}{\sqrt{2\omega_p}} b_p^r u^{s\dagger}(\vec p)u^r(\vec p)$

then use orthonormality of spinors $u^{s\dagger}(\vec p)u^r(\vec p) = 2p_0\delta^{rs}$ (i.e. $=0$ if $s \neq r$ and $2\omega_p$ if $s = r$)

$\Rightarrow \sqrt{2\omega_p}\, b_p^s$

So we have shown $b_p^s = \frac{1}{\sqrt{2\omega_p}}\int d^3x\, e^{ipx}u^\dagger(\vec p)\psi(x)$

conclude:
the $\int d^3x\, e^{ipx}$ gets rid of the $c_p^{s\dagger}$ term since it creates a $\delta(p+q)$ and the $u^{s\dagger}(\vec p)$ extracts b_p^s for us since it is orthonormal to $u^r(\vec p)$

Lecture 18 — 20.11.24

Recap: (I) Canonical Quantisation

$$\Pi = i\psi^\dagger$$

$$\Rightarrow \{\psi_a(\vec{x},t), \psi_b^\dagger(\vec{x}',t)\} = \delta_{ab}\delta^3(\vec{x}-\vec{x}')$$

$$\{\psi_a(\vec{x},t), \psi_b(\vec{x}',t)\} = 0$$

$$\Rightarrow \{b_p^s, b_{p'}^{s'\dagger}\} = (2\pi)^3\delta(\vec{p}-\vec{p}')\delta_{ss'} \qquad (1)$$

$$\{c_p^s, c_{p'}^{s'\dagger}\} = (2\pi)^3\delta(\vec{p}-\vec{p}')\delta_{ss'}$$

and all other combinations of anti-comm vanish

Hamiltonian (normal ordered) $H = \sum_s \int \frac{d^3p}{(2\pi)^3}\omega_p \left(b_p^{s\dagger} b_p^s + c_p^{s\dagger} c_p^s\right)$

$[H, b_p^{s\dagger}] = \omega_p b_p^{s\dagger} \qquad [H, c_p^{s\dagger}] = \omega_p c_p^{s\dagger}$

Comment: Normal ordering for fermions

$$:b_q c_q b_k^\dagger: = (-1)^2 b_k^\dagger b_q c_q$$

$$= (-1) b_k^\dagger c_q b_q$$

every time you pass a thing something you get a minus sign, since I passed it through twice I get two minus signs

1) Fock space

construct

Define a vacuum state $|0\rangle$, s.t. $b_p^s|0\rangle = c_p^s|0\rangle = 0$

$H|0\rangle = 0$: ground state

one particle states

now correctly normalised

$|b^s\rangle = \sqrt{2\omega_p}\, b_p^{s\dagger}|0\rangle \qquad H|b^s\rangle = \omega_p|b^s\rangle$

$|c^s\rangle = \sqrt{2\omega_p}\, c_p^{s\dagger}|0\rangle \qquad H|c^s\rangle = \omega_p|c^s\rangle$

U(1) Charge

$$Q = \sum_s \int \frac{d^3p}{(2\pi)^3}\left(b_p^{s\dagger} b_p^s - c_p^{s\dagger} c_p^s\right)$$

$Q|b^s\rangle = |b^s\rangle$

$Q|c^s\rangle = -|c^s\rangle$

note to self: (again helps to use $[AB,C] = A\{B,C\} - \{A,C\}B$)

Angular momentum

not going to write down explicitly is cumbersome

J_z at $\vec{p} = 0$

$J_z|b^s\rangle = \pm \frac{1}{2}|b^s\rangle$

$J_z|c^s\rangle = \pm \frac{1}{2}|c^s\rangle$

eval in rest frame

$s=1 \rightarrow +$

$s=2 \rightarrow -$

s determines whether particle is spin up or spin down

I think note to self

$\not{p}^2 = (\gamma^\mu p_\mu)^2 = (\gamma^0 p_0 + \gamma^i p_i)$ [illegible]

$(\gamma^0)^2 = \mathbb{1}$, $(\gamma^i)^2 = -\mathbb{1}$ [illegible]

$\Rightarrow \not{p}^2 = p_0^2 - p_i^2 = p^2$

this only is done in clifford algebra whereas [illegible] we did not use clifford algebra so had to explicitly include the metric. when we work in the clifford algebra we don't need to include the metric explicitly [illegible]

Then we have multi-particle states by acting S multiple times with $b^\dagger, c^\dagger$

$b_p^{s\dagger}|0\rangle = |p,s\rangle$?)

Note: Fermi-statistics

$$|p_1,s_1,p_2,s_2\rangle = b_{p_1}^{s_1\dagger} b_{p_2}^{s_2\dagger}|0\rangle = -b_{p_2}^{s_2\dagger} b_{p_1}^{s_1\dagger}|0\rangle = -|p_2,s_2,p_1,s_1\rangle$$

consequence of the anticommutator: antisymmetric upon particle exchange, which we know to be true for fermions

note also $\{b_p^{s\dagger}, b_p^{s\dagger}\} = b_p^{s\dagger}b_p^{s\dagger} + b_p^{s\dagger}b_p^{s\dagger} = 0$ requires $b_p^{s\dagger}b_p^{s\dagger} = 0$

hence $|p,s,p,s\rangle = b_p^{s\dagger}b_p^{s\dagger}|0\rangle = 0$ which we know is true from the pauli exclusion principle

2) Propagator

Inspect this first with the aim of getting

$$\langle 0|\psi_a(x)\bar\psi_b(y)|0\rangle$$

$$\rightarrow \langle 0|\underbrace{T\psi(x)\bar\psi(y)}_{\text{should be Lor. inv.}}|0\rangle$$

Feynman Propagator for Dirac Spinor

recall the state

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(b_p^s U^s(\vec p)e^{-ipx} + c_p^{s\dagger}V^s(\vec p)e^{ipx}\right)$$

only ones that survive basically $b, b^\dagger$ → everything else annihilated by vacuum (skipping steps here)

Replace in

$$\langle 0|\psi_a(x)\bar\psi_b(y)|0\rangle = \sum_{s,s'}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}e^{-ipx}e^{ip'y}U_a^s(\vec p)\bar U_b^{s'}(y)\langle 0|\underbrace{b_p^s b_{p'}^{s'\dagger}}_{\text{use (1)}}|0\rangle$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}(\not p+m)_{ab}e^{-ip(x-y)}$$

← use $\sum_s U_a^s(\vec p)\bar U_b^s(\vec p) = (\not p+m)_{ab}$

"write in way which is more illuminating"

$$\langle 0|\psi_a(x)\bar\psi_b(y)|0\rangle = (i\not\partial_x+m)_{ab}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{-ip(x-y)}$$

← (consider $i\partial_x e^{-ip(x-y)} = i(-ip)e^{-ip(x-y)} = p\, e^{-ip(x-y)}$ so $\not p \rightarrow i\not\partial_x$)

$$= (i\not\partial_x+m)_{ab}\,D(x-y) \longrightarrow \text{Scalar propagator}$$

using $\sum_s V^s(\vec p)\bar V^s(\vec p) = \not p - m$ (and $\bar V^s(\vec p)V^s(\vec p) = -2m\delta^{rs}$?)

Look at reverse

similar steps not shown here

$$\langle 0|\bar\psi_b(y)\psi_a(x)|0\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}(\not p-m)_{ba}e^{-ip(y-x)}$$

$$= -(i\not\partial_x+m)D(y-x)$$

KEY POINT: minus sign

If idea: time ordering as product of these I get problem b/c minus sign but if idea: time ordering with a minus sign then I get nice lor. inv. quantity.

The minus sign is an indication that we should define time ordering

$$T\psi_a(x)\bar\psi_b(y) = \begin{cases}\psi_a(x)\bar\psi_b(y) & x^0>y^0\\ -\bar\psi_b(y)\psi_a(x) & y^0>x^0\end{cases}$$

with this def. feynman propagator for fermions

$$\langle 0|T\psi(x)\bar\psi(y)|0\rangle = (i\not\partial_x+m)\Delta_F(x-y) \equiv S_F(x-y)$$

scalar feynman propagator; feynman propagator for fermions

MOMENTUM INTEGRAL REPRESENTATION FOR FEYNMAN PROPAGATOR: $S_F(x-y) = \int\frac{d^4p}{(2\pi)^4}(\not p+m)\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-y)}$

It follows

$$(i\not\partial_x - m)S_F(x-y) = i\delta^4(x-y) \rightarrow \text{Green's function for Dirac equation}$$

to show move integral out & drop the $i\epsilon$ then use $\not p\not p = p^2$

b/c

$$(\Box_x + m^2)\Delta_F(x-y) = -i\delta^4(x-y)$$

(or do same steps in diff order → $-(i\not\partial_x-m)(i\not\partial_x+m) = -(\Box_x+m^2)$ roughly)

$(i\gamma^\mu\partial_\mu\psi - m\psi = 0$, $(i\not\partial - m)\psi = 0$ dirac operator)

Also

$$S_F(x-y)(i\overleftarrow{\not\partial_y}+m) = -i\delta^4(x-y)$$

(again use $\not p^2 = p^2$)

"eq. for adjoint"

Wick Theorem for Spinors

$$T(\psi(x)\bar\psi(y)) = :\psi(x)\bar\psi(y): + \underbrace{\psi\bar\psi}_{S_F(x-y)}$$

(1) time order spinors same as normal order spinors + (2) with this in mind, the wick's theorem is the same as we'd expect given the bosonic case

(1) both time ordering and normal ordering are antisymmetric under fermion exchange!

Note: $T(\psi_1\psi_2\psi_3\psi_4) = \psi_3\psi_1\psi_4\psi_2(-1)^3$

$x_3^0 > x_1^0 > x_4^0 > x_2^0$

if want to order them in these times

remember this: every time you pass through something you get a minus sign done with free theory; now look at interactions

(II) Interactions : Yukawa theory

(the paper version of the baby scalar Yukawa theory; we will later contrast it with scalar Yukawa theory)

$$\mathcal{L}_0 = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

$$\mathcal{L}_{int} = -\lambda\phi\bar{\psi}\psi$$

mass of scalar field μ; mass of spinor m (fermion)

Inspect which type of theory we have by looking at units

→ be reassured no one ever asked this in lecture

(just thing we do when handed a new theory → inspect units!)

no exam question: state units!

$[\mathcal{L}] = 4 \quad , [\partial_\mu] = 1 \, , [\phi] = 1 \, , [\mu] = 1$

units of Dirac spinor – not same as scalar since it has 2 deriv.

$[\psi] = 3/2 \, , \quad [m] = 1 \quad , [\lambda] = 0$ marginal coupling

1) Feynman Rules

skip some steps here because lengthy but go over in lecture

Consider $\boxed{\text{nucleon-antinucleon scattering}}$ (note to self: $\psi\bar{\psi} \to \psi\bar{\psi}$)

initial state at $-\infty$: $|i;-\infty\rangle = \sqrt{2\omega_p}\sqrt{2\omega_q}\, b_p^{s\dagger}(-\infty)c_q^{r\dagger}(-\infty)|\Omega\rangle$

diff momenta, diff spins ↳ most general way to write

final state w/ diff momenta: $|f;+\infty\rangle = \sqrt{2\omega_{p'}}\sqrt{2\omega_{q'}}\, b_{p'}^{s'\dagger}(+\infty)c_{q'}^{r'\dagger}(+\infty)|\Omega\rangle$

LSZ argument: example of scattering, looks a bit of a mouthful to write

$$\langle f|S|i\rangle = \sqrt{2\omega_p}\sqrt{2\omega_q}\sqrt{2\omega_{p'}}\sqrt{2\omega_{q'}}\,\langle\Omega|T(c_{q'}^{r'}(+\infty)b_{p'}^{s'}(+\infty)b_p^{s\dagger}(-\infty)c_q^{r\dagger}(-\infty))|\Omega\rangle$$

need to know relationship between creation & annihilation operators at $+, -\infty$

we want $\quad b_p^{s\dagger}(-\infty) - b_p^{s\dagger}(+\infty) = ?$

→ to do this guide yourself w/ the free theory → gives a "well-educated" guess of what you consider

interacting field: shows up in full Lagrangian

Consider $\quad i\int d^4x\, \bar{\psi}(x)(i\overleftarrow{\not{\partial}} + m)U_s(p)e^{-ipx}$

(not sure why she started writing $U_s(p)$, not $U_s(\vec{p})$?)

$\hookrightarrow (\not{p} - m)U_s(p) = 0 \quad , \omega_p = \sqrt{\vec{p}^2 + m^2}$

massage the eq. argue how it's related to previous expression $=$ t.b.c. anyday :o

note to self: well-educated guess from eqn. for field $\psi(x)$ in free theory (as well as second order eq. for b_p^s seemed pulled from a "simple educated guess" ish?)

$U_s(p)$ is a spinor that obeys $(\not{p}-m)U_s(p)=0$ and the dispersion relation for ω_p

note to self:
- of scalar theory → again refer to ϕ as mesons (charge 0) & ψ as nucleons (charge ± 1) (or antinucleons $\psi^\dagger$)
- → but now nucleons ψ also have spin

also
$c_p^{s\dagger}$ acts on states to give antinucleons: $Q(c_p^{s\dagger}|0\rangle) = -c_p^{s\dagger}|0\rangle$
and $b_p^{s\dagger}$ acts on states to give nucleons: $Q(b_p^{s\dagger}|0\rangle) = b_p^{s\dagger}|0\rangle$ I think

normalisation $|b^s\rangle = \sqrt{2\omega_p}\, b_p^{s\dagger}|0\rangle$ ← basis $|\psi\rangle$

$|c^s\rangle = \sqrt{2\omega_p}\, c_p^{s\dagger}|0\rangle$ ← $|\bar{\psi}\rangle$ or $|\psi^\dagger\rangle$, ok?

Lecture 19

22.11.24

Interactions: Yukawa Theory cont.

Ex example of scattering process we considered

1) $\langle f|S|i\rangle = \sqrt{2\omega_p}\sqrt{2\omega_{p'}}\sqrt{2\omega_q}\sqrt{2\omega_{q'}}\,\langle\Omega|T(c_{q'}^{r'}(+\infty)\,b_{p'}^{s'}(+\infty)\,b_p^{s\dagger}(-\infty)\,c_q^{r\dagger}(-\infty))|\Omega\rangle$

Consider $i\int d^4x\,\bar\psi(x)(i\overleftarrow{\partial\!\!\!/}+m)U_s(p)e^{-ipx}$ where $U_s(p)$ obeys
$(p\!\!\!/-m)U_s(p)=0$
$\omega_p=\sqrt{\vec p^{\,2}+m^2}$

$= i\int d^4x\,\bar\psi(x)(i\gamma^0\overleftarrow{\partial}_0 + \underline{i\gamma^i\overleftarrow{\partial}_i} + m)U_s(p)e^{-ipx}$

note to self: integrate by parts for ∂_i term → flips arrow and introduces minus sign. (as usual ignoring boundary terms)

$= i\int d^4x\,\bar\psi(x)(i\gamma^0\overleftarrow{\partial}_0 - \underbrace{i\gamma^i(ip^i) + m}_{p^0\gamma^0})U_s(p)e^{-ipx}$ → use $(p^0\gamma^0 - p^i\gamma^i - m)U_s(p)=0$ [recall p190].

arrow shows in which direction this acts

remember $U_s(p)$ not affected by $\partial_i = \frac{\partial}{\partial x^i}$. remember the sign $\partial_i(e^{-ipx}) = ip^ie^{-ipx}$ → (for some reason now write p^i as $p^0 - p^i$ ... ?)

$= i\int d^4x\,\bar\psi\,(i\gamma^0\overleftarrow{\partial}_0 + p^0\gamma^0)U_s(p)e^{-ipx}$

$= i\int d^4x\,\bar\psi\,(i\gamma^0\overleftarrow{\partial}_0 + i\gamma^0\overrightarrow{\partial}_0)U_s(p)e^{-ipx}$

total deriv

$= -\int d^4x\,\partial_0(\underbrace{\bar\psi\gamma^0U_s(p)e^{-ipx}}) \to = -\int dt\,\partial_0\big(\underbrace{\int d^3x\,\gamma^0\bar\psi(x)e^{-ipx}U_s(p)}_{\frac{1}{\sqrt{2\omega_p}}b_p^{s\dagger}}\big)$

recall from F.T.: this is total deriv so eval at $t=\pm\infty$, at $t=\pm\infty$ they match w/ free theory

$= -\sqrt{2\omega_p}\,(b_p^{s\dagger}(+\infty) - b_p^{s\dagger}(-\infty))$

(note we have again used assumption that fields are the free fields at asymptotic times to arrive at this expression.)

write for all other combos b c

$$\left[\begin{aligned}
\sqrt{2\omega_p}\,(b_p^{s\dagger}(-\infty) - b_p^{s\dagger}(+\infty)) &= i\int d^4x\,\bar\psi(i\overleftarrow{\partial\!\!\!/}+m)U_s(p)e^{-ipx}\\
\sqrt{2\omega_p}\,(b_p^{s}(+\infty) - b_p^{s}(-\infty)) &= i\int d^4x\,e^{ipx}\bar U_s(p)(-i\partial\!\!\!/+m)\psi(x)\\
\sqrt{2\omega_p}\,(c_p^{s\dagger}(-\infty) - c_p^{s\dagger}(+\infty)) &= -i\int d^4x\,e^{-ipx}\bar V_s(p)(-i\partial\!\!\!/+m)\psi(x)\\
\sqrt{2\omega_p}\,(c_p^{s}(+\infty) - c_p^{s}(-\infty)) &= -i\int d^4x\,\bar\psi(x)(i\overleftarrow{\partial\!\!\!/}+m)V_s(p)e^{ipx}
\end{aligned}\right] \quad (*)$$

Finally using $(*)$ we get

note says the order ... is not important since i am scalar multiplying these two bracketed terms together (but if i had the spinors in the second bracket then order would matter)

$$\langle f|S|i\rangle = \prod_{j=1}^{4} i\int d^4x_j\,\big(\underbrace{e^{ip'x_3}\bar U_{s'}(p')(-i\partial\!\!\!/_3+m)}_{\text{outgoing } b^\dagger\ b}\big)\big(\underbrace{e^{-iqx_2}\bar V_r(q)(-i\partial\!\!\!/_2+m)}_{\text{ingoing } c^\dagger}\big)$$

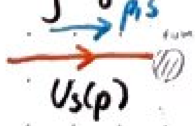
$$\langle\Omega|T\,\bar\psi(x_4)\psi(x_3)\bar\psi(x_1)\psi(x_2)|\Omega\rangle\,\underbrace{((i\overleftarrow{\partial\!\!\!/}_1+m)U_s(p)e^{-ipx_1})}_{\text{ingoing } b^\dagger}\,\underbrace{((i\overleftarrow{\partial\!\!\!/}_4+m)V_{r'}(q')e^{iq'x_4})}_{\text{outgoing } c^\dagger\ c}$$

Feynman Rules: $\langle f|S|i\rangle$ connected contribution

in principle also rules for scalars, save to before

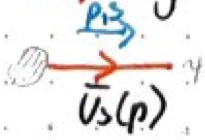
1) Incoming fermion (w/ momentum p & spin s)

assign momentum & spin

$b^\dagger$ ψ


$c^\dagger$ $\bar\psi$


2) Outgoing fermion



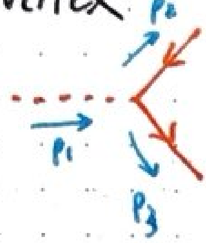
ψ b

$\bar\psi$ c

note to self: recall: particles ψ → b; antiparticles $\psi^\dagger$ ← $c^\dagger$

but before we had particles were from $c^\dagger$ but antiparticles $b^\dagger$ (?) is not?

3) Vertex



$$= (-i\lambda)(2\pi)^4 \delta(p_1 - p_2 - p_3)$$

the δ fn imposes conservation of momentum at a vertex.

4) Each internal line

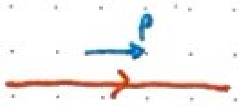
(both should be $\frac{d^4p}{(2\pi)^4}$ i think)

internal fermion only line to assign momentum, don't have to assign spin, because they're virtual particles / off shell so can be in any superposition of spin

scalars

$$= \int \frac{d^3p}{(2\pi)^3} \frac{i}{p^2 - \mu^2 + i\varepsilon}$$

fermions (note no arrow!)



$$= \int \frac{d^3p}{(2\pi)^3} \frac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon}$$

5) Indices contracted at vertex with other propagators + spinors

this means do $\bar{V}_s U_s$, $\bar{U}_s U$ etc. (i.e. thing with a bar goes on the left) since this makes a scalar, $V_s \bar{U}_s$ makes a 4×4 matrix but if any γ matrices are around eg if you need to sandwich these in the middle, the $\bar{V} V$ etc.!

be careful when computing corr. fn. → when swapping positions of fermions give - sign

6) Add extra minus sign for statistics of fermions

↳ why? Ex. Free theory — more examples in notes, consider free thing here

$$\langle \psi_1 \psi_2 \bar{\psi}_3 \bar{\psi}_4 \rangle$$

do this by Schw. D (can choose, why?) with things in same role

(Free thry.)

Schw. D $\quad (i\not{\partial}_x - m)\langle T\, \psi(x)\bar{\psi}(x_2) \ldots \psi(x_n)\rangle_0 = i \sum_j (x - x_j)(\text{signed contractions})$

can't have fermions go due to position get to that contraction

↳ remove $\psi(x)\bar{\psi}(x)$

see this from EOM $(\Box_y + m^2)\, v = (\ldots)\psi$?

$$\langle \psi_1 \psi_2 \bar{\psi}_3 \bar{\psi}_4 \rangle_0 = \int d^4y\, \underbrace{i\, \delta_F(x_1 - y)(i\overleftarrow{\not{\partial}}_y + m)}_{\delta(x_1 - y)} \langle \psi_y \psi_2 \bar{\psi}_3 \bar{\psi}_4 \rangle$$

note ours: remember contract ψ w/ $\bar{\psi}$

use Green's fn identities $S_F(x-y)(i\overleftarrow{\not{\partial}}_y + m) = -i\delta^4(x-y)$

ψ_y cannot contract w/ 2 but can contract w/ 3 → when in more than one place add minus sign

$$= \int d^4y\, (-i)\, S_F(x_1 - y)(i\overrightarrow{\not{\partial}}_y - m)\langle \psi_y \psi_2 \bar{\psi}_3 \bar{\psi}_4 \rangle$$

$$= (-i)^2 \int d^4y\, S_F(x_1 - y)\big(-\delta(x_3 - y) S_F(x_2 - x_4) + \delta(x_4 - y) S_F(x_2 - x_3)\big)$$

apparently there is an incorrect minus sign somewhere here?

$$= S_F(x_1 - x_4) S_F(x_2 - x_3) - S_F(x_1 - x_3) S_F(x_2 - x_4)$$

(6) this minus sign is what rule 6 reminds us of

2) Ex. $\boxed{\text{Nucleon to meson scattering}}$ $\quad \psi\bar{\psi} \to \phi\phi$

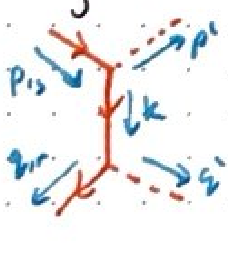
can get to handle expr. like for ex1 but here we apply Feynman rules

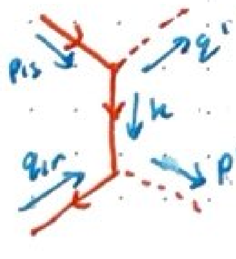
$$\underbrace{b_p^{s\dagger} c_q^{r\dagger}}_{\text{in}} \qquad \underbrace{a_{p'}^{\dagger} a_{q'}^{\dagger}}_{\text{out}}$$

To leading order + connected

switch positions of final states

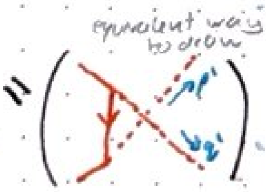
possible diagrams:





equivalent way to draw

$=$



only 2 poss diagrams to leading order

equivalent drawing w/ this crossing but tend to avoid this because we worry crossing is significant (it's not) lines don't intersect there is a jump!

write out digrs w/ rules, remember flow of arrows!

scales: external particles do not add anything

do Psys Mod follow arrows &

this is middle integral

new add vertices

$$\langle f|S|i\rangle = \int \frac{d^4k}{(2\pi)^4} \bar{V}_r(q) \frac{i(\not{k}+m)}{k^2-m^2} U_s(p)(-i\lambda)(2\pi)^4\delta(q+k-q')(-i\lambda)(2\pi)^4 \delta(p-p'-k)$$

(right order of V & U matters!)

to get 2nd diagram just switch p' & q' & don't bother writing twice since long

$$+ (p' \leftrightarrow q')$$

$$= i(2\pi)^4\delta(p+q-p'-q')\left(\frac{\bar{V}^r(q)(\not{p}-\not{p}'+m)U^s(p)}{(p-p')^2-m^2} + \frac{\bar{V}^r(q)(\not{p}-\not{q}'+m)U^s(p)}{(p-q')^2-m^2}\right)$$

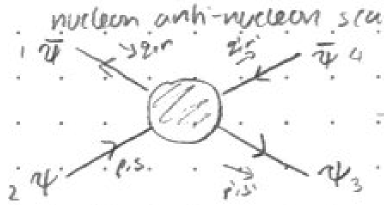
exchange role of p', q'

drop $i\varepsilon$ because not pole for physical values of p,q,p',q' don't give pole b/c it is T & U channel

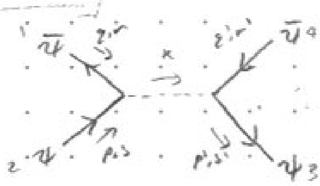
First Feynman diagram w/ Fermions!

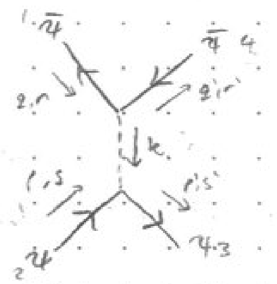
Note to self:

nucleon anti-nucleon scattering $\boxed{\psi\bar{\psi} \to \psi\bar{\psi}}$ (2 1 → 3 4)



→ diagrams are





scalar! but fig we've got

use feynman rules:

$$\langle f|S|i\rangle = \int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\varepsilon}[\bar{V}_r(q)U_s(p)](-i\lambda)(2\pi)^4\delta(q+p-k)[\bar{U}_{s'}(p')V_{r'}(q')](-i\lambda)(2\pi)^4\delta(k-q'-p')$$

$$+\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\varepsilon}[\bar{V}_r(q)V_{r'}(q')](-i\lambda)(2\pi)^4\delta(q-q'-k)[\bar{U}_{s'}(p')U_s(p)](-i\lambda)(2\pi)^4\delta(k+p-p')$$

$$= (-i\lambda)(2\pi)^4\delta(p+q-q'-p')\left[-\frac{i[\bar{V}_r(q)U_s(p)][\bar{U}_{s'}(p')V_{r'}(q')]}{(p+q)^2-\mu^2+i\varepsilon} + \frac{i[\bar{V}_r(q)V_{r'}(q')][\bar{U}_{s'}(p')U_s(p)]}{(p-p')^2-\mu^2+\cancel{i\varepsilon}}\right]$$

s channel so keep

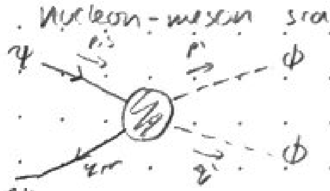
(apparently ignore $i\varepsilon$ for everything but s-channel)

note: ans in form $\langle f|S|i\rangle = i(2\pi)^4\delta^4(\sum p_i)A$ ← (scattering?) amplitude

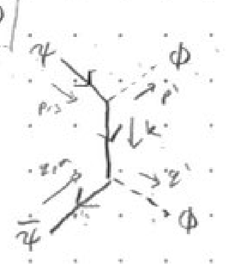
momentum conservation in whole interaction

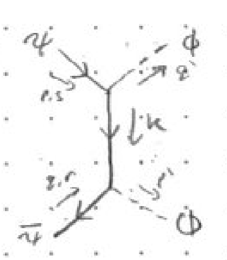
also important: minus sign for 1st diagram! we have $\psi_2\bar{\psi}_1\psi_3\bar{\psi}_4$, look at flow of arrows to understand contracting. in 1st diagram we have $\psi_2\bar{\psi}_1\ \bar{\psi}_4\psi_3$ → one swap so factor (−1). In 2nd diagram we have $\psi_2\psi_3\bar{\psi}_4\bar{\psi}_1$ → two swaps so factor 1. So will get different ans for $\psi\bar{\psi}\to\bar{\psi}\psi$ vs $\psi\bar{\psi}\to\psi\bar{\psi}$ for this reason i think.

nucleon-meson scattering $\boxed{\psi\bar{\psi} \to \phi\phi}$



→ diagrams are





sandwich!

remember internal terms are different (solid lines)

use feynman rules:

$$\langle f|S|i\rangle = \int\frac{d^4k}{(2\pi)^4}\bar{V}_r(q)\frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}U_s(p)(2\pi)^4\delta^4(p-k-p')(2\pi)^4\delta^4(k+q-q')(-i\lambda)^2$$

$$+\int\frac{d^4k}{(2\pi)^4}\bar{V}_r(q)\frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}U_s(p)(2\pi)^4\delta^4(p-k-q')(2\pi)^4\delta^4(q+k-p')(-i\lambda)^2$$

$$= (-i\lambda)^2(2\pi)^4\delta^4(p+q-p'-q')\left[\frac{\bar{V}_r(q)\,i(\not{p}-\not{p}'+m)U_s(p)}{(p-p')^2-m^2} + \frac{\bar{V}_r(q)\,i(\not{p}+\not{q}'+m)U_s(p)}{(p-q')^2-m^2}\right]$$

(drop $+i\varepsilon$ again)

Lecture 20 25.11.24

Note: about Schw-Dyson Dirac Field — last lecture incorrect

$$(i\partial\!\!\!/_x - m)\langle T\,\psi(x)\bar\psi(y)\rangle = \cdots + i\delta(x-y)$$

QUANTUM ELECTRO DYNAMICS

(I) Classical aspects
(II) Quantisation, free theory
(III) Interactions: QED

{theory of light interacting with matter}

(I) Classical Theory

seen gauge before $A^\mu = (\phi, \underline{A})$ local

Focus of classical aspects of Maxwell's theory (theory of light)

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu}$$ ← adv: the lagrangian for maxwell's eq. in the absence of any sources

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$ ← where def. of field strength tensor

review aspects of classical theory (Electromagnetism) to put into context what we are doing

def. $\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$ and $\vec{B} = \nabla\times\vec{A}$ apparently gives

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

A_μ: gauge field (potentials)

$F_{\mu\nu}$: field strength $(\vec{E}, \vec{B})$

(variation: $\frac{\partial}{\partial(\partial_\mu A_\nu)}\partial_\alpha A_\beta \partial^\alpha A^\beta = 2\partial^\mu A^\nu$)

E.O.M.: $\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0$

$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0$ since $\mathcal{L}$ depends only on derivatives of A_ν, 2nd term is zero ∴ 1st term is zero (EOM is just 1st term)

if you go through the working then 1st term $\partial_\mu F^{\mu\nu}$ so eq. (1) is also EOM (using this)

EOM? ↓

"$\partial_\mu F^{\mu\nu} = 0$" $\Rightarrow$ $\boxed{\partial_\mu F^{\mu\nu} = 0}$ (1)

these give Maxwell equations if written out (think component wise)

find from EoL, but also have eqns from ↓

Bianchi identity — field strength $F^{\mu\nu}$ also satisfies the Bianchi identity

$$\boxed{\varepsilon_{\mu\nu\rho\sigma}\partial^\rho F^{\mu\nu} = 0}$$

can also be written:

$$(\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda}) = 0$$

Gauge symmetry

write in components

Look at (1): $\partial_\mu\partial^\nu A_\nu - \partial_\nu\partial^\mu A_\mu = 0$

$(\nu = i)$ $\quad \Box A_i - \partial_i(\partial_0 A^0 + \partial_j A^j) = 0$

$(\nu = 0)$ $\quad \Box A_0 - \partial_0(\partial_0 A^0 + \partial_j A^j) = 0$

$$\vec{\nabla}^2 A_0 + \partial_0(\vec{\nabla}\cdot\vec{A}) = 0$$

can write down general solution b/c A does not have time derivative

$$\vec{A}_0 = \int d^3x' \frac{1}{4\pi|\underline{x}-\underline{x}'|}\left(\vec{\nabla}\cdot\frac{\partial\vec{A}}{\partial t}\right)(\vec{x}')$$

A_0 is fully determined by A.

Second note, we have also a redundancy

Summary: A^μ has 4 components but photon only has 2 dof in key?

- consider EOM 0 component $\partial_\mu F^{\mu 0} = 0$ expanding out gives $\nabla^2 A_0 + \partial_0(\nabla\cdot\vec{A}) = 0$
- a no $\partial_t A_0$ terms in lagrangian ⇒ A_0 is "not dynamical" so doesn't depend on time, it has solution $A_0(\vec{x}) = \int d^3y \frac{\nabla\cdot(\partial_t\vec{A}(\vec{y}))}{4\pi|\vec{x}-\vec{y}|}$ so we see
- A_0 is not a dof since it is fully determined by $\partial_t\vec{A}$. we still need to get rid of one more dof:
- lagrangian has symmetry $A_\mu(x) \to A_\mu(x) + \partial_\mu\lambda$ (easy to show that $F_{\mu\nu}$ is inv. under this ∴ lagrangian is inv) → inv. w.r.t. symmetry transformed
- but this is a gauge symmetry not a global symmetry, so we don't get an infinite no. of conservation laws. global symmetry takes one state to another state w/ the same physics but gauge symmetry is "just" a redundancy in our description
- how do we show this?
  - → write EOM $\partial_\mu F^{\mu\nu}$ as $(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu = 0$ but the operator in brackets annihilates any fn. of the form $\partial_\mu\lambda$, ∴ it is not invertible → i.e. there is no way to uniquely determine A_μ, we cannot distinguish between A_μ & $A_\mu + \partial_\mu\lambda$
  - → if we identify A_μ & $A_\mu + \partial_\mu\lambda$ as the same "physical" object then our problems disappear!
- another reason to see why this symmetry gives a redundancy is to look at Noether's theorem

GAUGE TRANSFORMATION

$$\boxed{A_\mu \to A_\mu + \partial_\mu \lambda = A'_\mu}$$

$\lambda(x)$: any regular function

note: any smooth fn. that dies off suitably quickly at spatial infinities / abort has compact support!

$$F_{\mu\nu} \to F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$$
$$= \partial_\mu (A_\nu + \partial_\nu \lambda) - \partial_\nu (A_\mu + \partial_\mu \lambda)$$
$$= F_{\mu\nu}$$

transf. A_i, F not affected and A not affected

have to be careful b/c this symmetry tells us we have a redundancy of the system

The existance of this gauge symmetry actually introduces a redundancy. One way to see this is from (+)

$$\underbrace{(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)}_{\text{not invertible operator}} A^\nu = 0$$

since $A^\nu = \partial^\nu \lambda$ solves automatically

$(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\partial^\nu\lambda = 0$ trivially for any smooth λ

Another reason: look at Noether's theorem

apply noether to this symmetry

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \delta A_\nu = -F^{\mu\nu}\partial_\nu \lambda$$
$$\overset{\text{eom}}{=} -\partial_\nu(\lambda F^{\mu\nu})$$

current computed on shell always so can use eom

$$Q = \int d^3x\, j^0 = -\int d^3x\, \partial_i(\lambda F^{0i})$$

total deriv so $= 0$

note to self: since we have total derivative and λ dies off at infinities, it is $= 0$

for any compact support λ $= 0$

no charge associated to gauge symmetry

(a bit of a lie but don't worry about it for this course)

note: conclude gauge symmetries do not take a physical state to another physical state w/ the same properties (as global symmetries do), they are instead redundancies in our description!

Gauss law:

$$Q = \int d\vec{A} \cdot \vec{E} \quad \vec{E} = F^{0i} \text{ — in this notation}$$

integrating by parts

$$= \int_{\partial V} d^3x\, n_i F^{0i} \lambda$$

normal, tells in which direction we do integration by parts — evaluated at boundary

(reconcile how we construct charge in EM and via noether theorem)

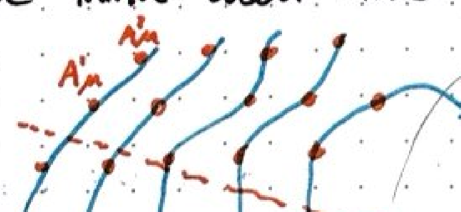
kind of unnecessary comment but don't want to leave this lie hanging

how do we fix redundancy?

The way we think about this: *PTO

pictures how we deal w/ gauge symmetry (gauge fixing)

A_μ] - solve for A_μ solutions to maxwell eqns.



solutions to maxwell eq
if solutions are related by a gauge transformation then draw in line direction solutions

gauge orbits
is solutions related via $\partial_\mu\lambda$ ← gauge transf

Gauge fixing — take slice of gauge space where we hit each of the gauge orbits once

what is a good choice for orange dotted line?

Fix a gauge:

we discuss 2 possibilities (both are terrible but they have pros & cons)

1) Lorentz gauge

how to gauge fix → get rid of ambiguity by imposing a constraint on A^μ itself; get rid of redundancy/fact it can move w/ λ

defined $\partial_\mu A^\mu = 0$

show that if we start w/ something that does not comply w/ this we can always bring it to satisfy this condition

Say A'_μ s.t. $\partial_\mu A'^\mu = f(x)$

$$A_\mu = A'_\mu + \partial_\mu \lambda$$

start w/ something that does not comply w/ gauge condition but we can bring it to gauge condition

$$\partial_\mu A^\mu = \partial_\mu A'^\mu + \Box\lambda$$
$$= f + \Box\lambda = 0$$

A, A' live in same gauge orbit b/c they are related via gauge transf.

Construct λ s.t. → start on blue line, find the λ that brings me to the orange dotted line

$$\Box\lambda = -f$$

(still having we can do)

notes: $\Box\lambda = -f$ always has a solution, so we can always find the λ that brings me to the orange line

A^μ, A'^μ, $\| A'^\mu + \partial^\mu\lambda$

(can always find the λ that brings A^μ onto the dotted line)

Downside: still residual transformations if $\Box\lambda = 0$ — can make further transformations w/ $\partial_\mu\partial^\mu\lambda = 0$ s.t. "this condition doesn't pick a unique representative from the gauge orbit" (downside)

Upside: it is Lorentz invariant

2) Coulomb gauges

defined $\vec{\nabla}\cdot\vec{A} = 0$

call it alpha b/c only spatial part (?)

→ can show we can always do gauge transform s.t. the spatial part is brought to comply w/ the condition $\vec{\nabla}\cdot\vec{A} = 0$ (consider $A_i \to A_i + \partial_i\alpha$ & same prog as for Lorentz gauge) but still have residual freedom in A_0? no since we showed that A_0 is fully determined by $\vec{\nabla}\cdot\vec{A}$ earlier so if that's 0 then $A_0 = 0$ → no residual dof remaining!

As before $A_i \to A_i + \partial_i\alpha$ to comply with condition. (transf s.t. spatial part complies w/ condition)

Still residual $\vec{\nabla}^2\lambda = 0$, however $\vec{\nabla}^0 A_0 = 0$

Use λ to set $A_0 = 0$

$$A_0' = A_0 + \partial_0\lambda$$

Upside: all freedom is gone — no residual transf. — only 2 dof since 3 components of $\vec{A}$ satisfy a single constraint $\nabla\cdot\vec{A} = 0$

$$\vec{\nabla}\cdot\vec{A} = 0 \;,\; A_0 = 0$$

w.l.o.g. can reduce to this form

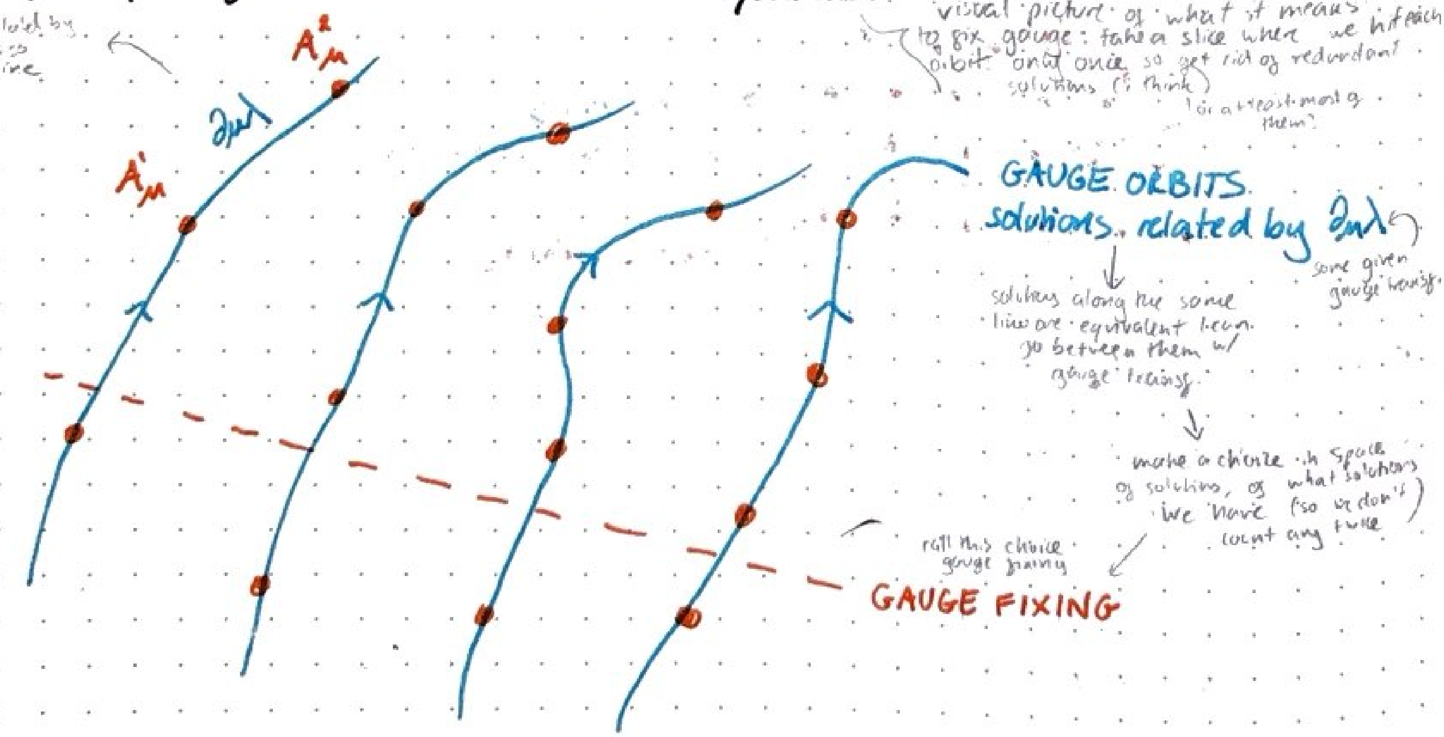
Downside: not Lorentz invariant.

---

* Want to find A_μ (solutions of Maxwell equations). Imagine I draw the space of solutions for A_μ i.e. space of solutions to Maxwell equations. (draw dot for every solution)

visual picture of what it means to fix gauge: take a slice where we hit each orbit only once, so get rid of redundant solutions (I think) (or at least most of them?)

A_μ, A'_μ related by gauge transf so draw line



write big

Lecture 21

27.11.24

Recap: Gauge fixing (eliminating "λ")

1) Lorentz gauge: $\partial_\mu A^\mu = 0$

2) Coulomb gauge: $\vec{\nabla}\cdot\vec{A} = 0$

this concludes classical thing → now we try to quantise this thing

(II) Quantisation

want to quantise the free maxwell theory!

"could have a whole lecture course on this" ↳ here we present short version

1) Canonical Quantisation

(can also do in lor. gauge but takes too long)

→ focus on Coulomb gauge (b/c here we get rid of redundancy)

$\vec{\nabla}\cdot\vec{A} = 0, \quad A_0 = 0$

EOM for A^i: $\Rightarrow \Box A_i = 0$ ($\partial_\mu\partial^\mu A_i = 0$)

EOM for $\vec{A}$ is: $\partial_\mu\partial^\mu \vec{A} = 0$

(show from EOM $\partial_\mu F^{\mu\nu} = 0$ in coulomb gauge. $\vec{\nabla}\cdot\vec{A} = 0, A_0 = 0$ consider i component: $\partial_\mu\partial^\mu A^i - \partial_\mu\partial^i A^\mu = 0$, $\partial_\mu\partial^\mu A^i - \partial_0\partial^i A^0 - \partial_j\partial^i A^j = 0$ $\Rightarrow \partial_\mu\partial^\mu A^i = 0$) $\partial_i A^i = 0$

klein gordon eq w/o mass term

solve in usual way, make ansatz: $A_\mu \sim \varepsilon_\mu(p) e^{-ipx}$ with $p^2 = 0$ → $(p^0)^2 - \vec{p}^2 = 0$

EOM is solved by this eq w/ $p^2 = 0$

also have constraints on ε_μ

The choice of gauge restricts $\varepsilon_\mu(p)$

For coulomb gauge:

since $A_0 = 0$, $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow$

$\varepsilon_0 = 0$

$\vec{p}\cdot\vec{\varepsilon} = 0$

if you want this is the analogue of vs & us from dirac eq

Choose a frame, e.g. $p_\mu = (E, 0, 0, E)$ → (put all spatial momentum in 1 direction) simple choice

two solutions are (form a basis)

$\varepsilon_\mu^1 = (0,1,0,0)$

$\varepsilon_\mu^2 = (0,0,1,0)$

→ two polarisation states (know from E-M)

We will write in general

in general we write $\vec{\varepsilon}$ as a linear combination of orthonormal vectors $\vec{\varepsilon}_r$ $r=1,2$ which satisfy

$\vec{\varepsilon}_r(\vec{p})\cdot\vec{p} = 0 \qquad r = 1,2$

$\vec{\varepsilon}_r\cdot\vec{\varepsilon}_s = \delta_{rs}$

2 dof. for two vectors i.e they correspond to the two polarisation states of the photon

Completeness relation

you can check that this setting the identity is true

sum over 2 poss. polarisation states

$$\sum_{r=1}^{2} \varepsilon_r^i(p)\varepsilon_r^j(p) = \delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2}$$

(can say: act on both sides with the polarisation vectors & the 3 momentum?)

The general solution is (in coulomb gauge)

$A(x) \sim \vec{\varepsilon}(p) e^{-ipx} + \vec{\varepsilon}(p) e^{ipx}$ $\sim \sum \vec{\varepsilon}^r(a_p^r e^{-ipx} + a_p^{r\dagger} e^{ipx})$ then integrate all momenta

$$\vec{A}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{r=1}^{2} \vec{\varepsilon}^r (a_p^r e^{-ipx} + a_p^{r\dagger} e^{ipx})$$

note to self: like superposing solutions w/ an orth. basis vectors ↓ $A \sim \vec{\varepsilon} e^{-ipx} + \vec{\varepsilon} e^{ipx}$ split $\vec{\varepsilon}$ into a linear combination of $\vec{\varepsilon}^1, \vec{\varepsilon}^2$ orthonormal vector, weights by a and then integrate over all p

Next, commutation relation $\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} \equiv E^i$$

If $\vec{\nabla}\cdot\vec{A} = 0$ it also implies $\vec{\nabla}\cdot\vec{E} = 0$ (1)

Naive guess: impose equal time comm. rel and state equal to RHS

$$[A_i(\vec{x},t), \pi_j(\vec{y},t)] = i\delta_{ij}\delta^3(\vec{x}-\vec{y})$$

notice: incompatible w/ $\vec{\nabla}\cdot\vec{E} = 0$ (1)

Wrong! because it doesn't comply with gauge condition.

$$[\vec{\nabla}\cdot\vec{A}, \vec{\nabla}\cdot\vec{E}] = i\nabla^2\delta^3(x-y)$$

incompatible with $\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{E} = 0$

$\neq 0$!!!

there is a way you can figure out the ans but in this course I just tell you what the ans. is

A better guess $E_j(\vec{y},t)$

$$[A_i(\vec{x},t), \pi_j(\vec{y},t)] = i\left(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\right)\delta^3(\vec{x}-\vec{y})$$

write in momentum space

$$= i\int \frac{d^3p}{(2\pi)^3}\left(\delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2}\right)e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \quad (\star)$$

Check if it complies with (1)

$$[\partial_i A_i, \pi_j] = i\int \frac{d^3p}{(2\pi)^3}\left(\delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2}\right) i p_i e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = 0$$

($\vec{\nabla}E_j$)

($\checkmark$ just complies w/ $\vec{\nabla}\cdot\vec{A} = 0$) easy to show complies w/ $\vec{\nabla}\cdot\vec{E} = 0$ by $[A_i, \partial_j\pi_j] = 0$ (I think)

($\star$) is the correct thing

we also have

$$[A_i(\vec{x},t), A_j(\vec{y},t)] = 0$$

& $[E_i(\vec{x},t), E_j(\vec{y},t)] = 0$

From here, straight forward to see that ($\star$) implies

$$[a^r_p, a^{s\dagger}_q] = (2\pi)^3\delta^{rs}\delta^3(\vec{p}-\vec{q})$$

$\Rightarrow [a^r_p, a^s_q] = [a^{r\dagger}_p, a^{s\dagger}_q] = 0$

and all other zero.

need to use completeness relation $\sum_{r=1}^{2} \epsilon_r^i(p)\epsilon_r^j(p) = \delta^{ij} - \frac{p_i p_j}{|\vec{p}|^2}$ to show this apparently

Hamiltonian

$$H = \int d^3x\,(\pi^i \dot{A}_i - \mathcal{L})$$

$$= \tfrac{1}{2}\int d^3x\,(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B})$$

$E^i = -F^{0i} \Rightarrow E^i = -\partial^0 A^i$

$\varepsilon^{ijk}B_k = F^{ij}$ for me easier to obj. $\vec{B} = \nabla\times\vec{A}$ i.e. $B_k = \varepsilon^{ijk}\partial_i A_j$

$\pi^i\dot{A}_i = -E^i E_i$

$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu)$

$= -\tfrac{1}{2}(\underbrace{\partial_0 A_i \partial^0 A^i}_{E^2} + \underbrace{\partial_j A_i \partial^j A^i - \partial_j A_i \partial^i A^j}_{B^2 \text{ using } (\star)})$

$\mathcal{L} = -\tfrac{1}{2}(E^2 + B^2) \Rightarrow H = \int d^3x\,(-E^2 + \tfrac{1}{2}E^2 + \tfrac{1}{2}B^2)$

after normal ordering

$$H = \int \frac{d^3p}{(2\pi)^3}|\vec{p}|\sum_{r=1}^{2} a^{r\dagger}_p a^r_p$$

write $\vec{E}, \vec{B}$ in terms of a_p^r using $A_i(x)$ seems long

$\Rightarrow$ Perfectly good fock space

we have done canonical quantization in coulomb gauge. you can also do in Lor. gauge. but

Propagator:

Coulomb gauge

very fine answer but not lor. inv. :(

(I don't exactly know how we get here) →

$$\langle 0|T A_i(x) A_j(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4} \left(\frac{i}{p^2+i\epsilon} \left(\delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2} \right) e^{-ip(x-y)} \right)$$

But I want

what trick do we use now? we know propagator = greens fn, use gre...

$$\langle 0|T A_\mu(x) A_\nu(y)|0\rangle$$

goal: want to compute this but not necessarily in coulomb gauge

1) Answer should be Lor. inv.
2) Strategy: use the fact that it is a Green's function

let's talk about greens fns

Green's functions:

think of trying to solve the problem w/ an arbitrary source

side note: talk about scalar field

→ scalar field: $(\Box + m^2) G(x) = J(x)$

way to solve this: move to fourier space

$$(\Box + m^2) \int \frac{d^4p}{(2\pi)^4} G(p) e^{ipx} = \int \frac{d^4p}{(2\pi)^4} J(p) e^{ipx}$$

Remind Green's fns:
$LG(x,s) = \delta(x-s)$
$Lu(x) = f(x)$
↑ is solved by
$u(x) = \int G(x,s) f(s) ds$
where G satisfies the eq. above

$$\hookrightarrow \int \frac{d^4p}{(2\pi)^4} \left((-p^2+m^2) G(p) - J(p) \right) e^{ipx} = 0$$

$$G(p) = \frac{J(p)}{-p^2+m^2}$$

$$\Rightarrow G(x) = \int \frac{d^4p}{(2\pi)^4} \frac{J(p)}{-p^2+m^2} e^{ipx}$$

(since $\int \frac{d^4x}{(2\pi)^4} e^{ipx}$ gives? $= \delta^4(p)$ cancelling fn? wh?)

when $J(x) = i\delta^4(x)$, $J(p) = -i$

$$G(x) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} e^{ipx}$$

Apply to Maxwell's equation

$$\partial^\mu F_{\mu\nu} = J_\nu(x)$$

EOM $\partial^\mu F_{\mu\nu}$ → (with a current added on rhs?)
can write as
$\partial^\mu F_{\mu\nu} = (\partial^\sigma \partial_\sigma \eta_{\mu\nu} - \partial_\mu \partial_\nu) A^\mu$

$$\Box A_\nu - \partial_\nu \partial^\mu A_\mu = J_\nu(x)$$

in momentum space

$$(-p^2 \eta_{\mu\nu} + p_\mu p_\nu) A^\mu(p) = J_\nu(p)$$

$$\cancel{A^\mu = (-p^2 \eta^{\mu\nu} + p^\mu p^\nu)^{-1}} J_\nu$$

→ can't do this b/c no inverse.

However we can't invert this

$$(-p^2 \eta^{\mu\nu} + p^\mu p^\nu) p_\nu = 0$$

↳ zero eigenvalue hence no inverse

$(M - \lambda\mathbb{I})v = 0$

? eigenvalues $Mv = \lambda v$
so eigenvalue zero since
$Mv = 0$
luca shows
$Mp^\mu = 0$ rather than $\partial^\mu \lambda$?

equivalent statements

Recall

$$(\eta_{\nu\mu} \Box - \partial_\nu \partial_\mu) \partial^\mu \lambda = 0$$

'basically i'm going to change my theory'

To get rid of this, we introduce a Lagrange multiplier

$$\mathcal{L} = \overbrace{-\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu}}^{\text{beautiful lagrangian}} \overbrace{- \tfrac{1}{2\alpha} (\partial_\mu A^\mu)^2}^{\text{'add something that might offend you'}}$$

↳ this term is not gauge inv

where α is a non-prop (const.) aux field which acts as a Lagrangian multiplier

e.o.m. α: $\partial_\mu A^\mu = 0$

can invert this so can use source trick

e.o.m. A_μ: $(\eta^{\nu\lambda}\Box + (1/\alpha - 1)\partial^\nu\partial^\lambda) A_\lambda = 0$

Now, going back to Green's function

$$\underbrace{\left(-\eta_{\lambda\nu}p^2 - (1/\alpha - 1)p_\nu p_\lambda\right)}_{\hat{\Pi}_{\lambda\nu}} \Delta^\lambda(p) = J_\nu(p)$$

now $\hat{\Pi}_{\lambda\nu}$ has an inverse!

$$\Pi_{\mu\nu} = -\frac{\eta_{\mu\nu} + (\alpha - 1)\, p_\mu p_\nu / p^2}{p^2}$$

where $\hat{\Pi}^{\lambda\nu}\Pi_{\nu\mu} = \delta^\lambda{}_\mu$. you can check this

remember this trick!
↳ always think of propagator as a greens fn.

Lecture 22

29.11.24

Recap: * Propagator $\langle 0|T A_\mu(x) A_\nu(y)|0\rangle$
we want to compute this but not neccarily in Coulomb gauge

* Trick

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \underbrace{\tfrac{1}{2\alpha}}_{\text{Lagrange multiplier}}(\partial_\mu A^\mu)^2$$

propose trick is introduce this term

(ensure gauge condition as eq of motion)

* The appropriate Green's function

add source which will basically be δ fn.

get propagator by understanding it as a Green's fn.

$$(\eta_{\lambda\nu}\Box + (\tfrac{1}{\alpha}-1)\partial_\nu\partial_\lambda)A^\lambda = J_\nu$$

write in mom. space

$$(-\eta_{\lambda\nu}p^2 - (\tfrac{1}{\alpha}-1)p_\nu p_\lambda)A^\lambda(p) = J_\nu(p)$$

Now, we need to invert

$$\hat{\Pi}_{\mu\nu} := -\eta_{\mu\nu}p^2 - (\tfrac{1}{\alpha}-1)p_\mu p_\nu$$

note: introduced lagrange multiplier as $1/\alpha$ so that here it turns up as α (convenient)

The inverse is

$$\Pi^{\mu\nu} = (\hat{\Pi}^{-1})^{\mu\nu} = \frac{-\eta_{\mu\nu} + (\alpha-1)\frac{p_\mu p_\nu}{p^2}}{p^2}$$

check

$$\hat{\Pi}^{\mu\nu}\Pi_{\mu\alpha} = \delta^\nu{}_\alpha$$

Therefore, what we expect from propagator

what I mean by I want propagator to obey greens eq

"eom" $\langle T A_\mu(x) A_\nu(y)\rangle$ = "delta"

$$\hookrightarrow \langle 0|T A_\mu(x)A_\nu(y)|0\rangle = i\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2+i\varepsilon}\underbrace{-}_{\text{minus sign}}\left(\eta_{\mu\nu} + (\alpha-1)\frac{p_\mu p_\nu}{p^2}\right)$$

remark on - sign: gauge field spatial components of A in coulomb gauge work like scalars so should expect same sign for propagator for scalar. This is what - sign does. Remember η gives - sign so get $+\delta_{ij}$

(more complicated case of what we did with scalar last lecture)

$$= \underbrace{i\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\Pi_{\mu\nu}}_{\text{Lorentz invariant ✓}}$$

yay

Notes:

i) The minus sign is correct, if $\mu=i$, $\nu=j$, will give positive sign for physical component

(compromise we made to get lor. invariance)

α is artificial, not a physical quantity → might worry that propagator depends on something that is not physical.

ii) Choices ~~α.~~ Why is α there? Should it bother us? – (in principle, yes)
α is unphysical ⇒ S-matrix should not depend on α – if it does, we made a mistake

iii) Choices for α

$\alpha = 1$: Feynman - t'Hooft gauge

nice choice, "Feynman was a bit lazy & simplify things". convenient for computation as propagator is just $\eta_{\mu\nu}$

$\alpha = 0$: Lorentz (Landau) gauge
(strong enforcement of Lor. gauge)

from part of lagrangian the gauge condition is super big so strongly enforces Lor. gauge.

$\alpha \to \infty$: Unitary gauge
(useful in non-Abelian/massive gauge fields, weak interactions)

convenient for later on when you do non abelian gauge theories for weak interactions

[mention for completeness: not useful here in QED (ignore)]

(III) Interactions : couple light to matter

(1) Actions

(we've coupled light to matter before ; add current in maxwell eqns.)

Maxwell equations $\partial_\mu F^{\mu\nu} = j^\nu$

one consequence, $\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu$

$\underset{\shortparallel}{}$ 0

$\rightarrow j^\nu$ is a conserved current

If j^ν is independent of A_μ itself, then

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu$$

Look at action and gauge transform

$$S = \int d^4x \mathcal{L}$$

$$\rightarrow \int d^4x \left(-\tfrac{1}{4}F^2 - j^\mu(A_\mu + \partial_\mu \lambda)\right)$$

$$= \int d^4x \mathcal{L} - \int d^4x \left(\underbrace{\partial_\mu(j^\mu \lambda)}_{\text{zero b/c total deriv}} - \partial_\mu j^\mu \lambda\right)$$

respects gauge symm. if current is conserved

Action is gauge invariant if current is conserved!

With this let's couple light to spinors

Dirac Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi$$

this theory has a current

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi \quad \text{due to} \quad \begin{aligned} \Psi &\to e^{i\alpha}\Psi \\ \bar{\Psi} &\to e^{-i\alpha}\bar{\Psi} \end{aligned}$$

$\hookrightarrow \partial_\mu j^\mu = 0$

So let's consider

maxwell part / dirac lagrangian part / part that makes them talk to eachother according to that rule

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\partial\!\!\!/ - m)\Psi - e j^\mu A_\mu$$

$$= -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\partial\!\!\!/ - eA_\mu\gamma^\mu - m)\Psi$$

$$= -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i \not{D} - m)\Psi$$

similar to in GR:
how do you parallel transport in presence of a gauge field
like Γ_μ

where

$$\boxed{D_\mu = \partial_\mu + ie A_\mu}$$

$\hookrightarrow \Gamma_\mu$

COVARIANT DERIVATIVE

$\not{D} = D_\mu \gamma^\mu$

check gauge invariance: arbitrary fn. of space

$$A_\mu \to A_\mu + \partial_\mu \lambda(x)$$
$$\Psi \to e^{-ie\lambda(x)} \Psi$$
$$\bar{\Psi} \to e^{ie\lambda(x)} \bar{\Psi}$$

now look at how covariant deriv acting on Ψ transforms.

now

$$D_\mu \Psi \to \partial_\mu (e^{-ie\lambda} \Psi) + ie(A_\mu + \partial_\mu \lambda) e^{-ie\lambda} \Psi$$
$$= e^{-ie\lambda}(\partial_\mu - \cancel{ie(\partial_\mu \lambda)} + ie A_\mu + \cancel{ie \partial_\mu \lambda})\Psi$$
$$= e^{-ie\lambda} D_\mu \Psi$$

Then

$$\bar{\Psi} \not{D} \Psi \to \bar{\Psi} \not{D} \Psi$$

$\Rightarrow$ Action is gauge invariant!

(so we have coupled a dirac spinor to a gauge field in a way that gauge inv. / conservation etc. are preserved

Look at Noether's theorem

$$Q = \int d^3x\, F^{0i} \partial_i \lambda$$
$$= -\int d^3x\, \partial_i F^{0i} \lambda$$

integrate by parts (ignore total deriv (?))

$\swarrow$ set $\lambda = 1$ — pick const. gauge transformation

$$= e \int d^3x\, j^0$$
$$= -e \int d^3x\, \bar{\Psi} \gamma^0 \Psi$$

$\lambda =$ const.
called this a large gauge transf. since it does not vanish at ∞

gauge symmetry themselves don't give us conserved charges but they can when coupled to matter!

note that we pick $\lambda = 1$. Gauge symmetries, if they contain global symmetries (internal) where $\lambda =$ const., then we have charge associated to global symmetry.
Also note that $\lambda =$ const. is a "large gauge transform".

conserved charge controlled by number e which tells me how light talks to matter, but important to remark in this context it only comes from the fact we had matter in it.

Lecture 23 2.12.24

derived this

Recap: • coupled light to matter
QED Lagrangian $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\Psi}(i\not{D} - m)\Psi$

• $D_\mu \Psi = \partial_\mu \Psi + ie A_\mu \Psi$ covariant derivative

• Now theory is invariant under
$$A_\mu \to A_\mu + \partial_\mu \lambda(x)$$
$$\Psi \to e^{-ie\lambda(x)} \Psi$$
$$\bar{\Psi} \to e^{ie\lambda(x)} \bar{\Psi}$$
promoted global symmetry of Ψ to local symmetry

$$\mathcal{L}_{int} = -e \bar{\Psi} A_\mu \gamma^\mu \Psi$$

units: e is coupling constant

homework: find dimension of e

HW: $[e] = 0$ marginal coupling

fine structure const

traditional to introduce α
$$\alpha = \frac{e^2}{4\pi\hbar c} \sim \frac{1}{137}$$ fine structure constant

conserved current

SQED: coupling gauge fields to scalars
$$\partial_\mu F^{\mu\nu} = j^\nu \quad \text{with} \quad \partial_\nu j^\nu = 0$$

no internal symmetry for real scalar fields

impossible to couple to real scalars (no internal symmetry)
↪ however for complex scalar we do!
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \Psi^*)(D^\mu \Psi) - m|\Psi|^2$$
$$D_\mu \Psi = \partial_\mu \Psi + ie A_\mu \Psi$$
invariance is under:
$$A_\mu \to A_\mu + \partial_\mu \lambda$$
$$\Psi \to e^{-ie\lambda(x)} \Psi$$
$$\Psi^* \to e^{ie\lambda(x)} \Psi$$

Interaction term: $\mathcal{L}_{int} = ie \underbrace{\left((\partial_\mu \Psi^*)\Psi - \Psi^* \partial_\mu \Psi\right)}_{J_\mu \text{ w/o } A^\mu} A^\mu + e^2 A_\mu A^\mu \Psi^* \Psi$

Revisit Noether's theorem with $\mathcal{L}_{int}$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi}\delta\psi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^*}\delta\psi^* \qquad \text{for } \lambda = \text{const.}$$

$$\underset{\lambda=1}{\overset{}{=}} (-ie)\Big((\partial_\mu\psi^*)\psi - \psi^*(\partial_\mu\psi) - \underbrace{2ieA_\mu\psi^*\psi}_{\text{Extra?}}\Big)$$

note here to be more careful w/ trick saying variation in action is just $j^\mu A_\mu$ here

lesson: if want to couple things to gauge fields: then promote partial deriv to covariant deriv

$\boxed{\text{Lesson:}}$ minimal coupling is done by promoting $\partial_\mu \to D_\mu$

↳ also good way of making sure theory is gauge inv.

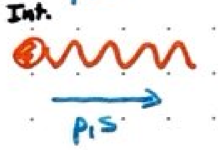
Feynman Rules QED

External lines:

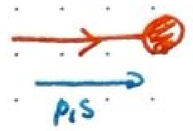
① Photons: add polarisation vector

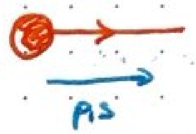
blob means where scattering is happening (where they interact)

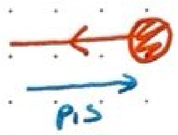

$= \varepsilon_\mu^s(p)$ incoming

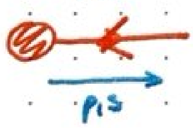

$= \varepsilon_\mu^{s*}(p)$ outgoing

② Fermions


$: u^s(p)$ incoming


$: \bar{u}^s(p)$ outgoing

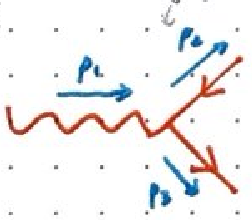

$: \bar{v}^s(p)$ incoming


$: v^s(p)$ outgoing

Internal lines:

① Vertex

2 spinors attach & gauge field

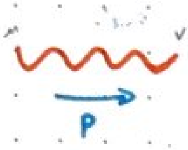


$$: -ie\gamma^\mu \underbrace{(2\pi)^4\delta(p_1 - p_2 - p_3)}_{\text{conservation of momenta}}$$

prop. i.e.

interaction has γ coupling → it is a matrix later comment: each vertex has its own index, don't repeat indices

② Photon propagator



$$: \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2+i\varepsilon} \underbrace{\left(-\eta_{\mu\nu} - (\alpha-1)\frac{p_\mu p_\nu}{p^2}\right)}_{\tilde{\Pi}_{\mu\nu}}$$

convenient shorthand

③ Fermion propagator

slash since fermion propagator is also a matrix (remember).

$$: \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} (\not{p}+m)$$

and we have minus signs if fermions are swapped.

(comment: relative minus signs are important but not as much overall minus sign)

note: just writing rules here, not deriving them, but you have the tools to see/derive this yourself (also on ex sheet 4)

With this we will evaluate S-matrix

$$\langle f | S - \mathbb{1} | i \rangle = iA\, (2\pi)^4 \delta^4\left(\sum_f p_f - \sum_i p_i\right)$$

↳ amplitude

We have two requirements on S-matrix:

1) Internal photon

$$A = A_{\mu\nu} \tilde{\Pi}^{\mu\nu}$$

can be multiple powers of this: here just show for one internal photon

↳ gauge invariant $\Rightarrow$ $\boxed{A_{\mu\nu} p^\mu p^\nu = 0}$ (1)

b/c part that deps on α is part which has $p^\mu p^\nu$ $\left(-\frac{p_\mu p_\nu}{p^2}\right)$

i.e. answer independent of α (equivalent to saying this)

2) External photon

$$A = \varepsilon^\mu_s A_\mu$$

↳ Lorentz invariant

$$A'_\mu = \Lambda_\mu{}^\nu A_\nu \qquad p'_\mu = \Lambda_\mu{}^\nu p_\nu$$

now polarisation vector

$$\varepsilon'^\mu_s = \Lambda^\mu{}_\nu \varepsilon^\nu_s + c p^\mu$$

a coefficient c transforms as something prop. to p^μ

(i.e. polarisation vectors don't transform under Lor. as vectors) we eg gaps theoretical reasons here don't worry about this here

board cut off: condition $\varepsilon \cdot p = 0$. adjust that whatever we do this eqn, the still satisfy $\varepsilon \cdot p$. basically since $p^2=0$ } reflection of gauge symmetry

Example: $$\Lambda^\mu{}_\nu = \begin{pmatrix} 3/2 & 1 & 0 & -1/2 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1/2 & 1 & 0 & 1/2 \end{pmatrix}$$

"example that will make you congested"

"an example for you since I am not going through gaps theoretical reasons"

$$p^\mu = (E,0,0,E)$$
$$\varepsilon_1^\mu = (0,1,0,0)$$
$$\varepsilon_2^\mu = (0,0,1,0)$$

$$\Lambda^\mu{}_\nu p^\nu = p^\nu$$
$$\Lambda^\mu{}_\nu \varepsilon_1^\nu = \varepsilon_1^\mu + \tfrac{1}{E} p^\mu \quad //$$

Then a requirement of Lorentz invariance of A is

$$\boxed{p^\mu A_\mu = 0} \quad (2) \text{ WARD IDENTITY}$$

"see this in courses next term" not as relevant to now?

These two conditions are sanity conditions (reasonable theories should obey them). we are not going to prove them now

Scattering in QED

scattering examples

T&U channel

1) $e^-e^- \to e^-e^-$

T&S channel

2) $e^+e^- \to e^+e^-$ $\Big\} \to (1)$

only have time to cover one of these examples here

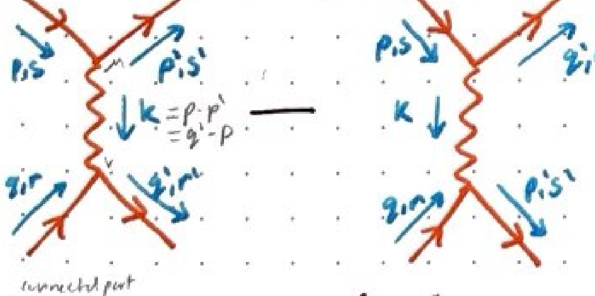
3) $e^-\gamma \to e^-\gamma$ $(Ex4) \to (2)$

1) $e^-e^- \to e^-e^-$

scattering at tree level

$$b^\dagger_{p,s}\, b^\dagger_{q,r} \to b^\dagger_{p',s'}\, b^\dagger_{q',r'}$$
$$\| $$
$$-b^\dagger_{q,r}\, b^\dagger_{p,s} \to \text{this gives an overall minus}$$

relative minus sign



only 2 diagrams at tree level we are interested to compute

reminder: defined Π

$$\left(\Pi_{\mu\nu}(k) = -\eta_{\mu\nu} - (\alpha-1)\frac{k_\mu k_\nu}{k^2} \right)$$

connected part

$$\langle f|S|i\rangle_c = (ie)^2 \Bigg(\bar{U}^{s'}(p')\gamma^\mu U^s(p)\, \frac{i\Pi_{\mu\nu}(p-p')}{(p-p')^2}\, \bar{U}^{r'}(q')\gamma^\nu U^r(q)$$

emphasise relative minus sign between diagrams because there is swap in fermions (similar to Wick thm reasons for getting minus sign)

$$- \bar{U}^{s'}(p')\gamma^\mu U^r(q)\, \frac{i\Pi_{\mu\nu}(p-q')}{(p-q')^2}\, \bar{U}^{r'}(q')\gamma^\nu U^s(p) \Bigg)$$
$$\cdot (2\pi)^4 \delta(p+q-p'-q')$$

Lecture 24

4.12.24

no big flashy finish to this course: because 'this subject never ends'

ans to question asked in lecture

this is what we have to compute:

$$\langle \Omega | b^{r'} b^{s'} b^{+s} b^{+r} | \Omega \rangle$$

LSZ ← LSZ tells us correlation involved in this process

$$\langle \psi_3 \psi_4 \bar{\psi}_1 \bar{\psi}_2 \rangle$$

(arbitrary positions 1 to 4) → scattering amplitude is controlled by this 4pt function

↓ Schw.-dyson

$$\overset{\text{to order } e^2}{} + e^2 \langle A_\mu A_\nu \rangle \langle \psi_3 \psi_4 \bar{\psi}_1 \bar{\psi}_2 \rangle + \ldots$$

2pt function of gauge fields times 4pt function of fermions. using schw.-D we get to order e^2; this is what drawings are showing

$$\| $$

$$- S_F(31) S_F(42) + S_F(41) S_F(32)$$

want leading order connected contribution: approximate using free theory

diagrams (?) come from understanding what is the appropriate correlation functions

continuing from before:

$$\bar{U}^{s'}(p')\gamma^\mu U^s(p)\, \tilde{\Pi}_{\mu\nu}(\underbrace{p-p'}_{k})\, \bar{U}^{r'}(q')\gamma^\nu U^r(q)$$

↓ want to see that this is independent of α

$$\boxed{\begin{aligned} k &= p-p' \\ &= q'-q \end{aligned}} \quad (1)$$

$$\bar{U}^{s'}(p')\gamma^\mu U^s(p)\, k_\mu k_\nu\, \bar{U}^{r'}(q')\gamma^\nu U^r(q) \overset{!}{=} 0 \qquad (A_{\mu\nu} k^\mu k^\nu = 0)$$

← equiv. to saying

we also know

$$\boxed{\begin{aligned} (-\not{p}+m)U(p) &= 0 \\ \bar{U}(p)(\not{p}-m) &= 0 \end{aligned}} \quad (2)$$

Then look at

(going to suppress spin index because it is not important to the point here but you can write s, s' if you wish)

$$\begin{aligned} \bar{U}(p')k_\mu\gamma^\mu U(p) &\overset{(1)}{=} \bar{U}(p')(\not{p}-\not{p}')U(p) \\ &\overset{(2)}{=} \bar{U}(p')(m-m)U(p) \\ &= 0 \end{aligned}$$

can do same w/ other spinors e.g. →

$$\begin{aligned} \bar{U}(q')k_\nu\gamma^\nu U(q) &= \bar{U}(q')(\not{q}-\not{q}')U(q) \\ &= 0 \end{aligned}$$

$\Rightarrow \langle f | S | i \rangle$ is independent of α !!

Note: in some cases each diagram is separately invariant but not always!

what do people actually mean in particle physics (LHC)?
↳ cross sections & decay rates

SPIN SUMS

Common quantity in particle physics is the cross section σ

events per unit time = # incoming particles per unit area per unit time × cross section fraction of particles that collide

$\downarrow$ unit flux

Diff cross section

$$\Rightarrow \quad d\sigma = \frac{\text{diff probability per unit time}}{\text{flux}}$$

Probability

$$P = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}$$

$$\langle f|S|i\rangle = i(2\pi)^4\delta(\Sigma p)A$$

$$|\langle f|S|i\rangle|^2 = (2\pi)^4\delta(\Sigma p)\underbrace{(2\pi)^4\delta^4(0)}_{VT}|A|^2$$

volume of space × time elapsed

Focus on this

When we have spin (e.g. fermions), it is common:

→ you can't measure outgoing spin

→ don't control the ingoing spin

$$|A|^2 \rightarrow \mathcal{P} = \langle |A|^2\rangle = \tfrac{1}{4}\sum_{\text{spins}}|A|^2$$: spin summed average squared matrix element

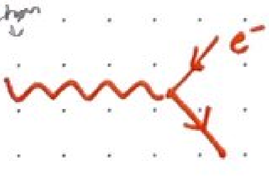
factor 1/4 because of 2 → something
½ × ½

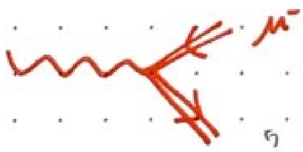
1/4 since 2 fermions in initial state

(½ chance could be spin up, ½ chance could be spin down)

Example: $e^-\mu^- \rightarrow e^-\mu^-$

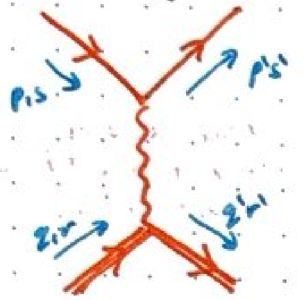
electron

vertex  $: -ie\gamma^\mu A_\mu$

 $: -ie\gamma^\mu A_\mu$

muon (double line to distinguish it)

one diagram



$$\rightarrow A = (-ie)^2 \frac{\overbrace{\bar{U}^{s'}(p')\gamma^\mu U^s(p)}^{e^-}\ \overbrace{\bar{U}^{r'}(q')\gamma_\mu U^r(q)}^{\mu^-}}{(p-p')^2}$$

$$|A|^2 = \frac{(-ie)^4}{(p-p')^4}\left(\bar{U}^{s'}(p')\gamma^\mu U^s(p)\right)\underbrace{\left(\bar{U}^{s'}(p')\gamma^\nu U^s(p)\right)^\dagger}_{U^{s\dagger}(p)\gamma^{\nu\dagger}\gamma^0 U^{s'}(p') = \bar{U}^s(p)\gamma^\nu U^{s'}(p')}\left(\bar{U}^{r'}(q')\gamma_\mu U^r(q)\right)\left(\bar{U}^{r'}(q')\gamma_\nu U^r(q)\right)^\dagger$$

(highlight sum over s,s')

$$P = \frac{1}{4}\sum_{\substack{s,s'\\r,r'}} |A|^2 = \frac{1}{4}(\dots)\sum_s\sum_{s'} \bar{U}^{s'}(p')\gamma^\mu U^s(p)\,\bar{U}^s(p)\gamma^\nu U^{s'}(p') \sum_{r,r'} \dots$$

$$\sum_s U^s_a(p)\bar{U}^s_b(p) = (\not{p}+m)_{ab} = \sum \bar{U}^s_b(p) U^s_a(p)$$

(trace when you have lots of gammas?)

$$= \frac{(-ie)^4}{4(p-p')^4}\,\mathrm{Tr}\left((\not{p}'+m_e)\gamma^\mu(\not{p}+m_e)\gamma^\nu\right)\mathrm{Tr}\left((\not{q}'+m_\mu)\gamma_\nu(\not{q}+m_\mu)\gamma_\nu\right)$$

(electron & muon have diff masses)

(we will ___ on ex sheet 3 (Clifford algebra))

$$= \frac{8e^4}{(p-p')^4}\left(p'\cdot q'\, p\cdot q + p\cdot q'\, p'\cdot q - m_e^2\, q\cdot q' - m_\mu^2\, p\cdot p' + 2m_\mu^2 m_e^2\right)$$

we are done :) with QFT
(now do ex sheet 4 :))

don't end on a bang, just end with an example; there is still lots to learn!