PfT Seminar 05/14/24: Amazin' capsaicin!

You can access a shared spreadsheet for today's workshop at

`https://tinyurl.com/PfT5-14-2024`

Opener (game version)

1. The coin game is a two-player game. A turn consists of naming an integer from 1 to 50 which cannot be written as a sum of nonnegative multiples of previously named integers. The first player to name the integer 1 loses.

With a partner, play a few rounds of the coin game at

`https://cshor.org/assets/html/coin_game.html`

What sort of strategy, if any, can you come up with?

This game is sometimes described in the context of minting coins, hence "coin game."

Opener (spicy version)

2. At *Peter Piper, Inc.*, you can purchase packs of pickled peppers! *Peter Piper's* peppers are available in packs of 6, 14, and 21.

Market research has shown that people don't really buy pecks of peppers anymore. A peck is two dry gallons, which is a lot of peppers!

a. For each of the following numbers p, decide if there's a way to purchase *exactly* p pickled peppers from *Peter Piper*.

(i) $p = 12$	(iv) $p = 26$	(vii) $p = 130$
(ii) $p = 19$	(v) $p = 31$	(viii) $p = 631$
(iii) $p = 22$	(vi) $p = 41$	(ix) $p = 6037$

Let's try these without any technology to get a feel for what's going on.

b. Suppose p is a positive integer. If you can purchase p peppers exactly, can you purchase $p + 6$ peppers exactly?

c. Suppose p is a positive integer. If you cannot purchase $p + 6$ peppers exactly, can you purchase p peppers exactly?

d. Among all positive integers, is there a largest integer you *can* purchase exactly?

e. Among all positive integers, is there a largest integer you *cannot* purchase exactly?

f. If the answer to d or e is 'yes', then find it. How many such integers are there?

If you've answered 'no' to both parts d and e, use this spacious margin to write a complete proof of Fermat's Last Theorem.

Some lingo

As everyone knows, pepper-based businesses are all the rage these days. Pepper entrepreneurs typically order their packaging materials from *Sgt. Pepper*. Are you thinking about getting into selling peppers? You too can order packaging. Just specify positive integer sizes $n_1, n_2, \ldots, n_k$.

Definition. The numbers $n_1, n_2, \ldots, n_k$ are called *pack numbers*. An integer s is *spicy* if there is a way add some combination of pack numbers to produce s. A positive integer is *bland* if it is not spicy.

←In other words, a number s is *spicy* if there are nonnegative integers $x_1, \ldots, x_k$ such that $s = n_1x_1 + \cdots + n_kx_k$.

Notation. Given pack numbers $n_1, \ldots, n_k$, let $W = \langle n_1, \ldots, n_k \rangle$ denote the set of all spicy numbers. This is a *spice world*. Let $\beta(W)$ be the set of all bland numbers. The spice world's *mildness*, denoted $m(W)$, is the number of bland numbers. The maximal bland number, if it exists, is denoted $B(W)$.

I'll tell you what I want, what I really really want.

←I.e., $m(W) = |\beta(W)|$ and $B(W) = \max(\beta(W))$.

Example. *Glenn's House of Peppers* has $n_1 = 3$ and $n_2 = 5$, so

$$W = \langle 3, 5 \rangle = \{3x_1 + 5x_2 : x_1, x_2 \in \mathbb{Z},\ x_1, x_2 \geq 0\}.$$

Scoville level 0–1000

3. For each of the following spice worlds W, compute $m(W)$ and $B(W)$.

Q: How do you tell how heavy a pepper is?

a. $W = \langle 3, 5 \rangle$ **c.** $W = \langle 3, 5, 7 \rangle$ **e.** $W = \langle 2, 101 \rangle$
b. $W = \langle 3, 5, 11 \rangle$ **d.** $W = \langle 2, 11 \rangle$ **f.** $W = \langle 5, 6, 7, 8, 9 \rangle$

4. At *Peter Piper*, $W = \langle 6, 14, 21 \rangle$. (Right?) Determine $\beta(W)$, $m(W)$, and $B(W)$.

Maybe you've already done this.

5. Al is thinking about starting his own pepper business. He wants the mildness to be 4. What are his possible max bland numbers? What are his packaging purchase options?

Obviously, this would be the *Cuoco Capsaicin Corporation*.

6. Tzippy is throwing her hat in the pepper ring, with plans to call her pepper store *Don't Sweat It!*. She's wants to have max bland number equal to 5. What are her packaging purchase options?

7. *Patty's Pepper Mints* sells jalapeño breath mints in packs of 4 and 7.

I.e., $W = \langle 4, 7 \rangle$ here.

a. Compute $B = B(W)$, the maximal bland number here.
b. Suppose x and y are non-negative integers such that $x + y = B$. Can x and y both be spicy? Can x and y both be bland?

Materials created from scratch for the 5/14/2024 PROMYS for Teachers workshop.

Scoville level 1,000 – 10,000

Notation. For any real number x,

A: Give it a weigh, give it a weigh, give it a weigh now.

- $\lfloor x \rfloor$ is the *floor* of x, the largest integer n such that $n \leq x$; and
- $\{x\}$ is the *fractional part* of x, defined by $\{x\} = x - \lfloor x \rfloor$.

Yeah, the curly bracket notation isn't great. But it's standard. We can tell that these aren't sets.

8. Compute $\lfloor 29/6 \rfloor$, $\{29/6\}$, $\lfloor \pi \rfloor$, and $\lfloor -\pi \rfloor$.

9. Here are a few problems that are maybe related to each other, but are definitely not about peppers.

a. Evaluate the sum

$$\frac{1}{14} + \frac{2}{14} + \frac{3}{14} + \cdots + \frac{13}{14}.$$

b. For n a positive integer, find a closed form for the sum

$$\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n}.$$

c. Evaluate the following sum of fractional parts:

$$\left\{\frac{1 \cdot 5}{14}\right\} + \left\{\frac{2 \cdot 5}{14}\right\} + \left\{\frac{3 \cdot 5}{14}\right\} + \cdots + \left\{\frac{13 \cdot 5}{14}\right\}.$$

10. Dorothy wants a closed form for $1^2 + 2^2 + 3^2 + \cdots + n^2$. What is it?

←Here's a hint, courtesy of Pascal. Evaluate $\sum_{k=1}^{n} (k+1)^3 - k^3$ in two ways.

Scoville level 10,000 – 50,000

11. The fine folks at *Sgt. Pepper* have created a web app to assist aspiring pepper entrepreneurs, available at

`https://cshor.org/assets/html/spicy.html`

Visualize *Patty's Pepper Mints* $W = \langle 4, 7 \rangle$ with different combinations of rows and columns. Are there "good" numbers of rows and/or columns?

Here's a fun one. Try it with 2 rows and 9 columns, snake style!

12. Visualize *Sarah's Serranos* $W = \langle 5, 9 \rangle$ with 5 columns. What do you see? How about with 9 columns? Are any other numbers nice?

Is there a fun one here?

13. Create your own company name and spice world W of the form $W = \langle a, b \rangle$. Then compute $m(W)$ and $B(W)$. Can you find a formula for $m(W)$, the number of bland numbers, in terms of the pack sizes?

Add your data to our shared spreadsheet!

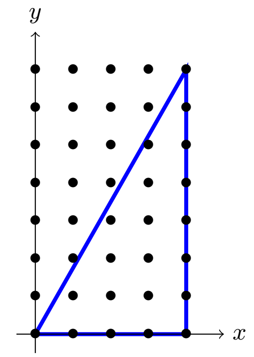
14. For each of the following triangles, determine the number of lattice points in the *interior* of the triangle.

a. The triangle in the plane with vertices $(0,0)$, $(4,0)$, and $(4,7)$. (Picture to the right.)

b. The triangle in the plane with vertices $(0,0)$, $(5,0)$, and $(5,9)$.

c. The triangle in the plane with vertices $(0,0)$, $(9,0)$, and $(9,5)$.

A *lattice point* in the plane is a point (x,y) where x and y are integers.



15. Still with the picture of the triangle to the right, write the equation of the line that the hypotenuse lies on. Freeman says the number of interior lattice points with $x=3$ is equal to $\lfloor 7 \cdot 3/4 \rfloor$. How'd he get that?

Definition. Integers a and b are *coprime* if $\gcd(a,b)=1$.

16. Suppose a and b are coprime positive integers. How many lattice points are in the interior of the triangle with vertices at $(0,0)$, $(a,0)$, and (a,b)?

Scoville level 50,000 – 500,000

17. Given coprime positive integers a and b, evaluate the following sum.

$$\sum_{i=0}^{a-1} \left\lfloor \frac{bi}{a} \right\rfloor$$

Geometry's cool. So is Problem 9, kinda sorta.

18. Brandon wonders about the bland numbers of a spice world of the form $W = \langle a, b \rangle$. Does it ever happen that the number of odd bland numbers is equal to the number of even bland numbers?

19. Cat wants to know about a formula for the *sum* of the bland numbers.

20. For $W = \langle a, b \rangle$, Kathleen decides to compute the following product.

$$\prod_{z \in \beta(W)} \left(1 + \frac{a}{z}\right)$$

What's up with that?

E.g., if $W = \langle 3, 5 \rangle$, then $\beta(W) = \{1, 2, 4, 7\}$, so you'd compute $(1+3/1) \cdot (1+3/2) \cdot (1+3/4) \cdot (1+3/7)$, which equals . . .

21. Let $W = \langle 4, 7 \rangle$. In the figure to the right, circle the spicy numbers. Then answer the questions below.

a. Do all of the bland numbers already appear in the table, or would you find more if you continued the table higher upward?

b. How many bland numbers are in the column that contains 1? How does the quantity $\lfloor 21/4 \rfloor$ relate?

c. Use this idea to find a formula for $m(W)$ involving floor functions.

24	25	26	27
20	21	22	23
16	17	18	19
12	13	14	15
8	9	10	11
4	5	6	7
0	1	2	3

Materials created from scratch for the 5/14/2024 PROMYS for Teachers workshop.

22. Suppose $W = \langle a, b\rangle$ for positive coprime integers a and b. Imagine visualizing W with a columns.

- **a.** Explain why each column contains at least one spicy number.
- **b.** Does each column contain a finite number of bland numbers?
- **c.** Find a formula for $m(W)$ using floor functions.
- **d.** Find a closed form for $m(W)$, the number of bland numbers.

Scoville level 500,000 – 1,500,000

Definition. For a spice world W and a spicy positive integer a, imagine visualizing W with a columns. Consider the set of *smallest spicy numbers mod* a, defined by

$$S(W; a) = \{n \in \mathbb{Z} : n \text{ is the smallest spicy number in its column}\}.$$

23. For $W = \langle 4, 7\rangle$, compute $S(W; 4)$. What's the structure here?

Problem 21 might be helpful.

24. Look back at the visualization of $W = \langle 5, 9\rangle$ with 5 columns in Problem 12. Compute $S(W; 5)$. Among the elements of $S(W; 5)$, which is largest? How does that largest element relate to $B(W)$?

25. Now, visualizing $W = \langle 5, 9\rangle$ with 9 columns, compute $S(W; 9)$, find its largest element, and explain how that relates to $B(W)$.

26. Suppose $W = \langle a, b\rangle$ for a and b coprime positive integers. How many elements are in the set $S(W; a)$? What are they?

27. Consider $W = \langle a, b\rangle$ for a and b coprime positive integers. Find a formula for $B(W)$, the max bland number.

Think back to Problem 24. How might you visualize this?

28. Visualize $W = \langle 4, 7\rangle$ with 4 columns. Ignore all of the spicy numbers *except* for the smallest spicy number in each column.

I.e., ignore all spicy numbers except for those in $S(W; 4)$.

- **a.** Explain why the following equality holds.

$$S(W; 4) \cup \beta(W) = \{0, 1, 2, 3\} \cup \{n + 4 : n \in \beta(W)\}.$$

- **b.** Explain why the sets $S(W; 4)$ and $\beta(W)$ are disjoint.
- **c.** Explain why $\{0, 1, 2, 3\}$ and $\{n + 4 : n \in \beta(W)\}$ are disjoint.
- **d.** Explain why, for any arithmetic function f, we have:

$$\sum_{n\in\beta(W)} (f(n + 4) - f(n)) = \sum_{n\in S(W;4)} f(n) - \sum_{i=0}^{3} f(i).$$

An *arithmetic* function is one which is defined on the set of positive integers with outputs in some subset of the complex numbers (like $\mathbb{Z}$). We'll assume our function is also defined at 0.

Scoville level 1,500,000 – 2,500,000

29. For coprime positive integers a and b, let $W = \langle a, b \rangle$. For any arithmetic function f, explain why

$$\sum_{n \in \beta(W)} (f(n+a) - f(n)) = \sum_{i=0}^{a-1} (f(bi) - f(i)) .$$

30. Using the function $f(n) = n$ with the identity from Problem 29, what can you conclude?

31. What if you use the function $f(n) = (-1)^n$?

Try it with a and b both odd first.

32. What if you use the function $f(n) = \log n$?

33. What if you use the function $f(n) = n^2$?

Scoville level 2,693,000. Proceed with caution!!

34. Here's an old math competition problem, taken from the 1971 Putnam Exam.
A player scores either a or b points at each turn, where a and b are unequal positive integers. He notices that his cumulative score can take any positive integer value except for those in a finite set S, where $|S| = 35$, and $58 \in S$. Find a and b.

35. Can you generalize Problem 29 to spice worlds W that have three pack numbers? I.e., if $W = \langle a, b, c \rangle$? Or maybe with a little more structure, if $W = \langle a^2, ab, b^2 \rangle$, or $W = \langle a, a+b, a+2b \rangle$, or $W = \langle ab, bc, ca \rangle$? Or spice worlds with more than three pack numbers?

36. Sometimes in a spice world $W = \langle a, b \rangle$, the number of odd bland numbers is equal to the number of even bland numbers. Does it ever happen that there are the same number of bland numbers in each congruence class modulo 3? Modulo 4? Modulo 5?

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