Carnegie Mellon
School of Computer Science

Monte Carlo Learning

CMU 10-03

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Used Materials

- **Disclaimer**: Much of the material and slides for this lecture were borrowed from Russ who in turn borrowed some materials from Rich Sutton's class and David Silver's class on Reinforcement Learning.

Summary so far

- So far, to estimate value functions we have been using dynamic programming with *known* rewards and dynamics functions:

$$v_{[k+1]}(s) = \sum_a \pi(a \mid s) \left(r(s,a) + \gamma \sum_{s'} p(s' \mid s, a) v_{[k]}(s') \right), \forall s$$

$$v_{[k+1]}(s) = \max_{a \in \mathcal{A}} \left(r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s' \mid s, a) v_{[k]}(s') \right), \forall s$$

Q: Was our agent interacting with the world? Was our agent *learning* something?

Coming up

- So far, to estimate value functions we have been using dynamic programming with *known* rewards and dynamics functions:

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- Next: estimate value functions and policies from interaction experience, without known rewards or dynamics.

- How? By sampling all the way. Instead of probabilities distributions to compute expectations, we will use empirical expectations by averaging sampled returns!

Monte Carlo (MC) Methods

- Monte Carlo methods are learning methods
  - Experience → values, policy
- Monte Carlo methods learn from complete sampled trajectories and their returns
  - Only defined for episodic tasks
  - All episodes must terminate
- Monte Carlo uses the simplest possible idea: value = mean return

Monte-Carlo Policy Evaluation

- Goal: learn $v_\pi(s)$ from episodes of experience under policy π

$$S_1, A_1, R_2, ..., S_k \sim \pi$$

- Remember that the return is the total discounted reward:

$$G_t = R_{t+1} + \gamma R_{t+2} + ... + \gamma^{T-1} R_T$$

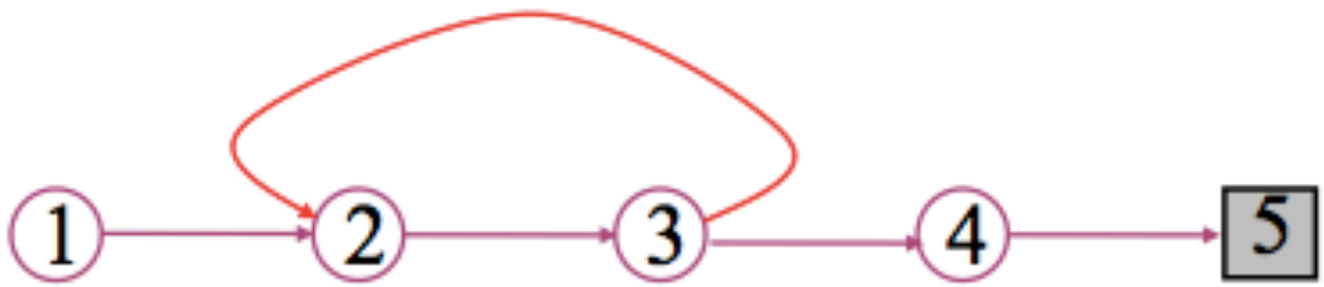
- Remember that the value function is the expected return:

$$v_\pi(s) = \mathbb{E}_\pi \left[G_t \mid S_t = s \right]$$

- Monte-Carlo policy evaluation **uses empirical mean return instead of expected return**

Monte-Carlo Policy Evaluation

- Goal: learn $v_\pi(s)$ from episodes of experience under policy π
- Idea: Average returns observed after visits to s:



- Every-Visit MC: average returns for every time s is visited in an episode
- First-visit MC: average returns only for first time s is visited in an episode
- Both converge asymptotically

First-Visit MC Policy Evaluation

- To evaluate state s
- The first time-step t that state s is visited in an episode,
  - Increment counter: $N(s) \leftarrow N(s) + 1$
  - Increment total return: $S(s) \leftarrow S(s) + G_t$
- Value is estimated by mean return $V(s) = S(s)/N(s)$
- By law of large numbers $V(s) \rightarrow v_\pi(s)$ as $N(s) \rightarrow \infty$

Law of large numbers

Every-Visit MC Policy Evaluation

- To evaluate state s
- Every time-step t that state s is visited in an episode,
  - Increment counter: $N(s) \leftarrow N(s) + 1$
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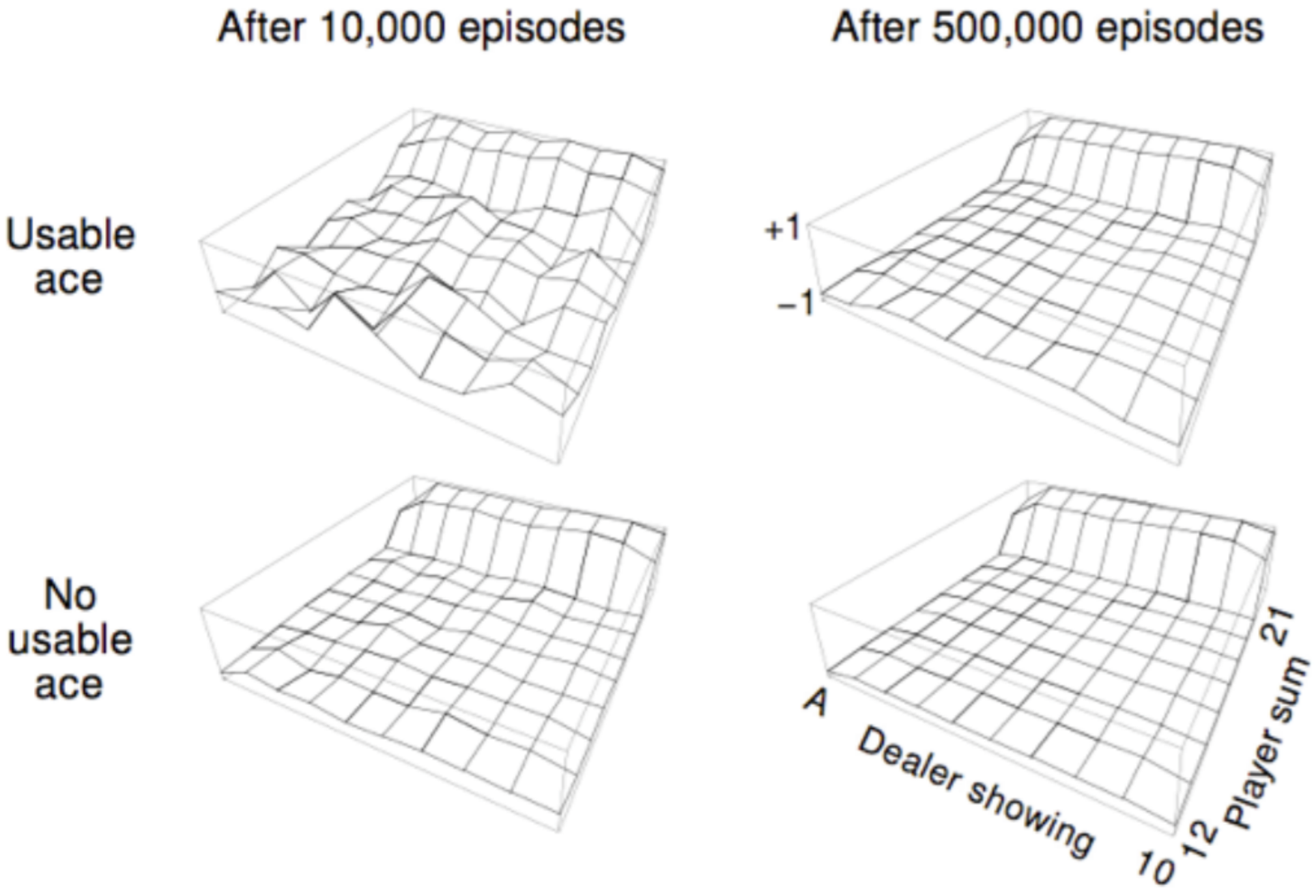
Blackjack Example

- Objective: Have your card sum be greater than the dealer's without exceeding 21.
- States (200 of them):
  - current sum (12-21)
  - dealer's showing card (ace-10)
  - do I have a useable ace?

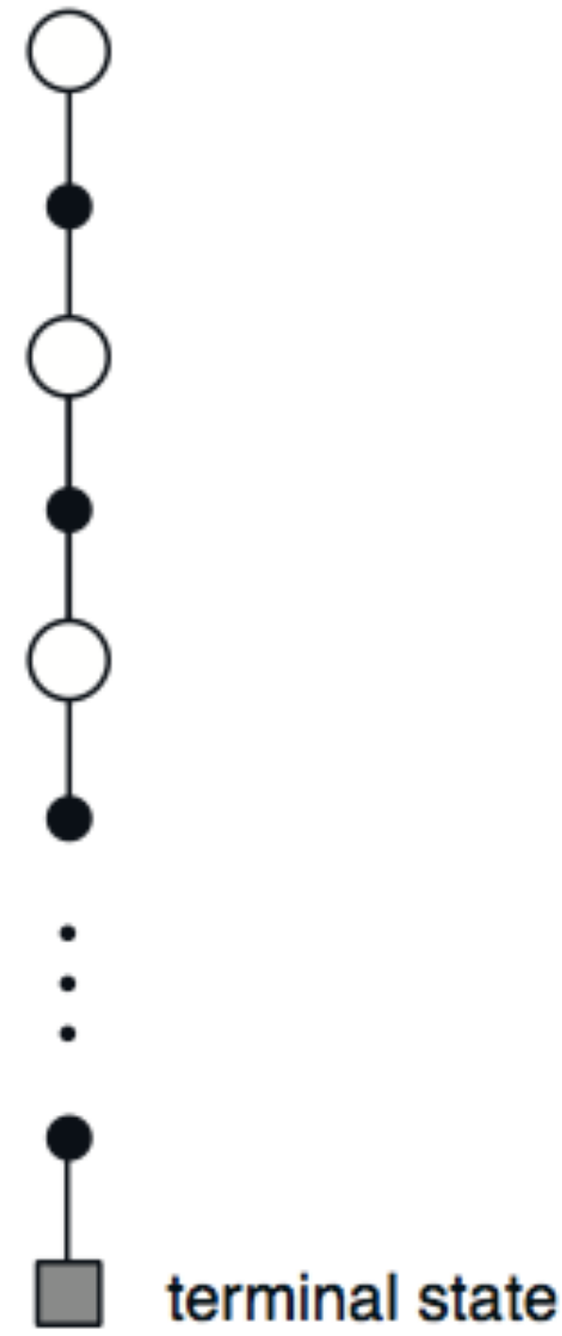
- Reward: +1 for winning, 0 for a draw, -1 for losing
- Actions: stick (stop receiving cards), hit (receive another card)
- Policy: Stick if my sum is 20 or 21, else hit
- No discounting ($\gamma=1$)

Learned Blackjack State-Value Functions



Backup Diagram for Monte Carlo

- Entire rest of episode included
- **Only one choice considered at each state (unlike DP)**
  - thus, there will be an explore/exploit dilemma
- Does not bootstrap from successor state's values (unlike DP)
- Value is estimated by mean return



Incremental Mean

- The mean μ_1, μ_2, ... of a sequence x_1, x_2, ... can be computed incrementally:

$$\mu_k = \frac{1}{k}\sum_{j=1}^{k} x_j$$

$$= \frac{1}{k}\left(x_k + \sum_{j=1}^{k-1} x_j\right)$$

$$= \frac{1}{k}\left(x_k + (k-1)\mu_{k-1}\right)$$

$$= \mu_{k-1} + \frac{1}{k}\left(x_k - \mu_{k-1}\right)$$

Incremental Monte Carlo Updates

- Update V(s) incrementally after episode $S_1, A_1, R_2, ..., S_T$
- For each state S_t with return G_t

$$N(S_t) \leftarrow N(S_t) + 1$$

$$V(S_t) \leftarrow V(S_t) + \frac{1}{N(S_t)}(G_t - V(S_t))$$

- In non-stationary problems, it can be useful to track a running mean, i.e. forget old episodes.

$$V(S_t) \leftarrow V(S_t) + \alpha(G_t - V(S_t))$$

Summary so far

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and policy greedification:

$$\pi'(s) = \operatorname*{argmax}_a q_\pi(s,a)$$

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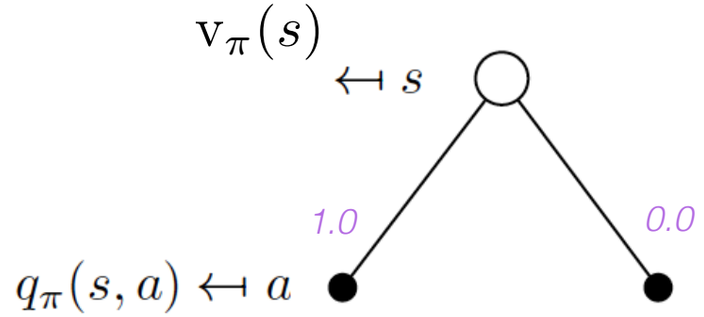
Q: After a **single** action update, does it suffice to update the value of the corresponding state for policy evaluation?



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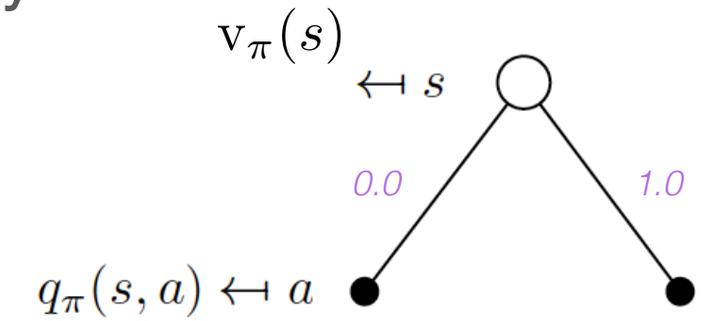

$$v_{\pi'}(s) = q(s, \pi'(s))$$

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    and policy greedification:

$$\pi'(s) = \operatorname*{argmax}_a q_\pi(s,a)$$

  - Initialize value function. Apply value iteration:

$$v_{[k+1]}(s) = \max_{a \in \mathcal{A}} \left(r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s' \mid s,a) v_{[k]}(s') \right), \forall s$$

    and greedify to obtain the optimal policy

Summary so far

- Unknown dynamics: estimate value functions and optimal policies using Monte Carlo
  - **Monte Carlo Prediction**: estimate the value function of a given policy by deploying it, collect episodes and average their returns.

Monte Carlo Prediction

- Update V(s) incrementally after episode $S_1, A_1, R_2, ..., S_T$
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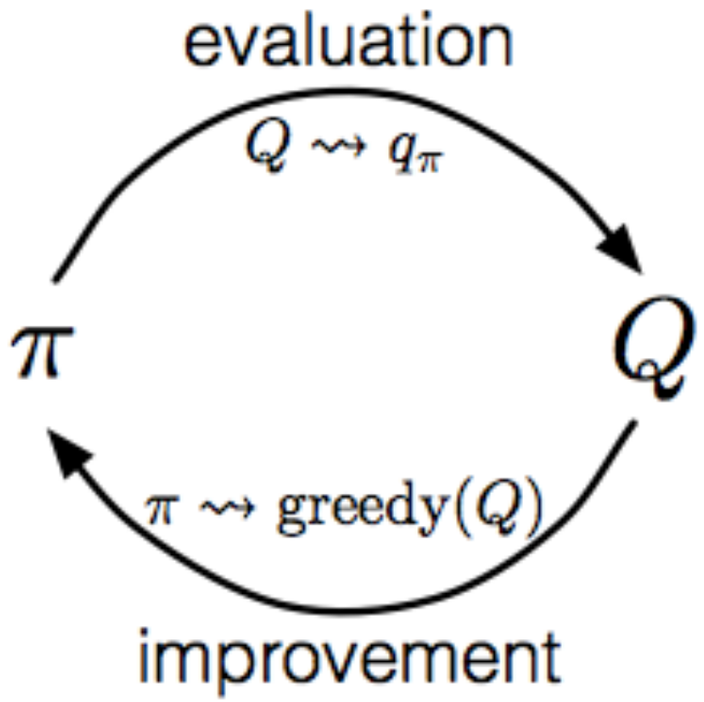
Summary so far

- Unknown dynamics: estimate value functions and optimal policies using Monte Carlo
    - **Monte Carlo Prediction:** estimate the value function of a given policy by deploying it, collect episodes and average their returns.

- **Monte Carlo control**: find optimal policies by interaction

Monte-Carlo Control

$$\pi_0 \xrightarrow{\text{E}} q_{\pi_0} \xrightarrow{\text{I}} \pi_1 \xrightarrow{\text{E}} q_{\pi_1} \xrightarrow{\text{I}} \pi_2 \xrightarrow{\text{E}} \cdots \xrightarrow{\text{I}} \pi_* \xrightarrow{\text{E}} q_*$$



- MC policy iteration step: Policy evaluation using MC methods followed by policy improvement
- Policy improvement step: greedify with respect to value (or action-value) function

Greedy Policy

- For any action-value function q, the corresponding greedy policy is the one that:
  - For each s, deterministically chooses an action with maximal action-value:

$$\pi(s) \doteq \arg\max_a q(s,a).$$

- Policy improvement then can be done by constructing each π_{k+1} as the greedy policy with respect to $q_{\pi k}$.

MC Estimation of Action Values (Q)

- Monte Carlo (MC) is most useful when a model is not available
  - We want to learn q*(s,a)

- $q_\pi(s,a)$ - average return starting from state s and action a following π

$$
\begin{aligned}
q_\pi(s,a) &= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big].
\end{aligned}
$$

- Converges asymptotically if every state-action pair is visited

Q: Is this possible if we are using a deterministic policy?

The Exploration problem

- If we always follow the deterministic policy to collect experience, we will never have the opportunity to see and evaluate (estimate q) of alternative actions…

- ALL learning methods face a dilemma: they seek to learn action values conditioned on subsequent optimal behaviour but they need to act suboptimally in order to explore all actions (to discover the optimal actions). The epxloration-exploitation dilemma.

- Q: Does a learning algorithm know when the optimal policy has been reached to stop exploring?

The Exploration problem

- If we always follow the deterministic policy to collect experience, we will never have the opportunity to see and evaluate (estimate q) of alternative actions…
- ALL learning methods face a dilemma: they seek to learn action values conditioned on subsequent optimal behaviour but they need to act suboptimally in order to explore all actions (to discover the optimal actions). The epxloration-exploitation dilemma.
- Solutions:
    1. exploring starts: Every state-action pair has a non-zero probability of being the starting pair
    2. Give up on deterministic policies and only search over ϵ-soft policies
    3. Off policy: use a different policy to collect experience than the one you care to evaluate

Monte Carlo Exploring Starts

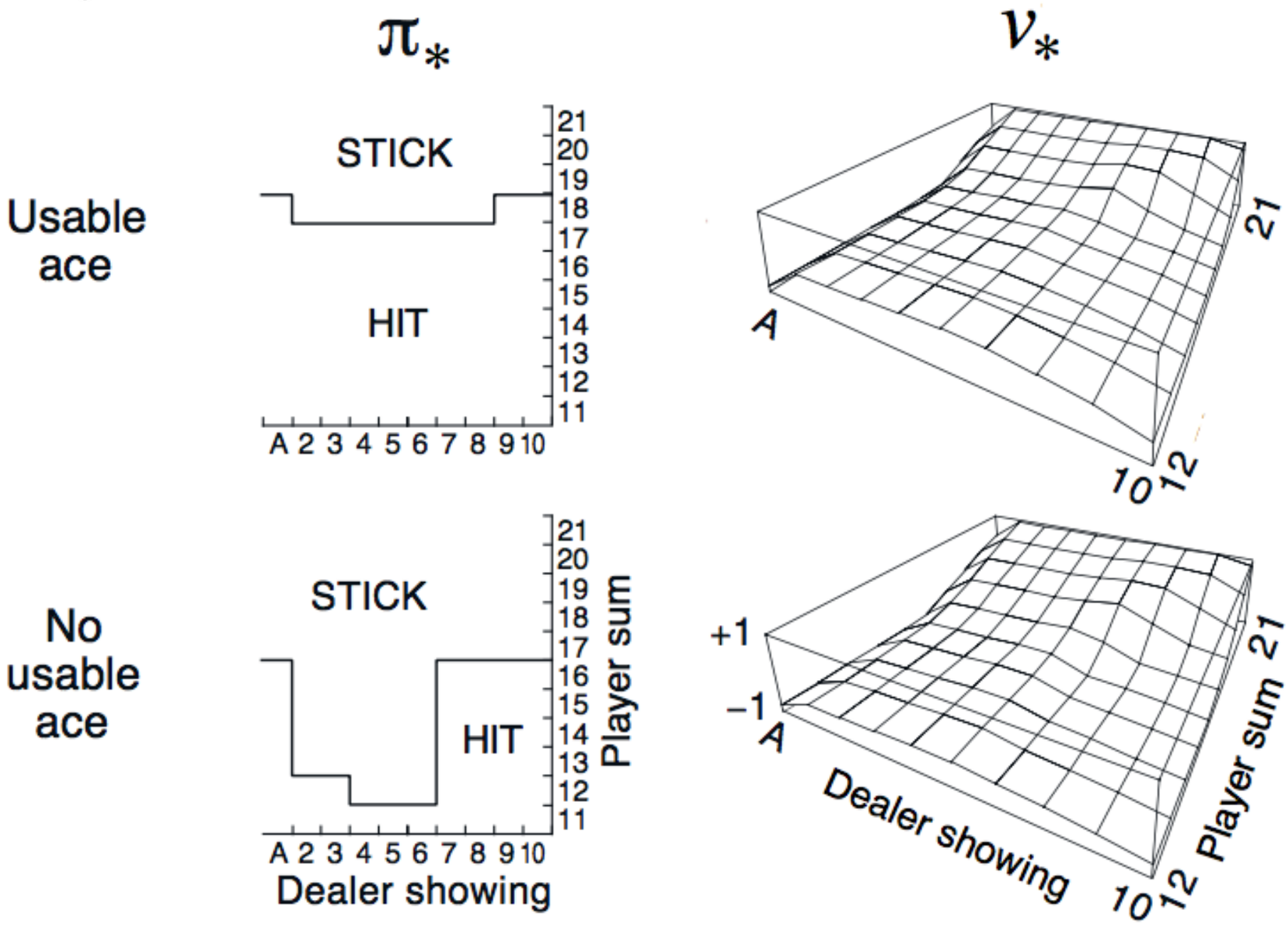
Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
$Q(s,a) \leftarrow$ arbitrary
$\pi(s) \leftarrow$ arbitrary
$Returns(s,a) \leftarrow$ empty list

Fixed point is optimal policy π^*

Repeat forever:
Choose $S_0 \in \mathcal{S}$ and $A_0 \in \mathcal{A}(S_0)$ s.t. all pairs have probability > 0
Generate an episode starting from S_0, A_0, following π
For each pair s, a appearing in the episode:
$G \leftarrow$ return following the first occurrence of s, a
Append G to $Returns(s,a)$
$Q(s,a) \leftarrow \text{average}(Returns(s,a))$
For each s in the episode:
$\pi(s) \leftarrow \arg\max_a Q(s,a)$

Blackjack example continued

- With exploring starts



Convergence of MC Control

- Greedified policy meets the conditions for policy improvement:

$$\begin{aligned} q_{\pi_k}(s, \pi_{k+1}(s)) &= q_{\pi_k}(s, \arg\max_a q_{\pi_k}(s, a)) \\ &= \max_a q_{\pi_k}(s, a) \\ &\geq q_{\pi_k}(s, \pi_k(s)) \\ &\geq v_{\pi_k}(s). \end{aligned}$$

- And thus must be ≥ π_k.

- This assumes exploring starts and infinite number of episodes for MC policy evaluation

On-policy Monte Carlo Control

- On-policy: learn about policy currently executing
- How do we get rid of exploring starts?
  - The policy must be eternally soft: $\pi(a|s) > 0$ for all s and a.
- For example, for ε-soft policy, probability of an action, $\pi(a|s)$,

$$= \underbrace{\frac{\epsilon}{|\mathcal{A}(s)|}}_{\text{non-max}} \quad \text{or} \quad \underbrace{1 - \epsilon + \frac{\epsilon}{|\mathcal{A}(s)|}}_{\text{max (greedy)}}$$

- Similar to GPI: move policy towards greedy policy
- Converges to the best ε-soft policy.

ϵ – soft Policies

- They keep choosing suboptimal actions even when the best one has been discovered
- The second best action is as bad as the worst action
- However, we will stick with them till we figure out better exploration methods later in the course

On-policy Monte Carlo Control

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
$Q(s,a) \leftarrow$ arbitrary
$Returns(s,a) \leftarrow$ empty list
$\pi(a|s) \leftarrow$ an arbitrary ε-soft policy

Repeat forever:
(a) Generate an episode using π
(b) For each pair s, a appearing in the episode:
$G \leftarrow$ return following the first occurrence of s, a
Append G to $Returns(s,a)$
$Q(s,a) \leftarrow \text{average}(Returns(s,a))$
(c) For each s in the episode:

$$A^* \leftarrow \arg\max_a Q(s,a)$$
For all $a \in \mathcal{A}(s)$:
$$\pi(a|s) \leftarrow \begin{cases} 1-\varepsilon+\varepsilon/|\mathcal{A}(s)| & \text{if } a = A^* \\ \varepsilon/|\mathcal{A}(s)| & \text{if } a \neq A^* \end{cases}$$

Off-policy methods

- Learn the value of the target policy π from experience due to behavior policy μ.
- For example, π is the greedy policy (and ultimately the optimal policy) while μ is exploratory (e.g., ε-soft) policy
- In general, we only require coverage, i.e., that μ generates behavior that covers, or includes, π

$$\mu(a|s) > 0 \quad \text{for every } s,a \text{ at which } \quad \pi(a|s) > 0$$

  - Q: can i average returns as before to obtain the value function of π?

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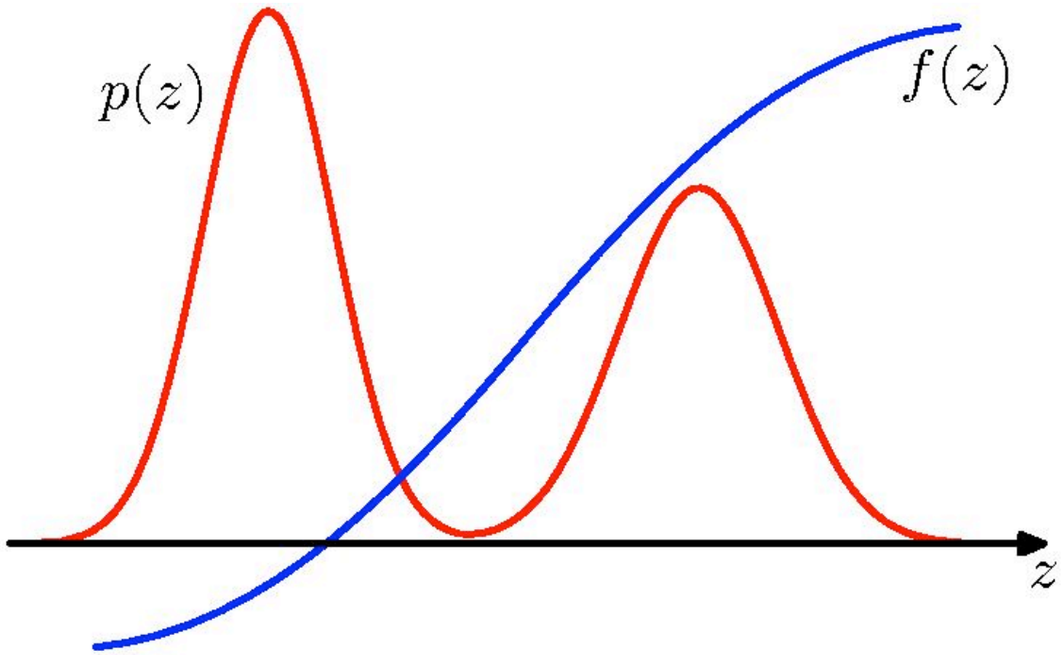
- Idea: Importance Sampling:
  - Weight each return by the ratio of the probabilities of the trajectory under the two policies.

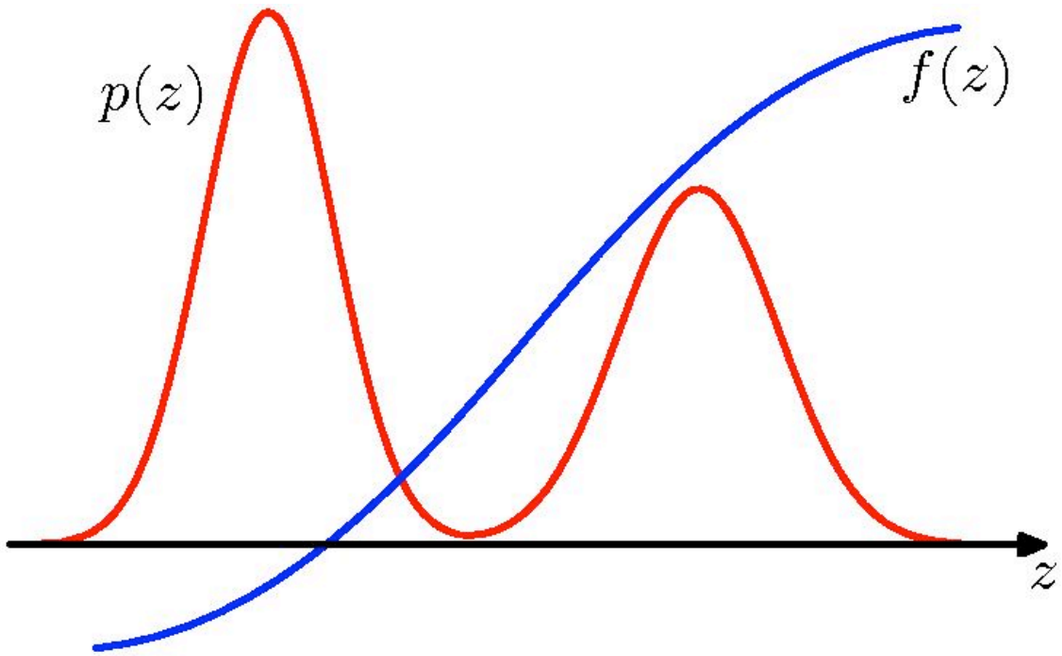
Expectations



$$\mathbb{E}[f] = \int f(z)p(z)dz$$

Estimating Expectations

• General Idea: Draw independent samples {z¹,..,zⁿ} from distribution p(z) to approximate expectation:



$$\mathbb{E}[f] = \int f(z)p(z)dz \approx$$

$$\frac{1}{N}\sum_{n=1}^{N} f(z^n) = \hat{f}.$$

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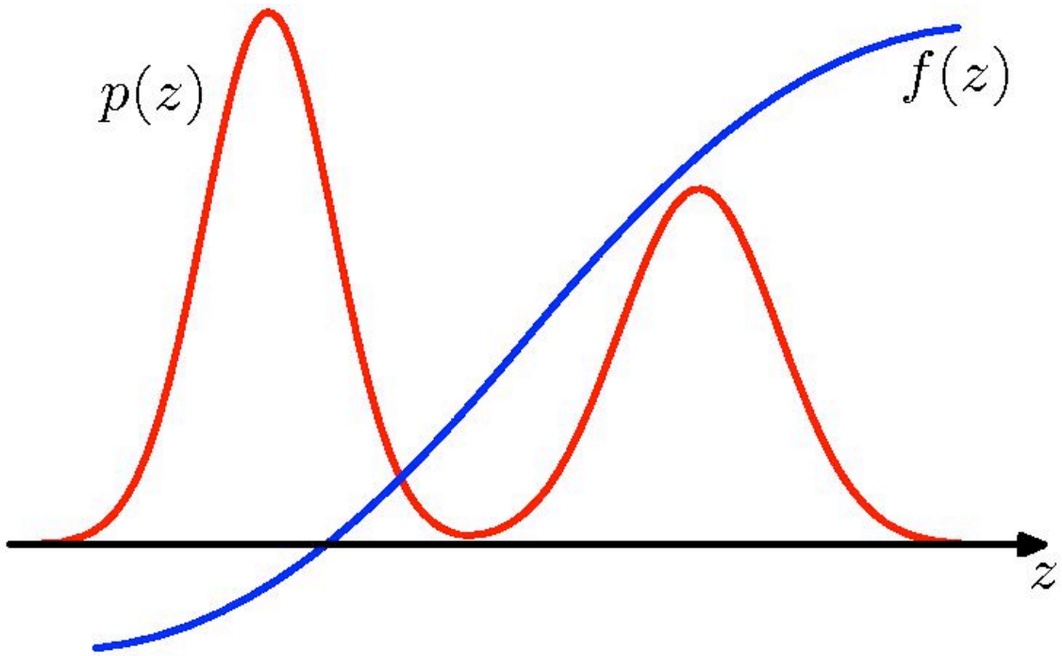

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Note that:

$$\mathbb{E}[f] = \mathbb{E}[\hat{f}].$$

so the estimator has correct mean (unbiased).

• The variance:

$$\mathrm{var}[\hat{f}] = \frac{1}{N}\mathbb{E}\left[(f - \mathbb{E}[f])^2\right].$$

• Variance decreases as 1/N.

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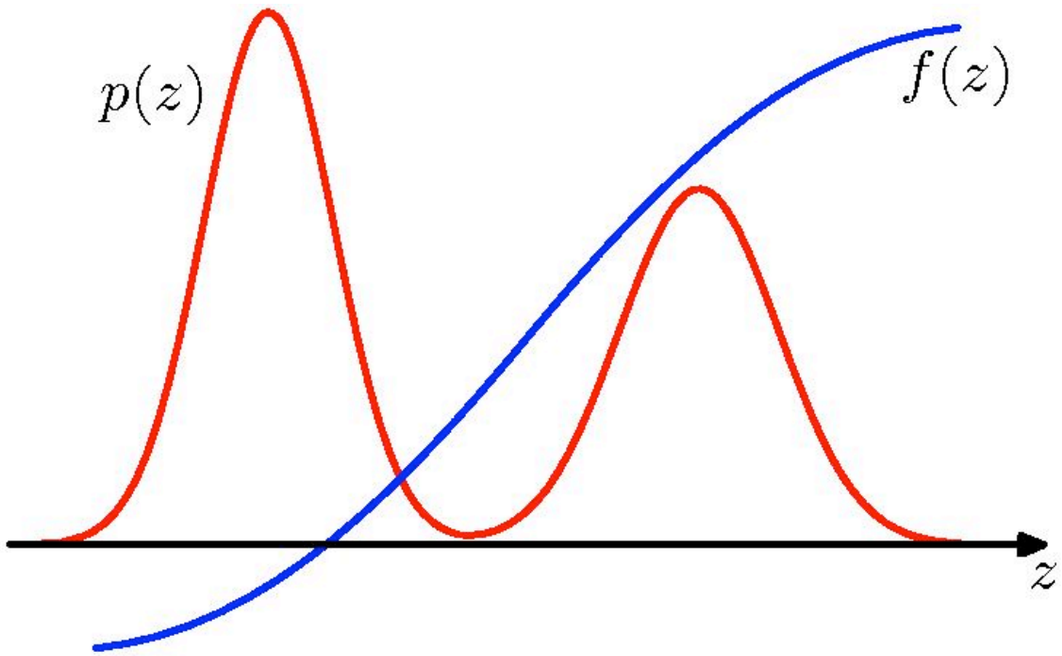

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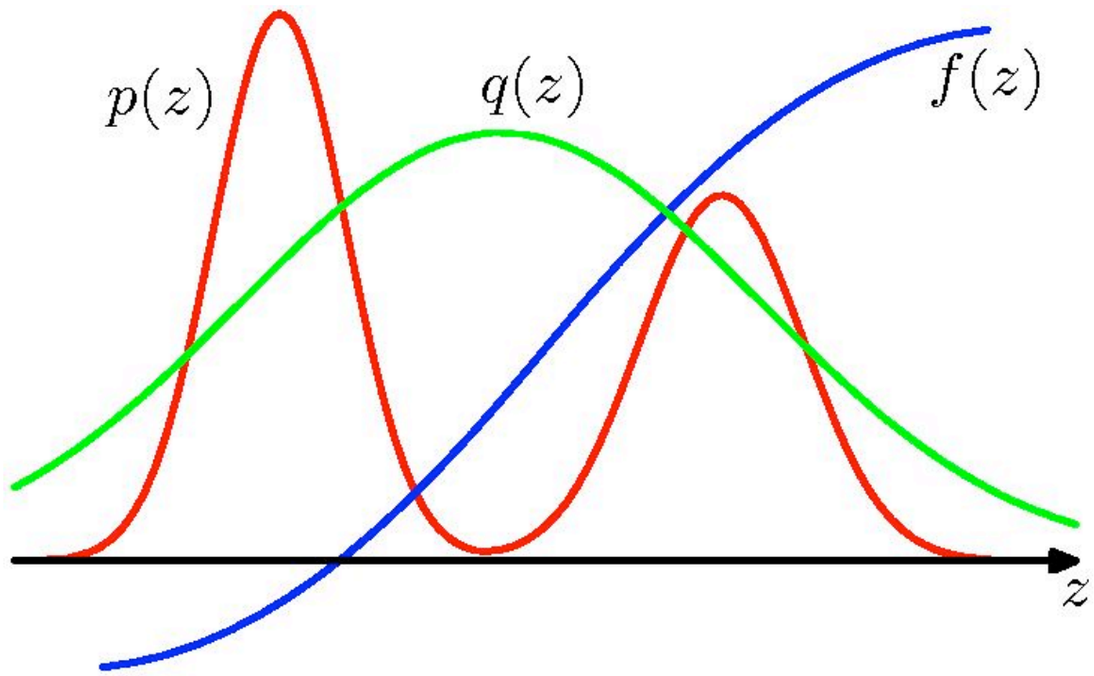
• Variance decreases as 1/N.

• **Remark**: The accuracy of the estimator does not depend on dimensionality of z.

Importance Sampling

- Suppose we have an easy-to-sample proposal distribution q(z), such that

$$q(z) > 0 \quad \text{if} \quad p(z) > 0.$$



$$\mathbb{E}[f] = \int f(z)p(z)dz$$

$$= \int f(z)\frac{p(z)}{q(z)}q(z)dz$$

$$\approx \frac{1}{N}\sum_n \frac{p(z^n)}{q(z^n)} f(z^n), \;\; z^n \sim q(z).$$

- The quantities

$$w^n = p(z^n)/q(z^n)$$

are known as **importance weights**.

This is useful when we can evaluate the probability p but is hard to sample from it

Importance Sampling

- Let our proposal be of the form: $q(z) = \tilde{q}(z)/\mathcal{Z}_q.$

$$\mathbb{E}[f] = \int f(z)p(z)dz = \int f(z)\frac{p(z)}{q(z)}q(z)dz = \frac{Z_q}{Z_p}\int f(z)\frac{\tilde{p}(z)}{\tilde{q}(z)}q(z)dz$$
$$\approx \frac{\mathcal{Z}_q}{\mathcal{Z}_p}\frac{1}{N}\sum_n \frac{\tilde{p}(z^n)}{\tilde{q}(z^n)}f(z^n) = \frac{\mathcal{Z}_q}{\mathcal{Z}_p}\frac{1}{N}\sum_n w^n f(z^n),$$

- But we can use the same weights to approximate $\mathcal{Z}_q/\mathcal{Z}_p$:

$$\frac{\mathcal{Z}_p}{\mathcal{Z}_q} = \frac{1}{\mathcal{Z}_q}\int \tilde{p}(z)dz = \int \frac{\tilde{p}(z)}{\tilde{q}(z)}q(z)dz \approx \frac{1}{N}\sum_n \frac{\tilde{p}(z^n)}{\tilde{q}^n(z)} = \frac{1}{N}\sum_n w^n.$$

- Hence:

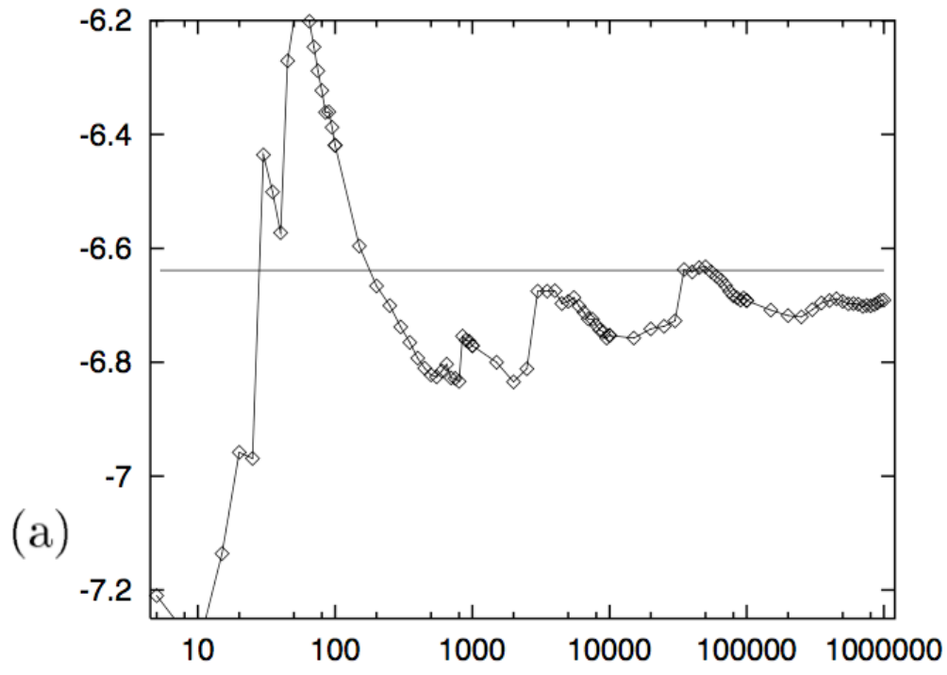
$$\mathbb{E}[f] \approx \sum_{n=1}^{N} \frac{w^n}{\sum_{m=1}^{N} w^m} f(z^n), \qquad z^n \sim q(z).$$

Importance Sampling: Example

- With importance sampling, it is hard to estimate how reliable the estimator is:

$$\hat{f} = \sum_{n=1}^{N} \frac{w^n}{\sum_{m=1}^{N} w^m} f(z^n), \qquad \mathbb{E}[f] = \int f(z) \frac{p(z)}{q(z)} q(z) dz$$

- Huge variance if the proposal density q(z) is small in a region where |f(z)p(z)| is large



- Example of using Gaussian distribution as a proposal distribution (1-d case).

- Even after 1 million samples, the estimator has not converged to the true value.

Importance Sampling Ratio

- Probability of the rest of the trajectory, after S_t, under policy π

$$\begin{aligned}
&\Pr\{A_t, S_{t+1}, A_{t+1}, \ldots, S_T \mid S_t, A_{t:T-1} \sim \pi\} \\
&\quad = \pi(A_t|S_t)p(S_{t+1}|S_t, A_t)\pi(A_{t+1}|S_{t+1}) \cdots p(S_T|S_{T-1}, A_{T-1}) \\
&\quad = \prod_{k=t}^{T-1} \pi(A_k|S_k)p(S_{k+1}|S_k, A_k),
\end{aligned}$$

- Importance Sampling: Each return is weighted by the relative probability of the trajectory under the target and behavior policies

$$\rho_t^T = \frac{\prod_{k=t}^{T-1} \pi(A_k|S_k)p(S_{k+1}|S_k, A_k)}{\prod_{k=t}^{T-1} \mu(A_k|S_k)p(S_{k+1}|S_k, A_k)} = \prod_{k=t}^{T-1} \frac{\pi(A_k|S_k)}{\mu(A_k|S_k)}$$

- This is called the Importance Sampling Ratio

Importance Sampling

- Ordinary importance sampling forms estimate

First time of termination following time t

return after t up through T(t)

$$V(s) \doteq \frac{\sum_{t \in \mathcal{T}(s)} \rho_t^{T(t)} G_t}{|\mathcal{T}(s)|}.$$

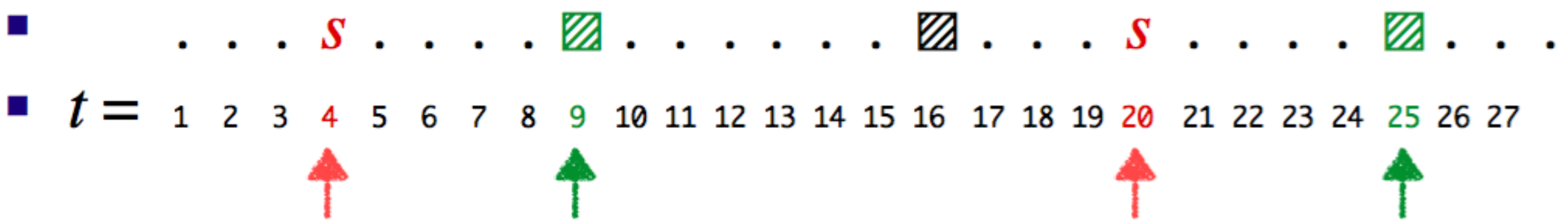
Every time: the set of all time steps in which state s is visited

Importance Sampling

- Ordinary importance sampling forms estimate

$$V(s) \doteq \frac{\sum_{t \in \mathcal{T}(s)} \rho_t^{T(t)} G_t}{|\mathcal{T}(s)|}.$$

- New notation: time steps increase across episode boundaries:



$\mathcal{T}(s) = \{4, 20\}$ set of start times

$T(4) = 9 \quad T(20) = 25$ next termination times

Importance Sampling Ratio

- All importance sampling ratios have expected value 1

$$\mathbb{E}_{A_k \sim \mu}\left[\frac{\pi(A_k|S_k)}{\mu(A_k|S_k)}\right] = \sum_a \mu(a|S_k)\frac{\pi(a|S_k)}{\mu(a|S_k)} = \sum_a \pi(a|S_k) = 1.$$

- Note: Importance Sampling can have high (or infinite) variance.

Importance Sampling

Two ways of averaging weighted returns:

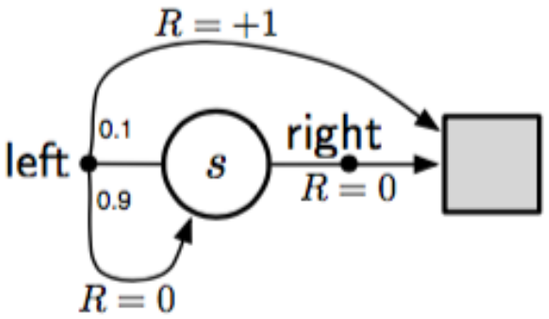
- Ordinary importance sampling forms estimate:

$$V(s) \doteq \frac{\sum_{t \in \mathcal{T}(s)} \rho_t^{T(t)} G_t}{|\mathcal{T}(s)|}.$$

- Weighted importance sampling forms estimate:

$$V(s) \doteq \frac{\sum_{t \in \mathcal{T}(s)} \rho_t^{T(t)} G_t}{\sum_{t \in \mathcal{T}(s)} \rho_t^{T(t)}}$$

Example of Infinite Variance under Ordinary Importance Sampling

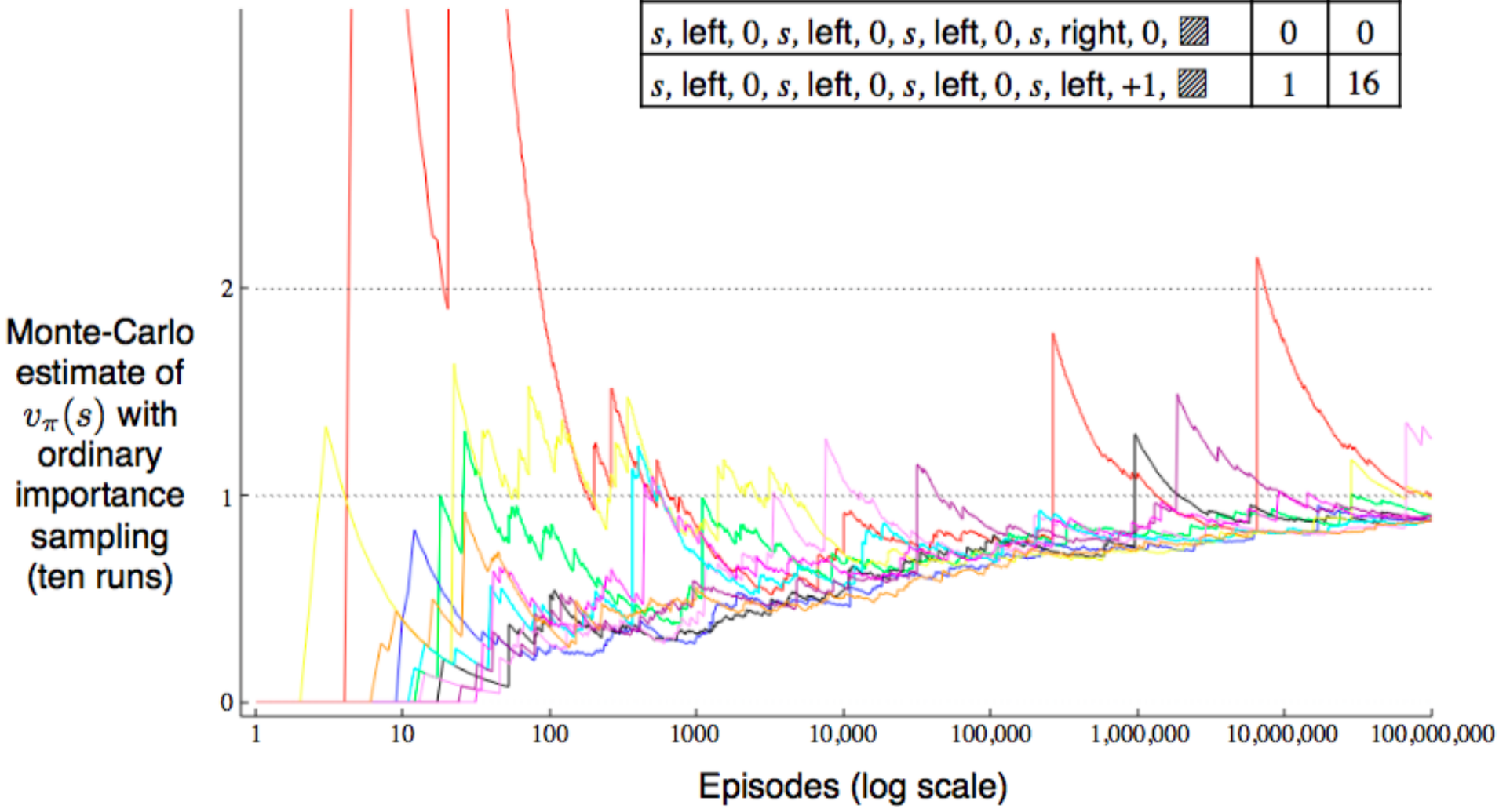


$\pi(\text{left}|s) = 1$ $\qquad$ $\gamma = 1$

$\mu(\text{left}|s) = \frac{1}{2}$ $\qquad$ $v_\pi(s) = 1$

$\frac{\pi(\text{right}|s)}{\mu(\text{right}|s)} = 0$ $\qquad$ $\frac{\pi(\text{left}|s)}{\mu(\text{left}|s)} = 2$

Trajectory	G_0	ρ_0^T
s, left, 0, s, left, 0, s, left, 0, s, right, 0, ▨	0	0
s, left, 0, s, left, 0, s, left, 0, s, left, +1, ▨	1	16



OIS:

$$V(s) \doteq \frac{\sum_{t \in \mathcal{T}(s)} \rho_t^{T(t)} G_t}{|\mathcal{T}(s)|}$$

WIS:

$$V(s) \doteq \frac{\sum_{t \in \mathcal{T}(s)} \rho_t^{T(t)} G_t}{\sum_{t \in \mathcal{T}(s)} \rho_t^{T(t)}}$$

So far

- MC has several advantages over DP:
  - Can learn directly from interaction with environment
  - No need for full models
- MC methods provide an alternate policy evaluation process
- One issue to watch for: maintaining sufficient exploration

- Looked at distinction between on-policy and off-policy methods
- Looked at importance sampling for off-policy learning
- Looked at distinction between ordinary and weighted IS

Coming up next

- MC methods are different than Dynamic Programming in that they:

    1. use experience in place of known dynamics and reward functions
    2. do not bootstrap

- Next lecture we will see temporal difference learning which

    3. use experience in place of known dynamics and reward functions
    4. bootstrap!

Carnegie Mellon

School of Computer Science

Deep Reinforcement Learning and Control

Temporal Difference Learning

CMU 10-703

Katerina Fragkiadaki

Used Materials

• **Disclaimer**: Much of the material and slides for this lecture were borrowed from Russ Salakhutdinov who in turn borrowed some material from Rich Sutton's class and David Silver's class on Reinforcement Learning.

MC and TD Learning

- Goal: learn $v_\pi(s)$ from episodes of experience under policy π

- Incremental every-visit Monte-Carlo:
  - Update value $V(S_t)$ toward actual return G_t

$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t - V(S_t)\right)$$

- Simplest Temporal-Difference learning algorithm: TD(0)
  - Update value $V(S_t)$ toward estimated returns $R_{t+1} + \gamma V(S_{t+1})$

$$V(S_t) \leftarrow V(S_t) + \alpha\left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\right)$$

- $R_{t+1} + \gamma V(S_{t+1})$ is called the TD target
- $\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$ called the TD error.

DP vs. MC vs. TD Learning

- Remember:

MC: sample average return approximates expectation

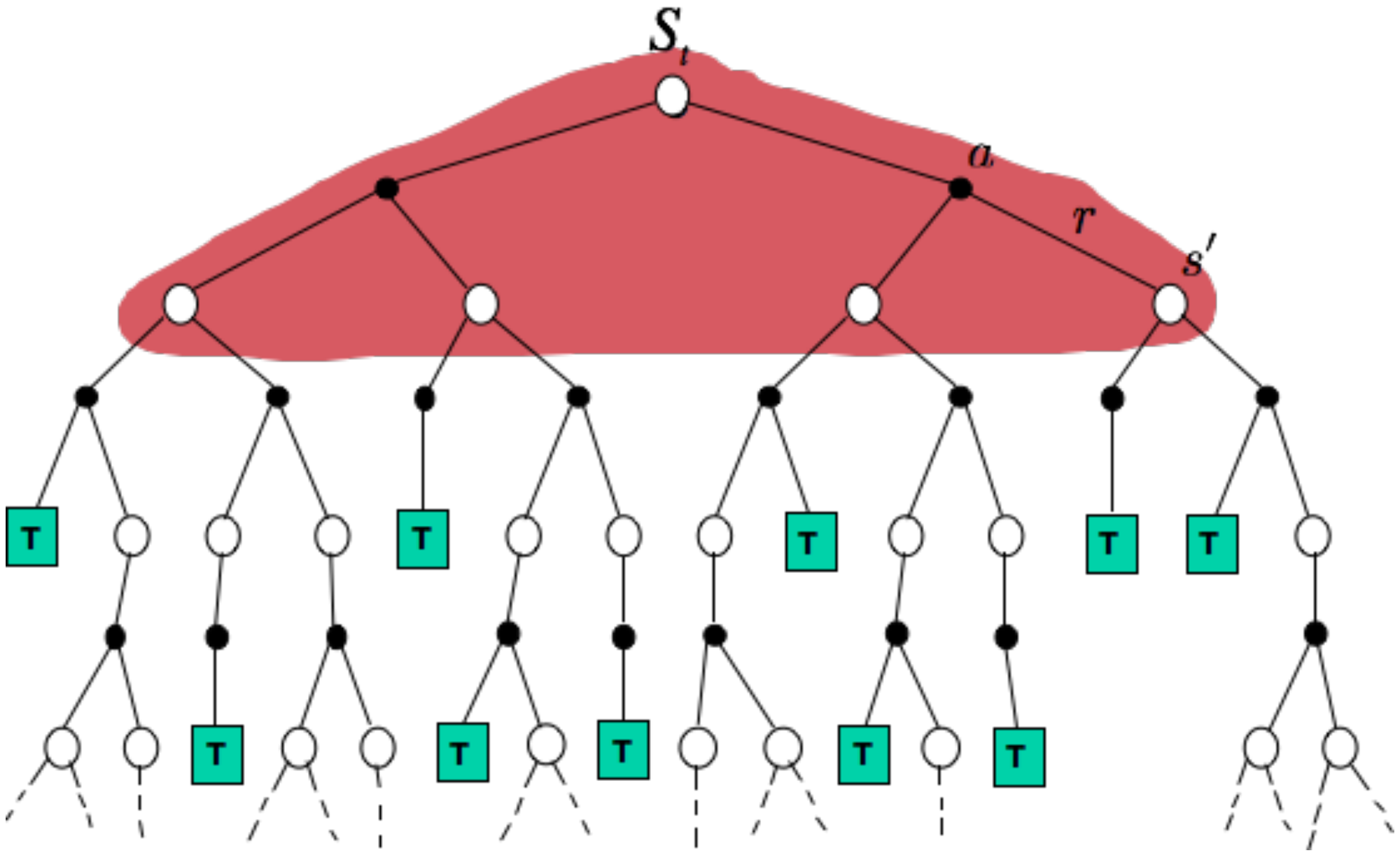
$$
\begin{aligned}
v_\pi(s) &\doteq \mathbb{E}_\pi[G_t \mid S_t = s] \\
&= \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s\right] \\
&= \mathbb{E}_\pi\left[R_{t+1} + \gamma \sum_{k=0}^{\infty} \gamma^k R_{t+k+2} \,\middle|\, S_t = s\right] \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s] \,.
\end{aligned}
$$

TD: combine both: Sample expected values *and* use a current estimate $V(S_{t+1})$ of the true $v_\pi(S_{t+1})$

DP: the expected values are provided by a model. But we use a current estimate $V(S_{t+1})$ of the true $v_\pi(S_{t+1})$

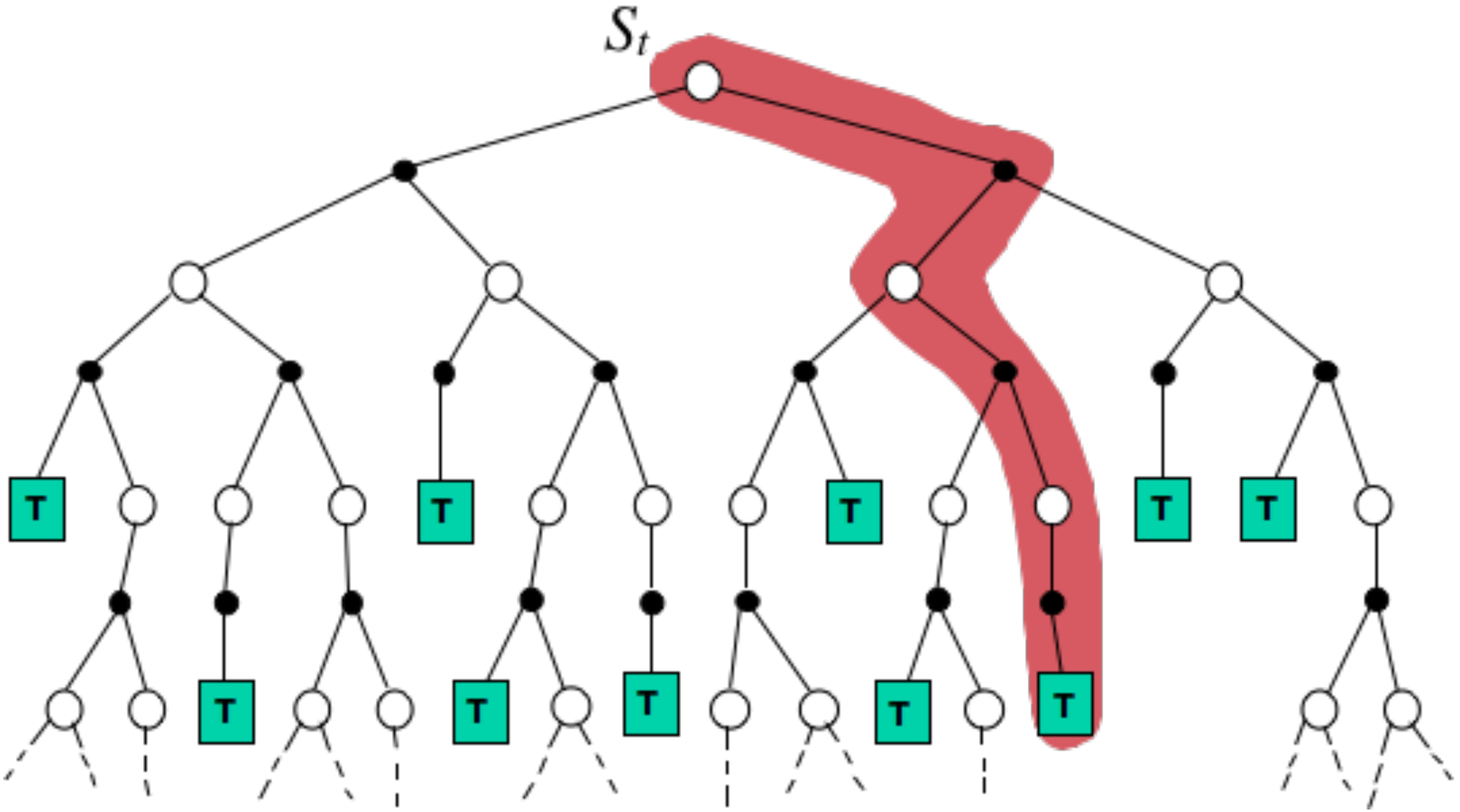
Dynamic Programming

$$V(S_t) \leftarrow E_\pi\left[R_{t+1} + \gamma V(S_{t+1})\right] = \sum_a \pi(a|S_t) \sum_{s',r} p(s', r|S_t, a)[r + \gamma V(s')]$$



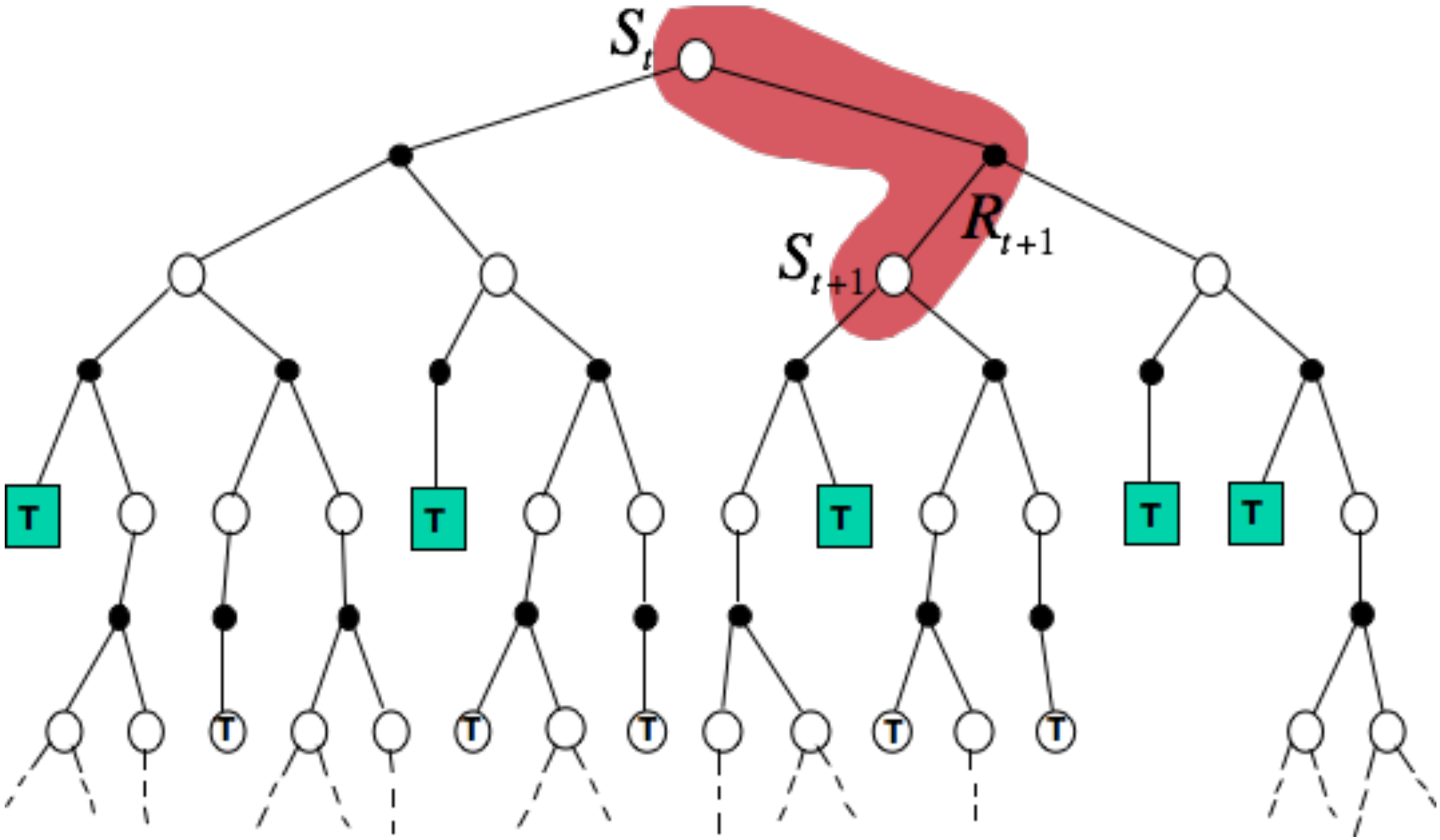
Monte Carlo

$$V(S_t) \leftarrow V(S_t) + \alpha\,(G_t - V(S_t))$$



Simplest TD(0) Method

$$V(S_t) \leftarrow V(S_t) + \alpha\left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\right)$$



TD Methods Bootstrap and Sample

- Bootstrapping: update involves an estimate
  - MC does not bootstrap
  - DP bootstraps
  - TD bootstraps

- Sampling: update does not involve an expected value
  - MC samples
  - DP does not sample
  - TD samples

TD Prediction

- Policy Evaluation (the prediction problem):
  - for a given policy π, compute the state-value function v_π

- Remember: Simple every-visit Monte Carlo method:

$$V(S_t) \leftarrow V(S_t) + \alpha\Big[G_t - V(S_t)\Big]$$

target: the actual return after time *t*

- The simplest Temporal-Difference method TD(0):

$$V(S_t) \leftarrow V(S_t) + \alpha\Big[R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\Big]$$

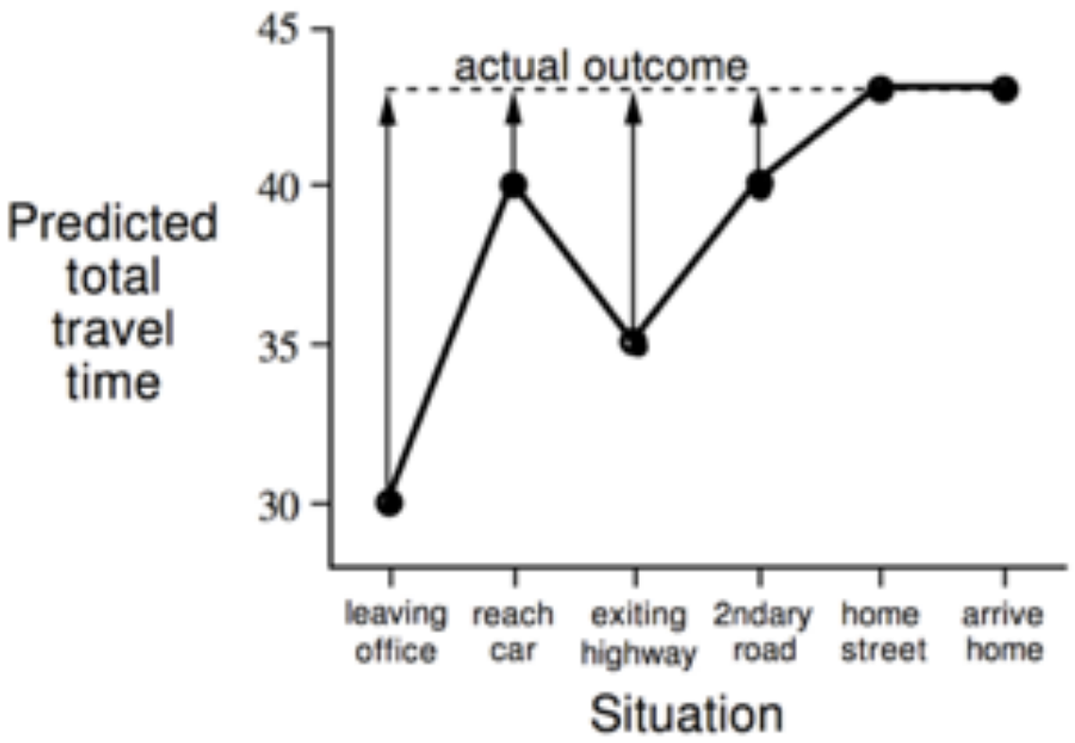
target: an estimate of the return

Example: Driving Home

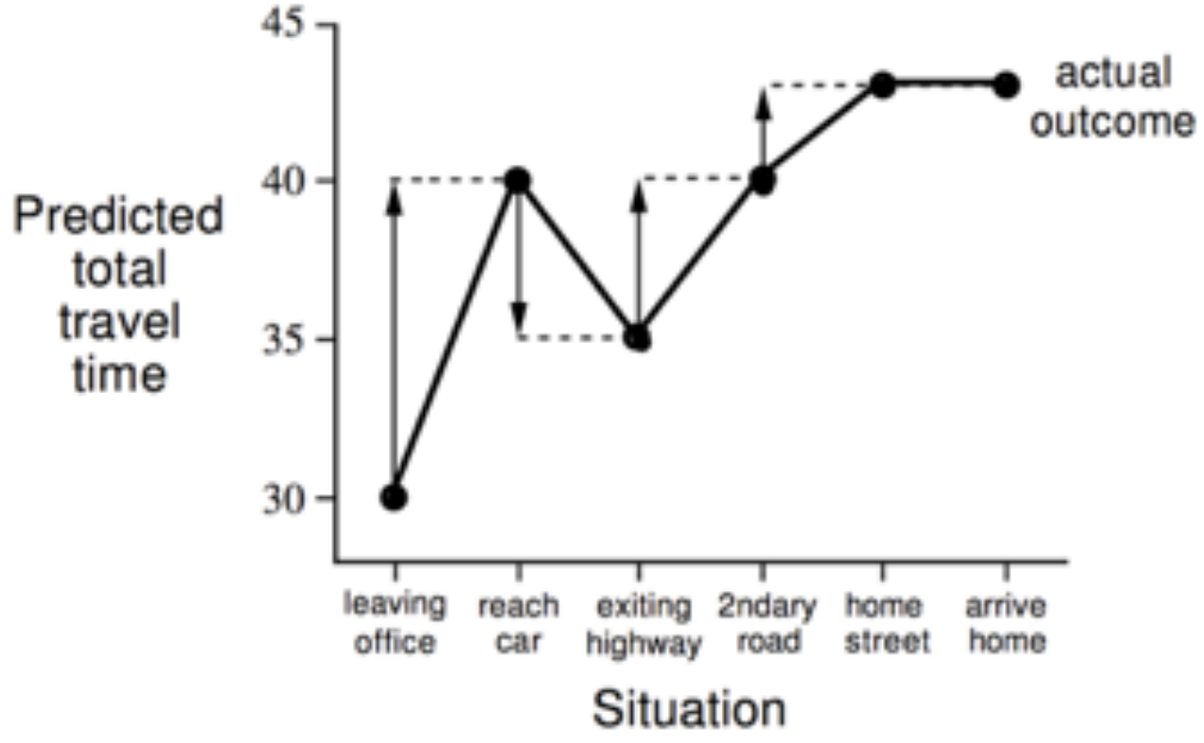
State	*Elapsed Time (minutes)*	*Predicted Time to Go*	*Predicted Total Time*
leaving office, friday at 6	0	30	30
reach car, raining	5	35	40
exiting highway	20	15	35
2ndary road, behind truck	30	10	40
entering home street	40	3	43
arrive home	43	0	43

Example: Driving Home

Changes recommended by Monte Carlo methods (α=1)



Changes recommended by TD methods (α=1)



Advantages of TD Learning

- TD methods do not require a model of the environment, only experience
- TD, but not MC, methods can be fully incremental
- You can learn before knowing the final outcome
  - Less memory
  - Less computation
- You can learn without the final outcome
  - From incomplete sequences
- Both MC and TD converge (under certain assumptions to be detailed later), but which is faster?

Bias-Variance Trade-Off

- Monte-Carlo: Update value $V(S_t)$ toward actual return G_t

$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t - V(S_t)\right)$$

- Return $G_t = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{T-1} R_T$ unbiased estimate of $v_\pi(S_t)$

- TD: Update value $V(S_t)$ toward estimated returns $R_{t+1} + \gamma V(S_{t+1})$

$$V(S_t) \leftarrow V(S_t) + \alpha\left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\right)$$

- True TD target: $R_{t+1} + \gamma v_\pi(S_{t+1})$ is unbiased estimate of $v_\pi(S_t)$

- TD target: $R_{t+1} + \gamma V(S_{t+1})$ is biased estimate of $v_\pi(S_t)$

- TD target is much lower variance than the return:
  - Return depends on many random actions, transitions, rewards
  - TD target depends on one random action, transition, reward

Bias-Variance Trade-Off

- MC has high variance, zero bias
  - Good convergence properties
  - Even with function approximation
  - Not very sensitive to initial value
  - Very simple to understand and use

- TD has low variance, some bias
  - Good Usually more efficient than MC
  - TD(0) converges to $v_\pi(s)$
  - More sensitive to initial value

Batch Updating in TD and MC methods

- Batch Updating: train completely on a finite amount of data,
  - e.g., train repeatedly on 10 episodes until convergence.
- Compute updates according to TD or MC, but only update estimates after each complete pass through the data.
- For any finite Markov prediction task, under batch updating, TD converges for sufficiently small α.
- Constant-α MC also converges under these conditions, but may converge to a different answer.

AB Example

- Suppose you observe the following 8 episodes:

A, 0, B, 0
B, 1
B, 1
B, 1
B, 1
B, 1
B, 1
B, 0

$V(\mathrm{B})?$ 0.75

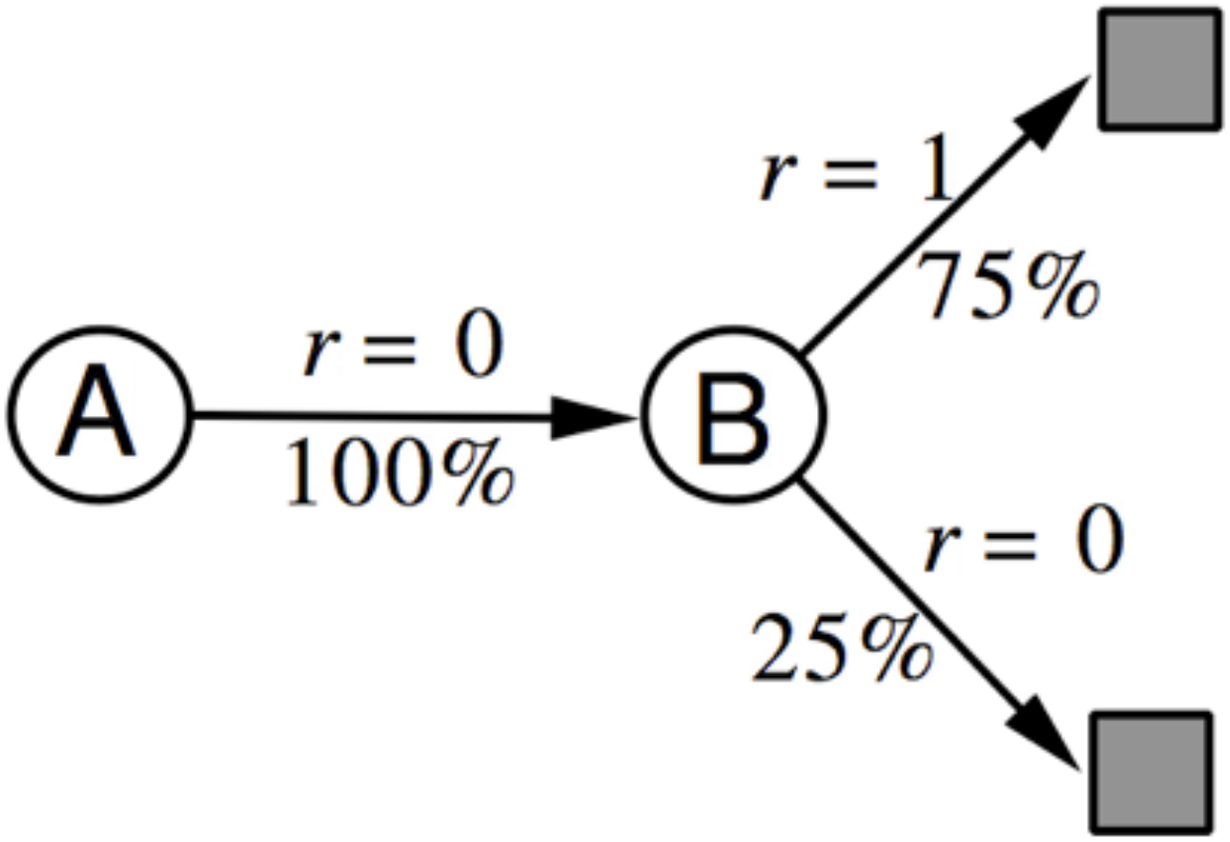
$V(\mathrm{A})?$ 0

- Assume Markov states, no discounting ($\gamma = 1$)

AB Example

- The prediction that best matches the training data is V(A)=0
  - This minimizes the mean-square-error on the training set
  - This is what a batch Monte Carlo method gets

AB Example



$V(\mathrm{A})?$ 0.75

AB Example

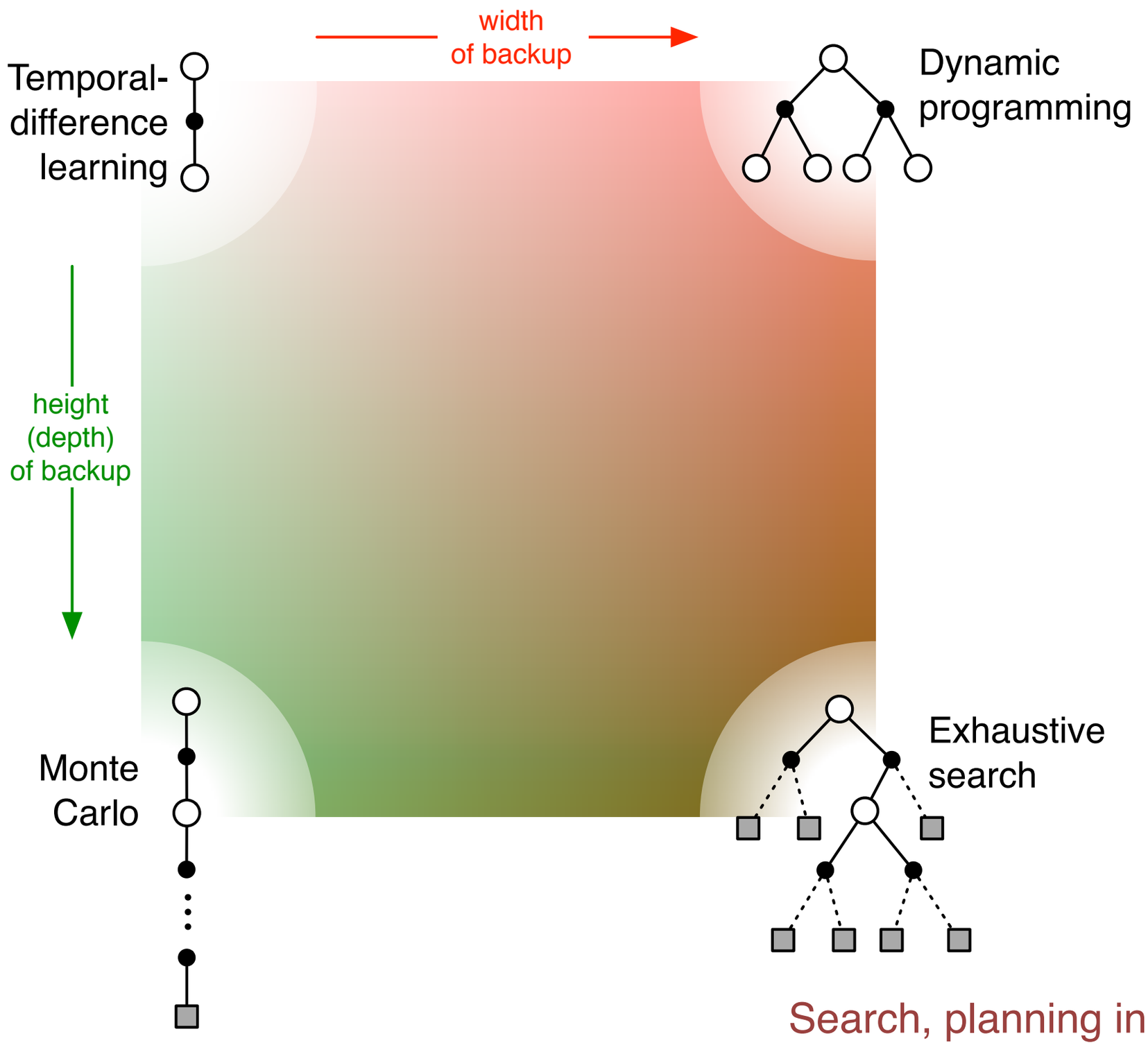
- The prediction that best matches the training data is V(A)=0
  - This minimizes the mean-square-error on the training set
  - This is what a batch Monte Carlo method gets

- If we consider the sequentiality of the problem, then we would set V(A)=.75
  - This is correct for the maximum likelihood estimate of a Markov model generating the data
  - i.e, if we do a best fit Markov model, and assume it is exactly correct, and then compute what it predicts.
  - This is called the certainty-equivalence estimate
  - This is what TD gets

Summary so far

- Introduced one-step tabular model-free TD methods
- These methods bootstrap and sample, combining aspects of DP and MC methods

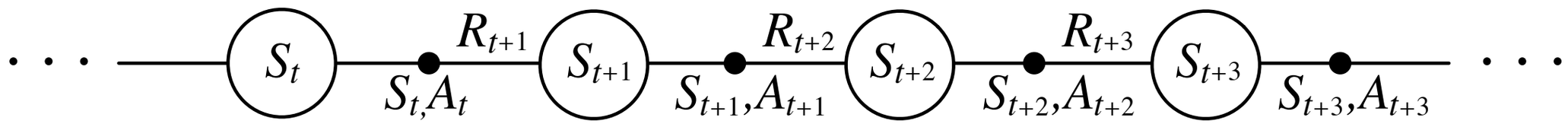
Unified View



Search, planning in a later lecture!

Learning An Action-Value Function

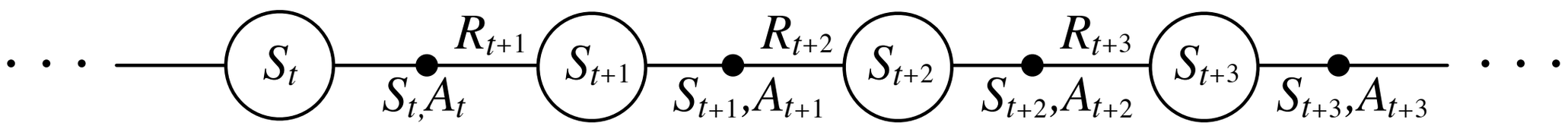
- Estimate q_π for the current policy π



- Can we come up with the TD update equation for Q values?

Learning An Action-Value Function

- Estimate q_π for the current policy π



After every transition from a nonterminal state, S_t, do this:

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\left[R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)\right]$$

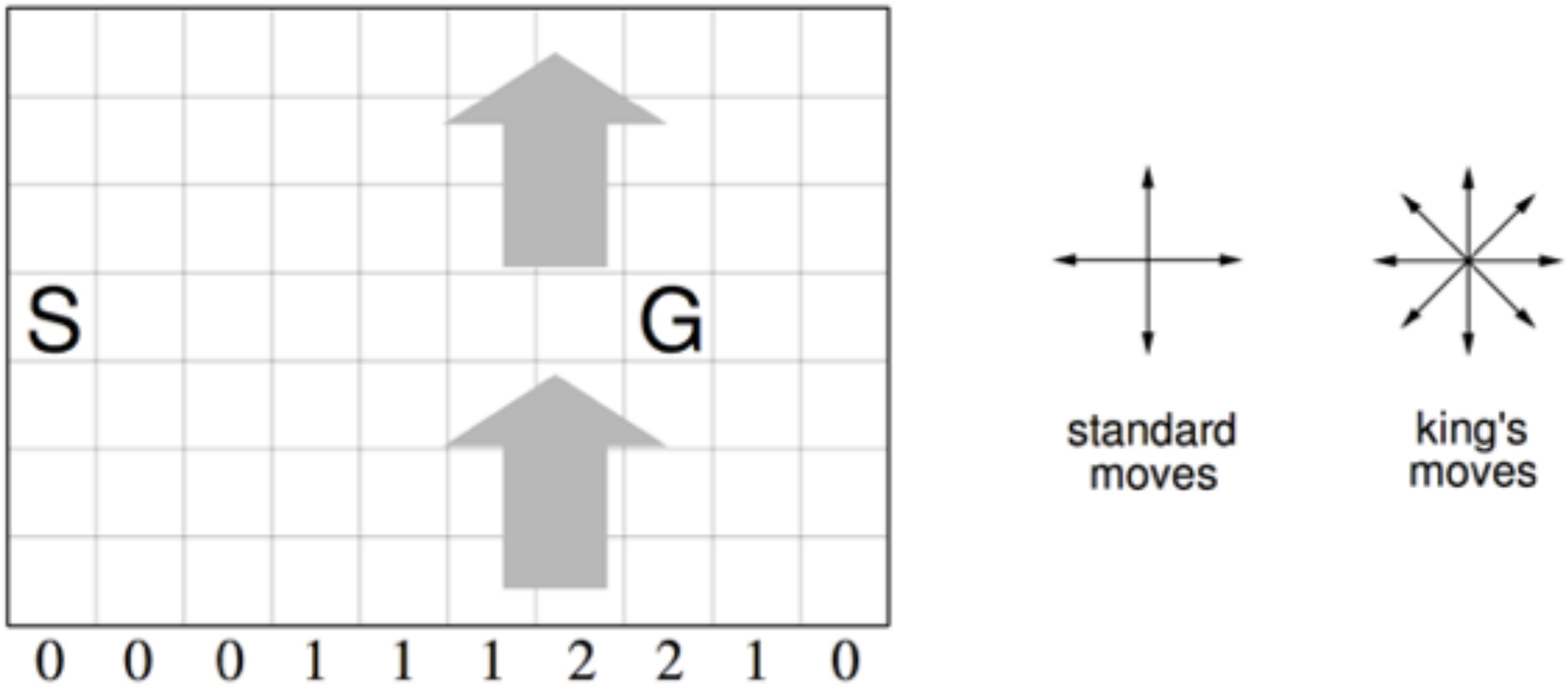
If S_{t+1} is terminal, then define $Q(S_{t+1}, A_{t+1}) = 0$

Sarsa: On-Policy TD Control

- Turn this into a control method by always updating the policy to be greedy with respect to the current estimate:

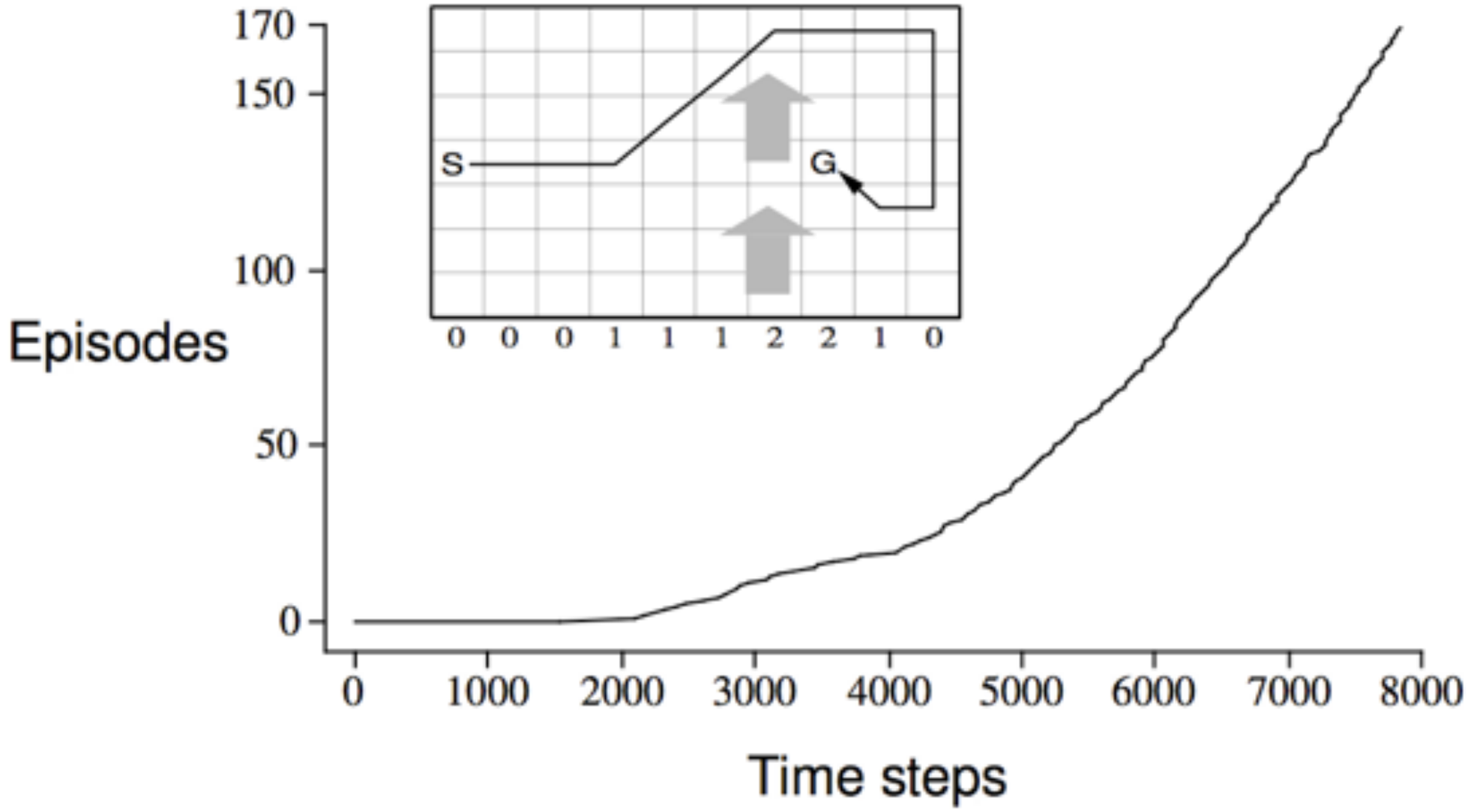
Initialize $Q(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
  Initialize S
  Choose A from S using policy derived from Q (e.g., ε-greedy)
  Repeat (for each step of episode):
    Take action A, observe R, S'
    Choose A' from S' using policy derived from Q (e.g., ε-greedy)
    $Q(S,A) \leftarrow Q(S,A) + \alpha[R + \gamma Q(S',A') - Q(S,A)]$
    $S \leftarrow S'$; $A \leftarrow A'$;
  until S is terminal

Windy Gridworld



- undiscounted, episodic, reward = –1 until goal

Results of Sarsa on the Windy Gridworld



Q: Can a policy result in infinite loops? What will MC policy iteration do then?

- If the policy leads to infinite loop states, MC control will get trapped as the episode will not terminate.
- Instead, TD control can update continually the state-action values and switch to a different policy.

Q-Learning: Off-Policy TD Control

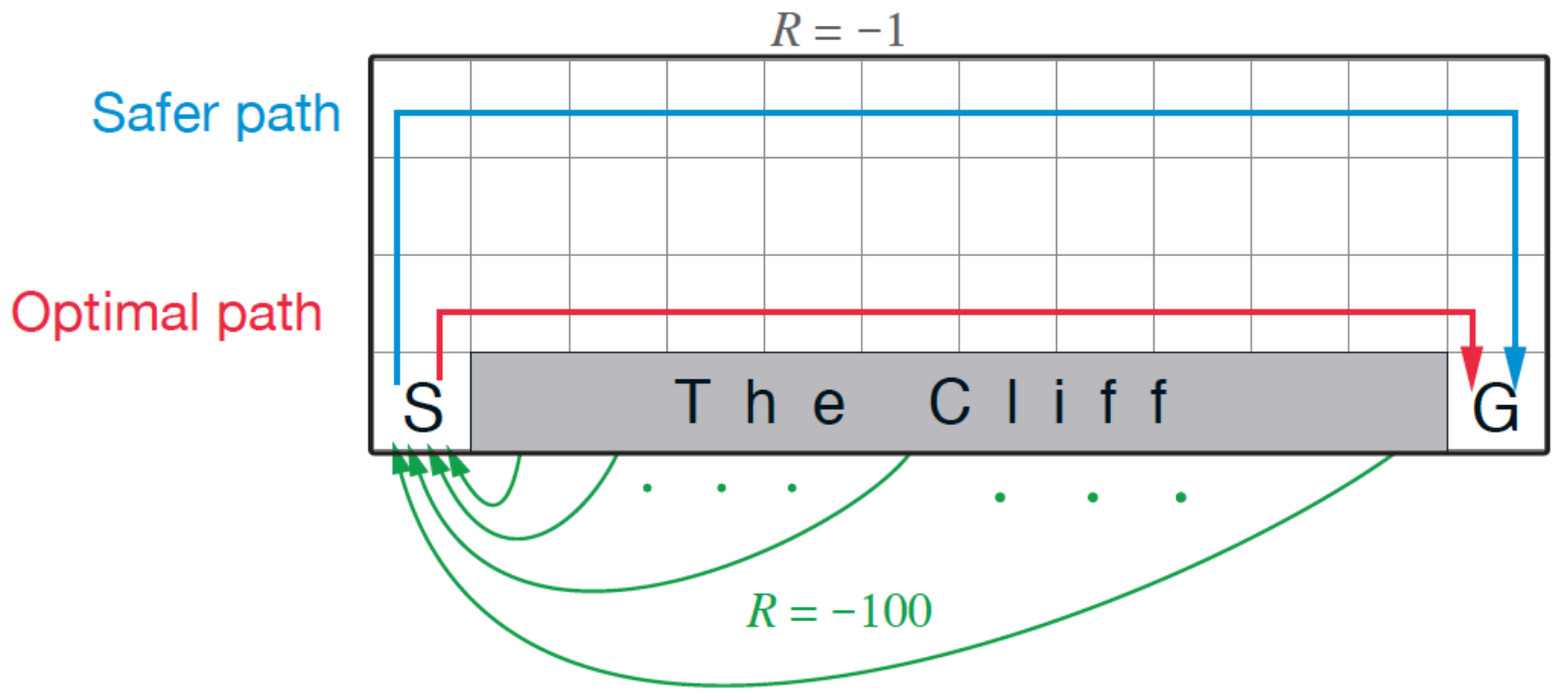
- One-step Q-learning:

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\Big[R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t)\Big]$$

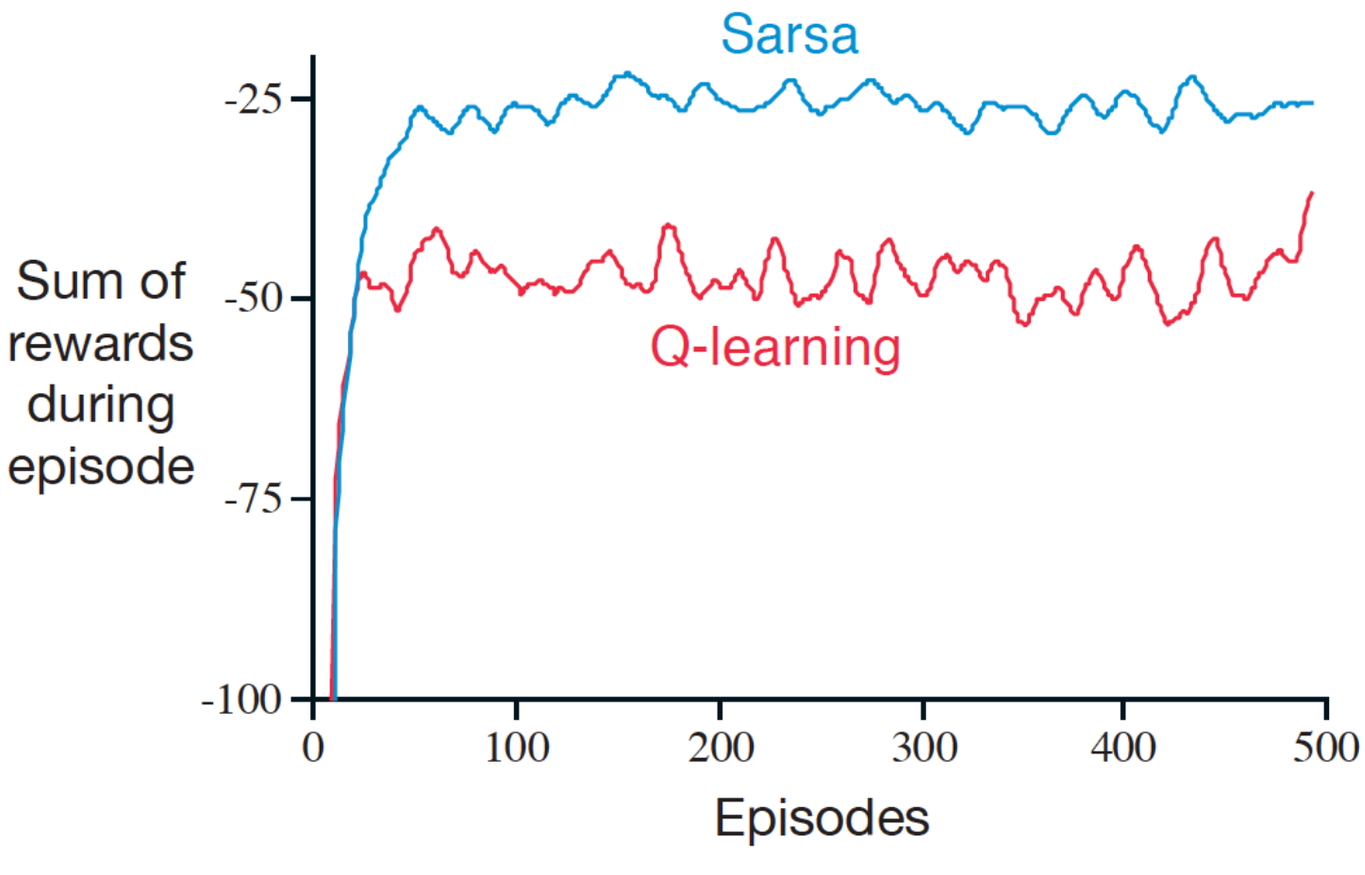
Initialize $Q(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
  Initialize S
  Repeat (for each step of episode):
    Choose A from S using policy derived from Q (e.g., ε-greedy)
    Take action A, observe R, S'
    $Q(S,A) \leftarrow Q(S,A) + \alpha[R + \gamma \max_a Q(S',a) - Q(S,A)]$
    $S \leftarrow S'$;
  until S is terminal

Remember SARSA: $Q(S,A) \leftarrow Q(S,A) + \alpha[R + \gamma Q(S',A') - Q(S,A)]$

Cliffwalking

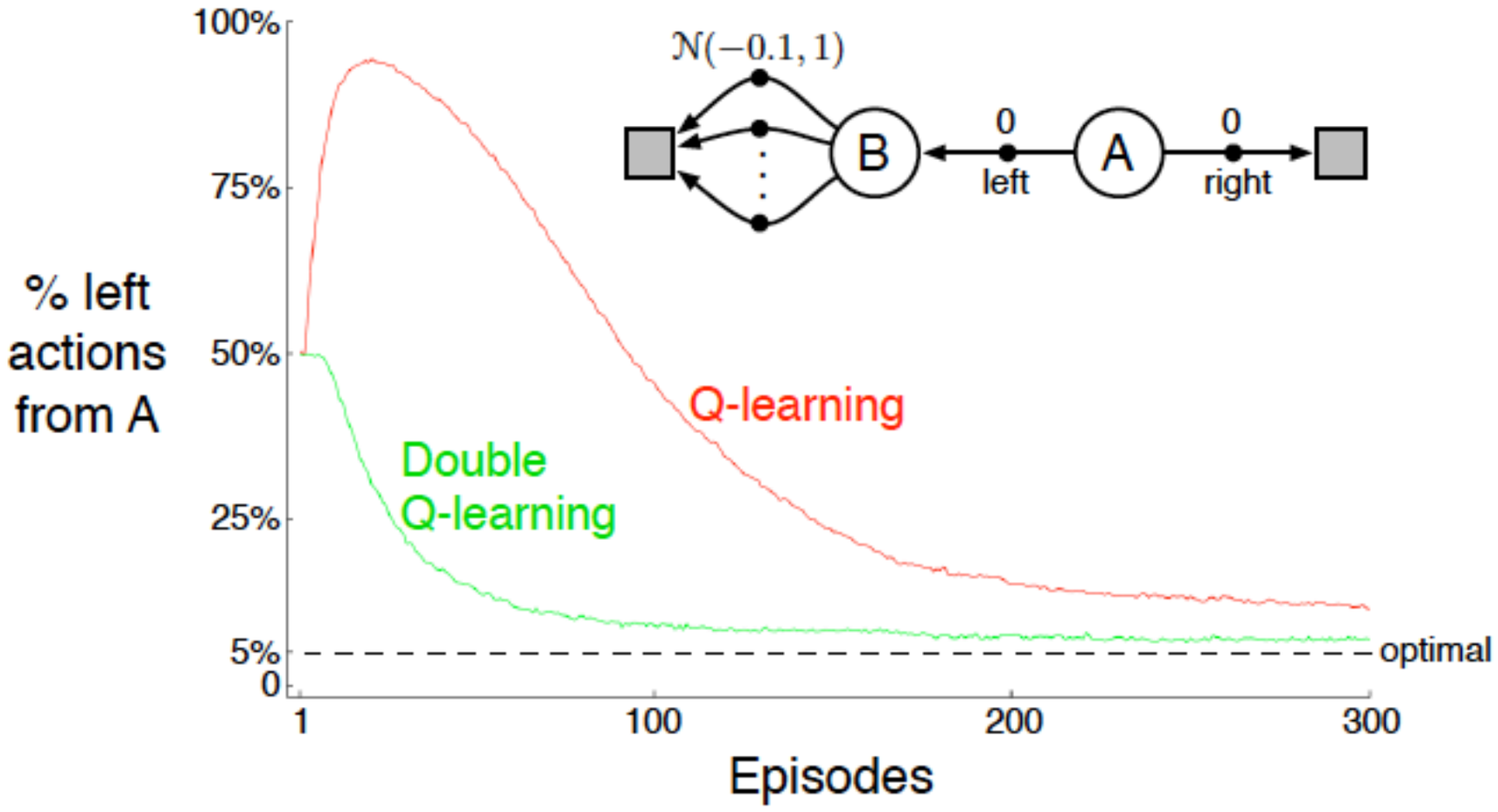


ϵ – greedy, $\epsilon = 0.1$



Maximization Bias

- We often need to maximize over our value estimates. The estimated maxima suffer from maximization bias
- Consider a state for which all ground-truth q(s,a)=0. Our estimates Q(s,a) are uncertain, some are positive and some negative. Q(s,argmax_a(Q(s,a)) is positive while q(s,argmax_a(q(s,a))=0.



Double Q-Learning

- Train 2 action-value functions, Q_1 and Q_2
- Do Q-learning on both, but
  - never on the same time steps (Q_1 and Q_2 are independent)
  - pick Q_1 or Q_2 at random to be updated on each step
- If updating Q_1, use Q_2 for the value of the next state:

$$Q_1(S_t, A_t) \leftarrow Q_1(S_t, A_t) + \\ + \alpha\Big(R_{t+1} + Q_2\big(S_{t+1}, \operatorname*{arg\,max}_a Q_1(S_{t+1}, a)\big) - Q_1(S_t, A_t)\Big)$$

- Action selections are ε-greedy with respect to the sum of Q_1 and Q_2

Double Q-Learning

- Train 2 action-value functions, Q_1 and Q_2
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- Action selections are ε-greedy with respect to the sum of Q_1 and Q_2

Double Tabular Q-Learning

Initialize $Q_1(s,a)$ and $Q_2(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily
Initialize $Q_1(\textit{terminal-state},\cdot) = Q_2(\textit{terminal-state},\cdot) = 0$
Repeat (for each episode):
  Initialize S
  Repeat (for each step of episode):
    Choose A from S using policy derived from Q_1 and Q_2 (e.g., ε-greedy in $Q_1 + Q_2$)
    Take action A, observe R, S'
    With 0.5 probabilility:

$$Q_1(S,A) \leftarrow Q_1(S,A) + \alpha\Big(R + \gamma Q_2\big(S', \operatorname{argmax}_a Q_1(S',a)\big) - Q_1(S,A)\Big)$$

    else:

$$Q_2(S,A) \leftarrow Q_2(S,A) + \alpha\Big(R + \gamma Q_1\big(S', \operatorname{argmax}_a Q_2(S',a)\big) - Q_2(S,A)\Big)$$

  $S \leftarrow S'$;
until S is terminal

Hado van Hasselt 2010

Expected Sarsa

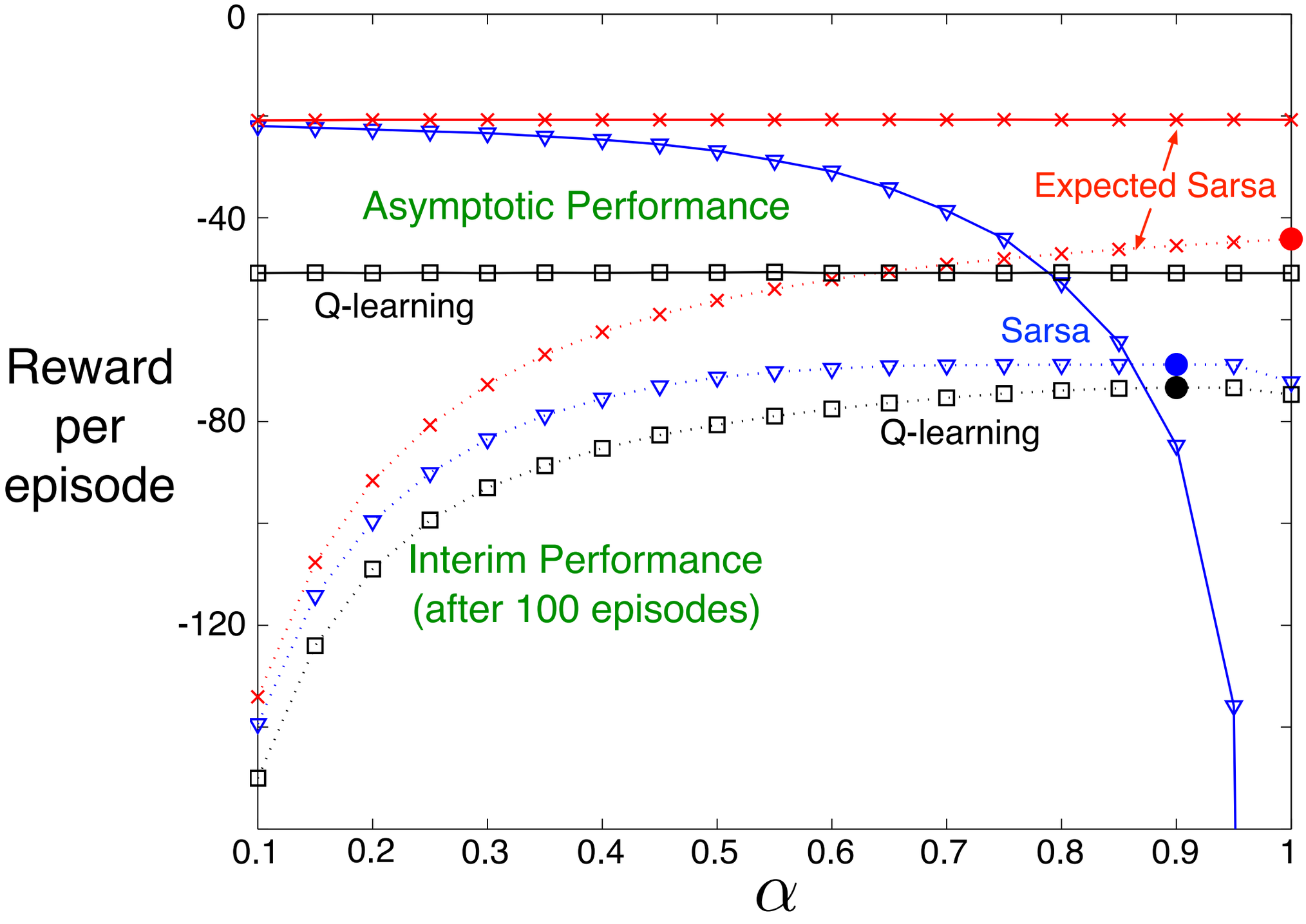
- Instead of the sample value-of-next-state, use the expectation!

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \Big[R_{t+1} + \gamma \mathbb{E}[Q(S_{t+1}, A_{t+1}) \mid S_{t+1}] - Q(S_t, A_t) \Big]$$
$$\leftarrow Q(S_t, A_t) + \alpha \Big[R_{t+1} + \gamma \sum \pi(a|S_{t+1}) Q(S_{t+1}, a) - Q(S_t, A_t) \Big]$$

- Expected Sarsa performs better than Sarsa (but costs more)
  - Q: why?

Q: Is expected SARSA on policy or off policy? What if \pi is the greedy deterministic policy?

Performance on the Cliff-walking Task



Summary

- Introduced one-step tabular model-free TD methods
- These methods bootstrap and sample, combining aspects of DP and MC methods
- TD methods are computationally congenial

- Extend prediction to control by employing some form of GPI
  - On-policy control: Sarsa, Expected Sarsa
  - Off-policy control: Q-learning