Solutions to Homework 1

1. (a) Poisson (b) exponential (c) geometric (d) Bernoulli (e) binomial (f) normal

2. We are given that $E[X] = 3$ and $Var(X) = 25$. Thus

   a) $E[6 - 4X] = 6 - 4E[X] = -6$ and $Var(6 - 4X) = 16Var(X) = 400$.

   b) $E[(X - 3)/5] = \frac{1}{5}E[X] - \frac{3}{5} = 0$ and $Var((X - 3)/5) = \frac{1}{25}Var(X) = 1$.

3. We know that

   a)
   $$1 = \sum_{k=1}^{5} P(X = 2k - 1) = \sum_{k=1}^{5} (2k - 1)c = 25c$$
   so it follows that c = 1/25.

   b) $E[X] = \sum_{i=1}^{5}(2k)P(X = 2k - 1) = (1/25)(1 + 9 + 25 + 49 + 81) = 33/5$.

   c) $E[X^2] = \sum_{i=1}^{5}(2k)^2P(X = 2k) = (1/25)(1 + 27 + 125 + 343 + 729) = 49$

   d) $Var(X) = E[X^2] - (E[X])^2 = 49 - (33/5)^2 = 136/25$.

   e) Note that
   $$(X-2)^+ = \begin{cases} 0 & \text{with probability } 1/25 \\ 1 & \text{with probability } 3/25 \\ 3 & \text{with probability } 5/25 \\ 5 & \text{with probability } 7/25 \\ 7 & \text{with probability } 9/25 \end{cases}$$
   Hence, $E[(X - 2)^+] = 1 * 3/25 + 3 * 5/25 + 5 * 7/25 + 7 * 9/25 = 116/25$.

4. a) $P(X = k) = 5^k e^{-5}/k!$

   b) $E(X) = 5$

   c) $Var(X) = 5$.

   d) Suppose $0 \leq k \leq 1$, k integer. Then
   $$P(Y = 2) = P(min(X, 2) = k) = P(X = k) = 5^k e^{-5}/k!$$
   $$P(Y = 6) = P(min(X, 2) = 2) = \sum_{k=2}^{\infty} P(X = k) = 1 - e^{-5} - 5e^{-5}.$$

   e)
   $$E[Y] = \sum_{k=0}^{2} kP(Y = k) = 5e^{-5} + 2(1 - e^{-5} - 5e^{-5}).$$

5. a) We know that

$$1 = \int_0^\infty ce^{-4s} ds = c/4$$

so it follows that $c = 4$ (thus, Y is an exponential random variable with rate 4).

b) Let $cv(Y)$ denote the squared coefficient of variation of Y. Then

$$E[Y] = 1/4,\ Var(Y) = 1/16$$

(since Y has an exponential distribution) and so

$$cv(Y) = (1/16)/(1/4)^2 = 1.$$

c)

$$P(Y > 4) = \int_4^\infty 4e^{-4t} dt = e^{-16}.$$

d)

$$P(Y > 6|Y > 2) = P(Y > 6)/P(Y > 2) = e^{-16}.$$

e) We know that x^* satisfies the following:

$$2/3 = P(Y > x^*) = e^{-4x^*}.$$

After simplifying, we see that $x^* = -ln(2/3)/4$.

6.

a) $P(X = Y) = 0$ (to see this, try setting up the limits of integration).

b)

$$P(min(X, Y) > 1/3) = P(X > 1/3, Y > 1/3) = \int_{1/3}^\infty \int_{1/3}^\infty 18e^{-3s} e^{-6t} dsdt = e^{-3}.$$

c)

$$P(X \leq Y) = \int_0^\infty \int_x^\infty 3e^{-3x} 6e^{-6y} dydx = 1/3.$$

d) Let $x \geq 0$. Then

$$f_X(x) = \int_0^\infty 3e^{-3x} 6e^{-6y} dy = 3e^{-3x}.$$

d)

$$E[XY] = \int_0^\infty \int_0^\infty xy 3e^{-3x} 6e^{-6y} dydx = 1/18.$$

7. Let X_k denote the processing time (measured in minutes) of the kth item, where $1 \leq k \leq 100$. Then the total processing time is $\sum_{k=1}^{100} X_k$. Then

$$P\left(\sum_{k=1}^{100} X_k \leq 375\right) = P\left(\frac{\sum_{k=1}^{100} X_k - (100 * 2)}{2 * \sqrt{100}} \leq \frac{375 - (100 * 2)}{2 * \sqrt{100}}\right) \approx P\left(Z \leq 8.75\right) \approx 1$$

where Z denotes a standard normal random variable (mean 0 and variance 1) (notice that the first approximation follows from the Central Limit Theorem; the second is just the number found in a standard normal table.