

Note on the time evolution sample for an adiabatic case

A. Initial condition and parameter settings

Here we consider the adiabatic perturbation generated by the initial curvature perturbation ζ . As is described in Ref. [1], once the spatial profile of ζ is specified, the growing mode solution can be described up through the next-to-leading order of the long-wavelength approximation. We use the analytic expressions for the geometrical variables with the constant-mean-curvature and zero-shift gauge up through the next-to-leading order as the initial data for the sample code.

In this sample code, we consider the numerical domain given by $0 \leq R \leq L$ with R being the radial coordinate of the isotropic coordinate system. The specific profile of the initial curvature perturbation is the same as in Ref. [2]:

$$\zeta = -\mu \exp\left(-\frac{1}{6}k^2 R^2\right) W(R), \quad (1)$$

where the function $W(R)$ is given by [3]

$$W(R; R_W, L) = \begin{cases} 1 & \text{for } 0 \leq R \leq R_W \\ 1 - \frac{((R_W - L)^6 - (L - R)^6)^6}{(R_W - L)^{36}} & \text{for } R_W \leq R \leq L \\ 0 & \text{for } L \leq R \end{cases} \quad (2)$$

with $R_W = 0.8L$. The parameters k and μ are set as $k = 10/L$ and $\mu = 0.83$. The amplitude μ is slightly larger than the threshold value $\mu_{\text{th}} \simeq 0.805$ for PBH formation reported in Ref. [2].

The background universe is assumed to be radiation-dominated, and the Hubble parameter at the initial time H_i is set to be $H_i = 50/L$. The coordinate length L is covered by 100 grid points with grids on both ends. Since this sample code is just for demonstration, we do not seriously consider the constraint violations.

B. CARTOON method

The spherically symmetric numerical code COSMOS-S is firstly used in Ref. [4]. We perform the spherically symmetric relativistic simulation by using the CARTOON method [5]. That is, we solve the evolution equations on the Z -axis and create the necessary data around it based on the spherical symmetry with the 3rd order spline interpolation. A schematic figure for the CARTOON method is shown in Fig. 1.

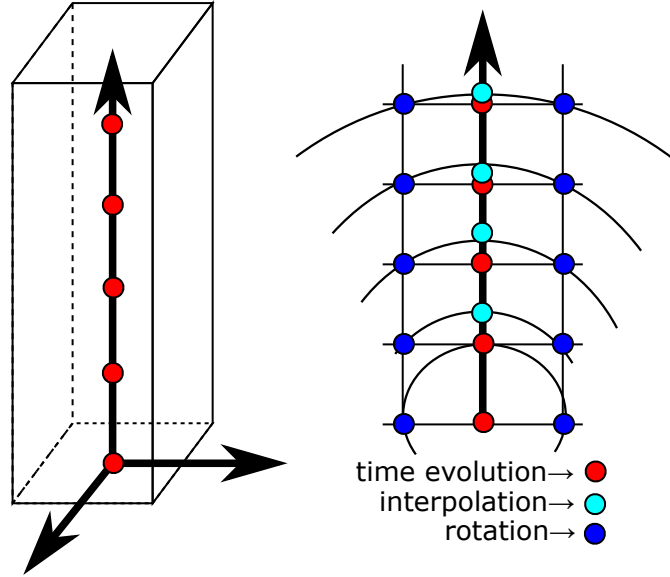


FIG. 1: A schematic figure for the CARTOON method.

C. Boundary condition

For the boundary condition, we adopt the asymptotically FLRW condition, but the constraints are significantly violated on the boundary through time evolution. The constraint violation at the boundary starts to occur when the effects of the inhomogeneity reach the boundary, and the origin of the constraint violation is due to unphysical reflected modes. As far as we know, no established procedure to reduce the unphysical reflected modes has been reported for the asymptotically FLRW boundary. Therefore the time evolution should be terminated before the boundary effects influence the dynamics of interest. We do not seriously consider this issue in this sample code. Because it is just for a demonstration. In order to avoid detecting a significant constraint violation due to the boundary effect, we exclude the outer $1/20$ region from the region for checking the constraint violation. The user who wants to change the region has to change the value of the parameter `exclude_boundary_rate` in `cosmos.s.h`.

D. Non-Cartesian coordinate

In this sample code, we employ the non-Cartesian coordinate system (x, y, z) firstly implemented in Ref. [3]. The non-Cartesian coordinate z is related to the Cartesian coordinate Z as follows:

$$Z = z - \frac{\eta}{1 + \eta} \frac{L}{\pi} \sin\left(\frac{\pi}{L} z\right). \quad (3)$$

This functional form satisfies $z = 0$ at $Z = 0$ and $z = L$ at $Z = L$. At the origin, the infinitesimal interval in the Cartesian coordinate ΔZ is covered by the non-Cartesian

coordinate interval $\Delta z = (1 + \eta)\Delta Z$. Therefore the central part is enlarged in this non-Cartesian coordinate system. In this sample code, the value of η is set to 10.

E. Mesh refinement

On top of the non-Cartesian coordinate system, we implement fixed mesh refinement. When the higher layer steps start, values on 6 outer grid points in the higher layer are calculated from the interpolation with the values in the lower layer. During the evolution steps, the values on 3 outer grid points in the higher layer are obtained from the interpolation with the values in the lower layer. At the final step, the given values on the outer 6 grid points are discarded when the lower layer data is updated. In the sample code, 3 additional layers for the mesh refinement are introduced when the value of the lapse function at the origin reduces to 0.1 0.075 and 0.05. The region covered by one higher layer is the region covered by 20 grids in the lower layer. These parameters for the mesh refinement can be controlled by the parameter file `par_fmr.d`.

F. Results of the time evolution

In the default setting, the time evolution starts from the initial data described above. Then one finds PBH formation at around the coordinate time $t \simeq 52L$.¹ It takes some time. We also attached the data file `ini_all.dat` which describes the system soon after the horizon formation. For instance, one can set the value of “*maximum step of the main loop*” in `par_ini.d` to 1 and set the value of the line indicated by “*0:no continue 1:continue*” to 1. When the numerical code is executed, first the data file `ini_all.dat`, in which 3 higher layers are already introduced, is loaded. After just one step of the evolution, all the data will be restored into `out_all.dat`, and the calculation stops. The value of the lapse function on the x -axis at $t = 50L$ is depicted as a function of x in Fig. 2. The result with the sample 3+1 code in the same setting is also shown for reference.

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- [1] T. Harada, C.-M. Yoo, T. Nakama, and Y. Koga, Phys. Rev. **D91**, 084057 (2015), arXiv:1503.03934, *Cosmological long-wavelength solutions and primordial black hole formation*.
 - [2] C.-M. Yoo, T. Harada, and H. Okawa, Phys. Rev. D **102**, 043526 (2020), arXiv:2004.01042, *Threshold of Primordial Black Hole Formation in Nonspherical Collapse*, [Erratum: Phys.Rev.D 107, 049901 (2023)].

¹ This value is different from the value given in the 3+1 dimensional sample code in the same setting. This would be because of the low resolution in the 3+1 dimensional sample code.

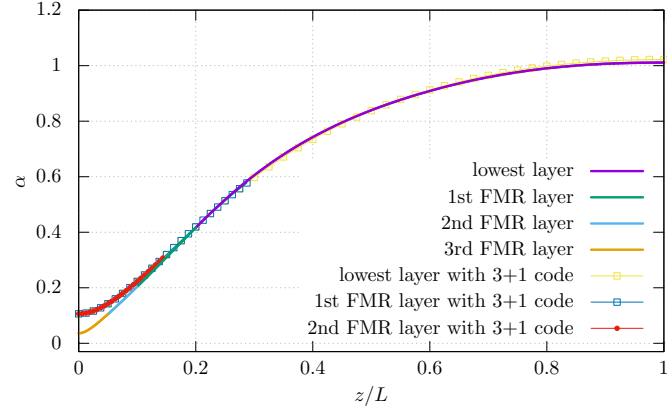


FIG. 2: Lapse function generated from 1s and 2nd columns in `out_jkz.dat`.

- [3] C.-M. Yoo, T. Ikeda, and H. Okawa, *Class. Quant. Grav.* **36**, 075004 (2019), arXiv:1811.00762, *Gravitational Collapse of a Massless Scalar Field in a Periodic Box*.
- [4] C.-M. Yoo, T. Harada, S. Hirano, H. Okawa, and M. Sasaki, *Phys. Rev. D* **105**, 103538 (2022), arXiv:2112.12335, *Primordial black hole formation from massless scalar isocurvature*.
- [5] M. Alcubierre *et al.*, *Int. J. Mod. Phys. D* **10**, 273 (2001), arXiv:gr-qc/9908012, *Symmetry without symmetry: Numerical simulation of axisymmetric systems using Cartesian grids*.