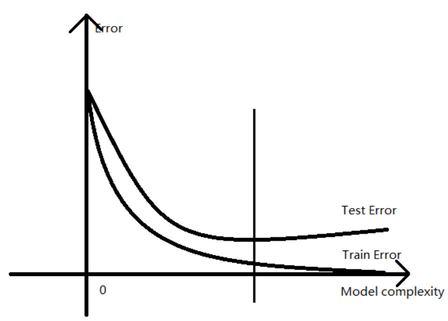
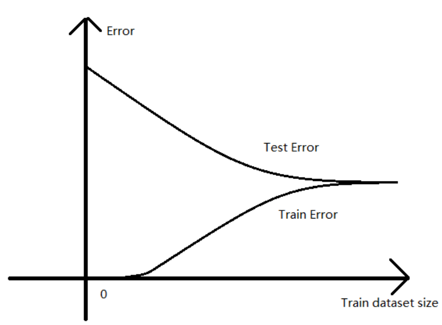
10-707: Deep Learning, Fall 2017	
Homework 1	
Lecturer: Russ Salakhutdinov	*Name: Yuan Liu(yuanl4)*

1 Problem 1

1.1 Error vs model complexity



1.2 Error vs size of training dataset



1.3 Early stopping

Early stopping is a reasonable regularization metric. Machine learning algorithms train a model based on a finite set of training data with an iterative method. The goal of machine learning is to predict the unseen observations. Up to a point, the iterative update methods improves the model's performance on unseen observations. Past that point, overfitting occurs. Early stopping rules provide guidance as to how many iterations can be run before the learner begins to over-fit.

2 Problem 2

- K-nearest-neighbor regression:

  For knn, the estimator is given by:

$$\hat{f}(x^*) = \frac{1}{k} \sum_{i \in N_k(x^*)} y_i$$

  $N_k(x^*)$ contains the indices of the k closest points of $x_1, \ldots, x_N$ to x^*. Then we can know:

$$l_i(x^*; \mathcal{X}) = \begin{cases} 1, & i \in N_k(x^*) \\ 0, & otherwise \end{cases} \tag{1}$$

- Linear regression: For linear regression, the estimator is given by:

$$\hat{f}(x^*) = {x^*}^T w$$

  where $w = (X^T X)^{-1} X^T y$, $y = (y_1, \ldots, y_N)^T$ and $X = (x_1, \ldots, x_N)^T$. Then we can get:

$$\hat{f}(x^*) = {x^*}^T (X^T X)^{-1} X^T y$$

  So

$$l_i(x^*; \mathcal{X}) = ({x^*}^T (X^T X)^{-1} X^T)_i$$

  $l_i(x^*; \mathcal{X})$ equals to the i_{th} component of ${x^*}^T (X^T X)^{-1} X^T$.

3 Problem 3

- Normalization: $p(x = 1|\mu) + p(x = -1|\mu) = \frac{1+\mu}{2} + \frac{1-\mu}{2} = 1$
- Mean: $E[x] = 1 * p(x = 1|\mu) + (-1) * p(x = -1|\mu) = \mu$
- Variance: $Var[x] = E[x^2] - (E[x])^2 = \frac{1+\mu}{2} + \frac{1-\mu}{2} - \mu^2 = 1 - \mu^2$
- Entropy: $Entropy = -\sum_{i \in \{-1,1\}} p(x = i|\mu) \log p(x = i|\mu) = -\frac{1+\mu}{2} \log \frac{1+\mu}{2} - \frac{1-\mu}{2} \log \frac{1-\mu}{2}$

4 Problem 4

Denote l is the correct label of x, and t is the label of x given by the dataset. So we can have the following formula:

$$p(l = 1|t = 1) = 1 - \epsilon$$

$$p(l = 1|t = 0) = \epsilon$$

$$
\begin{aligned}
p(l=1|x;w) &= p(t=1|x;w)p(l=1|t=1) + p(t=0|x;w)p(l=1|t=0) \\
&= y(x,w)(1-\epsilon) + (1-y(x,w))\epsilon
\end{aligned}
$$

Then we can get:

$$l \sim Bernoulli\left(y(x,w)(1-\epsilon) + (1-y(x,w))\epsilon\right)$$

$$p(l|x,w) = \left[y(x,w)(1-\epsilon) + (1-y(x,w))\epsilon\right]^l \left[1 - y(x,w)(1-\epsilon) - (1-y(x,w))\epsilon\right]^{1-l}$$

$$
\begin{aligned}
\text{cost function} &= -\log p(l|x,w) \\
&= -l\log(y(x,w)(1-\epsilon) + (1-y(x,w))\epsilon) - (1-l)log(1 - y(x,w)(1-\epsilon) - (1-y(x,w))\epsilon) \\
&= -l * \log(y - 2y\epsilon + \epsilon) - (1-l) * \log(1 - y + 2y\epsilon - \epsilon)
\end{aligned}
$$

Where l is the label of training dataset, y is the output of neural network.

If $\epsilon = 0$, then

$$\text{cost function} = -l * \log y - (1-l) * log(1-y)$$

which is the standard negative log likelihood of binary classification.

5 Problem 5

First represent two networks in the following form:

- Sigmoid network: Input $x = (x_1, \ldots, x_p)^T$, First Layer: $a^{sig} = W_1^{sig}x + b_1^{sig}$, Activation function: $h^{sig} = \sigma(a^{sig})$, Second Layer: $o^{sig} = W_2^{sig}h^{sig} + b_2^{sig}$
- Tanh network: Input $x = (x_1, \ldots, x_p)^T$, First Layer: $a^{tanh} = W_1^{tanh}x + b_1^{tanh}$, Activation function: $h^{tanh} = tanh(a)$, Second Layer: $o^{tanh} = W_2^{tanh}h^{tanh} + b_2^{tanh}$

By observation we can have:

$$\sigma(2a) = \frac{tanh(a) + 1}{2}$$

First, we can assume:

$$W_1^{sig} = 2W_1^{tanh}, \quad b_1^{sig} = 2b_1^{tanh}$$

Then

$$a^{sig} = W_1^{sig}x + b^{sig} = 2W_1^{tanh}x + 2b_1^{tanh} = 2a^{tanh}$$

$$h^{sig} = \sigma(2a^{tanh}) = \frac{h^{tanh} + 1}{2}$$

Second, we can assume:

$$W_2^{sig} = 2W_2^{tanh}, \quad b_2^{sig} = b_2^{tanh} - W_2^{tanh} \cdot \mathbf{1}$$

Where $\mathbf{1}$ is a vector and all its component is 1.Then

$$o^{sig} = W_2^{sig}h^{sig} + b_2^{sig} = 2W_2^{tanh}\frac{h^{tanh} + \mathbf{1}}{2} + b_2^{tanh} - W_2^{tanh} \cdot \mathbf{1} = o^{tanh}$$

As a result, we can have the following equation:

$$W_1^{sig} = 2W_1^{tanh}, \quad b_1^{sig} = 2b_1^{tanh}$$

$$W_2^{sig} = 2W_2^{tanh}, \quad b_2^{sig} = b_2^{tanh} - W_2^{tanh} \cdot \mathbf{1}$$