Lecture Notes for Chapter 7: Quicksort

Chapter 7 overview

[The treatment in the second edition differs from that of the first edition. We use a different partitioning method—known as "Lomuto partitioning"—in the second edition, rather than the "Hoare partitioning" used in the first edition. Using Lomuto partitioning helps simplify the analysis, which uses indicator random variables in the second edition.]

Quicksort

- Worst-case running time: $\Theta(n^2)$.
- Expected running time: $\Theta(n \lg n)$.
- Constants hidden in $\Theta(n \lg n)$ are small.
- Sorts in place.

Description of quicksort

Quicksort is based on the three-step process of divide-and-conquer.

- To sort the subarray $A[p \,..\, r]$:

  Divide: Partition $A[p \,..\, r]$, into two (possibly empty) subarrays $A[p \,..\, q-1]$ and $A[q+1 \,..\, r]$, such that each element in the first subarray $A[p \,..\, q-1]$ is $\le A[q]$ and $A[q]$ is $\le$ each element in the second subarray $A[q+1 \,..\, r]$.

  Conquer: Sort the two subarrays by recursive calls to QUICKSORT.

  Combine: No work is needed to combine the subarrays, because they are sorted in place.

- Perform the divide step by a procedure PARTITION, which returns the index q that marks the position separating the subarrays.

```
QUICKSORT(A, p, r)
if p < r
  then q ← PARTITION(A, p, r)
       QUICKSORT(A, p, q − 1)
       QUICKSORT(A, q + 1, r)
```

Initial call is QUICKSORT$(A, 1, n)$.

Partitioning

Partition subarray $A[p\,..\,r]$ by the following procedure:

```
PARTITION(A, p, r)
x ← A[r]
i ← p − 1
for j ← p to r − 1
    do if A[j] ≤ x
          then i ← i + 1
               exchange A[i] ↔ A[j]
exchange A[i + 1] ↔ A[r]
return i + 1
```

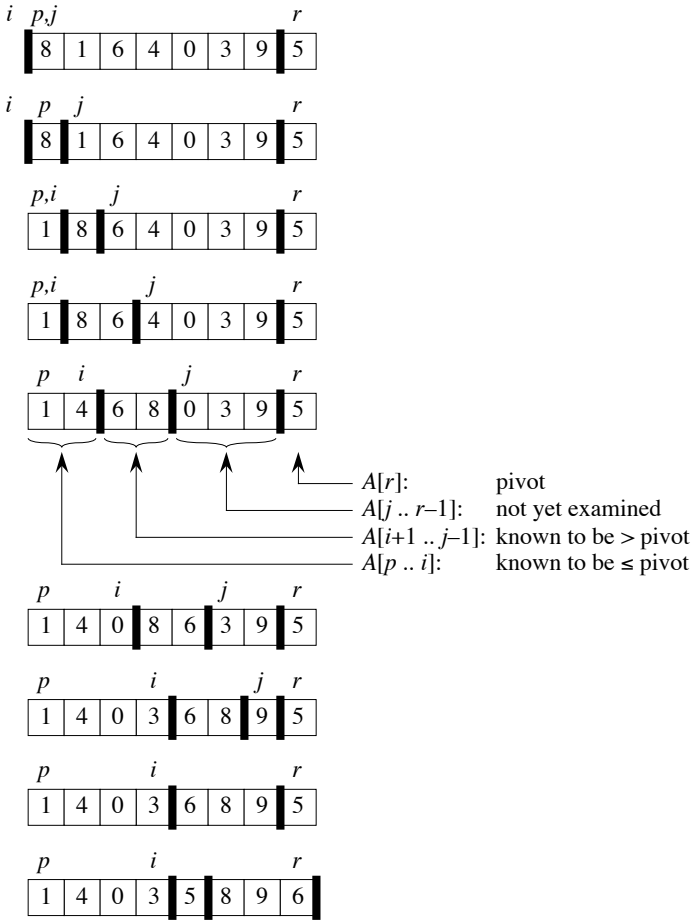
- PARTITION always selects the last element $A[r]$ in the subarray $A[p\,..\,r]$ as the ***pivot***—the element around which to partition.
- As the procedure executes, the array is partitioned into four regions, some of which may be empty:

  Loop invariant:

  1. All entries in $A[p\,..\,i]$ are $\le$ pivot.
  2. All entries in $A[i + 1\,..\,j - 1]$ are $>$ pivot.
  3. $A[r] =$ pivot.

  It's not needed as part of the loop invariant, but the fourth region is $A[j\,..\,r-1]$, whose entries have not yet been examined, and so we don't know how they compare to the pivot.

Example: On an 8-element subarray.



[The index j disappears because it is no longer needed once the **for** *loop is exited.]*

Correctness: Use the loop invariant to prove correctness of PARTITION:

Initialization: Before the loop starts, all the conditions of the loop invariant are satisfied, because r is the pivot and the subarrays $A[p\,..\,i]$ and $A[i+1\,..\,j-1]$ are empty.

Maintenance: While the loop is running, if $A[j] \le$ pivot, then $A[j]$ and $A[i+1]$ are swapped and then i and j are incremented. If $A[j] >$ pivot, then increment only j.

Termination: When the loop terminates, $j = r$, so all elements in A are partitioned into one of the three cases: $A[p\,..\,i] \le$ pivot, $A[i+1\,..\,r-1] >$ pivot, and $A[r] =$ pivot.

The last two lines of PARTITION move the pivot element from the end of the array to between the two subarrays. This is done by swapping the pivot and the first element of the second subarray, i.e., by swapping $A[i+1]$ and $A[r]$.

Time for partitioning: $\Theta(n)$ to partition an n-element subarray.

Performance of quicksort

The running time of quicksort depends on the partitioning of the subarrays:

- If the subarrays are balanced, then quicksort can run as fast as mergesort.
- If they are unbalanced, then quicksort can run as slowly as insertion sort.

Worst case

- Occurs when the subarrays are completely unbalanced.
- Have 0 elements in one subarray and $n-1$ elements in the other subarray.
- Get the recurrence
$$\begin{aligned} T(n) &= T(n-1) + T(0) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \ . \end{aligned}$$
- Same running time as insertion sort.
- In fact, the worst-case running time occurs when quicksort takes a sorted array as input, but insertion sort runs in $O(n)$ time in this case.

Best case

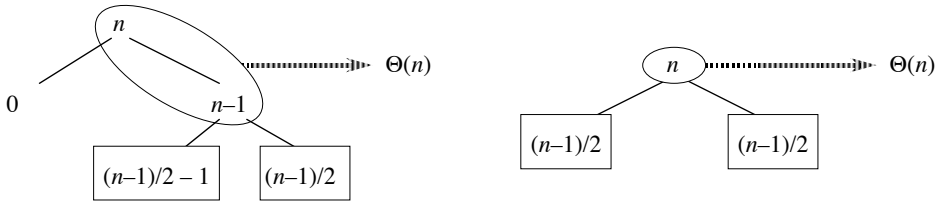
- Occurs when the subarrays are completely balanced every time.
- Each subarray has $\leq n/2$ elements.
- Get the recurrence
$$\begin{aligned} T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \lg n) \ . \end{aligned}$$

Balanced partitioning

- Quicksort's average running time is much closer to the best case than to the worst case.
- Imagine that PARTITION always produces a 9-to-1 split.
- Get the recurrence
$$\begin{aligned} T(n) &\leq T(9n/10) + T(n/10) + \Theta(n) \\ &= O(n \lg n) \ . \end{aligned}$$
- Intuition: look at the recursion tree.
  - It's like the one for $T(n) = T(n/3) + T(2n/3) + O(n)$ in Section 4.2.
  - Except that here the constants are different; we get $\log_{10} n$ full levels and $\log_{10/9} n$ levels that are nonempty.
  - As long as it's a constant, the base of the log doesn't matter in asymptotic notation.
  - Any split of constant proportionality will yield a recursion tree of depth $\Theta(\lg n)$.

Intuition for the average case

- Splits in the recursion tree will not always be constant.
- There will usually be a mix of good and bad splits throughout the recursion tree.
- To see that this doesn't affect the asymptotic running time of quicksort, assume that levels alternate between best-case and worst-case splits.



- The extra level in the left-hand figure only adds to the constant hidden in the Θ-notation.
- There are still the same number of subarrays to sort, and only twice as much work was done to get to that point.
- Both figures result in $O(n \lg n)$ time, though the constant for the figure on the left is higher than that of the figure on the right.

Randomized version of quicksort

- We have assumed that all input permutations are equally likely.
- This is not always true.
- To correct this, we add randomization to quicksort.
- We could randomly permute the input array.
- Instead, we use ***random sampling***, or picking one element at random.
- Don't always use $A[r]$ as the pivot. Instead, randomly pick an element from the subarray that is being sorted.

We add this randomization by not always using $A[r]$ as the pivot, but instead randomly picking an element from the subarray that is being sorted.

RANDOMIZED-PARTITION(A, p, r)
$i \leftarrow$ RANDOM(p, r)
exchange $A[r] \leftrightarrow A[i]$
return PARTITION(A, p, r)

Randomly selecting the pivot element will, on average, cause the split of the input array to be reasonably well balanced.

RANDOMIZED-QUICKSORT(A, p, r)
if $p < r$
  then $q \leftarrow$ RANDOMIZED-PARTITION(A, p, r)
    RANDOMIZED-QUICKSORT$(A, p, q-1)$
    RANDOMIZED-QUICKSORT$(A, q+1, r)$

Randomization of quicksort stops any specific type of array from causing worst-case behavior. For example, an already-sorted array causes worst-case behavior in non-randomized QUICKSORT, but not in RANDOMIZED-QUICKSORT.

Analysis of quicksort

We will analyze

- the worst-case running time of QUICKSORT and RANDOMIZED-QUICKSORT (the same), and
- the expected (average-case) running time of RANDOMIZED-QUICKSORT.

Worst-case analysis

We will prove that a worst-case split at every level produces a worst-case running time of $O(n^2)$.

- Recurrence for the worst-case running time of QUICKSORT:
  $T(n) = \max_{0 \le q \le n-1} (T(q) + T(n-q-1)) + \Theta(n)$.
- Because PARTITION produces two subproblems, totaling size $n-1$, q ranges from 0 to $n-1$.
- ***Guess:*** $T(n) \le cn^2$, for some c.
- Substituting our guess into the above recurrence:
  $$\begin{aligned} T(n) &\le \max_{0 \le q \le n-1} (cq^2 + c(n-q-1)^2) + \Theta(n) \\ &= c \cdot \max_{0 \le q \le n-1} (q^2 + (n-q-1)^2) + \Theta(n) \ . \end{aligned}$$
- The maximum value of $(q^2 + (n-q-1)^2)$ occurs when q is either 0 or $n-1$. (Second derivative with respect to q is positive.) This means that
  $$\begin{aligned} \max_{0 \le q \le n-1} (q^2 + (n-q-1)^2) &\le (n-1)^2 \\ &= n^2 - 2n + 1 \ . \end{aligned}$$
- Therefore,
  $$\begin{aligned} T(n) &\le cn^2 - c(2n-1) + \Theta(n) \\ &\le cn^2 \qquad \text{if } c(2n-1) \ge \Theta(n) \ . \end{aligned}$$
- Pick c so that $c(2n-1)$ dominates $\Theta(n)$.
- Therefore, the worst-case running time of quicksort is $O(n^2)$.
- Can also show that the recurrence's solution is $\Omega(n^2)$. Thus, the worst-case running time is $\Theta(n^2)$.

Average-case analysis

- The dominant cost of the algorithm is partitioning.
- PARTITION removes the pivot element from future consideration each time.
- Thus, PARTITION is called at most n times.
- QUICKSORT recurses on the partitions.
- The amount of work that each call to PARTITION does is a constant plus the number of comparisons that are performed in its **for** loop.
- Let $X =$ the total number of comparisons performed in all calls to PARTITION.
- Therefore, the total work done over the entire execution is $O(n + X)$.

We will now compute a bound on the overall number of comparisons.

For ease of analysis:

- Rename the elements of A as $z_1, z_2, \ldots, z_n$, with z_i being the ith smallest element.
- Define the set $Z_{ij} = \{z_i, z_{i+1}, \ldots, z_j\}$ to be the set of elements between z_i and z_j, inclusive.

Each pair of elements is compared at most once, because elements are compared only to the pivot element, and then the pivot element is never in any later call to PARTITION.

Let $X_{ij} = \mathrm{I}\{z_i \text{ is compared to } z_j\}$.
(Considering whether z_i is compared to z_j at any time during the entire quicksort algorithm, not just during one call of PARTITION.)

Since each pair is compared at most once, the total number of comparisons performed by the algorithm is

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \ .$$

Take expectations of both sides, use Lemma 5.1 and linearity of expectation:

$$\begin{aligned} \mathrm{E}[X] &= \mathrm{E}\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathrm{E}[X_{ij}] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \text{ is compared to } z_j\} \ . \end{aligned}$$

Now all we have to do is find the probability that two elements are compared.

- Think about when two elements are *not* compared.
  - For example, numbers in separate partitions will not be compared.
  - In the previous example, $\langle 8, 1, 6, 4, 0, 3, 9, 5\rangle$ and the pivot is 5, so that none of the set $\{1, 4, 0, 3\}$ will ever be compared to any of the set $\{8, 6, 9\}$.

- Once a pivot x is chosen such that $z_i < x < z_j$, then z_i and z_j will never be compared at any later time.
- If either z_i or z_j is chosen before any other element of Z_{ij}, then it will be compared to all the elements of Z_{ij}, except itself.
- The probability that z_i is compared to z_j is the probability that either z_i or z_j is the first element chosen.
- There are $j-i+1$ elements, and pivots are chosen randomly and independently. Thus, the probability that any particular one of them is the first one chosen is $1/(j-i+1)$.

Therefore,

$$\begin{aligned}
\Pr\{z_i \text{ is compared to } z_j\} &= \Pr\{z_i \text{ or } z_j \text{ is the first pivot chosen from } Z_{ij}\} \\
&= \Pr\{z_i \text{ is the first pivot chosen from } Z_{ij}\} \\
&\qquad + \Pr\{z_j \text{ is the first pivot chosen from } Z_{ij}\} \\
&= \frac{1}{j-i+1} + \frac{1}{j-i+1} \\
&= \frac{2}{j-i+1} .
\end{aligned}$$

[The second line follows because the two events are mutually exclusive.]

Substituting into the equation for $\mathrm{E}[X]$:

$$\mathrm{E}[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} .$$

Evaluate by using a change in variables ($k = j-i$) and the bound on the harmonic series in equation (A.7):

$$\begin{aligned}
\mathrm{E}[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} \\
&< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} \\
&= \sum_{i=1}^{n-1} O(\lg n) \\
&= O(n \lg n) .
\end{aligned}$$

So the expected running time of quicksort, using RANDOMIZED-PARTITION, is $O(n \lg n)$.

Solutions for Chapter 7: Quicksort

Solution to Exercise 7.2-3

PARTITION does a "worst-case partitioning" when the elements are in decreasing order. It reduces the size of the subarray under consideration by only 1 at each step, which we've seen has running time $\Theta(n^2)$.

In particular, PARTITION, given a subarray $A[p\,..\,r]$ of distinct elements in decreasing order, produces an empty partition in $A[p\,..\,q-1]$, puts the pivot (originally in $A[r]$) into $A[p]$, and produces a partition $A[p+1\,..\,r]$ with only one fewer element than $A[p\,..\,r]$. The recurrence for QUICKSORT becomes $T(n) = T(n-1) + \Theta(n)$, which has the solution $T(n) = \Theta(n^2)$.

Solution to Exercise 7.2-5

The minimum depth follows a path that always takes the smaller part of the partition—i.e., that multiplies the number of elements by α. One iteration reduces the number of elements from n to αn, and i iterations reduces the number of elements to $\alpha^i n$. At a leaf, there is just one remaining element, and so at a minimum-depth leaf of depth m, we have $\alpha^m n = 1$. Thus, $\alpha^m = 1/n$. Taking logs, we get $m \lg \alpha = -\lg n$, or $m = -\lg n / \lg \alpha$.

Similarly, maximum depth corresponds to always taking the larger part of the partition, i.e., keeping a fraction $1-\alpha$ of the elements each time. The maximum depth M is reached when there is one element left, that is, when $(1-\alpha)^M n = 1$. Thus, $M = -\lg n / \lg(1-\alpha)$.

All these equations are approximate because we are ignoring floors and ceilings.

Solution to Exercise 7.3-1

We may be interested in the worst-case performance, but in that case, the randomization is irrelevant: it won't improve the worst case. What randomization can do is make the chance of encountering a worst-case scenario small.

Solution to Exercise 7.4-2

To show that quicksort's best-case running time is $\Omega(n \lg n)$, we use a technique similar to the one used in Section 7.4.1 to show that its worst-case running time is $O(n^2)$.

Let $T(n)$ be the best-case time for the procedure QUICKSORT on an input of size n. We have the recurrence

$$T(n) = \min_{1 \le q \le n-1} (T(q) + T(n-q-1)) + \Theta(n) .$$

We guess that $T(n) \ge cn \lg n$ for some constant c. Substituting this guess into the recurrence, we obtain

$$\begin{aligned} T(n) &\ge \min_{1 \le q \le n-1} (cq \lg q + c(n-q-1) \lg(n-q-1)) + \Theta(n) \\ &= c \cdot \min_{1 \le q \le n-1} (q \lg q + (n-q-1) \lg(n-q-1)) + \Theta(n) . \end{aligned}$$

As we'll show below, the expression $q \lg q + (n-q-1) \lg(n-q-1)$ achieves a minimum over the range $1 \le q \le n-1$ when $q = n-q-1$, or $q = (n-1)/2$, since the first derivative of the expression with respect to q is 0 when $q = (n-1)/2$ and the second derivative of the expression is positive. (It doesn't matter that q is not an integer when n is even, since we're just trying to determine the minimum value of a function, knowing that when we constrain q to integer values, the function's value will be no lower.)

Choosing $q = (n-1)/2$ gives us the bound

$$\begin{aligned} &\min_{1 \le q \le n-1} (q \lg q + (n-q-1) \lg(n-q-1) \\ &\qquad \ge \frac{n-1}{2} \lg \frac{n-1}{2} + \left(n - \frac{n-1}{2} - 1\right) \lg \left(n - \frac{n-1}{2} - 1\right) \\ &\qquad = (n-1) \lg \frac{n-1}{2} . \end{aligned}$$

Continuing with our bounding of $T(n)$, we obtain, for $n \ge 2$,

$$\begin{aligned} T(n) &\ge c(n-1) \lg \frac{n-1}{2} + \Theta(n) \\ &= c(n-1) \lg(n-1) - c(n-1) + \Theta(n) \\ &= cn \lg(n-1) - c \lg(n-1) - c(n-1) + \Theta(n) \\ &\ge cn \lg(n/2) - c \lg(n-1) - c(n-1) + \Theta(n) \qquad \text{(since } n \ge 2\text{)} \\ &= cn \lg n - cn - c \lg(n-1) - cn + c + \Theta(n) \\ &= cn \lg n - (2cn + c \lg(n-1) - c) + \Theta(n) \\ &\ge cn \lg n , \end{aligned}$$

since we can pick the constant c small enough so that the $\Theta(n)$ term dominates the quantity $2cn + c \lg(n-1) - c$. Thus, the best-case running time of quicksort is $\Omega(n \lg n)$.

Letting $f(q) = q \lg q + (n-q-1) \lg(n-q-1)$, we now show how to find the minimum value of this function in the range $1 \le q \le n-1$. We need to find the value of q for which the derivative of f with respect to q is 0. We rewrite this function as

$$f(q) = \frac{q \ln q + (n-q-1)\ln(n-q-1)}{\ln 2} ,$$

and so

$$\begin{aligned} f'(q) &= \frac{d}{dq}\left(\frac{q \ln q + (n-q-1)\ln(n-q-1)}{\ln 2}\right) \\ &= \frac{\ln q + 1 - \ln(n-q-1) - 1}{\ln 2} \\ &= \frac{\ln q - \ln(n-q-1)}{\ln 2} . \end{aligned}$$

The derivative $f'(q)$ is 0 when $q = n-q-1$, or when $q = (n-1)/2$. To verify that $q = (n-1)/2$ is indeed a minimum (not a maximum or an inflection point), we need to check that the second derivative of f is positive at $q = (n-1)/2$:

$$\begin{aligned} f''(q) &= \frac{d}{dq}\left(\frac{\ln q - \ln(n-q-1)}{\ln 2}\right) \\ &= \frac{1}{\ln 2}\left(\frac{1}{q} + \frac{1}{n-q-1}\right) \\ f''\left(\frac{n-1}{2}\right) &= \frac{1}{\ln 2}\left(\frac{2}{n-1} + \frac{2}{n-1}\right) \\ &= \frac{1}{\ln 2} \cdot \frac{4}{n-1} \\ &> 0 \qquad \text{(since } n \ge 2) . \end{aligned}$$

Solution to Problem 7-4

a. QUICKSORT′ does exactly what QUICKSORT does; hence it sorts correctly.

QUICKSORT and QUICKSORT′ do the same partitioning, and then each calls itself with arguments $A, p, q-1$. QUICKSORT then calls itself again, with arguments $A, q+1, r$. QUICKSORT′ instead sets $p \leftarrow q+1$ and performs another iteration of its **while** loop. This executes the same operations as calling itself with $A, q+1, r$, because in both cases, the first and third arguments (A and r) have the same values as before, and p has the old value of $q+1$.

b. The stack depth of QUICKSORT′ will be $\Theta(n)$ on an n-element input array if there are $\Theta(n)$ recursive calls to QUICKSORT′. This happens if every call to PARTITION(A, p, r) returns $q = r$. The sequence of recursive calls in this scenario is

QUICKSORT′$(A, 1, n)$,
QUICKSORT′$(A, 1, n-1)$,
QUICKSORT′$(A, 1, n-2)$,
$\vdots$
QUICKSORT′$(A, 1, 1)$.

Any array that is already sorted in increasing order will cause QUICKSORT′ to behave this way.

c. The problem demonstrated by the scenario in part (b) is that each invocation of QUICKSORT′ calls QUICKSORT′ again with almost the same range. To avoid such behavior, we must change QUICKSORT′ so that the recursive call is on a smaller interval of the array. The following variation of QUICKSORT′ checks which of the two subarrays returned from PARTITION is smaller and recurses on the smaller subarray, which is at most half the size of the current array. Since the array size is reduced by at least half on each recursive call, the number of recursive calls, and hence the stack depth, is $\Theta(\lg n)$ in the worst case. Note that this method works no matter how partitioning is performed (as long as the PARTITION procedure has the same functionality as the procedure given in Section 7.1).

```
QUICKSORT''(A, p, r)
while p < r
    do ▷ Partition and sort the small subarray first
       q ← PARTITION(A, p, r)
       if q − p < r − q
         then QUICKSORT''(A, p, q − 1)
              p ← q + 1
         else QUICKSORT''(A, q + 1, r)
              r ← q − 1
```

The expected running time is not affected, because exactly the same work is done as before: the same partitions are produced, and the same subarrays are sorted.