Permutations (cont.)

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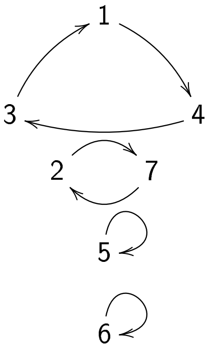
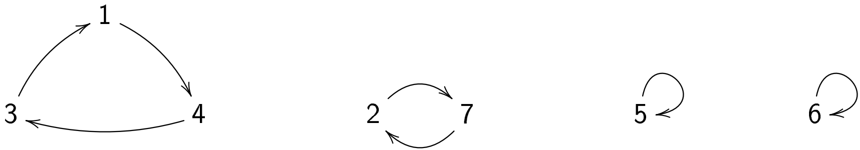
Cycle notation

Suppose we are given the following permutation and we wish to write it in cycle notation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 7 & 1 & 3 & 5 & 6 & 2 \end{pmatrix}$$



$$(1\ 4\ 3)(2\ 7)(5)(6) = (1\ 4\ 3)(2\ 7)$$



Note: $(1\ 4\ 3) = (4\ 3\ 1) = (3\ 1\ 4)$ all represent the same cycle
Convention: Start with the smallest number. After closing the loop, start with the next smallest number not in the loop.

Cycle notation

Let
$$\phi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$
Alternative cyclic view: $1 \to 3 \to 2 \to 1$
Cycle notation:
$$\phi = (1 \;\; 3 \;\; 2)$$

Cycle notation (cont.)

Let

$$\psi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}.$$

Then $1 \to 2 \to 5 \to 1$, and $3 \to 4 \to 3$

Cycle notation:

$$\psi = (1 \;\; 2 \;\; 5)(3 \;\; 4)$$

In general,

$$(a_1 \quad a_2 \quad \cdots \quad a_n)$$

represents the permutation with assignments

$$a_1 \to a_2 \to a_3 \to \cdots \to a_{n-1} \to a_n \to a_1.$$

It is a cycle of **length** n.

The cycle of length 1 represents a fixed element. They are not explicitly written but implicitly understood.
e.g. (2) fixes the element 2.

Question: Which elements in S_6 does the cycle $(1 \quad 3 \quad 4)$ fix?

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Question: Which elements in S_6 does the cycle $(1 \quad 3 \quad 4)$ fix? 2,5,6

Exercises

Write in cycle notation.

1. $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 4 & 3 & 1 & 5 & 2 \end{pmatrix}$$

2. $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 7 & 3 & 1 & 8 & 6 & 2 & 5 \end{pmatrix}$$

Inverses

The inverse of a permutation reverses the order by going in the opposite direction.

$$\sigma = (a_1 \quad a_2 \quad \cdots \quad a_n)$$

$$\sigma^{-1} = (a_1 \quad a_n \quad a_{n-1} \quad \cdots \quad a_3 \quad a_2).$$

Inverses (cont.)

So if

$$\sigma = (1 \;\; 2 \;\; 3 \;\; 4 \;\; 5),$$

$$\sigma^{-1} = (5 \;\; 4 \;\; 3 \;\; 2 \;\; 1),$$

and by convention, realign starting with the smallest number,

$$\sigma^{-1} = (1 \;\; 5 \;\; 4 \;\; 3 \;\; 2).$$

So if

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and by convention, realign starting with the smallest number,

$$\sigma^{-1} = (1 \;\; 5 \;\; 4 \;\; 3 \;\; 2).$$

Note: You can also think of writing this by keeping the first number the same and writing the rest of the numbers in reverse order.

Composing cycles

Let $f, g \in S_4$, $f = (1\ \ 4\ \ 3)$, $\quad g = (2\ \ 3)$. Determine fg.
(Remember: composition is right to left, and $(2\ \ 3)$ fixes 1.)

g	f	fg
$\boldsymbol{1} \mapsto 1$	$1 \mapsto \boldsymbol{4}$	$1 \mapsto 4$
$\boldsymbol{4} \mapsto 4$	$4 \mapsto \boldsymbol{3}$	$4 \mapsto 3$
$\boldsymbol{3} \mapsto 2$	$2 \mapsto \boldsymbol{2}$	$3 \mapsto 2$
$\boldsymbol{2} \mapsto 3$	$3 \mapsto \boldsymbol{1}$	$2 \mapsto 1$

Let $f, g \in S_4$, $f = (1 \;\; 4 \;\; 3)$, $\quad g = (2 \;\; 3)$. Determine fg.
(Remember: composition is right to left, and $(2 \;\; 3)$ fixes 1.)

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$\boldsymbol{1} \mapsto 1$	$1 \mapsto \boldsymbol{4}$	$1 \mapsto 4$
$\boldsymbol{4} \mapsto 4$	$4 \mapsto \boldsymbol{3}$	$4 \mapsto 3$
$\boldsymbol{3} \mapsto 2$	$2 \mapsto \boldsymbol{2}$	$3 \mapsto 2$
$\boldsymbol{2} \mapsto 3$	$3 \mapsto \boldsymbol{1}$	$2 \mapsto 1$

Hence

$$fg = (1 \;\; 4 \;\; 3)(2 \;\; 3) = (1 \;\; 4 \;\; 3 \;\; 2).$$

Let $\sigma, \tau \in S_7$, $\sigma = (1\ \ 4\ \ 3\ \ 7\ \ 5)$, $\quad \tau = (1\ \ 2\ \ 3\ \ 5\ \ 7)$. Determine $\sigma\tau$.

τ	σ	$\sigma\tau$
$\boldsymbol{1} \mapsto 2$	$2 \mapsto \boldsymbol{2}$	$1 \mapsto 2$
$\boldsymbol{2} \mapsto 3$	$3 \mapsto \boldsymbol{7}$	$2 \mapsto 7$
$\boldsymbol{7} \mapsto 1$	$1 \mapsto \boldsymbol{4}$	$7 \mapsto 4$
$\boldsymbol{4} \mapsto 4$	$4 \mapsto \boldsymbol{3}$	$4 \mapsto 3$
$\boldsymbol{3} \mapsto 5$	$5 \mapsto \boldsymbol{1}$	$3 \mapsto 1$
$\boldsymbol{5} \mapsto 7$	$7 \mapsto \boldsymbol{5}$	$5 \mapsto 5$
$\boldsymbol{6} \mapsto 6$	$6 \mapsto \boldsymbol{6}$	$6 \mapsto 6$

Let $\sigma, \tau \in S_7$, $\sigma = (1\ \ 4\ \ 3\ \ 7\ \ 5)$, $\quad \tau = (1\ \ 2\ \ 3\ \ 5\ \ 7)$. Determine $\sigma\tau$.

τ	σ	$\sigma\tau$
$\boldsymbol{1} \mapsto 2$	$2 \mapsto \boldsymbol{2}$	$1 \mapsto 2$
$\boldsymbol{2} \mapsto 3$	$3 \mapsto \boldsymbol{7}$	$2 \mapsto 7$
$\boldsymbol{7} \mapsto 1$	$1 \mapsto \boldsymbol{4}$	$7 \mapsto 4$
$\boldsymbol{4} \mapsto 4$	$4 \mapsto \boldsymbol{3}$	$4 \mapsto 3$
$\boldsymbol{3} \mapsto 5$	$5 \mapsto \boldsymbol{1}$	$3 \mapsto 1$
$\boldsymbol{5} \mapsto 7$	$7 \mapsto \boldsymbol{5}$	$5 \mapsto 5$
$\boldsymbol{6} \mapsto 6$	$6 \mapsto \boldsymbol{6}$	$6 \mapsto 6$

$$\sigma\tau = (1\ \ 4\ \ 3\ \ 7\ \ 5)(1\ \ 2\ \ 3\ \ 5\ \ 7) = (1\ \ 2\ \ 7\ \ 4\ \ 3).$$

Note: We started by following the path of 1 until it looped back. Sometimes we need to repeat this process until all numbers are accounted for. We may end up with more than one cycle. To be consistent, and to stick with convention, all our cycles will begin with the smallest number.

Let $f, g \in S_6$, $f = (3\ \ 5\ \ 4)$, $\quad g = (1\ \ 6\ \ 3\ \ 2\ \ 4\ \ 5)$. Determine gf.
(Remember: composition is right to left, and $(2\ \ 3)$ fixes 1.)

f	g	gf
$\boldsymbol{1} \mapsto 1$	$1 \mapsto \boldsymbol{6}$	$1 \mapsto 6$
$\boldsymbol{6} \mapsto 6$	$6 \mapsto \boldsymbol{3}$	$6 \mapsto 3$
$\boldsymbol{3} \mapsto 5$	$5 \mapsto \boldsymbol{1}$	$3 \mapsto 1$
$\boldsymbol{2} \mapsto 2$	$2 \mapsto \boldsymbol{4}$	$2 \mapsto 4$
$\boldsymbol{4} \mapsto 3$	$3 \mapsto \boldsymbol{2}$	$4 \mapsto 2$
$\boldsymbol{5} \mapsto 4$	$4 \mapsto \boldsymbol{5}$	$5 \mapsto 5$

Composing cycles

Let $f, g \in S_6$, $f = (3\ \ 5\ \ 4)$, $\quad g = (1\ \ 6\ \ 3\ \ 2\ \ 4\ \ 5)$. Determine gf. (Remember: composition is right to left, and $(2\ \ 3)$ fixes 1.)

f	g	gf
$\boldsymbol{1} \mapsto 1$	$1 \mapsto \boldsymbol{6}$	$1 \mapsto 6$
$\boldsymbol{6} \mapsto 6$	$6 \mapsto \boldsymbol{3}$	$6 \mapsto 3$
$\boldsymbol{3} \mapsto 5$	$5 \mapsto \boldsymbol{1}$	$3 \mapsto 1$
$\boldsymbol{2} \mapsto 2$	$2 \mapsto \boldsymbol{4}$	$2 \mapsto 4$
$\boldsymbol{4} \mapsto 3$	$3 \mapsto \boldsymbol{2}$	$4 \mapsto 2$
$\boldsymbol{5} \mapsto 4$	$4 \mapsto \boldsymbol{5}$	$5 \mapsto 5$

Hence

$$gf = (1\ \ 6\ \ 3\ \ 2\ \ 4\ \ 5)(3\ \ 5\ \ 4) = (1\ \ 6\ \ 3)(2\ \ 4).$$

Let $f, g, h \in S_5$, $f = (2\ \ 5\ \ 4)$, $\quad g = (1\ \ 3)$, $\quad h = (1\ \ 3\ \ 2)$.
Determine fgh.

h	g	f	fgh
$\boldsymbol{1} \mapsto 3$	$3 \mapsto 1$	$1 \mapsto \boldsymbol{1}$	$1 \mapsto 1$
$\boldsymbol{2} \mapsto 1$	$1 \mapsto 3$	$3 \mapsto \boldsymbol{3}$	$2 \mapsto 3$
$\boldsymbol{3} \mapsto 2$	$2 \mapsto 5$	$5 \mapsto \boldsymbol{4}$	$3 \mapsto 4$
$\boldsymbol{4} \mapsto 4$	$4 \mapsto 4$	$4 \mapsto \boldsymbol{2}$	$4 \mapsto 2$

Composing cycles

Let $f, g, h \in S_5$, $f = (2\ \ 5\ \ 4)$, $\quad g = (1\ \ 3)$, $\quad h = (1\ \ 3\ \ 2)$.
Determine fgh.

h	g	f	fgh
$\boldsymbol{1} \mapsto 3$	$3 \mapsto 1$	$1 \mapsto \boldsymbol{1}$	$1 \mapsto 1$
$\boldsymbol{2} \mapsto 1$	$1 \mapsto 3$	$3 \mapsto \boldsymbol{3}$	$2 \mapsto 3$
$\boldsymbol{3} \mapsto 2$	$2 \mapsto 5$	$5 \mapsto \boldsymbol{4}$	$3 \mapsto 4$
$\boldsymbol{4} \mapsto 4$	$4 \mapsto 4$	$4 \mapsto \boldsymbol{2}$	$4 \mapsto 2$

Hence

$$fgh = (2\ \ 3\ \ 4).$$

Exercise

Let $\alpha, \beta \in S_5$ where

$$\alpha = (1\ \ 5\ \ 3), \qquad \beta = (2\ \ 3\ \ 4\ \ 5).$$

Compute $\beta\beta^{-1}$ and α^3.

Order of a cycle

If π is any permutation, the least positive integer n such that $\pi^n = \varepsilon$ is called the order of π.

e.g. $\pi = (1 \;\; 2)$, then $\pi^2 = (1 \;\; 2)(1 \;\; 2) = \varepsilon$, so $|\pi| = 2$.

Odd and Even Permutations

Definition: A 2-cycle is called a transposition. e.g. $(a_1 \quad a_2)$

FACT 1: Every permutation is either the identity, a single cycle, or a product of disjoint cycles.

FACT 2: Every permutation can be written as a product of transpositions

$$(a_1 \quad a_2 \quad \cdots \quad a_n) = (a_1 \quad a_n)(a_1 \quad a_{n-1}) \cdots (a_1 \quad a_2)$$
$$(1 \quad 4 \quad 5 \quad 2) = (1 \quad 2)(1 \quad 5)(1 \quad 4)$$

Odd and Even Permutations

Odd permutation: odd number of transpositions.

$$(2\ \ 3), \qquad (1\ \ 4)(1\ \ 3)(2\ \ 3)(4\ \ 5)(3\ \ 6)$$

Even permutation: even number of transpositions.

$$(2\ \ 3)(1\ \ 3), \qquad (1\ \ 4)(1\ \ 3)(2\ \ 3)(4\ \ 5)$$

Recall that a cycle can be broken down into transpositions:

- odd-length cycle: $(1\ \ 2\ \ 3) = (1\ \ 3)(1\ \ 2)$, even permutation
- even-length cycle: $(1\ \ 2\ \ 3\ \ 4) = (1\ \ 4)(1\ \ 3)(1\ \ 2)$, odd permutation

Recall that a cycle can be broken down into transpositions:

- odd-length cycle: $(1\ \ 2\ \ 3) = (1\ \ 3)(1\ \ 2)$, even permutation
- even-length cycle: $(1\ \ 2\ \ 3\ \ 4) = (1\ \ 4)(1\ \ 3)(1\ \ 2)$, odd permutation

This is true in general.

Even permutations

The even permutations of the symmetric group S_n form a subgroup, called the alternating group A_n with half as many elements. That is

$$|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}.$$

S_3 and A_3

$$S_3 = \{(1),\ (1\ \ 2),\ (1\ \ 3),\ (2\ \ 3),\ (1\ \ 2\ \ 3),\ (1\ \ 3\ \ 2)\}$$

For even permutations, find odd-length cycles:

$$A_3 = \{(1),\ (1\ \ 2\ \ 3),\ (1\ \ 3\ \ 2)\}$$

Odd or even?

1. $(1\ \ 2)(3\ \ 4)$ in S_4
2. $(1\ \ 2)(3\ \ 4\ \ 5\ \ 6)$ in S_6
3. $(1\ \ 4\ \ 7\ \ 6)(2\ \ 3)$ in S_8
4. $(2\ \ 5\ \ 3)(1\ \ 3\ \ 4)$ in S_7.

Sign of a Permutation

Let σ be a permutation where if σ is even/odd then m is even/odd.

$$sgn(\sigma) = (-1)^m$$

In other words:

$$sgn(\sigma) = \begin{cases} 1, & \sigma \text{ is even} \\ -1, & \sigma \text{ is odd} \end{cases}$$

Determinant Formula

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$$

$$\det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

Homework

1. Let $\alpha = (1\ \ 3\ \ 2) \in S_3$. Find $\beta\alpha\beta^{-1}$ for each $\beta \in S_3$.
2. For ease of notation, let ε be the identity, $\alpha = (2\ 3)$, $\beta = (1\ 2)$, $\gamma = (1\ 3)$, $\sigma = (1\ 2\ 3)$, $\tau = (1\ 3\ 2)$. Construct the Cayley table for S_3 using cycle notation. State the inverse of each element.
3. What is the order of S_4? List all the elements of S_4. List all the even permutations in S_4.
4. Find the order of $\tau = (2\ \ 5\ \ 4\ \ 3)$ in S_5.
5. Which elements of S_4 fix 1?
6. Which elements of S_5 fix 1 and 2?
7. Let $\sigma = (1\ \ 3)(2\ \ 4\ \ 5)$. Determine $\langle\sigma\rangle$.