Computing with Events Part 2

CMSC 326 Simulations

First, we'll define a few terms

Experiment: An action that produces an outcome that we're interested in, which can be repeated under the same conditions

For example, **a coin flip**

First, we'll define a few terms

An **outcome** is an observable result of an experiment

For instance, **Heads**

First, we'll define a few terms

The **sample space** for an experiment is the set of all possible outcomes

For example, **{Heads, Tails}**

We use Ω to represent the sample space

Perhaps the most important term

An **event** is a subset of the sample space

→ <u>Any subset</u> is an event

Consider this problem

What is the probability of obtaining exactly **two heads** in **three flips** of a coin?

What is the sample space?

$$\Omega = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$$

Consider this problem

What is the probability of obtaining exactly **two heads** in **three flips** of a coin?

What is the *event of interest*?

$$E = \{(H,H,T),(H,T,H),(T,H,H)\}$$

Another problem

What is the probability of obtaining an **odd number** in a die roll?

What is the sample space?

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Another problem

What is the probability of obtaining an
odd number in a die roll?

What is the *event of interest*?

$$E = \{1, 3, 5\}$$

Another problem

What is the probability of obtaining an
odd number in a die roll?

The "occurrence" of an event:

Suppose you roll the die and a 3 shows up

Event E = {1, 3, 5} occurred
(because **one of its outcomes** occurred)

Probability Measure

A probability measure on a sample space Ω is:

A collection of numbers, one number for each event

$\Pr[E]$ is the "number" associated with event E

Probability Rules

(Axiom 1) $\Pr[\Omega] = 1$

(Axiom 2) $0 \leq \Pr[E] \leq 1$ for any event E

(Axiom 3) For disjoint events A and B

$$\Pr[\mathrm{A} \cup \mathrm{B}] = \Pr[\mathrm{A}] + \Pr[\mathrm{B}]$$

Probability Rules: Example

Sample space $\Omega = \{\mathrm{H}, \mathrm{T}\}$

Specify a number for each possible event:

$$\Pr[\mathrm{H}] = ?$$

$$\Pr[\mathrm{T}] = ?$$

$$\Pr[\Omega] = 1$$ (always 1, by definition)

$$\Pr[\emptyset] = ?$$ (empty subset)

Probability Rules: Example

Sample space $\Omega = \{\mathrm{H}, \mathrm{T}\}$

Specify a number for each possible event:

$$\Pr[\mathrm{H}] = 0.5$$

$$\Pr[\mathrm{T}] = 0.5$$

$$\Pr[\Omega] = 1$$ (always 1, by definition)

$$\Pr[\emptyset] = 0$$ (must be zero)

Note: we have specified the numbers as a list

- One number for each event.

Does it Satisfy the Axioms?

Axiom 1: $\Pr[\Omega]=1$ (Satisfied)

Axiom 2: $0 \leq \Pr[E] \leq 1$ for any event E (Satisfied)

Axiom 3: For disjoint events $\Pr[A \cup B] = \Pr[A] + \Pr[B]$

- Check for each combination of disjoint events

$$A = \{H\}, B = \{T\}$$

$$\Pr[A \cup B] = \Pr[\{H, T\}] = \Pr[\Omega] = 1 = 0.5 + 0.5 = \Pr[A] + \Pr[B]$$

Operations on Events

Set operators apply to events:

$$\mathrm{A}' = \Omega - \mathrm{A} = \text{complement of } \mathrm{A}$$

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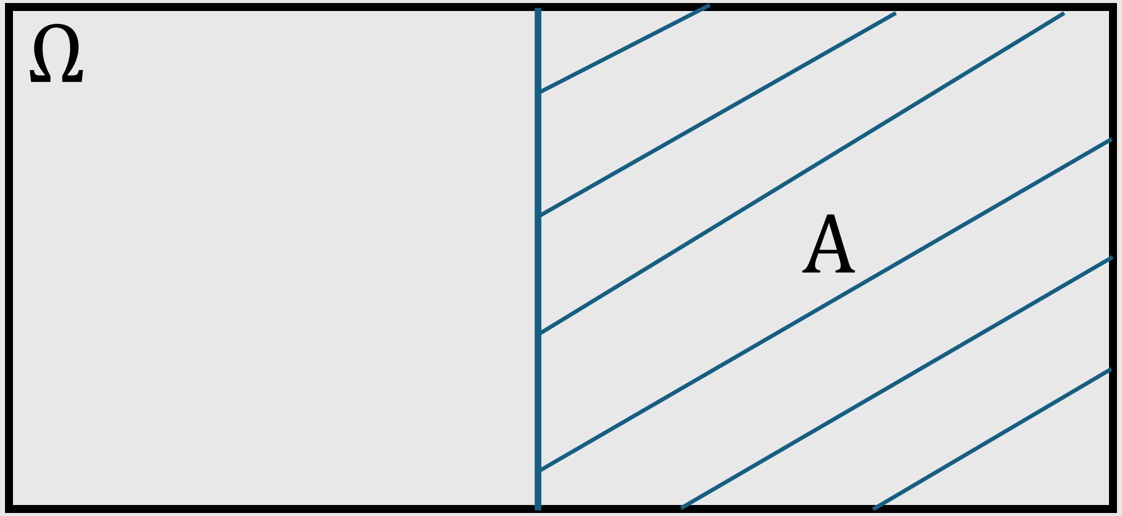

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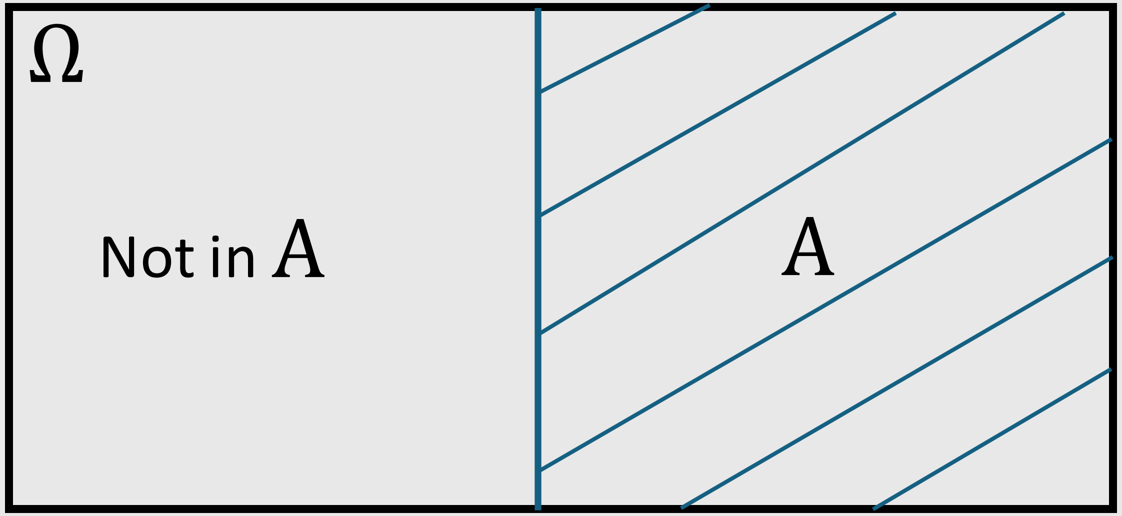

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Set operators apply to events:

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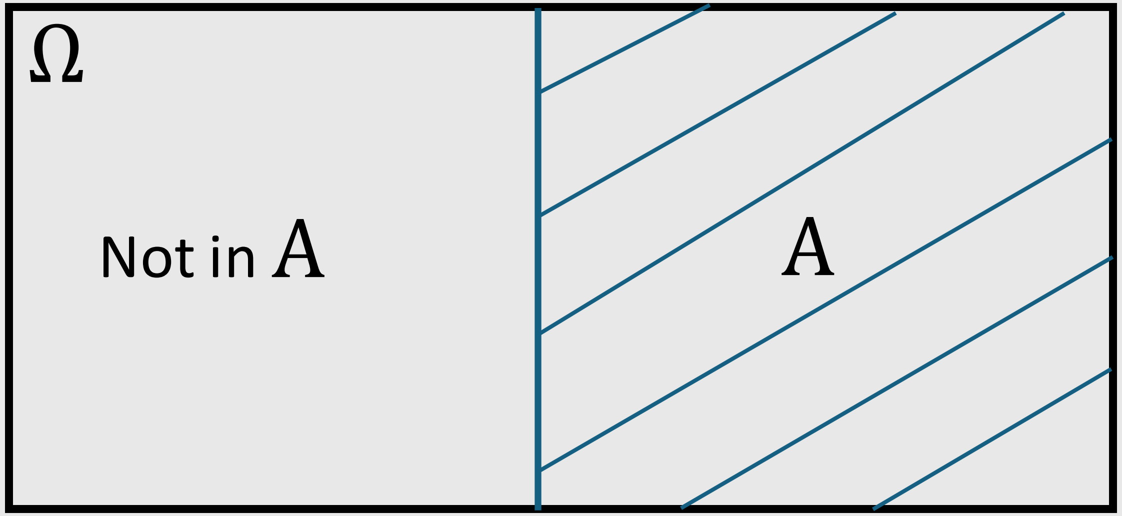
$$\Pr[Not\ in\ A] = 1 - \ \Pr[A]$$



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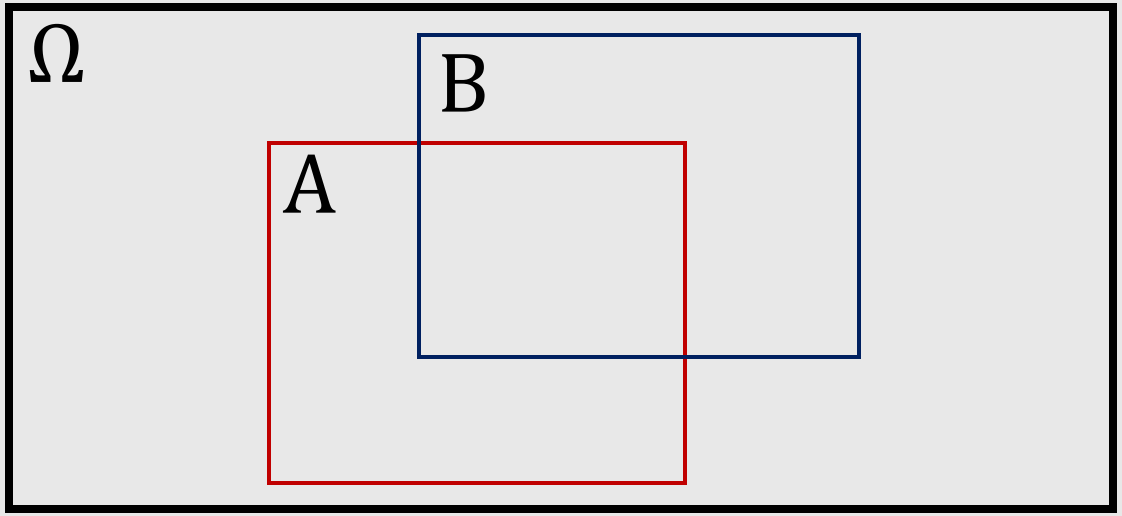

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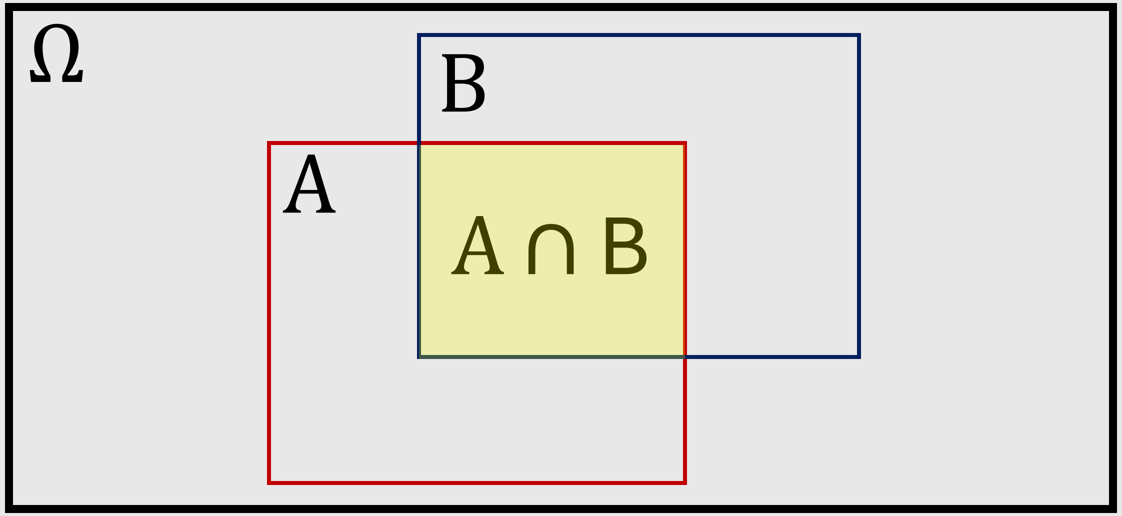


Specifying Probability Measures

For a die-roll, specify as follows:

$$\Pr[E] = \frac{|E|}{6} = \frac{1}{6} \text{ for any singleton event } \mathrm{E}$$

For the die-roll, the singleton events are

$$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$$

We can build the rest of the probability measure

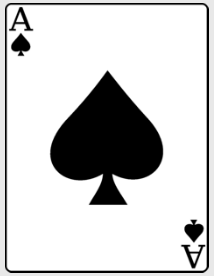
Specifying Probability Measures

For example, $\Pr[rolling\ an\ odd\ number]$:

Using Axiom 3

$$\Pr[\{1, 3, 5\}] = \Pr[\{1\}] + \Pr[\{3\}] + \Pr[\{5\}] = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$$

Specifying Probability Measures

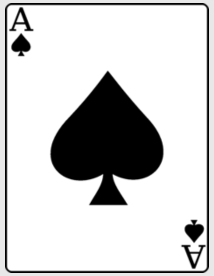


Similarly, for a single card drawing:

$$\Pr[E] = \frac{|E|}{52} = \frac{1}{52}$$ for any singleton event E

We can build the rest of the probability measure

Specifying Probability Measures



For a larger subset like $E = \{0, 1, \ \dots \ , \ \ 12\}$ (spades):

$$\Pr[E] = \Pr[\{0,1, \dots, 12\}]$$

$$= \Pr[\{1\}] + \Pr[\{2\}] + \cdots + \Pr[\{12\}]$$

$$= \frac{1}{52} + \frac{1}{52} + \cdots + \frac{1}{52} = \frac{13}{52}$$

Consider Two Die Rolls

Sample Space:

$$\begin{aligned}\Omega = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\ &(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\ &(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\ &(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\ &(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\ &(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}\end{aligned}$$

Thus, there are $2^{|\Omega|} = 2^{36}$ possible events (subsets of the sample space set)

Consider Two Die Rolls

The convenient way to specify:

$Pr[E] = \frac{|E|}{36}$, for each single event E

For example: $\Pr[\{4,3\}] = \frac{1}{36}$

Then, one can *calculate* other probabilities