Math 2602 K1-K3
Spring 2014
Midterm 1
2/4/14
Time Limit: 80 Minutes

Name (Print): Gagik

Section ________________

This exam contains 5 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes and calculators on this exam.

You are required to show your work on each problem on this exam.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

1. (10 points) Show that $n^4 - n^2$ is divisible by 4 for all integers n. (hint $n^4 - n^2 = n^2(n^2 - 1)$.)

$$n^4 - n^2 = n^2(n^2-1) = n^2 \cdot (n-1)(n+1)$$

Case 1: n is even $n = 2m$

$n^4 - n^2 = 4m^2 \cdot (2m-1)(2m+1)$ is divisible by 4.

Case 2: n is odd $n = 2m+1$

$n^4 - n^2 = (2m+1)^2 \cdot (2m) \cdot (2m+2) = 4(2m+1)^2 \cdot m \cdot (m+1)$

is divisible by 4.

2. (10 points) Prove that if a is a rational number and b is an irrational numbers then $a + b$ is an irrational number.

Proof by a contradiction.

Assume $a+b$ is a rational.

a is rational too, so we can write

$a+b = \frac{m}{n}$ m, n are integers,

$a = \frac{k}{l}$ k, l are integers.

$b = \frac{m}{n} - \frac{k}{l} = \frac{ml - kn}{nl}$ is a rational.

3. (10 points) Show that $p \wedge (\neg p \vee \neg q) \wedge (p \to q)$ is a contradiction.

If the statement above is true then

$p = T$, $\underbrace{\neg p}_{F} \vee \underbrace{\neg q}_{T} = T$ and $p \to q = T$

$q = F$, so $p \to q$ is false.

Hence the statement is always false. ∎

4. (10 points) Show the following logical equivalence $\neg(p \vee q) \vee (\neg p \wedge q) \Leftrightarrow \neg p$.

We have $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

So $\neg(p \vee q) \vee (\neg p \wedge q) \Leftrightarrow$

$(\neg p \wedge \neg q) \vee (\neg p \wedge q) \Leftrightarrow$

$\neg p \wedge (\neg q \vee q) \Leftrightarrow$

$\neg p \wedge 1 \Leftrightarrow$

$\neg p$. ∎

5. (10 points) The binary relation $\mathcal{R}$ on $\mathbb{R}$ is defined by $\mathcal{R} = \{(x,y) \in \mathbb{R}^2 \mid xy > 0\}$. Is $\mathcal{R}$ a) Reflexive? b) Symmetric ? c)Antisymmetric ? d)Transitive ? Justify your answer.

$xy > 0 \Rightarrow yx > 0$ so it's ~~reflexive~~ symmetric.

$0 \cdot 0 = 0 \Rightarrow$ it's not reflexive.

$1 \cdot 2 > 0$ $1 \cdot 2 > 0$ but $1 \neq 2$

it's not antisymmetric.

$xy > 0$, $y \cdot z > 0$, x and y have the same sign, y, z have the same sign hence x and z have the same sign, so $xz > 0$

it's transitive.

6. (10 points) For integers a and b define $a \sim b$ if $a - b$ is divisible by 3.

a) Show that $\sim$ defines an equivalence relation on $\mathbb{Z}$.

b) What are the equivalence classes for $\sim$?

a) $a - a = 0$ is divisible by 3 so $a \sim a$.

it's reflexive.

$a \sim b$, $b \sim c$ then $a - b = 3m$, $b - c = 3k$ then $a - c = 3(m+k)$ so $a \sim c$.

$\sim$ is transitive.

$a \sim b \Rightarrow a - b = 3m \Rightarrow b - a = 3 \cdot (-m)$ hence $b \sim a$. it's symmetric.

b) $\bar{0} = 3\mathbb{Z}$, $\bar{1} = 3\mathbb{Z} + 1$ $\bar{2} = 3\mathbb{Z} + 2$.

7. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$. Check if f is a) one-to-one b) onto.

a) $f(1) = f(-1) = 2$ Not one-to-one.

b) $x^2 + 1 \geqslant 1$ Not onto.

8. (10 points) Check if the sets have the same cardinality and justify your answer.

a) $\{n^2 + 8 |\ n \in \mathbb{N}\}$ and $\mathbb{N}$.

b) The intervals $(0, 1)$ and $(6, 8)$.

a) $A = \{n^2 + 8 | n \in N\}$

$f : N \to A$ $f(n) = n^2 + 8$ is a bijection

hence $|A| = |N|$

b) $f : (0,1) \to (6,8)$ $f(x) = 6 + 2x$

it's one-to-one and onto.

So $|(0,1)| = |(6,8)|$.