MATH 1552 QUIZ 3, FALL 2015, GRODZINSKY

Print Your Name: Key-1

T.A.: (circle one) Miheer Brandon Stephen Kabir

1. (18 points) Determine whether the integral below converges or diverges. If it converges, evaluate the value of the integral. SHOW ALL YOUR STEPS; make sure you mathematically explain any limits with appropriate rules. Points will be deducted for notational errors.

$$\int_1^\infty xe^{-4x}\,dx$$

First, find the general antiderivative:

$I = \int xe^{-4x}\,dx$ by parts: $u = x$ $dv = e^{-4x}\,dx$

$du = dx$ $v = -\frac{1}{4}e^{-4x}$

$$I = -\frac{1}{4}xe^{-4x} + \frac{1}{4}\int e^{-4x}\,dx$$

$$= -\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x} + C$$

So: $\int_1^\infty xe^{-4x}\,dx = \lim_{N\to\infty}\int_1^N xe^{-4x}\,dx$

$$= \lim_{N\to\infty}\left[-\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x}\right]\Big|_1^N$$

$$= \lim_{N\to\infty}\left[-\frac{1}{4}Ne^{-4N} - \frac{1}{16}e^{-4N} + \frac{1}{4}e^{-4} + \frac{1}{16}e^{-4}\right]$$

$$= \lim_{N\to\infty}\left[\overset{\to 0}{-\frac{N}{4e^{4N}}} \overset{\to 0}{-\frac{1}{16e^{4N}}} + \frac{5}{16}e^{-4}\right] = \frac{5}{16e^4},$$

[Note: $\lim_{N\to\infty} -\frac{N}{4e^{4N}} \overset{LH}{=} \lim_{N\to\infty} -\frac{1}{16e^{4N}} = 0$]

so the integral $\boxed{\text{converges}}$

2. (14 points) Use the integral comparison test to determine whether or not the integral below converges. Justify any inequalities you use in the comparison. You may quote properties derived in class without proof. DO NOT EVALUATE the integral.

$$\int_1^\infty \frac{1}{\sqrt{9x^3+1}}\,dx$$

Let's compare to $\int_1^\infty \frac{1}{x^{3/2}}\,dx$, which converges since $p = 3/2 > 1$.

Note: $9x^3 + 1 > 9x^3$, so $\sqrt{9x^3+1} > \sqrt{9x^3} = 3x^{3/2}$

and $\frac{1}{\sqrt{9x^3+1}} < \frac{1}{3x^{3/2}}$. Since $\int_1^\infty \frac{dx}{x^{3/2}}$ is finite,

so is $\frac{1}{3}\int_1^\infty \frac{dx}{x^{3/2}} = \int_1^\infty \frac{dx}{3x^{3/2}}$; therefore, $\int_1^\infty \frac{dx}{\sqrt{9x^3+1}}$

must also converge.

3. (18 points) For the sequence $\left\{\frac{2n}{n+3}\right\}$:
(i) find the l.u.b. and g.l.b., if they exist.
(ii) determine if the sequence is monotonic; if so, describe the type of monotonicity.
(iii) determine if the sequence converges. If so, find the limit. If not, explain why the limit does not exist.

$\left\{\frac{2n}{n+3}\right\} = \frac{2}{4}, \frac{4}{5}, \frac{6}{6}, \frac{8}{7}, \frac{10}{8}, \frac{12}{9}, \ldots$

Note that $\lim_{n\to\infty} \frac{2n}{n+3} = 2$, so the sequence

converges to 2. (iii)

The terms are increasing $\Rightarrow$ the sequence
(ii)
is monotonic.

Therefore, l.u.b. = 2 ← limit (i)
g.l.b. = $\frac{1}{2}$ ← 1st term

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1. (18 points) Determine whether the integral below converges or diverges. If it converges, evaluate the value of the integral. SHOW ALL YOUR STEPS; make sure you mathematically explain any limits with appropriate rules. Points will be deducted for notational errors.

$$\int_1^\infty xe^{-3x}\,dx$$

First, we will find a general antiderivative:

$I = \int xe^{-3x}\,dx$ By parts: $u = x$, $dv = e^{-3x}\,dx$; $du = dx$, $v = -\frac{1}{3}e^{-3x}$

Then: $I = \frac{-xe^{-3x}}{3} + \frac{1}{3}\int e^{-3x}\,dx$

$= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C$

So: $\int_1^\infty xe^{-3x}\,dx = \lim_{N\to\infty}\int_1^N xe^{-3x}\,dx$

$= \lim_{N\to\infty}\left[-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right]\Big|_1^N$

$= \lim_{N\to\infty}\left[-\frac{N}{3e^{3N}} - \frac{1}{9e^{3N}} + \frac{1}{3}e^{-3} + \frac{1}{9}e^{-3}\right]$

$\frac{N}{3e^{3N}} \to 0$, $\frac{1}{9e^{3N}} \to 0$

[Note: $\lim_{N\to\infty}\frac{N}{-3e^{3N}} \overset{L'H}{=} \lim_{N\to\infty}\frac{1}{-9e^{3N}} = 0$]

$= \frac{4}{9e^3}$, so the integral converges.

2. (18 points) For the sequence $\left\{\frac{3n}{n+5}\right\}$: $= \frac{3}{6}, \frac{6}{7}, \frac{9}{8}, \frac{12}{9}, ---$

(i) find the l.u.b. and g.l.b., if they exist.

(ii) determine if the sequence is monotonic; if so, describe the type of monotonicity.

(iii) determine if the sequence converges. If so, find the limit. If not, explain why the limit does not exist.

Note that $\lim_{n\to\infty} \frac{3n}{n+5} = 3$, so the sequence converges to 3. (iii)

The terms are increasing $\Rightarrow$ monotonic (ii),

so $\boxed{\begin{array}{l} \text{l.u.b.} = 3 \\ \text{g.l.b.} = 1/2 \end{array}}$ $\leftarrow$ limit, $\leftarrow$ 1st term (i)

3. (14 points) Use the integral comparison test to determine whether or not the integral below converges. Justify any inequalities you use in the comparison. You may quote properties derived in class without proof. DO NOT EVALUATE the integral.

$$\int_1^\infty \frac{1}{\sqrt{4x^5+1}}dx$$

Compare to $\int_1^\infty \frac{1}{x^{5/2}}dx$, which converges since $p = \frac{5}{2} > 1$.

Note that: $4x^5+1 > 4x^5$

so $\sqrt{4x^5+1} > \sqrt{4x^5} = 2x^{5/2}$, and thus,

$$\frac{1}{\sqrt{4x^5+1}} < \frac{1}{2x^{5/2}}.$$

Since $\int_1^\infty \frac{dx}{x^{5/2}}$ converges, $\frac{1}{2}\int_1^\infty \frac{dx}{x^{5/2}} = \int_1^\infty \frac{dx}{2x^{5/2}}$ also converges, so $\int_1^\infty \frac{1}{\sqrt{4x^5+1}}dx$ must converge.