Last Time:

- Dynamics
- Orbital Elements
- Sim / RK Methods

Today:

- Spacecraft Navigation
- Orbit Determination

Guidance - How to get there. Trajectory design / planning
Navigation - Where are we? State estimation
Control - Executing the plan. Tracking / Stabilizing

* Orbit Determination

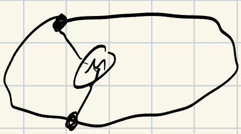
  - Special case of state estimation focusing on position + velocity of spacecraft

  - Very old problem: Originally Lambert's Problem (1761)

* Lambert's Problem

  - Given two position measurements at different times, fit a Keplerian orbit.

  - Reduces to basic geometry: fit an ellipse knowing 2 points + a focus.



* Issues:

  - Only uses 2 data points
  - Only handles position measurements
  - Can only deal with idealized Keplerian dynamics (no drag, J2, maneuvers, etc.)

- Still widely used for initial/coarse OD and maneuver planning.

- More modern solutions use nonlinear least-squares and/or filtering techiques with nonlinear dynamics + measurement models.

* Linear Least Squares

- Assume have measurements y related to some "hidden state" x by a linear function:

$$\underbrace{\begin{bmatrix} y \end{bmatrix}}_{\mathbb{R}^m} = \underbrace{\begin{bmatrix} C \end{bmatrix}}_{\mathbb{R}^{m \times n}} \underbrace{\begin{bmatrix} x \end{bmatrix}}_{\mathbb{R}^n} + w \quad , \quad w \sim N(0, W)$$

- We can't solve this exactly, so we'll look for a "least-squares" solution that minimizes squared error:

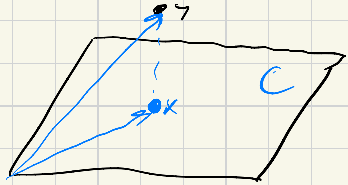
$$\min_x \frac{1}{2}(y - Cx)^T (y - Cx)$$

$$\min \frac{1}{2} x^T C^T C x - \frac{1}{2} y^T C x - \frac{1}{2} x^T C^T y + \frac{1}{2} y^T y$$

$$\Rightarrow \nabla_x (\ ) = C^T C x - C^T y = 0$$

$$\Rightarrow \boxed{x = (C^T C)^{-1} C^T y} \leftarrow \text{"Normal Equations" aka "Pseudo inverse"}$$

- Geometrically, this is just orthogonal projection onto columns of C ("feasible measurements")



- What is we know the noise covariance?

$$\min_x \frac{1}{2}(y-Cx)^T W^{-1}(y-Cx)$$

Inverse covariance $\Rightarrow$ trust measurements with less noise more

$$\Rightarrow \boxed{x = (C^T W^{-1} C)^{-1} C^T W^{-1} y}$$

$\leftarrow$ "Weighted least squares"

- How do I know if x is observable given y?

$$C^T C \text{ invertible} \iff C \text{ has full column rank}$$

* Nonlinear Least-Squares

  - Same thing, but now y is a nonlinear function of x:

$$y = g(x) + w$$

  - Solution technique: "Gauss-Newton Method"

    1) Make a guess for x
    2) Taylor expand to 1st order:

$$\cancel{y} + \Delta y = g(x+\Delta x) \approx \cancel{g(x)} + \underbrace{\frac{\partial g}{\partial x}}_{C} \Delta x$$

$$\Rightarrow \Delta y = C\Delta x$$

} Linear least-squares problem

3) Calculate Δx correction by solving linear least-squares problem

$$\Delta x = (C^T C)^{-1} C^T \Delta y$$

4) Update $x \leftarrow x + \Delta x$ and repeate until convergence

- Lots of extensions + tricks. Generally a good idea to use library (e.g. MATLAB lsqnonlin, Google Ceres)

* What if I have dynamics and I'm trying to estimate a trajectory $x(t)$ instead of just a single x?

$$\dot{x} = f(x) \longrightarrow x_{n+1} = f(x_n)$$

- We can turn this into standard least squares by just estimating the initial (or final) state:

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} g(x_0) \\ g(x_1) = g(f(x_0)) \\ g(x_2) = g(f(f(x_0))) \end{bmatrix}$$

$$\bar{y} = \bar{g}(x_0)$$

- This works if f is a very good model. What if it's bad?

* Batch Nonlinear State Estimation

- Assume a noisy dynamics model:

$$x_{n+1} = f(x_n) + v_n \quad , \quad v_n \sim N(0, V)$$ "Process noise"

- Assume a noisy measurement model:

$$y_n = g(x_n) + w_n \quad , \quad w_n \sim N(0, W)$$ "Measurement noise"

- Solve the following least-squares problem:

$$\min_{x_{1:N}} \sum_{n=1}^{N} \underbrace{\tfrac{1}{2}(y_n - g(x_n))^T W^{-1} (y_n - g(x_n))}_{\text{"Measurement loss"}}$$

$$+ \underbrace{\tfrac{1}{2}(x_{n+1} - f(x_n))^T V^{-1} (x_{n+1} - f(x_n))}_{\text{"dynamics loss"}}$$

- We can turn this into a standard least-squares problem:

$$r = \underbrace{\begin{bmatrix} \sqrt{W^{-1}}\,(y_0 - g(x_0)) \\ \sqrt{V^{-1}}\,(x_1 - f(x_0)) \\ \vdots \\ \sqrt{W^{-1}}\,(y_N - g(x_N)) \end{bmatrix}}_{\text{"residual"}} = r(X), \quad X = \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix}$$

$\sqrt{W} = \mathrm{chol}(W)$ ← "Cholesky factor"

$$\Rightarrow \min_{X} \tfrac{1}{2} r(X)^T r(X)$$

- Use Gauss-Newton to solve this

- Least-squares toolboxes like lsqnonlin and Ceres want $r(x)$ and $\nabla r(x)$

- For us, $f(x)$ is the orbital dynamics and $g(x)$ maps spacecraft state into unit vectors to landmarks in the camera frame.

* Extra Tips + Tricks

- Solving the normal equations is bad numerically. Instead use a QR factorization.

$$y = Cx = \underbrace{Q}_{\text{Orthogonal}}\underbrace{R}_{\text{Upper-triangular}}x$$

$$\min_x \tfrac{1}{2}(y-QRx)^T(y-QRx)$$

$$\min_x \tfrac{1}{2}x^T R^T \underbrace{\cancel{Q^T Q}}_{= I} R x - y^T QRx$$

$$\min_x \tfrac{1}{2}x^T R^T R x - y^T QRx$$

$$\Rightarrow \quad R^T R x = R^T Q^T y$$

$$\Rightarrow \quad \boxed{Rx = Q^T y}$$ - we avoided squaring C

- Often C is nearly rank deficient $\Rightarrow$ C^TC is poorly conditioned

$\Rightarrow$ add regularization

small scalar $\sim 10^{-4} - 10^{-5}$

$$\min_x \tfrac{1}{2}(y-Cx)^T(y-Cx) + \alpha \tfrac{1}{2}x^T x$$

- In words: If x is not unique, give me the smallest x that is consistent with the data.

$$\Rightarrow \quad x = (C^TC + \alpha I)^{-1} C^T y$$

- Can also do this with QR solution:

$$Q, R = qr\left(\begin{bmatrix} C \\ \sqrt{\alpha} I \end{bmatrix}\right)$$