Rounding sequences and counting by congruence classes

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Talk outline

Opener

The question

Let $n \in \mathbb{N}$. Consider the sequence of n fractions with numerator n:

$$n/1,\ n/2,\ n/3,\ \ldots,\ n/n.$$

Round each fraction to the nearest integer to form a *rounding sequence* of n integers.

How many of these integers are odd?

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Notation: $\lfloor x \rceil$

For $x \in \mathbb{R}$, the nearest integer to x is denoted $\lfloor x \rceil$. Half-integers are rounded up.
Claim: $\lfloor x \rceil = \lfloor x + 1/2 \rfloor$.

Examples

- $\lfloor \pi \rceil =$
- $\lfloor 4\pi \rceil =$
- $\lfloor 7.5 \rceil =$
- $\lfloor -7.5 \rceil =$

Opener, reframed

Notation ($\mathcal{R}_n$)

For $n \in \mathbb{N}$, let $\mathcal{R}_n$ denote the number of odd terms in the rounding sequence

$$\lfloor n/1 \rceil, \lfloor n/2 \rceil, \lfloor n/3 \rceil, \ldots, \lfloor n/n \rceil.$$

In other words, $\mathcal{R}_n = \#\{k \in \mathbb{Z} : 1 \leq k \leq n, \text{ and } \lfloor n/k \rceil \text{ is odd}\}$.

The question, reframed

Can we find a formula for $\mathcal{R}_n$? How big will it be in general?

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Example (Computing $\mathcal{R}_7$)

- Start with $7/1$, $7/2$, $7/3$, $7/4$, $7/5$, $7/6$, $7/7 = 7$, 3.5, $2.\overline{3}$, 1.75, 1.4, $1.1\overline{6}$, 1.
- Round each to obtain 7, 4, 2, 2, 1, 1, 1.
- There are four odd terms, so $\mathcal{R}_7 = 4$.

Data collection

Recall: $\mathcal{R}_n = \#$ odd terms in the sequence $\lfloor n/1 \rceil, \lfloor n/2 \rceil, \ldots, \lfloor n/n \rceil$.

Values of $\mathcal{R}_n$ for $n = 1, \ldots, 18$

n	1	2	3	4	5	6	7	8	9	10
$\mathcal{R}_n$	1						4	4	6	

n	11	12	13	14	15	16	17	18
$\mathcal{R}_n$	6	5		8	9	9	10	10

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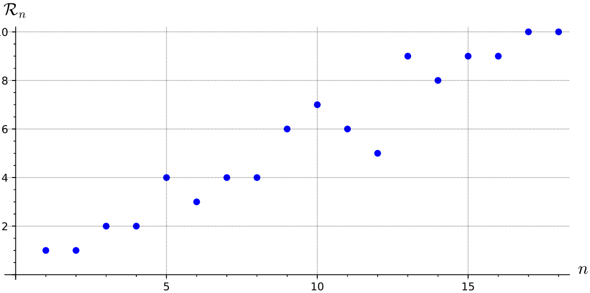
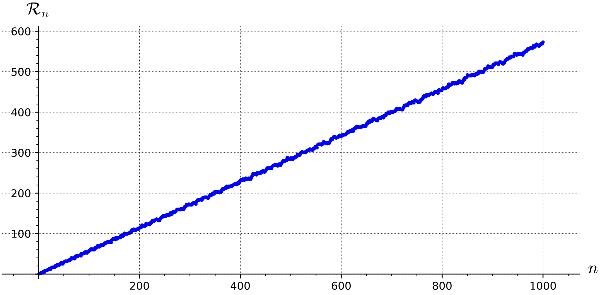


Data collection and linear regression (with Sage!)



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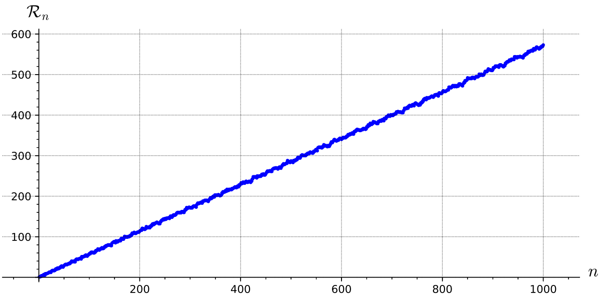
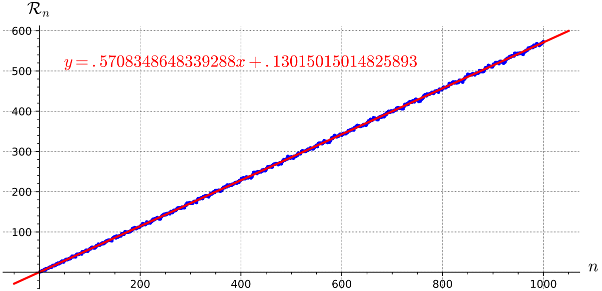
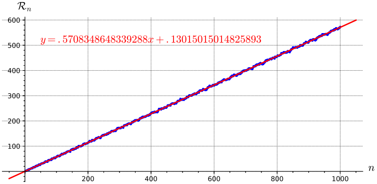




A conjecture for $\mathcal{R}_n$



The data suggest that $\mathcal{R}_n$ is roughly linear. The linear regression suggests that among values of n in the interval $[1, 1000]$,

$$\mathcal{R}_n \approx 0.5708348648339288n + 0.13015015014825893.$$

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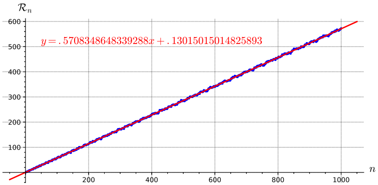

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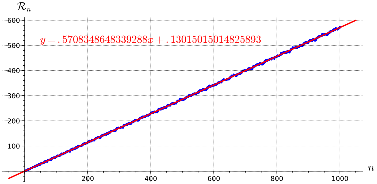
In other words, on average approximately 57.08% of integers in the sequence

$$\lfloor n/1 \rceil\,,\ \lfloor n/2 \rceil\,,\ \lfloor n/3 \rceil\,,\ \ldots,\ \lfloor n/n \rceil$$

are odd.

A conjecture for $\mathcal{R}_n$



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Questions

Does this hold for larger n? Where does 57.08% come from?

Talk outline

1. Introduction: Counting odd numbers in a rounding sequence
2. A formula for $\mathcal{R}_n$
3. Counting Gaussian integers
4. Putting it all together
5. Taking it further

A first observation for $\mathcal{R}_n$

Proposition

For $\ell \in \mathbb{N}$, let $f(\ell)$ equal the number of terms in the rounding sequence $\lfloor n/1 \rceil, \lfloor n/2 \rceil, \ldots, \lfloor n/n \rceil$ that are greater than or equal to ℓ. Then

$$f(\ell) = \begin{cases} n & \text{if } \ell = 1, \\ \left\lfloor \dfrac{2n}{2\ell - 1} \right\rfloor & \text{if } \ell > 1 \end{cases}$$

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Example

Back to $\mathcal{R}_7$, the rounding sequence is 7, 4, 2, 2, 1, 1, 1.

How many of these terms are ≥ 3? $\lfloor 2 \cdot 7/(2 \cdot 3 - 1) \rfloor = \lfloor 14/5 \rfloor = 2$ are.

How many of these terms are ≥ 2? $\lfloor 2 \cdot 7/(2 \cdot 2 - 1) \rfloor = \lfloor 14/3 \rfloor = 4$ are.

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Proof.

All terms in the rounding sequence are at least 1, so $f(1) = n$.

The rounding sequence is non-increasing, so for $\ell > 1$ the first k terms will be at least ℓ precisely when $\lfloor n/k \rceil \geq \ell$ and $\lfloor n/(k+1) \rceil < \ell$.

In other words, $n/k \geq \ell - 1/2$ and $n/(k+1) < \ell - 1/2$. Thus,

$$2n/(2\ell - 1) - 1 < k \leq 2n/(2\ell - 1).$$

Since $k \in \mathbb{Z}$, we have $k = \lfloor 2n/(2\ell - 1) \rfloor$. □

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Corollary

$$\mathcal{R}_n = n - \left\lfloor \frac{2n}{3} \right\rfloor + \left\lfloor \frac{2n}{5} \right\rfloor - \left\lfloor \frac{2n}{7} \right\rfloor + \cdots \pm \left\lfloor \frac{2n}{2n-1} \right\rfloor = n + \sum_{\ell=2}^{n} (-1)^{\ell+1} \left\lfloor \frac{2n}{2\ell - 1} \right\rfloor$$

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Proof sketch

The number of 1s is [the number of terms that are at least 1] minus [the number of terms that are at least 2]. That is $f(1) - f(2) = n - \lfloor 2n/3 \rfloor$.
The number of 3s is $f(3) - f(4)$. The number of 5s is $f(5) - f(6)$. Etc.

Adding and subtracting $2n$, we get the following:

Alternating sum formula for $\mathcal{R}_n$

For $n \in \mathbb{N}$,

$$\mathcal{R}_n = -n + \sum_{\ell=1}^{n} (-1)^{\ell+1} \left\lfloor \frac{2n}{2\ell - 1} \right\rfloor.$$

Example ($\mathcal{R}_7$ again!)

$$\begin{aligned}
\mathcal{R}_7 &= -7 + \sum_{\ell=1}^{7} (-1)^{\ell+1} \left\lfloor \frac{2 \cdot 7}{2\ell - 1} \right\rfloor \\
&= -7 + \left\lfloor \frac{14}{1} \right\rfloor - \left\lfloor \frac{14}{3} \right\rfloor + \left\lfloor \frac{14}{5} \right\rfloor - \left\lfloor \frac{14}{7} \right\rfloor + \left\lfloor \frac{14}{9} \right\rfloor - \left\lfloor \frac{14}{11} \right\rfloor + \left\lfloor \frac{14}{13} \right\rfloor \\
&= \\
&=
\end{aligned}$$

Talk outline

Now, thinking about $\mathbb{Z}[i]$...

Our next goal is to find a formula for the number of Gaussian integers with a particular norm.

norm n	elements z with $N(z) = n$	# of elements
0	$0 + 0i$	1
1	$\pm 1, \ \pm i$	4
2	$\pm 1 \pm i$	4
3	none	0
4	$\pm 2, \ \pm 2i$	4
5	$\pm 1 \pm 2i, \ \pm 2 \pm i$	8
6	none	0
7	none	0
8		
9		
10		
13		
16		
25		
65		

For $n \geq 1$, the number of Gaussian integers z with $N(z) = n$ is divisible by 4.

$$\text{Let } r(n) = \frac{1}{4} \cdot \#\{z \in \mathbb{Z}[i] : N(z) = n\}.$$

norm n	elements z with $N(z) = n$	$r(n)$
1	$\pm 1,\ \pm i$	1
2	$\pm 1 \pm i$	1
3	none	0
4	$\pm 2,\ \pm 2i$	1
5	$\pm 1 \pm 2i,\ \pm 2 \pm i$	2
6	none	0
7	none	0
8	$\pm 2 \pm 2i$	1
9	$\pm 3,\ \pm 3i$	1
10	$\pm 1 \pm 3i,\ \pm 3 \pm i$	2
13	$\pm 2 \pm 3i,\ \pm 3 \pm 2i$	2
16	$\pm 4,\ \pm 4i$	1
25	$\pm 3 \pm 4i,\ \pm 4 \pm 3i,\ \pm 5,\ \pm 5i$	3
65	$\pm 4 \pm 7i,\ \pm 7 \pm 4i,\ \pm 8 \pm i,\ \pm 1 \pm 8i$	4

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6	none	0
7	none	0
8	$\pm 2 \pm 2i$	1
9	$\pm 3,\ \pm 3i$	1
10	$\pm 1 \pm 3i,\ \pm 3 \pm i$	2
13	$\pm 2 \pm 3i,\ \pm 3 \pm 2i$	2
16	$\pm 4,\ \pm 4i$	1
25	$\pm 3 \pm 4i,\ \pm 4 \pm 3i,\ \pm 5,\ \pm 5i$	3
65	$\pm 4 \pm 7i,\ \pm 7 \pm 4i,\ \pm 8 \pm i,\ \pm 1 \pm 8i$	4

Predict $r(n)$ for other values. $r(325)$? Any conjectures?

Toward a conjecture for $r(n)$

Some more data, looking at powers:

- $r(1) = 1$
- $r(2) = 1$, $r(4) = 1$, $r(8) = 1$, $r(16) = 1$, ...
- $r(3) = 0$, $r(9) = 1$, $r(27) = 0$, $r(81) = 1$, ...
- $r(5) = 2$, $r(25) = 3$, $r(125) = 4$, $r(625) = 5$, ...
- $r(7) = 0$, $r(49) = 1$, $r(343) = 0$, $r(7^4) = 1$, ...
- $r(13) = 2$, $r(169) = 3$, $r(13^3) = 4$, ...

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Conjecture

For all $k \geq 0$, $r(2^k) =$

- For $p \equiv 1 \pmod 4$,

  $r(p^k) =$

- For $p \equiv 3 \pmod 4$,

  $r(p^k) =$

χ, a character

Definition (χ, pronounced "chi")

Define the function $\chi : \mathbb{Z} \to \{-1, 0, 1\}$ by

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv 3 \pmod 4 \\ 0 & \text{if } n \equiv 0, 2 \pmod 4 \end{cases}$$

Claim

The function χ is *completely multiplicative*. That is, for all $a, b \in \mathbb{Z}$,

$$\chi(a \cdot b) = \chi(a) \cdot \chi(b).$$

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Getting to know χ, for any $k \geq 1$ ...

- If $n \equiv 1 \pmod 4$, $\chi(n^k) =$
- If $n \equiv 0, 2 \pmod 4$, $\chi(n^k) =$
- If $n \equiv 3 \pmod 4$, $\chi(n^k) =$

Putting some ideas together

Thus far, we have conjectured for all $k \geq 0$:

- $r(2^k) = 1$
- if $p \equiv 1 \pmod 4$, $r(p^k) = k + 1$
- if $p \equiv 3 \pmod 4$, $r(p^k) = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$

And we have seen for all $k \geq 1$,

- $\chi(1) = 1$
- $\chi(2^k) = 0$
- if $p \equiv 1 \pmod 4$, $\chi(p^k) = 1$
- if $p \equiv 3 \pmod 4$, $\chi(p^k) = (-1)^k$

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Proposition

Let p be prime. For all $k \geq 0$,

$$r(p^k) = \chi(1) + \chi(p) + \chi(p^2) + \cdots + \chi(p^k) = \sum_{i=0}^{k} \chi(p^i).$$

$r(n)$ in terms of χ

Proposition

$r(n)$ is a multiplicative function. That is, for $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$,

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Corollary

For $n \in \mathbb{N}$,

$$r(n) = \sum_{d|n} \chi(d),$$

where the sum is over positive divisors d of n.

$r(n)$ in terms of χ

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For $n \in \mathbb{N}$,

$$r(n) = \sum_{d \mid n} \chi(d),$$

where the sum is over positive divisors d of n.

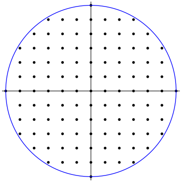
Proof in the case where $n = pq^2$

$$\begin{aligned} r(n) = r(pq^2) = r(p)r(q^2) &= (1 + \chi(p))(1 + \chi(q) + \chi(q^2)) \\ &= \chi(1) + \chi(q) + \chi(q^2) + \chi(p) + \chi(pq) + \chi(pq^2) = \sum_{d \mid pq^2} \chi(d). \end{aligned}$$

Counting Gaussian integers, version #1

Question

Let $N \in \mathbb{N}$. How many Gaussian integers are there in a circle of radius $\sqrt{N}$?



For example, when $N = 36$ there are 113 lattice points in the closed circle of radius $\sqrt{36} = 6$.

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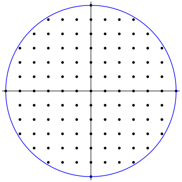

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Answer #1

Every Gaussian integer in a circle of radius $\sqrt{N}$ has a norm that is an integer somewhere between 0 and N.
Let $C(N)$ be the number of Gaussian integers in a circle of radius $\sqrt{N}$. Then

$$C(N) = 1 + 4\sum_{n=1}^{N} r(n).$$

Counting with χ (still answer #1)

Since we can write $r(n)$ in terms of χ, we can write $C(N)$ in terms of χ.

Proposition

For $C(N)$ the number of Gaussian integers in a circle of radius $\sqrt{N}$,

$$C(N) = 1 + 4\sum_{n=1}^{N}\sum_{d|n}\chi(d).$$

Question: Suppose $N = 20$. How many times does $\chi(3)$ show up? How many times does $\chi(8)$ show up?

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Corollary

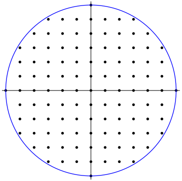
$$C(N) = 1 + 4\sum_{d=1}^{N}\left\lfloor\frac{N}{d}\right\rfloor\chi(d) = 1 + 4\left(\left\lfloor\frac{N}{1}\right\rfloor - \left\lfloor\frac{N}{3}\right\rfloor + \left\lfloor\frac{N}{5}\right\rfloor - \left\lfloor\frac{N}{7}\right\rfloor + \cdots\right)$$

$$= 1 + 4\sum_{\ell=1}^{\infty}\left\lfloor\frac{N}{2\ell-1}\right\rfloor.$$

Counting Gaussian integers, version #2

Question

Let $N \in \mathbb{N}$. How many Gaussian integers are there in a circle of radius $\sqrt{N}$?



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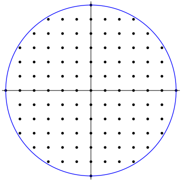

For example, when $N = 36$ there are 113 lattice points in the closed circle of radius $\sqrt{36} = 6$.

Answer #2 (wave hands!!)

Every Gaussian integer is the bottom left corner of a unit square with area 1. It stands to reason that the area of the circle is approximately the area of these squares. As a result, $C(n) \approx \pi(\sqrt{N})^2 = \pi N$.

(Note: $36\pi = 113.097\ldots$)

Big O notation

Notation (Big O)

Let $f(x)$ and $g(x)$ be defined on $\mathbb{R}$. We write $f(x) = \mathcal{O}(g(x))$, saying

"$f(x)$ is big O of $g(x)$"

if $f(x)$ is at most a positive constant multiple of $g(x)$. With limits, this means for some x_0, there is some constant M for which $|f(x)| \leq Mg(x)$ for all $x \geq x_0$.

Big O notation

Notation (Big O)

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Examples

- $x = \mathcal{O}(x^2)$
- $x^2 = \mathcal{O}(2^x)$
- $8x + 3 = \mathcal{O}(x)$
- $x + x^3 = \mathcal{O}(x^3)$

Gauss to the rescue!

We have $C(x)$ as the number of lattice points (or Gaussian integers) contained in a circle of radius $\sqrt{x}$.

Proposition (Gauss)

For real x, $C(x) = \pi x + \mathcal{O}(\sqrt{x})$.

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Gauss' circle problem

"Gauss' circle problem" is concerned with improving the error bound here. It is conjectured that the best bound is $\mathcal{O}(x^{1/4})$. Currently the best known is $\mathcal{O}(x^{131/416})$, and $131/416 = 0.3149\ldots$. (This is due to Huxley in 2000.)

Talk outline

Connecting the dots

Summary thus far:

- $\mathcal{R}_n = -n + \left\lfloor \frac{2n}{1} \right\rfloor - \left\lfloor \frac{2n}{3} \right\rfloor + \left\lfloor \frac{2n}{5} \right\rfloor - \left\lfloor \frac{2n}{7} \right\rfloor + \left\lfloor \frac{2n}{9} \right\rfloor - \left\lfloor \frac{2n}{11} \right\rfloor + \cdots$
- $C(N) = 1 + 4\left(\left\lfloor \frac{N}{1} \right\rfloor - \left\lfloor \frac{N}{3} \right\rfloor + \left\lfloor \frac{N}{5} \right\rfloor - \left\lfloor \frac{N}{7} \right\rfloor + \left\lfloor \frac{N}{9} \right\rfloor - \left\lfloor \frac{N}{11} \right\rfloor + \cdots \right)$
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Proposition

For $n \geq 0$, the number of lattice points in a circle of radius $\sqrt{2n}$ is

$$C(2n) = 4\mathcal{R}_n + 4n + 1.$$

Connecting the dots

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Proposition

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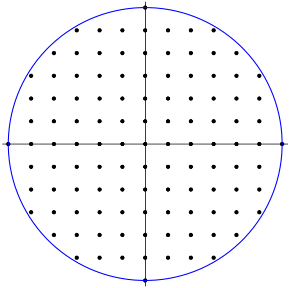
Example

The number of lattice points in a disc of radius $6 = \sqrt{36} = \sqrt{2 \cdot 18}$ is

$$4\mathcal{R}_{18} + 4 \cdot 18 + 1 = 4 \cdot 10 + 4 \cdot 18 + 1$$
$$= 113.$$



Connecting the dots

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- $C(N) = \pi N + \mathcal{O}(\sqrt{N})$

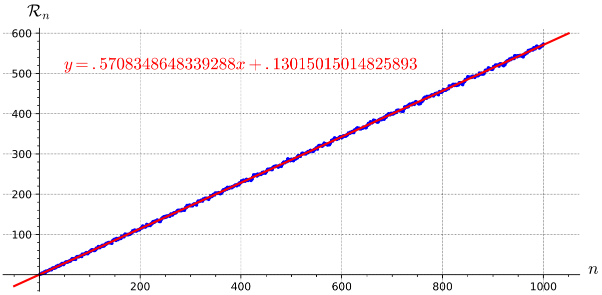
Finally, we can solve for $\mathcal{R}_n$:

$$\begin{aligned}
\mathcal{R}_n &= -n + (C(2n) - 1)/4 = -n - 1/4 + C(2n)/4 \\
&= -n - 1/4 + (1/4)(2n\pi + \mathcal{O}(\sqrt{2n})) \\
&= -n + \pi n/2 + \mathcal{O}(\sqrt{n})
\end{aligned}$$

Connecting the dots

Proposition

$\mathcal{R}_n = (-1+\frac{\pi}{2})n + \mathcal{O}(\sqrt{n})$. *In particular,* $\lim_{n\to\infty} \frac{1}{n}\mathcal{R}_n = -1 + \frac{\pi}{2} = 0.570796\ldots$



Talk outline

Taking it further

We can take this much further!

We have different types of rounding available. E.g., we can use the floor or the ceiling. Or we can view $\lfloor x \rceil$ as $\lfloor x + 1/2 \rfloor$ and replace the $1/2$ with some other real number α to get a sequence like

$$\left\lfloor \frac{n}{1} + \alpha \right\rfloor, \left\lfloor \frac{n}{2} + \alpha \right\rfloor, \ldots, \left\lfloor \frac{n}{n} + \alpha \right\rfloor.$$

Or, instead of counting odd integers (which are 1 mod 2), we can focus on other congruence classes.

And we can consider more general rounding sequences. E.g., for some $\nu \in \mathbb{R}$, start with

$$\left\lfloor \frac{n-\nu}{1} \right\rfloor, \left\lfloor \frac{n-\nu}{2} \right\rfloor, \ldots, \left\lfloor \frac{n-\nu}{n} \right\rfloor.$$

We'll look at two more examples here: one using the floor and counting how many terms are 1 mod 3; and another using the floor and counting how many terms are 1 mod 2.

Floor and 1 mod 3: $\mathcal{H}_n$ and hexagonal lattice points

For $n \in \mathbb{N}$, let $\mathcal{H}_n = \#\{1 \le k \le n : \lfloor n/k \rfloor \equiv 1 \pmod 3\}$.

Proposition

The number of lattice points in a hexagonal lattice contained in a disc of radius $\sqrt{n}$ is $1 + 6\mathcal{H}_n$.

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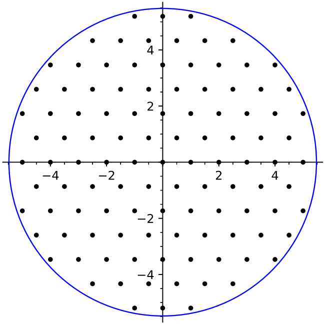
Example ($n = 30$)

$$\left\lfloor \frac{30}{1} \right\rfloor, \left\lfloor \frac{30}{2} \right\rfloor, \ldots, \left\lfloor \frac{30}{30} \right\rfloor$$

$$= 30, 15, 10, 7, 6, 5, 4, 3, 3, 3, 2, 2, 2, 2, 2,$$
$$1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1.$$

18 terms are 1 modulo 3. The circle of radius $\sqrt{30}$ contains $1 + 6 \cdot 18 = 109$ hexagonal lattice points.



Floor & counting odds: $\mathcal{F}_n$ via Dirichlet divisor problem

Let $\mathcal{F}_n = \#\{k \in \mathbb{Z} : 1 \leq k \leq n,\ \lfloor n/k \rfloor \text{ is odd}\}$. Let $\mathrm{D}(n)$ be the number of lattice points beneath the hyperbola $xy = n$.

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$\mathcal{F}_n = \mathrm{D}(n) - \mathrm{D}(n/2)$.

Floor & counting odds: $\mathcal{F}_n$ via Dirichlet divisor problem

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Proposition

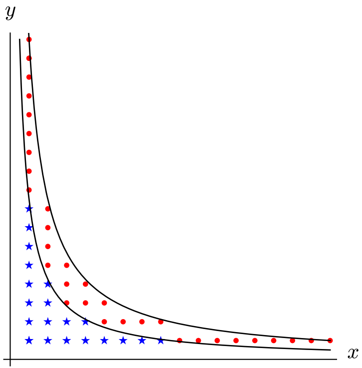
$\mathcal{F}_n = \mathrm{D}(n) - \mathrm{D}(n/2)$.

Example (Interpreting $\mathcal{F}_{17}$)

For $n = 17$, the graphs of the hyperbolas $y = 17/x$ and $y = (17/2)/x$ are to the right.

- Lattice points between the hyperbolas are circles. There are 32 circles.
- Lattice points on or below the lower hyperbola are stars. There are 20 stars.

For $\mathcal{F}_{17}$, we count the number of odd integers in $\lfloor 17/1 \rfloor, \lfloor 17/2 \rfloor, \dots, \lfloor 17/17 \rfloor$
$= 17, 8, 5, 4, 3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1$.
Thus, $\mathcal{F}_{17} = 12$.



For more information...

- Tom M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York-Heidelberg, 1976, Undergraduate Texts in Mathematics.
- Nicholas Dent and Caleb M. Shor, *On the residues of rounded fractions with a common numerator*, Journal of Integer Sequences **27** (2024), Article 24.2.5.
- M. N. Huxley, *Integer points, exponential sums and the Riemann zeta function*, Number theory for the millennium, II, A K Peters, Natick, MA, 2002, pp. 275–290.

Also, check out sequence A363341 in the On-Line Encyclopedia of Integer Sequences!

Thank you!