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Solutions to Homework 2

1. (a) To find the arrival rates λ, we set up the following traffic equations (with units of arrivals per hour):

$$\begin{aligned}\lambda_E &= 10 + 0.4\lambda_M,\\ \lambda_M &= 5 + 0.25\lambda_E,\\ \lambda_C &= 0.75\lambda_E + 0.6\lambda_M.\end{aligned}$$

Where E stands for the English table, M for the math table and C for the cashier. Solving this system yields $\lambda(E, M, C) = \left(\frac{40}{3}, \frac{25}{3}, 15\right)$.
We have $\mu(E, M, C) = (25, 30, 20)$, so each queue is stable. Now we need to find the joint distribution for the (long run) number of customers in the system. Since this is an open Jackson network, we know that

$$P(X_1(t) = n_1, X_2(t) = n_2, X_3(t) = n_3) = P(X_1(t) = n_1) \cdot P(X_2(t) = n_2) \cdot P(X_3(t) = n_3).$$

We also know that station each station is a $M/M/1$ queue. Thus we apply relevant queueing theory results to determine (letting $\alpha_i = \frac{\lambda_i}{\mu_i}$)

$$\begin{aligned}p_{X_1(t),X_2(t),X_3(t)}(n_1, n_2, n_3) &= (1-\alpha_1)(\alpha_1)^{n_1}(1-\alpha_2)(\alpha_2)^{n_2}(1-\alpha_3)(\alpha_3)^{n_3}\\ &= \left(1-\frac{8}{15}\right)\left(\frac{8}{15}\right)^{n_1}\left(1-\frac{5}{18}\right)\left(\frac{5}{18}\right)^{n_2}\left(1-\frac{3}{4}\right)\left(\frac{3}{4}\right)^{n_3}\end{aligned}$$

for any $n_1, n_2, n_3 \geq 0$.

(b) Letting N be expected number of people in the system, and N_i be the expected number in station i, we know that $N = N_1 + N_2 + N_3$. We use the results for $M/M/1$ queues to calculate that the expected number in each station. We have

$$N_i = \frac{\lambda_i}{\lambda_i - \mu_i}$$

so

$$\begin{aligned}N &= \frac{\lambda_1}{\lambda_1-\mu_1} + \frac{\lambda_2}{\lambda_2-\mu_2} + \frac{\lambda_3}{\lambda_3-\mu_3}\\ &= \frac{8}{7} + \frac{5}{13} + 3\\ &\approx 4.527\end{aligned}$$

(c) So by Little's Law applied to the entire system, we get

$$T = \frac{N}{1} = \frac{4.527}{15} \approx 0.3018 \text{ hours.}$$

2. (a) Note that if you model this as a CTMC you get the state space $S = \{0, 1, 2....\} = \{\text{Number of taxis waiting}\}$ and rate matrix

$$r_{ij} = \begin{cases}\lambda & j = i+1\\ \mu & j = i-1\\ 0 & \text{otherwise}\end{cases}$$

for all $i, j \geq 0$, where $\lambda = 2$ and $\mu = 3$. Here we recognize that this is the exact same rate structure as an $M/M/1$ queue with arrival rate λ and service rate μ. So it follows that

$$P\{\text{Customer finds taxi}\} = 1 - \pi_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^0 = \frac{2}{3}$$

(b) Using the formulas for the $M/M/1$ queue, the expected number of taxis is

$$N = \frac{\lambda}{\lambda - \mu} = 2 \text{ taxis}$$

3. (a) To find the arrival rates λ, we set up the following traffic equations (with units of arrivals per hour):

$$\begin{aligned}
\lambda_1 &= 1 + 0.1\lambda_2 + 0.05\lambda_3, \\
\lambda_2 &= \lambda_1 + 0.1\lambda_3, \\
\lambda_3 &= 0.9\lambda_2.
\end{aligned}$$

Solving this system yields $\lambda = \left(\frac{182}{153}, \frac{200}{153}, \frac{20}{17}\right) \approx (1.1895, 1.3072, 1.1765)$.
We have $\mu = (1.25, 1.43, 1.25)$, so each queue is stable. Now we need to find the joint distribution for the (long run) number of customers in the system. Since this is an open Jackson network, we know that

$$P(X_1(t) = n_1, X_2(t) = n_2, X_3(t) = n_3) = P(X_1(t) = n_1) \cdot P(X_2(t) = n_2) \cdot P(X_3(t) = n_3).$$

We also know that station each station is a $M/M/1$ queue. Thus we apply relevant queueing theory results to determine (letting $\alpha_i = \frac{\lambda_i}{\mu_i}$)

$$\begin{aligned}
&p_{X_1(t),X_2(t),X_3(t)}(2, 1, 4) = (1 - \alpha_1)(\alpha_1)^2 (1 - \alpha_2)(\alpha_2)^1 (1 - \alpha_3)(\alpha_3)^4 \\
&= \left(1 - \frac{1.1895}{1.25}\right)\left(\frac{1.1895}{1.25}\right)^2 \left(1 - \frac{1.3072}{1.43}\right)\left(\frac{1.3072}{1.43}\right)\left(1 - \frac{1.1765}{1.25}\right)\left(\frac{1.1765}{1.25}\right)^4 \approx 0.0001572,
\end{aligned}$$

(b) Letting N be expected number of people in the system, and N_i be the expected number in station i, we know that $N = N_1 + N_2 + N_3$. We use the results for $M/M/1$ queues to calculate that the expected number in each station. Again, let $\alpha_i = \frac{\lambda_i}{\mu_i}$, and we have

$$N_i = \frac{\lambda_i}{\lambda_i - \mu_i}$$

so

$$N_3 = \frac{20/17}{20/17 - 5/4} = 16$$

We can do the same for each station, so we have

$$\begin{aligned}
N &= \frac{\lambda_1}{\lambda_1 - \mu_1} + \frac{\lambda_2}{\lambda_2 - \mu_2} + \frac{\lambda_3}{\lambda_3 - \mu_3} \\
&= 19.66 + 10.64 + 16 \\
&\approx 46.3
\end{aligned}$$

So by Little's Law applied to the entire system, we get

$$T = \frac{N}{1} \approx 46.3 \text{ hours.}$$

We re-do the traffic equations with the new probabilities

$$\begin{aligned}\lambda_1 &= 1 + 0.05\lambda_2 + 0.05\lambda_3,\\ \lambda_2 &= \lambda_1 + 0.1\lambda_3,\\ \lambda_3 &= 0.95\lambda_2.\end{aligned}$$

Solving this system yields $\lambda = \left(\frac{362}{323}, \frac{400}{323}, \frac{20}{17}\right) \approx (1.1207, 1.2384, 1.1765)$. Then

$$\begin{aligned}N &= \frac{\lambda_1}{\lambda_1 - \mu_1} + \frac{\lambda_2}{\lambda_2 - \mu_2} + \frac{\lambda_3}{\lambda_3 - \mu_3}\\ &\approx 8.67 + 6.48 + 16\\ &= 31.15\end{aligned}$$

So by Little's Law applied to the entire system, we get

$$T = \frac{N}{1} \approx 31.13 \text{ hours.}$$

So reducing the number of jobs that need re-work after station 2 by half, cuts total system time for each job by 1/3.