A Set, an Operation, and its Properties

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See: Pinter textbook, Chapter 2

Notation: If A is a set, A^+ is its positive elements, A^* is its nonzero elements.

Exercises: A Mission of Properties

0.1 Operation Closure

Your mission, should you choose to accept it, is to determine which examples satisfy the definition of closure.

*The following operations should be embarked upon only if you can complete Operation Closure.

0.2 Operation Associativity

Your mission, should you choose to accept it, is to determine which examples satisfy the associative property.

0.3 Operation Identity

Your mission, should you choose to accept it, is to determine which sets have an identity element under the operation.

0.4 Operation Inverses

Your mission, should you choose to accept it, is to determine which sets have an inverse for each element under the operation. Make sure you can complete Operation Identity.

1. $(\mathbb{N}, +)$
2. $(\mathbb{Z}, +)$
3. $(\mathbb{Q}, +)$
4. $(\mathbb{R}, +)$
5. $(\mathbb{N}, \cdot)$
6. $(\mathbb{Z}, \cdot)$
7. $(\mathbb{Q}, \cdot)$
8. $(\mathbb{R}, \cdot)$
9. $(\mathbb{C}, +)$
10. $(\mathbb{C}, \cdot)$
11. $(\mathbb{Q}^+, +)$
12. $(\mathbb{Q}^+, \cdot)$
13. $(\mathbb{R}^+, \cdot)$
14. $(\mathbb{R}^*, +)$
15. $(\mathbb{R}^*, \cdot)$
16. $(H, \cdot)$ where $H = \{n^2 | n \in \mathbb{Z}^+\}$
17. $(A, \cdot)$ where $A = \{3^n 5^m | n, m \in \mathbb{Z}\}$
18. $(\mathbb{N}, *)$ where $a * b := a$

19. $(\mathbb{Z}, *)$ where $a * b := a$

20. $(\mathbb{N}, *)$ where $a * b := -a$

21. $(\mathbb{Z}, *)$ where $a * b := -a$

22. $(\mathbb{R}, *)$ where $a * b := |a - b|$

23. $(\mathbb{R}, *)$ where $a * b := |a + b|$

24. $(\mathbb{R}, *)$ where $a * b := a + b + 3$

25. $(\mathbb{R}, *)$ where $a * b := a + b + ab$

26. $(\mathbb{Z}, *)$ where $a * b := a - b$

27. $(\mathbb{Q}, *)$ where $a * b := ab + 1$

28. $(\mathbb{Q}, *)$ where $a * b := ab/2$

29. $(\mathbb{Z}^+, *)$ where $a * b := 2^{ab}$

30. $(\mathbb{Z}^+, *)$ where $a * b := a^b$

31. $(a, b) + (c, d) := (a + c, b + d)$ on $\mathbb{R}^2$

32. $(a, b) * (c, d) := (ac - bd, ad + bc)$ on the set $\mathbb{R} \times \mathbb{R}$ without the origin

Integers mod n

33. $(\mathbb{Z}_n, +)$, addition mod n

34. $(\mathbb{Z}_n, \cdot)$, multiplication mod n

35. $A = \mathbb{Z}_3 \times \mathbb{Z}_4$, where $(a, b) + (c, d) := (a + c \mod 3, b + d \mod 4)$

36. $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where $(a,b,c) + (x,y,z) := (a + x \mod 2,\ b + y \mod 2,\ c + z \mod 2)$

Matrices

37. $(M_{r\times s}(\mathbb{R}), +)$, $r \times s$ matrices with matrix addition

38. $(M_{r\times s}(\mathbb{R}),\ \cdot)$, $r \times s$ matrices with matrix multiplication

39. $(M_n(\mathbb{R}), +)$, square $n \times n$ matrices

40. $(M_n(\mathbb{R}),\ \cdot)$, matrix multiplication

41. $(M_{r\times s}(T), +)$, $r \times s$ matrices with matrix addition (T is a set closed under addition)

42. $(M_{r\times s}(T)\ \cdot)$, $r \times s$ matrices with matrix multiplication (T is a set closed under addition and multiplication)

43. $A = M_2(\mathbb{R})$, where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x & y \\ z & w \end{bmatrix} := \begin{bmatrix} a+x & 0 \\ 0 & d+w \end{bmatrix}$

44. Let H the subset of $M_2(\mathbb{R})$ defined as:

$$H = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \ : \ a, b \in \mathbb{R} \right\}.$$

The operation is matrix addition.

45. Let H the subset of $M_2(\mathbb{R})$ defined as:

$$H = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \ : \ a, b \in \mathbb{R} \right\}.$$

The operation is matrix multiplication.

46. Let D the subset of $M_2(\mathbb{R})$ defined as:

$$D = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \ : \ a, b \in \mathbb{R} \right\}.$$

The operation is matrix multiplication.

Functions

Recall from algebra/precalculus the ways we can operate on functions:

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$
$$(f \circ g)(x) = f(g(x))$$

Let S be a set and $\mathcal{F}(S) = \{f : S \to S\}$ the set of functions from S to S. In particular, $\mathcal{F}(\mathbb{R})$ is the set of real-valued functions having as domain the set $\mathbb{R}$ of all real numbers.

47. $(\mathcal{F}(\mathbb{R}), +)$

48. $(\mathcal{F}(\mathbb{R}), -)$

49. $(\mathcal{F}(\mathbb{R}), \cdot)$

50. $(\mathcal{F}(\mathbb{R}), \circ)$

Units modulo n

51. $U(8)$, multiplication mod 8 (for any $n \geq 2$)

52. $U(n)$, multiplication mod n (for any $n \geq 2$)

53. $U(3) \times U(4)$, where $(a, b) \cdot (c, d) := (ac \mod 3, bd \mod 4)$

54. $U(m) \times U(n)$, where $(a, b) \cdot (c, d) := (ac \mod m, bd \mod n)$