MATH 2602 **Exam III** November 21, 2011

Printed Name: Solutions

GT ID #: ____________________

Section (circle one): (G1 - Geehoon Hong) (G2 - Ke Yin)

Instructions:

- There are 9 problems. Point values for each problem are as indicated.
- You may use scratch paper that I provide but ONLY THE WORK WRITTEN IN THIS BOOKLET WILL BE GRADED.
- On each question you must show all appropriate legible work to receive full credit.
- Box or circle your final answer.
- Calculators are not allowed.
- SCHOLASTIC DISHONESTY WILL NOT BE TOLERATED.

Good Luck!

1. (7 points) For which values of m and n ($m \leq n$) does the complete bipartite graph $K_{m,n}$ have a Hamiltonian cycle? Explain your answer.

A possible Hamiltonian cycle would alternate between the two partition sets of $K_{m,n}$, so in order to be able to traverse all vertices without repetition and close the cycle we must have $m=n$. The case $m=n=1$ is excluded though because there are not enough edges. So, the answer is $m=n\geq 2$

2. (8 points) If G is a graph with 40 vertices and 5 components, all of which are trees, how many edges does G have? Explain your answer.

Suppose the 5 components of G have $n_1, n_2, n_3, n_4,$ and n_5 vertices respectively, where $n_1+n_2+n_3+n_4+n_5=40$. Since the components are trees they have $n_1-1, n_2-1, n_3-1, n_4-1, n_5-1$ edges respectively, and the total number of edges is $(n_1-1)+(n_2-1)+\ldots+(n_5-1)=40-5=35$.

3. (a) (6 points) How many edges does K_9 have?

K_9 has an edge ~~for~~ between every two vertices, so it has $\binom{9}{2}$ edges.

(b) (6 points) How many subgraphs isomorphic to $K_{3,4}$ does K_9 have?

Answer: $\binom{9}{7}\cdot\binom{7}{3}$

To form a subgraph isomorphic to $K_{3,4}$ we need to choose:

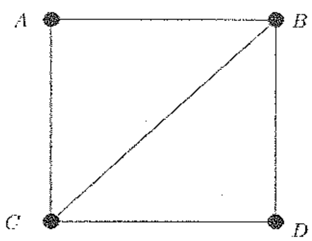
7 vertices to belong to the subgraph $\rightarrow \binom{9}{7}$ ways

3 of those 7 vertices to be in one partition set $\rightarrow \binom{7}{3}$ ways

the remaining 4 vertices to belong to the other partition set $\rightarrow \binom{4}{4}$

Total	/100

4. (a) (10 points) Use the tree-matrix theorem to compute the number of spanning trees of the following graph G.

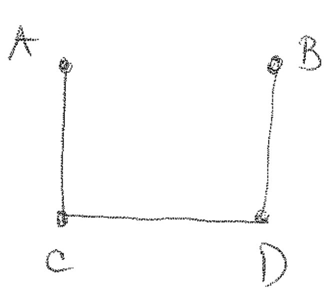


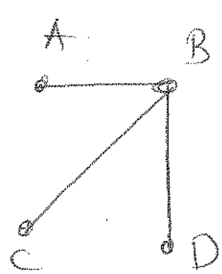
$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \end{array}$$

spanning trees = (1,1)-cofactor of M

$$= (-1)^{1+1} \det \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{pmatrix} = 18-1-1-3-3-2 = \boxed{8}$$

(b) (5 points) Draw two non-isomorphic spanning trees of G. Explain why the trees you drew are not isomorphic.



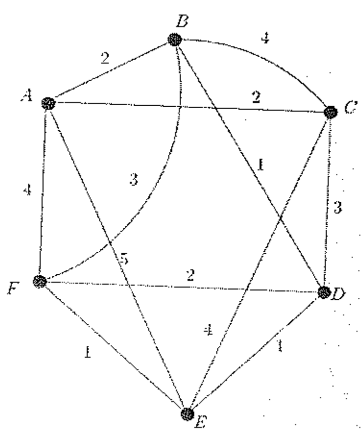


These 2 spanning trees are not isomorphic Because the left one doesn't have a vertex of degree 3 and the right one does.

5. (a) (4 points) What is the Floyd-Warshall algorithm used for?

To find the weights of the shortest paths between all pairs of vertices in a weighted graph.

(b) (5 points) Write the initial matrix for the Floyd-Warshall algorithm for the following graph G.



$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 2 & 2 & \infty & 5 & 4 \\ 2 & 0 & 4 & 1 & \infty & 3 \\ 2 & 4 & 0 & 3 & 4 & \infty \\ \infty & 1 & 3 & 0 & 1 & 2 \\ 5 & \infty & 4 & 1 & 0 & 1 \\ 4 & 3 & \infty & 2 & 1 & 0 \end{bmatrix} \end{array}$$

(c) (5 points) Suppose the initial matrix in the Floyd-Warshall algorithm for some graph G is

$$\begin{bmatrix} 0 & 3 & 2 & 4 & 1 \\ 3 & 0 & 2 & 9 & \infty \\ 2 & 2 & 0 & \infty & 2 \\ 4 & 9 & \infty & 0 & 6 \\ 1 & \infty & 2 & 6 & 0 \end{bmatrix}.$$

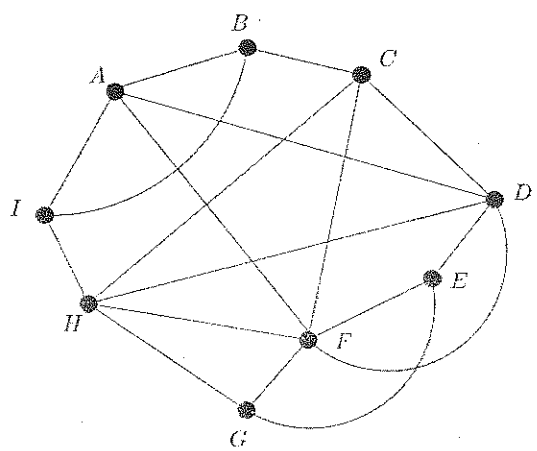
Write the matrix after $k = 1$.

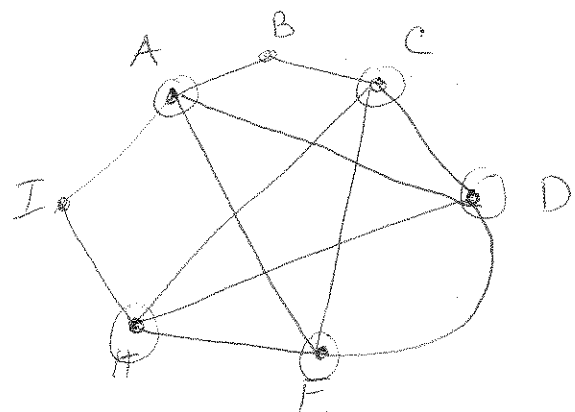
$$\begin{bmatrix} 0 & 3 & 2 & 4 & 1 \\ 3 & 0 & 2 & 7 & 4 \\ 2 & 2 & 0 & 6 & 2 \\ 4 & 7 & 6 & 0 & 5 \\ 1 & 4 & 2 & 5 & 0 \end{bmatrix}$$

6. (a) (4 points) State Kuratowski's theorem.

A graph is planar if and only if it doesn't contain a subgraph homeomorphic to $K_{3,3}$ or K_5

(b) (10 points) Use Kuratowski's theorem to show that G is not planar. Explain which subgraph you found.





This is a subgraph of G which is a homeomorphic to K_5, so G is not planar.

7. (10 points) Suppose that a connected planar graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

$$\deg(v_1) + \deg(v_2) + \cdots + \deg(v_{20}) = 2 \cdot E$$

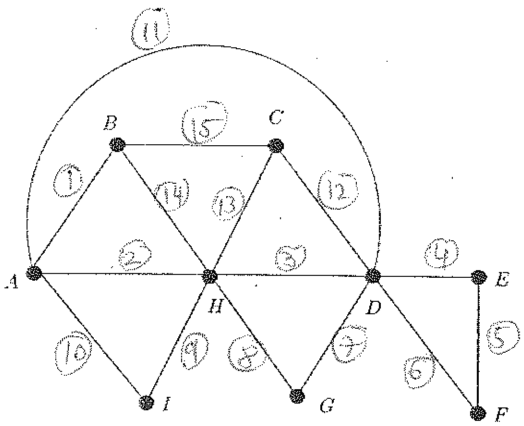
$$20 \cdot 3 = 2 \cdot E$$

$$\Rightarrow \quad E = 30$$

$$V - E + R = 2 \rightarrow R = E - V + 2 = 30 - 20 + 2 = 12$$

$$\boxed{R = 12}$$

8. (10 points) Does the following graph have an Eulerian circuit or an Eulerian trail? Explain. If the answer to either of the questions is yes, number the edges to exhibit an Eulerian circuit/trail.

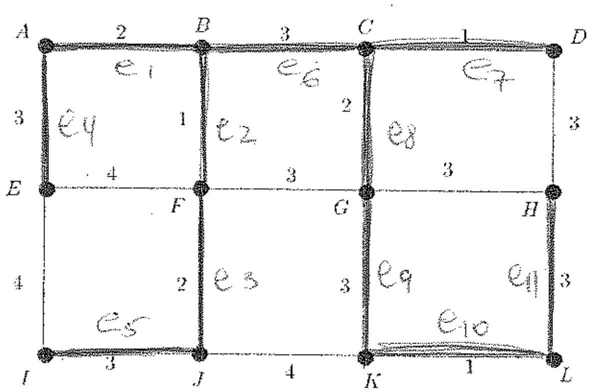


B and C are odd vertices, all other vertices are even.

A connected graph has an Eulerian circuit if and only if all its vertices are even, and it has an Eulerian trail if and only if ~~all its verti~~ exactly 2 of its vertices are odd.

So, this graph has no Eulerian circuit and it does have an Eulerian trail between B and C.

9. (10 points) Use Prim's algorithm to find a minimal spanning tree for the following graph. Label the edges $e_1, e_2, \ldots$ in the order in which you add them to the spanning tree. What is the weight of the spanning tree you found?



Example, start at vertex A.

weight = 2+1+2+3+3+3+1+2+3+1+3

$= \boxed{24}$