MATH 1502 TEST 1, PAGE 1, FALL 2013, GRODZINSKY

Print Your Name: Key- Form 1

T.A. or Section Number: ____________________

1. (14 points) Evaluate the improper integral. Does it converge or diverge?

$$\int_0^\infty x^2 e^{-2x^3}\,dx$$

Let $u = -2x^3$

$du = -6x^2\,dx$

$= \lim_{b\to\infty} \int_0^b x^2 e^{-2x^3}\,dx$

$= -\frac{1}{6} \lim_{b\to\infty} \int_0^{-2b^3} e^u\,du$

$= -\frac{1}{6} \lim_{b\to\infty} \left[e^u\right]_0^{-2b^3}$

$= -\frac{1}{6} \lim_{b\to\infty} \left[\underbrace{e^{-2b^3}}_{\to 0} - 1\right] = \left(-\frac{1}{6}\right)(-1) = \boxed{\frac{1}{6}}$

The integral $\boxed{\text{converges}}$.

2. (15 points) Find the general solution to the differential equation:

$$\frac{dy}{dx} = 3x^2y - 7xy.$$

$\frac{dy}{dx} = (3x^2 - 7x)y$

$\int \frac{1}{y}\,dy = \int (3x^2 - 7x)\,dx$

$\ln|y| = x^3 - \frac{7}{2}x^2 + C$

$|y| = e^{x^3 - \frac{7}{2}x^2 + C}$ Let $C^* = e^C$

$\boxed{|y| = C^* e^{x^3 - \frac{7}{2}x^2}}$

3. (14 points) Evaluate the limit:

$$\lim_{x\to\infty} [\ln(3x) - \ln(5x+6)]$$

$$= \lim_{x\to\infty} \ln\left(\frac{3x}{5x+6}\right)$$

$$= \ln\left(\lim_{x\to\infty} \frac{3x}{5x+6}\right)$$

$$= \boxed{\ln\left(\frac{3}{5}\right)}$$

4. (14 points) Evaluate the improper integral. Does it converge or diverge?

$$\int_1^4 \frac{1}{(x-3)^2}\,dx$$

$$= \int_1^3 \frac{dx}{(x-3)^2} + \int_3^4 \frac{dx}{(x-3)^2}$$

$$= \lim_{b\to 3^-} \int_1^b \frac{dx}{(x-3)^2} + \lim_{a\to 3^+} \int_a^4 \frac{dx}{(x-3)^2}$$

$$= \lim_{b\to 3^-} \left[-\frac{1}{x-3}\right]\Big|_1^b + \lim_{a\to 3^+} \left[-\frac{1}{x-3}\right]\Big|_a^4$$

$$= \lim_{b\to 3^-} \left[-\underbrace{\frac{1}{b-3}}_{+\infty} + \frac{1}{1-3}\right] + \lim_{a\to 3^+} \left[-\frac{1}{4-3} + \underbrace{\frac{1}{a-3}}_{+\infty}\right]$$

$$= \infty - \frac{1}{2} - 1 + \infty = \infty$$

, so the integral diverges

Print Your Name: Key- Form 1

T.A. or Section Number: ____________

5. (15 points) Solve the initial value problem:

$$\frac{dy}{dx} + \frac{y}{x} = \sin x, \quad y(\pi/2) = 0, \quad x > 0$$

I.F.: $e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

So $x\left[\frac{dy}{dx} + \frac{y}{x}\right] = x\sin x$

$\int \frac{d}{dx}[xy]\,dx = \int x\sin x\,dx$

$u = x \quad dv = \sin x\,dx$

$du = dx \quad v = -\cos x$

$xy = -x\cos x + \int \cos x\,dx$

$xy = -x\cos x + \sin x + C \Rightarrow y = -\cos x + \frac{\sin x}{x} + \frac{c}{x}$

$y(\frac{\pi}{2}) = \frac{2}{\pi} + \frac{2c}{\pi} \stackrel{\text{set}}{=} 0 \quad C = -1$, so $\boxed{y = -\cos x + \frac{\sin x}{x} - \frac{1}{x}}$

6. (14 points) Evaluate the limit:

$$\lim_{x\to\infty} (x + 3e^x)^{\frac{1}{x}}$$

Let $y = (x+3e^x)^{1/x}$, then $\ln y = \frac{1}{x}\ln(x+3e^x)$

$= \frac{\ln(x+3e^x)}{x}$

So $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(x+3e^x)}{x} \left[\frac{\infty}{\infty}\right]$

$\stackrel{L'H}{=} \lim_{x\to\infty} \frac{\frac{1+3e^x}{x+3e^x}}{1} \left[\frac{\infty}{\infty}\right] \stackrel{L'H}{=} \lim_{x\to\infty} \frac{3e^x}{1+3e^x} \left[\frac{\infty}{\infty}\right]$

$\stackrel{L'H}{=} \lim_{x\to\infty} \frac{3e^x}{3e^x} = 1$, so $\boxed{\lim_{x\to\infty} (x+3e^x)^{1/x} = e}$

7. (14 points) Evaluate the limit:

$$\lim_{x\to 0}\left(\frac{5x^2}{1-\cos(3x)}\right) \quad \left[\frac{0}{0}\right]$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{10x}{3\sin(3x)} \quad \left[\frac{0}{0}\right]$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{10}{9\cos(3x)} = \boxed{\frac{10}{9}}$$

BONUS: (6 points) Use L'Hopital's Rule to show that:

$$\lim_{x\to\infty}\left(\frac{a^{1/x}+b^{1/x}}{2}\right)^x = \sqrt{ab}.$$

You may assume both a and b are positive. (The left-hand side is called the *geometric mean* of the a and b).

Let $y = \left(\frac{a^{1/x}+b^{1/x}}{2}\right)^x$, then $\ln y = x \ln\left(\frac{a^{1/x}+b^{1/x}}{2}\right)$

$$= \frac{\ln(a^{1/x}+b^{1/x}) - \ln 2}{1/x}.$$

Then: $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(a^{1/x}+b^{1/x}) - \ln 2}{1/x} \quad \left[\frac{0}{0}\right]$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{a^{1/x}+b^{1/x}}\right)\left(a^{1/x}\ln a + b^{1/x}\ln b\right)\left(-1/x^2\right)}{-1/x^2}$$

$$= \lim_{x\to\infty} \frac{a^{1/x}\ln a + b^{1/x}\ln b}{a^{1/x}+b^{1/x}} = \frac{\ln a + \ln b}{2} = \frac{1}{2}\ln ab$$

$$= \ln\sqrt{ab}$$

So $\lim_{x\to\infty} y = e^{\ln\sqrt{ab}} = \sqrt{ab}.$ qed

Print Your Name: Key-2

T.A. or Section Number: ____________

1. (14 points) Evaluate the limit:

$$\lim_{x\to 0}\left(\frac{3x^2}{1-\cos(2x)}\right) \quad \left[\frac{0}{0}\right]$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{6x}{2\sin(2x)} \quad \left[\frac{0}{0}\right]$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{6}{4\cos(2x)} = \boxed{\frac{3}{2}}$$

2. (15 points) Solve the initial value problem:

$$\frac{dy}{dx} + \frac{y}{x} = \cos x, \quad y(\pi/2) = 0, \quad x > 0$$

I.F: $e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

So: $x\frac{dy}{dx} + y = x\cos x$

$\int \frac{d}{dx}[xy]\,dx = \int x\cos x\,dx$ $\quad u = x \quad dv = \cos x\,dx$

$\quad du = dx \quad v = \sin x$

$xy = x\sin x - \int \sin x\,dx$

$xy = x\sin x + \cos x + C$

$y = \sin x + \frac{\cos x}{x} + \frac{C}{x}$

$y(\pi/2) = 1 + \frac{2C}{\pi} \Rightarrow C = -\pi/2 \Rightarrow \boxed{y = \sin x + \frac{\cos x}{x} + \frac{-\pi}{2x}}$

3. (14 points) Evaluate the limit:

$$\lim_{x\to\infty} [\ln(5x) - \ln(3x+8)]$$

$$= \lim_{x\to\infty}\left[\ln\left(\frac{5x}{3x+8}\right)\right]$$

$$= \ln\left(\lim_{x\to\infty}\left(\frac{5x}{3x+8}\right)\right)$$

$$= \boxed{\ln(5/3)}$$

4. (14 points) Evaluate the improper integral. Does it converge or diverge?

$$\int_2^5 \frac{1}{(x-4)^2}\,dx$$

$$= \int_2^4 \frac{dx}{(x-4)^2} + \int_4^5 \frac{dx}{(x-4)^2}$$

$$= \lim_{b\to 4^-}\int_2^b \frac{dx}{(x-4)^2} + \lim_{a\to 4^+}\int_a^5 \frac{dx}{(x-4)^2}$$

$$= \lim_{b\to 4^-}\left(-\frac{1}{x-4}\right)\Big|_2^b + \lim_{a\to 4^+}\left(-\frac{1}{x-4}\right)\Big|_a^5$$

$$= \lim_{b\to 4^-}\left[\underbrace{-\frac{1}{b-4}}_{\infty} + \frac{1}{2-4}\right] + \lim_{a\to 4^+}\left[-\frac{1}{5-4} + \underbrace{\frac{1}{a-4}}_{\infty}\right]$$

$$= \infty - \frac{1}{2} + 1 + \infty = \infty$$, so the integral $\boxed{\text{diverges}}$

Print Your Name: Key- Form 2

T.A. or Section Number: ____________

5. (14 points) Evaluate the improper integral. Does it converge or diverge?

$$\int_0^\infty x^3 e^{-3x^4}\, dx$$

$$= \lim_{b\to\infty} \int_0^b x^3 e^{-3x^4}\, dx \qquad u = -3x^4, \quad du = -12x^3\, dx$$

$$= -\frac{1}{12} \lim_{b\to\infty} \int_0^{-3b^4} e^u\, du$$

$$= -\frac{1}{12} \lim_{b\to\infty} e^u \Big|_0^{-3b^4} = -\frac{1}{12} \lim_{b\to\infty} \left[e^{-3b^4} - e^0 \right]$$

$$= -\frac{1}{12} [0 - 1] = \boxed{\frac{1}{12}}$$

Converges

6. (15 points) Find the general solution to the differential equation:

$$\frac{dy}{dx} = 4x^2y - 5xy.$$

$$\frac{dy}{dx} = (4x^2 - 5x)y$$

$$\int \frac{1}{y}\, dy = \int (4x^2 - 5x)\, dx$$

$$\ln|y| = \frac{4}{3}x^3 - \frac{5}{2}x^2 + C$$

$$|y| = e^{4/3 x^3 - 5/2 x^2 + C} \qquad C^* = e^C$$

$$\boxed{|y| = C^* e^{4/3 x^3 - 5/2 x^2}}$$

7. (14 points) Evaluate the limit:

$$\lim_{x \to \infty} (x + 5e^x)^{\frac{1}{x}}$$

Let $y = (x+5e^x)^{1/x}$, then $\ln y = \frac{1}{x}\ln(x+5e^x)$

So: $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(x+5e^x)}{x}$ $\left[\frac{\infty}{\infty}\right]$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1+5e^x}{x+5e^x}}{1}$ $\left[\frac{\infty}{\infty}\right]$ $\overset{L'H}{=} \lim_{x\to\infty} \frac{5e^x}{1+5e^x}$ $\left[\frac{\infty}{\infty}\right]$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{5e^x}{5e^x} = 1$, so $\boxed{\lim_{x\to\infty} (x+5e^x)^{1/x} = e}$

BONUS: (6 points) Use L'Hopital's Rule to show that:

$$\lim_{x \to \infty} \left(\frac{a^{1/x} + b^{1/x}}{2}\right)^x = \sqrt{ab}.$$

You may assume both a and b are positive. (The left-hand side is called the *geometric mean* of the a and b).

See Form 1.