Permutations

Carmen M. Wright, Ph.D.

Jackson State University

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In this section, we will mainly consider permutations on finite subsets of $\mathbb{N}$ of the form $X_n = \{1, 2, \ldots, n\}$.

Let $X_3 = \{1, 2, 3\}$. We consider all the possible mappings $X_3 \to X_3$:

$$\varepsilon : \begin{array}{c} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \end{array} \qquad \alpha : \begin{array}{c} 1 \mapsto 1 \\ 2 \mapsto 3 \\ 3 \mapsto 2 \end{array} \qquad \beta : \begin{array}{c} 1 \mapsto 2 \\ 2 \mapsto 1 \\ 3 \mapsto 3 \end{array}$$

$$\sigma : \begin{array}{c} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 1 \end{array} \qquad \tau : \begin{array}{c} 1 \mapsto 3 \\ 2 \mapsto 1 \\ 3 \mapsto 2 \end{array} \qquad \gamma : \begin{array}{c} 1 \mapsto 3 \\ 2 \mapsto 2 \\ 3 \mapsto 1 \end{array}$$

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Notice ε fixes every element, so essentially it is the identity mapping on X_3.

For a permutation on 3 objects, we determine possible assignments for each element.

- 1 has three choices of an assignment
- 2 has two choices of an assignment
- 3 has one choice of an assignment

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Question: How many different mappings do we get for a set of n objects, $X_n = \{1, 2, \ldots, n\}$?

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Matrix notation

We can write the mappings

$$\varepsilon : \begin{array}{c} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \end{array} \qquad \alpha : \begin{array}{c} 1 \mapsto 1 \\ 2 \mapsto 3 \\ 3 \mapsto 2 \end{array} \qquad \beta : \begin{array}{c} 1 \mapsto 2 \\ 2 \mapsto 1 \\ 3 \mapsto 3 \end{array}$$

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in the following special matrix notation:

$$\varepsilon = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

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Read top-down to know what each element gets assigned.

Composition

In this class, we choose the convention Right-to-Left.
For example, in $\sigma \circ \beta = \sigma\beta$, apply β then apply σ.

$$\sigma : \begin{array}{c} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 1 \end{array} \qquad \beta : \begin{array}{c} 1 \mapsto 2 \\ 2 \mapsto 1 \\ 3 \mapsto 3 \end{array}$$

$$\sigma\beta(1) = \sigma(2) = 3$$

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Notice $\sigma\beta = \gamma$.
What is $\beta\sigma$?

Compositions (cont.)

Function composition is always associative,

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma$$

Function composition is not commutative,

$$\alpha\beta \neq \beta\alpha$$

Composition of bijective functions is bijective,

$$\alpha, \beta \text{ bijective} \implies \alpha \circ \beta \text{ bijective} \quad (\textit{closure})$$

Notation: A mapping α composed with itself 3 times,

$$\alpha^3 = \alpha\alpha\alpha.$$

There is an identity permutation ε such that $\sigma\varepsilon = \sigma = \varepsilon\sigma$ for all permutations $\sigma \in S_n$. The identity permutation fixes each element; it has a different form in each S_n but acts the same way. In S_3,

$$\varepsilon = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

In S_5,

$$\varepsilon = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

We know inverse permutations exist because permutations are bijective mappings, which means they are invertible.

If $\sigma, \tau \in S_n$ and $\sigma\tau = \tau\sigma = \varepsilon$, then σ and τ are *inverses* of each other. That is,

$$\tau = \sigma^{-1}, \qquad \sigma = \tau^{-1}.$$

The Symmetric Group S_n

We define S_n as the set of all permutations on $\{1, 2, \ldots, n\}$. With the operation of (mapping) composition, it is a group. Then the order of the group is

$$|S_n| = n!$$