Equivalence Relations and Classes

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Equivalence relation

A relation is said to be an equivalence relation if it is reflexive, symmetric, and transitive.

If $\sim$ is an equivalence relation on A, then $\sim$ partitions A into disjoint subsets called equivalence classes:

$$[a] = \{x \in A \mid x \sim a\}$$

- $\displaystyle\bigcup_{a \in A} [a] = A$
- If $b \in [a]$, then $[b] = [a]$.
- If $[a] \neq [b]$, then $[a] \cap [b] = \emptyset$.

Representatives

If $[a] = \{a, b, c, ...\}$, then $[a] = [b] = [c] = \cdots$.
Any one of the elements may serve as a class representative since they all represent the same equivalence class.

Example

$\frac{a}{b} \sim \frac{c}{d}$ on $\mathbb{Q}$ defined as $ad = bc$:
For any $\frac{a}{b} \in \mathbb{Q}$,

$$\begin{aligned}
\left[\frac{a}{b}\right] &= \left\{ \frac{x}{y} \in \mathbb{Q} \mid \frac{x}{y} \sim \frac{a}{b} \right\} \\
&= \left\{ \frac{x}{y} \in \mathbb{Q} \mid xb = ya \right\} \\
&= \left\{ \frac{x}{y} \in \mathbb{Q} \mid \frac{x}{y} = \frac{a}{b} \right\}
\end{aligned}$$

Example

$\ell_1 \parallel \ell_2$ on $\mathcal{L}$, the set of lines:

$$[\ell] = \{\ell' \in \mathcal{L} \ : \ \ell' \parallel \ell\}$$

A line can be defined as $\ell : mx + b$, where $m, b \in \mathbb{R}$. Let's abbreviate this as $\ell_{(m,b)}$.

$$\begin{aligned} [\ell_{(m,b)}] &= \{\ell' \in \mathcal{L} \ : \ \ell' \parallel \ell_{(m,b)}\} \\ &= \{\ell_{(m,c)} \ : \ c \in \mathbb{R}\} \end{aligned}$$

Example

$x \sim y$ on $\mathbb{R}$ iff $x = y$

$$\begin{aligned}[x] &= \{y \in \mathbb{R} \mid y \sim x\} \\ &= \{y \in \mathbb{R} \mid y = x\} \\ &= x\end{aligned}$$

Example

$(a, b) \sim (c, d)$ on $\mathbb{R}^2$ means $a^2 + b^2 = c^2 + d^2$

$$
\begin{aligned}
[(a, b)] &= \{(x, y) \in \mathbb{R}^2 \ : \ (x, y) \sim (a, b)\} \\
&= \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = a^2 + b^2\} \\
&= \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = R \text{ where } R = a^2 + b^2\} \\
&= \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = r^2 \text{ where } r^2 = a^2 + b^2\} \\
&= \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = r^2 \text{ where } r = \sqrt{a^2 + b^2}\}
\end{aligned}
$$

On circle of same radius centered at origin