15-131 Homework 1

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Question 1.

Prove that $A \cup (B \cap C) = (A \cup B) \cap C$, where A, B, and C are sets such that $A \subseteq C$.

Proof. Let $x \in A \cup (B \cap C)$ be arbitrary and fixed. Then:

$$\begin{aligned}
x \in A \cup (B \cap C) &\iff x \in A \vee x \in (B \cap C) && \text{(Definition of } \cup) \\
&\iff x \in A \vee (x \in B \wedge x \in C) && \text{(Definition of } \cap) \\
&\iff (x \in A \wedge x \in C) \vee (x \in B \wedge x \in C) && (A \subseteq C) \\
&\iff (x \in A \vee x \in B) \wedge x \in C && \text{(Distributivity)} \\
&\iff (x \in (A \cup B) \wedge x \in C && \text{(Definition of } \cup) \\
&\iff x \in (A \cup B) \cap C && \text{(Definition of } \cap)
\end{aligned}$$

Therefore, since we have shown that an arbitrary element of $A \cup (B \cap C)$ is in $A \cup (B \cap C)$ and vice versa, we have proven that these two sets are equal. □

Question 2.

For any $n \in \mathbb{N}$ define $S_n = \sum_{i=1}^{n} i^2$. Prove

$$\forall n \in N.\, S_n = \frac{n(n+1)(2n+1)}{6}$$

Proof. Let $P(n) \iff S_n = \frac{n(n+1)(2n+1)}{6}$. We will prove $P(n)$ by induction on $n \in \mathbb{N}$.

Base Case

When $n = 0$, $P(0) \iff \frac{(0)(1)(1)}{6} = S_0$. Since $S_0 = 0$, $P(n)$ is true.

Induction Hypothesis

Assume $P(k)$ for some $k \in \mathbb{N}$.

Induction Step

Note that $S_k = \frac{k(k+1)(2k+1)}{6}$ and that

$$S_{k+1} = \sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = S_k + (k+1)^2$$

Therefore, we can substitute and rewrite the expression as follows:

$$\begin{aligned}
S_{k+1} &= S_k + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left(\frac{k(2k+1) + (k+1)}{6}\right) \\
&= \frac{(k+1)}{6}\left(k(2k+1) + 6(k+1)\right) \\
&= \frac{(k+1)}{6}(2k^2 + 7k + 6) \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

Thus we can conclude that $P(k+1)$ is true.

Therefore, because the base case and the induction step hold, $P(n)$ is true for all $n \in \mathbb{N}$ by induction. $\square$

Question 3.

(a) *Count the rectangles of all sizes and of all positions that are formed using segments in a grid with m horizontal and n vertical lines.*

Solution. A rectangle is uniquely described by two distinct vertical lines and two distinct horizontal lines. Therefore we can just select these two lines and multiply these quantities by rule of product:

$$\binom{n}{2}\binom{m}{2}$$

(b) *How many ways are there to rearrange the letters in the word "anagram"?*

Solution. We can choose an arrangement of the letters in "anagram" in two steps. We first choose 3 of the 7 positions to be a's, then permute "ngrm" in the remaining positions. Thus, we have

$$\binom{7}{3}4!$$

ways to choose an arrangement.