Homework 8: solutions

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Remark In these solutions we use Ω to denote the sample space (set of elementary events).

Problem 1

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$\Pr[\text{one head}] = \sum_{x \in E} \Pr[x] \quad \text{where } E = \{HTT, THT, TTH\}.$$

Each elementary event has probability $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. So we just sum this probability over the set E, i.e. multiply the probability by the size of the set:

$$\Pr[\text{one head}] = \sum_{x \in E} \frac{1}{8} = \frac{3}{8}.$$

Problem 2

$$\Omega = \{(x_1, x_2, x_3, x_4, x_5) | x_i \in \{1, \dots, 6\} \text{ for } i = 1, \dots, 5\}$$

That is, our sample space (set of elementary events) is the set of 5-tuples of die rolls. Each elementary event $(x_1, x_2, x_3, x_4, x_5)$ has probability $\left(\frac{1}{6}\right)^5$, since each die has 6 possible outcomes and the 5 die rolls are independent. We must compute $\Pr[\text{exactly two 3s}]$. Since all elementary events are equally likely, we can just find the size of the event set:

$$E = \{(x_1, x_2, x_3, x_4, x_5) \in \Omega | \text{ exactly two of the } x_i \text{ are equal to } 3\}$$

How do we count this set? Well, of the five rolls, exactly two must have value 3. So for each the other three rolls, there are 5 possibilities: 1, 2, 4, 5, or 6. Since there are three such rolls, we get $5 \times 5 \times 5$ possibilities for the values of the other three rolls. The other degree of freedom is the ordering of the rolls that have value 3. They could be 1st and 5th, or 3rd and 4th, and so on. Therefore, we have an additional factor of $\binom{5}{2}$ ways to arrange the rolls.

Remark The binomial coefficient $\binom{n}{k}$ is defined to be the number of ways of choosing k items from a total of n, and is readily seen to be

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{where } m! = m \times (m-1) \times \cdots \times 2 \times 1.$$

Hence the size of E is

$$\begin{aligned} |E| &= \binom{5}{2} \times 5^3 \\ &= \frac{5 \times 4}{2 \times 1} \times 5^3 \\ &= 5^4 \times 2. \end{aligned}$$

Therefore the probability of the event E is

$$\begin{aligned} \Pr[E] &= 5^4 \times 2 \times \frac{1}{6^5} \\ &= \frac{5^4 \times 2}{6^5}. \end{aligned}$$

Problem 3

$$\Omega = \{(i,j) | i,j \in \{1,2,3,4,5\}\}$$

Each elementary events $x \in \Omega$ has probability $\frac{1}{25}$. We want to find $\Pr[E]$ where E is the event that i is strictly greater than j, i.e. $E = \{(i,j) \in \Omega | i < j\}$. To count this set, we can just consider the possibilities for i as we vary j.

- If $j = 1$, we can have $i = 2, 3, 4, 5$.
- If $j = 2$, we can have $i = 3, 4, 5$.
- If $j = 3$, we can have $i = 4, 5$.
- If $j = 4$, we can have $i = 5$.
- If $j = 5$, no i works.

We just count the number of possibilities we found: $4+3+2+1+0 = 10$. Therefore, the probability that i is strictly greater than j is

$$\Pr[E] = \sum_{x \in E} \Pr[x] = \sum_{x \in E} \frac{1}{25} = \frac{10}{25} = \frac{2}{5}.$$

Problem 4

$$\begin{aligned}\Omega &= \{1,2,3,4,5,6\}\\ A &= \{2,4,6\}\\ B &= \{4,5,6\}\end{aligned}$$

Each elementary event has equal probability, therefore the probability of an event is proportional to how many elementary events it contains. To see whether A and B are independent, we must see if $\Pr[A]\Pr[B] = \Pr[A,B]$. [Recall: $\Pr[A,B]$ is defined to be $\Pr[A \cap B]$.] We evaluate them separately:

$$\begin{aligned}\Pr[A]\Pr[B] &= \frac{|A|}{|\Omega|} \times \frac{|B|}{|\Omega|}\\ &= \frac{3}{6} \times \frac{3}{6}\\ &= \frac{1}{2} \times \frac{1}{2}\\ &= \frac{1}{4}.\end{aligned}$$

$$\begin{aligned}\Pr[A,B] &= \Pr[A \cap B]\\ &= \frac{|A \cap B|}{|\Omega|}\\ &= \frac{|\{4,6\}|}{6}\\ &= \frac{2}{6}\\ &= \frac{1}{3}.\end{aligned}$$

We can see that these quantities are not equal, therefore the events are not independent.

Problem 5

$$\Omega = \{(i, j) | i, j \in \{1, 2, 3, 4, 5, 6\}\}$$

We want to find the probability that the minimum of i and j is 3, i.e. $\Pr[E]$ where

$$E = \{(3, 3), (3, 4), (3, 5), (3, 6), (4, 3), (5, 3), (6, 3)\}$$

Each elementary event has equal probability, therefore $\Pr[E]$ is proportional to its size. Hence

$$\Pr[E] = \frac{|E|}{|\Omega|} \tag{1}$$

$$= \frac{7}{36}. \tag{2}$$

Problem 6

$$\Omega = \{\text{graphs on 4 vertices}\}$$

Each graph occurs with equal probability, since each edge is present with probability $1/2$ independent of the other edges. Hence the probability of an event is proportional to the number of graphs that are in that event.

For each triplet t of vertices in the graph (such as $\{1, 2, 3\}$), we can ask: do the vertices in t form a triangle or not? Imagine if we counted up the positive answers to all these questions – it would sum to X, the number of triangles in the graph. Therefore, for each triplet t, we define the *indicator variable* Y_t (indicator just means that it is 0 or 1):

$$Y_t = \begin{cases} 1 & \text{if } t \text{ is a triangle} \\ 0 & \text{otherwise} \end{cases}$$

Rephrasing our plain English above into math, we know that

$$X = \sum_t Y_t.$$

Therefore, we can take expectations of both sides:

$$\begin{aligned} \mathrm{Ex}[X] &= \mathrm{Ex}\left[\sum_t Y_t\right] \\ &= \sum_t \mathrm{Ex}[Y_t] \qquad \text{by linearity of expectation.} \end{aligned}$$

So we just need to compute the expectation of Y_t for each triplet t and then add them up to get the expectation of X. Now let's compute the expectation of a specific Y_t – say, $Y_{\{1,2,3\}}$. We will soon see that the analysis works for *any* triplet, but for the sake of argument we can focus on a concrete instance – the vertices 1, 2 and 3. By definition, it is

$$\begin{aligned} \mathrm{Ex}\left[Y_{\{1,2,3\}}\right] &= \sum_y y \cdot \Pr\left[Y_{\{1,2,3\}} = y\right] \\ &= 1 \cdot \Pr\left[Y_{\{1,2,3\}} = 1\right] + 0 \cdot \Pr\left[Y_{\{1,2,3\}} = 0\right] \\ &= \Pr\left[Y_{\{1,2,3\}} = 1\right]. \end{aligned}$$

Remark This is a neat property of indicator variables – their expectation is just the probability they are 1.

Since the edges are erased independently and uniformly at random, each of the three possible edges – (1,2), (2,3) and (3, 1) – is present with probability $1/2$. So with probability $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$, all the edges will exist, and we get a triangle.

We have shown that $\Pr\left[Y_{\{1,2,3\}}\right] = \frac{1}{8}$. But there was nothing special about (1,2,3) – we could have said (2,3,4), or (1,3,4), or (1,2,4). So in fact, $\Pr\left[Y_t\right] = \frac{1}{8}$ for any triplet t.

Now we can go back to the main thing we were calculating, the expectation of X:

$$\mathrm{Ex}\left[X\right] = \sum_t \frac{1}{8}$$

How many triplets of vertices t can we make from a graph on 4 vertices? We have to choose 3 from 4, so the answer is $\binom{4}{3} = 4$. Therefore, we have determined the expectation of X:

$$\mathrm{Ex}\left[X\right] = 4 \times \frac{1}{8} = \frac{1}{2}.$$