Mappings: Basic Definitions

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Introduction

In Calculus, you saw real-valued functions. In fact, pretty much all your the functions you've worked with so far in your life took in real numbers as input and output real numbers. So your domain and range were the set of real numbers, or they were some subset of the real numbers.

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = x^2$$

$$g : \{x \in \mathbb{R} \mid x > 0\} \to \mathbb{R}, \qquad g(x) = \ln x$$

Maybe you've even seen functions on several variables, $\mathbb{R}^n \to \mathbb{R}^m,\ n, m \in \mathbb{N}$ such as

$$h : \mathbb{R} \to \mathbb{R}^3, \qquad h(x) = (\cos x, \sin x, x)$$

$$j : \mathbb{R}^2 \to \mathbb{R}^2, \qquad j(r, \theta) = (r\cos\theta, r\sin\theta)$$

Formal definition

Subsets of $A \times B$ are called relations. We will define a **mapping/map/function** $f \subset A \times B$ from a set A to a set B to be the special type of relation in which for each element $a \in A$ there is a unique element $b \in B$ such that $(a, b) \in f$. We usually write $f : A \to B$ and $f(a) = b$; or $f : a \to b$, i.e. "a maps to b".
Note: The word *unique* is important. A mapping can't assign one-to-many but it can assign many-to-one (e.g. the constant function). Note: If the mapping is not many-to-one either, there is a **one-to-one correspondence** and we say it is **one-to-one** or **injective**. For example $f(x) = x^2$ is not one-to-one since both 2 and -2 map to 4.

A mapping is **well-defined** if

(i) every assignment is unique (one element can't have two assignments), and

(ii) every element in the domain has an assignment.

Property (i) really states: If $a_1 = a_2$, then $f(a_1) = f(a_2)$.

The following functions are not well-defined. Why not?

(a) $\alpha : \mathbb{Q} \to \mathbb{Z}, \qquad \alpha(\frac{p}{q}) = p$, since $\alpha(1/2) \neq \alpha(2/4)$.

(b) $\beta : \mathbb{Q} \to \mathbb{Q}, \qquad \beta(\frac{p}{q}) = \frac{p-1}{q-1}$, since not defined for integers $p = \frac{p}{1}$

(c) $\gamma : \mathbb{R} \to \mathbb{R}, \qquad \gamma(x) = \log_3(x - 5)$, since not defined for $x \leq 5$

Identity mapping

The identity mapping on a set A is: $id_A(a) = id(a) = a \quad \forall a \in A$.

Image definitions

- If $a \in A$, then the image of a under f is $f(a) = b$ (the value of f when applied to a.)
- If $S \subseteq A$, the image of a S under f is

$$f(S) = \{f(x) \mid x \in S\}$$

- The image of f is the image $f(A)$ of the entire domain A of f.

Recall: The set B is the codomain; the set $f(A)$ is the image. We always have that the image is a subset of the codomain, i.e. $f(A) \subseteq B$. Note: In the special case that $f(A) = B$, we say that f is **onto**, or **surjective**. For example, if $f : \mathbb{R} \to \mathbb{R}$, then $f(x) = x^2$ is not onto since $f(\mathbb{R}) = [0, \infty) \neq \mathbb{R}$, but $f(x) = x^3$ is onto since in this case $f(\mathbb{R}) = \mathbb{R}$.

Inverse set of an element or subset vs. Inverse mapping

- Inverse/Preimage set of an element:

$$f^{-1}(b) = \{a \in A \mid f(a) = b\}$$

  This does not mean that the inverse mapping f^{-1} exists.
  $f^{-1}(b)$ is a subset of A; it is all the elements in the domain that get mapped to that particular b in the codomain.

- Inverse/Preimage set of a subset: If $T \subseteq B$,

$$f^{-1}(T) = \{a \in A \mid f(a) \in T\}$$

  This does not mean that the inverse mapping f^{-1} exists.
  $f^{-1}(T)$ is a subset of A; it is all the elements in the domain whose images are all in the subset T in the codomain.

- A map $g : B \to A$ is an **inverse mapping** of $f : A \to B$ if

  (i) $g \circ f = id_A$

  (ii) $f \circ g = id_B$

  So g "reverse" f, and vice versa. We say f is **invertible** if it has an inverse, and $g = f^{-1}$ and $f = g^{-1} = f$.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$. Then

- Suppose $b = 4$. Then $f^{-1}(b) = \{-2, 2\}$
- Suppose $T = \{4, 9\}$. Then $f^{-1}(T) = \{-3, -2, 2, 3\}$
- One reason why the inverse mapping f^{-1} does not exist is because the image $f(\mathbb{R})$ is $[0, \infty]$, which is not equal to the codomain $\mathbb{R}$.

Injective and Surjective Mappings

Injectivity

Informal: An element in the codomain can only be hit once.
Formal: A mapping f is said to be **injective** if for all $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$. Or the contrapositive holds: For all $a_1, a_2 \in A$, if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$

Surjectivity

Informal: Every element in the codomain gets hit, possibly more than once. That is, every element in the codomain has at least one preimage.
Formal: A mapping f is said to be **surjective** if for all $b \in B$, there exists $a \in A$ such that $f(a) = b$. Equivalently, $f(A) = B$.

Bijective/Invertible mappings

A **bijective** mapping is both injective and surjective.

Theorem 1. *A mapping is invertible if and only if it is both injective and surjective.*

Theorem 2. *Let $f : A \to B,\ g : B \to C$, and $h : C \to D$. Then*

1. *Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$*
2. *If f and f are injective, then $g \circ f$ is injective.*
3. *If f and g are bijective, then $g \circ f$ is surjective.*
4. *If f and g are bijective, then $g \circ f$ is bijective.*

Examples and Exercises

Determine whether the following mappings are injective, surjective, or bijective.

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$
2. $h : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$
3. $g : \mathbb{R} \to \mathbb{R},\ g(x) = 2x - 1$
4. Restriction of g to $\mathbb{Z}$: $g|_{\mathbb{Z}} = \hat{g} : \mathbb{Z} \to \mathbb{Z},\ \hat{g}(x) = 2x - 1$
5. $k : \mathbb{R} \to \mathbb{R},\ k(t) = 5$
6. $\alpha : \mathbb{R}^2 \to \mathbb{R},\ \alpha(x, y) = xy$
7. $\varphi : A \times B \to A,\ \varphi(a, b) = a$
8. $\gamma : S \to S \times \{0\},\ \gamma(s) = (s, 0)$
9. $\psi_1 : A^3 \to A^3,\ \psi_1(a, b, c) = (a, b, a)$
10. $\psi_2 : A^3 \to A^3,\ \psi_2(a, b, c) = (c, a, b)$

Special mappings

Linear transformations

Linear transformations are mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$, $n, m \in \mathbb{N}$.
Example: $T : \mathbb{R}^2 \to \mathbb{R}^2,\ T(x, y) = (x + 4y,\ 2x - 3y)$ This can also be represented my matrix multiplication of some coefficient matrix A by a vector of unknowns $(x, y)^T$:

$$\begin{pmatrix} 1 & 4 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 4y \\ 2x - 3y \end{pmatrix}$$

Permutations

A permutation of a set A is bijective mapping on itself.
Example: Define the set $X_4 = \{1, 2, 3, 4\}$. Let $\pi : X_4 \to X_4$ be defined as: $\pi(1) = 2,\ \pi(2) = 4,\ \pi(3) = 3,\ \pi(4) = 1$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \pi(1) & \pi(2) & \pi(3) & \pi(4) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$