Mappings

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Give some examples of functions.

In Calculus, you saw **real-valued functions**, where the range was a subset of real values. If you have made it past Calculus II, you may have seen vector-valued functions, as well.

(In fact, most if not all the functions you've worked with so far in your life took in real numbers as input, as well. So your *domain* and *range* were the set of real numbers, or they were at least some subset of the real numbers.)

Representing functions

- Graph ($\mathbb{R}^2$ or $\mathbb{R}^3$)
- Definition

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = x^2$$

$$g : \mathbb{R}^+ \to \mathbb{R}, \qquad g(x) = \ln x$$

Maybe you've even seen functions on several variables, $\mathbb{R}^n \to \mathbb{R}^m,\ n, m \in \mathbb{N}$ such as

$$h : \mathbb{R}^2 \to \mathbb{R}^3, \qquad h(x, y) = (x, \cos x^2, \sin xy)$$

$$j : \mathbb{R}^2 \to \mathbb{R}^2, \qquad j(r, \theta) = (r\cos\theta, r\sin\theta)$$

Formal Definition

Recall: Relations defined on subsets of $A \times B$

mapping/map/function: A relation $f \subset A \times B$ is said to be a mapping from a set A to a set B if for each element $a \in A$ there is a unique element $b \in B$ such that $(a, b) \in f$.

Notation:
"f maps a to b"
$f : A \to B$, $f(a) = b$
$f : a \to b$
"a maps to b"
$a \mapsto b$

$$f : A \to B$$

- Domain: A
- Codomain: B
- Image: $f(A)$

- Invalid: one-to-many
- Valid: many-to-one (examples?)

A mapping is **well-defined** if

(i) every assignment is unique (one element can't have two assignments), and

(ii) it is defined for every element in the domain.

Property (i) really states: If $a_1 = a_2$, then $f(a_1) = f(a_2)$.

The following functions are not well-defined. Why not?

(a) $\alpha : \mathbb{Q} \to \mathbb{Z}, \quad \alpha(\frac{p}{q}) = p,$

(b) $\beta : \mathbb{Q} \to \mathbb{Q}, \quad \beta(\frac{p}{q}) = \frac{p-1}{q-1},$

(c) $\gamma : \mathbb{R} \to \mathbb{R}, \quad \gamma(x) = \log_3(x-5),$

The identity mapping on a set A is: $id_A(a) = id(a) = a \quad \forall a \in A$.

- If $a \in A$, then the image of a under f is $f(a) = b$ (the value of f when applied to a.)
- If $S \subseteq A$, the image of a S under f is

$$f(S) = \{f(x) \mid x \in S\}$$

- The image of f is the image $f(A)$ of the entire domain A of f.

$$f : A \to B$$

- Domain: A
- Codomain: B
- Image: $f(A)$
- $f(A) \subseteq B$
- **Onto/Surjective**: $f(A) = B$

Are $f(x) = x^2$ and $f(x) = x^3$ surjective when $f : \mathbb{R} \to \mathbb{R}$?

- Inverse/Preimage set of an element:

$$f^{-1}(b) = \{a \in A \mid f(a) = b\}$$

- Inverse/Preimage set of a subset: If $T \subseteq B$,

$$f^{-1}(T) = \{a \in A \mid f(a) \in T\}$$

These sets can be defined even when f^{-1} does not exist.

A map $g : B \to A$ is an **inverse mapping** of $f : A \to B$ if

(i) $g \circ f = id_A$

(ii) $f \circ g = id_B$

So g "reverses" f, and vice versa. We say f is **invertible** if it has an inverse, and $g = f^{-1}$ and $f = g^{-1} = f$.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$. Then

- Suppose $b = 4$. Then $f^{-1}(b) =$
- Suppose $T = \{4, 9\}$. Then $f^{-1}(T) =$
- One reason why f^{-1} does not exist is because f is not surjective. Why?

Definition of injective

Informal: An element in the codomain can only be hit once. Must be one-to-one.
Formal: A mapping f is said to be **injective** if for all $a_1, a_2 \in A$, if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$. Or the contrapositive holds:

$$\text{For all } a_1, a_2 \in A, \text{ if } f(a_1) = f(a_2), \text{ then } a_1 = a_2.$$

Definition of surjective

Informal: Every element in the codomain gets hit, possibly more than once. That is, every element in the codomain has at least one preimage.
Formal: A mapping f is said to be **surjective** if

$$\text{For all } b \in B, \text{ there exists } a \in A \text{ such that } f(a) = b.$$

Equivalently, $f(A) = B$.

A **bijective** mapping is both injective and surjective.

Theorem

A mapping is invertible if and only if it is both injective and surjective.

Theorem

Let $f : A \to B$, $g : B \to C$, and $h : C \to D$. Then

1. *Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$*
2. *If f and g are injective, then $g \circ f$ is injective.*
3. *If f and g are surjective, then $g \circ f$ is surjective.*
4. *If f and g are bijective, then $g \circ f$ is bijective.*

Determine whether the following mappings are injective, surjective, or bijective.

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$
2. $h : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$
3. $g : \mathbb{R} \to \mathbb{R},\ g(x) = 2x - 1$
4. Restriction of g to $\mathbb{Z}$: $g|_{\mathbb{Z}} = \hat{g} : \mathbb{Z} \to \mathbb{Z},\ \hat{g}(x) = 2x - 1$
5. $k : \mathbb{R} \to \mathbb{R},\ k(t) = 5$

6. $\alpha : \mathbb{R}^2 \to \mathbb{R},\ \alpha(x, y) = xy$
7. $\varphi : A \times B \to A,\ \varphi(a, b) = a$
8. $\gamma : S \to S \times \{0\},\ \gamma(s) = (s, 0)$
9. $\psi_1 : A^3 \to A^3,\ \psi_1(a, b, c) = (a, b, a)$
10. $\psi_2 : A^3 \to A^3,\ \psi_2(a, b, c) = (c, a, b)$

Linear transformations

Linear transformations are mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$, $n, m \in \mathbb{N}$.
Example: $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x + 4y,\ 2x - 3y)$ This can also be represented my matrix multiplication of some coefficient matrix A by a vector of unknowns $(x, y)^T$:

$$\begin{pmatrix} 1 & 4 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 4y \\ 2x - 3y \end{pmatrix}$$

Permutations

A **permutation** of a set A is bijective mapping on itself.

Example: Define the set $X_4 = \{1, 2, 3, 4\}$. Let $\pi : X_4 \to X_4$ be defined as: $\pi(1) = 2,\ \pi(2) = 4,\ \pi(3) = 3,\ \pi(4) = 1.$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \pi(1) & \pi(2) & \pi(3) & \pi(4) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$