MATH 1502 TEST 3, PAGE 1, FALL 2013, GRODZINSKY

Print Your Name: Key-form 1

T.A. or Section Number: ____________

1. (16 points) Find the radius and interval of convergence of the power series

$$\sum_{k=0}^{\infty} \frac{2^k}{\sqrt{k+3}}(x-5)^k.$$

Using the ratio test to find R:

$$L = \lim_{n\to\infty}\left|\frac{2^{n+1}}{\sqrt{(n+1)+3}} \cdot \frac{\sqrt{n+3}}{2^n}\right| = \lim_{n\to\infty} \frac{2\sqrt{n+3}}{\sqrt{n+4}}$$

$$= 2\lim_{n\to\infty}\sqrt{\frac{n+3}{n+4}} = 2\sqrt{\lim_{n\to\infty}\frac{n+3}{n+4}} = 2\cdot 1 = 2,$$

so $\boxed{R = \frac{1}{2}}$.

Thus, the series converges absolutely when

$|x-5| < \frac{1}{2}$, or $\frac{9}{2} < x < 11/2$.

Check the endpoints:

$x = \frac{11}{2}$: $\sum \frac{2^k}{\sqrt{k+3}}\left(\frac{1}{2}\right)^k = \sum \frac{1}{\sqrt{k+3}}$ diverges (p-series, $\frac{1}{2} < 1$)

$x = \frac{9}{2}$: $\sum \frac{2^k}{\sqrt{k+3}}\left(-\frac{1}{2}\right)^k = \sum \frac{(-1)^k}{\sqrt{k+3}}$ converges conditionally

So: $\boxed{I.C. = \left[\frac{9}{2}, \frac{11}{2}\right)}$

2. (a) (16 points) Find a Taylor polynomial of degree $n = 3$ for the function $f(x) = \frac{1}{x-3}$ in powers of $x - 4$.

k	$f^{(k)}(x)$	$f^{(k)}(4)$	$\frac{f^{(k)}(4)}{k!}(x-4)^k$
0	$(x-3)^{-1}$	$1^{-1} = 1$	1
1	$-(x-3)^{-2}$	$-(1)^{-2} = -1$	$-(x-4)$
2	$2(x-3)^{-3}$	$2(1)^{-3} = 2$	$\frac{2}{2!}(x-4)^2 = (x-4)^2$
3	$-6(x-3)^{-4}$	$-6(1)^{-4} = -6$	$\frac{-6}{3!}(x-4)^3 = -(x-4)^3$

So $\boxed{P_3(x) = 1 - (x-4) + (x-4)^2 - (x-4)^3}$

(b) (10 points) For the problem in part (a), find the maximum value of $|f^{(4)}(c)|$, where c lies between 4 and 4.5.

$$f^{(4)}(x) = 24(x-3)^{-5} = \frac{24}{(x-3)^5}, \quad \text{so } |f^{(4)}(c)| = \left|\frac{24}{(c-3)^5}\right|.$$

Between 4 and 4.5, $f^{(4)}$ is largest when the denominator is smaller $\Rightarrow$ largest at $x = 4$, so

$$|f^{(4)}(c)| \le \frac{24}{(4-3)^5} = 24.$$

(c) (10 points) Use your answer to part (b) to estimate the maximum error in approximating $\frac{1}{1.5} = \frac{2}{3}$ using your Taylor polynomial in part (b).
Recall: $|R_n(x)| \le \max|f^{(n+1)}(c)| \frac{|x-a|^{n+1}}{(n+1)!}$.   Here, $x = 4.5$.

$$|R_3(4.5)| \le 24 \cdot \frac{|4.5-4|^4}{4!}$$

$$= 24 \cdot \frac{(0.5)^4}{4!}$$

$$= \left(\frac{1}{2}\right)^4 = \boxed{\frac{1}{16}}$$ maximum error

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3. (a) (12 points) Find a MacLaurin series for the function $f(x) = \cos(x^3)$.

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \text{ so}$$

$$\cos(x^3) = \sum_{k=0}^{\infty} (-1)^k \frac{(x^3)^{2k}}{(2k)!}$$

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{6k}}{(2k)!}}$$

(b) (12 points) Use your answer to part (a) to estimate the value of $\int_0^1 f(x)dx$ using a polynomial of degree ~~no more than~~ seven.

$$\int_0^1 \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{6k}}{(2k)!} \right) dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{6k+1}}{(2k)!(6k+1)} \Big|_0^1$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!(6k+1)}$$

(7th degree polynomial when k=1)

$$\approx \sum_{k=0}^{1} (-1)^k \frac{1}{(2k)!(6k+1)}$$

$$= (-1)^0 \cdot \frac{1}{0!(1)} + (-1)^1 \frac{1}{2!(7)}$$

$$= 1 - \frac{1}{14} = \boxed{\frac{13}{14}}$$

4. (12 points each) Find a MacLaurin series for the functions below. For what values of x is your series valid?

$$f(x) = \frac{2x}{3-x}, \quad g(x) = \frac{e^{-x^2}}{x}.$$

$$f(x) = \frac{2}{3}x\left(\frac{1}{1-x/3}\right) = \frac{2}{3}x\sum_{k=0}^{\infty}\left(\frac{x}{3}\right)^k$$

$$= \frac{2}{3}x\sum_{k=0}^{\infty}\frac{x^k}{3^k} = 2\sum_{k=0}^{\infty}\frac{x^{k+1}}{3^{k+1}}$$, valid when $\left|\frac{x}{3}\right| < 1$ or $|x| < 3$

$$g(x) = \frac{1}{x}e^{-x^2} = \frac{1}{x}\sum_{k=0}^{\infty}\frac{(-x^2)^k}{k!} = \frac{1}{x}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{k!}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k-1}}{k!}$$, valid when $x \neq 0$

(0 is not in the domain)

BONUS: (5 points) Derive a series expansion for $\ln\left(\frac{1+x}{1-x}\right)$ using the standard MacLaurin series for $\ln(1+x)$.

Since $\ln(1+x) = \sum_{k=0}^{\infty}(-1)^k\frac{x^{k+1}}{k+1}$, then

$$\ln(1-x) = \ln(1+(-x)) = \sum_{k=0}^{\infty}(-1)^k\frac{(-x)^{k+1}}{k+1} = \sum_{k=0}^{\infty}\frac{(-1)^k(-1)^{k+1}x^{k+1}}{k+1}$$

$$= \sum_{k=0}^{\infty}(-1)^{2k+1}\frac{x^{k+1}}{k+1} = -\sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1}.$$

Then: $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{x^{k+1}}{k+1} - \left(-\sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1}\right)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) + \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) = \boxed{2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}}$$

The expansion is valid for $|x| < 1$.

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T.A. or Section Number: ____________

1. (12 points each) Find a MacLaurin series for the functions below. For what values of x is your series valid?

$$f(x) = \frac{4x}{5-x}, \quad g(x) = \frac{e^{-x^3}}{x}.$$

$$f(x) = \frac{4x}{5} \cdot \frac{1}{1 - x/5} = \frac{4x}{5} \sum_{k=0}^{\infty} \left(\frac{x}{5}\right)^k$$

$$= 4 \sum_{k=0}^{\infty} \left(\frac{x}{5}\right)^{k+1}, \quad \text{valid for } \left|\frac{x}{5}\right| < 1$$

$$\Rightarrow |x| < 5$$

$$g(x) = \frac{1}{x} \cdot e^{-x^3} = \frac{1}{x} \sum_{k=0}^{\infty} \frac{(-x^3)^k}{k!}$$

$$= \frac{1}{x} \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k}}{k!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{3k-1}}{k!}, \quad \text{valid for } x \neq 0$$

2. (a) (16 points) Find a Taylor polynomial of degree $n = 3$ for the function $f(x) = \frac{1}{x-2}$ in powers of $x - 3$.

K	$f^{(K)}(x)$	$f^{(K)}(3)$	$\frac{f^{(K)}(3)}{K!}(x-3)^K$
0	$(x-2)^{-1}$	1	1
1	$-(x-2)^{-2}$	-1	$-(x-3)$
2	$2(x-2)^{-3}$	2	$\frac{2}{2!}(x-3)^2 = (x-3)^2$
3	$-6(x-2)^{-4}$	-6	$\frac{-6}{3!}(x-3)^3 = -(x-3)^3$

So $\boxed{P_3(x) = 1-(x-3)+(x-3)^2-(x-3)^3}$

(b) (10 points) For the problem in part (a), find the maximum value of $|f^{(4)}(c)|$, where c lies between 3 and 3.5.

$f^{(4)}(x) = \frac{24}{(x-2)^5}$ (a decreasing function on $[3, 3.5]$)

When $3 \le c \le 3.5$, $\max|f^{(4)}(c)| = f^{(4)}(3) = \boxed{24}$

(c) (10 points) Use your answer to part (b) to estimate the maximum error in approximating $\frac{1}{1.5} = \frac{2}{3}$ using your Taylor polynomial in part (b).
Recall: $|R_n(x)| \le \max|f^{(n+1)}(c)|\frac{|x-a|^{n+1}}{(n+1)!}$.

$$|R_3(3.5)| = \max|f^{(4)}(c)| \frac{|3.5-3|^4}{4!}$$

$$= 24 \cdot \frac{(1/2)^4}{4!}$$

$$= 24 \cdot \frac{1}{2^4 \cdot 4!}$$

$$= \boxed{\frac{1}{16}}$$

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T.A. or Section Number: ____________

3. (a) (12 points) Find a MacLaurin series for the function $f(x) = \cos(x^4)$.

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \quad \text{so}$$

$$\cos(x^4) = \sum_{k=0}^{\infty} (-1)^k \frac{(x^4)^{2k}}{(2k)!}$$

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{8k}}{(2k)!}}$$

(b) (12 points) Use your answer to part (a) to estimate the value of $\int_0^1 \cos(x^4)dx$ using a polynomial of degree nine.

$$\int_0^1 \cos(x^4)\,dx = \int_0^1 \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{8k}}{(2k)!} \right) dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{8k+1}}{(8k+1)(2k)!} \Big|_0^1$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(8k+1)(2k)!}$$

the polynomial has degree 9 when $8k+1=9 \Rightarrow k=1$

$$\approx \sum_{k=0}^{1} (-1)^k \frac{1}{(8k+1)(2k)!}$$

$$= 1 - \frac{1}{9 \cdot 2!} = 1 - \frac{1}{18} =$$

$$= \boxed{\frac{17}{18}}$$

4. (16 points) Find the radius and interval of convergence of the power series

$$\sum_{k=0}^{\infty} \frac{4^k}{\sqrt{k+2}}(x-6)^k.$$

Ratio test:

$$L = \lim_{n\to\infty} \left| \frac{4^{n+1}}{\sqrt{n+3}} \cdot \frac{\sqrt{n+2}}{4^n} \right| = \lim_{n\to\infty} 4\sqrt{\frac{n+2}{n+3}}$$

$$= 4\sqrt{\lim_{n\to\infty} \frac{n+2}{n+3}} = 4, \quad \text{so } \boxed{R = 1/4}$$

The series converges absolutely when $|x-6| < 1/4$, or $23/4 < x < 25/4$.

Check endpoints:

$x = 25/4$: $\sum \frac{4^k}{\sqrt{k+2}} \cdot \left(\frac{1}{4}\right)^k = \sum \frac{1}{\sqrt{k+2}}$ which diverges (p-series, $p = 1/2 < 1$)

$x = 23/4$: $\sum \frac{4^k}{\sqrt{k+2}}\left(-\frac{1}{4}\right)^k = \sum (-1)^k \frac{1}{\sqrt{k+2}}$ which converges conditionally

$$\Rightarrow \boxed{\text{I.C.} = \left[23/4, 25/4\right)}$$

BONUS: (5 points) Derive a series expansion for $\ln\left(\frac{1+x}{1-x}\right)$ using the standard MacLaurin series for $\ln(1+x)$.

See Form 1.