1 Set the Stage

1. List the elements in each set.

   (a) $\{y \in \mathbb{R} \mid y^4 = 1\}$

   (b) $\{y \in \mathbb{C} \mid y^4 = 1\}$

   (c) $\left\{ \sum_{i=1}^{k} 3i - 2 \mid k = 1, 2, 3 \right\}$

   (d) $\left\{ \prod_{k=0}^{n-1} (q^n - q^k) \mid n = 1, 2, 3, 4 \right\}$ (Hint: Results will be expressions, not numbers.)

2. List at least 3 elements in each set, if possible.

   (a) $A = \{x^3 \ : \ -8 \leq x \leq 8, \ x \in \mathbb{R}\}$

   (b) $B = \{x^3 \ : \ |x| \leq 8, \ x \in \mathbb{Z}\}$

   (c) $C = \{x \in \mathbb{R} \ : \ |x^3| \leq 8\}$

   (d) $D = \{x \in \mathbb{Z} \ : \ |x^3| \leq 8\}$

   (e) $A \setminus B$, with A, B defined as above.

   (f) $A \cap D$, where A, D are defined above.

3. List at least 3 elements in each set, if possible.

   (a) $4 + 3\mathbb{Z} = \{4 + 3k \ : \ k \in \mathbb{Z}\}$

   (b) $\{x \in \mathbb{Z} \mid x - 2 \text{ is a multiple of } 5\}$

   (c) $\{(a, b, a^2 + b^2) \mid a, b \in \mathbb{N}\}$

   (d) $\{2^n 5^m \mid n, m \in \mathbb{Z}\}$

   (e) $\{(x, y) \in \mathbb{R}^2 \ : \ y - x^2 = 0\}$

   (f) $\{f \in \mathcal{F}(\mathbb{R}) \mid f'(x) = 2x + 5\}$, where $\mathcal{F}(\mathbb{R})$ is the set of real-valued functions, $f : \mathbb{R} \to \mathbb{R}$ (think Calculus)

   (g) $\left\{\frac{a}{b} \in \mathbb{Q} \ : \ b \neq 1, \ \exists n \in \mathbb{Z} \ s.t. \ b = na\right\}$

   (h) $\mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3}\}$

   (i) $\mathbb{Z}[\sqrt{-1}] = \{a + b\sqrt{-1}\}$

2 Relations: Find your Relatives

1. $\{(x, 2x) : x \in \mathbb{R}\}$

2. $\{(x, \frac{1}{x+1}) : x \in \mathbb{R}\}$

3. $\{(x^3, x) : x \in \mathbb{R}\}$

2.1 "Who you with?" - *Bernie Mac*

List elements in the relation.

2.2 Not even distant cousins...

List elements not in the relation.

3 Functions: We're relations too

3.1 A rose by another other name

What functions do these represent?

1. $\{(x, 2x) : x \in \mathbb{R}\}$

2. $\{(x, \frac{1}{x+1}) : x \neq -1,\ x \in \mathbb{R}\}$

3. $\{(x^3, x) : x \in \mathbb{R}\}$

3.2 In-and-Out

1. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 3x + 1$

2. $g : \mathbb{Z} \to \mathbb{Z}, \quad g(x) = 3x + 1$

3. $f : \mathbb{R} \to \mathbb{R}, \quad f(s) = 2s^4$

4. $\psi : \mathbb{R} \to \mathbb{R}, \quad \psi(x) = 7x^3 - 1$

5. $\beta : \mathbb{R} \to \mathbb{R}, \quad \beta(t) = \cos t$

6. $f : (7, \infty) \to \mathbb{R}, \quad f(x) = \log_5(x - 7)$

7. $h : \mathbb{Q} \to \mathbb{Q}, \quad h\left(\frac{a}{b}\right) = \frac{a}{b^2}$

8. $j : \mathbb{Q} \to \mathbb{Q}, \quad j\left(\frac{a}{b}\right) = \frac{b}{a}$

9. $id : \mathbb{N} \to \mathbb{N}, \quad id(x) = x$

10. $k : \mathbb{Z} \to \mathbb{Z}, \quad k(n) = -n$

11. $k : \mathbb{Z} \to \mathbb{Z}, \quad k(t) = 0$

12. $\rho : \mathbb{R}^2 \to \mathbb{R}, \quad \rho\,(x, y) = x + y$

13. $\pi_2 : \mathbb{R}^4 \to \mathbb{R}, \quad \pi_2\,(x_1, x_2, x_3, x_4) = x_2$

14. $g : \mathbb{C} \to \mathbb{R}, \quad g\,(a + bi) = \sqrt{a^2 + b^2}$

15. $h : \mathbb{C}^* \to \mathbb{C}^*, \quad h\,(a + bi) = \dfrac{a}{a^2 + b^2} - \dfrac{a}{a^2 + b^2} i$

16. $f : \mathbb{Z}^2 \to \mathbb{Z}^2, \quad f\,(r, s) = (s, r)$

17. $\theta : \mathbb{N}^3 \to \mathbb{N}^3, \quad \theta\,(r, s, t) = (t, r, s)$

18. $\alpha : \mathbb{R} \to \mathbb{R}, \quad \alpha\,(x) = e^{-x^2}$

19. $d : \mathbb{R}^2 \to \mathbb{R}^2, \quad d\,((x, y), (a, b)) = \sqrt{(x - a)^2 + (y - b)^2}$

20. $\psi : \mathbb{R}^2 \to \mathbb{R}, \quad \psi\,(x, y) = xy$

21. $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2, \quad f\,((a, b), (c, d)) = (ac - bd, ad + bc)$

22. $N : \mathbb{Z}[\sqrt{5}] \to \mathbb{Z}, \quad N\left(a + b\sqrt{5}\right) = a^2 + 5b^2$

3.2.1 Determine the domain and codomain.

3.2.2 Play with the function by evaluating a variety of elements from the domain.

3.2.3 Check that the function is well-defined.

3.2.4 Check injectivity.

3.2.5 Check surjectivity.

3.2.6 If the function is non-surjective, list some elements that are in the codomain, but not in the image of the function.

3.2.7 Choose some elements in the image of the function and find their pre-images (it may be a set with more than one element.)

3.2.8 If bijective, find the inverse.

3.3 Take it back!

1. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 3x + 1$. Determine

   $f^{-1}(15),\ f^{-1}(-3),\ f^{-1}(3.2),\ f^{-1}(\frac{\pi}{4}),\ f^{-1}(\{-1, 1\}),\ f^{-1}([-1, 1])$

2. $g : \mathbb{Z} \to \mathbb{Z}, \quad g(x) = 3x + 1$. Determine $g^{-1}(7),\ g^{-1}(14),\ g^{-1}(\{7, 10\}),\ g^{-1}([0, 2])$.

3. $\theta : \mathbb{N}^3 \to \mathbb{N}^3, \quad \theta(r, s, t) = (t, r, s)$. Determine $\theta^{-1}(3, 4, 5),\ \theta^{-1}(2, 1, 2)$.

4. $k : \mathbb{Z} \to \mathbb{Z}, \quad k(n) = -n$. Determine $k^{-1}(14),\ k^{-1}(\{-100, 1100\})$.

5. $k : \mathbb{Z} \to \mathbb{Z}, \quad k(t) = 0$. Determine $k^{-1}(0),\ k^{-1}(5)$.

6. Define a function $\varphi : \mathbb{Z} \to \mathbb{Z}$ as

$$\varphi(n) = \begin{cases} \dfrac{n+1}{2}, & n \text{ odd} \\ \dfrac{n}{2}, & n \text{ even} \end{cases}$$

Determine $\varphi^{-1}(79),\ \varphi^{-1}(-79),\ \varphi^{-1}(\{-1, 0, 1\})$.

3.4 Manipulation

1. $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2, \quad f\left((a, b), (c, d)\right) = (ac - bd, ad + bc)$

   (a) Determine $f((a, b), (1, 0))$.

   (b) Determine $f((1, 0), (a, b))$.

   (c) Find (x, y) such that $f((x, y), (3, 4)) = (1, 0)$.

   (d) Find (x, y) such that $f((3, 4), (x, y)) = (1, 0)$.

   (e) Find (x, y) such that $f((a, b), (x, y)) = (1, 0)$. Your answer should be in terms of a and b. (Hint: How do you solve a linear system of equations?)

4 Not your average function

1. $\pi_i : \mathbb{R}^n \to \mathbb{R}, \quad \pi_i\left(x_1, x_2, \ldots, x_i, \ldots, x_n\right) = x_i, \quad$ for some $1 \leq i \leq n$

2. $\pi_i : \mathbb{R}^n \to \mathbb{R}^n, \quad \pi_i\left(x_1, x_2, \ldots, x_i, \ldots, x_n\right) = (0, 0, \ldots, x_i, \ldots, 0), \quad$ for some $1 \leq i \leq n$

3. $\theta : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R}), \quad \theta\left(f, g\right) = f + g$, where $(f + g)(x) = f(x) + g(x)$.

4. $\beta : \mathbb{C} \to M_2(\mathbb{R}), \quad \beta\left(a + bi\right) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

5. $\alpha : M_2(\mathbb{R}) \to M_2(\mathbb{R}), \quad \alpha\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

6. $\delta : M_2(\mathbb{R}) \to \mathbb{R}, \quad \delta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

7. $\sigma : M_2(\mathbb{R}) \to M_2(\mathbb{R}), \quad \sigma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

8. $\sigma : M_2(\mathbb{R}) \to M_2(\mathbb{R}), \quad \sigma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

9. $\sigma : M_2(\mathbb{R}) \to M_2(\mathbb{R}), \quad \sigma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

10. $Tr : M_2(\mathbb{R}) \to \mathbb{R}, \quad Tr\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$

11. $p : M_2(\mathbb{R}) \to \mathbb{R}[x], \quad p\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = x^2 - (a+d)x + (ad - bc)$

5 Abstract Algebra

Let G be a group.

1. $f : G \to G, \quad f(a) = a^{-1}$
2. $f_x : G \to G, \quad f_x(a) = xa$
3. $\sigma_x : G \to G, \quad \sigma_x(a) = xax^{-1}$

5.1 Homomorphism

5.1.1 Homomorphism?

5.1.2 Isomorphism?

(bijective homomorphism)

5.1.3 Automorphism?

(bijective homomorphism from a set back to itself)

5.1.4 Monomorphism?

(injective homomorphism)

5.1.5 Epimorphism?

(surjective homomorphism)

1. Norm: $N(ab) = N(a)N(b)$.