Note for the time evolution sample for radiation fluid

The notation is complicatedly mixed from Refs. [? ? ?]. The relation is written as much as possible.

I. FLUID QUANTITIES AND DYNAMICAL EQS IN 3+1 FORM

We consider the perfect fluid whose energy-momentum tensor is given by

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \tag{1}$$

where u^μ is the fluid four-velocity and the Lorentz factor Γ is given by[1]

$$\Gamma = -u^\mu n_\mu \tag{2}$$

with n_μ being the normal one-form for the time slice. We also introduce the velocity relative to the Eulerian observer U^μ as[2]

$$u^\mu = \Gamma(n^\mu + U^\mu) \tag{3}$$

with $n^\mu U_\mu = 0$ and

$$\Gamma = \left(1 - U^i U_i\right)^{-1/2}. \tag{4}$$

The fluid energy density measured by the Eulerian observer is given by

$$E = T^{\mu\nu} n_\mu n_\nu = \Gamma^2(\rho + P) - P. \tag{5}$$

Let us write the proper rest mass energy density as ρ_0. Then, the relativistic specific enthalpy is defined by[3]

$$h = \frac{\rho + P}{\rho_0} = \frac{\rho_0(1+\varepsilon) + P}{\rho_0}, \tag{6}$$

where $\rho_0\varepsilon$ is the internal energy. Let D denote the baryon rest mass density measured by the Eulerian observer as[4]

$$D = \rho_0 \Gamma. \tag{7}$$

For later convenience, we also introduce the fluid momentum density measured by the Eulerian observer p_μ as

$$p_\mu = (E + P)U_\mu. \tag{8}$$

Then, the fluid equations are written as [?]

$$(\partial_t - \mathcal{L}_\beta)\, D \;+\; D_i(\alpha D U^i) - \alpha K D = 0, \tag{9}$$

$$(\partial_t - \mathcal{L}_\beta)\, E \;+\; \alpha\left[D_i p^i - (E+P)(K + K_{ij}U^i U^j)\right] + p^i D_i \alpha = 0, \tag{10}$$

$$(\partial_t - \mathcal{L}_\beta)\, p_i \;+\; \alpha D_j(P\delta_i^j + p_i U^j) + \left[P\delta_i^j + p_i U^j\right] D_j \alpha - \alpha K p_i + E D_i \alpha = 0. \tag{11}$$

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[1] Γ is W in Font [?] and w in Shibata [?].
[2] U^i is v^i in Font [?].
[3] ρ_0 is ρ in Shibata [?].
[4] D is $m_{\rm B}\mathcal{N}_{\rm B}$ in Gourgoulhon [?].

These equations can be rewritten as

$$\partial_t(\sqrt{\gamma}D) \ + \ \partial_i\left[\alpha\sqrt{\gamma}D\left(U^i - \frac{\beta^i}{\alpha}\right)\right] = 0, \tag{12}$$

$$\partial_t(\sqrt{\gamma}E) \ + \ \partial_i\left[\alpha\sqrt{\gamma}\left(p^i - \frac{\beta^i}{\alpha}E\right)\right] + \sqrt{\gamma}\left(p^i\partial_i\alpha - \alpha S_{ij}K^{ij}\right) = 0, \tag{13}$$

$$\begin{aligned}\partial_t(\sqrt{\gamma}p_i) \ + \ \partial_j\left[\alpha\sqrt{\gamma}\left\{p_i\left(U^j - \frac{\beta^j}{\alpha}\right) + \delta_i^j P\right\}\right] \\ +\sqrt{\gamma}\left(E\partial_i\alpha - p_j\partial_i\beta^j + \frac{1}{2}\alpha S_{jk}\partial_i\gamma^{jk}\right) = 0,\end{aligned} \tag{14}$$

where

$$S_{ij} = (E+P)U_iU_j + P\gamma_{ij}. \tag{15}$$

Let us define the following variables:

$$\rho_* \ = \ \sqrt{\gamma}D, \tag{16}$$

$$S_0 \ = \ \sqrt{\gamma}E, \tag{17}$$

$$S_i \ = \ \sqrt{\gamma}p_i. \tag{18}$$

Then, the equations can be rewritten as

$$\partial_t\rho_* \ + \ \partial_i\left[\rho_* V^i\right] = 0, \tag{19}$$

$$\partial_t S_0 \ + \ \partial_i\left[S_0V^i + P\sqrt{\gamma}(V^i + \beta^i)\right] + S^iD_i\alpha - \alpha\sqrt{\gamma}S_{ij}K^{ij} = 0, \tag{20}$$

$$\partial_t S_i \ + \ \partial_j\left[S_iV^j + \alpha\sqrt{\gamma}\delta_i^j P\right] + S_0D_i\alpha - S_j\partial_i\beta^j + \frac{1}{2}\alpha\sqrt{\gamma}S_{jk}\partial_i\gamma^{jk} = 0, \tag{21}$$

where we have introduced V^i as

$$V^i = \alpha U^i - \beta^i, \tag{22}$$

and used the following relations

$$\alpha p^i - E\beta^i = EV^i + P(V^i + \beta^i). \tag{23}$$

Note that $V^i = u^i/u^0$.[5] We solve the above equations for the dynamical variables ρ_*, S_0 and S_i. The so-called primitive variables are ρ, V^i and ε. The fluxes are given by

$$f^i_{\rho_*} \ = \ \rho_* V^i, \tag{24}$$

$$f^i_{S_0} \ = \ S_0V^i + \sqrt{\gamma}P(V^i + \beta^i), \tag{25}$$

$$f^i_{S_j} \ = \ S_jV^i + \alpha\sqrt{\gamma}\delta^i_j P. \tag{26}$$

We note that the variable γ in the fluxes can be evaluated by $\gamma = \mathrm{e}^{12\psi}\tilde{\gamma} = \mathrm{e}^{12\psi}f$ with f being the determinant of the reference flat metric.

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[5] V^i is v^i in Shibata [**?**].

From the Jacobi matrix of the fluxes, we obtain the following expression for the three characteristic speeds for three directions:

$$\lambda_0^i = V^i, \tag{27}$$
$$\lambda_\pm^i = \frac{\alpha}{1-U^2c_\mathrm{s}^2}\left\{U^i(1-c_\mathrm{s}^2) \pm c_\mathrm{s}\sqrt{(1-U^2)\left[\gamma^{ii}(1-U^2c_\mathrm{s}^2)-(1-c_\mathrm{s}^2)U^iU^i\right]}\right\}, \tag{28}$$

where c_s is the sound velocity defined by

$$c_\mathrm{s}^2 = \left(\frac{\partial P}{\partial \rho}\right)_s \tag{29}$$

with fixed specific entropy s.

II. PRIMITIVE VARIABLES FROM DYNAMICAL VARIABLES

The equations between the primitive variables and the conserved variables are given by

$$P = P(\rho, \varepsilon), \tag{30}$$
$$\rho_* = \sqrt{\gamma}\Gamma\frac{\rho}{1+\varepsilon}, \tag{31}$$
$$S_0 = \sqrt{\gamma}\left[\Gamma^2(\rho+P)-P\right], \tag{32}$$
$$S_i = \sqrt{\gamma}\,(E+P)\,U_i = \frac{1}{\alpha}\,(S_0+\sqrt{\gamma}P)\,\gamma_{ij}\left(V^j+\beta^j\right). \tag{33}$$

We need to invert these equations to obtain the primitive variables from the dynamical variables.

From Eqs. (4) and (8), we obtain

$$\Gamma^2 p^2 = (E+P)^2(\Gamma^2-1), \tag{34}$$

where $p^2 = p^\mu p_\mu$. From Eq. (5), we find

$$\Gamma^2 = \frac{E+P}{\rho+P}. \tag{35}$$

Substituting Eq. (35), into Eq. (34), we obtain

$$p^2 - E^2 - (P-\rho)E + \rho P = 0. \tag{36}$$

A. Barotropic EoM

Let us assume $P = P(\rho)$. In this case, we need not solve the continuity equation (19) unless we are interested in the value of the internal energy. Since the values of E and p_i can be calculated from the dynamical variables S_0 and S_i, Eq. (36) can be regarded as an equation to get the value of ρ with a given equation of state $P = P(\rho)$. Once the value of ρ is calculated, the value of Γ is given by Eq. (35). From Eq. (33), the value of V^i is given by

$$V^i = \alpha U^i - \beta^i = \alpha \frac{\gamma^{ij} S_j}{S_0 + \sqrt{\gamma} P} - \beta^i. \tag{37}$$

The value of ε is given by

$$\varepsilon = \frac{\rho - \rho_0}{\rho_0} = \frac{\sqrt{\gamma}\Gamma\rho - \rho_*}{\rho_*}. \tag{38}$$

For the case $P = w\rho$, we obtain

$$\rho = \frac{1}{2w}\left[-(1-w)E + \sqrt{E^2(1-w)^2 + 4w(E^2 - p^2)}\right]. \tag{39}$$

From Eq. (35), the value of Γ is given by

$$\Gamma^2 = \frac{E + w\rho}{(1+w)\rho}. \tag{40}$$

From the conservation equation $u^\mu \nabla_\nu T^\nu_\mu = 0$ and the continuity equation $\nabla_\mu(\rho_0 u^\mu) = 0$, we obtain

$$\mathrm{d}\ln(1+\varepsilon) = w\,\mathrm{d}\ln\rho_0. \tag{41}$$

Then, the value of ε is given by

$$\varepsilon = C\rho_0^w - 1, \tag{42}$$

or, equivalently,

$$\varepsilon = \varepsilon_\rho := C'\rho^{w/(1+w)} - 1, \tag{43}$$

where C and C' are integration constants fixed by the initial condition. If we solve Eq. (19), the value of ε calculated by Eq. (38) should be consistent with Eq. (43).

III. FLUX CALCULATION SCHEME

For the flux calculation, we use a central scheme with MUSCL (Mono Upstream-centered Scheme for Conserva-tion Laws) [**? ?**].

For simplicity, let us consider the following 1-dimensional equation in a conserved form:

$$\partial_t u + \partial_x f(u) = 0. \tag{44}$$

We evaluate the flux in a finite difference formula through the values at mid-points between two grid points as follows:

$$\partial_t u = -\frac{1}{\Delta x}\left(f_{i+1/2} - f_{i-1/2}\right). \tag{45}$$

In order to evaluate $f_{i\pm1/2}$, we first introduce the following variables for the value of u at the point specified by $i + 1/2$:

$$(u_L)_{i+1/2} = u_i + \frac{1}{4}\left((1-\kappa)(\bar{\Delta}_-)_i + (1+\kappa)(\bar{\Delta}_+)_i\right), \tag{46}$$

$$(u_R)_{i+1/2} = u_{i+1} - \frac{1}{4}\left((1-\kappa)(\bar{\Delta}_+)_{i+1} + (1+\kappa)(\bar{\Delta}_-)_{i+1}\right), \tag{47}$$

where κ is a parameter to specify the way to the interpolation, and $(\bar{\Delta}_\pm)_i$ is defined by

$$(\bar{\Delta}_+)_i = \mathrm{minmod}((\Delta_+)_i, b(\Delta_-)_i), \tag{48}$$
$$(\bar{\Delta}_-)_i = \mathrm{minmod}((\Delta_-)_i, b(\Delta_+)_i) \tag{49}$$

with

$$(\Delta_+)_i = u_{i+1} - u_i, \tag{50}$$
$$(\Delta_-)_i = u_i - u_{i-1}. \tag{51}$$

The function minmod(a,b) returns 0 if a and b have different signs, and if not, the value of one of the arguments which has a smaller absolute value. That is,

$$\mathrm{minmod}(a, b) = \mathrm{sign}(a)\max(0, \min(|a|, \mathrm{sign}(a)b)). \tag{52}$$

Then, the flux is evaluated as

$$f = \frac{1}{2}\left(f(u_L) + f(u_R) - a^*(u_R - u_L)\right), \tag{53}$$

where a^* is the value of the fastest characteristic speed given by

$$a^* = \max\{|\lambda_{0L}|, |\lambda_{+L}|, |\lambda_{-L}|, |\lambda_{0R}|, |\lambda_{+R}|, |\lambda_{-R}|\}. \tag{54}$$

IV. CONSISTENCY CHECK

We solve the hydro-dynamic equations (19), (20) and (21) by a central scheme briefly explained in the previous section. As a first check, let us compare our non-linear calculation with a linear perturbation solution. The linear perturbation equations and useful gauge invariant variables are summarized in Appendix. For simplicity, we consider the following single-mode perturbation:

$$\Phi(t, \boldsymbol{x}) = \mathcal{A} f(t) g(\boldsymbol{x}) = \mathcal{A} f(t) \sum_{i=1}^{3} \cos(k x_i), \tag{55}$$

where each component x_i denotes the spatial coordinates x, y and z, and $f(t)$ gives the growing mode solution of the master equation (B23). In the following calculation, we set $w = 1/3$, $k = \pi/L$, $H_\mathrm{i} = 2/L$ and $\mathcal{A} = 0.01$, where H_i and L are the initial Hubble parameter and the size of the numerical box. The initial scale factor is set to be 1. The calculation is performed in the domain $0 \leq x_i \leq L$. The number of grids in each side is 40. The gauge is fixed by $\alpha = \beta^i = 0$ for convenience in comparison with linear perturbation. We note that the initial configurations of the fluid dynamical variables are set so that the constraint equations (A5) and (A6) are satisfied. That is, we need not solve the constraint equations but calculate the value of $S_0 = \sqrt{\gamma}E$ and $S_i = \sqrt{\gamma}p_i$ from the geometrical variables through constraint equations (A5) and (A6).

Let us compare our result with that given by the linearized equations. In Fig. 1, we show the value of κ defined by

$$\kappa = -\frac{K + 3H}{3H}. \tag{56}$$

The deviation between the numerical solution and the linearized solution can be regarded as the effect of the non-linearity.

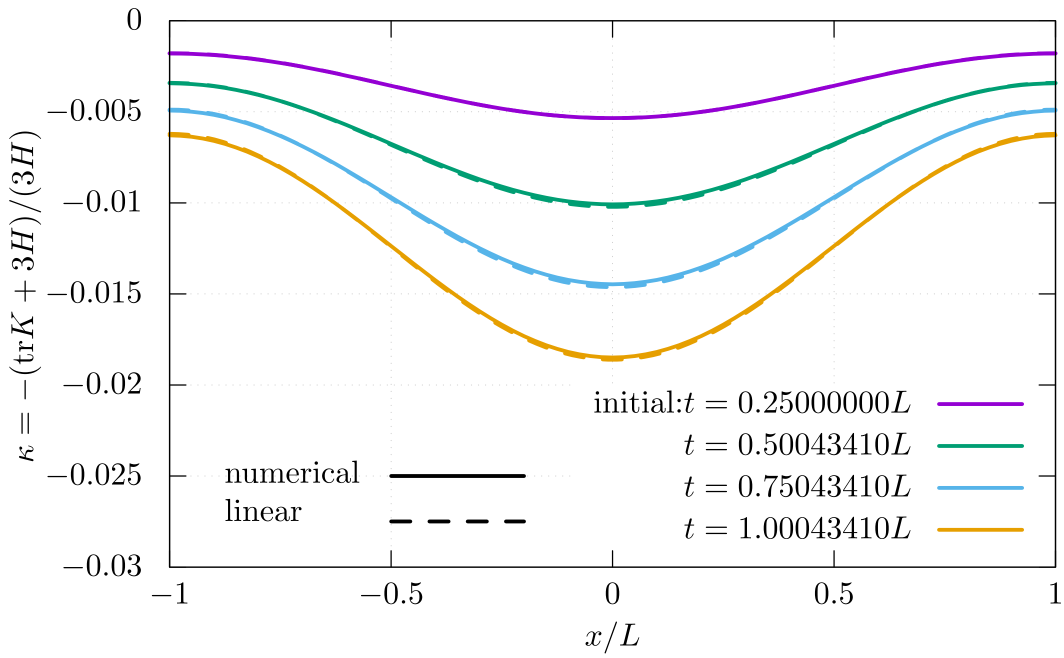

FIG. 1: The value of κ on x axis is depicted as a function of x for each time.

To obtain the same plot, the output file `out_xkl.dat` must be used. As the default, 5 data blocks are generated. The 4th and 5th data blocks are the same one. The four independent blocks correspond to the data sets at $t = 0.25L$ (initial time), $t = 0.50L + \delta t$, $t = 0.75L + \delta t$ and $t = 1.00L + \delta t$ with $\delta t = 0.00043410L$, respectively. The values of $x[L]$ and $K[1/L]$ on the x-axis are given in the first and 22nd columns, respectively. Therefore plotting the value of $\kappa = -(K(t, x) + 3H(t))/(3H(t))$ as a function of x for each block, one can obtain the same plot as Fig. 1, where $H(t) = 1/(2t)$.

Appendix A: Cosmological conformal 3+1 decomposition

We refer to [?] for this section. Following Ref.[?], we decompose the spatial metric and the extrinsic curvature K_{ij} as follows:

$$\gamma_{ij} = \psi^4 a(t)^2 \tilde{\gamma}_{ij} \tag{A1}$$

$$K_{ij} = \psi^4 a^2 \tilde{A}_{ij} + \frac{\gamma_{ij}}{3} K, \tag{A2}$$

where $a(t)$ is the scale factor of the background homogeneous and isotropic universe, K is the trace of the extrinsic curvature, and we choose $\tilde{\gamma}_{ij}$ such that $\det(\tilde{\gamma}_{ij})$ is time independent and equal to $\det(f_{ij})$ with f_{ij} being flat metric components. Hereafter, we raise and lower the indices of an object with tilde by using $\tilde{\gamma}_{ij}$ and its inverse $\tilde{\gamma}^{ij}$. From Eq. (A2), we find $\tilde{\gamma}^{ij}\tilde{A}_{ij}=0$. We define three covariant derivatives D_i, $\tilde{D}_i$ and $\mathcal{D}_i$ as the covariant derivatives associated with γ_{ij}, $\tilde{\gamma}_{ij}$ and f_{ij}, respectively.

From the definition of the extrinsic curvature, we obtain

$$(\partial_t-\mathcal{L}_\beta)\psi \;=\; -\frac{\dot{a}}{2a}\psi+\frac{\psi}{6}\left(-\alpha K+\mathcal{D}_k\beta^k\right), \tag{A3}$$

$$(\partial_t-\mathcal{L}_\beta)\tilde{\gamma}_{ij} \;=\; -2\alpha\tilde{A}_{ij}-\frac{2}{3}\tilde{\gamma}_{ij}\mathcal{D}_k\beta^k. \tag{A4}$$

The Hamiltonian and momentum constraints are given by

$$\tilde{\triangle}\psi=\frac{\tilde{R}^i_i}{8}\psi-2\pi\psi^5a^2E-\frac{\psi^5a^2}{8}\left(\tilde{A}_{ij}\tilde{A}^{ij}-\frac{2}{3}K^2+2\Lambda\right), \tag{A5}$$

$$\tilde{D}^j(\psi^6\tilde{A}_{ij})-\frac{2}{3}\psi^6\tilde{D}_iK=8\pi p_i\psi^6, \tag{A6}$$

where $\tilde{R}_{ij}$ is the Ricci curvature tensor with respect to the metric $\tilde{\gamma}_{ij}$. The evolution equations are written as

$$\begin{aligned}(\partial_t-\mathcal{L}_\beta)\tilde{A}_{ij} \;=\;& \frac{1}{a^2\psi^4}\left[\alpha\left(R_{ij}-\frac{8\pi}{a^2\psi^4}S_{ij}\right)-D_iD_j\alpha\right]_{\rm TL}\\ &+\alpha\left(K\tilde{A}_{ij}-2\tilde{A}_{ik}\tilde{A}^k_j\right)-\frac{2}{3}\tilde{A}_{ij}\mathcal{D}_k\beta^k,\end{aligned} \tag{A7}$$

$$(\partial_t-\mathcal{L}_\beta)K \;=\; \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij}+\frac{1}{3}K^2-\Lambda\right)-D_kD^k\alpha+4\pi\alpha(E+S^k_k), \tag{A8}$$

where R_{ij} is the Ricci curvature tensor with respect to the metric γ_{ij}, TL denotes the trace-less part of the object, and $\mathcal{L}_\beta$ is the Lie derivative along β^i.

The hydrodynamical equations are written in the form

$$\begin{aligned}&\left[\psi^6a^3\left\{(\rho+P)\Gamma^2-P\right\}\right]_{,t}+\frac{1}{\sqrt{f}}\left[\sqrt{f}\psi^6a^3\left\{(\rho+P)\Gamma^2-P\right\}V^l\right]_{,l}\\ =\;& -\frac{1}{\sqrt{f}}\left[\sqrt{f}\psi^6a^3P(V^l+\beta^l)\right]_{,l}+\alpha\psi^6a^3PK-\alpha^{-1}\alpha_{,l}\psi^6a^3\Gamma^2(\rho+P)(V^l+\beta^l)\\ &+\alpha^{-1}\psi^{10}a^5\Gamma^2(\rho+P)(V^l+\beta^l)(V^m+\beta^m)\left(\tilde{A}_{lm}+\frac{\tilde{\gamma}_{lm}}{3}K\right),\end{aligned} \tag{A9}$$

$$\begin{aligned}&(\Gamma\psi^6a^3(\rho+P)u_j)_{,t}+\frac{1}{\sqrt{f}}(\sqrt{f}\Gamma\psi^6a^3(\rho+P)V^ku_j)_{,k}\\ &=-\alpha\psi^6a^3P_{,j}+\Gamma\psi^6a^3(\rho+P)\left(-\Gamma\alpha_{,j}+u_k\beta^k_{,j}-\frac{u_ku_l}{2u^t}\gamma^{kl}_{,j}\right).\end{aligned} \tag{A10}$$

Appendix B: Linear perturbation equations

For simplicity, we focus on the case $P=w\rho$ hereafter. In this section, we consider the linear perturbation on a background homogeneous and isotropic spatially flat spacetime.

We denote the small parameter with respect to the perturbation amplitude as ϵ. Then, the order of each variable introduced in Sec. A is given as follows:

$$\psi = 1 + \mathcal{O}(\epsilon),\ \gamma_{ij} = f_{ij} + \mathcal{O}(\epsilon),\ K = -3H + O(\epsilon),\ \tilde{A}_{ij} = O(\epsilon),\ \rho = \rho_{\rm b} + O(\epsilon), \tag{B1}$$

where H is the background Hubble expansion rate and $\rho_{\rm b} = 3H^2/(8\pi)$.

We introduce the followig perturbation variables of $\mathcal{O}(\epsilon)$:

$$\xi := \psi - 1, \tag{B2}$$

$$\chi := \alpha - 1, \tag{B3}$$

$$\kappa := -\frac{K + 3H}{3H}, \tag{B4}$$

$$\delta := \frac{\rho - \rho_{\rm b}}{\rho_{\rm b}}, \tag{B5}$$

$$h_{ij} := \tilde{\gamma}_{ij} - f_{ij}. \tag{B6}$$

Since we are interested in the scalar perturbation, introducing the two scalar variables h and A, we assume that h_{ij} and $\tilde{A}_{ij}$ are respectively given as

$$h_{ij} = \left(\mathcal{D}_i\mathcal{D}_j - \frac{1}{3}f_{ij}\triangle\right)h, \tag{B7}$$

$$\tilde{A}_{ij} = \left(\mathcal{D}_i\mathcal{D}_j - \frac{1}{3}f_{ij}\triangle\right)A. \tag{B8}$$

Then, in the linear order, EoM can be written as follows:

$$\triangle\xi = \frac{1}{12}\triangle\triangle h - \frac{3}{4}a^2(H^2 - 3\Lambda)\delta + \frac{3}{2}a^2H^2\kappa, \tag{B9}$$

$$\triangle A = -3H\kappa + \frac{3}{2}(3H^2 - \Lambda)(1+w)(v+\beta), \tag{B10}$$

$$\partial_t A = \frac{1}{a^2}\left(-2\xi + \frac{1}{6}\triangle h - \chi\right) - 3HA, \tag{B11}$$

$$\partial_t\xi = \frac{1}{2}H(\chi+\kappa) + \frac{1}{6a^2}\triangle\beta, \tag{B12}$$

$$\begin{aligned}\partial_t\kappa =& -\frac{1}{2}(1+w)\left(3H - \frac{\Lambda}{H}\right)\chi + \frac{1}{3a^2H}\triangle\chi \\ &-\frac{1}{2}\left[(1-3w)H + (1+w)\frac{\Lambda}{H}\right]\kappa - \frac{1}{6}(1+3w)\left(3H - \frac{\Lambda}{H}\right)\delta,\end{aligned} \tag{B13}$$

$$\partial_t h = -2A + \frac{2}{a^2}\beta, \tag{B14}$$

$$6\partial_t\xi + \partial_t\delta = -(1+w)\frac{1}{a^2}\triangle v - w\frac{1}{a^2}\triangle\beta - 3wH(\chi+\kappa), \tag{B15}$$

$$\partial_t v + \partial_t\beta = 3wH(v+\beta) - \frac{w}{1+w}\delta - \chi. \tag{B16}$$

A useful set of gauge invariant variables is given by

$$\Psi := 2\xi + H\left(\beta - \frac{1}{2}a^2\partial_t h\right) - \frac{1}{6}\triangle h, \tag{B17}$$

$$\Phi := \chi + \partial_t\left(\beta - \frac{1}{2}a^2\partial_t h\right), \tag{B18}$$

$$\Delta := \delta - 3(1+w)H(\beta + v), \tag{B19}$$

$$V := \frac{v}{a} + \frac{1}{2}a\partial_t h, \tag{B20}$$

$$\mathcal{K} := \kappa - \left[\frac{1}{2}(1+w)\left(3H - \frac{\Lambda}{H}\right) - \frac{1}{3a^2H}\triangle\right]\left(\beta - \frac{1}{2}a^2\partial_t h\right), \tag{B21}$$

$$\Xi := A - \frac{1}{a^2}\left(\beta - \frac{1}{2}a^2\partial_t h\right). \tag{B22}$$

EoM can be reduced to the following equations:

$$0 = a\partial_t\left(a\partial_t\Phi\right) + 3(1+w)a^2H\partial_t\Phi - w\triangle\Phi + (1+w)a^2\Lambda\Phi, \tag{B23}$$

$$\Psi = -\Phi, \tag{B24}$$

$$\Delta = \frac{2}{a^2(3H^2 - \Lambda)}\triangle\Phi, \tag{B25}$$

$$\mathcal{K} = -\frac{1}{aH}\partial_t(a\Phi), \tag{B26}$$

$$V = -\frac{2}{(1+w)a^2(3H^2-\Lambda)}\partial_t(a\Phi), \tag{B27}$$

$$\Xi = 0. \tag{B28}$$

Therefore, a linear solution is described by a solution of Eq. (B23).

Appendix C: Solution of the perturbation equations with the synchronous gauge

In the synchronous gauge given by $\alpha = \beta^i = 0$, for the single mode perturbation in the radiation-dominated universe, the master equation (B23) can be rewritten as

$$\frac{3}{H^2}\partial_t^2\Phi + \frac{15}{H}\partial_t\Phi + \frac{k^2}{a^2H^2}\Phi = 0. \tag{C1}$$

The growing mode solution is given by Eq. (55) with

$$f(t) = -\frac{1}{a}j_1\left(\frac{k}{\sqrt{3}aH}\right), \tag{C2}$$

where j_1 is the spherical Bessel function of the first kind. Then the gauge invariant variables Ψ, Δ, $\mathcal{K}$ and V can be explicitly written through Φ by using Eqs. (B24, B25, B26, B27). Since $\chi = \beta = 0$ in the synchronous gauge, from Eq. (B18), we obtain

$$\frac{h(t,\boldsymbol{x})}{\mathcal{A}g(\boldsymbol{x})} = -2\int_0^t \mathrm{d}t' \frac{1}{a^2(t')}\int_0^{t'} \mathrm{d}t'' f(t''). \tag{C3}$$

This integration can be numerically performed. Then, by using the numerical solution of $h(t, \boldsymbol{x})$, we obtain the perturbation variables ξ, κ, A, δ and v through Eqs. (B17, B19, B20, B21, B22). Specifically, for $H_{\rm i} = 2/L$ and $t = t_{\rm i} := 1/(2H_{\rm i})$, we obtain the values of the geometrical variables as follows:

$$\frac{h(t_{\rm i}, \boldsymbol{x})}{\mathcal{A}g(\boldsymbol{x})} = 0.0370205697380725L^2, \tag{C4}$$

$$\frac{\kappa(t_{\rm i}, \boldsymbol{x})}{\mathcal{A}g(\boldsymbol{x})} = -0.178126451216426, \tag{C5}$$

$$\frac{\xi(t_{\rm i}, \boldsymbol{x})}{\mathcal{A}g(\boldsymbol{x})} = 0.181157064844222, \tag{C6}$$

$$\frac{v(t_{\rm i}, \boldsymbol{x})}{\mathcal{A}g(\boldsymbol{x})} = -0.0147059776435973 1L, \tag{C7}$$

$$\frac{A(t_{\rm i}, \boldsymbol{x})}{\mathcal{A}g(\boldsymbol{x})} = -0.0725272477763359L^2. \tag{C8}$$

Appendix D: Other useful gauge invariant variables

Another useful set of gauge invariant variables is given by replacing $\mathcal{K}$ and Ξ by $\widetilde{\mathcal{K}}$ and $\widetilde{\Xi}$ defined as

$$\widetilde{\mathcal{K}} := \kappa - \left[\frac{1}{2}(1+w)\left(3H - \frac{\Lambda}{H}\right) - \frac{1}{3a^2H}\triangle\right](\beta + v), \tag{D1}$$

$$\widetilde{\Xi} := A - \frac{1}{a^2}(\beta + v). \tag{D2}$$

By definition, these varables are equivalent to the perturbation of the trace and the traceless part of the extrinsic curvature in the comoving gauge $v + \beta = 0$ or equivalently $u^\mu = n^\mu$. Therefore, these two variables correspond to the expansion and shear of the fluid four velocity. From EoM, we obtain

$$\widetilde{\mathcal{K}} = -\frac{2}{3}\frac{\partial_t(a\triangle\Phi)}{(1+w)a^3H\left(3H^2 - \Lambda\right)}, \tag{D3}$$

$$\widetilde{\Xi} = 2\frac{\partial_t(a\Phi)}{(1+w)a^3\left(3H^2 - \Lambda\right)}. \tag{D4}$$

Yet another useful vriable is $\widetilde{\Psi}$ defined as follows:

$$\widetilde{\Psi} := 2\xi + H(\beta + v) - \frac{1}{6}\triangle h. \tag{D5}$$

In the comoving gauge, we obtain

$$\widetilde{\Psi} = -\frac{1}{4}a^2R, \tag{D6}$$

where R is the Ricci scalar with respect to the 3-metric γ_{ij}. It can be shown that $\partial_t\widetilde{\Psi} = 0$ for $w = 0$.