

Advanced Materials Modeling Forces, DFPT and phonons

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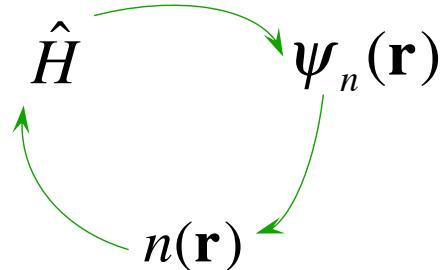
- + Computing forces and phonons (in need for perturbations)
- + Perturbations (adiabatic)
- + Perturbations in ordinary quantum mechanics
- + Density Functional Perturbation Theory (DFPT)
- + Phonon band structures from DFPT
- + Thermodynamic properties
- + Electron-phonon effects on electronic energies
- + Electron-phonon effects on transport properties

Computing the forces

Basic equations in DFT

Solve self-consistently the Kohn-Sham equation

$$\left\{ \begin{array}{l} \hat{H} |\Psi_n\rangle = \varepsilon_n |\Psi_n\rangle \\ \hat{H} = \hat{T} + \hat{V} + \hat{V}_{Hxc}[n] \\ n(\mathbf{r}) = \sum_n^{occ} \Psi_n^*(\mathbf{r}) \Psi_n(\mathbf{r}) \end{array} \right.$$



$$\delta_{mn} = \langle \Psi_m | \Psi_n \rangle \text{ for } m, n \in \text{occupied set}$$

or minimize

$$E_{el}\{\Psi\} = \sum_n^{occ} \langle \Psi_n | \hat{T} + \hat{V} | \Psi_n \rangle + E_{Hxc}[n]$$

with

$$\hat{V}(\mathbf{r}) = \sum_{\kappa} -\frac{Z_{\kappa}}{|\mathbf{r} - \mathbf{R}_{\kappa}|}$$

Exchange-correlation functional might be LDA, GGA (e.g. PBE, PBESol), or hybrids, van der waals, etc

Computing the forces (I)

Born - Oppenheimer approx. \Rightarrow find electronic ground state in potential created by nuclei.

A starting configuration of nuclei $\{\mathbf{R}_\kappa\}$ is usually NOT in equilibrium geometry.

$$F_{\kappa,\alpha} = - \frac{\partial E}{\partial R_{\kappa,\alpha}} \Bigg|_{\{\mathbf{R}_\kappa\}} \quad (\text{principle of virtual works})$$

Forces are first derivatives of total energy.
Can be computed by finite differences.

Better approach : compute the response to a perturbation

\Rightarrow What is the energy change ?

$$\{\mathbf{R}_{\kappa,\alpha}\} \rightarrow \{\mathbf{R}_{\kappa,\alpha} + \lambda \delta \mathbf{R}_{\kappa,\alpha}\}$$

Small parameter

Computing the forces (II)

To simplify, let's compute the derivative of an electronic eigenvalue

Perturbation theory : Hellmann - Feynman theorem

$$\hat{H}(\lambda) \quad \varepsilon_n(\lambda) \quad \psi_n(\lambda) \text{ normalized} \quad \hat{H}|\psi_n\rangle = \varepsilon_n|\psi_n\rangle \quad \text{for all } \lambda$$

$$\begin{aligned}\frac{d\varepsilon_n}{d\lambda} &= \frac{d}{d\lambda} \left(\langle \psi_n | \hat{H} | \psi_n \rangle \right) \\ &= \left\langle \frac{d\psi_n}{d\lambda} | \hat{H} | \psi_n \right\rangle + \left\langle \psi_n \left| \frac{d\hat{H}}{d\lambda} \right| \psi_n \right\rangle + \left\langle \psi_n | \hat{H} | \frac{d\psi_n}{d\lambda} \right\rangle \\ &= \left\langle \frac{d\psi_n}{d\lambda} | \varepsilon_n | \psi_n \right\rangle + \left\langle \psi_n \left| \frac{d\hat{H}}{d\lambda} \right| \psi_n \right\rangle + \left\langle \psi_n | \varepsilon_n | \frac{d\psi_n}{d\lambda} \right\rangle \quad = \left\langle \psi_n^{(0)} \left| \frac{d\hat{H}}{d\lambda} \right| \psi_n^{(0)} \right\rangle\end{aligned}$$

Indeed $\left\langle \frac{d\psi_n}{d\lambda} \left| \psi_n \right. \right\rangle + \left\langle \psi_n \left| \frac{d\psi_n}{d\lambda} \right. \right\rangle = \frac{d}{d\lambda} \langle \psi_n | \psi_n \rangle = \frac{d}{d\lambda} (1) = 0$

$\frac{d\psi_n}{d\lambda}$ not needed !

Computing the forces (III)

Application to the derivative of
with respect to an atomic displacement :

$$\hat{H} = \hat{T} + \hat{V}_{ext}\{\hat{R}\} \Rightarrow \frac{\partial \hat{H}}{\partial R_{\kappa,\alpha}} = \frac{\partial \hat{V}_{ext}}{\partial R_{\kappa,\alpha}}$$

$$\frac{\partial \varepsilon_n}{\partial R_{\kappa,\alpha}} = \left\langle \Psi_n \left| \frac{\partial \hat{H}}{\partial R_{\kappa,\alpha}} \right| \Psi_n \right\rangle = \int \frac{\partial \hat{V}_{ext}(\mathbf{r})}{\partial R_{\kappa,\alpha}} n(\mathbf{r}) d\mathbf{r}$$

Computing the forces (IV)

Generalisation to density functional theory

Reminder : $E[\psi] = \sum_n \langle \psi_n | \hat{T} | \psi_n \rangle + \int n(\mathbf{r}) V_{\text{ext}}(\mathbf{r}) d\mathbf{r} + E_{\text{Hxc}}[n]$

If change of atomic positions ...

$$V_{\text{ext}}(\mathbf{r}) = \sum_k -\frac{Z_k}{|\mathbf{r} - \mathbf{R}_k|} \quad (\text{can be generalized to pseudopotential case})$$

$$\frac{\partial V_{\text{ext}}(\mathbf{r})}{\partial R_{k,\alpha}} = +\frac{Z_k}{|\mathbf{r} - \mathbf{R}_k|^2} \cdot \frac{\partial |\mathbf{r} - \mathbf{R}_k|}{\partial R_{k,\alpha}} = -\frac{Z_k}{|\mathbf{r} - \mathbf{R}_k|^3} \cdot (\mathbf{r} - \mathbf{R}_k)_\alpha$$

$$\boxed{\frac{\partial E}{\partial R_{k,\alpha}} = \int n(\mathbf{r}) \frac{\partial V_{\text{ext}}(\mathbf{r})}{\partial R_{k,\alpha}} d\mathbf{r} = - \int \frac{n(\mathbf{r})}{|\mathbf{r} - \mathbf{R}_k|^3} \cdot (\mathbf{r} - \mathbf{R}_k)_\alpha d\mathbf{r}}$$

Forces can be computed directly from the density !

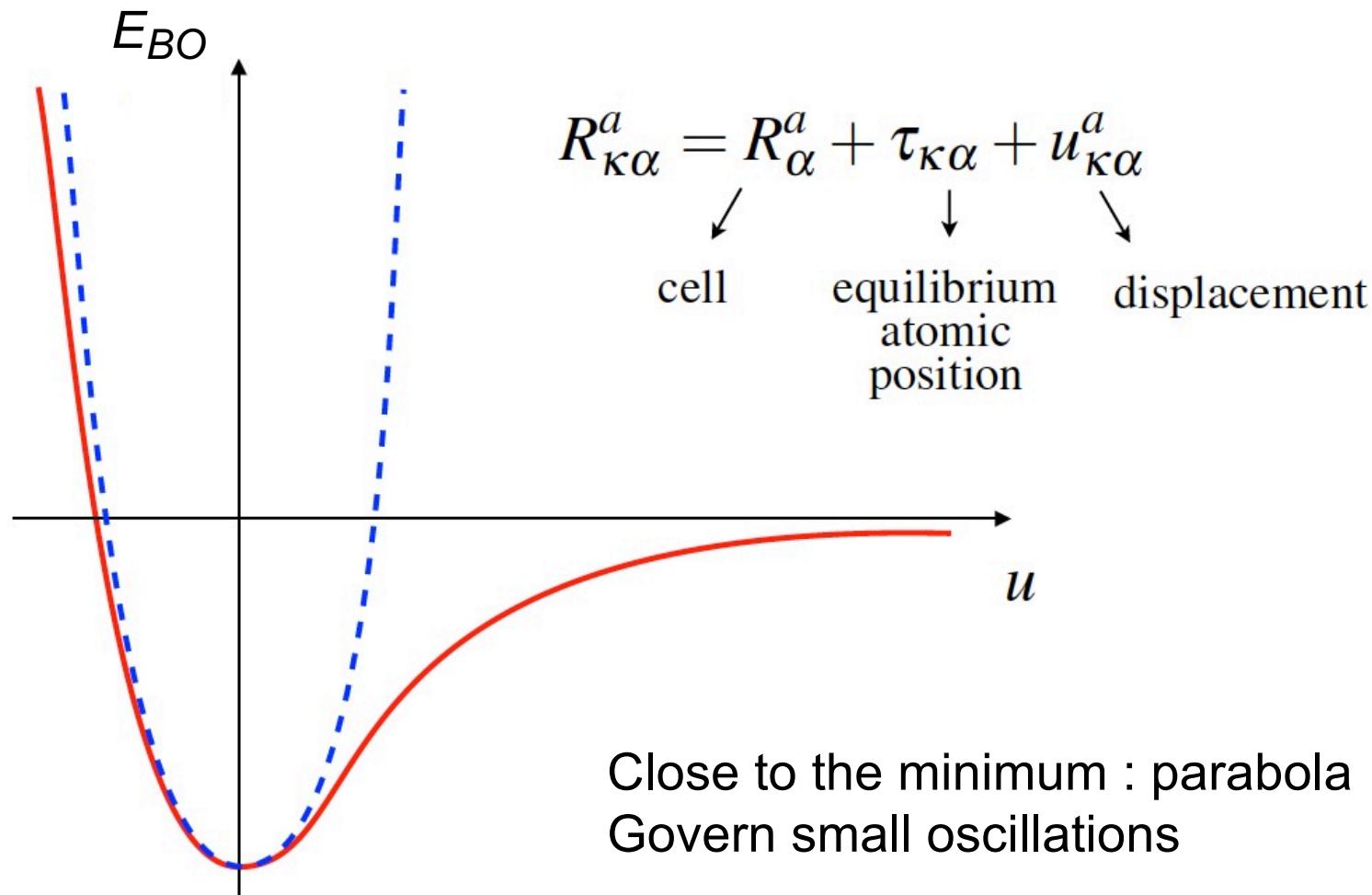
Also, similarly, stresses.

=> Optimization of geometry (lattice parameters and atomic positions)

Computing phonons

Changing atomic positions

Born-Oppenheimer approximation ...



Harmonic oscillator

Classical mechanics

Potential : $V(x) = \frac{1}{2} kx^2$

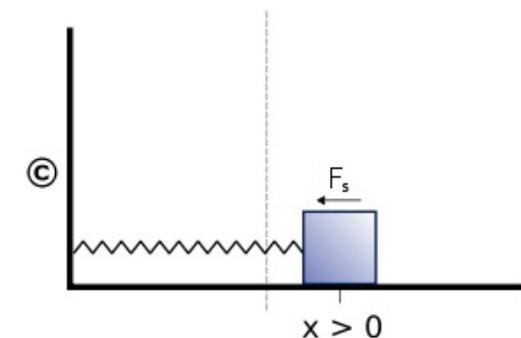
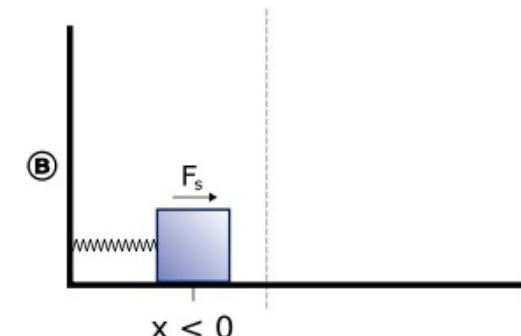
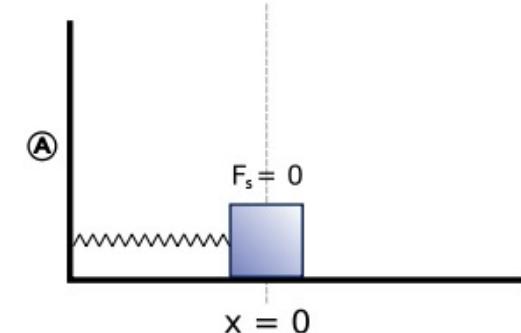
Restoring force: $F(x) = -kx$

$$-kx = m \frac{d^2x}{dt^2}$$

Solution : $x(t) = A \sin(\omega t) + B \cos(\omega t)$

with

$$\omega = \sqrt{k/m}$$



Phonon frequencies from force constants

Matrix of interatomic force constants :

$$C_{\kappa\alpha,\kappa'\alpha'}(a,a') = \frac{\partial^2 E_{BO}}{\partial R_{\kappa\alpha}^a \partial R_{\kappa'\alpha'}^{a'}}$$

Fourier Transform (using translational invariance) :

$$\tilde{C}_{k\alpha,k'\alpha'}(\vec{q}) = \sum_{q'} C_{k\alpha,k'\alpha'}(0,a') e^{i\vec{q}\cdot\vec{R}^{a'}}$$

Computation of phonon frequencies and eigenvectors =
solution of generalized eigenvalue problem

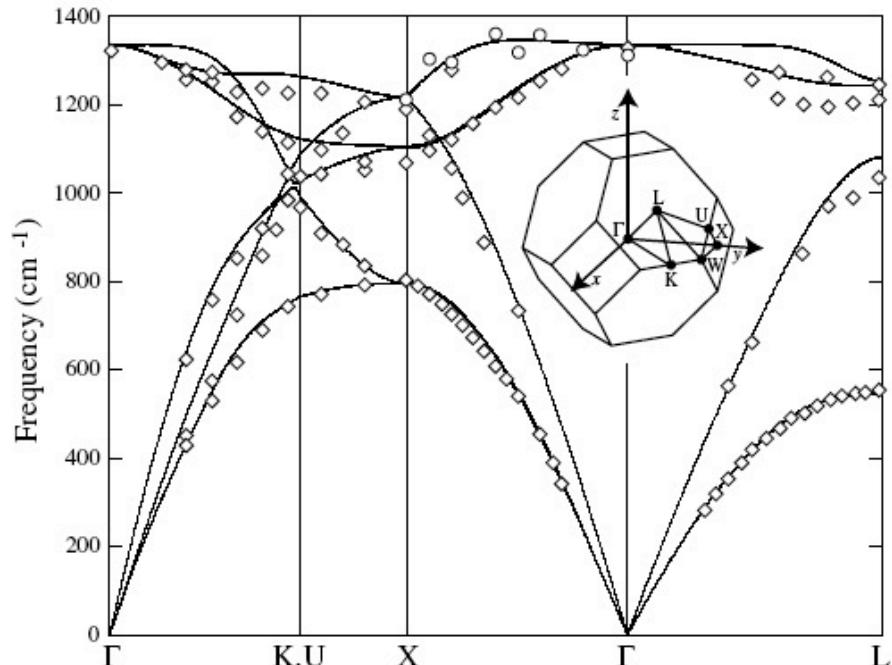
$$\sum_{k'\alpha'} \tilde{C}_{k\alpha, k'\alpha'}(\vec{q}) \cdot u_{m\vec{q}}(k'\alpha') = M_k \cdot \omega_{m\vec{q}}^2 \cdot u_{m\vec{q}}(k\alpha)$$

↑ phonon displacement pattern
↑ masses ↑ square of phonon frequencies

How to get second derivatives of the energy ?

Density Functional Perturbation Theory...

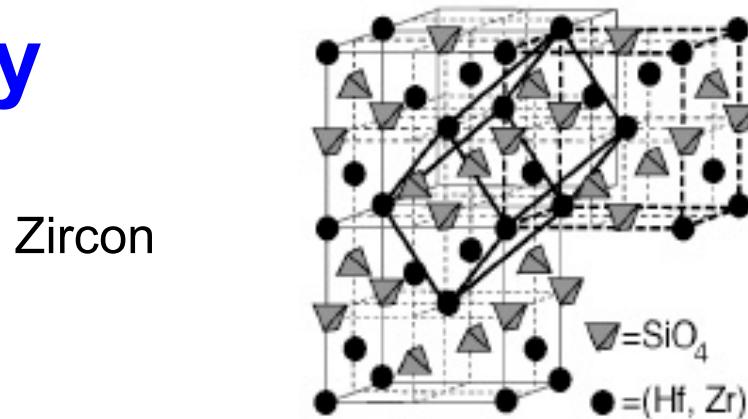
Phonons : exp vs theory



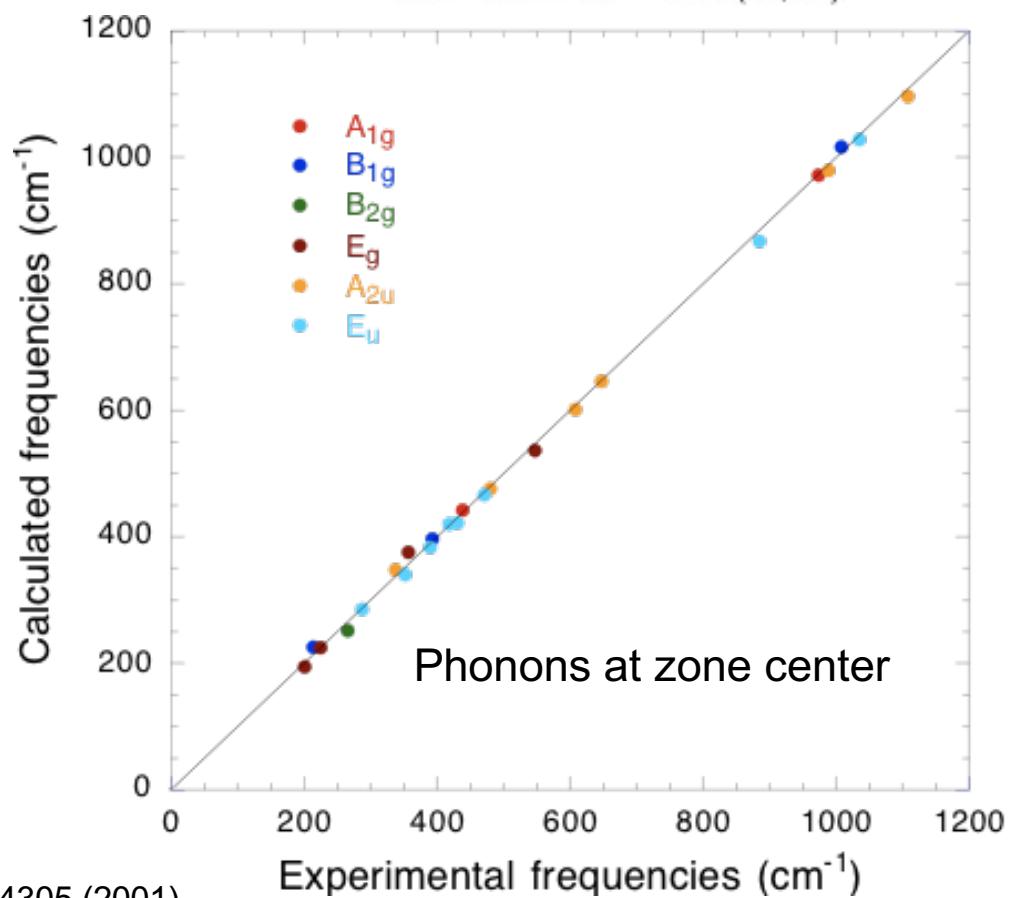
Diamond

XG, G.-M. Rignanese and R. Caracas.
Zeit. Kristall. 220, 458-472 (2005)

Rignanese, XG and Pasquarello. *Phys. Rev. B* 63, 104305 (2001)



Zircon



Challenges for periodic materials ?

In addition of being able to compute derivatives of BO energy :

Treating phonons of **different wavelengths** ?
(Not only periodic ones)

Treating **electric field** ?
Electric field => linear potential,
incompatible with periodicity

Even for phonons at zero wavevector (Gamma),
treating **LO-TO splitting**
(longitudinal optic – transverse optic)

Perturbations (adiabatic)

Why perturbations ?

Many physical properties = derivatives of total energy
(or suitable thermodynamic potential) with respect to perturbations.

Consider :

- atomic displacements (phonons)
- dilatation/contraction of primitive cell
- homogeneous external field (electric field, magnetic field ...)

Derivatives of total energy (electronic part + nuclei-nuclei interaction) :

1st order derivatives : forces, stresses, dipole moment ...

2nd order derivatives : dynamical matrix, elastic constants, dielectric susceptibility
atomic polar tensors or Born effective charge tensors
piezoelectricity, internal strains ...

3rd order derivatives : non-linear dielectric susceptibility, Raman susceptibilities
electro-optic effect, phonon - phonon interaction, Grüneisen parameters, ...

Further properties obtained by integration over phononic degrees of freedom :
entropy, thermal expansion, phonon-limited thermal conductivity ...

Perturbations

- * Variation of energy and density around fixed potential

$$E_{el}(\lambda) = \sum_{\alpha,occ} \langle \Psi_\alpha(\lambda) | \hat{T} + \hat{V}_{ext}(\lambda) | \Psi_\alpha(\lambda) \rangle + E_{Hxc}[\rho(\lambda)]$$

$$\rho(\vec{r};\lambda) = \sum_{\alpha,occ} \Psi_\alpha^*(\vec{r};\lambda) \Psi_\alpha(\vec{r};\lambda)$$

- * Perturbations (assumed known through all orders)

$$\hat{V}_{ext}(\lambda) = \hat{V}_{ext}^{(0)} + \lambda \hat{V}_{ext}^{(1)} + \lambda^2 \hat{V}_{ext}^{(2)} + \dots$$

i.e. : to investigate phonons, parameter of perturbation governs linearly nuclei displacement, but change of potential is non-linear in this parameter.

$$\Delta V_{ph}(\vec{r}) = \sum_{\kappa: \text{nuclei+cell}} V_\kappa(\vec{r} - (\vec{R}_\kappa^{(0)} + \vec{u}_\kappa)) - V_\kappa(\vec{r} - \vec{R}_\kappa^{(0)})$$

$$\vec{u}_\kappa = \lambda \vec{e}_\kappa \cos(\vec{q} \cdot \vec{R}_\kappa^{(0)})$$

parameter 'polarisation' phonon wavevector

How to get energy derivatives ?

$$E = E^{(0)} + \lambda E^{(1)} + \lambda^2 E^{(2)} + \dots \quad \psi = \psi^{(0)} + \lambda \psi^{(1)} + \lambda^2 \psi^{(2)} + \dots$$

- * Finite Differences

Compare $E \{ \psi; V_{ext} \}$ and $E' \{ \psi'; V'_{ext} \}$

'Direct' Approach (Frozen phonons ... Supercells ...)
[Note problem with commensurability]

- * Hellman - Feynman theorem (for $E^{(1)}$)

Due to variational character : $\frac{\partial E}{\partial \psi} = 0$

$$\frac{dE}{d\lambda} = \frac{\partial E}{\partial V_{ext}} \frac{\partial V_{ext}}{\partial \lambda} + \frac{\partial E}{\partial \psi} \cdot \frac{\partial \psi}{\partial \lambda} = \frac{\partial E}{\partial V_{ext}} V_{ext}^{(1)}$$

\Downarrow $\psi^{(1)}$

In order to get $E^{(1)}$ we do not need $\psi^{(1)}$

General framework of perturbation theory

* $A(\lambda) = A^{(0)} + \lambda A^{(1)} + \lambda^2 A^{(2)} + \lambda^3 A^{(3)} \dots$

* $E\{\psi; V_{ext}\}$

Hypothesis : we know $V_{ext}(\lambda) = V_{ext}^{(0)} + \lambda V_{ext}^{(1)} + \lambda^2 V_{ext}^{(2)} + \dots$

through all orders, as well as $\psi^{(0)}, p_\alpha^{(0)}, E^{(0)}$

Should calculate :

$$E^{(1)}, E^{(2)}, E^{(3)} \dots$$

2nd order derivatives of BO energy :
dynamical matrix, dielectric susceptibility,
elastic constants, ...

$$p_\alpha^{(1)}, p_\alpha^{(2)}, p_\alpha^{(3)} \dots$$

$$\psi_\alpha^{(1)}, \psi_\alpha^{(2)}, \psi_\alpha^{(3)} \dots$$

$$\varepsilon_\alpha^{(1)}, \varepsilon_\alpha^{(2)}, \varepsilon_\alpha^{(3)} \dots$$

will be needed for
T-dependence of
electronic structure

Ordinary quantum mechanics

Perturbation theory for ordinary quantum mechanics

$$(\hat{H} - \varepsilon_\alpha) |\Psi_\alpha\rangle = 0 \quad (\text{Schrödinger equation})$$

$$\langle \Psi_\alpha | \Psi_\alpha \rangle = 1 \quad (\text{normalisation condition})$$

$$\langle \Psi_\alpha | \hat{H} - \varepsilon_\alpha | \Psi_\alpha \rangle = 0$$

$$\text{or } \varepsilon_\alpha = \langle \Psi_\alpha | \hat{H} | \Psi_\alpha \rangle \quad (\text{expectation value})$$

Hamiltonian supposed known through all orders

$$\hat{H} = \hat{H}^{(0)} + \lambda \hat{H}^{(1)} + \lambda^2 \hat{H}^{(2)} + \dots = \sum_n \lambda^n \hat{H}^{(n)}$$

Perturbation expansion of Schrödinger equation

Suppose $\hat{H}(\lambda) |\psi_n(\lambda)\rangle = \varepsilon_n(\lambda) |\psi_n(\lambda)\rangle$ valid for all λ

with
$$\begin{cases} \hat{H}(\lambda) = \hat{H}^{(0)} + \lambda \hat{H}^{(1)} \\ \psi_n(\lambda) = \psi_n^{(0)} + \lambda \psi_n^{(1)} + \lambda^2 \psi_n^{(2)} + \dots \\ \varepsilon_n(\lambda) = \varepsilon_n^{(0)} + \lambda \varepsilon_n^{(1)} + \lambda^2 \varepsilon_n^{(2)} + \dots \end{cases}$$

Expand Schrödinger equation:

$$\begin{aligned} & \hat{H}^{(0)} |\psi_n^{(0)}\rangle + \lambda \left(\hat{H}^{(1)} |\psi_n^{(0)}\rangle + \hat{H}^{(0)} |\psi_n^{(1)}\rangle \right) + \lambda^2 \left(\hat{H}^{(1)} |\psi_n^{(1)}\rangle + \hat{H}^{(0)} |\psi_n^{(2)}\rangle \right) + \dots \\ &= \varepsilon_n^{(0)} |\psi_n^{(0)}\rangle + \lambda \left(\varepsilon_n^{(1)} |\psi_n^{(0)}\rangle + \varepsilon_n^{(0)} |\psi_n^{(1)}\rangle \right) + \lambda^2 \left(\varepsilon_n^{(2)} |\psi_n^{(0)}\rangle + \varepsilon_n^{(1)} |\psi_n^{(1)}\rangle + \varepsilon_n^{(0)} |\psi_n^{(2)}\rangle \right) + \dots \end{aligned}$$

Perturbation expansion of Schrödinger equation

$$\begin{aligned} & \hat{H}^{(0)} |\psi_n^{(0)}\rangle + \lambda \left(\hat{H}^{(1)} |\psi_n^{(0)}\rangle + \hat{H}^{(0)} |\psi_n^{(1)}\rangle \right) + \lambda^2 \left(\hat{H}^{(1)} |\psi_n^{(1)}\rangle + \hat{H}^{(0)} |\psi_n^{(2)}\rangle \right) + \dots \\ &= \varepsilon_n^{(0)} |\psi_n^{(0)}\rangle + \lambda \left(\varepsilon_n^{(1)} |\psi_n^{(0)}\rangle + \varepsilon_n^{(0)} |\psi_n^{(1)}\rangle \right) + \lambda^2 \left(\varepsilon_n^{(2)} |\psi_n^{(0)}\rangle + \varepsilon_n^{(1)} |\psi_n^{(1)}\rangle + \varepsilon_n^{(0)} |\psi_n^{(2)}\rangle \right) + \dots \end{aligned}$$

If $\lambda = 0$, one gets $\hat{H}^{(0)} |\psi_n^{(0)}\rangle = \varepsilon_n^{(0)} |\psi_n^{(0)}\rangle$ no surprise ...

Derivative with respect to λ , then $\lambda = 0$ (=first order of perturbation)

$$\Rightarrow \hat{H}^{(1)} |\psi_n^{(0)}\rangle + \hat{H}^{(0)} |\psi_n^{(1)}\rangle = \varepsilon_n^{(1)} |\psi_n^{(0)}\rangle + \varepsilon_n^{(0)} |\psi_n^{(1)}\rangle$$

2 derivatives with respect to λ , then $\lambda = 0$ (=second order of perturbation)

$$\Rightarrow \hat{H}^{(1)} |\psi_n^{(1)}\rangle + \hat{H}^{(0)} |\psi_n^{(2)}\rangle = \varepsilon_n^{(2)} |\psi_n^{(0)}\rangle + \varepsilon_n^{(1)} |\psi_n^{(1)}\rangle + \varepsilon_n^{(0)} |\psi_n^{(2)}\rangle$$

Perturbation expansion of the normalisation

If $\forall \lambda : \langle \psi_n(\lambda) | \psi_n(\lambda) \rangle = 1$

with $\psi_n(\lambda) = \psi_n^{(0)} + \lambda \psi_n^{(1)} + \lambda^2 \psi_n^{(2)} + \dots$

Same technique than for Schrödinger equation, one deduces :

$$\langle \psi_n^{(0)} | \psi_n^{(0)} \rangle = 1$$

$$\langle \psi_n^{(1)} | \psi_n^{(0)} \rangle + \langle \psi_n^{(0)} | \psi_n^{(1)} \rangle = 0$$

$$\langle \psi_n^{(2)} | \psi_n^{(0)} \rangle + \langle \psi_n^{(1)} | \psi_n^{(1)} \rangle + \langle \psi_n^{(0)} | \psi_n^{(2)} \rangle = 0$$

no surprise ...

Hellmann & Feynman theorem : $\varepsilon_n^{(1)}$

Start from first-order Schrödinger equation

$$\hat{H}^{(1)} |\psi_n^{(0)}\rangle + \hat{H}^{(0)} |\psi_n^{(1)}\rangle = \varepsilon_n^{(1)} |\psi_n^{(0)}\rangle + \varepsilon_n^{(0)} |\psi_n^{(1)}\rangle$$

Premultiply by $\langle \psi_n^{(0)} |$

$$\langle \psi_n^{(0)} | \hat{H}^{(1)} | \psi_n^{(0)} \rangle + \underbrace{\langle \psi_n^{(0)} | \hat{H}^{(0)} | \psi_n^{(1)} \rangle}_{\substack{\parallel \\ \langle \psi_n^{(0)} | \varepsilon_n^{(0)} \\ \curvearrowleft}} = \varepsilon_n^{(1)} \underbrace{\langle \psi_n^{(0)} | \psi_n^{(0)} \rangle}_{=1} + \varepsilon_n^{(0)} \langle \psi_n^{(0)} | \psi_n^{(1)} \rangle$$

So : $\boxed{\varepsilon_n^{(1)} = \langle \psi_n^{(0)} | \hat{H}^{(1)} | \psi_n^{(0)} \rangle}$ = Hellmann & Feynman theorem

$\varepsilon_n^{(1)}$ OK !

- $\psi_n^{(0)}$ and $\hat{H}^{(1)}$ supposed known
- $\psi_n^{(1)}$ not needed
- $\langle \psi_n^{(0)} | \hat{H}^{(1)} | \psi_n^{(0)} \rangle$ = expectation of the Hamiltonian for the non-perturbed wavef.

Second-order derivative of total energy $\varepsilon_{\alpha}^{(2)}$

Start from second-order Schrödinger equation

$$\hat{H}^{(1)} |\psi_n^{(1)}\rangle + \hat{H}^{(0)} |\psi_n^{(2)}\rangle = \varepsilon_n^{(2)} |\psi_n^{(0)}\rangle + \varepsilon_n^{(1)} |\psi_n^{(1)}\rangle + \varepsilon_n^{(0)} |\psi_n^{(2)}\rangle$$

Premultiply by $\langle \psi_n^{(0)} |$

$$\varepsilon_{\alpha}^{(2)} = \langle \psi_{\alpha}^{(0)} | \hat{H}^{(1)} - \varepsilon_{\alpha}^{(1)} | \psi_{\alpha}^{(1)} \rangle \text{ or } \varepsilon_{\alpha}^{(2)} = \langle \psi_{\alpha}^{(1)} | \hat{H}^{(1)} - \varepsilon_{\alpha}^{(1)} | \psi_{\alpha}^{(0)} \rangle$$

Both can be combined :

$$\varepsilon_{\alpha}^{(2)} = \frac{1}{2} \left(\langle \psi_{\alpha}^{(0)} | \hat{H}^{(1)} - \varepsilon_{\alpha}^{(1)} | \psi_{\alpha}^{(1)} \rangle + \langle \psi_{\alpha}^{(1)} | \hat{H}^{(1)} - \varepsilon_{\alpha}^{(1)} | \psi_{\alpha}^{(0)} \rangle \right)$$

and, using $\langle \psi_n^{(1)} | \psi_n^{(0)} \rangle + \langle \psi_n^{(0)} | \psi_n^{(1)} \rangle = 0$

$$= \frac{1}{2} \left(\langle \psi_{\alpha}^{(0)} | \hat{H}^{(1)} | \psi_{\alpha}^{(1)} \rangle + \langle \psi_{\alpha}^{(1)} | \hat{H}^{(1)} | \psi_{\alpha}^{(0)} \rangle \right)$$

No knowledge of $\psi_{\alpha}^{(2)}$ is needed, but needs $\psi_{\alpha}^{(1)}$! How to get it ?

In search of $|\psi_n^{(1)}\rangle$

Again first-order Schrödinger equation :

$$\hat{H}^{(1)} \left| \psi_n^{(0)} \right\rangle + \hat{H}^{(0)} \left| \psi_n^{(1)} \right\rangle = \varepsilon_n^{(1)} \left| \psi_n^{(0)} \right\rangle + \varepsilon_n^{(0)} \left| \psi_n^{(1)} \right\rangle$$

known

Terms containing $\left| \psi_n^{(1)} \right\rangle$ are gathered :

$$\left(\hat{H}^{(0)} - \varepsilon_n^{(0)} \right) \left| \psi_n^{(1)} \right\rangle = - \left(\hat{H}^{(1)} - \varepsilon_n^{(1)} \right) \left| \psi_n^{(0)} \right\rangle \quad (\text{called Sternheimer equation})$$

Equivalence with matrix equation (systeme of linear equations)

$$\underline{\mathbf{A}} \cdot \underline{\mathbf{x}} = \underline{\mathbf{y}}$$

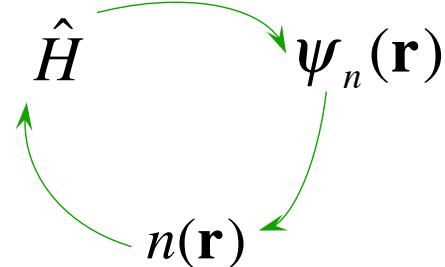
usually solved by $\underline{x} = \underline{\underline{A}}^{-1} \underline{y}$ if $\underline{\underline{A}}^{-1}$ exist.

Generalisation: Density-functional perturbation theory (DFPT)

Basic equations in DFT

Solve self-consistently Kohn-Sham equation

$$\left\{ \begin{array}{l} \hat{H} |\Psi_n\rangle = \varepsilon_n |\Psi_n\rangle \\ \hat{H} = \hat{T} + \hat{V} + \hat{V}_{Hxc}[n] \\ n(\vec{r}) = \sum_n^{occ} \Psi_n^*(\vec{r}) \Psi_n(\vec{r}) \end{array} \right.$$



$$\delta_{mn} = \langle \Psi_m | \Psi_n \rangle \text{ for } m, n \in \text{occupied set}$$

or minimize

$$E_{el}\{\Psi\} = \sum_n^{occ} \langle \Psi_n | \hat{T} + \hat{V} | \Psi_n \rangle + E_{Hxc}[n]$$

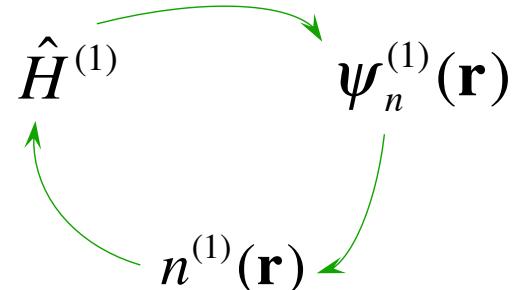
What is \hat{V} ?

$$\hat{V}(\vec{r}) = \sum_{ak} -\frac{Z_k}{|\vec{r} - \vec{R}_k|}$$

Basic equations in DFPT

Solve self-consistently Sternheimer equation

$$\left\{ \begin{array}{l} (\hat{H}^{(0)} - \varepsilon_n^{(0)}) |\Psi_n^{(I)}\rangle = - (\hat{H}^{(I)} - \varepsilon_n^{(I)}) |\Psi_n^{(0)}\rangle \quad 0 = \langle \Psi_m^{(0)} | \Psi_n^{(I)} \rangle \text{ for } m \in \text{occupied set} \\ \varepsilon_n^{(I)} = \langle \Psi_n^{(0)} | \hat{H}^{(I)} | \Psi_n^{(0)} \rangle \\ \hat{H}^{(I)} = \hat{V}^{(I)} + \int \frac{\delta^2 E_{Hxc}}{\delta \rho(r) \delta \rho(r')} n^{(I)}(r') dr' \\ n^{(I)}(\vec{r}) = \sum_n^{occ} \Psi_n^{(I)*}(\vec{r}) \Psi_n^{(0)}(\vec{r}) + \Psi_n^{(0)*}(\vec{r}) \Psi_n^{(I)}(\vec{r}) \end{array} \right.$$



or minimize

$$E_{el}^{(2)} \left\{ \Psi^{(I)}; \Psi^{(0)} \right\} = \sum_n^{occ} \left\langle \Psi_n^{(I)} \left| \hat{H}^{(0)} - \varepsilon_n^{(0)} \right| \Psi_n^{(I)} \right\rangle + \left\langle \Psi_n^{(I)} \left| \hat{V}^{(I)} \right| \Psi_n^{(0)} \right\rangle + \left\langle \Psi_n^{(0)} \left| \hat{V}^{(I)} \right| \Psi_n^{(I)} \right\rangle + \left\langle \Psi_n^{(0)} \left| \hat{V}^{(2)} \right| \Psi_n^{(0)} \right\rangle + \frac{1}{2} \iint \frac{\delta^2 E_{Hxc}}{\delta \rho(\vec{r}) \delta \rho(\vec{r}')} n^{(I)}(\vec{r}) n^{(I)}(\vec{r}') d\vec{r} d\vec{r}'$$

What is $\hat{V}^{(I)}$, $\hat{V}^{(2)}$?

The potential and its 1st derivative

Derivative with respect to $R_{\kappa\alpha}^a$

$$V^{(0)}(\vec{r}) = \sum_{\alpha\kappa} -\frac{Z_\kappa}{|\vec{r}-\vec{R}_\kappa^a|}$$

$$V^{(1)}(\vec{r}) = \frac{\partial V(\vec{r})}{\partial R_{\kappa,\alpha}^a} = \frac{Z_\kappa}{|\vec{r}-\vec{R}_\kappa^a|^2} \cdot \frac{\partial |\vec{r}-\vec{R}_\kappa^a|}{\partial u_{\kappa,\alpha}^a} = -\frac{Z_\kappa}{|\vec{r}-\vec{R}_\kappa^a|^3} \cdot (\vec{r}-\vec{R}_\kappa^a)_\alpha$$

Generalisation to pseudopotentials can be worked out ...

Collective displacement with wavevector \vec{q}

$$V_{\vec{q},\kappa,\alpha}^{(1)}(\vec{r}) = \sum_a e^{i\vec{q}\vec{R}_a} \frac{\partial V(\vec{r})}{\partial R_{\kappa,\alpha}^a}$$

Factorization of the phase

Suppose unperturbed system periodic $V^{(0)}(\vec{r} + \vec{R}_a) = V^{(0)}(\vec{r})$

If perturbation characterized by a wavevector : $V^{(I)}(\vec{r} + \vec{R}_a) = e^{i\vec{q}\cdot\vec{R}_a} V^{(I)}(\vec{r})$

all responses, at linear order, will be characterized by a wavevector :

$$n^{(I)}(\vec{r} + \vec{R}_a) = e^{i\vec{q}\cdot\vec{R}_a} n^{(I)}(\vec{r}) \quad \Psi_{m,\vec{k},\vec{q}}^{(I)}(\vec{r} + \vec{R}_a) = e^{i(\vec{k} + \vec{q})\cdot\vec{R}_a} \Psi_{m,\vec{k},\vec{q}}^{(I)}(\vec{r})$$

Now, define related periodic quantities

$$\bar{n}^{(I)}(\vec{r}) = e^{-i\vec{q}\cdot\vec{r}} n^{(I)}(\vec{r}) \quad u_{m,\vec{k},\vec{q}}^{(I)}(\vec{r}) = (N\Omega_0)^{1/2} e^{-i(\vec{k} + \vec{q})\cdot\vec{r}} \Psi_{m,\vec{k},\vec{q}}^{(I)}(\vec{r})$$

In equations of DFPT, only these periodic quantities appear:

phases $e^{-i\vec{q}\cdot\vec{r}}$ and $e^{-i(\vec{k} + \vec{q})\cdot\vec{r}}$ can be factorized

Treatment of perturbations **incommensurate** with unperturbed system periodicity is thus mapped onto the original periodic system.

Computing mixed derivatives

How to get $E^{j_1 j_2}$ from $\psi_\alpha^{(0)}, \psi_\alpha^{j_1}, \psi_\alpha^{j_2}$?

$$E_{el}^{(2)} \left\{ \Psi^{(I)}; \Psi^{(0)} \right\} = \sum_n^{occ} \left\langle \Psi_n^{(I)} \left| \hat{H}^{(0)} - \epsilon_n^{(0)} \right| \Psi_n^{(I)} \right\rangle + \left\langle \Psi_n^{(I)} \left| \hat{V}^{(I)} \right| \Psi_n^{(0)} \right\rangle \\ + \left\langle \Psi_n^{(0)} \left| \hat{V}^{(I)} \right| \Psi_n^{(I)} \right\rangle + \left\langle \Psi_n^{(0)} \left| \hat{V}^{(2)} \right| \Psi_n^{(0)} \right\rangle \\ + \frac{1}{2} \iint \frac{\delta^2 E_{Hxc}}{\delta \rho(\vec{r}) \delta \rho(\vec{r}')} n^{(I)}(\vec{r}) n^{(I)}(\vec{r}') d\vec{r} d\vec{r}'$$

Generalization to $E_{el}^{j_1 j_2} = \frac{1}{2} (\tilde{E}_{el}^{j_1 j_2} + \tilde{E}_{el}^{j_2 j_1})$

with $\tilde{E}_{el}^{j_1 j_2} \left\{ \Psi^{j_1}, \Psi^{j_2}; \Psi^{(0)} \right\} = \sum_n^{occ} \left\langle \Psi_n^{j_1} \left| \hat{H}^{(0)} - \epsilon_n^{(0)} \right| \Psi_n^{j_2} \right\rangle + \left\langle \Psi_n^{j_1} \left| \hat{V}^{j_2} \right| \Psi_n^{(0)} \right\rangle \\ + \left\langle \Psi_n^{(0)} \left| \hat{V}^{j_1} \right| \Psi_n^{j_2} \right\rangle + \left\langle \Psi_n^{(0)} \left| \hat{V}^{j_1 j_2} \right| \Psi_n^{(0)} \right\rangle \\ + \frac{1}{2} \iint \frac{\delta^2 E_{Hxc}}{\delta \rho(\vec{r}) \delta \rho(\vec{r}')} n^{j_1}(\vec{r}) n^{j_2}(\vec{r}') d\vec{r} d\vec{r}'$

being a stationary expression, leading to the non-stationary expression

$$E_{el}^{j_1 j_2} \left\{ \Psi^{j_1}; \Psi^{(0)} \right\} = \sum_n^{occ} \left\langle \Psi_n^{j_1} \left| \hat{V}^{j_2} \right| \Psi_n^{(0)} \right\rangle + \left\langle \Psi_n^{(0)} \left| \hat{V}^{j_1 j_2} \right| \Psi_n^{(0)} \right\rangle$$

Independent of Ψ^{j_2}

Order of calculations in DFPT

(1) Ground-state calculation $V^{(0)} \rightarrow \psi_n^{(0)}, n^{(0)}$

(2) Do for each perturbation j_1

use $\psi_n^{(0)}, n^{(0)}$

$V^{j_1} \rightarrow \psi_n^{j_1}, n^{j_1}$

using minimization of second-order energy
or
Sternheimer equation

Enddo

(3) Do for each $\{j_1, j_2\}$

get $E^{j_1 j_2}$ from $\psi_n^{(0)}, \psi_n^{j_1}, \psi_n^{j_2}$

Enddo

(4) Post-processing : from ‘bare’ $E^{j_1 j_2}$ to physical properties

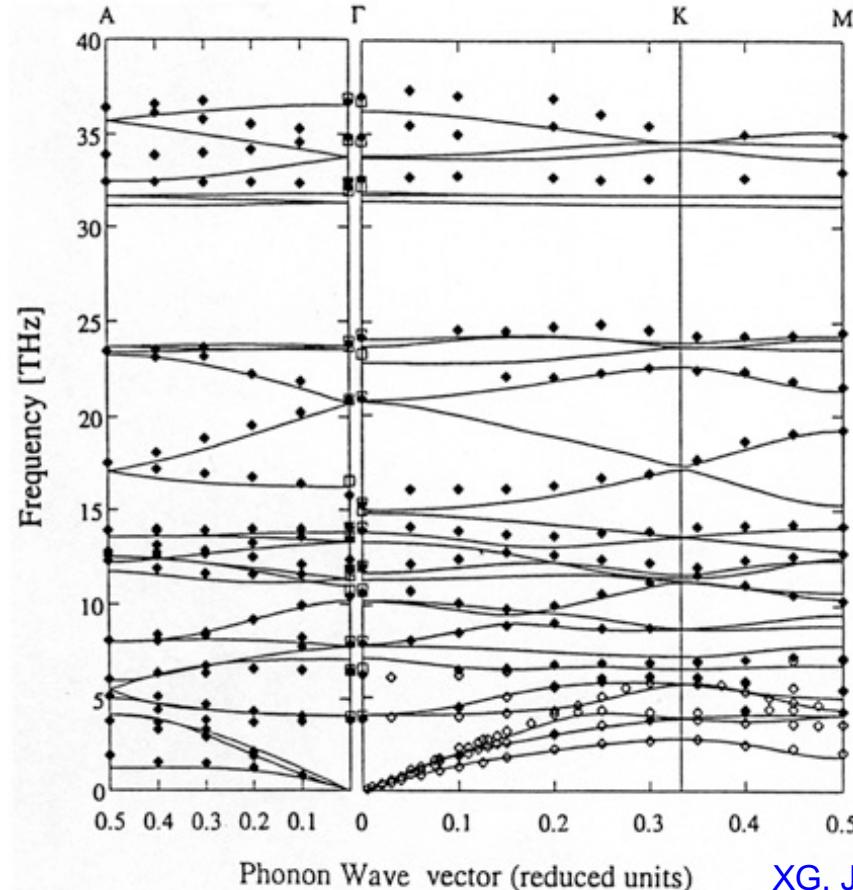
Phonon band structures from DFPT

Phonon band structure

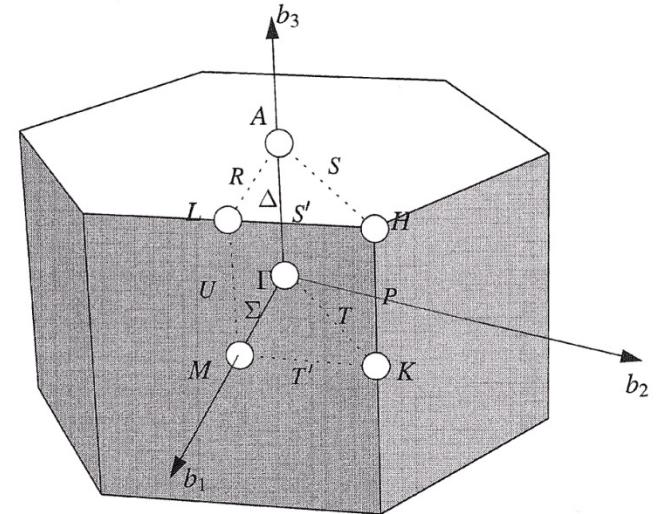
From DFPT : straightforward, although lengthy (self-consistent calculation) to compute, for one wavevector :

$$\tilde{C}_{k\alpha,k'\beta}(\vec{q})$$

Full band structure needs values for many wavevectors ...



SiO₂ alpha-quartz



XG, J.-C.Charlier, D.C.Allan, M.P.Teter, *Phys. Rev. B* 50, 13055 (1994)

Fourier Interpolation

If IFCs were available, dynamical matrices could be obtained easily for **any** number of wavevectors

$$\tilde{C}_{\kappa\alpha,\kappa'\beta}(\vec{q}) = \sum_b C_{\kappa\alpha,\kappa'\beta}(0,b) e^{i\vec{q}\cdot\vec{R}^b}$$

IFCs are generated by

$$C_{\kappa\alpha,\kappa'\beta}(0,b) = \frac{(2\pi)^3}{\Omega_0} \int_{BZ} \tilde{C}_{\kappa\alpha,\kappa'\beta}(\vec{q}) e^{-i\vec{q}\cdot\vec{R}^b} d\vec{q}$$

= Fourier interpolation of dynamical matrices.

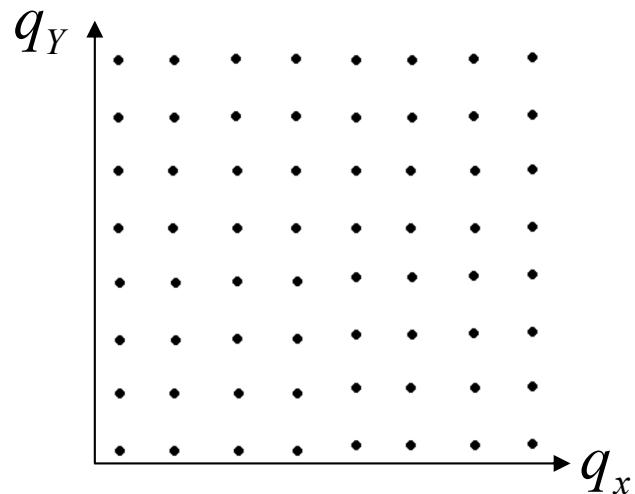
Numerical Fourier Interpolation

Key of the interpolation : replace the integral

$$C_{\kappa\alpha,\kappa'\beta}(0,b) = \frac{(2\pi)^3}{\Omega_0} \int_{BZ} \tilde{C}_{\kappa\alpha,\kappa'\beta}(\vec{q}) e^{-i\vec{q}\cdot\vec{R}^b} d\vec{q}$$

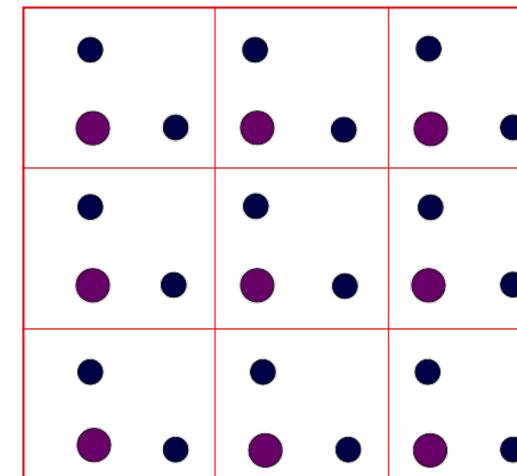
by summation on a few wavevectors (=“q-points”).

Grid of (l,m,n) points

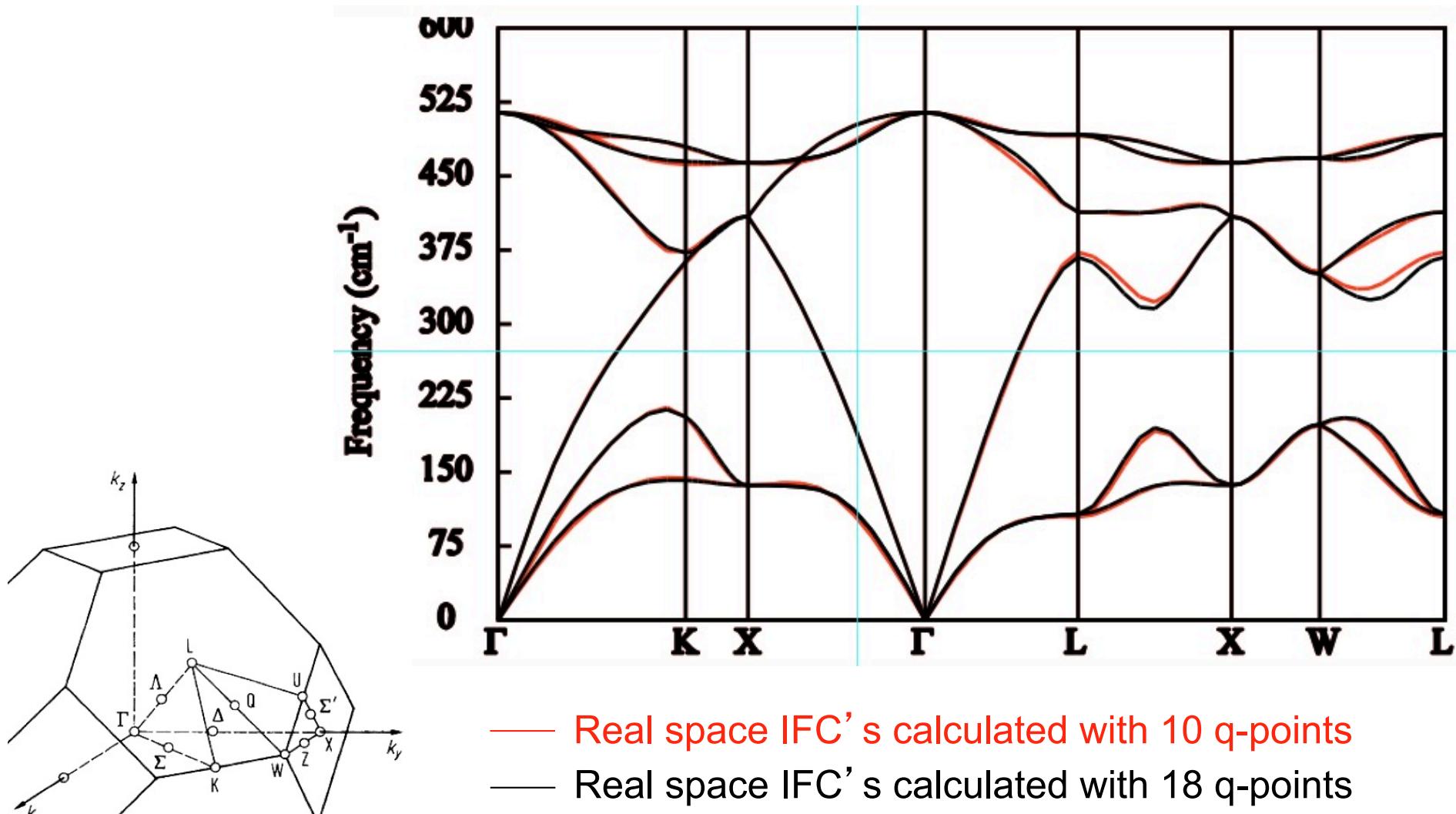


IFC's in box of (l,m,n) periodic cells

Fourier
↔

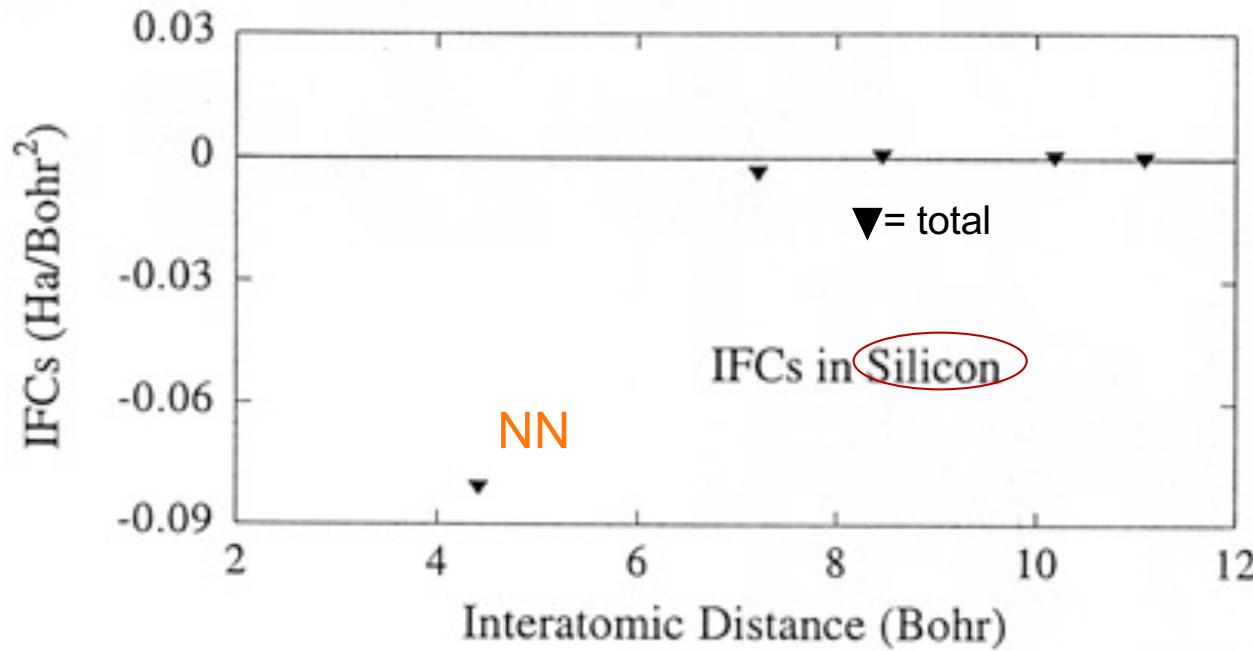


Fourier interpolation : Silicon



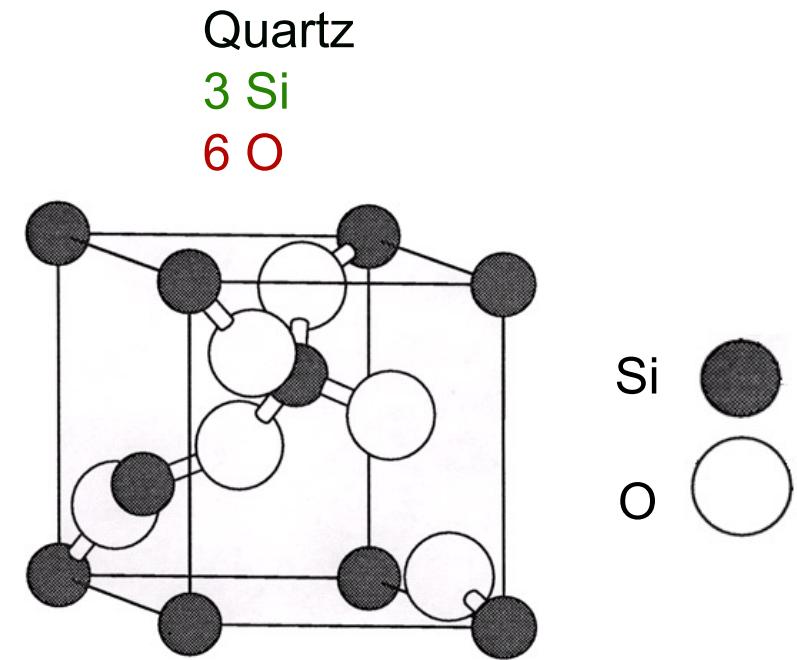
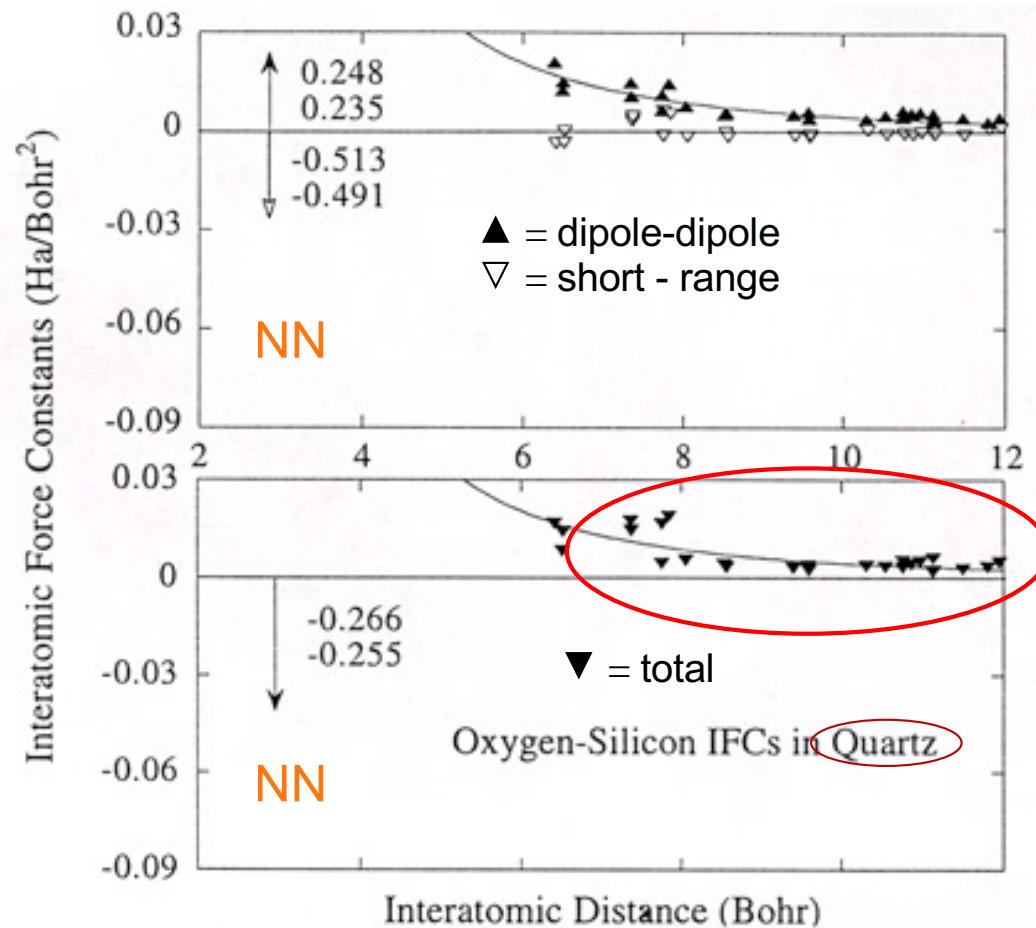
Interatomic force constants for silicon

IFC's are short range, i.e. falling to zero quickly after the nearest-neighbors (NN).



XG, *Adv. in Quantum Chemistry* 33, 225 (1999)

Interatomic force constants for silica quartz



Long-ranged
interatomic forces !

XG, *Adv. in Quantum Chemistry* 33, 225 (1999)

Understanding the long-range behaviour

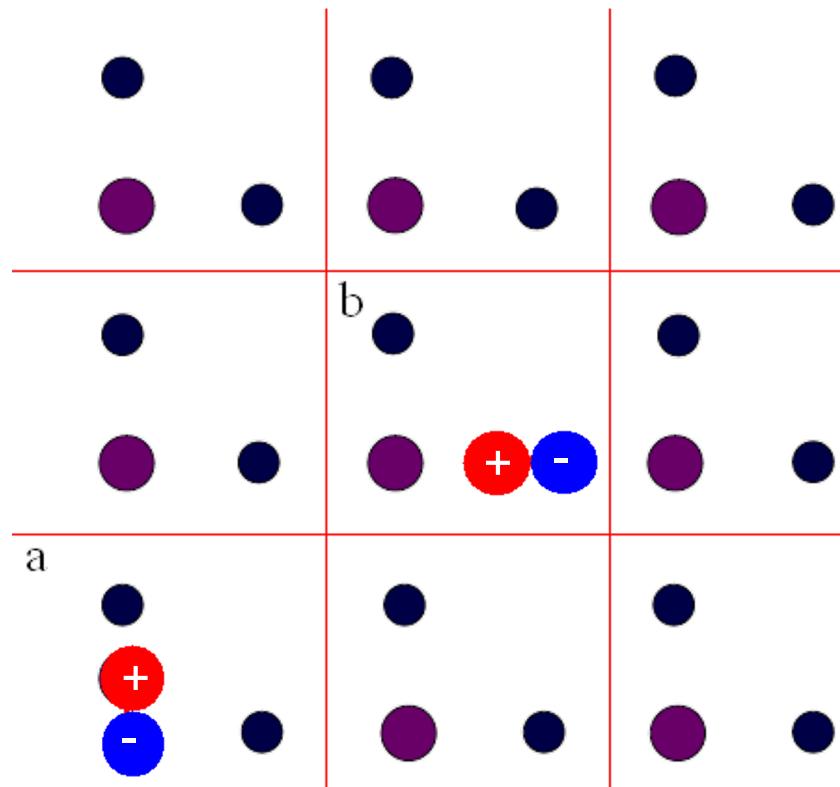
When a ion with charge Z is displaced from its equilibrium position, a **dipolar electric field** is created. Its effect on other ions is described by a **dipole - dipole interaction** appearing in IFC's.

Suppose : homogeneous material with isotropic dielectric tensor $\epsilon \delta_{\alpha\beta}$, ions with charges Z_k and $Z_{k'}$, then

$$C_{k\alpha,k'\beta}(0,b) = \frac{Z_k Z_{k'}}{\epsilon} \left(\frac{\delta_{\alpha\beta}}{d^3} - 3 \frac{d_\alpha d_\beta}{d^5} \right)$$

$$d = r_k^0 - r_{k'}^a = R^a + \tau_{k'} - \tau_k$$

Long range decay
of the IFC's : $1/d^3$



Effect of the long-range interaction

The dynamical matrix exhibit a non-analytical (**na**) behavior, mediated by the long-wavelength electric field

$$\tilde{C}_{\kappa\alpha,\kappa'\beta}^{\text{na}}(\vec{q} \rightarrow 0) = \frac{4\pi e^2}{\Omega_0} \frac{\sum_{\gamma} Z_{\kappa,\alpha\gamma}^* q_{\gamma} \sum_{\nu} Z_{\kappa',\beta\nu}^* q_{\nu}}{\sum_{\gamma,\nu} q_{\gamma} \epsilon_{\gamma\nu}^{\infty} q_{\nu}}$$

$$Z_{\kappa,\alpha\beta}^* = \Omega_0 \left. \frac{\partial P_{\alpha}}{\partial u_{\kappa,\beta}} \right|_{\delta \vec{E}=0} = \frac{\partial F_{\kappa\beta}}{\partial \mathcal{E}_{\beta}}$$

Born effective charge tensor for atom κ

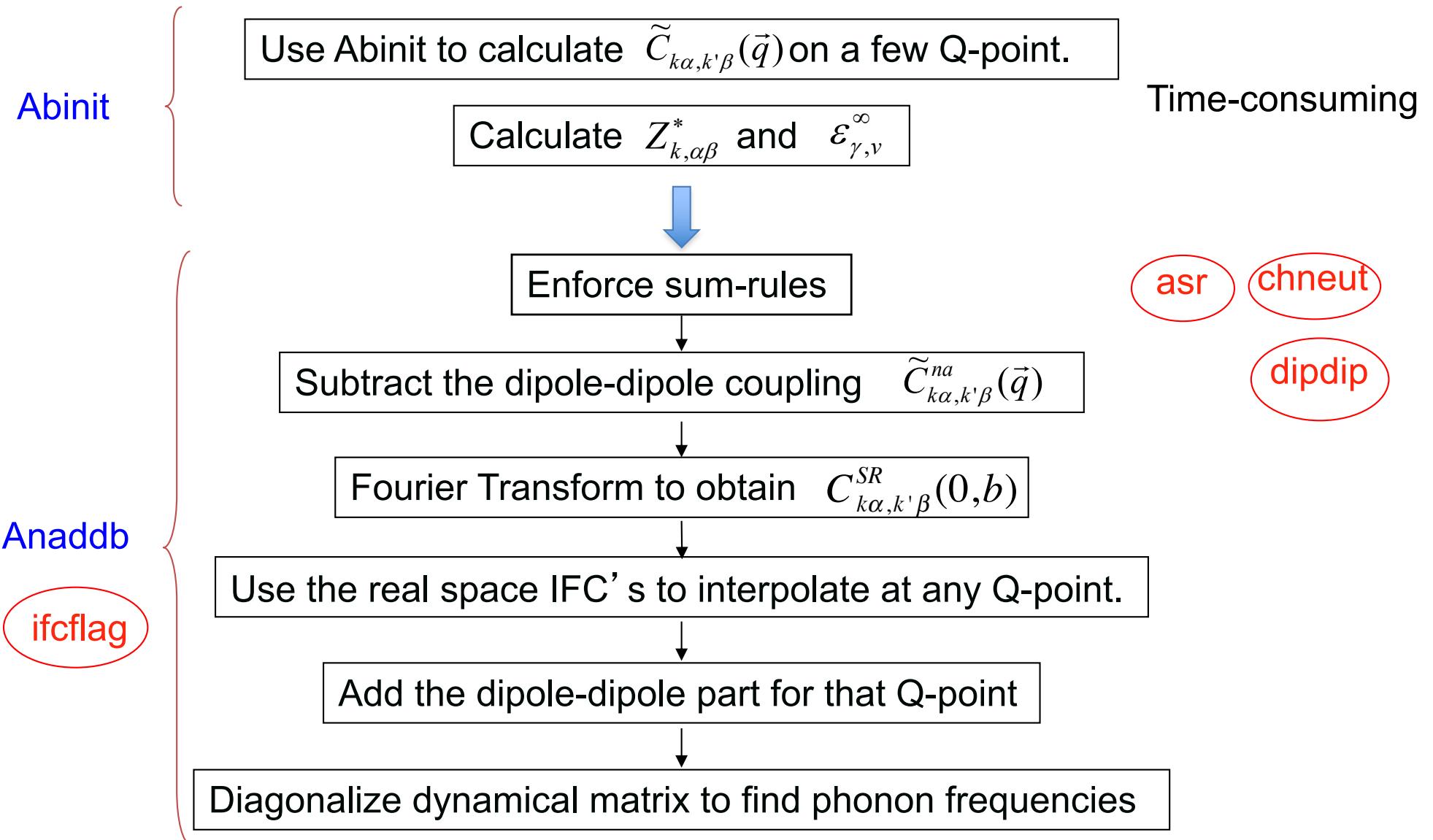
(Proportionality coefficient between polarisation and displacement, also between force and electric field)

$$\epsilon_{\gamma\nu}^{\infty} = \delta_{\gamma\nu} + 4\pi \frac{\partial P_{\gamma}}{\partial \mathcal{E}_{\nu}}$$

electronic dielectric tensor
(electronic contribution to the screening of the charges)

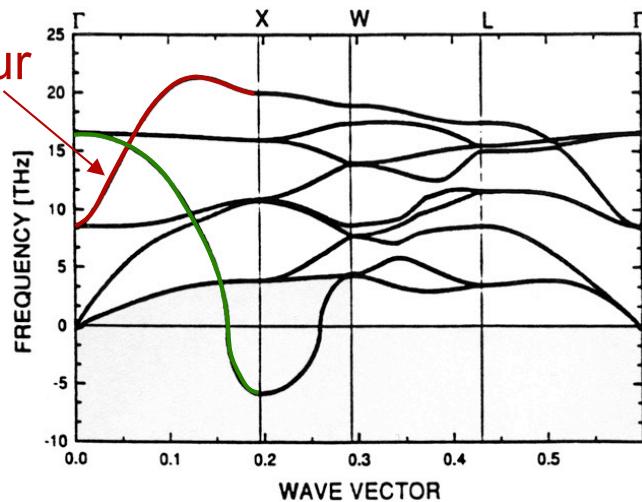
Both can be linked to a second derivative of total energy

Interpolation Scheme



Phonon dispersion curves of ZrO_2

Wrong behaviour

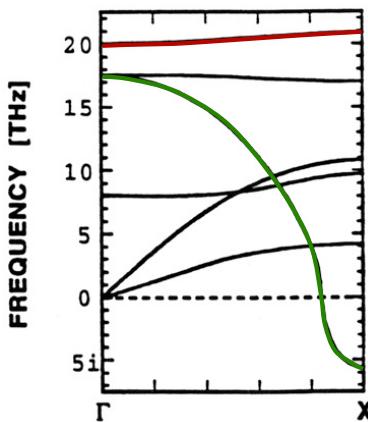


High - temperature : Fluorite structure
($Fm\bar{3}m$, one formula unit per cell)

Supercell calculation + interpolation
⚠ Long-range dipole-dipole
interaction not taken into account

ZrO_2 in the cubic structure at the equilibrium
lattice constant $a_0 = 5.13 \text{ \AA}$.

(From Parlinski K., Li Z.Q., and Kawazoe Y.,
Phys. Rev. Lett. 78, 4063 (1997))



DFPT (Linear-response)
with $Z_{\text{Zr}}^* = 5.75$
 $Z_0^* = -2.86$
 $\epsilon_\infty = 5.75$

LO - TO splitting 11.99 THz
Non-polar mode is OK

(From Detraux F., Ghosez Ph. and Gonze X., *Phys. Rev. Lett.* 81, 3297
(1998) - Comment to the Parlinski & al paper)

Analysis of instabilities

MgSiO₃

CUBIC

(5at/cell)

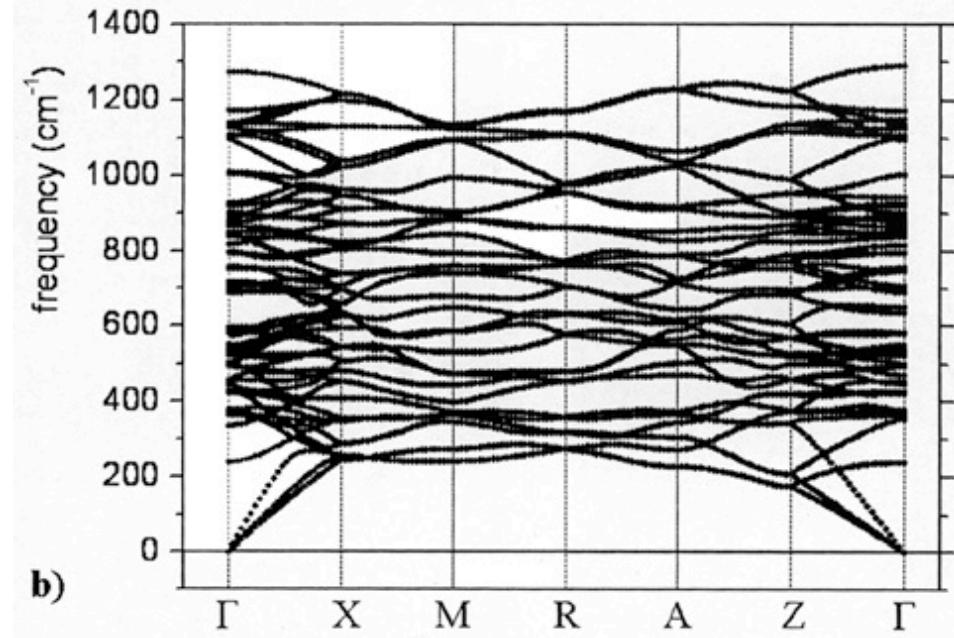
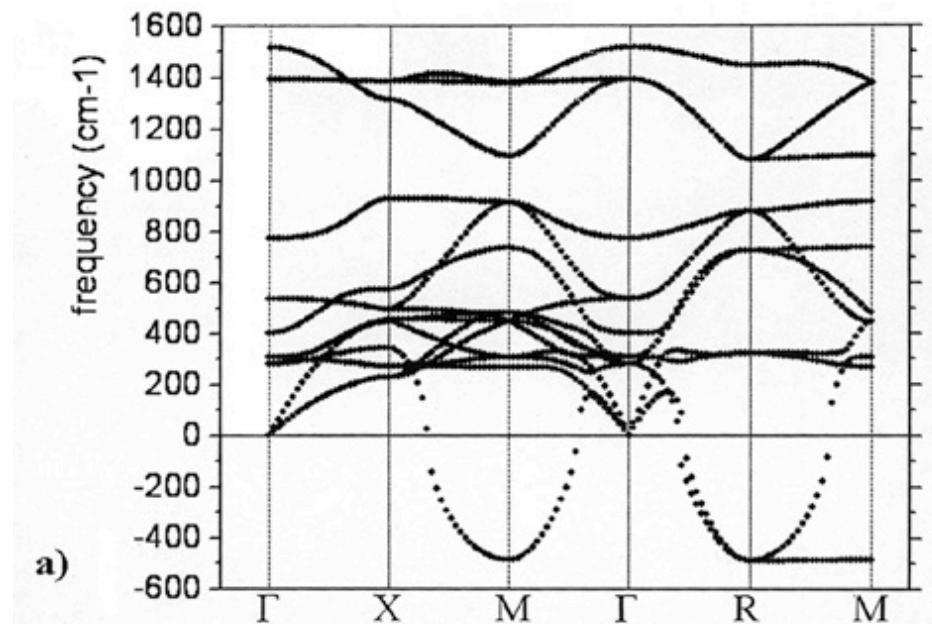
Phonon dispersion relations.

(a) Ideal cubic phase : unstable.

(b) Condensations of the unstable phonon modes generate a (meta) stable orthorhombic phase

ORTHORHOMBIC

(20at/cell)



Thermodynamic properties from DFPT

Statistical physics : phonons = bosons

Harmonic approximation :
phonons are independent particles,
obeying Bose-Einstein statistics

$$n(\omega) = \frac{1}{e^{\frac{\omega}{k_B T}} - 1}$$

Internal energy

$$U_{phon} = \int_0^{\omega_{\max}} \hbar\omega \left(n(\omega) + \frac{1}{2} \right) g(\omega) d\omega$$

Energy of the harmonic oscillator Phonon density of states

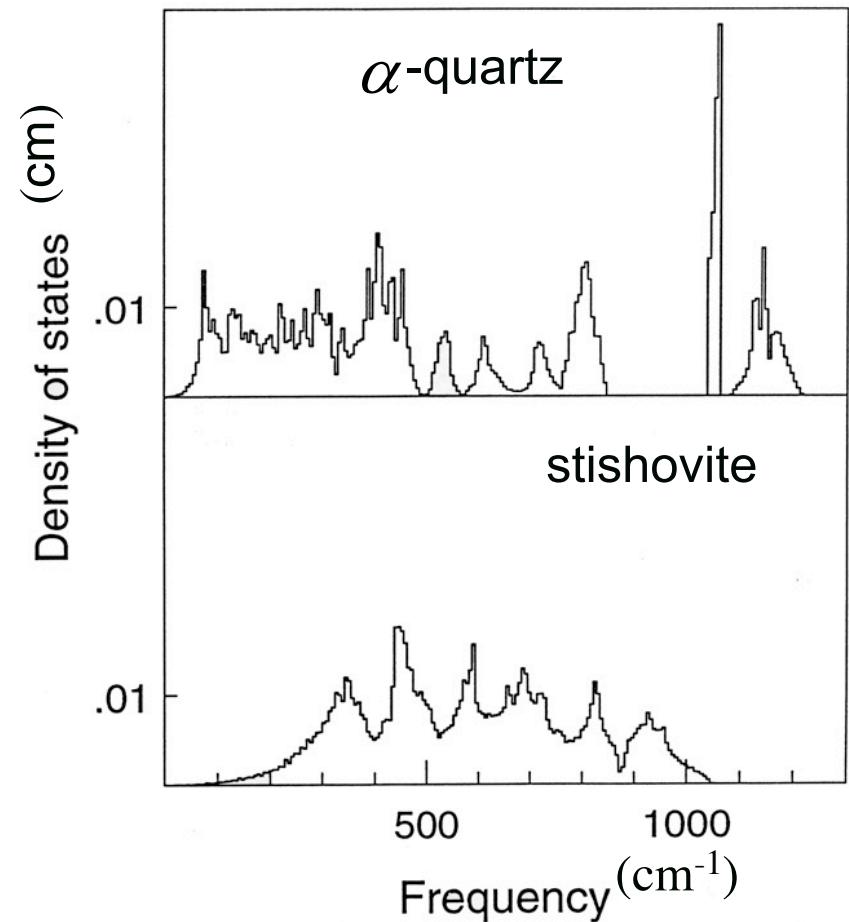
All vibrational contributions to thermodynamic properties,
in the harmonic approximation, can be calculated
in this manner.

Phonon density of states

For each frequency channel,
count the “number” of
phonon modes

$$g_{norm}(\omega) = \frac{1}{3n_{at}N} \sum_{mq} \delta(\omega - \omega_{mq})$$

m = index of pattern of vibration,
 \vec{q} = a crystalline momentum
(\Rightarrow velocity of the vibrational wave)



Helmoltz free energy and specific heat

$$F = U - TS$$

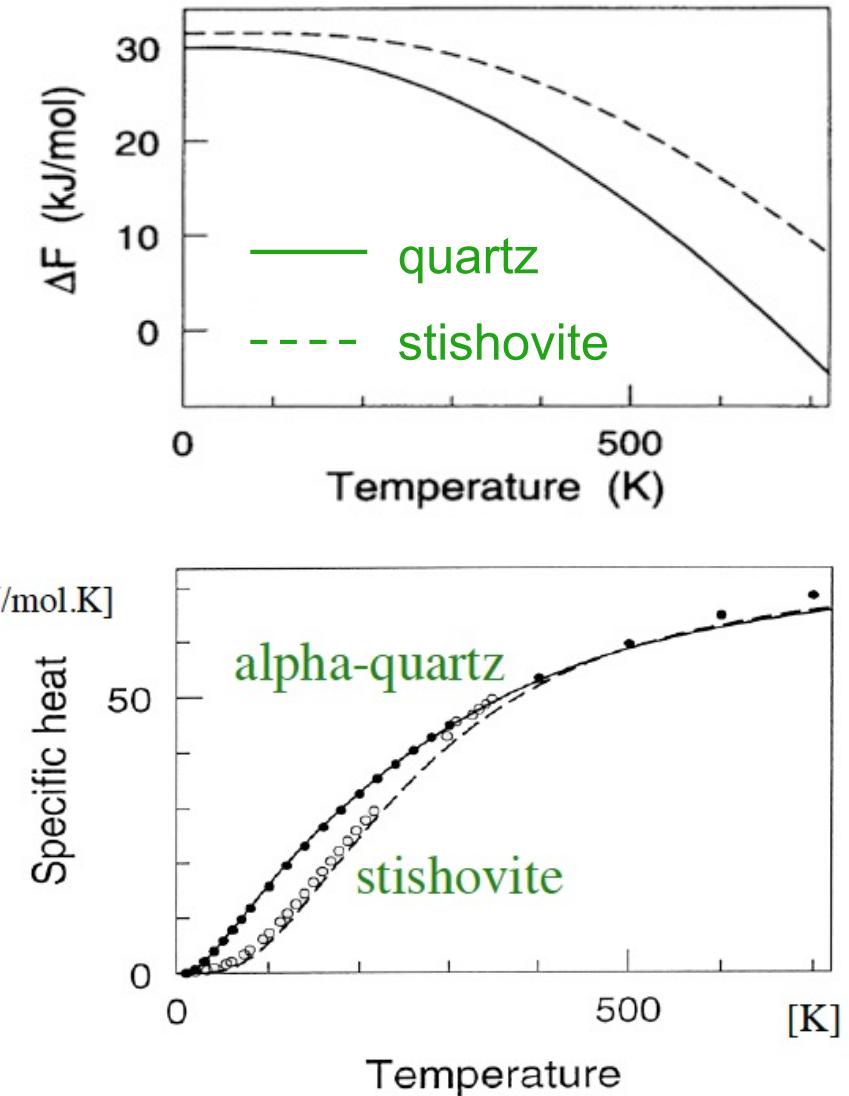
$$C_V = \left(\frac{\partial U}{\partial T} \right)_V = T \left(\frac{\partial S}{\partial T} \right)_V = -T \left(\frac{\partial^2 F}{\partial T^2} \right)_V$$

Vibrational contribution to F :

$$\Delta F = 3n_{at} N k_B T \int_0^{\omega_{\max}} \ln \left\{ 2 \sinh \left(\frac{\omega}{2k_B T} \right) \right\} g(\omega) d\omega$$

Vibrational contribution to C_V :

$$C_V = 3n_{at} N k_B \int_0^{\omega_{\max}} \left(\frac{\omega}{2k_B T} \right)^2 \operatorname{csch}^2 \left(\frac{\omega}{2k_B T} \right) g(\omega) d\omega$$

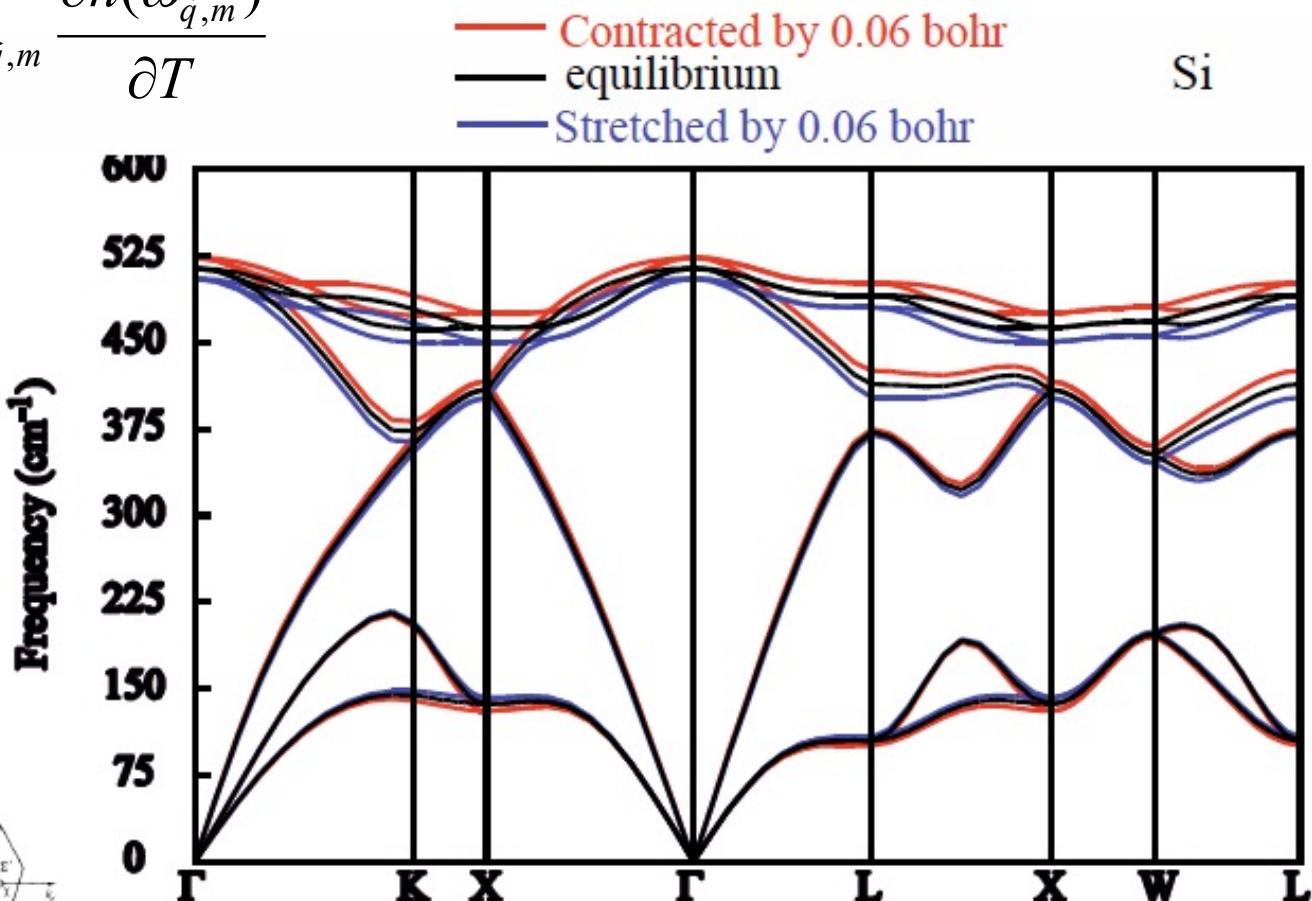
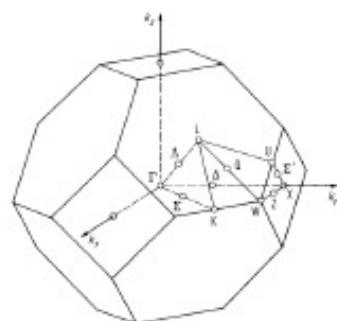


Ab initio thermal expansion

$$\alpha(T) = \frac{V}{3B} \sum_{\vec{q},m} \frac{1}{\hbar\omega_{\vec{q},m}} \gamma_{\vec{q},m} \frac{\partial n(\omega_{\vec{q},m})}{\partial T}$$

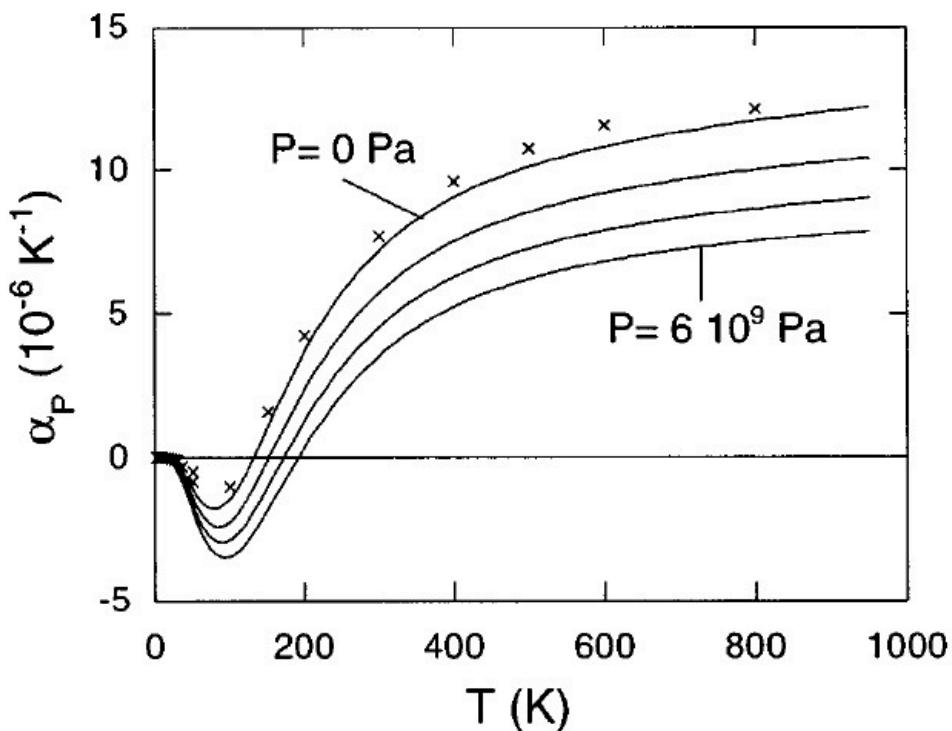
$$\gamma_{m,\vec{q}} = -\frac{\partial(\ln \omega_{m,\vec{q}})}{\partial(\ln V)}$$

Alternative path :
minimisation of
free energy

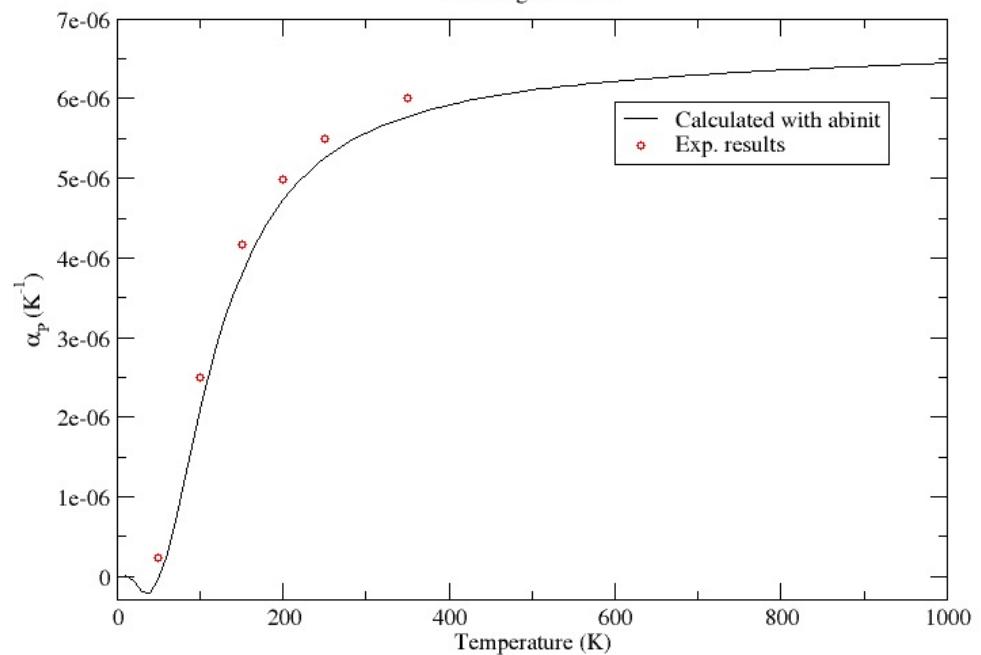


Ab initio thermal expansion

Linear thermal expansion coefficient
of bulk silicon



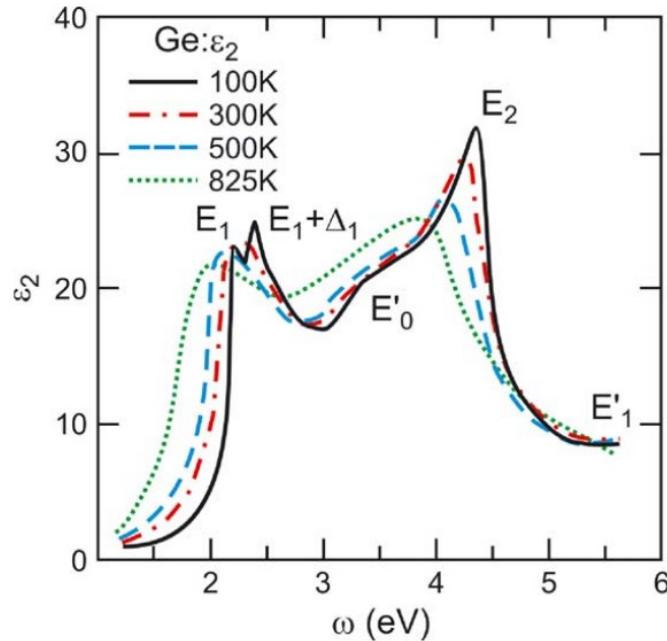
Linear thermal expansion coefficient
of bulk germanium



G.-M. Rignanese, J.-P. Michenaud and XG
Phys. Rev. B **53**, 4488 (1996)

Electron-phonon effects on electronic energies

T-dependence of electronic/optical properties

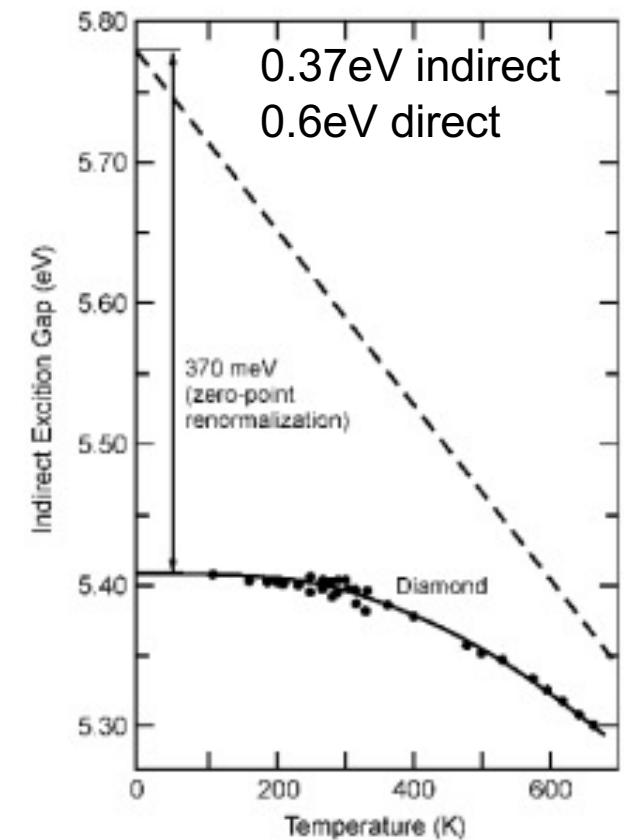


- peaks **shift** in energy
- peaks **broaden** with increasing temperature : decreased electron lifetime

L. Viña, S. Logothetidis and M. Cardona,
Phys. Rev. B **30**, 1979 (1984)

- even at 0K, vibrational effects are important, due to **Zero-Point Motion**

Usually, not included in first-principles
(DFT or beyond) calculations !



M. Cardona, *Solid State Comm.* **133**, 3 (2005)

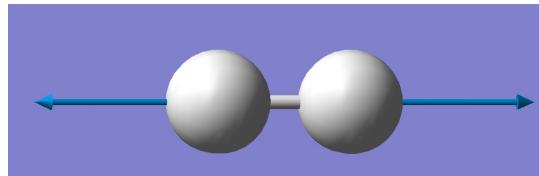
Phonon population effects in solids

Concepts ...

... can be explained with diatomic molecules

Simple :

- discrete levels, simple molecular orbitals
- only one relevant vibration mode.



(6 modes decouple as 3 translations, 2 rotations + the stretch.)

Average eigenenergies in the BO approx.

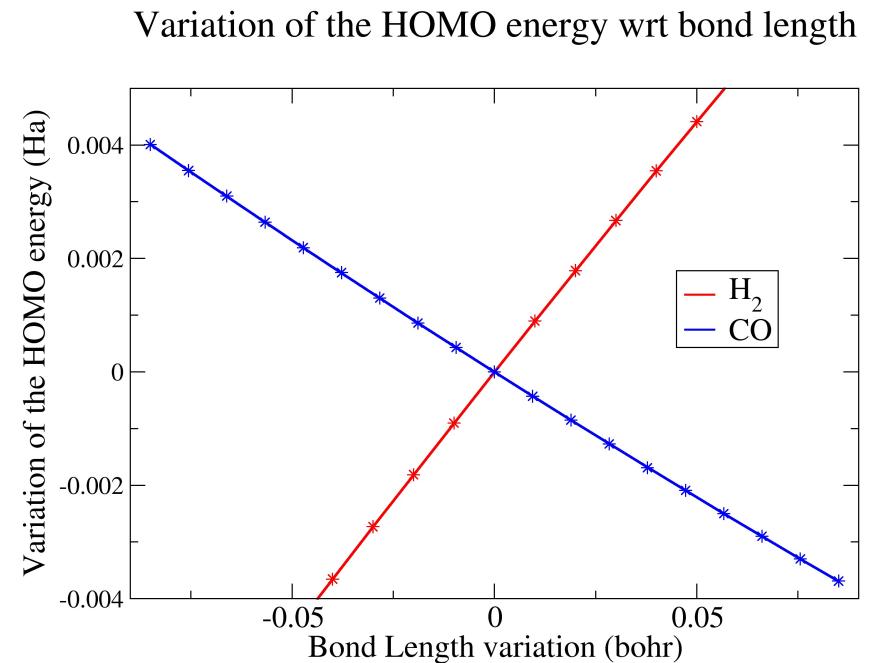
Electronic eigenenergies,
function of the bond length $\varepsilon_n(\Delta R) \Rightarrow$
 \Rightarrow broadening and shift !

- (1) Time-average of eigenenergies
from Molecular Dynamics trajectories,
 $\Delta R(t)$ at average T, with

$$\varepsilon_n(T) = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \varepsilon_n(\Delta R(t)) dt$$

Pros : well-defined procedure ; compatible with current implementations
and computing capabilities ; $\varepsilon_n(\Delta R(t))$ from DFT or GW ;
anharmonicities

Cons : if classical dynamics \Rightarrow no zero-point motion ; adiabatic
(vibrations, but no exchange of energy !) ; hard for solids (supercell)
also supercell mix eigenstates, need unfolding



Average eigenenergies in the BO approx.

Electronic eigenenergies
function of the bond length $\varepsilon_n(\Delta R)$

(2) Thermal average with accurate quantum vibrational states,

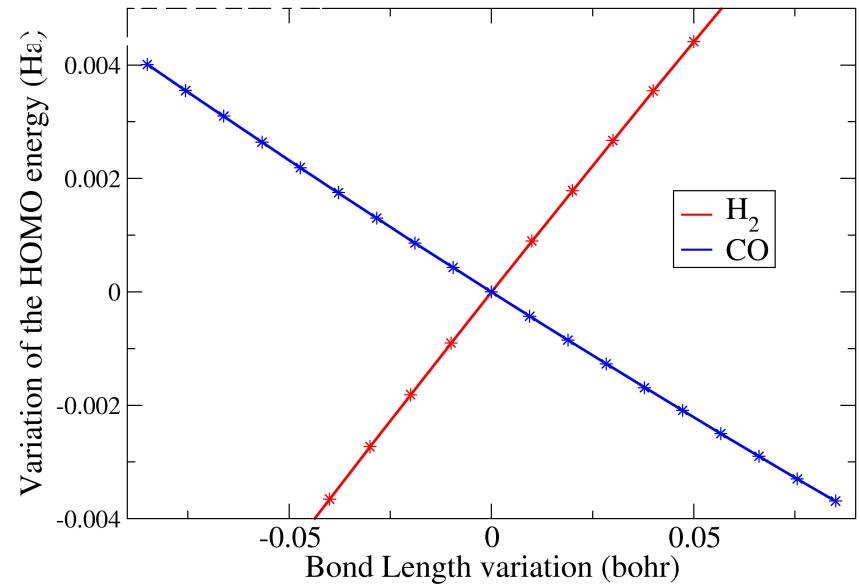
$$\varepsilon_n(T) = \frac{1}{Z} \sum_m e^{-\frac{E_{ph}(m)}{k_B T}} \left(\int \chi_m^*(\Delta R) \varepsilon_n(\Delta R) \chi_m(\Delta R) d\Delta R \right) \quad Z = \sum_m e^{-\frac{E_{ph}(m)}{k_B T}}$$

Pros : zero-point motion ; $\varepsilon_n(\Delta R(t))$ from DFT or GW ;
anharmonicities

Cons : hard to sample more than a few vibrational degrees of freedom ;
adiabatic (vibrations, but no exchange of energy !); hard for solids
(supercell), also supercell mix eigenstates, need unfolding

Alternative: one very large supercell with prepared atomic displacements

Variation of the HOMO energy wrt bond length



Average eigenenergies : BO and harmonic approx.

(3) Thermal average with quantum vibrational states in the **harmonic approximation**, **and** expansion of $\varepsilon_n(\Delta R)$ to second order

$$E_{ph}(m) = \hbar\omega(m + \frac{1}{2})$$

$$n_{vib}(T) = \frac{1}{e^{\frac{-\hbar\omega}{k_B T}} - 1}$$

T-dependent phonon occupation number (Bose-Einstein)

$$\varepsilon_n = \varepsilon_n^0 + \cancel{\frac{\partial \varepsilon_n}{\partial R} \Delta R} + \frac{1}{2} \boxed{\frac{\partial^2 \varepsilon_n}{\partial R^2}} \Delta R^2$$

$$\delta \varepsilon_n(T) = \frac{\partial \varepsilon_n}{\partial n_{vib}} \left(n_{vib}(T) + \frac{1}{2} \right)$$

Pros : zero-point motion ; $\varepsilon_n(\Delta R)$ from DFT or GW ;
tractable ... for molecules ...

Cons : hard for solids (supercells) ; no anharmonicities ;
adiabatic (vibrations, but no exchange of energy !); supercell mix
eigenstates, need unfolding

Allen-Heine-Cardona (AHC) formalism

Allen + Heine, J. Phys. C 9, 2305 (1976).

Allen + Cardona, Phys. Rev. B 24, 7479 (1981); 27, 4760 (1983).

Second-order (time-dependent) perturbation theory

(no average contribution from first order)

* Formulas for solids (phonons have crystalline momentum)

* If adiabatic approximation ... neglect the phonon frequencies with respect to the electronic gap, no transfer of energy :

$$\delta\epsilon_{\vec{k}n}(T, V = \text{const}) = \frac{1}{N_{\vec{q}}} \sum_{\vec{q}j} \frac{\partial\epsilon_{\vec{k}n}}{\partial n_{\vec{q}j}} \left(n_{\vec{q}j}(T) + \frac{1}{2} \right)$$

occupation number
from Bose-Einstein
statistics

$$\frac{\partial\epsilon_{\vec{k}n}}{\partial n_{\vec{q}j}} = \frac{1}{2\omega_{\vec{q}j}} \sum_{\kappa a \kappa' b} \frac{\partial^2 \epsilon_{\vec{k}n}}{\partial R_{\kappa a} \partial R_{\kappa' b}} \frac{\xi_{\kappa a}(\vec{q}j) \xi_{\kappa' b}(-\vec{q}j)}{\sqrt{M_{\kappa} M_{\kappa'}}} e^{iq.(R_{\kappa' b} - R_{\kappa a})}$$

Electron-phonon
coupling energy
(EPCE)

“Phonon mode factor”

$\xi_{\kappa a}(\vec{q}j)$ phonon eigenmodes

κ = atom label

$a=x, y, \text{ or } z$

Eigenvalue changes

$$\left(\frac{\partial^2 \varepsilon_{\vec{k}n}}{\partial R_{\kappa a} \partial R_{\kappa' b}} \right) ?$$

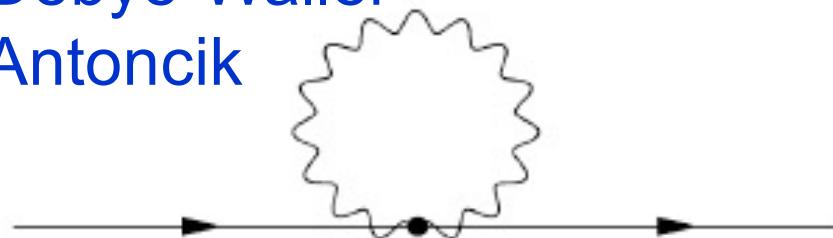
$$\varepsilon_{\vec{k}n} = \langle \phi_{\vec{k}n} | \hat{H}_{\vec{k}} | \phi_{\vec{k}n} \rangle \quad \hat{H} = \hat{T} + \hat{V}_{\text{nucl}} + \int \frac{\rho(r')}{|r-r'|} dr' + \frac{dE_{xc}}{d\rho(r)}$$

Hellman-Feynman theorem : $\varepsilon_{\vec{k}n}^{(1)} = \langle \phi_{\vec{k}n}^{(0)} | \hat{H}_{\vec{k}}^{(1)} | \phi_{\vec{k}n}^{(0)} \rangle$

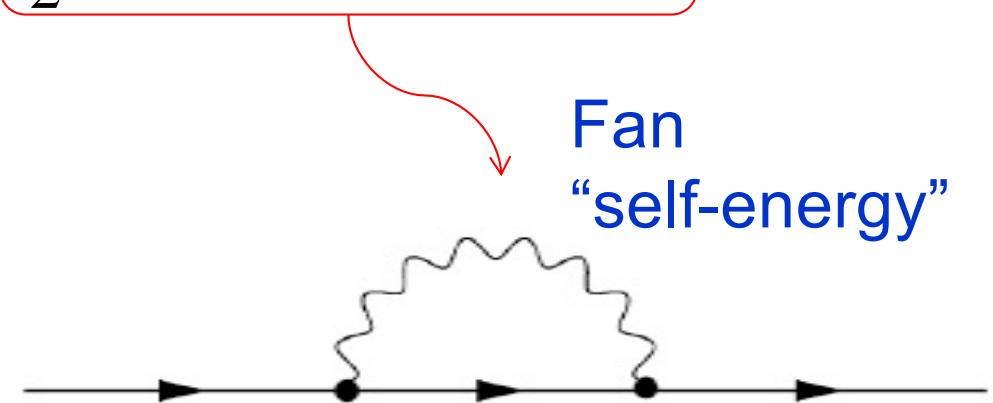
One more derivative :

$$\varepsilon_{\vec{k}n}^{(2)} = \boxed{\langle \phi_{\vec{k}n}^{(0)} | \hat{H}_{\vec{k}}^{(2)} | \phi_{\vec{k}n}^{(0)} \rangle} + \boxed{\frac{1}{2} \left(\langle \phi_{\vec{k}n}^{(0)} | \hat{H}_{\vec{k}+\vec{q}}^{(1)} | \phi_{\vec{k}+\vec{q}n}^{(1)} \rangle + (\text{c.c}) \right)}$$

Debye-Waller
Antoncik



Fan
“self-energy”



Non-adiabatic AHC theory

Beyond adiabatic perturbation theory ... Many-body perturbation theory !

Fan self-energy (also called Migdal self-energy) :

$$\Sigma_{\lambda\lambda'}^{Fan}(\omega) = \sum_{\nu} \frac{1}{2\omega_{\nu}} \sum_{\lambda''} \langle \psi_{\lambda} | H_{\mathcal{V}}^{(1)} | \psi_{\lambda''} \rangle \langle \psi_{\lambda''} | H_{\mathcal{V}}^{(1)*} | \psi_{\lambda'} \rangle$$
$$\left[\frac{n_{\nu}(T) + f_{\lambda''}(T)}{\omega - \varepsilon_{\lambda''}^0 + \omega_{\nu} + i\eta \operatorname{sgn}(\omega)} + \frac{n_{\nu}(T) + 1 - f_{\lambda''}(T)}{\omega - \varepsilon_{\lambda''}^0 - \omega_{\nu} + i\eta \operatorname{sgn}(\omega)} \right]$$

Different levels :

On-the-mass shell approximation $\varepsilon_{\lambda} = \varepsilon_{\lambda}^0 + \Sigma_{\lambda}^{ep}(\varepsilon_{\lambda}^0)$

Quasi-particle approximation $\varepsilon_{\lambda} = \varepsilon_{\lambda}^0 + \Sigma_{\lambda}^{ep}(\varepsilon_{\lambda})$ $\varepsilon_{\lambda} = \varepsilon_{\lambda}^0 + Z_{\lambda} \Sigma_{\lambda}^{ep}(\varepsilon_{\lambda}^0)$

$$Z_{\lambda} = \left(1 - \Re \frac{\partial \Sigma_{\lambda}^{ep}(\omega)}{\partial \omega} \Big|_{\omega=\varepsilon_{\lambda}^0} \right)^{-1}$$

Or even spectral functions

$$A_{\lambda}(\omega) = \frac{1}{\pi} \frac{|\Im \Sigma_{\lambda}^{ep}(\omega)|}{[\omega - \varepsilon_{\lambda}^0 - \Re \Sigma_{\lambda}^{ep}(\omega)]^2 + |\Im \Sigma_{\lambda}^{ep}(\omega)|^2}$$

*S. Poncé, Y. Gillet, J. Laflamme Janssen, A. Marini,
M. Verstraete & XG, J. Chem. Phys. 143, 102813 (2015)*

Reminder : Many-body perturbation theory

Historically older than DFT (from the 40-50' s) !

Big names : Feynman, Schwinger, Hubbard, Hedin, Lundqvist

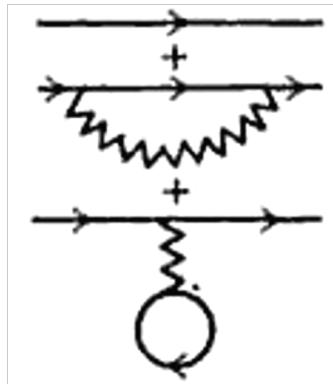
Heavy formalism ! Not amenable to 1-hour presentation ...

Diagrammatic representation of perturbative terms

Based on Green's functions = propagators

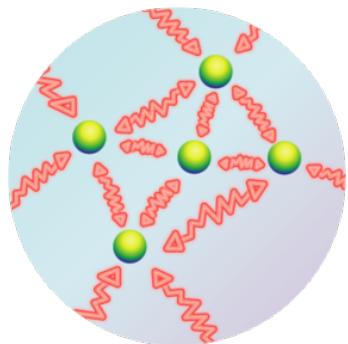
1-particle

$G(\mathbf{r}, t; \mathbf{r}', t')$

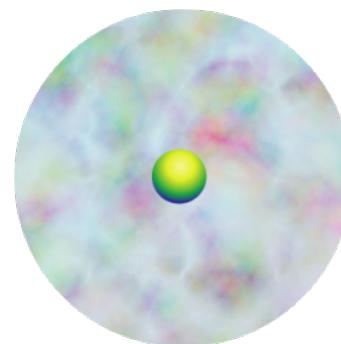


Propagation without Coulomb interactions
+
Propagation with polarization of medium
+
Propagation with mean-field electrostatic
interaction with other electrons

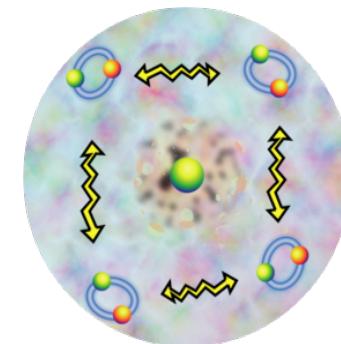
Reminder : GW approximation



Wavefunctions



DFT



GW

$$\left(-\frac{1}{2} \nabla^2 + V_{ext}(\mathbf{r}) + V_H(\mathbf{r}) \right) \psi_i(\mathbf{r}) + V_{xc}(\mathbf{r}) \psi_i(\mathbf{r}) = \varepsilon_i^{KS} \psi_i(\mathbf{r}) \quad \text{DFT}$$

$$\left(-\frac{1}{2} \nabla^2 + V_{ext}(\mathbf{r}) + V_H(\mathbf{r}) \right) \psi_i(\mathbf{r}) + \int \Sigma(\mathbf{r}, \mathbf{r}'; \varepsilon_i^{QP}) \psi_i(\mathbf{r}') d\mathbf{r}' = \varepsilon_i^{QP} \psi_i(\mathbf{r})$$

$$\Sigma(\mathbf{r}, \mathbf{r}'; \omega) = \lim_{\delta \rightarrow 0+} \frac{i}{2\pi} \int d\omega' e^{i\omega' \delta} G(\mathbf{r}, \mathbf{r}'; \omega + \omega') W(\mathbf{r}, \mathbf{r}'; \omega') \quad \text{GW}$$

Self energy

Green's function

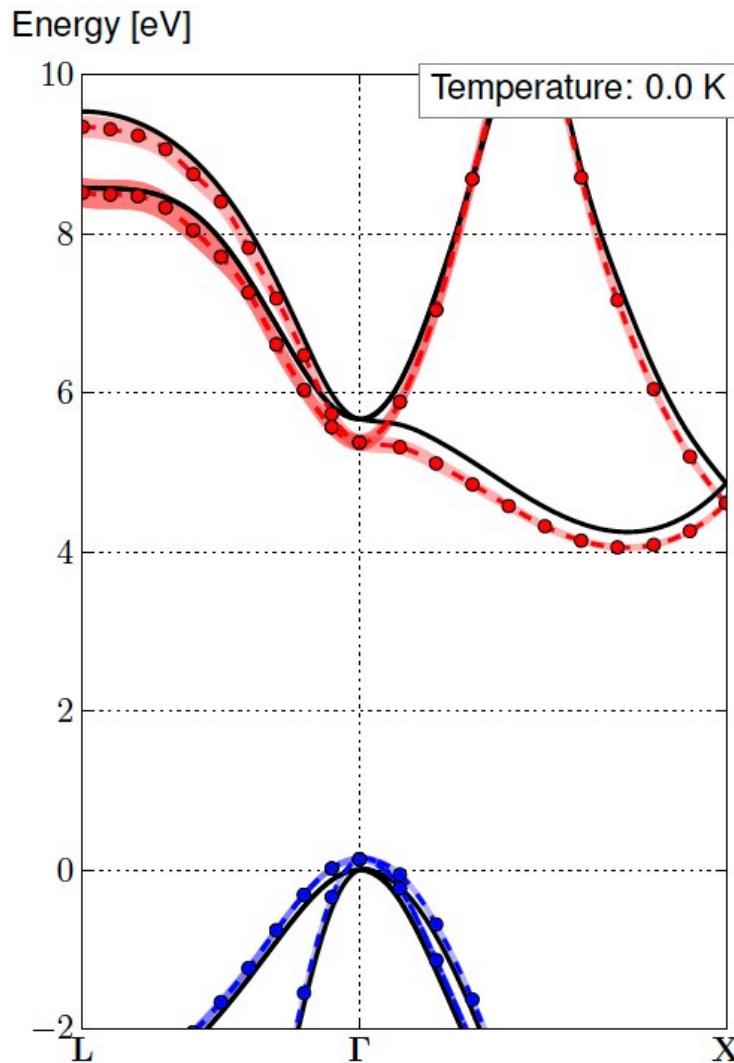
Screened interaction

Reminder : Linearized quasi-particle equation

$$\epsilon_i^{QP} = \epsilon_i^{KS} + Z_i \left\langle \psi_i^{KS} \left| \Sigma(\epsilon_i^{KS}) - V_{xc} \right| \psi_i^{KS} \right\rangle$$

with $Z_i^{-1} = 1 - \left\langle \psi_i^{KS} \left| \frac{\partial \Sigma}{\partial \epsilon} \right|_{\epsilon_i^{KS}} \psi_i^{KS} \right\rangle$

DFT T-dependent band structure

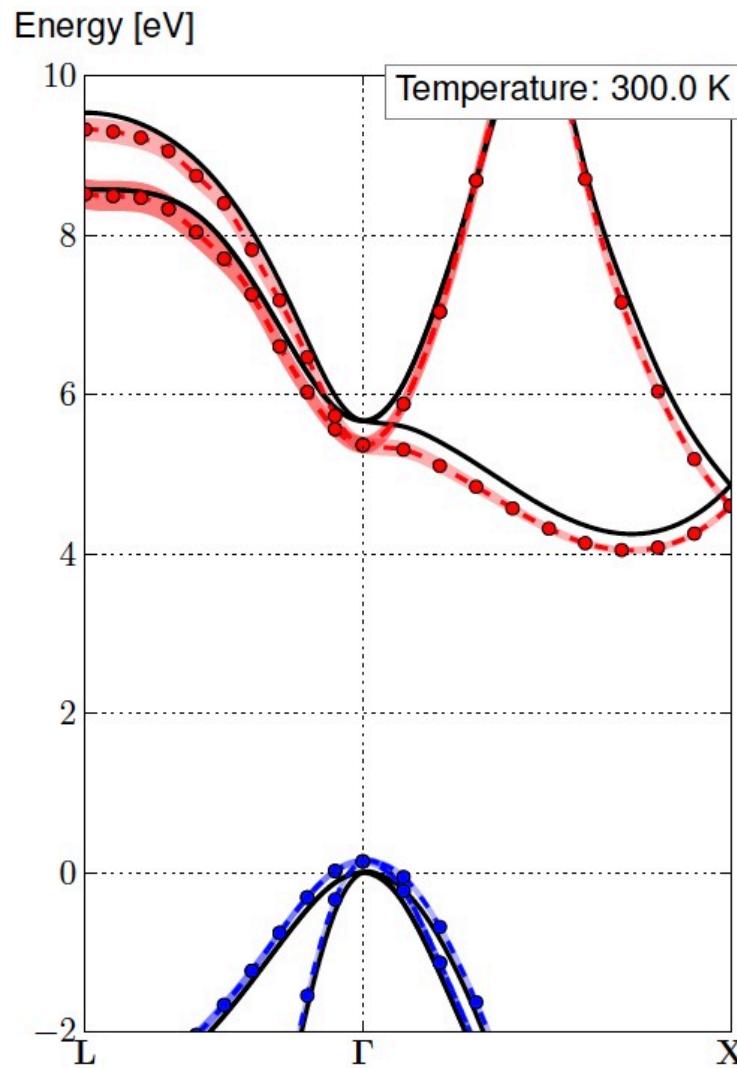


Diamond 0 Kelvin
(incl. Zero-point motion)

Note the widening of
the bands = lifetime

S. Poncé, Y. Gillet, J. Laflamme Janssen, A. Marini, M. Verstraete & XG, J. Chem. Phys. 143, 102813 (2015)

DFT T-dependent band structure

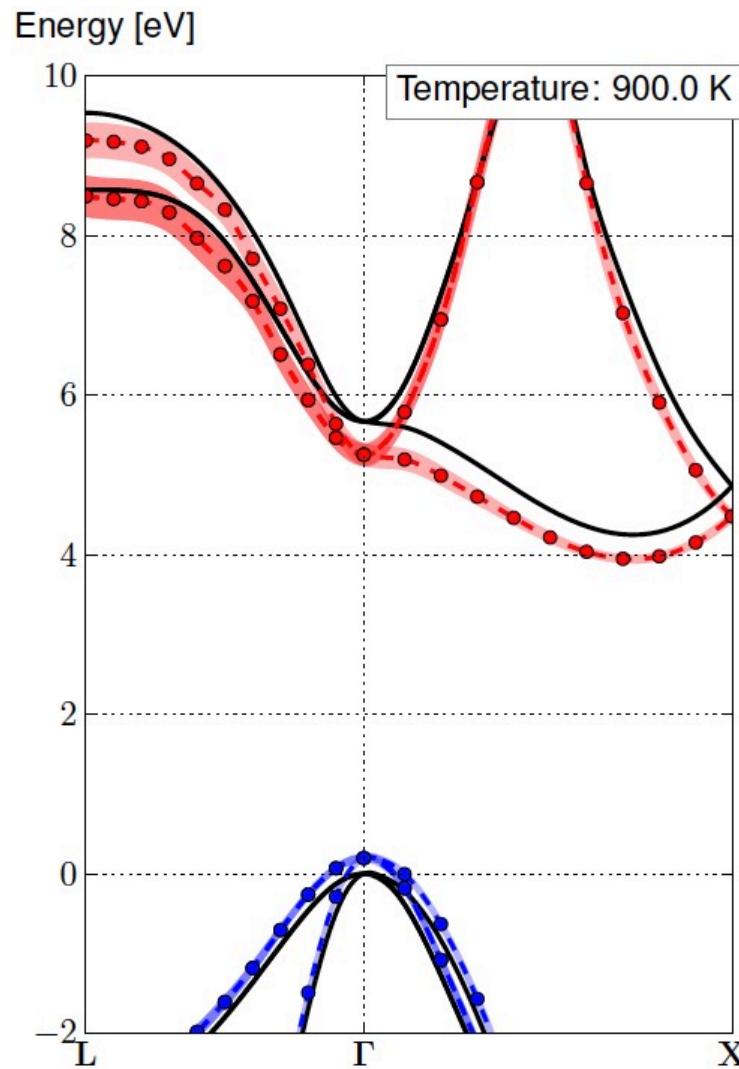


Diamond 300 Kelvin

Note the widening of
the bands = lifetime

S. Poncé, Y. Gillet, J. Laflamme Janssen, A. Marini, M. Verstraete & XG, J. Chem. Phys. 143, 102813 (2015)

DFT T-dependent band structure

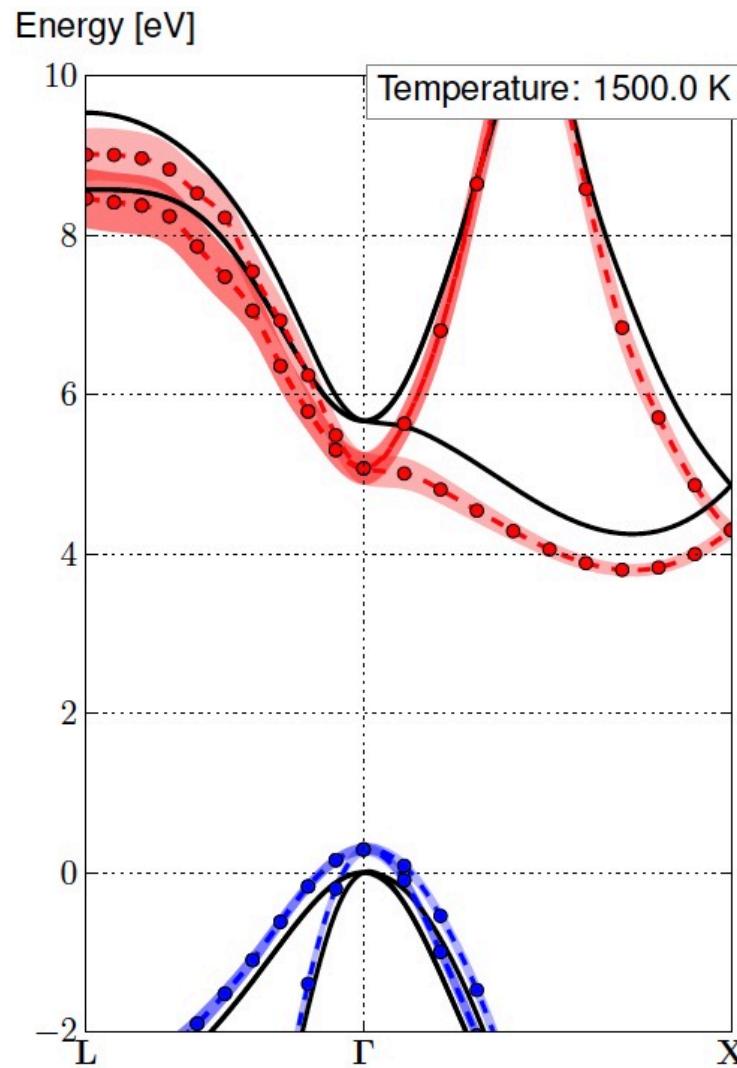


Diamond 900 Kelvin

Note the widening of
the bands = lifetime

S. Poncé, Y. Gillet, J. Laflamme Janssen, A. Marini, M. Verstraete & XG, J. Chem. Phys. 143, 102813 (2015)

DFT T-dependent band structure

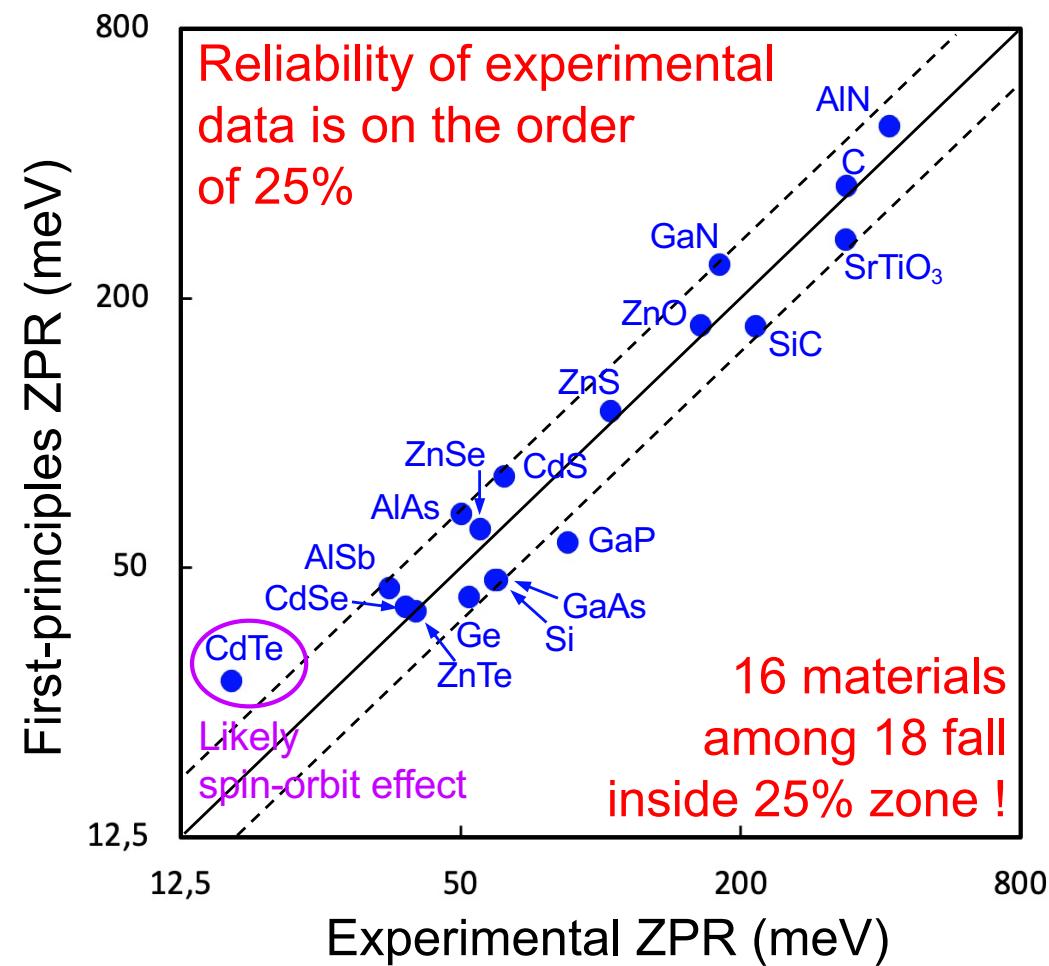


Diamond 1500 Kelvin

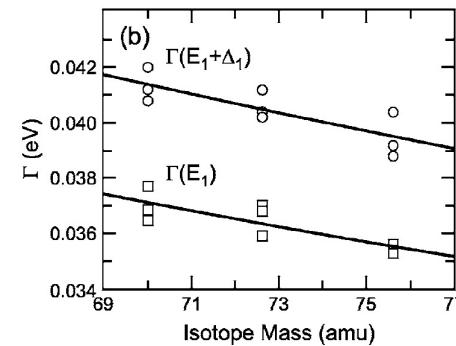
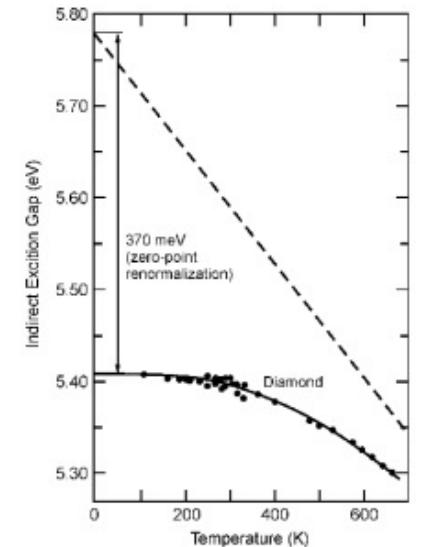
Note the widening of
the bands = lifetime

S. Poncé, Y. Gillet, J. Laflamme Janssen, A. Marini, M. Verstraete & XG, J. Chem. Phys. 143, 102813 (2015)

Non-adiabatic AHC theory vs experiment



ZPR of the band gap
Two different exp techniques

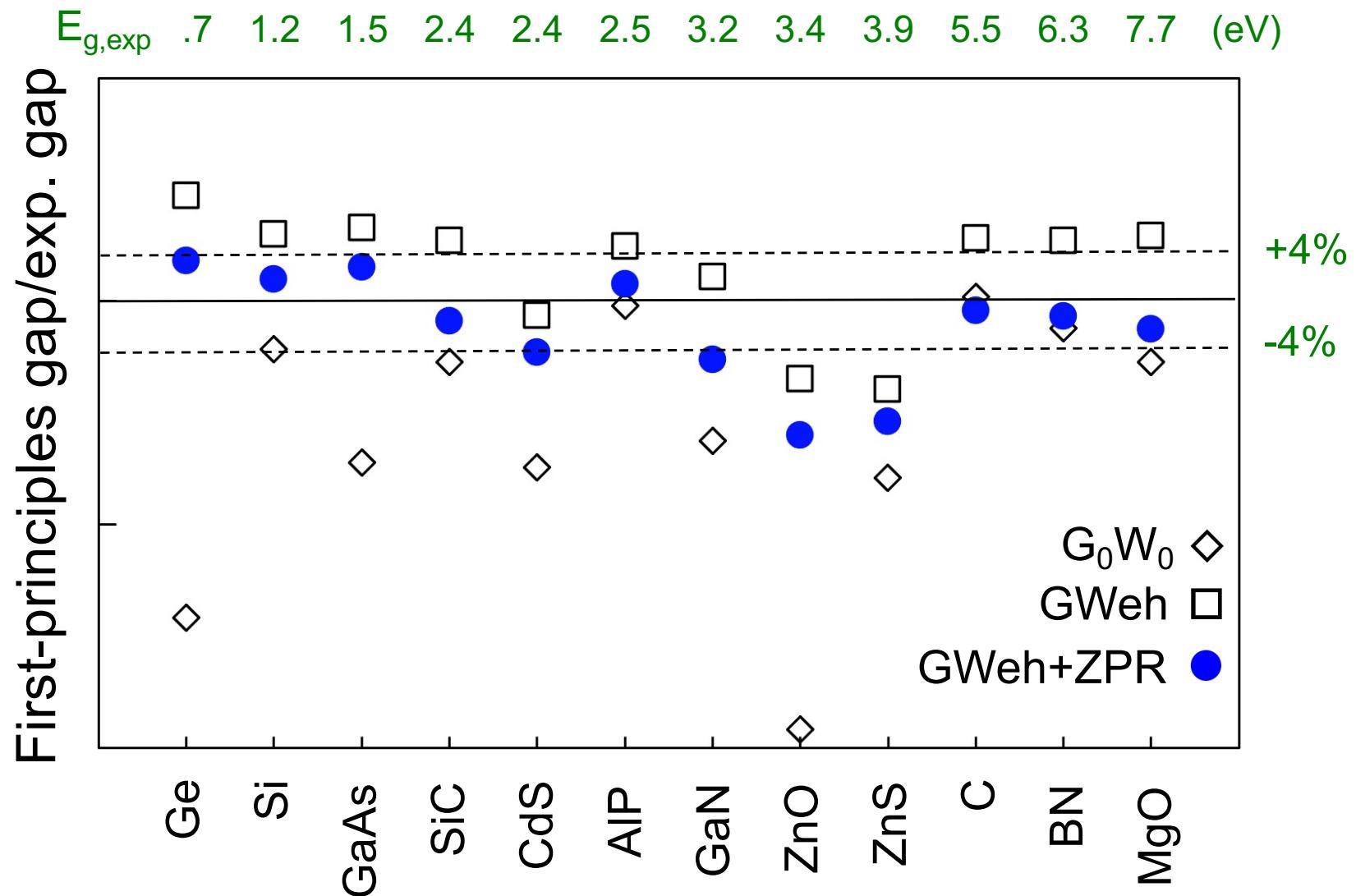


$$E_M = E_\infty + BM^{-1/2}$$

Miglio, Brousseau, Godbout, Côté, Antonius, Chan, Louie,
Giantomassi and XG, *npj Comput Materials* 6, 167, 2020

Cardona & Thewalt, *Rev. Mod. Phys.*, 77, 1173 (2005)

Band gap : theory vs experiment



Miglio, Brousseau, Godbout, Côté, Antonius, Chan, Louie, Giantomassi and XG, *npj Comput Materials* 6, 167, 2020

Electron-phonon effects on transport properties

Transport properties

For a metal, Ohm's law : $\mathbf{j} = \sigma \mathbf{E}$

How to compute the **conductivity tensor** σ ?

Conductivity in doped semiconductors.

Suppose we know the density of carriers (n_e or n_h).

$$\sigma = n_e \mu_e + n_h \mu_h$$

How to compute the **mobility tensors** μ_e and μ_h ?

Seebeck coefficient in thermoelectrics.

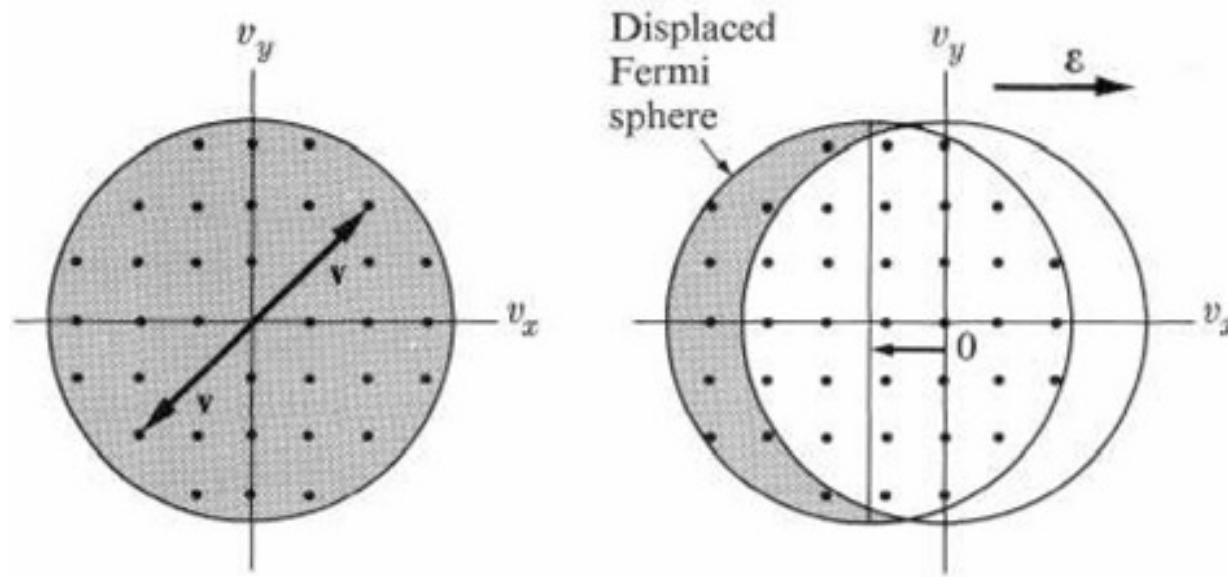
A difference of temperature creates a difference of electric potential.

$$\Delta V = S \cdot \Delta T$$

Superconducting **critical temperature** T_c

Resistivity of a normal metal drops to 0 below the critical temperature

Effect of a steady electric field on a metal



Fermi surface shift due to electric field

Group velocity of an electronic state : $v_{n\mathbf{k},\alpha} = \frac{1}{\hbar} \frac{\partial \epsilon_{n\mathbf{k}}}{\partial k_\alpha} = \langle u_{n\mathbf{k}} | \frac{\hat{p}_\alpha}{m_e} | u_{n\mathbf{k}} \rangle$

Without electric field: cancellation of velocities between opposite momentum electrons => no net current

With electric field: unbalance => net current

Boltzmann transport equation

Ensemble of particles distributed in space and evolving in time.

Distribution function $f(\mathbf{r}, \mathbf{p}, t)$

How this distribution evolve with time ?

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t} \right)_{force} + \left(\frac{\partial f}{\partial t} \right)_{diffusion} + \left(\frac{\partial f}{\partial t} \right)_{collisions}$$

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\partial f}{\partial t} \right)_{collisions}$$

In steady state, and with homogeneous medium

$$\mathbf{F} \frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\partial f}{\partial t} \right)_{collisions}$$

Electronic Boltzmann transport equation

Hypotheses : steady state, homogeneous medium, small electric field, electron-phonon collisions.

$f_{n\mathbf{k}}$ occupation number for state $n\mathbf{k}$

$$\mathbf{F} \frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\partial f}{\partial t} \right)_{collisions}$$

becomes

$$-ev_{n\mathbf{k},\beta} \frac{\partial f_{n\mathbf{k}}^0}{\partial \varepsilon_{n\mathbf{k}}} = \sum_m \int \frac{d\mathbf{q}}{\Omega_{BZ}} \left[\tau_{m\mathbf{k}+\mathbf{q} \rightarrow n\mathbf{k}}^{-1} \frac{\partial f_{m\mathbf{k}+\mathbf{q}}}{\partial E} - \tau_{n\mathbf{k} \rightarrow m\mathbf{k}+\mathbf{q}}^{-1} \frac{\partial f_{n\mathbf{k}}}{\partial E} \right]$$

with the Fermi-Dirac distribution $f_{n\mathbf{k}}^0 = \frac{1}{e^{(\varepsilon_{n\mathbf{k}} - \varepsilon_F)/k_B T} + 1}$

and $\tau_{n\mathbf{k} \rightarrow m\mathbf{k}+\mathbf{q}}^{-1}$ is the **electron-phonon partial decay rate**

Electronic partial decay rate

Electronic partial decay rate due to electron-phonon interaction

$$\tau_{n\mathbf{k} \rightarrow m\mathbf{k+q}}^{-1} = \sum_v \frac{2\pi}{\hbar} |g_{mnj}(\mathbf{k}, \mathbf{q})|^2 \\ \left[(n_{j\mathbf{q}} + f_{m\mathbf{k+q}}^0) \delta(\Delta\epsilon_{\mathbf{k}, \mathbf{k+q}}^{nm} + \hbar\omega_{j\mathbf{q}}) + (n_{j\mathbf{q}} + 1 - f_{m\mathbf{k+q}}^0) \delta(\Delta\epsilon_{\mathbf{k}, \mathbf{k+q}}^{nm} - \hbar\omega_{j\mathbf{q}}) \right]$$

with the electron-phonon matrix element obtained from DFPT

$$g_{mnj}(\mathbf{k}, \mathbf{q}) = \langle u_{m\mathbf{k+q}} | \Delta_{\mathbf{q}\nu} V_{KS} | u_{n\mathbf{k}} \rangle$$

$$\Delta_{j\mathbf{q}} V_{KS} = \frac{1}{\sqrt{2\omega_{j\mathbf{q}}}} \sum_{p\kappa\alpha} \frac{\partial V_{KS}}{\partial R_{\kappa\alpha}} \frac{\xi_{\kappa\alpha, j}(\mathbf{q})}{\sqrt{M_\kappa}} e^{i\mathbf{q}\mathbf{R}_p}$$

This is connected to the imaginary part of the Fan self-energy.

Imaginary part of the Fan self-energy

Fan self-energy (also called Migdal self-energy) :

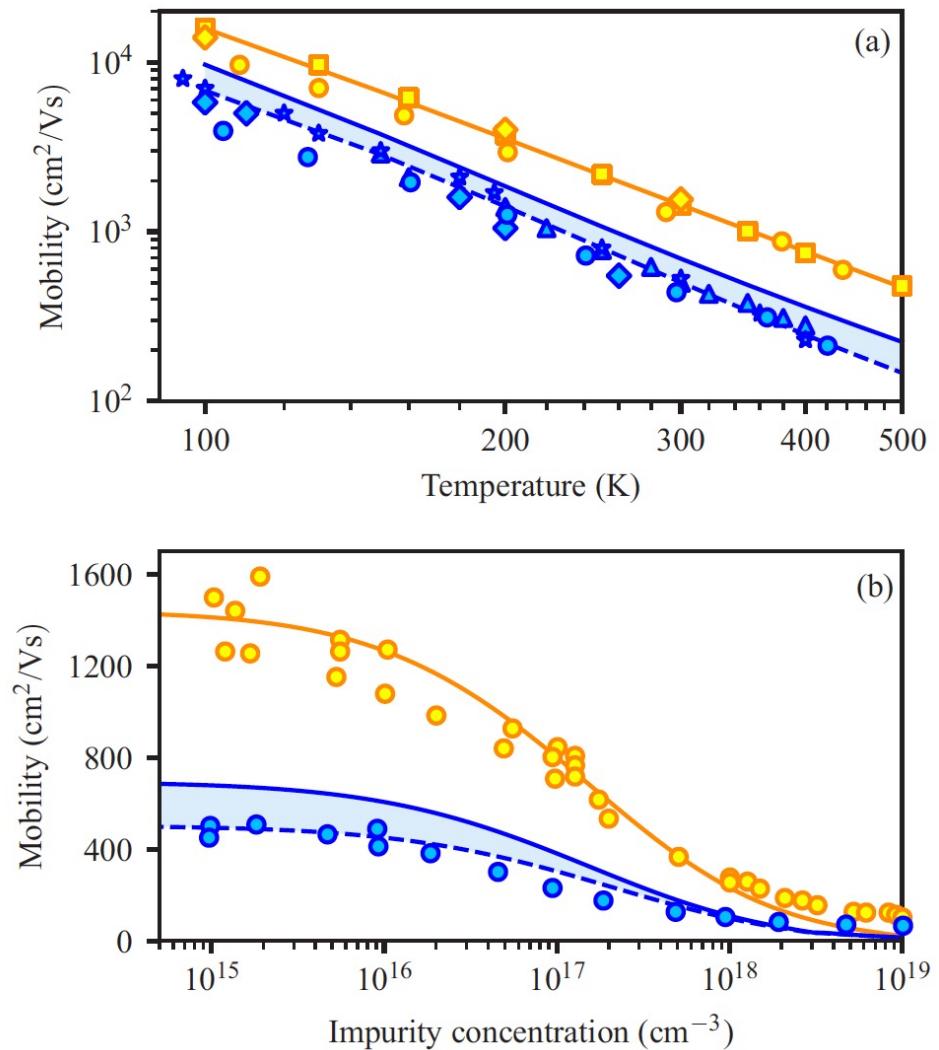
$$\Sigma_{\lambda\lambda'}^{Fan}(\omega) = \sum_{\nu} \frac{1}{2\omega_{\nu}} \sum_{\lambda''} \langle \psi_{\lambda} | H_{\mathcal{V}}^{(1)} | \psi_{\lambda''} \rangle \langle \psi_{\lambda''} | H_{\mathcal{V}}^{(1)*} | \psi_{\lambda'} \rangle$$
$$\left[\frac{n_{\nu}(T) + f_{\lambda''}(T)}{\omega - \varepsilon_{\lambda''}^0 + \omega_{\nu} + i\eta \operatorname{sgn}(\omega)} + \frac{n_{\nu}(T) + 1 - f_{\lambda''}(T)}{\omega - \varepsilon_{\lambda''}^0 - \omega_{\nu} + i\eta \operatorname{sgn}(\omega)} \right]$$

where

$$\lim_{\substack{\delta^+ \rightarrow 0 \\ \delta^+ > 0}} \frac{1}{\omega_{mn} - \omega - i\delta^+} = \frac{1}{\omega_{mn} - \omega} + i\pi\delta(\omega_{mn} - \omega)$$

$$\lim_{\substack{\delta^+ \rightarrow 0 \\ \delta^+ > 0}} \frac{1}{\omega_{mn} + \omega + i\delta^+} = \frac{1}{\omega_{mn} + \omega} - i\pi\delta(\omega_{mn} + \omega)$$

Mobility in Silicon

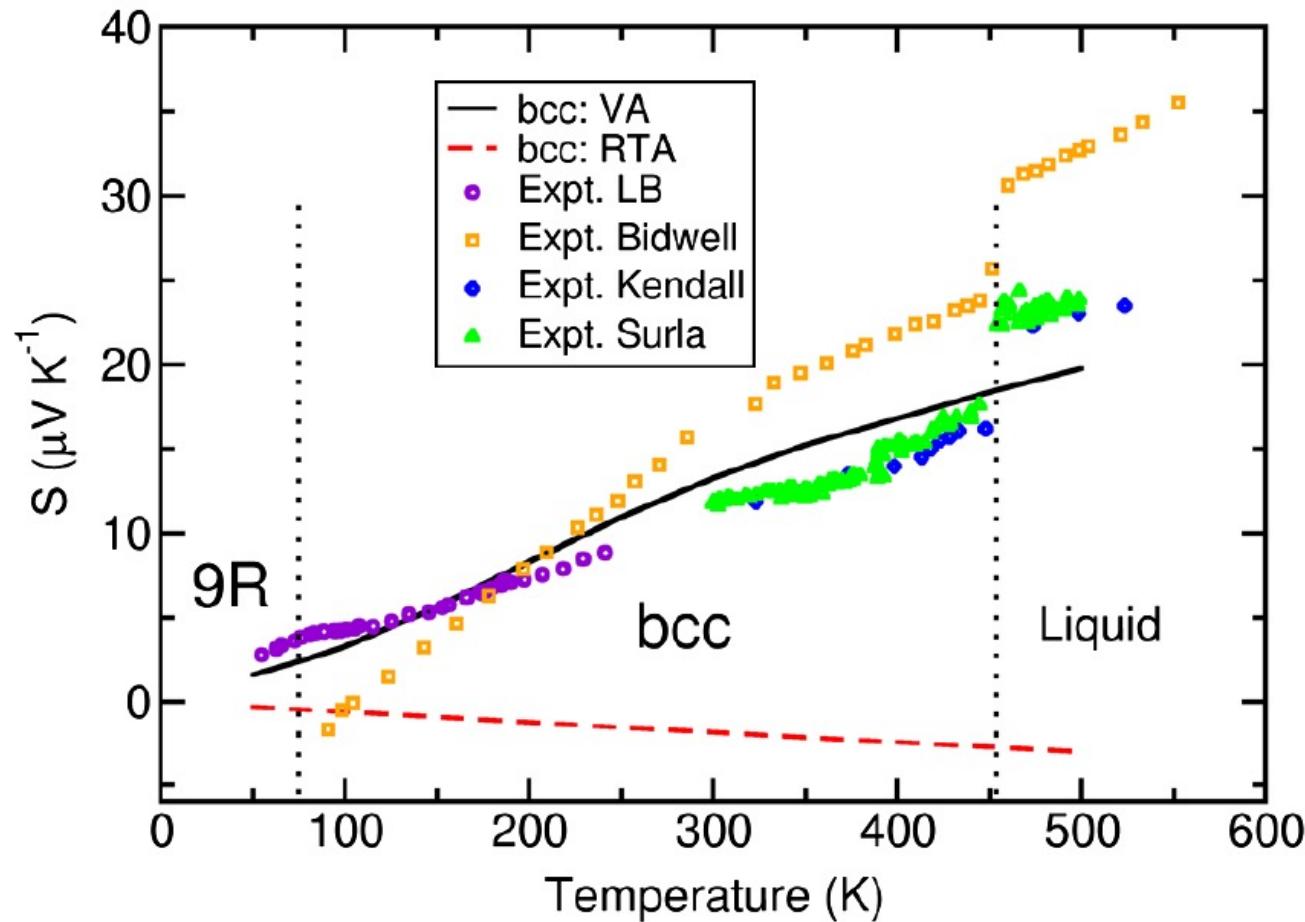


Orange : electrons
Blue : holes

Lines : Theory
Signs : Experiment

S. Poncé, E.R. Margine and F. Giustino, Phys. Rev. B 97, 121201(R) (2018)

Seebeck coefficient of Li



Bin Xu and M. Verstraete, Phys. Rev. Lett. 112, 196693 (2014)

Pressure-dependent superconducting temperature of disilane

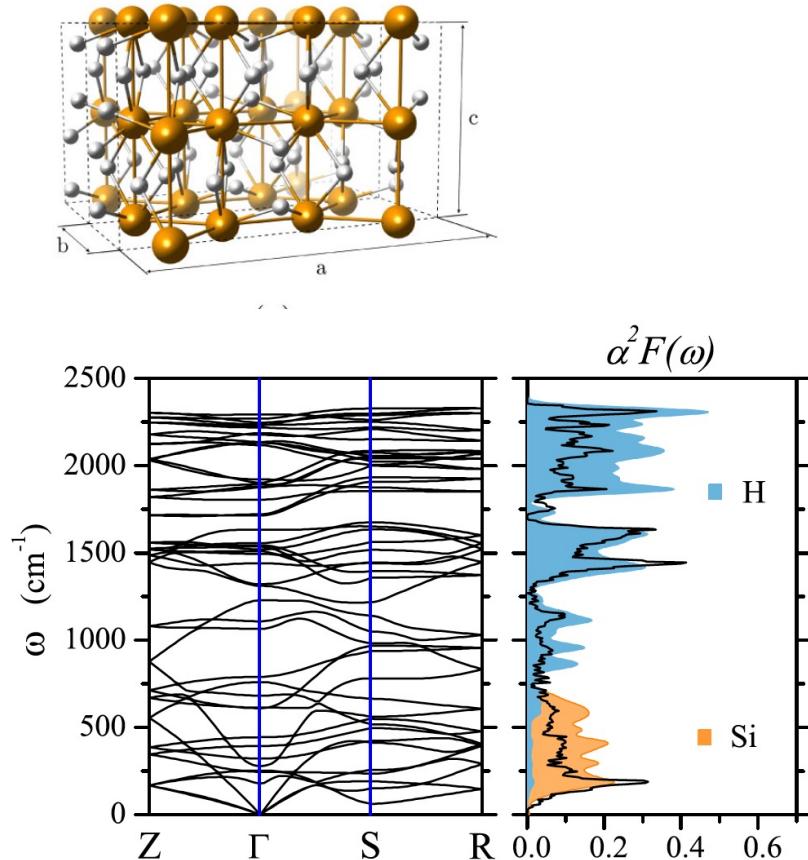


TABLE I. Superconducting properties of the *Cmcm* phase at different pressures. The transition temperatures were calculated by using Allen-Dynes modified McMillan's formula (see Supplemental Materials for details [39]).

Pressure (GPa)	λ	Ω_{\log}	T_c (K)	
			$\mu^* = 0.1$	$\mu^* = 0.13$
100	0.84	478	24.6	20.2
140	0.68	553	17.9	13.5
160	0.66	556	16.7	12.4
200	0.68	501	16.2	12.2
220	0.76	384	16.1	12.7

J. Flores-Livas et al, Phys. Rev. Lett. 108, 117004 (2012)