MATH 3012 A, Midterm 3

07/10/2013

Name: ______________________________ GTID:__________________________

Key

Problem No.	Points
1	10
2	10
3	10
4	10
5	15
6	20
7	15
8	10

TOTAL:______________

Please do show all your work including intermediate steps. Partial credit is available.

Problem 1 (10 points).

Find the number of positive integers less than or equal to 300 that are divisible by 7, 10, or 15.

A_1 : set of positive intergers ≤ 300 and divisible by 7.

A_2 : .. .. .. 10.

A_3 : .. .. 15.

... 3pts

So $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|$

$+ |A_1 \cap A_2 \cap A_3|$ ... 4pts

$= \lfloor \frac{300}{7} \rfloor + \lfloor \frac{300}{10} \rfloor + \lfloor \frac{300}{15} \rfloor - \lfloor \frac{300}{70} \rfloor - \lfloor \frac{300}{105} \rfloor - \lfloor \frac{300}{30} \rfloor$

$+ \lfloor \frac{300}{210} \rfloor$ ... 2pts

$= 42 + 30 + 20 - 4 - 2 - 10 + 1$

$= 77$ ... 1pts

Problem 2 (10 points).

5pts (a) A careless mailman is delivering mail to ten homes. After delivering the mail, he realizes that he made a few mistakes, and exactly five of the ten homes got the correct mail, while the other five did not. In how many ways could he have made this errant delivery?

$$\binom{10}{5} D_5$$

5pts (b) Let φ be Euler function in number theory. Determine $\varphi(20)$ by listing the integers it counts as well as by using the formula associated with it.

$$\varphi(20) = \varphi(2^2 \cdot 5) = 2^2 \cdot 5 \cdot \left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right) = 8 \quad \cdots \quad 3\text{pts.}$$

$$1, 3, 7, 9, 11, 13, 17, 19 \quad \cdots \quad 2\text{pts}$$

Problem 3 (10 points).

Interpret the coefficients of the function $(1+x)(1+x^2)(1+x^5)/(1-x^3)$ in terms of partitions of an integer. Then write all the partitions of the integer 10 that correspond to this interpretation.

$(1+x)$: partition contains at most one "1"

$(1+x^2)$: " " " "2"

$(1+x^5)$: " " " "5"

$\frac{1}{1-x^3}$: partition contains arbitrarily many "3"

... 8pts

$10 = 3+3+3+1$

$= 3+2+5$ ... 2 pts

Turn over for more problems

Problem 4 (10 points).

Set up the appropriate generating function for the following problem, indicate what coefficient you are looking for. You don't need to calculate the answer.

In how many ways can a total of 20 be obtained if 4 distinct six-sided dice are rolled?

$$(x+x^2+x^3+x^4+x^5+x^6)^4$$

looking for coefficient of x^{20}.

Problem 5 (15 points).
Solve the following recurrence relation:

$$a_n = 5a_{n-1} - 6a_{n-2} + n; \quad a_0 = 2, a_1 = 3.$$

$x^2 - 5x + 6 = 0$

$x_1 = 2, \ x_2 = 3$

$q_n = c_1 \cdot 2^n + c_2 \cdot 3^n$ ... 5pts

Assume $P_n = An + B$ ... 3pts

$P_n = 5P_{n-1} - 6P_{n-2} + n$

$An + B = 5(A(n-1) + B) - 6(A(n-2) + B) + n$

$\Rightarrow \begin{cases} A = -A + 1 \\ B = -5A + 5B + 12A - 6B \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \\ B = \frac{7}{4} \end{cases}$ ... 3pts

$a_n = P_n + q_n$ ... 2 pts

$= c_1 \cdot 2^n + c_2 \cdot 3^n + \frac{1}{2}n + \frac{7}{4}$

plug in $a_0 = 2, \ a_1 = 3$

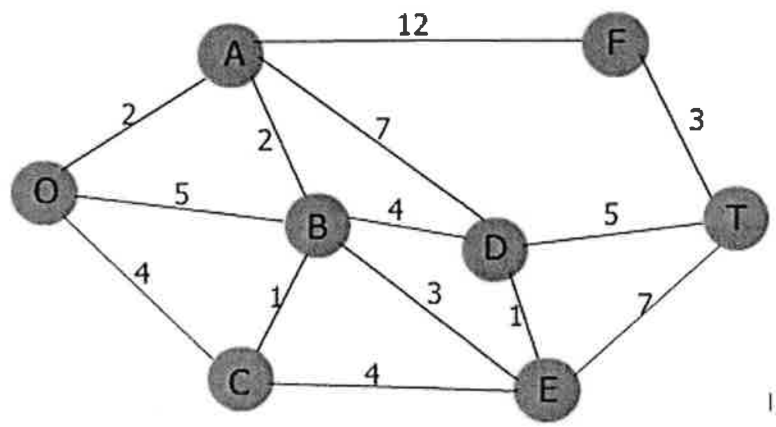
$\begin{cases} c_1 + c_2 + \frac{7}{4} = 2 \\ 2c_1 + 3c_2 + \frac{1}{2} + \frac{7}{4} = 3 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = \frac{1}{4} \end{cases}$

$\Rightarrow a_n = \frac{1}{4} \cdot 3^n + \frac{1}{2}n + \frac{7}{4}$ ... 2pts

Turn over for more problems

Problem 6 (20 points).

Consider the following weighted graph. List in order the edges that would be selected in carrying out Kruskal's algorithm and Prim's algorithm to find a minimum weight spanning tree. For Prim, use vertex A as the root.

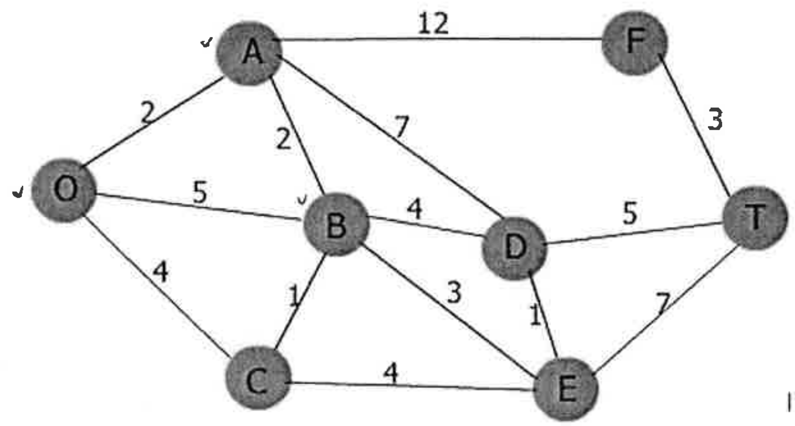


Kruskal.		Prim.			
BC	1	AB	2		
DE	1	~~AO~~	~~2~~		
AB	2	BC	1	AO	2
AO	2	BE	3		
BE	3	DE	1		
FT	3	DT	5		
DT	5	FT	3		

Turn over for more problems

Problem 7 (15 points).

Apply Dijkstra's Algorithm to find the shortest path from vertex O to all other vertices. Please do show all your work including intermediate steps.



vertex	label	path
$\boxed{O}$	0	
✓ $\boxed{A}$	2	OA
B	5	OB
C	4	OC
D	∞	
E	∞	
F	∞	
T	∞	

⇒

$\boxed{O}$	0	
$\boxed{A}$	2	OA
✓ $\boxed{B}$	4	OAB
C	4	OC
D	9	OAD
E	∞	
F	14	OAF
T	∞	

⇒

$\boxed{O}$	0	
$\boxed{A}$	2	OA
$\boxed{B}$	4	OAB
✓ $\boxed{C}$	4	OC
D	8	OABD
E	7	OABE
F	14	OAF
T	∞	

⇒

$\boxed{O}$	0	OA
$\boxed{A}$	2	OAB
$\boxed{B}$	4	OC
$\boxed{C}$	4	OABD
$\boxed{D}$	8	
E		
F		
T		

$\boxed{O}$	0	
$\boxed{A}$	2	OA
$\boxed{B}$	4	OAB
$\boxed{C}$	4	OC
D	8	OABD
✓ $\boxed{E}$	7	OABE
F	14	OAF
T	14	OABET

⇒

$\boxed{O}$	0	
$\boxed{A}$	2	OA
$\boxed{B}$	4	OAB
$\boxed{C}$	4	OC
✓ $\boxed{D}$	8	OABD
$\boxed{E}$	7	OABE
F	14	OAF
$\boxed{T}$	13	OABDT

Turn over for more problems

Problem 8 (10 points).

Determine whether each of the following statements is true-or-false. If the statement is true, circle the "T"; if false, circle the "F".

[**T** **F**] $1/(1-x^2)$ is the generating function of sequence $\{1, 0, 1, 0, 1, 0, \ldots\}$.

[**T** **F**] Dijkstra's algorithm runs in $O(n)$, where n is the number of vertices.

[**T** **F**] Kruskal's algorithm runs in $O(n)$, where n is the number of vertices.

[**T** **F**] Prim's algorithm runs in $O(n)$, where n is the number of vertices.

[**T** **F**] Given a weighted connected graph G, the minimum weight spanning tree of G is not necessarily unique.

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