ISYE 3232A Fall 2015 Test 2 - A

I, Keys, do swear that I abide by the Georgia Tech Honor Code. I understand that any honor code violations will result in a failure (an F).

Signature: ____________________

- You will have 1 hour 15 minutes.
- This quiz is closed book and closed notes. Calculators are not allowed. No scrap paper is allowed. Make sure that there is nothing on your desk except pens and erasers.
- If you need extra space, use the back of the page and indicate that you have done so.
- **Do not remove any page from the original staple**. Otherwise, there will be 5 points off.
- **Show your work on the test sheet**. If you do not show your work for a problem, we will give zero point for the problem even if your answer is correct.
- **We will not select among several answers**. Make sure it is clear what part of your work you want graded. If two answers are given, zero point will be given for the problem.
- Throughout, you will receive full credit (i) if the work is correct and (ii) if someone with no understanding of probability, set theory, and calculus could simplify your answer to obtain the correct numerical answer. **However, you must give a numerical answer where asked.**

1. $\underline{P}^{(n)} = \underline{P}^n$ and $\underline{a}^{(n)} = \underline{a}^{(0)}\underline{P}$.
2. Stationary distribution $\underline{\pi}$ is the solution to $\underline{\pi} = \underline{\pi}\,\underline{P}$ and $\sum_{i \in S} \pi_i = 1$.
3. An irreducible, positive recurrent DTMC has a unique stationary distribution. If it is also aperiodic, then limiting distributions exist.
4. Exponential with rate λ has mean $1/\lambda$, pdf $f(x) = \lambda e^{-\lambda x}$, and CDF $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$.
5. Suppose X_1 and X_2 are independent exponential with rate λ_1 and λ_2. Then $\Pr(X_1 < X_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$ and $\min(X_1, X_2) \sim$ expo with rate $(\lambda_1 + \lambda_2)$ and $\mathsf{E}[X_1 + X_2] = \mathsf{E}[\min(X_1, X_2)] + \mathsf{E}[\max(X_1, X_2)]$.
6. A Poisson process with rate λ has probability
$$\Pr(N(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \text{ for } t > s,$$
and its inter-arrival times are iid exponentially distributed with rate λ. The nth customer's arrival time T_n is Erlang(n, λ).
7. A non-homogeneous Poisson process with rate function $\lambda(t)$ has probability
$$\Pr(N(t) - N(s) = n) = \frac{e^{-\int_s^t \lambda(w)dw}(\int_s^t \lambda(w)dw)^n}{n!} \text{ for } t > s.$$

1. (10 points) Let $X_0, X_1, \ldots$ be a Markov chain with state space $S = \{0,1,2\}$, initial distribution $\underline{a}^{(0)} = (0.5, 0.1, 0.4)$, and transition matrix

$$\begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \underline{P} = \begin{pmatrix} .1 & .4 & .5 \\ .2 & .7 & .1 \\ .6 & .1 & .3 \end{pmatrix} \end{array}$$

(a) (5 points) Calculate $\Pr\{X_{11} = 1, X_{13} = 0 \mid X_{10} = 2\}$.

$$P_{21}^{(1)} \cdot P_{10}^{(2)} = (.1) \times \{(.2)(.1) + (.7)(.2) + (.1)(.6)\} = (.1)(.22)$$

$$= \underline{\underline{0.022}} \quad //$$

$$\underline{P}^2 = \begin{bmatrix} .1 & .4 & .5 \\ .2 & .7 & .1 \\ .6 & .1 & .3 \end{bmatrix} \begin{bmatrix} .1 & .4 & .5 \\ .2 & .7 & .1 \\ .6 & .1 & .3 \end{bmatrix} = \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} & & \\ \cdot & \cdots & \cdots \\ & & \end{bmatrix} \end{array} \cdots P_{10}^{(2)}$$

(b) (5 points) Calculate $\Pr\{X_1 X_2 = 1\}$.

$$Pr(X_1 = 1, X_2 = 1) = Pr(X_2 = 1 \mid X_1 = 1)\, Pr(X_1 = 1)$$

$$= (0.7)\{(.5)(.4) + (.1)(.7) + (.4)(.1)\}$$

$$= (0.7)(0.31) = \underline{\underline{0.217}}$$

2. (16 points) For each Markov chain with the following transition matrices, answer questions next each matrix.

(a) $S = \{1,2,3,4\}$

$$\underline{P} = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.7 & 0 & 0.3 \\ 0.8 & 0 & 0.2 & 0 \end{pmatrix}$$

i. Is it (A)Positive recurrent; or (B) not positive recurrent but recurrent; or (C) transient?

Positive reccurent

ii. What is the period for state 2 ?

2

iii. Does $\underline{P}^{(\infty)}$ exist?

No

iv. Does $\underline{\pi}$ exist?

Yes

(b) $S = \{1,2,3\}$

$$\underline{P} = \begin{pmatrix} 0 & 1/3 & 2/3 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

i. Is it (A)positive recurrent; or (B)not positive recurrent but recurrent; or (C)transient?

Positive recurrent

ii. What is the period for state 2?

1

iii. Does $\underline{P}^{(\infty)}$ exist?

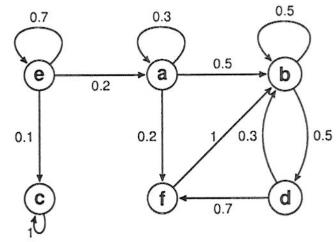
Yes

iv. Does $\underline{\pi}$ exist?

Yes

3. (20 points) Let X be a Markov chain with state space $\{a, b, c, d, e, f\}$ and transition probabilities given by

$$\underline{P} = \begin{pmatrix} .3 & .5 & 0 & 0 & 0 & .2 \\ 0 & .5 & 0 & .5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & .3 & 0 & 0 & 0 & .7 \\ .2 & 0 & .1 & 0 & .7 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$



(a) (8 points) Compute $\Pr\{X_\infty = f \mid X_0 = b\}$.

$\pi_f = 0.7\pi_d \longrightarrow \pi_f = 0.35\pi_b$

$\pi_d = 0.5\pi_b$

$\pi_b + \pi_d + \pi_f = 1 \longrightarrow \pi_b + 0.5\pi_b + 0.35\pi_b = 1$

$(\pi_b, \pi_d, \pi_f) = \left(\frac{100}{185}, \frac{50}{185}, \frac{35}{185}\right)$

$$\boxed{\therefore\ \pi_f = \frac{35}{185}}$$

(b) (3 points) Compute $\Pr\{X_\infty = c \mid X_0 = c\}$.

1

(c) (3 points) Compute $\Pr\{X_\infty = a \mid X_0 = e\}$.

0

(d) (6 points) Compute $\Pr\{X_\infty = f \mid X_0 = e\}$.

$\frac{0.2}{0.2+0.1} \times 1 \times \pi_f = \underline{\underline{\frac{2}{3} \times \frac{35}{185}}}$ //

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$\lambda_1 = \frac{1}{10}$ ① A ② B $\lambda_2 = \frac{1}{20}$

C

4. (30 points) Suppose there are two tellers taking customers in a bank. Service times at a teller are independent, exponentially distributed random variables, but the first teller has a mean service time of 10 minutes while the second teller has a mean of 20 minutes. There is a single queue for customers awaiting service. Suppose at noon, 3 customers enter the system. Customer A goes to the first teller, B to the second teller, and C queues.

   (a) (5 points) What is the probability that Customer A will still be in service at time 12:30?

$$Pr(X_1 > 30) = e^{-\frac{1}{10}\cdot 30} = \underline{\underline{e^{-3}}}_{\#}$$

   (b) (5 points) What is the expected clock time that A is in the system if A is still in the system at 12:30? Your answer must be in a form of clock time such as 12:01 or 12:11. An answer in a different from receives 0 point.

$$12{:}30 + 10 \text{ mins} = \underline{\underline{12{:}40}}$$

   (c) (6 points) What is the average length of time in minutes until the system is empty assuming no additional customers arrive?

$$E[\min(X_1, X_2)] + E[\max(X_1, X_2)]$$

$$= 10 + 20 = \underline{\underline{30 \text{ mins}}}$$

(d) (9 points) What is the expected length of time in minutes that C is in the system?

$$E[\min(X_1, X_2)] + \frac{\lambda_1}{\lambda_1+\lambda_2} 10 + \frac{\lambda_2}{\lambda_1+\lambda_2} 20$$

$$= \frac{1}{\frac{1}{10}+\frac{1}{20}} + \frac{\frac{1}{10}}{\frac{1}{10}+\frac{1}{20}} 10 + \frac{\frac{1}{20}}{\frac{1}{10}+\frac{1}{20}} 20$$

$$= \frac{20}{3} + \frac{20}{3} + \frac{20}{3} = \underline{\underline{20 \text{ mins}}}$$

(e) (5 points) What is the probability that C leaves last?

$$2 \frac{\frac{1}{10}}{\frac{1}{10}+\frac{1}{20}} \frac{\frac{1}{20}}{\frac{1}{10}+\frac{1}{20}} = 2 \times \frac{\cancel{4}\,2}{3} \times \frac{1}{3} = \underline{\underline{\frac{4}{9}}} //$$

$\lambda = 0.009/\text{meter}$

5. (19 points) Suppose we are producing wire which is extruded as a long, continuous strand. After each 100 meters, the wire is cut and placed on a spool. Defects in the wire appear at random and on the average there are 9 defects per kilometer of wire. Assume that the spacing between defects is exponentially distributed.

(a) (5 points) What is the probability that the fifth spool contains two defects?

$$Pr(N(500) - N(400) = 2) = Pr(N(100) = 2)$$

$$= \frac{e^{-0.009 \times 100}(0.9)^2}{2!} = \frac{e^{-0.9}(0.9)^2}{2!} \quad //$$

(b) (4 points) What is the expected location of the first defect?

$$\frac{1000}{9} \text{ meters}$$

(c) (5 points) Suppose that a defect can either be minor or major, and each defect is minor with probability 2/3 and major with probability 1/3, independent of the location and status of the other defects. What is the expected number of major defects in the first 5 spools?

$$E[N_{major}(500)] = \frac{1}{3} \times 0.009 \times 500 = 0.003 \times 500$$

$$= \underline{\underline{1.5 \text{ defects}}}$$

(d) (5 points) What is the probability of at least one major defect among the first 10 spools given that there were no minor defects?

$$Pr(N_{major}(1000) \geq 1 \mid N_{minor}(1000) = 0)$$

$$= Pr(N_{major}(1000) \geq 1) = 1 - Pr(N_{major}(1000) = 0)$$

$$= 1 - e^{-0.003 \times 1000} = \underline{\underline{1 - e^{-3}}} \quad //$$

(e) (**Bonus** 5 points) Suppose that one inspects each spool. Each minor defect in a spool can be fixed at a cost of \$0.50, but a spool with any major defects must be recycled at a cost of \$5. What is the expected cost due to defects of the first spool?

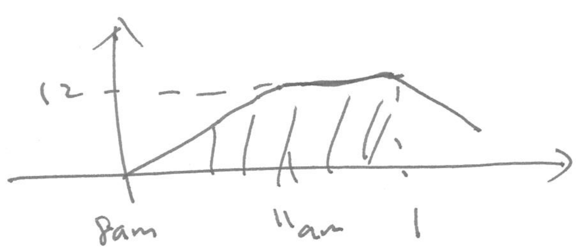
$$\$5 P_r\left(N_{major}(100) \geq 1\right) + \left\{\$0.5 \times E\left[N_{minor}(100)\right]\right\} P_r\left(N_{major}(100) = 0\right)$$

$$= 5\left(1 - e^{-0.3}\right) + 0.5 \times 0.006 \times 100 \times e^{-0.3}$$

$$= 5\left(1 - e^{-0.3}\right) + 0.3 e^{-0.3}$$

6. (5 points) Siegbert runs a hot dog stand that opens at 8 am. From 8 am until 11 am customers seem to arrive, on the average, at a steadily increasing rate that starts with an initial rate of 0 customer per hour at 8 am and reaches a maximum of 12 customers per hour at 11 am. From 11 am until 1 pm the (average) rate seems to remain constant at 12 customers per hour. However, the (average) arrival rate then drops steadily from 1 pm until closing time at 5 pm at which time it has the value of 4 customers per hour. We assume that the numbers of customers arriving at Siegbert's stand during disjoint time periods are independent.

What is the probability that five customers arrive between 8 am and 1 pm Monday morning?



$$\text{Area} = \frac{1}{2} \times 3 \times 12 + 2 \times 12$$

$$= 42$$

$$\boxed{\frac{e^{-42} (42)^5}{5!}}$$