Vector addition: Translation and vector addition

- With complex numbers, translation achieved by adding a complex number, e.g. $f(z) = z + (1 + 2\mathbf{i})$
- Let's do the same thing with vectors...
  - **Definition of vector addition:**

$$[u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n] = [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n]$$

  - For 2-vectors represented in Python as 2-element lists, addition procedure is

```
def add2(v,w): return [v[0]+w[0], v[1]+w[1]]
```

Vector addition: Translation and vector addition

Quiz: Suppose we represent n-vectors by n-element lists. Write a procedure `addn(v, w)` to compute the sum of two vectors so represented.

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```
def addn(v, w): return [v[i]+w[i] for i in range(len(v))]
```

Vector addition: The zero vector

The D-vector whose entries are all zero is the *zero vector*, written $\mathbf{0}_D$ or just $\mathbf{0}$

$$\mathbf{v} + \mathbf{0} = \mathbf{v}$$

Vector addition: Vector addition is associative and commutative

- *Associativity*

$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$$

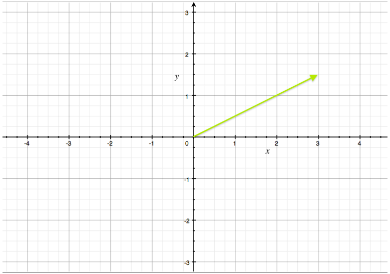
- *Commutativity*

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

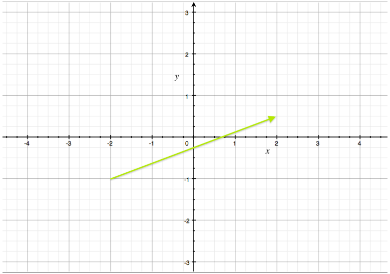
Vector addition: Vectors as arrows

Like complex numbers in the plane, n-vectors over $\mathbb{R}$ can be visualized as *arrows* in $\mathbb{R}^n$.

The 2-vector $[3, 1.5]$ can be represented by an arrow with its tail at the origin and its head at $(3, 1.5)$.



or, equivalently, by an arrow whose tail is at $(-2, -1)$ and whose head is at $(1, 0.5)$.

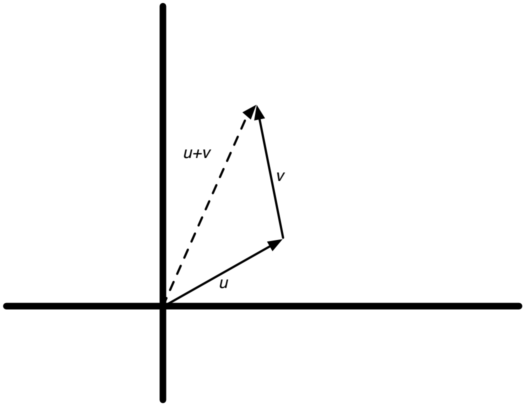


Vector addition: Vectors as arrows

Like complex numbers, addition of vectors over $\mathbb{R}$ can be visualized using arrows.

To add $\mathbf{u}$ and $\mathbf{v}$:

- place tail of $\mathbf{v}$'s arrow on head of $\mathbf{u}$'s arrow;
- draw a new arrow from tail of $\mathbf{u}$ to head of $\mathbf{v}$.



Scalar-vector multiplication

With complex numbers, *scaling* was multiplication by a real number $f(z) = r\,z$

For vectors,

- we refer to field elements as *scalars*;
- we use them to scale vectors:

$$\alpha\,\mathbf{v}$$

  Greek letters (e.g. α, β, γ) denote scalars.

Scalar-vector multiplication

Definition: Multiplying a vector **v** by a scalar α is defined as multiplying each entry of **v** by α:

$$\alpha\,[v_1, v_2, \ldots, v_n] = [\alpha\, v_1, \alpha\, v_2, \ldots, \alpha\, v_n]$$

Example: $2\,[5, 4, 10] = [2 \cdot 5, 2 \cdot 4, 2 \cdot 10] = [10, 8, 20]$

Scalar-vector multiplication

Quiz: Suppose we represent n-vectors by n-element lists. Write a procedure `scalar_vector_mult(alpha, v)` that multiplies the vector `v` by the scalar `alpha`.

Answer:
```
def scalar_vector_mult(alpha, v):
```

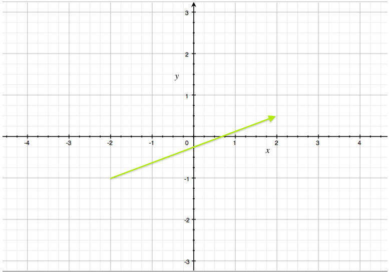
Scalar-vector multiplication

Quiz: Suppose we represent n-vectors by n-element lists. Write a procedure `scalar_vector_mult(alpha, v)` that multiplies the vector `v` by the scalar `alpha`.

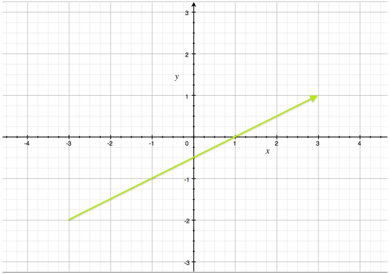
Answer: `def scalar_vector_mult(alpha, v):   return [alpha*x for x in v]`

Scalar-vector multiplication: Scaling arrows

An arrow representing the vector $[3, 1.5]$ is this:



and an arrow representing two times this vector is this:



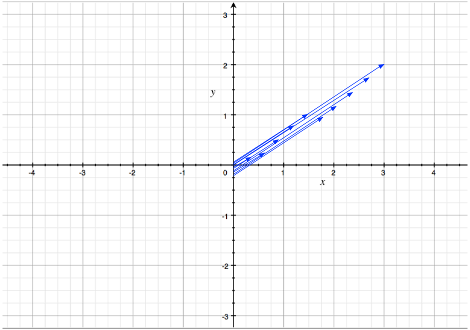
Scalar-vector multiplication: Associativity of scalar-vector multiplication

Associativity: $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$

Scalar-vector multiplication: Line segments through the origin

Consider scalar multiples of $\mathbf{v} = [3, 2]$:
$\{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$

For each value of α in this set,
$\alpha\,\mathbf{v}$ is shorter than $\mathbf{v}$ but in same direction.



Scalar-vector multiplication: Line segments through the origin

Conclusion: The set of points

$$\{\alpha\,\mathbf{v} \;:\; \alpha \in \mathbb{R}, 0 \leq \alpha \leq 1\}$$

forms the line segment between the origin and $\mathbf{v}$

Scalar-vector multiplication: Lines through the origin

What if we let α range over all real numbers?

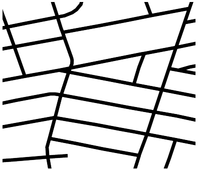
- Scalars bigger than 1 give rise to somewhat larger copies
- Negative scalars give rise to vectors pointing in the opposite direction

The set of points

$$\{\alpha\,\mathbf{v} \,:\, \alpha \in \mathbb{R}\}$$

forms the line through the origin and $\mathbf{v}$

Combining vector addition and scalar multiplication

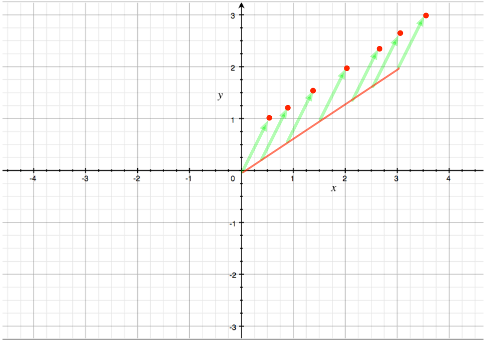


We want to describe the set of points forming an arbitrary line segment (not necessarily through the origin).

Idea: Use the idea of translation.

Start with line segment from $[0,0]$ to $[3,2]$:

$$\{\alpha\,[3,2] \;:\; 0 \leq \alpha \leq 1\}$$



Translate it by adding $[0.5,1]$ to every point:

$$\{[0.5,1] + \alpha\,[3,2] \;:\; 0 \leq \alpha \leq 1\}$$

Get line segment from $[0,0]+[0.5,1]$ to $[3,2]+[0.5,1]$

Combining vector addition and scalar multiplication: Distributive laws for scalar-vector multiplication and vector addition

Scalar-vector multiplication distributes over vector addition:

$$\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$$

Example:

- On the one hand,

$$2\,([1, 2, 3] + [3, 4, 4]) = 2\,[4, 6, 7] = [8, 12, 14]$$

- On the other hand,

$$2\,([1, 2, 3] + [3, 4, 4]) = 2\,[1, 2, 3] + 2\,[3, 4, 4] = [2, 4, 6] + [6, 8, 8] = [8, 12, 14]$$

Combining vector addition and scalar multiplication: First look at convex combinations

Set of points making up the the [0.5, 1]-to-[3.5, 3] segment:

$$\{\alpha\,[3,2] + [0.5,1] \;:\; \alpha \in \mathbb{R}, 0 \leq \alpha \leq 1\}$$

Not symmetric with respect to endpoints ☹

Use distributivity:

$$\begin{aligned}
\alpha\,[3,2] + [0.5,1] &= \alpha\,([3.5,3] - [0.5,1]) + [0.5,1] \\
&= \alpha\,[3.5,3] - \alpha\,[0.5,1] + [0.5,1] \\
&= \alpha\,[3.5,3] + (1-\alpha)\,[0.5,1] \\
&= \alpha\,[3.5,3] + \beta\,[0.5,1]
\end{aligned}$$

where $\beta = 1 - \alpha$

New formulation:

$$\{\alpha\,[3.5,3] + \beta\,[0.5,1] \;:\; \alpha, \beta \in \mathbb{R}, \alpha, \beta \geq 0, \alpha + \beta = 1\}$$

Symmetric with respect to endpoints ☺

Combining vector addition and scalar multiplication: First look at convex combinations

New formulation:

$$\{\alpha\,[3.5, 3] + \beta\,[0.5, 1] \;:\; \alpha, \beta \in \mathbb{R}, \alpha, \beta \geq 0, \alpha + \beta = 1\}$$

Symmetric with respect to endpoints ☺

An expression of the form

$$\alpha\,\mathbf{u} + \beta\,\mathbf{v}$$

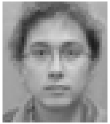
where $0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$, and $\alpha + \beta = 1$ is called a *convex combination* of **u** and **v**

$$\boxed{\text{The } \mathbf{u}\text{-to-}\mathbf{v} \text{ line segment consists of the set of convex combinations of } \mathbf{u} \text{ and } \mathbf{v}.}$$

Combining vector addition and scalar multiplication: First look at convex combinations

$\mathbf{u} =$  and $\mathbf{v} =$ 

Use scalars $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$:

$\frac{1}{2}$  $+$ $\frac{1}{2}$  $=$ 

"Line segment" between two faces:

 $1\mathbf{u} + 0\mathbf{v}$

 $\frac{7}{8}\mathbf{u} + \frac{1}{8}\mathbf{v}$

 $\frac{6}{8}\mathbf{u} + \frac{2}{8}\mathbf{v}$

 $\frac{5}{8}\mathbf{u} + \frac{3}{8}\mathbf{v}$

 $\frac{4}{8}\mathbf{u} + \frac{4}{8}\mathbf{v}$

 $\frac{3}{8}\mathbf{u} + \frac{5}{8}\mathbf{v}$

 $\frac{2}{8}\mathbf{u} + \frac{6}{8}\mathbf{v}$

 $\frac{1}{8}\mathbf{u} + \frac{7}{8}\mathbf{v}$

 $1\mathbf{u} + 0\mathbf{v}$

Combining vector addition and scalar multiplication: First look at convex combinations

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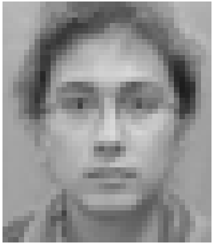

Combining vector addition and scalar multiplication: First look at convex combinations

Combining vector addition and scalar multiplication: First look at convex combinations

Combining vector addition and scalar multiplication: First look at convex combinations

Combining vector addition and scalar multiplication: First look at affine combinations

Infinite line through $[0.5, 1]$ and $[3.5, 3]$?
Our formulation so far ☹

$$\{[0.5, 1] + \alpha\,[3, 2] \ : \ \alpha \in \mathbb{R}\}$$

Nicer formulation ☺:

$$\{\alpha\,[3.5, 3] + \beta\,[0.5, 1] \ : \ \alpha \in \mathbb{R}, \beta \in R, \alpha + \beta = 1\}$$

An expression of the form $\alpha\,\mathbf{u} + \beta\,\mathbf{v}$ where $\alpha + \beta = 1$ is called an *affine* combination of $\mathbf{u}$ and $\mathbf{v}$.

The line through **u** and **v** consists of the set of affine combinations of **u** and **v**.