ISyE 2027C Probability with Applications

Solutions for Homework 6 for Quiz March 10 **March 5, 2015**

Simple Calculation Problems

1. $X = -1$ w.p. 0.5; $X = 0$ w.p. 0.2; $X = 3$ w.p. 0.3. Calculate $\sigma(X)$. Use Chebyshev's inequality to find an upper bound on $P(X \geq 3)$. $E[X] = -.5 + .9 = 0.4; E[X^2] = .5 + 2.7 = 3.2$; $\sigma(X) = \sqrt{3.2 - .4^2} = \sqrt{3.04}$. $|3 - E[X]| = 2.6 = \frac{\sigma}{sigma} 2.6 = \frac{2.6}{\sqrt{3.04}} \sigma \approx 1.49\sigma$ Answer $1/(1.49)^2 \approx 0.45$ which sure enough is ≥ 0.3. $\sqrt{3.04}; 0.45$.

2. Continuous random variable Y has uniform distribution on the interval $[0, 3]$. Calculate $\sigma(Y)$. Use Chebyshev's inequality to find an upper bound on the probability that $|Y - 1.5| > 1.25$ $E[Y] = 1.5$ obviously; $E[Y^2] = \int_0^3 y^2/3dy = \frac{1}{9} 27 = 3$; $\sigma(Y) = \sqrt{3 - 1.5^2} = \sqrt{3/4}$. $1.25 = \frac{1.25}{\sqrt{3/4}} \sigma(Y) = (2.5/sqrt3)\sigma(Y)$. Hence $P(|Y - E[Y]) > 1.25) \leq P(|Y - E[Y]) \geq 1.25) \leq 12/25$. $\sqrt{3}/2; 0.48$.

3. Continuous random variable Y has uniform distribution on the interval $[-11, 11]$. Use your answer to the previous question and properties of expectation and variance to find $\sigma(Y)$. The interval is 22/3 times bigger so $\sigma^2(Y) = (22/3)^2 \frac{3}{4}$ and $\sigma(Y) = 11/\sqrt{3}$. $11/\sqrt{3}$.

4. $X_i : i = 1, 2, 3$ are independent Bernoulli variables equal to 1 with probabilities $1/3, 1/2, 2/3$ respectively, and equal to 0 otherwise. Calculate $\sigma(Y)$ if
$$Y = \min_{1 \leq i \leq 3} X_i$$
. Y has distribution $P(Y = 1) = \frac{1}{3}\frac{1}{2}\frac{2}{3} = 1/9$ and $P(Y = 0) = 8/9$. $E[Y] = 1/9; E[Y^2] = 1/9; \sigma^2(Y) = 8/81$ $2\sqrt{2}/9$.

5. Discrete random variables $X_i : i = 1, 2, \ldots, 10$ are Bernoulli variables with parameter $p = P(X_i = 1) = 0.25$. Discrete random variables $Y_i : i = 1, 2, \ldots, 10$ are Bernoulli variables with parameter $p = P(X_i = 1) = 0.75$. All 20 variables are jointly independent. Let $Z = \sum_{i=1}^{10} X_i + Y_i$. Calculate $\sigma(Z)$. For all i, $\sigma^2(X_i) = p(1-p) = 3/16$. Similarly $\sigma^2(Y_i) = 3/16 \forall i$. By independence, $\sigma^2(Z) = 30/16 + 30/16 = 15/4$. $\sqrt{15}/2$

6. Continuous random variable Y has density 1/6 on the interval $[2, 4]$ and density 1/3 on the interval $[6, 8]$. Calculate $\sigma(Y)$.
$E[Y] = \int_2^4 t/6dt + \int_6^8 t/3dt = 1 + 14/3 = 17/3$ $E[Y^2] = \int_2^4 t^2/6dt + \int_6^8 t^2/3dt = 28/9 + (512 - 216)/9 = 36$ $\sigma^2(Y) = 3 + 8/9 = 35/9$ $\sqrt{35}/3$

7. Continuous random variable Y has density αy in the range $0 \leq y \leq 2$. Find α. Find $\sigma(Y)$.

Qualitative Problems

1. Let X and Y be independent random variables. Then $\sigma(X) + \sigma(Y) - \sigma(X + Y)$ is:
   (a) < 0
   (b) ≤ 0 and can be < 0
   (c) $= 0$
   (d) ≥ 0 and can be > 0
   (e) > 0
   (f) sometimes 0, sometimes < 0 and sometimes > 0

   Hint: try this on a couple of very simple random variables, or remember that by independence $\sigma^2(X) + \sigma^2(Y) = \sigma^2(X + Y)$ *and think about what the square root function does.*

2. Let X and Y be dependent random variables. Then $\sigma(X) + \sigma(Y) - \sigma(X + Y)$ is:
   (a) < 0

(b) ≤ 0 and can be < 0

(c) $= 0$

(d) ≥ 0 and can be > 0

(e) > 0

(f) sometimes 0, sometimes < 0 and sometimes > 0

Hint: The two extreme cases ought to be when $X = Y$ (positive correlation) and when $X = -Y$ (negative correlation). Figure out both extreme cases.

3. In Problem **??** above, suppose all 20 variables changed to be Bernoulli with parameter $p = \frac{1}{2}.25 + \frac{1}{2}.75 = .5$. Would $\sigma^2(Z)$ (the variance of Z, not the standard deviation of Z) change to a smaller, equal, or larger value?

   Hint: Consider the extreme case where $p = 0$ for the X_i variables and (you fill in the rest).

Problems

1. A Georgia Tech degree is worth \$100K today. Each day the value of the Tech degree increases by 1% with probability .5 and decreases by $\frac{100}{101}$ with probability .5. Let X be the number of days until the degree is again worth exactly \$100K. Prove that you can't calculate $\sigma^2(X)$. *Hint: Try to calculate $E[X]$, or to find lower bounds on $E[X]$.* This is logically identical to setting X to the number of steps a drunkard takes until returning to her starting position if she moves east w.p. .5 and west w.p. .5 at each step, independent of other steps. Then $E[X] \geq M]$ for all numbers M. Why? By symmetry assume the first step is east. Then $X = 1+Y$ where $Y =$ the number of steps until reaching the spot one west of her present location. Since the east-west probabilities never change, $E[Y]$ is the same for all locations. So $E[X] = 1 + E[Y] = .5(2) + .5(2 + 2E[Y])$. Then $E[Y] = 1 + E[Y]$ and so it can't equal any finite number.

2. Random variables X and Y are independent with $E[X] = 5, E[X^2] = 49, E[Y] = 30, E[Y^2] = 1000$. Use Chebyshev's inequality to find a number β (the smallest value you can get) such that $P(|X + Y - 35| \geq \beta) \leq 0.04$. *Hint: use the independence of X and Y, and observe that $E[X + Y] = 35$.*