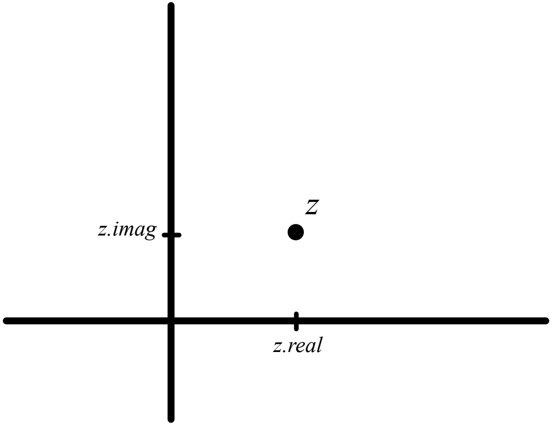
Complex numbers as points in the complex plane

Can interpret *real* and *imaginary* parts of a complex number as x and y coordinates.
Thus can interpret a complex number as a *point* in the plane

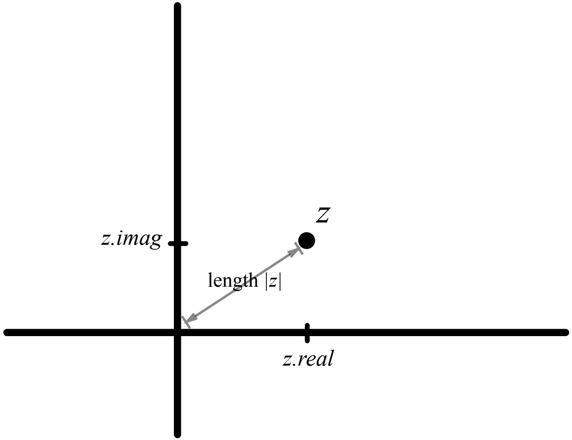


(the *complex plane*)

Playing with $\mathbb{C}$

Playing with $\mathbb{C}$: The absolute value of a complex number

Absolute value of z = distance from the origin to the point z in the complex plane.

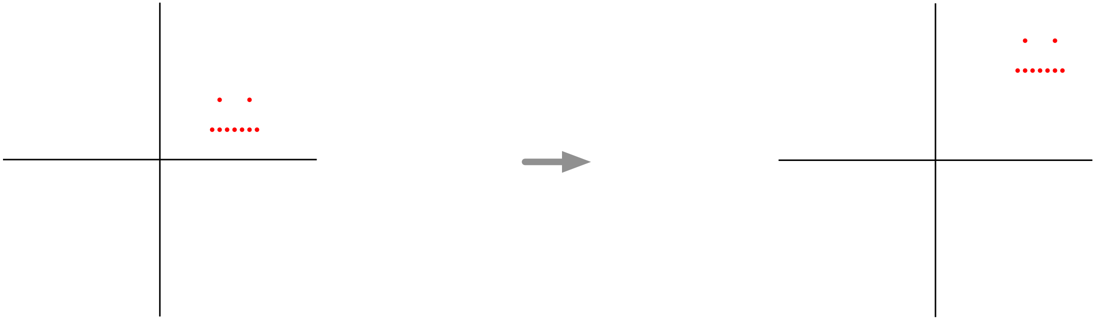


- In Mathese, written $|z|$.
- In Python, written `abs(z)`.

Playing with $\mathbb{C}$: Adding complex numbers

Geometric interpretation of $f(z) = z + (1 + 2\mathbf{i})$?

Increase each real coordinate by 1 and increases each imaginary coordinate by 2.



$f(z) = z + (1 + 2\mathbf{i})$ is called a *translation*.

Playing with $\mathbb{C}$: Adding complex numbers

- *Translation in general:*

  $$f(z) = z + z_0$$

  where z_0 is a complex number.

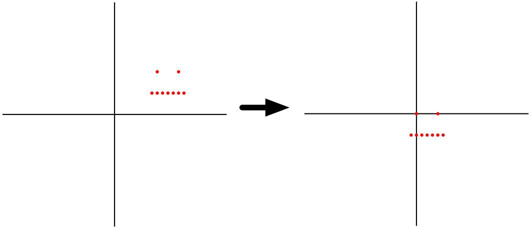
- A translation can "move" the picture anywhere in the complex plane.

Playing with $\mathbb{C}$: Adding complex numbers

- *Quiz:* The "left eye" of the list L of complex numbers is located at $2 + 2\mathbf{i}$. For what complex number z_0 does the translation

$$f(z) = z + z_0$$

move the left eye to the origin $0 + 0\mathbf{i}$?



Playing with $\mathbb{C}$: Adding complex numbers

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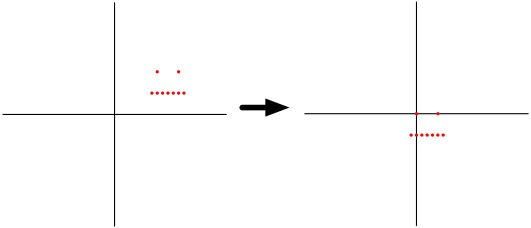
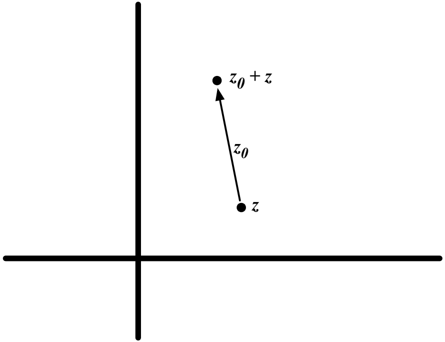


- *Answer:* $z_0 = -2 - 2\mathbf{i}$

Playing with $\mathbb{C}$: Adding complex numbers: Complex numbers as arrows

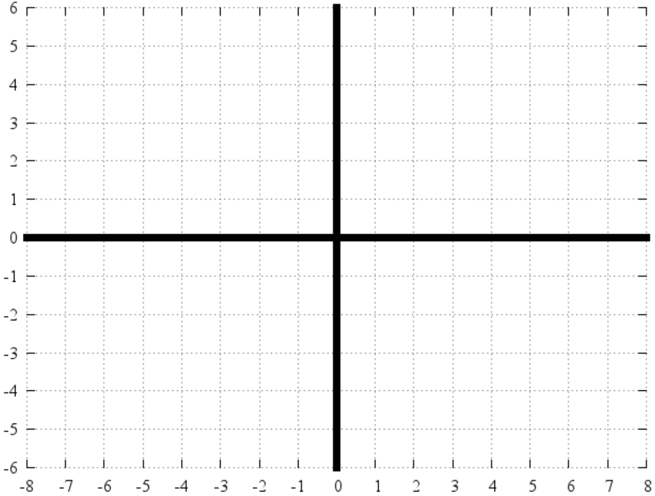
Interpret z_0 as representing the translation $f(z) = z + z_0$.

- Visualize a complex number z_0 as an arrow.
- Arrow's tail located an any point z
- Arrow's head located at $z + z_0$
- Shows an example of what the translation $f(z) = z + z_0$ does



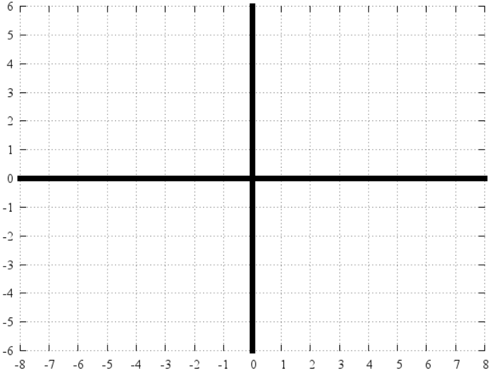
Playing with $\mathbb{C}$: Adding complex numbers: Complex numbers as arrows

Example: Represent $-6 + 5\mathbf{i}$ as an arrow.



Playing with $\mathbb{C}$: Adding complex numbers: Composing translations, adding arrows

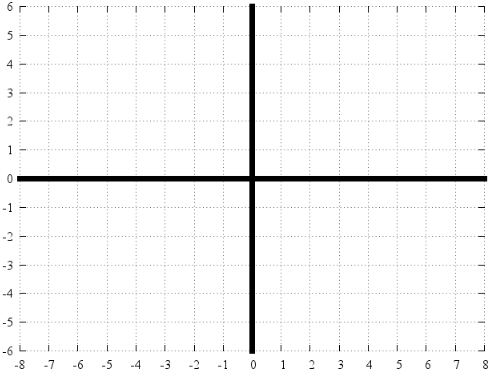
- Consider two complex numbers z_1 and z_2.
- They correspond to translations $f_1(z) = z + z_1$ and $f_2(z) = z + z_2$
- Functional composition: $(f_1 \circ f_2)(z) = z + z_1 + z_2$
- Represent functional composition by adding arrows.
- *Example:* $z_1 = 2 + 3\mathbf{i}$ and $z_2 = 3 + 1\mathbf{i}$



Playing with $\mathbb{C}$: Multiplying complex numbers by a positive real number

Multiply each complex number by 0.5

$f(z) = 0.5\,z$



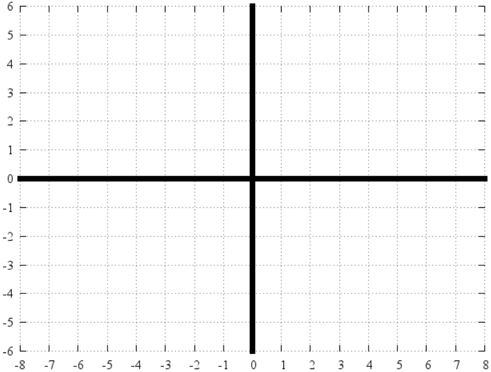
Arrow in same direction but half the length.

Scaling

Playing with $\mathbb{C}$: Multiplying complex numbers by a negative number

$f(z) = (-1)\, z$

Multiply each complex number by -1



Arrow in opposite direction

Reflection

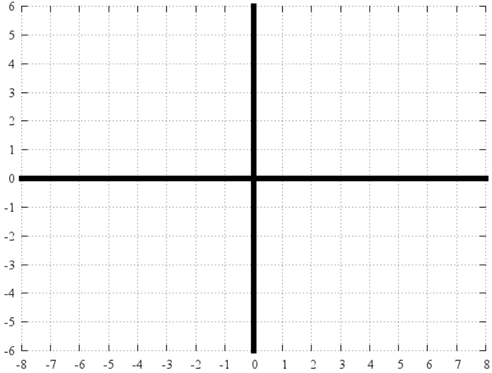
Playing with $\mathbb{C}$: Multiplying by $\mathbf{i}$: rotation by 90 degrees

How to rotate counterclockwise by 90°?

Need $x + y\,\mathbf{i} \mapsto -y + x\,\mathbf{i}$

Use $\mathbf{i}(x + y\mathbf{i}) = x\,\mathbf{i} + y\,\mathbf{i}^2 = x\,\mathbf{i} - y$

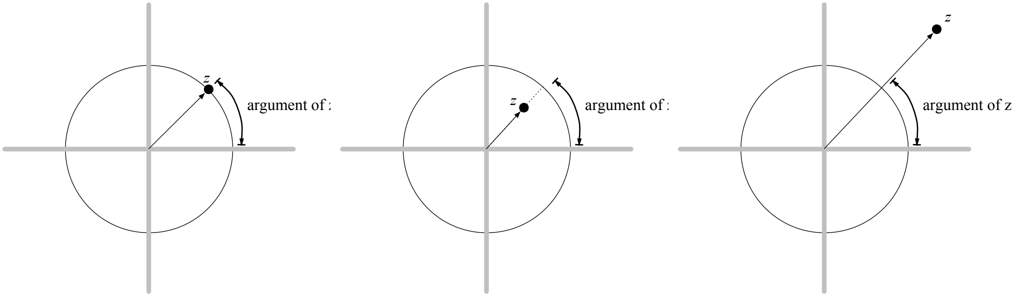
$f(z) = \mathbf{i}\,z$



Playing with $\mathbb{C}$: The unit circle in the complex plane: *argument* and angle

What about rotating by another angle?

Definition: *Argument* of z is the angle in radians between z arrow and $1 + 0\mathbf{i}$ arrow.



$\boxed{\text{Rotating a complex number } z \text{ means } \textit{increasing its argument}.}$

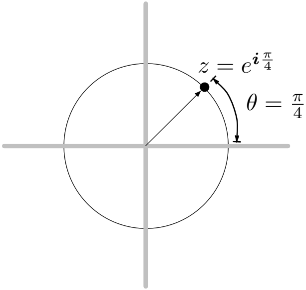
Playing with $\mathbb{C}$: Euler's formula

"He calculated just as men breathe, as eagles sustain themselves in the air."
Said of Leonhard Euler

Euler's formula: For any real number θ,

$$e^{\theta \mathbf{i}}$$

is the point z on the unit circle with argument θ.



$e = 2.718281828...$

Playing with $\mathbb{C}$: Euler's formula

Euler's formula: For any real number θ,

$$e^{\theta \mathbf{i}}$$

is the point z on the unit circle with argument θ.

Plug in $\theta = \pi$....

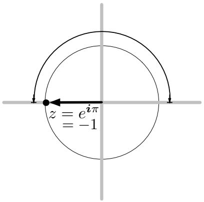
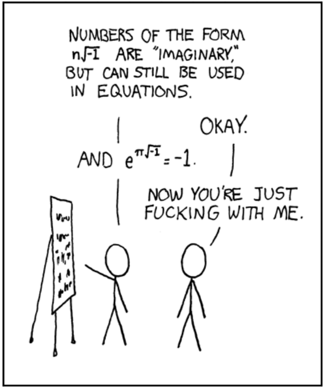


photo by Cory Doctorow



e to the π times $\mathbf{i}$ (`http://xkcd.com/179/`)

Playing with $\mathbb{C}$: Euler's formula

Plot

$$e^{0\cdot\frac{2\pi\mathbf{i}}{20}},\ e^{1\cdot\frac{2\pi\mathbf{i}}{20}}, e^{2\cdot\frac{2\pi\mathbf{i}}{20}}, e^{3\cdot\frac{2\pi\mathbf{i}}{20}}, \ldots, e^{19\cdot\frac{2\pi\mathbf{i}}{20}}$$

Playing with $\mathbb{C}$: Rotation by τ radians

Back to question of rotation by any angle τ.

- Every complex number can be written in the form $z = re^{\theta\mathbf{i}}$
  - r is the absolute value of z
  - θ is the argument of z
- Need to increase the argument of z
- Use exponentiation law $e^a \cdot e^b = e^{a+b}$
- $re^{\theta\mathbf{i}} \cdot e^{\tau\mathbf{i}} = re^{\theta\mathbf{i}+\tau\mathbf{i}} = re^{(\theta+\tau)\mathbf{i}}$
- $f(z) = z \cdot e^{\tau\mathbf{i}}$ does rotation by angle τ.

Playing with $\mathbb{C}$: Rotation by τ radians

Rotation by $3\pi/4$