Equivalence relation $\sim$ on set A satisfies the following properties:

i) $\forall a \in A,\ a \sim a.$ (Reflexive)

ii) $\forall a, b \in A,\ a \sim b \Rightarrow b \sim a.$ (Symmetric)

iii) $\forall a, b, c \in A,\ a \sim b$ and $b \sim c \Rightarrow a \sim c.$ (Transitive)

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1 $m \sim n$ in $\mathbb{Z}$ if $mn > 0$

☆ WTS (Want to show)

Reflexive:

WTS: $\forall m \in \mathbb{Z},\ m \sim m$

i.e. $\forall m \in \mathbb{Z},\ m \cdot m > 0.$

Let $m \in \mathbb{Z}$. Then $m^2 > 0$, so $m \sim m$.

Symmetric:

WTS: $\forall m, n \in \mathbb{Z},\ m \sim n \Rightarrow n \sim m.$

i.e. $\forall m, n \in \mathbb{Z},\ mn > 0 \Rightarrow nm > 0.$

Let $m, n \in \mathbb{Z}$ such that $mn > 0$.

Since multiplication on the integers is commutative,

$nm = mn > 0$, so $nm > 0$ and $n \sim m$.

Transitive:

WTS: $\forall m, n, p \in \mathbb{Z},\ m \sim n$ and $n \sim p \Rightarrow m \sim p.$

i.e. $\forall m, n, p \in \mathbb{Z}$, if $mn > 0$ and $np > 0$ then $mp > 0$.

Let $m, n, p \in \mathbb{Z}$ such that $mn > 0$ and $np > 0$.

Recall that a product of two integers is positive if either both integers are positive, or both are negative.

Case 1: Assume $m > 0$ and $n > 0$.

~~Th~~ Since $np > 0$, we must have $p > 0$. Therefore $mp > 0$.

Case 2: Assume $m < 0$ and $n < 0$.

Since $np > 0$, we must have $p < 0$. Therefore $mp > 0$.

(2) $a \sim b$ in $\mathbb{R}$ if $a - b \in \mathbb{Z}$

Reflexive:

WTS: $\forall a \in \mathbb{R},\ a \sim a$

i.e. $\forall a \in \mathbb{R},\ a - a \in \mathbb{Z}$

Let $a \in \mathbb{R}$. Then $a - a = 0 \in \mathbb{Z}$.

Symmetric.

WTS: $\forall a, b \in \mathbb{R},\ a \sim b \rightarrow b \sim a$

i.e. $\forall a, b \in \mathbb{R}$, if $a - b \in \mathbb{Z}$ then $b - a \in \mathbb{Z}$

Let $a, b \in \mathbb{R}$ such that $a - b \in \mathbb{Z}$.

Then there is an integer t such that $a - b = t$.

So $b - a = -(a-b) = -t$ is still an integer.

Transitive:

WTS: $\forall a, b, c \in \mathbb{R},\ a \sim b \land b \sim c \rightarrow a \sim c$

i.e. $\forall a, b, c \in \mathbb{R}$, if $a - b \in \mathbb{Z}$ and $b - c \in \mathbb{Z}$, then $a - c \in \mathbb{Z}$.

Let $a, b, c \in \mathbb{R}$ such that $a - b \in \mathbb{Z}$ and $b - c \in \mathbb{Z}$.

Then $a - b = t$ and $b - c = s$ for some $r, s \in \mathbb{Z}$.

Adding both sides of these two equations, we get

$$(a-b) + (b-c) = t + s$$

$$a - c = t + s.$$

~~But $t+s \in \mathbb{Z}$~~, So $a - c \in \mathbb{Z}$ since t and s are integers.

③ $a \sim b$ in $\mathbb{Z}$ if $3a+b$ is a multiple of 4

Reflexive:

$\boxed{\begin{array}{l}\text{WTS: } \forall a \in \mathbb{Z},\ a \sim a \\ \text{i.e. } \forall a \in \mathbb{Z},\ 3a+a \text{ is a multiple of } 4\end{array}}$

Let $a \in \mathbb{Z}$. Then $3a+a = 4a$ is a multiple of 4.

Symmetric:

$\boxed{\begin{array}{l}\text{WTS: } \forall a,b \in \mathbb{Z},\ a \sim b \rightarrow b \sim a \\ \text{i.e. } \forall a,b \in \mathbb{Z}, \text{ if } 3a+b \text{ is a multiple of } 4, \\ \qquad\qquad \text{then } 3b+a \text{ is a multiple of } 4.\end{array}}$

Let $a,b \in \mathbb{Z}$ such that $3a+b$ is a multiple of 4.
So $3a+b = 4k$ for some integer k. By applying algebraic techniques we have

$$3a+b = 4k$$
$$3(3a+b) = 3(4k)$$
$$9a+3b = 12k$$
$$8a+(a+3b) = 12k$$
$$a+3b = 12k-8a = 4(3k-2a).$$

Hence $3b+a$ is a multiple of 4.

Transitive:

$\boxed{\begin{array}{l}\text{WTS: } \forall a,b,c \in \mathbb{Z},\ a \sim b \wedge b \sim c \rightarrow a \sim c \\ \text{i.e. } \forall a,b,c \in \mathbb{Z}, \text{ if } 3a+b \in 4\mathbb{Z} \text{ and } 3b+c \in 4\mathbb{Z}, \text{ then } 3a+c \in 4\mathbb{Z}.\end{array}}$

Let $a,b,c \in \mathbb{Z}$ such that $3a+b \in 4\mathbb{Z}$ and $3b+c \in 4\mathbb{Z}$.
Thus $3a+b = 4t$ and $3b+c = 4s$ for some $t,s \in \mathbb{Z}$.
Adding both sides of the two equations, we get

$$(3a+b)+(3b+c) = 4t+4s$$
$$3a+4b+c = 4(t+s)$$
$$3a+c = 4(t+s-b).$$

Hence $3a+c$ is a multiple of 4.