You can answer all questions on this sheet, but may use extra sheets (from your personal notepad) if needed.

Solution
_____   _____
        Name   GT IDnumber

Problem 1. (100 points)

You are given a list of $n + 1$ natural numbers, $n \geq 1$.

a. Show that there is a pair of numbers whose difference is divisible by n.

There exists n congruence classes modulo n $(\bar{0}, \bar{1}, \ldots, \overline{n-1})$ and any of the given $n+1$ natural numbers must belong to one of the classes

$\to$ By Pigeon Hole Principle, there are at least $\left\lceil \dfrac{n+1}{n} \right\rceil = 2$ numbers in the same congruence class, which by definition means their difference is divisible by n.

b. Suppose next that the input list consists of $3n + 1$ natural number. What is the largest lower bound on the number of elements, each pair of which has a difference divisible by n? (Give your best possible answer.)

As above, there are still n congruence classes modulo n and each of the $3n+1$ natural numbers belong to one of the mentioned classes

$\to$ By Pigeon Hole Principle, there are at least

$\left\lceil \dfrac{3n+1}{n} \right\rceil = 4$ numbers out of these given numbers

that belong to the same class so the different between any 2 of these 4 must be divisible by n by definition

$\to$ the largest lower bound is 4