Computing with Events Part 3

CMSC 326 Simulations

Conditional Probability

How do we deal with questions of this sort:

Given that the outcome of a die-roll is odd, what is the probability that the outcome is 3?

Conditional Probability

Define the events:

$$
\begin{aligned}
A &= \{3\} \\
B &= \{1, 3, 5\}
\end{aligned}
$$

What is the probability that
A occurred *given* that B occurred?

```python
number_of_odd_rolls = 0
number_of_odd_AND_three_rolls = 0

for _ in range(number_of_trials):
    outcome_of_roll = die.roll()

    # Count if the roll was odd
    if outcome_of_roll in [1, 3, 5]:
        number_of_odd_rolls += 1

    # Count if the roll was odd and the roll was a three
    if outcome_of_roll in [1, 3, 5] and outcome_of_roll == 3:
        number_of_odd_AND_three_rolls += 1

# Estimate the probability
probability = number_of_odd_AND_three_rolls / number_of_odd_rolls
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```
Pr[3 given odd] = 0.33296, theory = 0.3333333333333333
```

Conditional Probability

Consider n repetitions of the experiment and let:

$$f_B(n) = \#\ \text{occurances of}\ B$$
$$f_{AB}(n) \ = \#\ \text{occurences of}\ B\ \text{when}\ A\ \text{also occurs}$$

Then, for large n, the desired conditional probability is approximately

$$\frac{f_{AB}(n)}{f_B(n)}$$

Conditional Probability

$$\frac{f_{AB}(n)}{f_B(n)}$$

Divide both top and bottom by n

$$\frac{\frac{f_{AB}(n)}{n}}{\frac{f_B(n)}{n}}$$

Conditional Probability

$$\frac{f_{AB}(n)}{n} \to \Pr[A \text{ and } B]$$

$$\frac{f_B(n)}{n} \to \Pr[B]$$

Conditional Probability

$$\frac{\frac{f_{AB}(n)}{n}}{\frac{f_B(n)}{n}} = \frac{\Pr[A \text{ and } B]}{\Pr[B]}$$

Conditional Probability

$$\frac{\frac{f_{AB}(n)}{n}}{\frac{f_B(n)}{n}} = \frac{\Pr[A \text{ and } B]}{\Pr[B]}$$

Thus,

$$\Pr[\text{observe } A \text{ given } B] = \frac{\Pr[A \text{ and } B]}{\Pr[B]}$$

Conditional Probability

We write this as,

$$\Pr[A \mid B] = \frac{\Pr[A \text{ and } B]}{\Pr[B]}$$

Conditional Probability

We write this as,

$$\Pr[A \mid B] = \frac{\Pr[A \text{ and } B]}{\Pr[B]}$$

$$\Pr[A \mid B] = \frac{\Pr[A \ \cap B]}{\Pr[B]}$$

Important Variation

$$\Pr[A \mid B] = \frac{\Pr[A \text{ and } B]}{\Pr[B]}$$

By re-arranging the definition:

$$\Pr[A \mid B]\ \Pr[B] = \Pr[A\ \cap B]$$

Important Variation

$$\Pr[A \mid B] \; \Pr[B] = \Pr[A \cap B]$$

By symmetry,

$$\Pr[B \mid A] \; \Pr[A] = \Pr[A \cap B]$$

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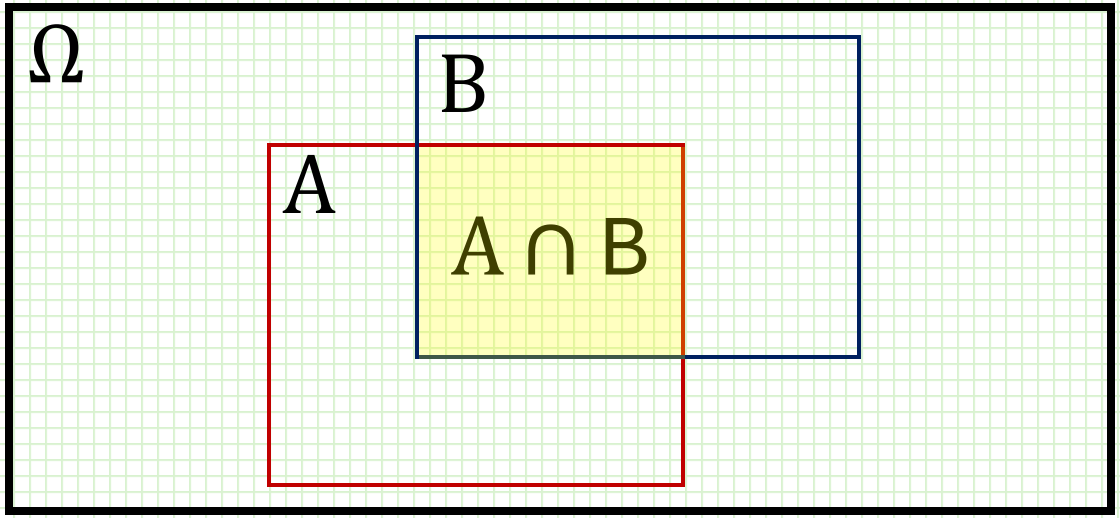


Important Variation

Thus, for any two events A, B:

$$\Pr[A \mid B]\ \Pr[B] = \Pr[B \mid A]\ \Pr[A]$$

Important Variation

$$\Pr[A \mid B] \ \Pr[B] = \Pr[B \mid A] \ \Pr[A]$$

Notice we can re-arrange this as:

$$\Pr[A \mid B] = \frac{\Pr[B \mid A] \ \Pr[A]}{\Pr[B]}$$

Bayes' Rule

$$\Pr[A \mid B] = \frac{\Pr[B \mid A] \; \Pr[A]}{\Pr[B]}$$

Bayes' Rule

What is the probability of the first card is a club *given* that the second card is a club

Define these events:

$$C_1 = \text{first card is a club}$$
$$C_2 = \text{second card is a club}$$

Bayes' Rule

What is the probability of the first card is a club *given* that the second card is a club

Define these events:

$$C_1 = \text{first card is a club}$$
$$C_2 = \text{second card is a club}$$

Then,

$$\Pr[C_1 \mid C_2] = \frac{\Pr[C_2 \mid C_1]\ \Pr[C_1]}{\Pr[C_2]}$$

Bayes' Rule

$$\Pr[C_1 \mid C_2] = \frac{\Pr[C_2 \mid C_1]\ \Pr[C_1]}{\Pr[C_2]}$$

Observe that,

$$\Pr[C_1] = \frac{13}{52}$$

$$\Pr[C_2] = \frac{13}{52}$$

$$\Pr[C_2 \mid C_1] = \frac{12}{51}$$

Bayes' Rule

$$\Pr[C_1 \mid C_2] = \frac{\Pr[C_2 \mid C_1]\ \Pr[C_1]}{\Pr[C_2]}$$

Substituting, we get

$$\Pr[C_1 \mid C_2] = \frac{\Pr[C_2 \mid C_1]\ \Pr[C_1]}{\Pr[C_2]} = \frac{\frac{12}{51} \times \frac{13}{52}}{\frac{13}{52}} = \frac{12}{51}$$

A Variation on Bayes' Rule

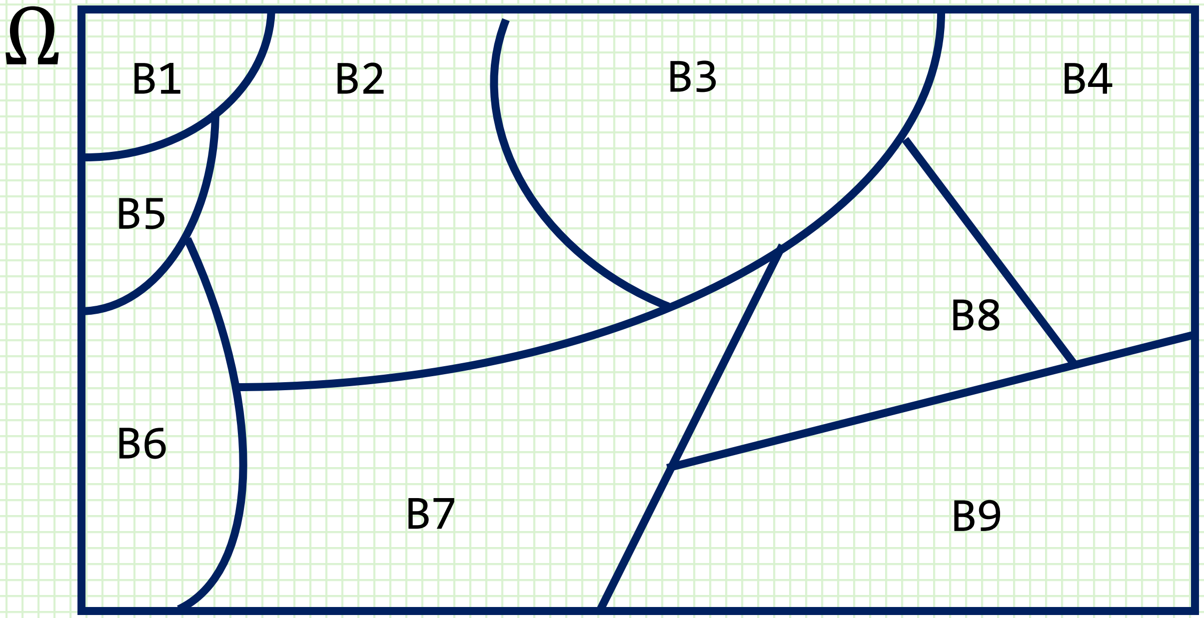
Recall that

$$\Pr[A \ \cap B] = \Pr[A \mid B] \ \Pr[B]$$

Suppose the sample space can be partitioned into disjoint sets $B_1, B_2, \dots, B_n$

Law of Total Probability

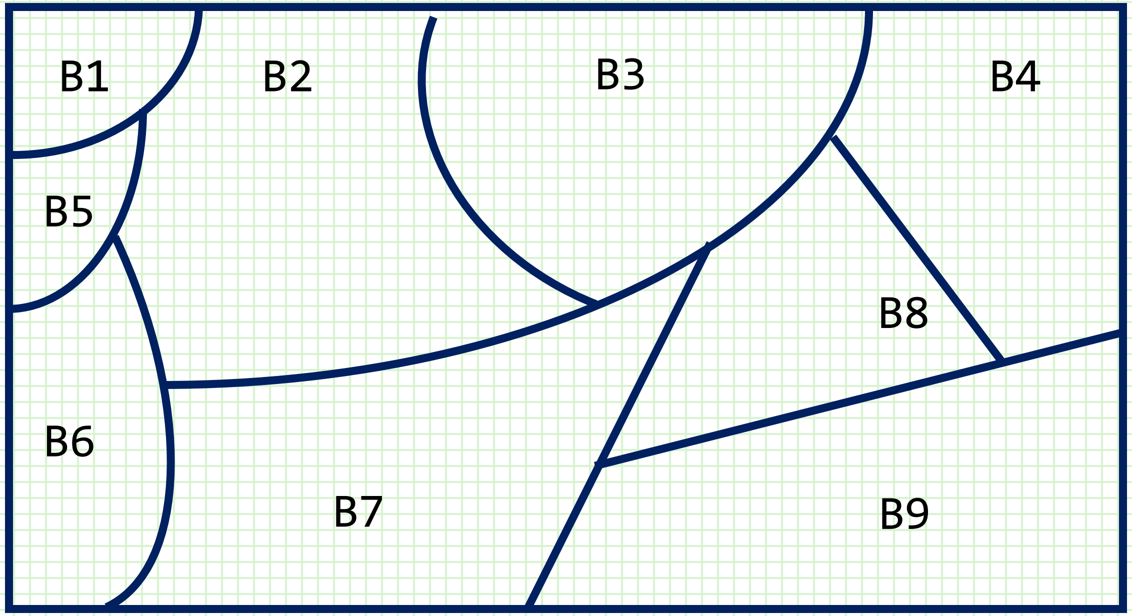
Suppose the sample space can be partitioned into disjoint sets $B_1, B_2, \dots, B_n$



Law of Total Probability

Suppose the sample space can be partitioned into disjoint sets $B_1, B_2, \ldots, B_n$

Ω

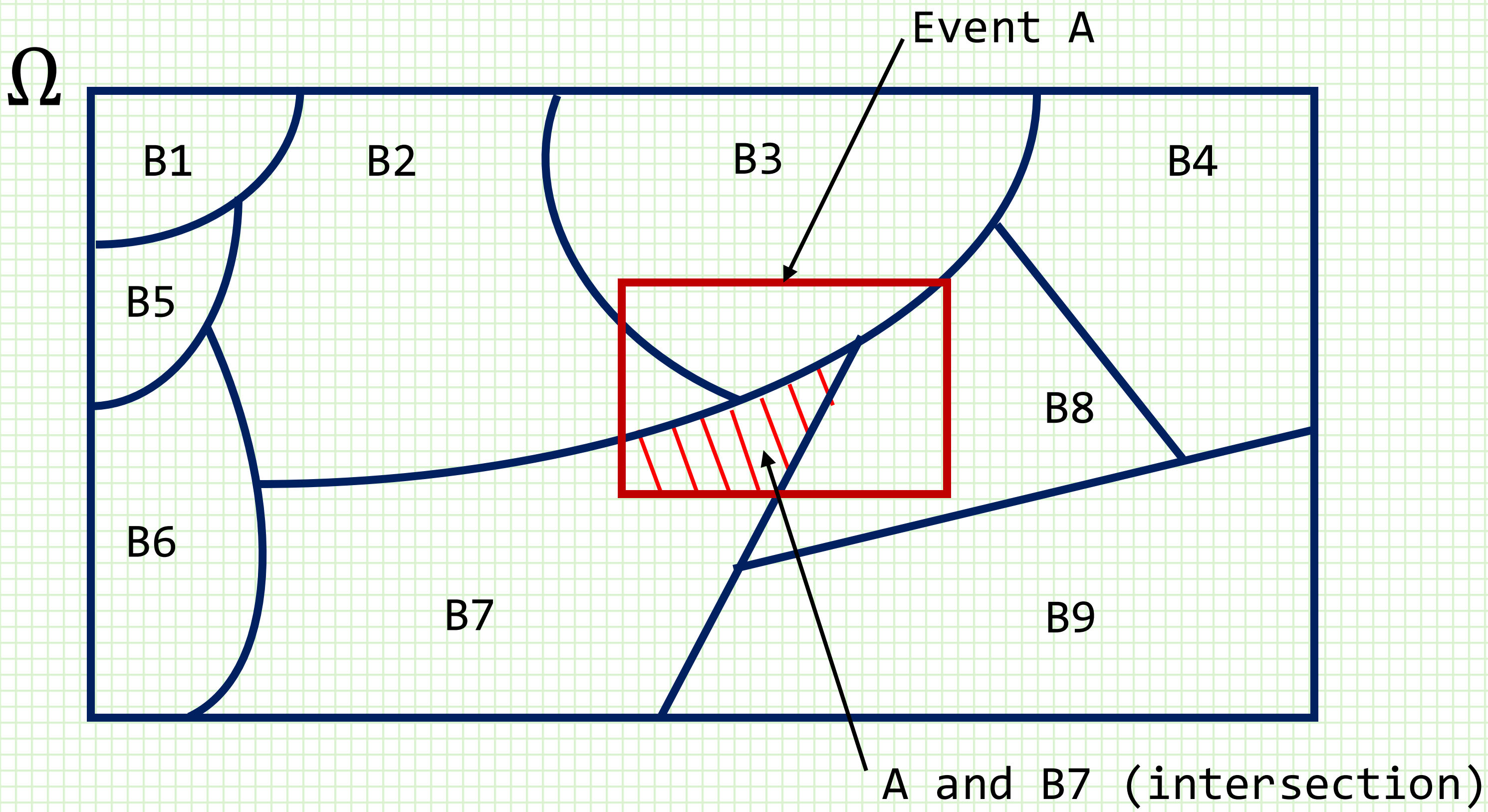


The B_i's are disjoint
$\Omega =$ union of B_i's

Law of Total Probability

Then for any event A:

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$$



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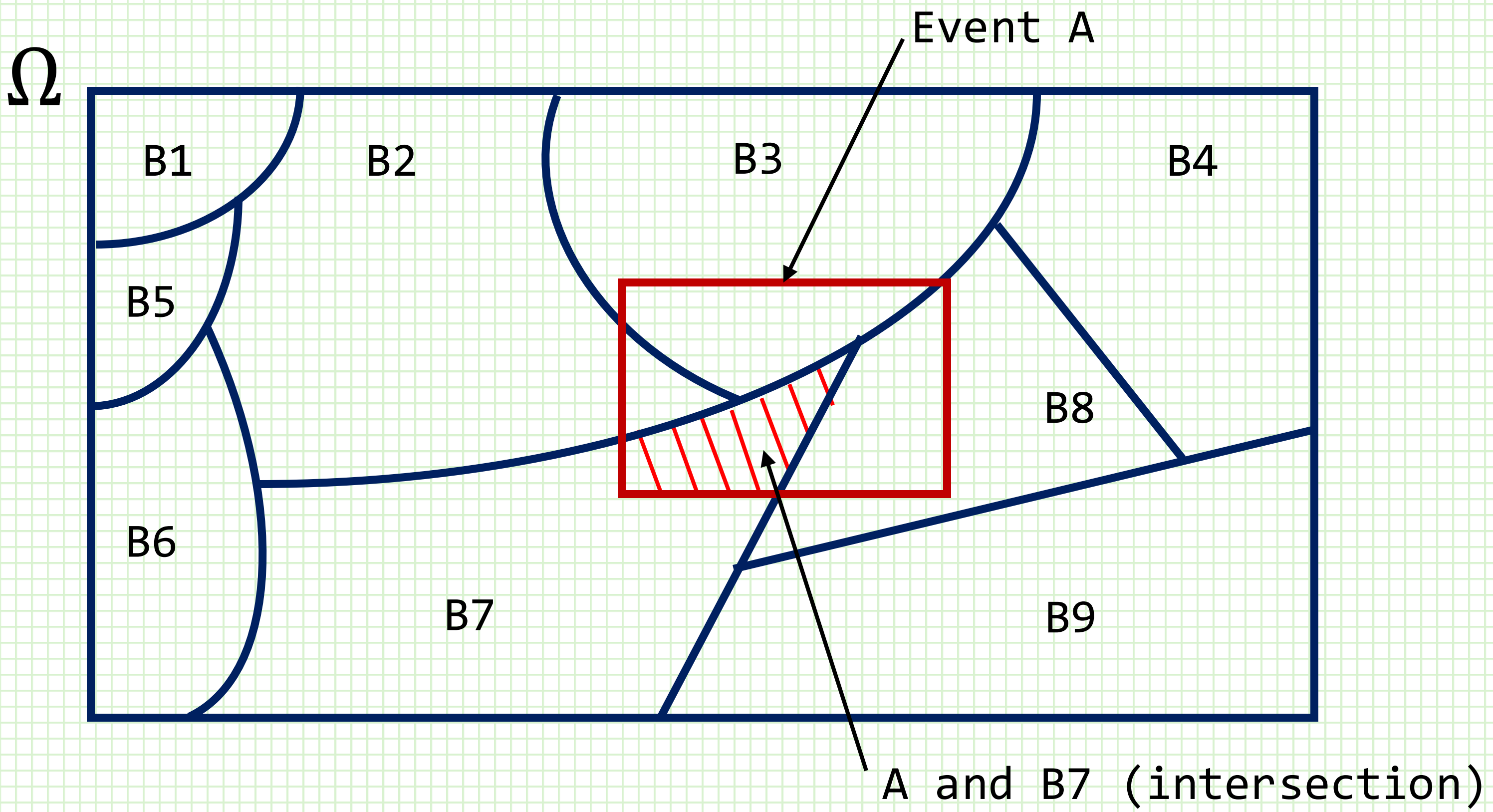
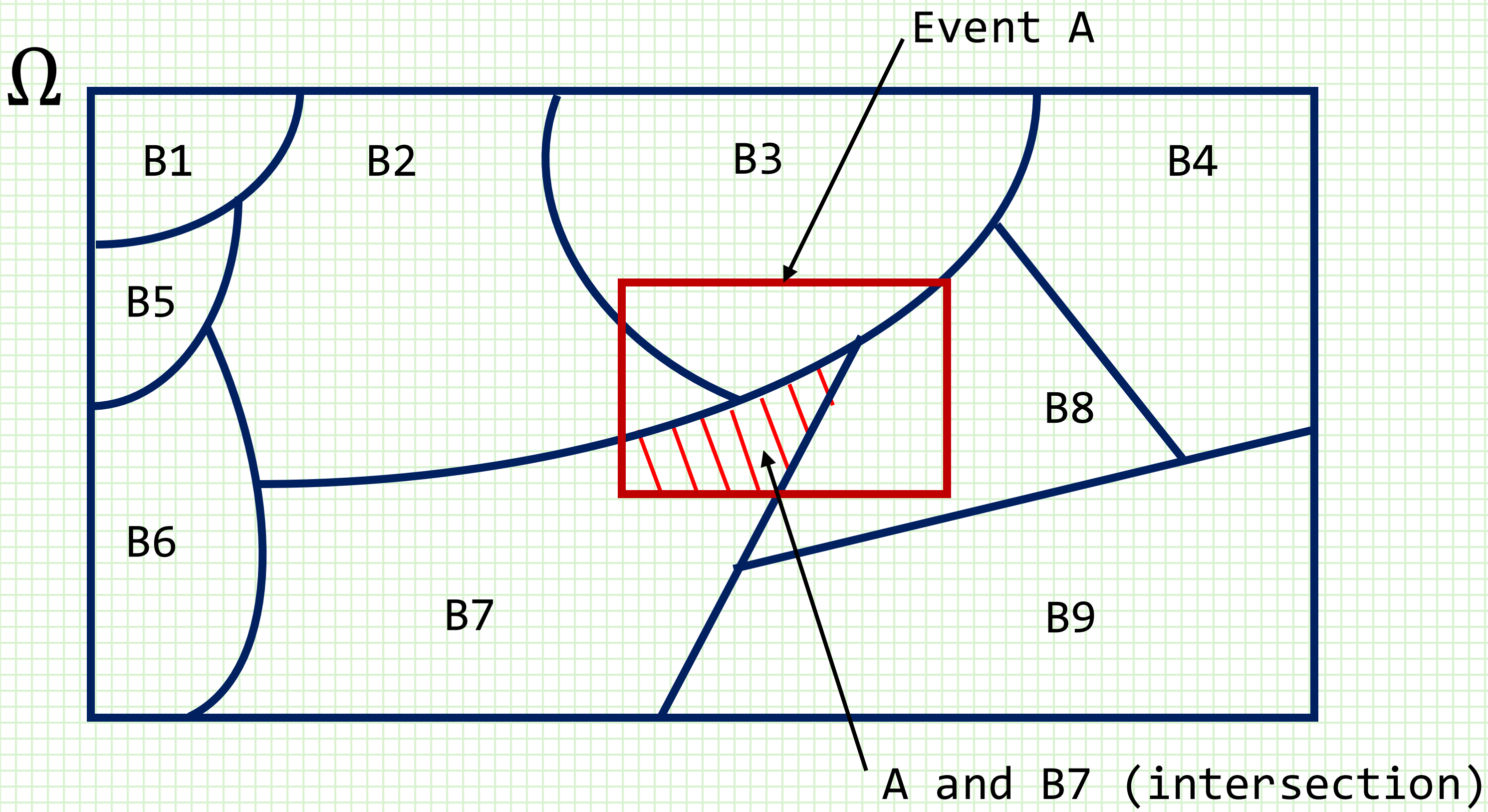


Law of Total Probability

Then for any event A:

$$Pr[A] = \Pr[(A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)]$$



Law of Total Probability

Recall that:

$$\Pr[A \ \cap B] = \Pr[A \mid B] \ \Pr[B]$$

Then for any event A:

$$Pr[A] = \Pr[(A \cap B_1)] + \Pr[(A \cap B_2)] + \cdots + \Pr[(A \cap B_n)]$$

$$= \Pr[A \mid B_1] \ \Pr[B_1]] + \Pr[A \mid B_2] \ \Pr[B_2]] + \cdots + \Pr[A \mid Bn] \ \Pr[Bn]]$$

Law of Total Probability

Recall that:

$$\Pr[A \ \cap B] = \Pr[A \mid B] \ \Pr[B]$$

Then for any event A:

$$Pr[A] = \Pr[(A \cap B_1)] + \Pr[(A \cap B_2)] + \cdots + \Pr[(A \cap B_n)]$$

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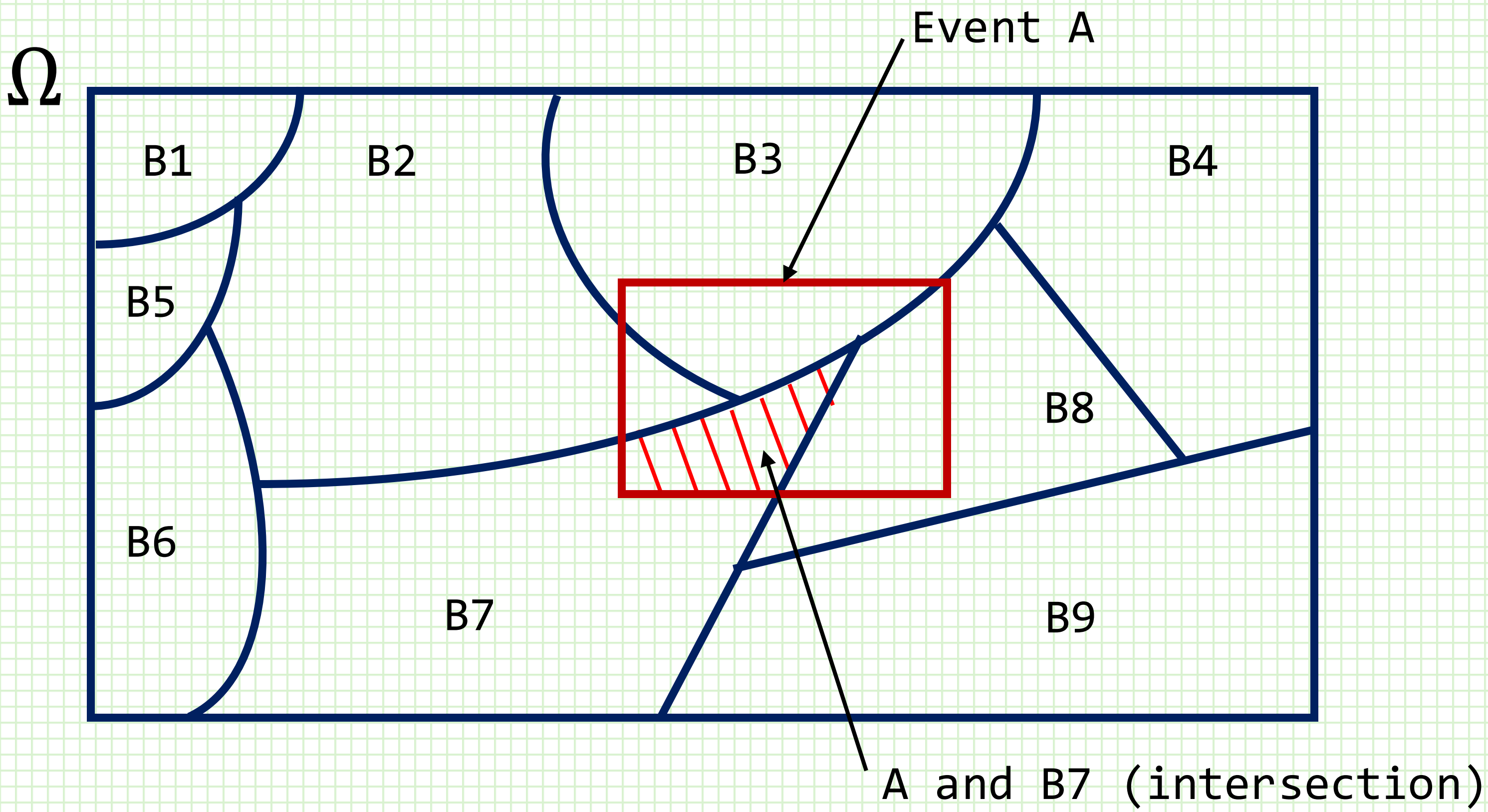
Law of Total Probability

Then for any event A:

$$Pr[A] = \Pr[A \mid B_2] \Pr[B_2] + \Pr[A \mid B_3] \Pr[B_3] + \Pr[A \mid B_7] \Pr[B_7] + \Pr[A \mid B_8] \Pr[B_8]$$



A Variation on Bayes' Rule

$$\Pr[A \mid B] = \frac{\Pr[B \mid A]\ \Pr[A]}{\Pr[B]}$$

$$\Pr[A \mid B] = \frac{\Pr[B \mid A]\ \Pr[A]}{\sum_i \Pr[B \mid A_i]\ \Pr[A_i]}$$

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Example

A lab performs a blood test to reveal the presence or absence of a certain infection

Example

The test is not perfect

- If a person is infected, the test may not always work
- An uninfected person may have a test turn out positive (false positive)

Example

Assume we have the following model:

1. Currently 5% of the population is infected

2. Probability that the test works for an infected person is 99% (true positive rate)

3. Probability of a false positive is 3%

Example

We want the following probabilities:

1. Probability that if a test is positive, the person is infected

2. Probability that if a test is positive, the person is well

Example

Another way to say that:

1. Probability that a person is sick *given* a positive test

2. Probability that a person is well *given* a positive test

Example

Define these events:

$$\begin{aligned} S &= \text{person is sick} \\ T &= \text{test is positive} \end{aligned}$$

Example

Define these events:

$$\begin{aligned} S &= \text{person is sick} \\ T &= \text{test is positive} \end{aligned}$$

Also,

$$S' = \text{person is well}$$

Example

What we know:

$$\Pr[S] = 0.05$$
$$\Pr[S'] = 1 - 0.05 = 0.95$$
$$\Pr[T \mid S] = 0.99 \ \ \text{(true positive rate)}$$
$$\Pr[T \mid S'] = 0.03 \ \ \text{(false positive rate)}$$

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What we want:

$$\Pr[S \mid T] \rightarrow \text{probability person is sick given positive test}$$
$$\Pr[S' \mid T] \rightarrow \text{probability person is well given positive test}$$

Example

What we know:

$$\Pr[S] = 0.05$$
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$$\Pr[T \mid S] = 0.99 \ \ \text{(true positive rate)}$$
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Apply Bayes' rule:

$$\Pr[S \mid T] = \frac{\Pr[T \mid S] \Pr[S]}{\Pr[T]}$$

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What we know:

$$\Pr[S] = 0.05$$
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$$\Pr[T \mid S] = 0.99 \quad \text{(true positive rate)}$$
$$\Pr[T \mid S'] = 0.03 \quad \text{(false positive rate)}$$

Apply Bayes' rule (using law of total probability):

$$\Pr[S \mid T] = \frac{\Pr[T \mid S]\Pr[S]}{\Pr[T]} = \frac{\Pr[T \mid S]\Pr[S]}{\Pr[T \mid S]\Pr[S] + \Pr[T \mid S']\Pr[S']}$$

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$$\Pr[S \mid T] = \frac{\Pr[T \mid S] \Pr[S]}{\Pr[T \mid S] \Pr[S] + \Pr[T \mid S'] \Pr[S']}$$
$$= \frac{0.99 \times 0.05}{0.99 \times 0.05 + 0.03 \times 0.95}$$

Example

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What we want:

$$\Pr[S \mid T] \cong 0.6346$$
$$\Pr[S' \mid T] \cong 0.3654$$