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Pricing CDO tranches in an intensity based model with the mean reversion approach

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ABSTRACT

We discuss the phenomenon of mean reversion in credit risk market and propose a class of models, in the framework of intensity based model, where the default intensity is composed of a common component and a idiosyncratic component which are specified by independent mean reverting stochastic processes of the following Markovian type

$$\mathrm{d}X(t) = (\theta + \sigma\alpha(X(t), t))X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t)$$

where $\theta \geq 0$ is the long-term mean value, the parameter $\sigma \geq 0$ stands for the scaling of the volatility, and $\alpha(X(t), t)$ is the mean correction with the function $\alpha : \mathbb{R} \times [0, \infty) \mapsto \alpha(x, t) \in \mathbb{R}$ being twice differentiable in x and differentiable in t, and $W(t)$ is a Brownian motion. We demonstrate how this class of models can be used to price synthetic CDOs and present a closed-form solution of tranche spreads in synthetic CDOs.

1. Introduction

In recent years, the market for credit derivatives has developed rapidly with the introduction of new contracts and the standardized documentation. Collateralized debt obligations (CDOs) have been among the fastest-developing investment vehicles in the financial industry. Along with the rapid development of the market of CDOs comes the issue of how to price them.

The market standard model is the so-called one factor Gaussian copula model. Its origins can be found in [1,2]. The fact that there is no dynamic behavior for default risk present in the model and the assumptions about the characteristics of the underlying portfolio simplify the analytical derivation of CDOs spreads make the model lacks realism. Thereafter more and more CDOs pricing models are proposed.

In one line of thinking, to relax the assumption of Gaussian distribution in the one factor Gaussian copula model, student-*t* copula [3–10], double-*t* copula [11,12], Clayton copula [13–17,10,18], Archimedian copula [19,20], and Marshall Olkin copula [21–24] are studied. And default correlations are made stochastic and correlated with the systematic factor in [25,26] to relax the assumption that default correlations are constant through time and independent of the firms default probabilities. Hull and White propose the implied copula method in [27]. An empirical analysis of the pricing of CDOs can be found in [28]. Although some of these models match CDO tranche spreads better, they are in essence still static models.

In the other line of thinking, many dynamic models are presented, where many stochastic processes are incorporated in CDOs pricing models to describe the default dependence. Markov chains are used to represent the distance to default of single obligor (eg. [29–31]). Correlation among obligors is introduced with non-recombining trees [29] or via a common time change of affine type [30].

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Some researchers incorporate jumps in CDOs pricing model. Duffie and Gârleanu [32], for example, propose an approach based on affine processes with both a diffusion and a jump component. To improve tractability, Chapovsky et al. [33] suggest a model in which default intensities are modelled as the the sum of a compensated common random intensity driver with tractable dynamics (e.g. CIR with jumps) and a deterministic name depended function.

Motivated by the possibility that price processes could be pure jump, several authors have focused attention on pure jump models in the Lévy class (cf. e.g. [34]). Firstly, we have the normal inverse Gaussian (NIG) model of Barndorff–Nielsen [35], and its generalization to the generalized hyperbolic class by Eberlein et al. [36]. Kalemanova et al. [37] and Guégan and Houdain [38] work with NIG factor model. Secondly, we have the symmetric variance gamma (VG) model studied by Madan and Seneta [39] and its asymmetric extension studied by Madan and Milne [40], Madan et al. [41]. Boxter [42] introduces the B-VG model where has both a continuous Brownian motion and a discontinuous variance—Gamma jump term. Finally, we have the model developed by Carr, Geman, Madan, and Yor (CGMY) [43], which further generalizes the VG model. Most of these models are special cases of the generic one-factor Lévy model supposed in [44]. Lévy models bring more flexibility into the dependence structure and allow tail dependence.

In this paper, under the assumption of equilibrium market, we consider a class of underlying mean reverting stochastic processes in an intensity based model. Intensity based models were introduced by Jarrow and Turnbull [45], Lando [46,47], Schönbucher [48] (see also [49]) and Duffie and Singleton [50]. Relative to the copula approach, the intensity based model has the advantage that the parameters have economic interpretations. Furthermore, the model, by nature, delivers stochastic credit spreads and is therefore well-suited for the pricing of CDOs tranches.

The variables representing defaults take different forms in literature. Let's give an outline of different presentations of default variables both in the copula setting and in the intensity based setting.

In the copula setting, we may illustrate three different descriptions of default variables in three copula models. First, in the one-factor Gaussian copula, the default variable is defined as

$$X_i = a_i Z + \sqrt{1 - a_i^2}\epsilon_i$$

where $Z, \epsilon_1, \ldots, \epsilon_N$ are independent standard normal random variables and the factor loadings are constant. The correlation between any pair of default variables, X_i and X_j, is then $\rho_{i,j} = a_i a_j$. The conditional and unconditional default probabilities in the Gaussian copula are given in terms of the standard normal distribution function.

Second, in the stochastic factor loading copula, the default variable is defined as

$$X_i = a_i(Z)Z + \upsilon_i\epsilon_i + m_i$$

again with independent standard normal common and idiosyncratic factors. The factor loadings are stochastic, and υ_i and m_i are constants chosen to ensure zero mean and unit variance. For more details, see [25].

Finally, in the double-t copula, the default variable is

$$X_i = a_i \frac{Z}{std(Z)} + \sqrt{1 - a_i^2}\frac{\epsilon_i}{std(\epsilon_i)}$$

where $std(x)$ stands for standard deviation of x; the common factor and idiosyncratic factors follow t distributions with d degrees of freedom, which have heavier tails than the normal distribution, and the loadings are constant. For more details, see [11].

In the intensity based model, default is defined as the first jump of a pure jump process, and it is assumed that the jump process has an intensity process. More formally, it is assumed that a non-negative process λ exists such that the process

$$M(t) := 1_{\{\tau \le t\}} - \int_0^t 1_{\{\tau > s\}}\lambda(s)\mathrm{d}s$$

is a martingale. And the default correlation is generated through dependence of firms' intensities on the common factor. Roughly, default intensities are represented in the following different forms.

Duffie and Singleton [51] formulate an intensity process as a mean reverting process with jumps, namely, the intensity process λ has independently distributed jumps at Poisson-arrival times, with independent jump sizes drawn from a specified probability distribution. Between jumps, λ reverts deterministically at rate k to a constant γ, i.e.,

$$\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} = k[\gamma - \lambda(t)].$$

An another simple parametric intensity model often used for modeling interest rates is the so-called CIR process,

$$\mathrm{d}\lambda(t) = k(\theta - \lambda(t)) + \sigma\sqrt{\lambda(t)}W(t)$$

where θ is the long-run mean of λ, k is the mean rate of reversion to the long-run mean, σ is a volatility coefficient and $W(t)$ is a standard Brownian motion.

Further, in [32], Duffie and Gârleanu extent this CIR process to a jump diffusion process, namely, the intensity process $(\lambda = \lambda(t))_{t\in[0,T]}$ for $T > 0$ solves a stochastic differential equation of the form

$$\mathrm{d}\lambda(t) = k(\theta - \lambda(t))\mathrm{d}t + \sigma\sqrt{\lambda(t)}\mathrm{d}W(t) + \triangle J(t) \tag{1.1}$$

where $\triangle J(t)$ denotes any jump that occurs at time t of a pure-jump process J, independent of $W(t)$, whose jump sizes are independent and exponentially distributed with mean μ and whose jump times are those of an independent Poisson process with mean jump arrival rate ℓ. The process λ is called a basic *affine process* with parameters $(k, \theta, \sigma, \mu, \ell)$.

In the multi-issuer default model of a CDO, Duffie and Gârleanu in [32] introduce correlation in the following way

$$\lambda_i(t) = X_c(t) + X_i(t), \tag{1.2}$$

where X_c and X_i are independent basic affine processes with respective parameters $(k, \theta_c, \sigma, \mu, \ell_c)$ and $(k, \theta_i, \sigma, \mu, \ell_i)$. Here k is the mean reversion rate, θ_c and θ_i are long-term means, σ is the diffusive volatility, μ is the mean jump size, ℓ_c and ℓ_i are mean jump arrival rates. One may view X_c as a state process governing common aspects of economic performance in an industry, sector, or currency region, and X_i as a state variable governing the idiosyncratic default risk specific to obligor i.

In [52], Mortensen assumes that default of obligor i is modelled as the first jump of a Cox process with a default intensity composed of a common and an idiosyncratic component in the following way:

$$\lambda_i(t) = a_i X_c(t) + X_i(t) \tag{1.3}$$

where $a_i > 0$ is a constant and the two processes, X_c and X_i, are independent affine processes of the form (1.1). This is a simple but relevant modification of the specification in (1.2).

In [53], Herbertsson specifies the default intensity in a Markov jump setting with the form

$$\lambda_i(t) = a_i + \sum_{j \neq i} b_{i,j} 1_{\{\tau_j \leq t\}}, \quad \tau_i \geq t,$$

and $\lambda_i(t) = 0$ for $t > \tau_i$. Further, $a_i \geq 0$ and $b_{i,j}$ are constants such that $\lambda_i(t)$ is non-negative.

In [54], Schönbucher assumes the loss process follows a Poisson process with stochastic intensity, a process referred to as a Cox process

$$\mathrm{d}\lambda_i(t) = \mu_i(t)\mathrm{d}t + \sigma_i(t)\mathrm{d}W(t).$$

To keep the stochastic processes consistent with the loss process L_t, the following conditions must be satisfied

$$P_{L_t}(t)\mu_i(t) = -\sigma_i(t) v_{L_t,i}(t), \quad 0 \leq i \leq N-1,$$

where $P_{L_t}(t)$ is the loss distribution and $v_{n,m}(t)$'s are given explicitly therein.

In [33], Chapovsky et al. assume the stochastic intensity $\lambda_i(t)$ satisfies the so-called single name equation

$$\lambda_i(t) = \lambda_i' - \lambda_i^c(t) + \lambda_i^f(t),$$

where $\lambda_i'(t)$ is an auxiliary random process, $\lambda_i^c(t)$ is the compensator and $\lambda_i^f(t)$ is the forward intensity. $\lambda_i^c(t)$ and $\lambda_i^f(t)$ are deterministic functions of time which solve equations

$$\mathbb{E}\left[\exp\left\{-\int_o^t \lambda_i' \mathrm{d}t\right\}\right] = \exp\left\{-\int_o^t \lambda_i^c \mathrm{d}t\right\}, \qquad \exp\left\{-\int_o^t \lambda_i^f \mathrm{d}t\right\} = p_i(t).$$

where $p_i(t)$ is implied survival probability of the credit.

The remainder of the paper is organized as follows. In Section 2, we specify the form of mean reversion which will be used throughout the rest of the paper. In Section 3, first we present a brief introduction to synthetic CDOs which will be priced in our model; then, we describe the cash flows in a synthetic CDO and derive marginal default probabilities and certain related characteristics. We summarize the main result in Section 4, where a semi-analytical solution to expected tranche losses is presented. Section 5, the final section, draws then the conclusion.

2. Mean reversion in credit risk

In this section and in the sequel, all analysis and computation are assumed to be made under a risk-neutral martingale measure Q. Typically, such a Q exists if we rule out arbitrage opportunities. Let $(\Omega, \mathbb{F}, P; \{\mathcal{F}_t\}_{t\geq 0})$ be a given filtered probability set-up with the filtration $(\{\mathcal{F}_t\}_{t\geq 0})$ satisfying the usual conditions (see e.g. [55] for details).

In intensity based models, default intensities are often specified with affine jump processes (e.g. in [32,52]), CIR with jumps (e.g. in [33]) or Markov chains (e.g. in [56,53]), to name a few. We consider a class of stochastic processes with a feature of mean reversion. The phenomenon of mean reversion is a tendency for a stochastic process to remain near, or return over time to a long-run average. Usually, a decision to model a quantity with a mean reverting stochastic process is based both on empirical observation of that quantity over time, as well as some theoretical argument as to why it should be mean reverting.

On one hand, the assumption of market efficiency (in other words, the markets are assumed to be equilibrium) has provided the theoretical basis. In the efficient (or equilibrium) markets, at any given time, the values of each parameter fully reflect all available information, thus, no one will have the ability to outperform the market. Values of parameters may be over-average or under-average only in random occurrences and eventually revert back to their long-term mean values.

On the other hand, the existence of the mean reversion phenomena in credit risk market can be found in [57,58], among other studies. In [58], An shows large variations of credit risk over time in the commercial mortgage market and demonstrates that these variations are explained by two mean reverting latent risk factors: the macroeconomic factor and a commercial property market-specific factor. More impressively, in the technical documentation of the rating system RiskCalc for private companies [57], Moody's KMV confirms the evidence that obligors appear to exhibit mean reversion in their credit quality. In other words, good credits today tend to become somewhat worse credits over time and bad credits (conditional upon survival) tend to become better credits over time. Default studies on rated bonds by Moody's Investors Services support this assertion. And they found further evidence for mean reversion in both proprietary public firm default databases and in the Credit Research Database data on private firms. Motivated by the empirical evidence, as well as the assumption of market efficiency, we incorporate the mean reversion in our model setting.

Mean reversion exists in many different forms within financial market. In this paper, we assume that the common riskiness factor $X_c \in \mathbb{R}$ and idiosyncratic riskiness factors $X_1, \ldots, X_N \in \mathbb{R}$ are independent and are solutions of the following Markovian type stochastic differential equation

$$\mathrm{d}X(t) = (\theta + \sigma\alpha(X(t), t))X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t) \tag{2.1}$$

where $\theta \geq 0$ is the long-term mean value, the parameter $\sigma \geq 0$ stands for the scaling of the volatility is a constant, and $\alpha(X(t), t)$ is the mean correction with the function $\alpha : \mathbb{R} \times [0, \infty) \mapsto \alpha(x, t) \in \mathbb{R}$ being twice differentiable in x and differentiable in t, and $W(t)$ is a Brownian motion on $(\Omega, \mathbb{F}, P; \{\mathcal{F}_t\}_{t\geq 0})$. Clearly, the risk factors are scalars.

In [59], the authors work on a stochastic process of the form (2.1) presenting credit rating of bond issuers. They have proved that the function $\alpha(x, t)$ satisfies the following viscous Burgers equation

$$\frac{\partial}{\partial t}\alpha(x, t) = -\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\alpha(x, t) - \sigma\alpha(x, t)\frac{\partial}{\partial x}\alpha(x, t) \tag{2.2}$$

for $(x, t) \in \mathbb{R} \times [0, \infty)$. The viscous Burgers equation (2.2) is a famous nonlinear PDE which is solvable via the celebrated Cole-Hopf transformation. [59] gives a numerical approximation to the Burgers equation and more detailed computation may be found therein.

3. Intensity based default risk model

In this section, we first introduce synthetic CDOs in Section 3.1 and then formulate the cash flows of a synthetic CDO in Section 3.2. In Section 3.3, we present derivations of marginal default probabilities and certain related characteristics.

3.1. Introduction to a synthetic CDOs

A CDO can be initiated by one or more of the followings: banks, non-bank financial institutions, and asset management companies, which are referred to as the *sponsors*. The sponsors of a CDO create a company so-called the *special purpose vehicle* (SPV). The SPV works as an independent entity and is usually bankruptcy remote. The sponsors can earn serving fees, administration fees and hedging fees from the SPV, but otherwise has no claim on the cash flow of the assets in the SPV.

According to how the SPV gains credit risks, two kinds of CDOs can be classified: *cashflow CDOs* and *synthetic CDOs*. If the SPV of a CDO owns the underlying debt obligations (the underlying portfolio), that is, the SPV obtains the credit risk exposure by purchasing debt obligations (eg. bonds, residential and commercial loans), the CDO is referred to as a cashflow CDO, which is the basic form in the CDOs market in their formative years. In contrast, if the SPV of a CDO does not own the debt obligations, instead obtaining the credit risk exposure by selling CDSs on the debt obligations of reference entities, the CDO is referred to as a synthetic CDO; the synthetic structure allows bank originators in the CDOs market to ensure that client relationships are not jeopardized, and avoids the tax-related disadvantages existing in cashflow CDOs.

After acquiring credit risks, SPV sells these credit risks in *tranches* to investors who, in return for an agreed payment (usually a periodic fee), will bear the losses in the portfolio derived from the default of the instruments in the portfolio. Therefore, the tranches holders have the ultimate credit risk exposure to the underlying reference portfolio. Tranching, a common characteristic of all securisations, is the structuring of the product into a number of different classes of notes ranked by the seniority of investor's claims on the instruments assets and cashflows. For example, in synthetic CDOs, the tranches typically have the following different seniorities: *super senior tranche*, the least risky tranche in CDOs, followed by *senior tranche, mezzanine tranche*, *junior mezzanine tranche*, and finally the first loss piece or *equity tranche* [60].

A synthetic CDO is written on CDS contracts instead of actual debt securities. CDSs are like insurance contracts that a bank signs with other banks or institutions, which provide security to lenders against the risk of default on assets of questionable quality. Therefore the sellers of CDSs may be viewed as protection sellers, and the buyers of CDSs may be viewed as protection buyers.

Let us end up this subsection with a simple example of a synthetic CDO.

Example. A CDO issuer, called **A**, first gains underlying CDSs portfolio by selling protection with notional $1 million in 125 5-year CDS contracts for a total notional of $100 million. Each CDS contract is written on a specific corporate bond, and A receives quarterly a CDS premium until the contract expires or the bond defaults. In case of default, the loss of **A** is the difference between face value and market value of the defaulted bond.

A then issues a CDO tranche on the first 3% losses in his CDS portfolio and **B** buys this tranche with the principal $3 million. Here we may regard **A** as the protection buyer and **B** as the protection seller. No money is exchanged at time 0. If the premium on the tranche is, say, 2000 basis points, **A** pays a quarterly premium of 500 basis points to **B** on the remaining principal. If a default occurs on any of the underlying CDS contracts in this tranche, the loss is covered by **B** and his/her principal is reduced accordingly. **B** continues to receive the premium on the remaining principal until either the CDO tranche matures or the remaining principal is exhausted. The 0%–3% tranche is called equity tranche, that is the first loss tranche. Similarly, **A** sells 3%–8% (junior mezzanine) tranche, 8%–15% (mezzanine) tranche, 15%–30% (senior) tranche, and 30%–100% (super senior) tranches such that the total principal of these tranches equals the principal in the CDS contracts.

3.2. The cash flows in a synthetic CDO

We consider a synthetic CDO, with $q \geq 1$ tranches and maturity T, is defined for a portfolio consisting of N single-name CDSs on obligors with default times $\tau_1, \tau_2, \dots, \tau_N$ and with the same constant recovery rate $\phi \in [0, 1)$. We assume that the nominal values, which are the same for all obligors, equal to 1. Further, we assume that the risk-free interest rate r_t is deterministic. Then the accumulated loss L_t, at time t for this portfolio is

$$L_t = \frac{1}{N} \sum_{i=1}^{N} (1 - \phi) 1_{\{\tau_i \leq t\}}. \tag{3.1}$$

Without loss of generally we express the accumulated loss L_t in percent of the nominal portfolio value at $t = 0$.

If a tranche γ, $1 \leq \gamma \leq q$, with m_γ obligors ($1 \leq m_\gamma \leq N$), covers losses between $k_{\gamma-1}$ and k_γ ($k_{\gamma-1} < k_\gamma$), $k_{\gamma-1}$ is called the attachment point and k_γ the detachment point. A CDO is specified by the points

$$0(= 0\%) = k_0 < k_1 < k_2 < \cdots < k_q = 1(= 100\%)$$

with corresponding tranches $[k_{\gamma-1}, k_\gamma]$. Namely, in a tranche r, the protection seller **B** agrees to pay the protection buyer **A** all losses that occur in the interval $[k_{\gamma-1}, k_\gamma]$ derived from L_t up to T. The payments are made at the corresponding default time, if they arrive before T, then the contract ends. The expected value of this payment is called the *protection leg*, denoted by $\mathbb{V}_\gamma(T)$. As compensation for this, protection buyer **A** pays protection seller **B** a periodic fee proportional to the current outstanding value on tranche γ up to time T. The expected value of this payment scheme constitutes the *premium leg* denoted by $\mathbb{W}_\gamma(T)$. The accumulated loss $L_t^{(\gamma)}$ of tranche γ at time t is

$$L_t^{(\gamma)} = (L_t - k_{\gamma-1}) 1_{\{L_t \in [k_{\gamma-1}, k_\gamma]\}} + (k_\gamma - k_{\gamma-1}) 1_{\{L_t \geq k_\gamma\}}. \tag{3.2}$$

Let $B_t = \exp(-\int_0^t r_s \mathrm{d}s)$ denote the discount factor where r_t is the risk-free interest rate. The protection leg for tranche γ is then given by

$$\mathbb{V}_\gamma(T) = \mathbb{E}\left[\int_0^T B_t \mathrm{d}L_t^{(\gamma)}\right] = B_T \mathbb{E}\left[L_T^{(\gamma)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(\gamma)}\right] \mathrm{d}t, \tag{3.3}$$

where we have used integration by parts for Lebesgue–Stieltjes measures together with Fubini–Tonelli and the fact that r_t is deterministic. Further, if the premiums are paid at $0 < t_1 < t_2 < \cdots < t_{n_T} = T$ and if we ignore the accrued payments at defaults, then the premium leg is given by

$$\mathbb{W}_\gamma(T) = S_\gamma(T) \sum_{n=1}^{n_t} B_{t_n} \left(\triangle k_\gamma - \mathbb{E}\left[L_{t_n}^{(\gamma)}\right]\right) \triangle_n \tag{3.4}$$

where $\triangle_n = t_n - t_{n-1}$ denotes the times between payments (measured in fractions of a year), $\triangle k_\gamma = k_\gamma - k_{\gamma-1}$ is the nominal size of tranche γ (as a fraction of the total nominal value of the portfolio), and $S_\gamma(T)$ is called the *spread* of tranche γ which is determined by equating the value of the premium leg to the value of the corresponding protection leg, i.e., $S_\gamma(T)$ is determined at $t = 0$ so that $\mathbb{V}_\gamma(T) = \mathbb{W}_\gamma(T)$. The tranche spreads $S_\gamma(T)$, for $1 \leq \gamma \leq q$, are functions of time horizon T. For a simple case where the maturity of a CDO tranche is 5 years, i.e., $T = 5$, $S_\gamma(T) = S_\gamma T = 5S_\gamma$ is a linear function of T, where S_γ is the annual spread of tranche γ.

Unlike a cashflow CDO, where the equity tranche is typically held by the CDO sponsor, the equity tranche of a synthetic CDO is also traded in the market. To reduce the credit risk of the equity tranche, a portion of the equity tranche premium is paid upfront. For example, for the CDX NA IG index, the equity tranche has a contractually set running spread of 500 basis points and an up-front fee that is negotiated in the market. The higher (lower) the upfront fee, the more (less) the seller of protection is compensated. Therefore upfront fee plus 500bp ongoing is the most common way to sell equity tranche. Specifically, an up-front fee of 20% means that the investor receives 20% of the tranche notional at time 0 plus a premium of 500bps per year paid quarterly, e.g. [61,62].

Therefore, for the first loss tranche (equity tranche), $S_1(T)$ is set to 500bp and a so called *up-front* fee $S_1^u(T)$ is added to the premium leg so that $\mathbb{V}_1(T) = S_1^u(T)k_1 + \mathbb{W}_1(T)$. Hence, we get that

$$S_\gamma(T) = \frac{B_T\mathbb{E}[L_T^{(\gamma)}] + \int_0^T r_t B_t \mathbb{E}[L_t^{(\gamma)}]\mathrm{d}t}{\sum_{n=1}^{n_T} B_{t_n}\left(\triangle k_\gamma - \mathbb{E}[L_{t_n}^{(\gamma)}]\right)\triangle_n} \quad \gamma = 2, \ldots, k \tag{3.5}$$

and

$$S_1^{(u)}(T) = \frac{1}{k_1}\left[B_T\mathbb{E}[L_T^{(1)}] + \int_0^T r_t B_t \mathbb{E}[L_t^{(1)}]\mathrm{d}t - 0.05\sum_{n=1}^{n_T} B_{t_n}\left(\triangle k_1 - \mathbb{E}[L_{t_n}^{(1)}]\right)\triangle_n\right]. \tag{3.6}$$

The spreads $S_\gamma(T)$ are quoted in bp per annum while $S_1^{(u)}(T)$ is quoted in percent per annum. The spreads (3.5) and (3.6) are pointed out in [53]. Note that spreads are independent of the nominal size of the portfolio. To find analytical expressions for expected tranche losses, and thus for tranche spreads is the main objective in this paper.

3.3. Derivation of marginal default probabilities and certain characteristics

From the representations of (3.5) and (3.6), we can easily see that to compute tranche spreads we have to compute $\mathbb{E}[L_t^{(\gamma)}]$, the expected loss of the tranche $[k_{\gamma-1}, k_\gamma]$ at time t. Let $F_{L_t}(x) = P[L_t \le x]$, then (3.2) implies that

$$\mathbb{E}[L_t^{(\gamma)}] = (k_\gamma - k_{\gamma-1})P[L_t > k_\gamma] + \int_{k_{\gamma-1}}^{k_\gamma} (x - k_{\gamma-1})\mathrm{d}F_{L_t}(x). \tag{3.7}$$

Hence, in order to compute $\mathbb{E}[L_t^{(\gamma)}]$, we must know the loss distribution $F_{L_t}(x)$ at time t. Further we denote

$$D_i(t) = 1_{\{\tau_i \le t\}} \tag{3.8}$$

the default indicator up to time $t \in [0, T]$ for $i = 1, \ldots, N$, where τ_i is the default time for each obligor i, and

$$D_t = \sum_{i=1}^N D_i(t) \tag{3.9}$$

the total number of defaults. From (3.1), we may get $L_t = \frac{(1-\phi)}{N}D_t$. Then

$$F_{L_t}(x) = P[L_t \le x] = P\left[D_t \le \frac{xN}{1-\phi}\right] = \sum_{j=0}^{[\frac{xN}{1-\phi}]} P[D_t = j] \tag{3.10}$$

where $x \in [0, 1] \cap [0, 1-\phi]$, i.e., $x \in [0, 1-\phi]$. Therefore, to find tranche spreads leads us to find default probability $P[D_t = j], 0 \le j \le [\frac{xN}{1-\phi}]$.

In this section, we assume that the default intensity $\lambda_i(t) > 0, t \ge 0$, is composed of the common and idiosyncratic components,

$$\lambda_i(t) = a_i X_c(t) + X_i(t) \quad 1 \le i \le N, \tag{3.11}$$

where $a_i > 0$ is a constant and presents the weight on common component for all firms belonging to the tranche γ; $X_c(t) \in \mathbb{R}$, the scaled common risk component, and $X_i(t) \in \mathbb{R}$, the scaled idiosyncratic riskiness component are solutions to the system of the $N+1$ independent, mean reverting (Markovian type) stochastic differential equation (2.1), that is,

$$\mathrm{d}X_c(t) = (\theta_c + \sigma_c\alpha_c(X_c(t), t))X_c(t)\mathrm{d}t + \sigma_c X_c(t)\mathrm{d}W_c(t) \tag{3.12}$$

and

$$\mathrm{d}X_i(t) = (\theta_i + \sigma_i\alpha_i(X_i(t), t))X_i(t)\mathrm{d}t + \sigma_i X_i(t)\mathrm{d}W_i(t), \quad 1 \le i \le N, \tag{3.13}$$

where $\theta_c, \theta_i \ge 0$ are the long-term mean values (constants), the volatility scaling parameter $\sigma_c, \sigma_i \ge 0$ are constants, and $\alpha_c(X_c(t), t), \alpha_i(X_i(t), t)$ are the mean corrections with the function $\alpha : \mathbb{R} \times [0, \infty) \mapsto \alpha(x, t) \in \mathbb{R}$ being twice differentiable in x and differentiable in t, and $W_c(t), W_i(t)$ are independent Brownian motions on $(\Omega, \mathbb{F}, P; \{\mathcal{F}_t\}_{t\ge0})$.

Based on these equations, without loss of generality, we consider the risk factors satisfying the following Markovian type mean reversion SDE

$$\mathrm{d}X(t) = (\theta + \sigma\alpha(X(t), t))X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t) \tag{3.14}$$

where $\theta \ge 0$ is the long-term mean value, the parameter $\sigma \ge 0$ stands for the scaling of the volatility is a constant, and $\alpha(X(t), t)$ is the mean correction with the function $\alpha : \mathbb{R} \times [0, \infty) \mapsto \alpha(x, t) \in \mathbb{R}$ being twice differentiable in x and differentiable in t, and $W(t)$ is a Brownian motion on $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\ge0})$.

Then we define the common factor Z_c as the integrated common process

$$Z_c(t) := \int_0^t X_c(s)\mathrm{d}s. \tag{3.15}$$

Note that conditional on the common factor, the default times of the obligors are independent. So unconditional joint default probabilities $P(D_t = j)$ can be written as integrals of the conditional probabilities over the common factor distribution, that is,

$$P(D_t = j) = \int_{-\infty}^{\infty} P(D_t = j|Z_c(t) = z) f_{Z_c(t)}(z)\mathrm{d}z \tag{3.16}$$

where $f_{Z_c(t)}(\cdot)$ is the density function of the common factor.

One one hand, given the common factor, the conditional joint default probabilities $P(D_t = j|Z_c(t) = z)$ can be calculated from the marginal default probabilities

$$p_t^i(z) := P(\tau_i \le t|Z_c = z)$$

for each obligor i, $1 \le i \le N$, where $z \in [0, \infty)$ is a constant. The number of defaults given the common factor is binomially distributed in a homogeneous portfolio pool, i.e.,

$$P(D_t = j|Z_c(t) = z) = \binom{N}{j} \left[p_t^i(z)\right]^j \left[(1 - p_t^i(z))\right]^{N-j}. \tag{3.17}$$

In a heterogeneous pool, joint default probabilities can be obtained through the following recursive algorithm due to [3]. Let D_t^K denote the number of defaults at time t in the pool consisting of the first K entities. Since defaults are conditionally independent, the conditional probability of observing j defaults in a K-pool can be written as

$$P(D_t^K = j|Z(t) = z) = P(D_t^{K-1} = j|Z(t) = z) \times (1 - p_K(t|z)) + P(D_t^{K-1} = j-1|Z(t) = z) \times p_K(t|z) \tag{3.18}$$

for $j = 1, \ldots, K$. For $j = 0$ the last term obviously disappears. The recursion starts from $P(D_t^0 = j|Z(t) = z) = 1_{\{j=0\}}$ and runs for $K = 1, \ldots, N$ with $P(D_t = j|Z(t) = z) = P(D_t^N = j|Z(t) = z)$.

On the other hand, by definition, the characteristic function $\varphi_{Z_c(t)}(\cdot)$ of the integrated common factor $Z_c(t)$ is given by

$$\varphi_{Z_c(t)}(u) := \mathbb{E}\left[\mathrm{e}^{\xi u Z_c(t)}\right] = \int_{-\infty}^{\infty} \mathrm{e}^{\xi uz} f_{Z_c(t)}(z)\mathrm{d}z \tag{3.19}$$

which is the Fourier transform of the density function $f_{Z_c(t)}(\cdot)$. Hence, the density $f_{Z_c(t)}(\cdot)$ can be recovered by the inverse Fourier transform,

$$f_{Z_c(t)}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\xi uz} \varphi_{Z_c(t)}(u)\mathrm{d}u. \tag{3.20}$$

Clearly, it is equivalently important to derive the characteristic function $\varphi_{Z_c(t)}(u)$.

Therefore, to derive an analytical expression for the expected tranche losses is equivalent to derive marginal default probability $p_t^i(z)$ and the density function of the common factor $\varphi_{Z_c(t)}(u)$.

In order to calculate the marginal default probabilities

$$p_t^i(z) = P(\tau_i \le t|Z_c(t) = z), \quad 1 \le i \le N,$$

we need stochastic calculus derivation for the risk component $X(t)$ from (3.14). We have the following

Proposition 3.1. *Given the initial data X_0, the risk factor $(X(t))_{t\ge 0}$ is determined by the following Itô process*

$$X(t) = X_0 + X_0 \int_0^t \exp[M(r)] \left(\theta + \sigma\alpha(X(r), r)\right) \mathrm{d}r + \sigma X_0 \int_0^t \exp[M(r)]\mathrm{d}W(r)$$

where

$$M(t) = \left(\theta - \frac{\sigma^2}{2} + \sigma\alpha(X(t), t)\right) t + \sigma W(t).$$

Proof. Starting from (3.14) and applying Itô formula to $\ln X(t)$, we get

$$\begin{aligned}\ln X(t) - \ln X_0 &= \int_0^t \left[(\theta + \sigma\alpha(X(s), s))\mathrm{d}s + \sigma \mathrm{d}W(s)\right] + \frac{1}{2}\int_0^t (-\sigma^2)\mathrm{d}s \\ &= \left(\theta - \frac{\sigma^2}{2} + \sigma\alpha(X(t), t)\right) t + \sigma W(t).\end{aligned}$$

Define the process $M(t)$ by

$$M(t) := \left(\theta - \frac{\sigma^2}{2} + \sigma\alpha(X(t), t)\right) t + \sigma W(t)$$

we can then have

$$X(t) = X_0 \exp[M(t)]. \tag{3.21}$$

We use Itô formula again to $\exp[M(t)]$ and get

$$\begin{aligned}\exp[M(t)] - \exp[M(0)] &= \int_0^t \exp[M(r)] \left(\theta - \frac{\sigma^2}{2} + \sigma\alpha(X(r), r)\right) \mathrm{d}r \\ &\quad + \sigma \int_0^t \exp[M(r)] \mathrm{d}W_r + \frac{1}{2}\sigma^2 \int_0^t \exp[M(r)]\mathrm{d}r \\ &= \int_0^t \exp[M(r)] \left(\theta + \sigma\alpha(X(r), r)\right) \mathrm{d}r + \sigma \int_0^t \exp[M(r)]\mathrm{d}W(r).\end{aligned}$$

Clearly with $M(0) = 0$ and $\exp[M(0)] = 1$, we get

$$X(t) = X_0 + X_0 \int_0^t \exp[M(r)] \left(\theta + \sigma\alpha(X(r), r)\right) \mathrm{d}r + \sigma X_0 \int_0^t \exp[M(r)]\mathrm{d}W(r). \quad \square$$

Next, we derive an expression for the integrated process $Z(t)$. We have

Proposition 3.2. *With the same preamble as in Proposition 3.1, the integrated process $Z(t) = \int_0^t X(s)\mathrm{d}s$ is given by*

$$Z(t) = X_0 t + X_0 \int_0^t (t - r) \exp[M(r)] \left(\theta + \sigma\alpha(X(r), r)\right) \mathrm{d}r + \sigma X_0 \int_0^t (t - r) \exp[M(r)]\mathrm{d}W(r). \tag{3.22}$$

Proof. From definition, $Z(t) = \int_0^t X(s)\mathrm{d}s$, so

$$Z(t) = X_0 t + X_0 \int_0^t \int_0^s \exp[M(r)] \left(\theta + \sigma\alpha(X(r), r)\right) \mathrm{d}r\mathrm{d}s + \sigma X_0 \int_0^t \int_0^s \exp[M(r)]\mathrm{d}W(r)\mathrm{d}s.$$

Further, by (stochastic) Fubini theorem (cf. e.g. [63] or [64]), we can get

$$Z(t) = X_0 t + X_0 \int_0^t (t - r) \exp[M(r)] \left(\theta + \sigma\alpha(X(r), r)\right) \mathrm{d}r + \sigma X_0 \int_0^t (t - r) \exp[M(r)]\mathrm{d}W(r). \quad \square$$

Now we are in the position to derive the marginal default probability $p_t^i(z)$. We obtain the following

Theorem 3.3. *The marginal default probabilities $p_t^i(z)$, $1 \le i \le N$, are given by the following formulae*

$$\begin{aligned}p_t^i(z) &= 1 - \mathrm{e}^{-a_i z} + \mathrm{e}^{-a_i z} X_i(0) \mathbb{E}\Big[\int_0^t \mathrm{e}^{-Z_i(r)} \big[1 + (t - r)\mathrm{e}^{M_i(r)} \left(\theta_i + \sigma_i\alpha_i(X_i(r), r)\right)\big]\mathrm{d}r\Big] \\ &\quad - \mathrm{e}^{-a_i z} \frac{\sigma_i^2 X_i^2(0)}{2} \mathbb{E}\left[\int_0^t \mathrm{e}^{-Z_i(r)} (t - r)^2 \mathrm{e}^{M_i(r)} \mathrm{d}r\right]\end{aligned}$$

where

$$Z_i(t) = X_i(0)t + X_i(0) \int_0^t (t - r) \exp[M_i(r)] \left(\theta_i + \sigma_i\alpha_i(X_i(r), r)\right) \mathrm{d}r + \sigma_i X_i(0) \int_0^t (t - r) \exp[M_i(r)]\mathrm{d}W_i(r)$$

and

$$M_i(t) = \left(\theta_i - \frac{\sigma_i^2}{2} + \sigma_i\alpha_i(X_i(t), t)\right) t + \sigma_i W_i(t).$$

Proof. By definition, we have

$$p_t^i(z) := P(\tau_i \le t | Z(t) = z) = 1 - \mathbb{E}\left[\mathrm{e}^{-\int_0^t \lambda_i(s)\mathrm{d}s}\right].$$

Next by (3.11) and independence of $X_c(t)$ and $X_i(t)$, we then have

$$p_t^i(z) = 1 - \mathrm{e}^{-a_i z}\mathbb{E}\left(\mathrm{e}^{-\int_0^t X_i(s)\mathrm{d}s}\right). \tag{3.23}$$

Set $Z_i(t) = \int_0^t X_i(s)\mathrm{d}s$ and $f(x) := \mathrm{e}^{-x}$, the we can rewrite (3.23) as

$$p_t^i(z) = 1 - \mathrm{e}^{-a_i z}\mathbb{E}(f(Z_i(t))). \tag{3.24}$$

From Proposition 3.2, we get

$$Z_i(t) = X_i(0)t + X_i(0)\int_0^t (t-r)\exp[M_i(r)]\,(\theta_i + \sigma_i\alpha_i(X_i(r), r))\,\mathrm{d}r + \sigma_i X_i(0)\int_0^t (t-r)\exp[M_i(r)]\mathrm{d}W_i(r) \tag{3.25}$$

where

$$M_i(t) = \left(\theta_i - \frac{\sigma_i^2}{2} + \sigma_i\alpha_i(X_i(t), t)\right)t + \sigma_i W_i(t).$$

Now by applying Itô formula to $f(Z_i(t)) = \mathrm{e}^{-Z_i(t)}$, we have

$$\begin{aligned} f(Z_i(t)) - f(Z_i(0)) &= -X_i(0)\int_0^t \mathrm{e}^{-Z_i(r)}\big[1 + (t-r)\mathrm{e}^{M_i(r)}\,(\theta_i + \sigma_i\alpha_i(X_i(r), r))\big]\mathrm{d}r \\ &\quad + \frac{\sigma_i^2 X_i^2(0)}{2}\int_0^t \mathrm{e}^{-Z_i(r)}(t-r)^2\mathrm{e}^{2M_i(r)}\mathrm{d}r - \sigma_i X_i(0)\int_0^t \mathrm{e}^{-Z_i(r)}(t-r)\mathrm{e}^{M_i(r)}\mathrm{d}W_i(r). \end{aligned} \tag{3.26}$$

Noticing that $f(Z_i(0)) = 1$ and the fact that the expectation of martingale parts are zero, we have

$$\begin{aligned} \mathbb{E}(f(Z_i(t))) &= 1 - X_i(0)\mathbb{E}\left[\int_0^t \mathrm{e}^{-Z_i(r)}\big[1 + (t-r)\mathrm{e}^{M_i(r)}\,(\theta_i + \sigma_i\alpha_i(X_i(r), r))\big]\mathrm{d}r\right] \\ &\quad + \frac{\sigma_i^2 X_i^2(0)}{2}\mathbb{E}\left[\int_0^t \mathrm{e}^{-Z_i(r)}(t-r)^2\mathrm{e}^{M_i(r)}\mathrm{d}r\right]. \end{aligned} \tag{3.27}$$

Finally, we obtain the following

$$\begin{aligned} p_t^i(z) &= 1 - \mathrm{e}^{-a_i z}\mathbb{E}(f(Z_i(t))) \\ &= 1 - \mathrm{e}^{-a_i z} + \mathrm{e}^{-a_i z}X_i(0)\mathbb{E}\Big[\int_0^t \mathrm{e}^{-Z_i(r)}\big[1 + (t-r)\mathrm{e}^{M_i(r)}\,(\theta_i + \sigma_i\alpha_i(X_i(r), r))\big]\mathrm{d}r\Big] \\ &\quad - \mathrm{e}^{-a_i z}\frac{\sigma_i^2 X_i^2(0)}{2}\mathbb{E}\left[\int_0^t \mathrm{e}^{-Z_i(r)}(t-r)^2\mathrm{e}^{M_i(r)}\mathrm{d}r\right]. \quad \square \end{aligned} \tag{3.28}$$

Now, it only remains to find the characteristic function $\varphi_{Z_c(t)}(u)$ of the common factor $Z_c(t)$. We have the following result

Theorem 3.4. *The characteristic function $\varphi_{Z_c(t)}(u)$ of the integrated common factor $Z_c(t)$ is given by*

$$\begin{aligned} \varphi_{Z_c(t)}(u) &= 1 + \xi u X_c(0)\mathbb{E}\Big[\int_0^t \mathrm{e}^{\xi u Z_c(r)}\big[1 + (t-r)\mathrm{e}^{M_c(r)}\,(\theta_c + \sigma_c\alpha_c(X_c(r), r))\big]\mathrm{d}r\Big] \\ &\quad - \frac{u^2\sigma_i^2 X_c^2(0)}{2}\mathbb{E}\left[\int_0^t \mathrm{e}^{\xi u Z_c(r)}(t-r)^2\mathrm{e}^{2M_c(r)}\mathrm{d}r\right] \end{aligned} \tag{3.29}$$

where

$$\begin{aligned} Z_c(t) &= X_c(0)t + X_c(0)\int_0^t (t-r)\exp[M_c(r)]\,(\theta_c + \sigma_c\alpha_c(X_c(r), r))\,\mathrm{d}r \\ &\quad + \sigma_c X_c(0)\int_0^t (t-r)\exp[M_c(r)]\mathrm{d}W_c(r) \end{aligned}$$

and

$$M_c(t) = \left(\theta_c - \frac{\sigma_c^2}{2} + \sigma_c\alpha_c(X_c(t), t)\right)t + \sigma_c W_c(t).$$

Proof. First set $g(x) := e^{\xi ux}$. From Proposition 3.2, we have

$$Z_c(t) = X_c(0)t + X_c(0)\int_0^t (t-r)\exp[M_c(r)]\,(\theta_c + \sigma_c\alpha_c(X_c(r), r))\,dr + \sigma_c X_c(0)\int_0^t (t-r)\exp[M_c(r)]dW_c(r)$$

where

$$M_c(t) = \left(\theta_c - \frac{\sigma_c^2}{2} + \sigma_c\alpha_c(X_c(t), t)\right)t + \sigma_c W_c(t).$$

Now applying Itô formula to $g(Z_c(t)) = e^{\xi u Z_c(t)}$, we have

$$\begin{aligned} g(Z_c(t)) - g(Z_c(0)) &= \xi u X_c(0)\int_0^t e^{\xi u Z_c(r)}\big[1 + (t-r)e^{M_c(r)}\,(\theta_c + \sigma_c\alpha_c(X_c(r), r))\big]dr \\ &\quad - \frac{u^2\sigma_i^2 X_c^2(0)}{2}\int_0^t e^{\xi u Z_c(r)}(t-r)^2 e^{2M_c(r)}dr + \xi u\sigma_c X_c(0)\int_0^t e^{\xi u Z_c(r)}(t-r)e^{M_c(r)}dW_c(r). \end{aligned} \tag{3.30}$$

Noticing that $Z_c(0) = 0$ so $g(Z_c(0)) = 1$ and the fact that the expectation of martingale parts are zero, we have

$$\begin{aligned} \mathbb{E}[g(Z_c(t))] &= 1 + \xi u X_c(0)\mathbb{E}\Big[\int_0^t e^{\xi u Z_c(r)}\big[1 + (t-r)e^{M_c(r)}\,(\theta_c + \sigma_c\alpha_c(X_c(r), r))\big]dr\Big] \\ &\quad - \frac{u^2\sigma_i^2 X_c^2(0)}{2}\mathbb{E}\left[\int_0^t e^{\xi u Z_c(r)}(t-r)^2 e^{2M_c(r)}dr\right]. \quad \square \end{aligned} \tag{3.31}$$

4. Semi-analytical expected tranche losses and our main result

In this paper, a synthetic CDO, with $q \geq 1$ tranches and maturity T, is defined for a portfolio consisting of N single-name CDSs on obligors with default times $\tau_1, \tau_2, \ldots, \tau_N$ and with the same constant recovery rate $\phi \in [0, 1)$. We assume that the nominal values, which are the same for all obligors, equal to 1 and that the risk-free interest rate r_t is deterministic. Let $B_t = exp(-\int_0^t r_s ds)$ denote the discount factor. In addition, a tranche γ consists of m_γ obligors such that $\sum_{\gamma=1}^q m_\gamma = N$ and premiums are paid at $0 < t_1 < t_2 < \cdots < t_{n_T} = T$. Then the tranche spreads are given by (3.5) and (3.6).

The expected tranche losses $\mathbb{E}[L_t^{(\gamma)}]$ are needed to find tranche spreads. We summarize our result for expected tranche losses $\mathbb{E}[L_t^{(\gamma)}]$ among a homogeneous pool in the following

Theorem 4.1. *The expected tranche losses of a synthetic CDO are given by*

$$\begin{aligned} \mathbb{E}[L_t^{(\gamma)}] &= (k_\gamma - k_{\gamma-1}) - k_\gamma \sum_{j=0}^{\frac{Nk_\gamma}{1-\phi}} \int_{-\infty}^{\infty} P(D_t = j|Z(t) = z)f_{Z(t)}(z)dz \\ &\quad + k_{\gamma-1}\sum_{j=0}^{\frac{Nk_{\gamma-1}}{1-\phi}} \int_{-\infty}^{\infty} P(D_t = j|Z(t) = z)f_{Z(t)}(z)dz + \sum_{j=\frac{Nk_{\gamma-1}}{1-\phi}}^{\frac{Nk_\gamma}{1-\phi}} j\int_{-\infty}^{\infty} P(D_t = j|Z(t) = z)f_{Z(t)}(z)dz \end{aligned}$$

where $P(D_t = j|Z(t) = z)$, the default probability conditional on the common factor, can be calculated from (3.17) and (3.18), where the marginal default probability $p_t^i(z)$ are given in Theorem 3.3; the density function of the common factor $f_{Z_c(t)}(z)$ can be derived from the Fourier inversion of $\varphi_{Z_c(t)}(u)$ given in Theorem 3.4.

Proof. Starting from (3.7), we have

$$\begin{aligned} \mathbb{E}[L_t^{(\gamma)}] &= (k_\gamma - k_{\gamma-1})P[L_t > k_\gamma] + \int_{k_{\gamma-1}}^{k_\gamma} (x - k_{\gamma-1})dF_{L_t}(x) \\ &= (k_\gamma - k_{\gamma-1})(1 - F_{L_t}(k_\gamma)) + \int_{k_{\gamma-1}}^{k_\gamma} x dF_{L_t}(x) - k_{\gamma-1}[F_{L_t}(k_\gamma) - F_{L_t}(k_{\gamma-1})] \\ &= (k_\gamma - k_{\gamma-1}) - k_\gamma F_{L_t}(k_\gamma) + k_{\gamma-1}F_{L_t}(k_{\gamma-1}) + \int_{k_{\gamma-1}}^{k_\gamma} x dF_{L_t}(x). \end{aligned}$$

We notice that

$$\int_{k_{\gamma-1}}^{k_\gamma} x\mathrm{d}F_{L_t}(x) = \mathbb{E}(L_t \cap [k_{\gamma-1}, k_\gamma]).$$

From definition in (3.9), we know $L_t = \frac{1-\phi}{N} D_t$, then $L_t \in [k_{\gamma-1}, k_\gamma]$ is equivalent to $D_t \in [\frac{Nk_{\gamma-1}}{1-\phi}, \frac{Nk_\gamma}{1-\phi}]$. Therefore,

$$\int_{k_{\gamma-1}}^{k_\gamma} x\mathrm{d}F_{L_t}(x) = \sum_{j=\frac{Nk_{\gamma-1}}{1-\phi}}^{\frac{Nk_\gamma}{1-\phi}} jP\,(D_t = j).$$

Then together with (3.10) and (3.16), we get

$$\mathbb{E}[L_t^{(\gamma)}] = (k_\gamma - k_{\gamma-1}) - k_\gamma \sum_{j=0}^{\frac{Nk_\gamma}{1-\phi}} \int_{-\infty}^{\infty} P(D_t = j|Z(t) = z) f_{Z(t)}(z)\mathrm{d}z$$
$$+ k_{\gamma-1} \sum_{j=0}^{\frac{Nk_{\gamma-1}}{1-\phi}} \int_{-\infty}^{\infty} P(D_t = j|Z(t) = z) f_{Z(t)}(z)\mathrm{d}z + \sum_{j=\frac{Nk_{\gamma-1}}{1-\phi}}^{\frac{Nk_\gamma}{1-\phi}} j \int_{-\infty}^{\infty} P(D_t = j|Z(t) = z) f_{Z(t)}(z)\mathrm{d}z. \quad \square$$

5. Conclusion

We propose a class of models, in the framework of intensity based models, where the default intensity is composed of a scaled common component and a scaled idiosyncratic component which are specified by independent Markovian type mean reverting stochastic processes. We get closed-form solutions for expected tranche losses, thus for tranche spreads in synthetic CDOs. We expect that the formula could be efficiently simulated and this will be our research topic in the future.

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