PVD Beam-Column Element

- Review of truss element stiffness matrix using PVD
- 6 DOF flexural element
- Higher order beam-column element

REVIEW: PVD Truss element

$$
\begin{aligned}
u(x) &= a_1 + a_2 x \\
\frac{d\,u(x)}{dx} &= a_2 \\
u(x) &= N_1(x)\,u_1 + N_2(x)\,u_2 \\
\frac{du(x)}{dx} &= N_1'(x)\,u_1 + N_2'(x)\,u_2
\end{aligned}
$$

$$
\begin{aligned}
u(x) &= a_1 + a_2 x + a_3 x^2 \\
\frac{du}{dx} &= a_2 + 2a_3 x \\
u(x) &= N_1(x)\,u_1 + N_2(x)\,u_2 + N_3(x)\,u_3 \\
\frac{du(x)}{dx} &= N^{1}_{\prime}(x)\,u_1 + N_2'(x)\,u_2 + N_3'(x)\,u_3
\end{aligned}
$$

$$N_1 = \left(1 - \frac{x}{L}\right) \qquad N_2 = \left(\frac{x}{L}\right)$$

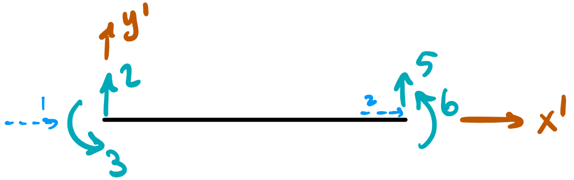
$$N_1' = -\frac{1}{L} \qquad N_2' = \frac{1}{L}$$

The solution interpolated by the shape fnc's
must satisfy compatibility
ideally close to satisfy equilibrium.

PVD $\rightarrow$ $\delta u_i F_{ij} = \delta u_i \int_L B_i (EA) B_j \, dx \; u_j$

Hence; $F_{ij} = \underbrace{\int_L B_i (EA) B_j \, dx}_{K_{ij}} u_j$

Flexural member calculations



transverse def

$$\nu(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$

RBM

$$\theta(x) = \nu'(x) = a_2 + 2a_3 x + 3a_4 x^2$$

RBR

BC

$x = 0$

$$\nu = v_2 \quad \rightarrow \quad a_1 = v_2$$

$$\nu' = \theta_3 \quad \rightarrow \quad a_2 = \theta_3$$

$x = L$

$$\nu = v_5$$

$$\nu' = \theta_6$$

SOLVE FOR a_3 & a_4 AND REGROUP.

$$\nu(x) = N_2 v_2 + N_3 \theta_3 + N_5 v_5 + N_6 \theta_6$$

$$N_2(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \qquad N_3(x) = x\left(1 - \frac{x}{L}\right)^2$$

$$N_5(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 \qquad N_6(x) = x\left(\left(\frac{x}{L}\right)^2 - \frac{x}{L}\right)$$

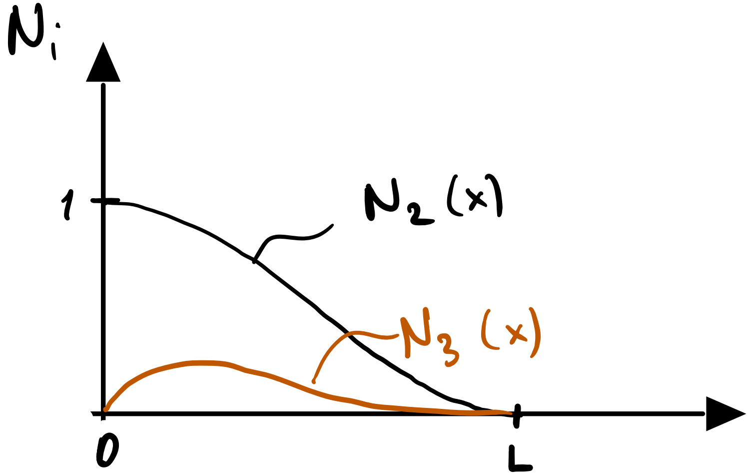
CUBIC HERMITIAN SHAPE FUNCTIONS

COMPATIBILITY OF SHAPE FUNCTIONS

$$\nu\left(x\right) = N_2 v_2 + N_3 \theta_3 + N_5 v_5 + N_6 \theta_6$$

$$N_2\left(x\right) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \qquad N_3\left(x\right) = x\left(1 - \frac{x}{L}\right)^2$$

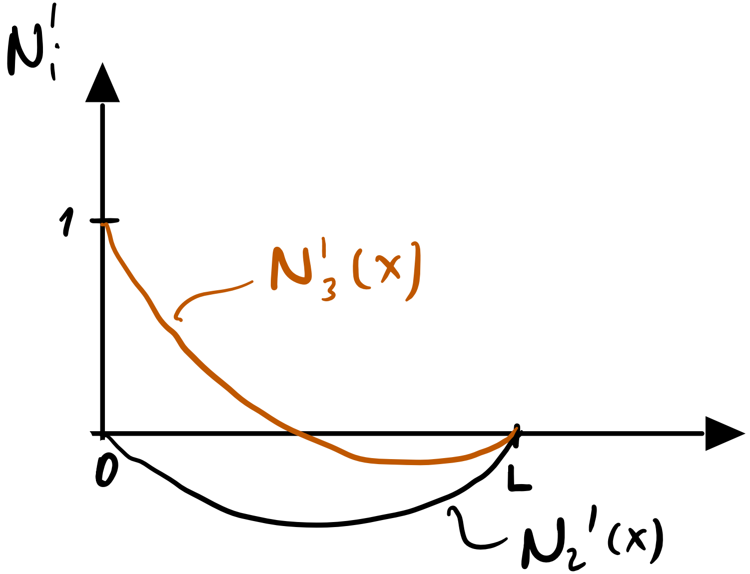
$$N_5\left(x\right) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 \qquad N_6\left(x\right) = x\left(\left(\frac{x}{L}\right)^2 - \frac{x}{L}\right)$$



$$\theta(x) = \nu'\left(x\right) = N_2{}' v_2 + N_3{}' \theta_3 + N_5{}' v_5 + N_6{}' \theta_6$$

$$N_2'\left(x\right) = -\frac{6x}{L^2} + \frac{6x^2}{L^3} \qquad N_3'\left(x\right) = 1 - \frac{4x}{L} + \frac{3x^2}{L^2}$$

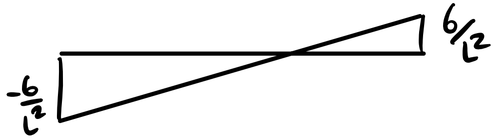
$$N_5'\left(x\right) = \frac{6x}{L^2} - \frac{6x^2}{L^3} \qquad N_6'\left(x\right) = \frac{3x^2}{L^2} - \frac{x^2}{L}$$



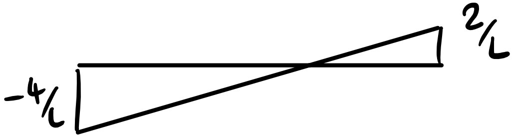
EVALUATE EQUILIBRIUM: $M(x) = EI(x)\,\phi(x)$

$$\phi(x) \approx \theta'(x) = \nu''(x) = N_2''v_2 + N_3''\theta_3 + N_5''v_5 + N_6''\theta_6$$

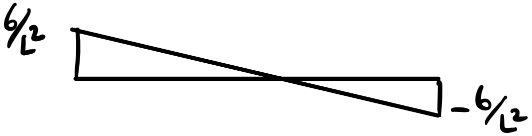
$$N_2'' = B_2 = -\frac{6}{L^2} + \frac{12 \cdot x}{L^3}$$



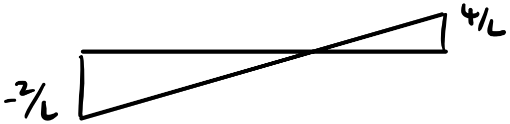
$$N_3'' = B_3 = -\frac{4}{L} + \frac{6x}{L^2}$$



$$N_5'' = B_5 = \frac{6}{L^2} - \frac{12x}{L^3}$$



$$N_6'' = B_6 = \frac{6x}{L^2} - \frac{2}{L}$$



EVALUATE EQUILIBRIUM: cont.

$M(x) = EI(x)\,\phi(x)$

$$\phi(x) \approx \theta'(x) = \nu''(x) = N_2''v_2 + N_3''\theta_3 + N_5''v_5 + N_6''\theta_6$$

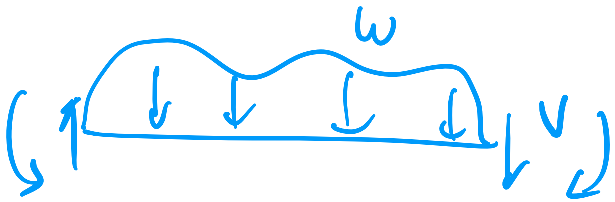
FOR EXAMPLE: $\nu(x) = N_2 \cdot u_2$

$$N = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

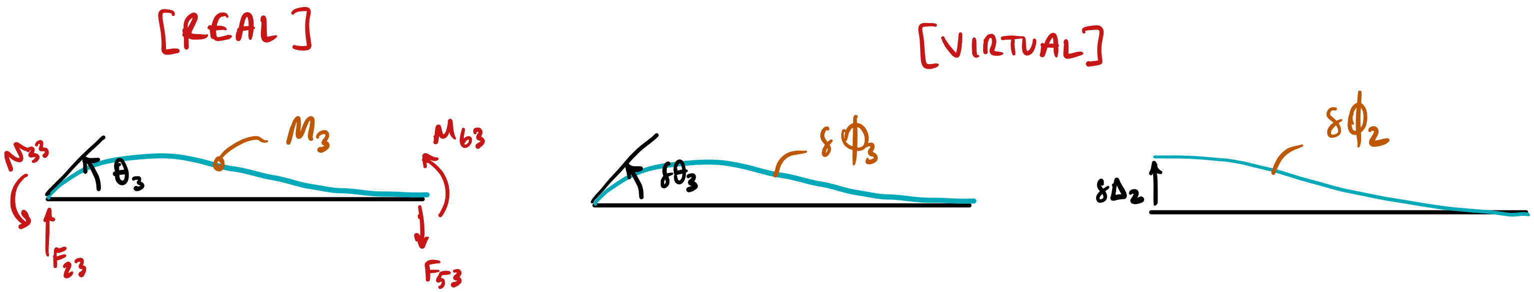
$$M = EI\phi = EI\frac{d^2v}{dx^2} \longrightarrow N'' = -\frac{6}{L^2} + \frac{12\cdot x}{L^3}$$

$$V = \frac{dM}{dx} = EI\frac{d^3v}{dx^3} \longrightarrow N''' = \frac{12}{L^3}$$

$$w = \frac{dV}{dx} = EI\frac{d^4v}{dx^4} \longrightarrow N^{IV} = 0$$



Calculate K_33:



$$\delta\theta_3 \, M_{33} = \int_L \delta\phi_3 \, M_3 \, dx = \int_L \delta\phi_3 EI \phi_3 \, dx$$

$$= \delta\theta_3 \int_2 \overline{B}_3 \, EI \, B_3 \; dx \, \theta_3$$

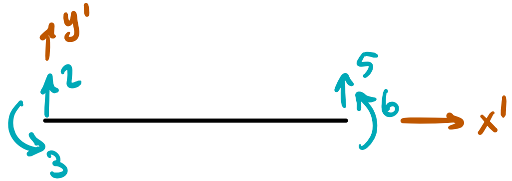
Hence ;

$$M_{33} = \underbrace{\int_L \overline{B}_3 \, EI \, B_3 \; dx}_{K_{33}} \, \theta_3$$

$$K_{33} = \int_L \left(-\frac{4}{L} + \frac{6x}{L^2} \right)^2 EI \, dx = \frac{4EI}{L}$$

$$
\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ F_4 \\ F_5 \\ M_6 \end{Bmatrix} =
\begin{bmatrix}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \theta_3 \\ \Delta_4 \\ \Delta_5 \\ \theta_6 \end{Bmatrix}
$$

THE ELEMENT STIFFNESS MATRIX [Ke']



$$[Ke'] = \int_L \underset{4\times1}{\{B\}}\; \underset{1\times1}{[EI(x)]}\; \underset{1\times4}{\lfloor B \rfloor}\, dx$$

$$\{B\} \longrightarrow \text{Using } v(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 \;\; \& \;\; \text{BCs} \rightarrow v_2;\ \theta_3;\ v_5;\ \theta_6$$

$$\{B\} = \begin{Bmatrix} N_2'' \\ N_3'' \\ N_5'' \\ N_6'' \end{Bmatrix} = \begin{Bmatrix} -\dfrac{6}{L^2} + \dfrac{12x}{L^3} \\ \dfrac{-4}{L} + \dfrac{6x}{L^2} \\ \dfrac{6}{L^2} - 12\dfrac{x}{L^3} \\ \dfrac{6x}{L^2} - \dfrac{2}{L} \end{Bmatrix}$$

$$\& \qquad \phi(x) = \lfloor B(x) \rfloor \begin{Bmatrix} v_2 \\ \theta_3 \\ v_5 \\ \theta_6 \end{Bmatrix}$$

- Let's try a different set of shape functions:

$$v(x) = x\left(1-\left(\frac{x}{L}\right)^2\right)^2 \theta_3$$

$$v'(x) = \left(1 - 6\,\frac{x^2}{L^2} + \frac{5x^4}{L^4}\right)\theta_3$$

$$v''(x) = \left(-\frac{12x}{L^2} + 20\,\frac{x^3}{L^4}\right)\theta_3$$

$$v'''(x) = \left(-\frac{12}{L^2} + 60\,\frac{x^2}{L^4}\right)\theta_3$$

$$v^{IV}(x) = 120\,\frac{x}{L^4}\,\theta_3$$

Check BC's : $v(0) = v(L) = 0$ & $v'(0) = \theta_3$ $v'(L) = 0$

Check Equil. : $M \rightarrow v''$, $V \rightarrow v'''$, $w \rightarrow v^{IV}$

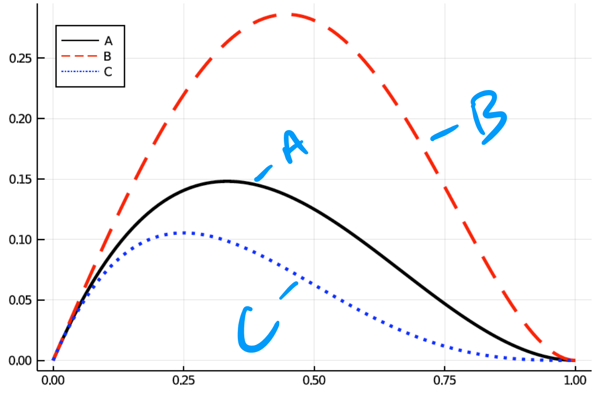
- $K_{33} = \int_L \bar{B}_3\, EI\, B_3\, dx$

$\bar{B}_3$: Higher order B_3 : linear $\rightarrow$ $K_{33} = \frac{4EI}{L}$

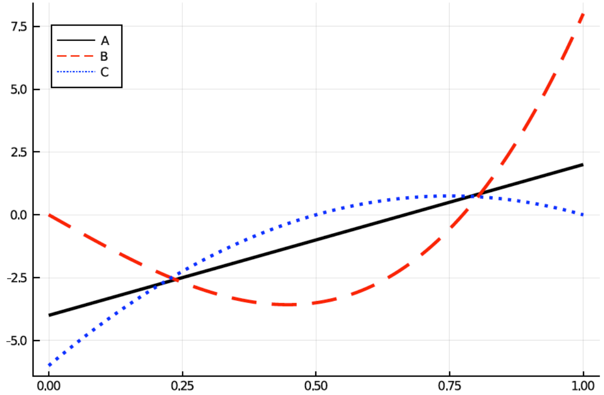
$\bar{B}_3$: Higher order B_3 : Higher order $\rightarrow$ $K_{33} = 9.1\frac{EI}{L}$

COMPARE THREE DIFFERENT SHAPE FUNCTIONS

$A \rightarrow N_3 = x\left(1-\left(\frac{x}{L}\right)\right)^2$ $\quad C \rightarrow N_3 = x\left(1-\left(\frac{x}{2}\right)\right)^3$ $\quad B \rightarrow N_3 = x\left(1-\left(\frac{x}{L}\right)^2\right)^2$

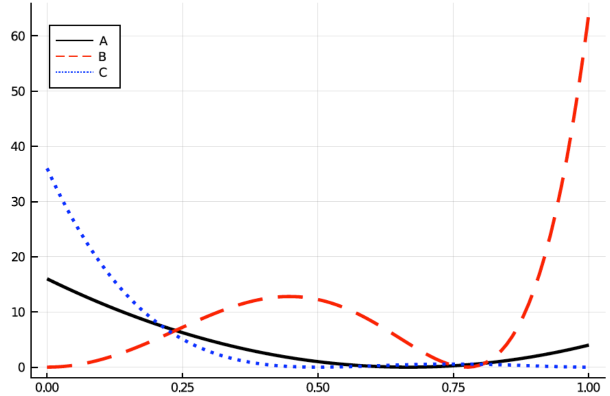


Shape functions



Curvature

$N_3''(x)$



Curvature^2 (~energy)

$N_3''(x) \times N_3''(x)$

OBSERVATION:

- Using non-exact shape functions results in higher computed internal strain energy, leading to a stiffer predicted structural response and underestimation of displacements. !!

REMARKS:

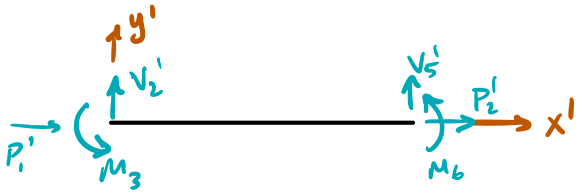
1) The global solution for $\{\Delta_F\}$ & $\{F^{e'}\}$

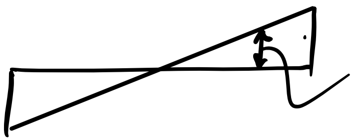
$$[K] = \sum_{e=1}^{N_{elem}} [\Gamma]^T [K^{e'}][\Gamma]$$

$$\{\Delta_F\} = [K_{FF}]^{-1}\{P\}$$

$$\{F^{e'}\} = [K^{e'}]\{\Delta_F^{e'}\}$$

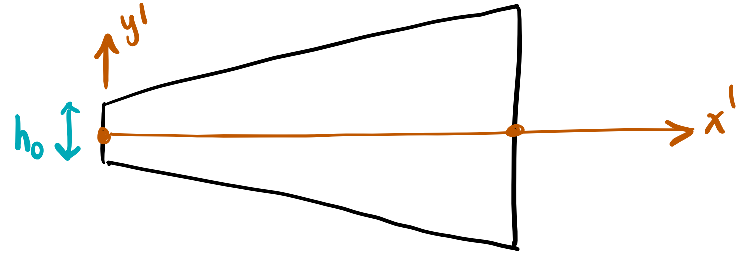
2) Calculation of internal forces: $P'(x)$ & $M'(x)$

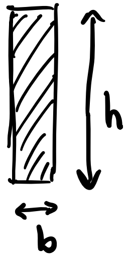




$$M(x) = EI\, \underbrace{B(x)_i}_{\cong\, \phi_i}\, \Delta_i$$

TAPERED BEAM EXAMPLE:





$$h(x) = h_0\left(1 + \frac{x}{L}\right)$$

$$I(x) = \frac{bh_0^3}{12}\left(1 + \frac{x}{L}\right)^3$$

$$= I_0\left(1 + \frac{x}{L}\right)^3$$

$\underline{K_{33}}$:

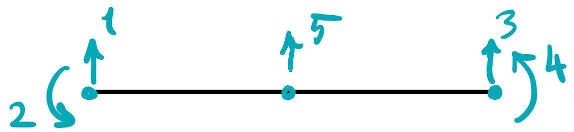
$$v(x) = N_3\theta_3 = x\left(1 - \frac{x}{L}\right)^2 \theta_3$$

$$\phi(x) = B_3\theta_3 = \left(6x/L^2 - 4/L\right)\theta_3$$

Equilibrium ? $\longrightarrow$ $M = EI\phi = EI_0 \underbrace{\left(1 + \frac{x}{L}\right)^3 \cdot \left(\frac{6x}{L^2} - \frac{4}{L}\right)}_{4^{th}\text{ order}} \theta_3$

$$K_{33} = \int_L \bar{B}_3\, EI(x)\, B_3\, dx = 9\, EI_0/L \qquad \text{vs.} \quad \text{exact} = 6.9\, EI_0/L$$

HIGHER ORDER ELEMENTS



$$\begin{Bmatrix} F_1 \\ M_2 \\ F_3 \\ M_4 \\ F_5 \end{Bmatrix} = \begin{bmatrix} & & \\ & k^{e'} & \\ & & \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_2 \\ v_3 \\ \theta_4 \\ v_5 \end{Bmatrix}$$

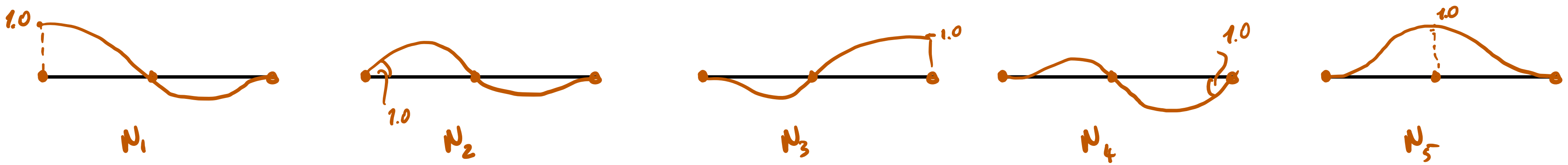
$$v(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \underbrace{a_5 x^4}_{4^{th}\text{ order}}$$

$$\phi(x) = \frac{d^2 v}{dx^2} = 2a_3 + 6a_4 x + 12 a_5 x^2$$

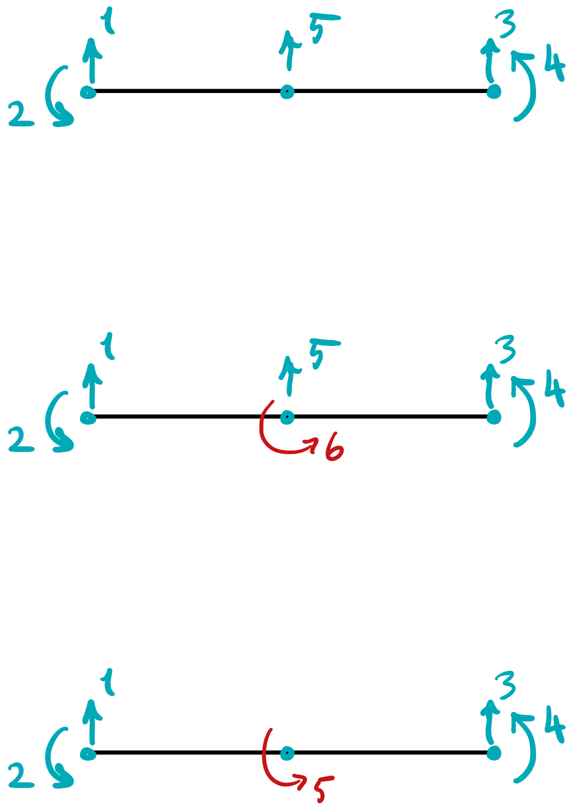
BCs: $v(0) = v_1$ $\quad v(L/2) = v_5$ $\quad v(L) = v_3$

$v'(0) = \theta_2$ $\quad v'(L) = \theta_4$

Regroup: $v(x) = N_1 v_1 + N_2 \theta_2 + N_3 v_3 + N_4 \theta_4 + N_5 v_5$



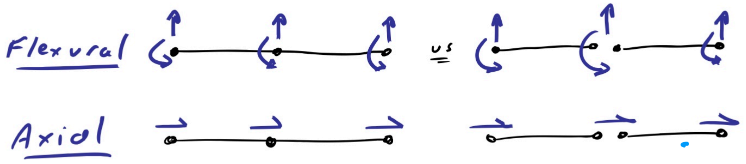
HIGHER ORDER ELEMENTS



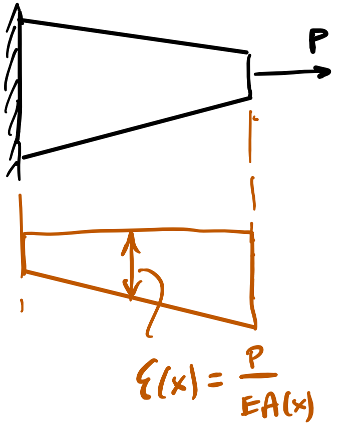
More elements vs. H.O. elements

- More elements increases size of [K]
- H.O. elements increases HBW

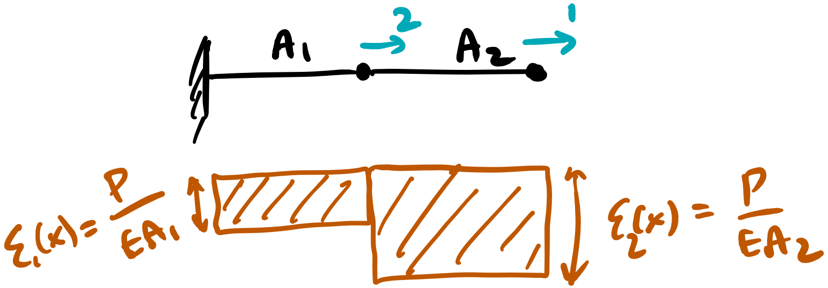
Which is more efficient?



For example: AXIAL DEFORMATION OF A TAPERED MEMBER



- Use two linear elements:



- Use one H.O. element: