Proceedings of the 2006 International Workshop on Variable Structure Systems
Alghero, Italy, June 5-7, 2006
WedC.3

Feedback Linearization and High Order Sliding Mode Observer For A Quadrotor UAV

A. Benallegue, A. Mokhtari, and L. Fridman

Abstract— **In this paper, a feedback linearization-based controller with a high order sliding mode observer running parallel is applied to a quadrotor unmanned aerial vehicle. The high order sliding mode observer works as an observer and estimator of the effect of the external disturbances such as wind and noise. The whole observer-estimator-control law constitutes an original approach to the vehicle regulation with minimal number of sensors. Performance issues of the controller-observer are illustrated in a simulation study that takes into account parameter uncertainties and external disturbances.**

I. Introduction

Small UAV Quadrotors are designed to easily move in different environments while following specific tasks and providing good performance as well as a great autonomy. Affected by aerodynamic forces, the quadrotor dynamics is nonlinear, multivariable, and is subject to parameter uncertainties and external disturbances. In turn, controlling of the quadrotor is required i) to meet the stability, robustness and desired dynamic properties; ii) to be able to handle nonlinearity; iii) to be adaptive to changing parameters and environmental disturbances.

Main difficulties of the motion control are thus parametric uncertainties, unmodeled dynamics, and external disturbances [1], which result in further complication in the design of controllers for actual systems [3]. However various advanced control methods such as feedback linearization method [4], have been developed to meet increasing demands on the performance, however, they required full information on the state that may limit their practical utility. Indeed, even if all the state measurements are possible they are typically corrupted by noise. Moreover, the increased number of sensors makes the overall system more complex in implementation and expensive in realization. In order to decrease the number of sensors in [5] the use only a rotational motion sensors is proposed in order to control tilt angles and evaluate translational motion. However, aerodynamic forces still cause difficulties to overcome. Thus motivated, an observer-based feedback design becomes an attractive approach to robotic control.

The use of state observers appears to be useful in not only system monitoring and regulation but also detecting as well as identifying failures in dynamic systems. Almost all observer designs are based on the mathematical model of the plant, is no linearized and has consequently have uncertain inputs. From the other hand the relative degree of the model with respect to the known outputs heavily dependent on the accuracy of the mathematical model of the plant [6].

So the main motivation of the paper are:

- Feedback linearization controller of the quadrotor needs the third derivatives of measured states in order to reconstruct tilt angles and to fulfill the controller requirement.
- When quadrotor is subjected to external disturbances, it would be suitable to compensate them through an observer based controller.
- The observers should be robust with respect to external perturbations (wind and noise).
- Observers based identification perturbation allow to reduce the number of sensors required for control design.

Methodology. The relative degree of the UAV Quadrotors model w.r.t. to unknown inputs is more than one and the standard necessary and sufficient conditions for observation of the systems with unknown inputs are not fulfilled [2]. To solve the problem of observation for UAV Quadrotors the higher order sliding mode observers will be used.

Sliding mode observers (see, for example, the corresponding chapters in the textbooks [13], [22], and the recent tutorials [7], [9], [10]) are widely used due to their attractive features: a) insensitivity (more than robustness!) with respect to unknown inputs; b) possibilities to use the values of the equivalent output injection for the unknown inputs identification; c) finite time convergence to exact values of the state vectors. In [14], [22] and [8] a step by step form of sliding mode observers were proposed. Such observers based on the transformation of a given system to a block observable form and the sequential estimation of each state by using of the value of the equivalent output injection. On the one hand, this schemes allows to formulate some observability conditions for linear time invariant systems with unknown inputs. Such conditions were formulated in [22], [8] for the scalar case. From the other hand, realization of this scheme caused obligatory filtration due to the non-idealities.
In [18], [19] and [21] a robust exact arbitrary order differentiator was designed ensuring finite time convergence to the values of the corresponding derivatives, and applications of higher order sliding algorithms were considered.

Basing on the second-order sliding-mode super twisting algorithm in [20].an observer for uncertain mechanical systems with only position measurements was proposed ensuring best possible approximation for the velocities.

A. Benallegue is with Robotics Laboratory of Versailles, France. benalleg@robot.uvsq.fr
A. Mokhtari is with the University of Science and Technology of Oran, Algeria. mokhtari@robot.uvsq.fr
L. Fridman is with the university of Mexico, Mexico. lfridman@servidor.unam.mx

1-4244-0208-5/06/$20 ©2006 IEEE

Main contribution. In the paper the model of UAV Quadrotor and feedback linearization-based controller is suggested. To realize this with a high order sliding mode observer running parallel is applied to a quadrotor unmanned aerial vehicle. The high order sliding mode observer works as an observer and estimator of the effect of the external disturbances such as wind and noise. The whole observer-estimator-control law constitutes an original approach to the vehicle regulation with minimal number of sensors. Performance issues of the controller-observer are illustrated in a simulation study that takes into account parameter uncertainties and external disturbances UAV Quadrotor is suggested. To realize the control algorithm and identify the uncertainties a fourth order sliding mode observer based on fourth order differentiator [21] is suggested This observer converge in finite time ensuring the identification of the effect of the external disturbances such as wind and noise. The whole observer-estimator-control law constitutes an original approach to the vehicle regulation with minimal number of sensors. Performance issues of the controller-observer are illustrated in a simulation study that takes into account parameter uncertainties and external disturbances.

*Paper structure.*The rest of the paper is outlined as follows. UAV dynamics is deduced in section 2. The inner outer controller is developed in section 3. The observer design is presented in section 4. Simulation results are given in section 5. Section 6 yields some conclusions.

II. QUADROTOR DYNAMICS

The quadrotor is composed of 4 rotors. Two diagonal motors (1 and 3) are running in the same direction whereas the others (2 and 4) in the other direction to eliminate the anti-torque. On varying the rotor speeds altogether with the same quantity the lift forces will change affecting in this case the altitude z of the system and enabling vertical take-off/on landing. Yaw angle ψ is obtained by speeding up\slowing down the diagonal motors depending on desired direction. Roll angle ϕ axe allows the quadrotor to move toward y direction. Pitch angle θ axe allows the quadrotor to move toward x direction. The rotor is the primary source of control and propulsion for the UAV. The Euler angle orientation to the flow provides the forces and moments to control the altitude and position of the system. The absolute position is described by three coordinates (x_0, y_0, z_0), and its attitude by Euler angles (ψ, θ, ϕ), under the conditions $(-\pi \leq \psi < \pi)$ for yaw, $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$ for pitch and $(-\frac{\pi}{2} < \phi < \frac{\pi}{2})$ for roll. The derivatives with respect to time of the angles (ψ,θ,ϕ) can be expressed in the form:

$$col(\dot{\psi}, \dot{\theta}, \dot{\phi}) = M(\psi, \theta, \phi)\omega \tag{1}$$

where $\omega = col(p, q, r)$ is the angular velocity expressed with respect to a body reference frame and $M(\psi, \theta, \phi)$ is the 3x3 matrix given by:

$$M(\psi, \theta, \phi) = \begin{bmatrix} 0 & S\phi S_e\theta & C\phi S_e\theta \\ 0 & C\phi & -S\phi \\ 1 & S\phi T\theta & C\phi T\theta \end{bmatrix} \tag{2}$$

with $S. = \sin(.),\ C. = \cos(.),\ T. = \tan(.),\ S_e. = \sec(.)$

This matrix, as shown, depends only on (ψ, θ, ϕ) and it is invertible if the above conditions on (ψ, θ, ϕ) hold.

Similarly, the time derivative of the position (x_0, y_0, z_0) is given by:

$$col(\dot{x}_0, \dot{y}_0, \dot{z}_0) = V_0 \tag{3}$$

where $V_0 = col(u_0, v_0, w_0)$ is the absolute velocity of the UAV expressed with respect to an earth fixed inertial reference frame. Let $V = col(u, v, w)$ be the absolute velocity of the UAV expressed in a body fixed reference frame. Then V and V_0 relate according to

$$V_0 = R(\psi, \theta, \phi)V$$

where $R(\psi, \theta, \phi)$ is the rotation matrix given by

$$R = \begin{bmatrix} C\theta C\psi & C\psi S\theta S\phi - C\phi S\psi & C\phi C\psi S\theta + S\phi S\psi \\ C\theta S\psi & S\theta S\phi S\psi + C\phi C\psi & C\phi S\theta S\psi - C\psi S\phi \\ -S\theta & C\theta S\phi & C\theta C\phi \end{bmatrix}$$

Equations (1) and (3) are the kinematic equations. The dynamic equations are now expressed. Using the Newton's laws about the center of mass one obtains the dynamic equations for the miniature four rotors helicopter

$$m\dot{V}_0 = \sum F_{ext} \tag{4}$$

$$J\dot{\omega} = -\omega \times J\omega + \sum T_{ext} \tag{5}$$

where the symbol $\times$ denotes the usual vector product, m is the mass, J is the inertia matrix which is given by

$$J = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

Due to the symmetry of the geometric form of the quadrotor the coupling inertia is assumed to be zero. The notations $\sum F_{ext}$, $\sum T_{ext}$ stand for the vector of external forces and that of external torques, respectively. They contain the helicopter's weight, the aerodynamic forces vector, the thrust and the torque developed by the four rotors. It is straightforward to compute that

$$\sum F_{ext} = \begin{bmatrix} A_x - (C\phi C\psi S\theta + S\phi S\psi)u_1 \\ A_y - (C\phi S\theta S\psi - C\psi S\phi)u_1 \\ A_z + mg - (C\theta C\phi)u_1 \end{bmatrix} \tag{6}$$

$$\sum T_{ext} = \begin{bmatrix} A_p + u_2 d \\ A_q + u_3 d \\ A_r + u_4 \end{bmatrix}$$

where

- $(A_x, A_y, A_z)^T$ and $col(A_p, A_q, A_r)^T$ are the resulting aerodynamic forces and moments acting on the UAV and are computed from the aerodynamic coefficients C_i as $A_i = \frac{1}{2}\rho_{air} C_i W^2$ [11],[12] (ρ_{air} is the air density, W is the velocity of the UAV with respect to the air) [15]. (C_i depend on several parameters like the angle between airspeed and the body fixed reference system, the aerodynamic and geometric form of the wing);
- g is the gravity constant ($g = 9.81ms^{-2}$);

- d is the distance from the center of mass to the rotors;
- u_1 is the resulting thrust of the four rotors defined as $u_1 = (F_1 + F_2 + F_3 + F_4)$
- u_2 is the difference of thrust between the left rotor and the right rotor defined as $u_2 = d\,(F_4 - F_2)$
- u_3 is the difference of thrust between the front rotor and the back rotor defined as $u_3 = d(F_3 - F_1)$
- u_4 is the difference of torque between the two clockwise turning rotors and the two counter-clockwise turning rotors defined as $u_4 = C(F_1 - F_2 + F_3 - F_4)$
- C is the force to moment scaling factor

Assuming that the electric motors are velocity controlled, then (u_1, u_2, u_3, u_4) may be viewed as control inputs. The dynamic model of the quadrotor has been developed in many experimental works but in different manner, like S. Bouabdallah ([16]). Referring to [17], the real control signals (u_1, u_2, u_3, u_4) have been replaced by $(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4)$ to avoid singularity in Lie transformation matrices when using exact linearization. In that case u_1 has been delayed by double integrator. The other control signals will keep unchanged

$$\begin{aligned} &u_1 = \zeta; \quad \dot{\zeta} = \xi; \quad \dot{\xi} = \bar{u}_1 \\ &u_2 = \bar{u}_2 \\ &u_3 = \bar{u}_3 \\ &u_4 = \bar{u}_4 \end{aligned} \tag{7}$$

The obtained extended system is described by state space equations of the form:

$$\begin{aligned} \dot{x} &= \bar{f}(x) + \sum_{i=1}^{4} \bar{g}_i(x)\bar{u}_i \\ y &= h(x) \end{aligned} \tag{8}$$

where

$$x = [x_0, y_0, z_0, \psi, \theta, \phi, u_0, v_0, w_0, \zeta, \xi, p, q, r]^T$$
$$y = [x_0, y_0, z_0, \psi]^T$$

$$f = \begin{bmatrix} u_0 \\ v_0 \\ w_0 \\ qS\phi S_e\theta + rC\phi S_e\theta \\ qC\phi - rS\phi \\ p + qS\phi T\theta + rC\phi T\theta \\ \frac{Ax}{m} - \frac{1}{m}(C\phi C\psi S\theta + S\phi S\psi)\zeta \\ \frac{A_y}{m} - \frac{1}{m}\left(C\phi S\theta S\psi - C\psi S\phi\right)\zeta \\ \frac{A_z}{m} + g - \frac{1}{m}(C\theta C\phi)\zeta \\ \xi \\ 0 \\ \frac{I_y - I_z}{I_x} qr + \frac{A_p}{Ix} \\ \frac{I_z - I_x}{I_y} pr + \frac{A_q}{Iy} \\ \frac{I_x - I_y}{I_z} pq + \frac{A_r}{Iz} \end{bmatrix}$$

$$\bar{g}_1(x) = [0,0,0,0,0,0,0,0,0,1,0,0,0]^T$$
$$\bar{g}_2(x) = \left[0,0,0,0,0,0,0,0,0,0,0,\tfrac{d}{I_x}0,0\right]^T$$
$$\bar{g}_3(x) = \left[0,0,0,0,0,0,0,0,0,0,0,0,\tfrac{d}{I_y},0\right]^T$$
$$\bar{g}_4(x) = \left[0,0,0,0,0,0,0,0,0,0,0,0,0\tfrac{1}{I_z}\right]^T$$

The purpose of the next section is to design a feedback controller for the four rotor miniature helicopter which exhibits robustness properties against neglected effects and parametric uncertainties.

III. Feedback linearization controller

The feedback linearization technique is based on inner and outer loops of the controller. The Input-Output linearization-based inner loop uses the full state feedback to globally linearize the nonlinear dynamics of selected controlled outputs. Each of the output channels is differentiated sufficiently many times until a control input component appears in the resulting equation. Using the Lie derivative, Input-Output linearization will transform the nonlinear system into a linear and non-interacting system in the Brunovsky form. The outer controller adopts a classical polynomial control law for the new input variable of the resulting linear system.

A. Structure of the inner controller

The input-output decoupling problem is solvable for the nonlinear system (8) by means of static feedback. The vector relative degree $\{r_1, r_2, r_3, r_4\}$ is given by

$$r_1 = r_2 = r_3 = 4;\ r_4 = 2$$

and we have

$$col(y_1^{(r_1)}, y_2^{(r_2)}, y_3^{(r_3)}, y_4^{(r_4)}) = b(x) + \Delta(x)\bar{u} \tag{9}$$

where $\Delta(x)$ and $b(x)$ are computed as follows:

$$\Delta(x) = \begin{bmatrix} L_{g_1} L_f^{r_1 - 1} h_1(x) & \ldots & L_{g_4} L_f^{r_1 - 1} h_1(x) \\ \cdots & \ddots & \cdots \\ L_{g_1} L_f^{r_4 - 1} h_4(x) & \cdots & L_{g_4} L_f^{r_4 - 1} h_4(x) \end{bmatrix}$$

$$b(x) = \begin{bmatrix} L_f^{r_1} h_1(x) \\ \vdots \\ L_f^{r_4} h_4(x) \end{bmatrix} \tag{10}$$

where

$$L_f h(x) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i(x); \quad L_f^k h(x) = L_f(L_f^{k-1} h(x))$$

The matrix $\Delta(x)$ is non singular everywhere in the region $\zeta \neq 0$, $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Therefore, the input-output decoupling problem is solvable for system (8) by means of a control law of the form:

$$\bar{u} = \alpha(x) + \beta(x)v \tag{11}$$

where $\alpha(x)$ and $\beta(x)$ are given by

$$\begin{aligned} \alpha(x) &= -\Delta^{-1}(x)b(x) \\ \beta(x) &= \Delta^{-1}(x) \end{aligned} \tag{12}$$

Taking into account relation (7), we derive the structure (Figure 1) of the control law of system (8). Moreover, since system (8) has dimension $n = 14$, the condition

$$r_1 + r_2 + r_3 + r_4 = n$$

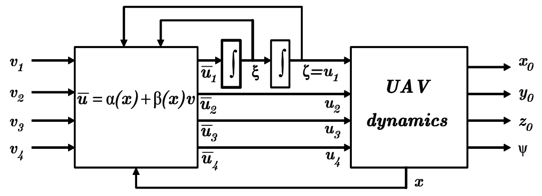

Fig. 1. Block diagram of the inner loop.

is fulfilled and therefore, the system can be transformed via static feedback into a system which, in suitable coordinates, is fully linear and controllable. However, due to the presence of external disturbances the Input-Output linearization is not exact and the inner closed loop system in that case is composed into a linear part and a nonlinear disturbance part:

$$\begin{pmatrix} y_1^{(4)} \\ y_2^{(4)} \\ y_3^{(4)} \\ y_4^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{d^4x_0}{dt^4} \\ \frac{d^4y_0}{dt^4} \\ \frac{d^4z_0}{dt^4} \\ \frac{d^2\psi}{dt^2} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} + \begin{pmatrix} \xi_1(x,t) \\ \xi_2(x,t) \\ \xi_3(x,t) \\ \xi_4(x,t) \end{pmatrix} \tag{13}$$

with

$$\begin{pmatrix} \xi_1(x,t) \\ \xi_2(x,t) \\ \xi_3(x,t) \\ \xi_4(x,t) \end{pmatrix} = \begin{pmatrix} \frac{\ddot{A}_x}{m} + a_{14}A_p + a_{15}A_q \\ \frac{A_y}{m} + a_{24}A_p + a_{25}A_q \\ \frac{A_z}{m} + a_{34}A_p + a_{35}A_q \\ a_{45}A_q + a_{46}A_r \end{pmatrix}$$

where

$$\begin{aligned} a_{14} &= (\zeta S\phi C\psi S\theta - \zeta C\phi S\psi)/(mI_x); \\ a_{15} &= -(\zeta C\psi C\theta)/(mI_y) \\ a_{24} &= (\zeta S\phi S\psi S\theta + \zeta C\phi C\psi)/(mI_x); \\ a_{25} &= -(\zeta S\psi C\theta)/(mI_y) \\ a_{34} &= (\zeta S\phi C\theta)/(mI_x);\ a_{35} = (\zeta S\theta)(mI_y) \\ a_{45} &= S\phi/(I_yC\theta);\ a_{46} = C\phi/(I_zC\theta) \end{aligned}$$

v_1, v_2, v_3, v_4, represent the new input control signals. The controller compares the primary state (x_0, y_0, z_0, ψ) and their successive derivatives to the desired state trajectory.

B. Structure of the outer controller

While adapting a classical polynomial control law for the new input variable v with disturbance compensation, one obtains the following equations:

$$\begin{aligned} v_1 &= x_d^{(4)} - \lambda_3\dddot{e}_{11} - \lambda_2\ddot{e}_{11} - \lambda_1\dot{e}_{11} - \lambda_0 e_{11} - z_{41}^f \\ v_2 &= y_d^{(4)} - \lambda_3\dddot{e}_{12} - \lambda_2\ddot{e}_{12} - \lambda_1\dot{e}_{12} - \lambda_0 e_{12} - z_{42}^f \\ v_3 &= z_d^{(4)} - \lambda_3\dddot{e}_{13} - \lambda_2\ddot{e}_{13} - \lambda_1\dot{e}_{13} - \lambda_0 e_{13} - z_{43}^f \\ v_4 &= \ddot{\psi}_d - \lambda_5\dot{e}_5 - \lambda_4 e_5 - z_6^f \end{aligned} \tag{14}$$

where x_d, y_d, z_d, ψ_d represent the desired output signals, corresponding to x_0, y_0, z_0, ψ, respectively, the errors signals $e_{11} = [x_0 - x_{0d}]$ $e_{12} = [y_0 - y_{0d}]$ $e_{13} = [z_0 - z_{0d}]$ and $e_5 = [\psi - \psi_d]$ and the coefficients λ_i, $i = 0, \cdots, 5$ are to be specified in the sequel. The variables z_{41}^f, z_{42}^f, z_{43}^f and z_6^f are the filtered signals of z_{41}, z_{42}, z_{43} and z_6 given in the observer section. The closed-loop system (13), (14) can be rewritten in the form

$$\dot{e} = Ae + \tilde{\xi}(x,t) \tag{15}$$

$$\xi = [\xi_1,\ \xi_2,\ \xi_3,\ \xi_4]^T \tag{16}$$

$$z^f = \left[z_{41}^f,\ z_{42}^f,\ z_{43}^f,\ z_6^f\right]^T \tag{17}$$

where e represents the tracking error between the desired value and the actual one, i.e.,

$$e = [e_1, e_2, e_3, e_4, e_5, e_6]^T$$

and

$$\begin{aligned} e_1 &= [e_{11}, e_{12}, e_{13}]^T \\ e_2 &= \dot{e}_1;\ e_3 = \ddot{e}_1;\ e_4 = \dddot{e}_1 \\ e_6 &= \dot{e}_5 \end{aligned} \tag{18}$$

$\tilde{\xi}(x,t)$ is the wind parameter errors of the disturbances The matrix A is then given by

$$A = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ -\lambda_0 I & -\lambda_1 I & -\lambda_2 I & -\lambda_3 I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\lambda_4 & -\lambda_5 \end{bmatrix}$$

where I is an identity matrix of dimension 3×3 and the the control gains λ_i, $i = 0, \ldots, 5$ are such that the eigenvalues of the matrix A have desired locations.

It is very important to know the domain of attraction of an equilibrium point that is the set of initial states from which the system converges to the equilibrium point itself [24],[25]. Actually, such problem arises in both system analysis and synthesis, in order to guarantee a stable behavior in a certain region of the state space.

IV. High Order Sliding Mode observer

Motivated by practice, the measured UAV variables are the absolute position x_0, y_0, z_0 and the orientation ψ which represent the translational motion and rotation around z axis, respectively. Although non measurable signals can be obtained by successive differentiation, however, they are contaminated by the measurement noise to such a degree that the differentiation can no longer be used. To avoid differentiation let us construct an observer based on arbitrary order high order sliding mode differentiator [21]..

A. Observer model

The linearized dynamic model of the quadrotor with the measured signals $x_1 = [x_0, y_0, z_0]^T$, and $x_5 = \psi$ can be represented in the following state space form

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= [v_1, v_2, v_3]^T + [\xi_1, \xi_2, \xi_3]^T \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= v_4 + \xi_4
\end{aligned}
\tag{19}
$$

Let us propose the observer based on high order differentiation for the state variables $x_1, x_2, x_3, x_4, x_5, x_6$ of the form:

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + z_1 \\
\dot{\hat{x}}_2 &= \hat{x}_3 + z_2 \\
\dot{\hat{x}}_3 &= \hat{x}_4 + z_3 \\
\dot{\hat{x}}_4 &= [v_1, v_2, v_3]^T + z_4 \\
\dot{\hat{x}}_5 &= \hat{x}_6 + z_5 \\
\dot{\hat{x}}_6 &= v_4 + z_6
\end{aligned}
\tag{20}
$$

where

$$
\begin{aligned}
z_1 &= \gamma_1 \left|x_1 - \hat{x}_1\right|^{3/4} sign(x_1 - \hat{x}_1) \\
z_2 &= \gamma_2 \left|\mu_2 - \hat{x}_2\right|^{2/3} sign(\mu_2 - \hat{x}_2) \\
z_3 &= \gamma_3 \left|\mu_3 - \hat{x}_3\right|^{1/2} sign(\mu_3 - \hat{x}_3) \\
z_4 &= \alpha_4 sign(\mu_4 - \hat{x}_4) \\
z_5 &= \gamma_4 \left|x_5 - \hat{x}_5\right|^{1/2} sign(x_5 - \hat{x}_5) \\
z_6 &= \alpha_6 sign(\mu_6 - \hat{x}_6)
\end{aligned}
\tag{21}
$$

and

$$
\mu_2 = \hat{x}_2 + z_1;\ \mu_3 = \hat{x}_3 + z_2;\ \mu_4 = \hat{x}_4 + z_3; \\
\mu_6 = \hat{x}_6 + z_5
$$

Theorem 1: The observer (20),(21) for the system (19) ensures in finite time the convergence of the estimated states to the real states, i.e $(\hat{x}_1,\ \hat{x}_2,\ \hat{x}_3,\ \hat{x}_4,\ \hat{x}_5,\ \hat{x}_6) \longrightarrow (x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)$ and the convergence of the filtered $z_4^f = \left[z_{41}^f, z_{42}^f, z_{43}^f\right]^T$ to $\xi_{123} = [\xi_1, \xi_2, \xi_3]^T$ and the filtered z_6^f to ξ_4. *Proof:* The finite time convergence of observers for variables $\tilde{x}_5, \tilde{x}_6$ is proved in [20].Taking $\tilde{x}_i = x_i - \hat{x}_i$ the estimation error can be written as:

$$
\begin{aligned}
\dot{\tilde{x}}_1 &= \tilde{x}_2 - \gamma_1 \left|\tilde{x}_1\right|^{3/4} sign(\tilde{x}_1) \\
\dot{\tilde{x}}_2 &= \tilde{x}_3 - \gamma_2 \left|\mu_2 - \hat{x}_2\right|^{2/3} sign(\mu_2 - \hat{x}_2) \\
\dot{\tilde{x}}_3 &= \tilde{x}_4 - \gamma_3 \left|\mu_3 - \hat{x}_3\right|^{1/2} sign(\mu_3 - \hat{x}_3) \\
\dot{\tilde{x}}_4 &= \xi_{123} - \alpha_4 sign(\mu_4 - \hat{x}_4)
\end{aligned}
\tag{22}
$$

To proof of finite time convergence of the error of observer (20) for $\tilde{x}_1, \tilde{x}_2,\ \tilde{x}_3, \tilde{x}_4$ we need just to rewrite first four equations of (22) in the form of differential inclusion

$$
\begin{aligned}
\dot{\tilde{x}}_1 &= \tilde{x}_2 - \gamma_1 \left|\tilde{x}_1\right|^{3/4} sign(\tilde{x}_1) \\
\dot{\tilde{x}}_2 &= \tilde{x}_3 - \gamma_2 \left|\mu_2 - \hat{x}_2\right|^{2/3} sign(\mu_2 - \hat{x}_2) \\
\dot{\tilde{x}}_3 &= \tilde{x}_4 - \gamma_3 \left|\mu_3 - \hat{x}_3\right|^{1/2} sign(\mu_3 - \hat{x}_3) \\
\dot{\tilde{x}}_4 &\in [-f_4^+, f_4^+] - \alpha_4 sign(\mu_4 - \hat{x}_4)
\end{aligned}
\tag{23}
$$

This inclusion is understood in Filippov sense [23]. The proof finite time convergence now is follows from Lemma 8 in [21]. ■

B. Output states reconstruction

The sliding observer presented above is in fact a state estimator with partial state feedback (x_0, y_0, z_0, ψ) taken as measured variables. The observer estimates the state needed by the control law to calculate the tracking error between the desired trajectories $(x_{1d}, x_{2d}, x_{3d}, x_{4d}, x_{5d}, x_{6d})$ and the estimated ones $(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5, \hat{x}_6)$. Unfortunately, the estimated state does not involve all the output states. In that case, to complete the full state output, the missed variables (θ, ϕ, p, q, r) of the state vector x (8) have been calculated through the estimated values and from the nonlinear system of equation (8), without taking the perturbation into account. So, from (8) θ and ϕ are deduced as follows:

$$
\begin{aligned}
\hat{\phi} &= \arcsin\left(\frac{-m(\hat{\ddot{x}}_0 S\psi - \hat{\ddot{y}}_0 C\psi)}{\zeta}\right) \\
\hat{\theta} &= \frac{1}{C\hat{\phi}} \arcsin\left(\frac{-m(\hat{\ddot{x}}_0 C\psi + \hat{\ddot{y}}_0 S\psi)}{\zeta}\right)
\end{aligned}
\tag{24}
$$

The variables $(\hat{p}, \hat{q}, \hat{r})$ can be found from the transformation matrix (2) which needs the variables $(\dot{\hat{\psi}}, \dot{\hat{\theta}}, \dot{\hat{\phi}})$. The latter can be evaluated from (24) and the third derivatives $(\hat{\dddot{x}}_0, \hat{\dddot{y}}_0)$ i.e. :

$$
\dot{\hat{\theta}} = -\frac{1}{C\hat{\theta}C^2\hat{\phi}\zeta} \left\{ \begin{array}{l} m\hat{\dddot{x}}_0(S\hat{\phi}S\hat{\theta}S\psi + C\psi C\hat{\phi}) + \\ m\hat{\dddot{y}}_0(C\hat{\phi}S\psi - S\hat{\phi}C\psi S\hat{\theta}) \\ +\hat{\dot{\psi}}\zeta C\hat{\phi}S\hat{\phi}C^2\hat{\theta} - S\hat{\theta}\zeta \end{array} \right\} \tag{25}
$$

$$
\dot{\hat{\phi}} = \frac{1}{\zeta C(\hat{\phi})} \left\{ -m\hat{\dddot{x}}_0 S\psi + \psi\zeta C\hat{\phi}S\hat{\theta} + \zeta S\hat{\phi} + mC\psi\hat{\dddot{y}}_0 \right\} \tag{26}
$$

So from the following matrix equation, the estimation of the variables $(\hat{p}, \hat{q}, \hat{r})$ can be deduced:

$$
\begin{bmatrix} \hat{p} \\ \hat{q} \\ \hat{r} \end{bmatrix} = \begin{bmatrix} 0 & S\hat{\phi}S_e\hat{\theta} & C\hat{\phi}S_e\hat{\theta} \\ 0 & C\hat{\phi} & -S\hat{\phi} \\ 1 & S\hat{\phi}T\hat{\theta} & C\hat{\phi}T\hat{\theta} \end{bmatrix}^{-1} \begin{bmatrix} \dot{\hat{\psi}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\phi}} \end{bmatrix} \tag{27}
$$

The over-all controller-observer closed-loop system is presented in figure 2. The stability proof for this over-all closed-loop system is similar to those of Theorem 1 and Theorem 2 and it is therefore omitted. Instead, simulation evidences will be provided in the next section.

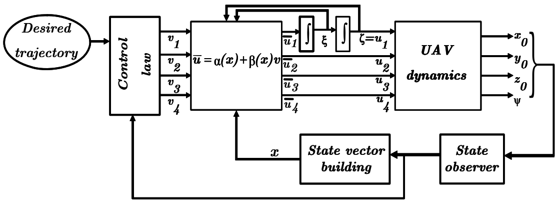

Fig. 2. The overall closed loop system

V. Simulation results

The constant quadrotor parameters, used in the simulation run, are:

$$m = 2Kg; \; I_x = I_y = I_z = 1.2416N.m/rad/s^2;$$
$$d = 0.1m; \; g = 9.81m/s^2$$

The gain values of $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ and (λ_4, λ_5) represent the coefficients of the polynomial $(s+5)^4$ and $(s+5)^2$ respectively. For a specific f_i^+ and α_i, the values of γ_i are chosen as $\gamma_1 = 3$, $\gamma_2 = 2.5$,$\gamma_3 = \gamma_4 = 1.5$ and $\alpha_4 = \alpha_6 = 1.1$. An application has been established without and with disturbances and with uncertainties to see the performance and robustness of the sliding mode observer.

a) Without disturbance: Taking for this case $(A_x = A_y = A_z = 0)$; $(A_p = A_q = A_r = 0)$; the following results are obtained (figures-(3, 4).

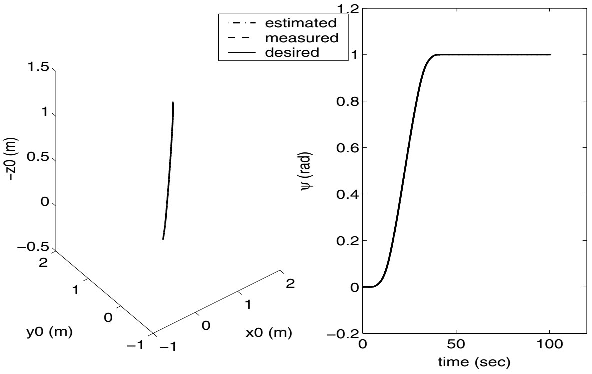

Fig. 3. Reference trajectories

b) With aerodynamic force disturbances: For $A_x = 2\sin 0.1t$, $A_y = 2\sin 0.1t$, $A_z = 2\sin 0.1t$ occurring at 10 sec, 20 sec and 40 sec respectively the following results are obtained (figures(5 to 11).

c) With aerodynamic moment disturbances: For $A_p = 0.09\sin 0.1t$, $A_q = 0.01\sin 0.1t$, $A_r = 0.2\sin 0.1t$ occurring at 10 sec, 20 sec and 40 sec respectively the following results are obtained (figures(12 to 13).

It is concluded from the simulations, made without perturbation, that the high order sliding mode observer gives satisfactory results. The results of estimation errors given in figure (4) show the efficiency of the observer. The same conclusion follows from the tracking errors which vanish after a finite time with a perfect convergence. When wind

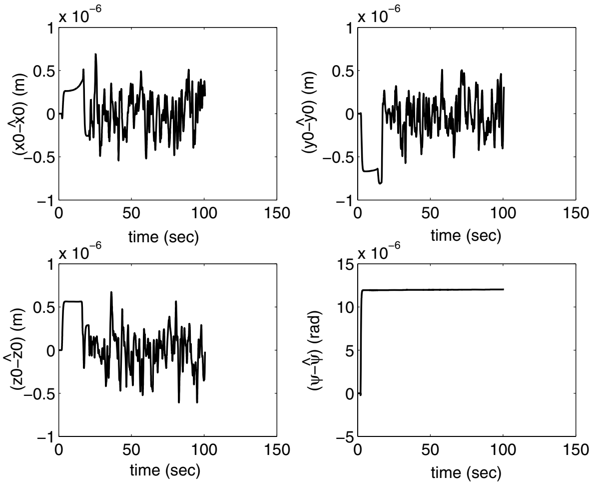

Fig. 4. Estimation errors

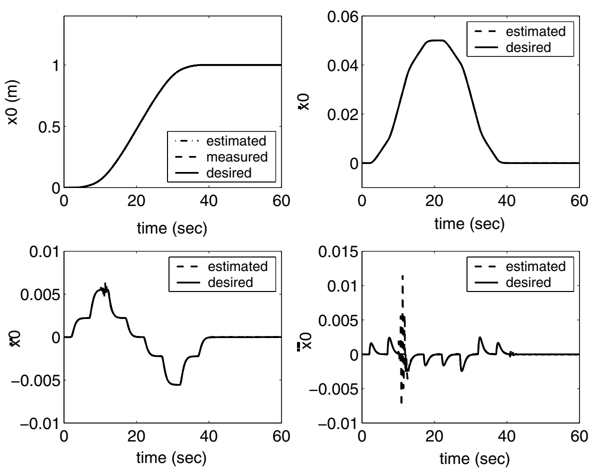

Fig. 5. Trajectories x_0, $\dot{x}_0$, $\ddot{x}_0$, $\dddot{x}_0$

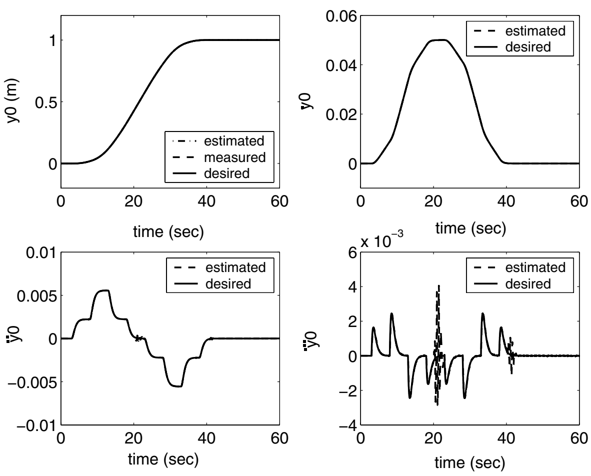

Fig. 6. Trajectories y_0, $\dot{y}_0$, $\ddot{y}_0$, $\dddot{y}_0$

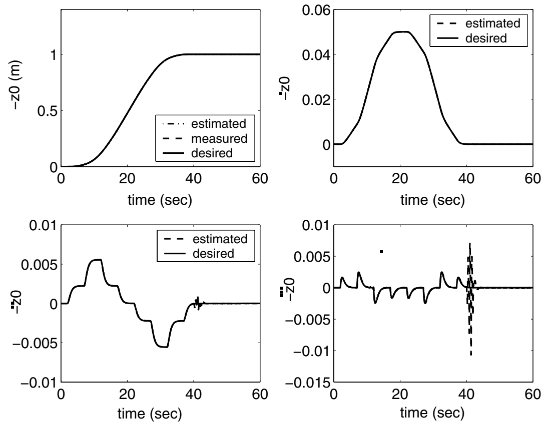

Fig. 7. Trajectories z_0, $\dot{z}_0$, $\ddot{z}_0$, $\dddot{z}_0$

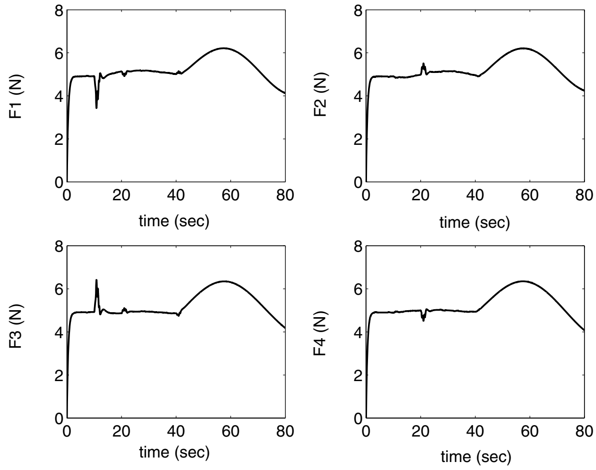

Fig. 10. Applied forces

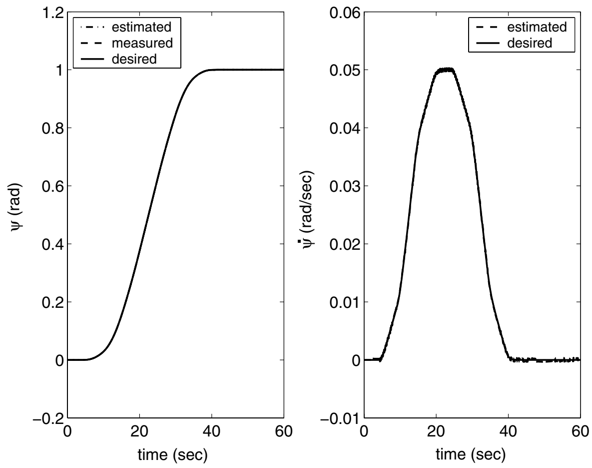

Fig. 8. Trajectories ψ, $\dot{\psi}$

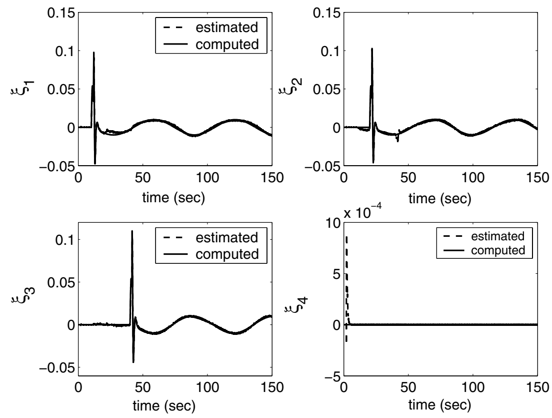

Fig. 11. Disturbance estimation

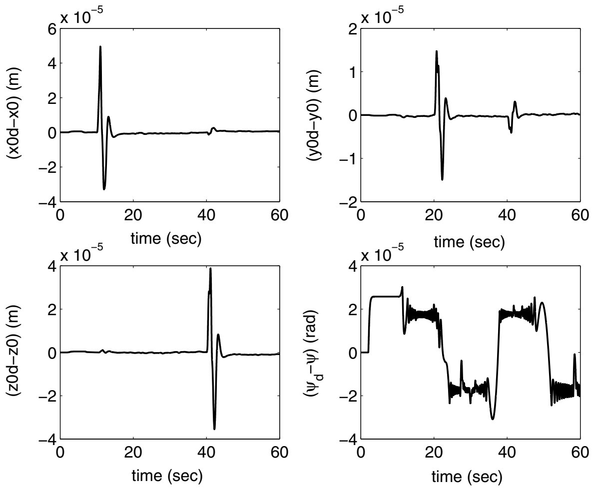

Fig. 9. Tracking errors with A_x, A_y, A_z

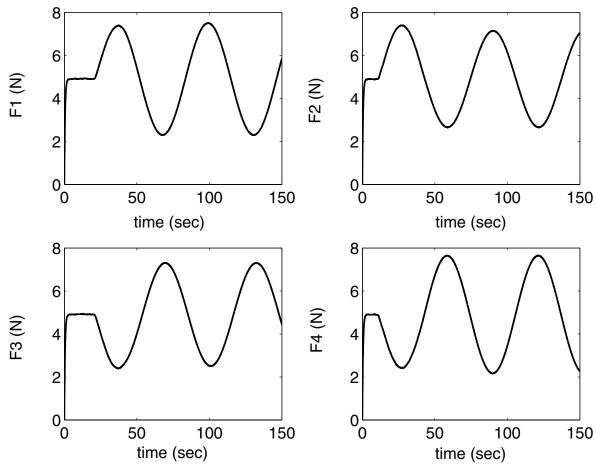

Fig. 12. Applied forces with A_p, A_q, A_r

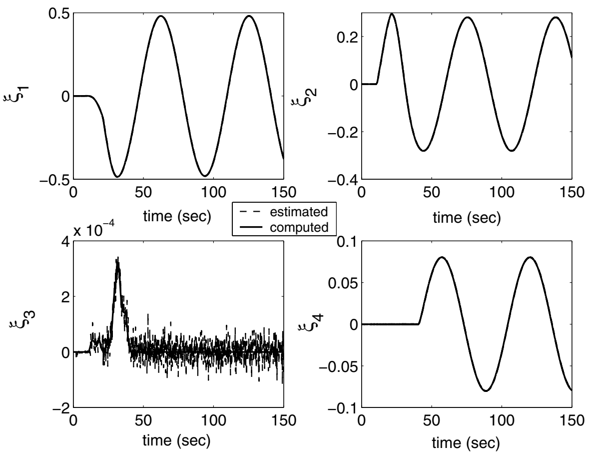

Fig. 13. Disturbance estimation

disturbances are introduced the results in figures (5 to 8) reflect the robustness of the mixed observer-controller, also confirmed by the tracking error convergence (figure-9), without need of an external estimation procedure.. The estimation of force and moment disturbances are presented in figures (11, 13), it shows that the estimated disturbances follow exactly the computed ones . However It appears that the system dynamic behavior is more sensitive toward aerodynamic moment disturbances. This is also confirmed by variation of forces F_1, F_2, F_3 and F_4 in figure (12) which exactly reflects the movement of the quadrotor in x, y, and z directions in the presence of disturbances. The convergence of the output state vector is obtained in spite of the non-robust exact linearization against uncertainties on system parameters. On the other side excessive chattering around desired trajectories is avoided by using high order sliding mode.

VI. CONCLUSION

A feedback linearization controller using high order sliding mode observer has been applied to a quadrotor Unmanned Aerial Vehicle (UAV). Although the behavior of the UAV, affected by aerodynamic forces and moments, is non linear and high coupled, the feedback linearization coupled to HOSM observer and applied to the UAV, turns out to be a good starting point to avoid complex nonlinear control solutions and excessive chattering. However, in the presence of nonlinear disturbances the system after linearization remains nonlinear. The observer used here overcomes easily this non-linearities by an inner estimation of the external disturbances to impose desired stability and robustness properties on the global closed loop system. The unmeasured states and their derivatives have been successfully reconstructed through the sliding mode observer design.

Theoretical results have been supported by numerical simulations that demonstrated efficiency of the proposed controller design. It is hoped that further investigation carries out robust controllers that would compensate noise effects and initial condition problems.

REFERENCES

[1] B. K. Kim, "Disturbance Observer Based Approach to the Design of Sliding Mode Controller for High Performance Positing Systems ", 15th Triennial World Congress (IFAC), Barcelona, 2002.

[2] M. Hautus, "Strong detectability and observers," Linear Algebra and its Applications, vol. 50, pp. 353—368, 1983.

[3] Jong-Rae. Kim, "Model-Error Control Synthesis: A New Approach to Robust Control", PhD Dissertation of Texas A&M University, August 2002.

[4] J.-J. E. Slotine and J. K. Hedrick, "Robust Input-Output Feedback Linearization", International Journal of Control, Vol. 57, pp. 1133-1139, 1993.

[5] A. Mokhtari and A. Benallegue, "Dynamic feedback controller of Euler angles and wind parameters estimation for a quadrotor unmanned aerial vehicle", IEEE Int. Conf. on Robotics and Automation (ICRA'04), 2004.

[6] W. Wang, Z. G. Weiwen "A Comparison Study of Advanced State Observer Design Techniques", American Control Conference (ACC'03), Denver, June 2003.

[7] J. Barbot, M. Djemai, and T. Boukhobza, "Sliding mode observers," in Sliding Mode Control in Engineering (W. Perruquetti and J. Barbot, eds.), Control Engineering, pp. 103—130, New York: Marcel Dekker, 2002.

[8] J. Barbot and T. Floquet, "A sliding mode approach of unknown input observers for linear systems," in 43th IEEE Conference on Decision and Control, pp. 1724—1729, 2004.

[9] C. Edwards, S. Spurgeon, and R. G. Hebden, "On development and applications of sliding mode observers,"in Variable Structure Systems:Towards XXIst Century (J. Xu and Y. Xu, eds.), Lecture Notes in Control and Information Science pp. 253—282, Berlin, Germany: Springer Verlag, 2002.

[10] A. Poznyak, "Stochastic output noise effects in sliding mode estimations," International Journal of Control,vol. 76, pp. 986—999, 2003.

[11] A. Gessow, G.C. Myers, "Aerodynamics of the Helicopter", New York: F. Ungar, 1967.

[12] W. Johnson, "Helicopter Theory", New York: Dover, 1994.

[13] C. Edwards and S. Spurgeon, Sliding Mode Control. London: Taylor and Francis, 1998.

[14] H. Hashimoto, V. Utkin, J. X. Xu, H. Suzuki, and F. Harashima, "Vss observer for linear time varying system," in Procedings of IECON'90, (Pacific Grove CA), pp. 34—39, 1990.

[15] S. Gomes and J. Ramos, "Airship dynamic modeling fot autonomous operation", IEEE Int. Conf. on Robotics and Automation (ICRA'98), 1998.

[16] S. Bouabdallah, P. Murrieri, R. Siegwart, "Design and Control of an Indoor Micro Quadrotor", IEEE Int. Conf. on Robotics and Automation (ICRA'04), 2004.

[17] V. Mistler, A. Benallegue, N. K. M'Sirdi, "Exact Linearization and Non-interacting Control of a 4 Rotors Helicopter via Dynamic Feedback", 10th IEEE Int. Workshop on Robot-Human Interactive Communication (Roman'2001), Paris, September 2001..

[18] A. Levant, "Sliding order and sliding accuracy in sliding mode control," International Journal of Control, vol. 58, pp. 1247—1263, 1993.

[19] A. Levant, "Robust exact differentiation via sliding mode technique," Automatica, vol. 34, no. 3, pp. 379—384,1998.

[20] J. Davila, L.Fridman, A.Levant, "Second Order Sliding Mode Observer for Mechanical Systems"' *IEEE Transaction on Automatic Control,* Vol. 50,(11):1785-1791, 2005.

[21] A. Levant. High-order sliding modes: differentiation and output-feedback control International Journal of Control,76(9-10):924—941, 2003.

[22] V. Utkin, J. Guldner, and J. Shi, Sliding Modes in Electromechanical Systems. London: Taylor and Francis,1999.

[23] A. Filippov, Differential Equations with Discontinuous Right-hand Sides. Dordrecht: Kluwer Academic Publishers, 1988.

[24] J. J. E. Slotine, "Sliding controller design for nonlinear systems", Int. Journal of Control, Vol. 38, pp. 465-492, Feb., 1984.

[25] J.-J. Slotine, J. K. Hedrick, E. A. Misawa, "On sliding observers for nonlinear systems", Journal of Dynamic Systems, Measurement and Control, Vol. 109, pp. 245, 1987.