MATH 1552 - SPRING 2016
TEST 3 - SHOW YOUR WORK

NAME: ____________________________ TA: ________________________________

1. (20 points) Find the limits of the following **sequences**. Show your work and justify your answer.

a. (6 pts) $a_n = \sqrt[n]{3^{2n+1}} = \left[3^{2n} \cdot 3\right]^{\frac{1}{n}} = 3^2 \cdot 3^{\frac{1}{n}} \to \mathbf{9}$

because $3^{\frac{1}{n}} \to 1$ - one of the common (or important) limits

b. (7 pts) $b_n = \left[\dfrac{3n+1}{3n-1}\right]^n$. Simplify the inside.

$$\frac{3n+1}{3n-1} = 1 + \frac{2}{3n} = 1 + \frac{\left(\frac{2}{3}\right)}{n} \Rightarrow$$

$$\left[\frac{3n+1}{3n-1}\right]^n = \left[1 + \frac{\left(\frac{2}{3}\right)}{n}\right]^n \to \boldsymbol{e}^{\frac{\mathbf{2}}{\mathbf{3}}}$$ one of the common (or important) limits

c. (7 pts) Show that $d_n = 1 + \dfrac{1}{n}$ converges using the Monotonic Sequence Theorem. You do not have to find the limit, but you must use MST.

1. Is $d_n = 1 + \dfrac{1}{n}$ is nonincreasing?

$$n < n+1 \Rightarrow \frac{1}{n+1} < \frac{1}{n} \Rightarrow 1 + \frac{1}{n+1} < 1 + \frac{1}{n}$$

That is, $d_{n+1} < d_n$ and $\boldsymbol{d_n}$ **is nonincreasing**

OR

let $f(x) = 1 + \dfrac{1}{x} \Rightarrow f'(x) = -\dfrac{1}{x^2} < 0 \Rightarrow f(x)$ *is decreasing*

$\Rightarrow$ ***d_n is nonincreasing because $d_n = f(n)$***

2. Is d_n *is bounded below*? **Yes, because $1 < d_n = 1 + \dfrac{1}{n}$**

1 & 2 $\Rightarrow d_n$ converges by the MST

2. (20 points) Find the radius of convergence and the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(5x-7)^n}{n^2}$.

Use the root test and **test the endpoints**. On the endpoints, just state the reason for your answer .

** Root Test: $\sqrt[n]{|a_n|} = \left[\left(\frac{|5x-7|^n}{(n^2)}\right)\right]^{\frac{1}{n}} = \frac{|5x-7|}{\left(n^{\frac{1}{n}}\right)^2} \to |5x-7| < 1$

because $n^{\frac{1}{n}} \to 1$ one of the common (or important) limits

$|5x-7| < 1 \leftrightarrow -1 < 5x-7 < 1 \Rightarrow$

$$\mathbf{\frac{6}{5} < x < \frac{8}{5} \text{ and } R = \frac{1}{5}}$$

** Test the endpoints:

$x = \frac{6}{5} \Rightarrow 5x-7 = -1$. *The series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ ***converges*** **by** ***AST***

$x = \frac{8}{5} \Rightarrow 5x-7 = 1$. *The series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ***converges*** **b/c it is a p - series, with p = 2**

3. (20 points) Use a known series to find the interval of convergence (do not test the endpoints) and the sum of $\sum_{n=0}^{\infty} (e^x - 4)^n$. Write down the known series , together with it's sum and interval of convergence. **DO NOT USE ANY TEST FOR CONVERGENCE.** Simplify you sum and make sure you show your IOC as $a < x < b$.

Known series (GS): $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ **for** $|x| < 1$

$$\Rightarrow \sum_{n=0}^{\infty} \left(e^x - 4\right)^n = \frac{1}{1 - \left(e^x - 4\right)} = \frac{1}{5 - e^x}$$

for $|e^x - 4| < 1 \Leftrightarrow -1 < e^x - 4 < 1 \Leftrightarrow 3 < e^x < 5$

$\ln(3) < x < \ln(5)$ is the IOC

5. (20 points) Let $f(x) = \sqrt{1+x}$, $a = 0$. Find $P_4(x)$. The Taylor polynomial of order n for $f(x)$ is

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} + \ldots$$

$$f(x) = (1+x)^{\frac{1}{2}} \qquad \frac{f^{(0)}(0)}{0!} = 1$$

$$f^{(1)}(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \qquad \frac{f^{(1)}(0)}{1!} = \frac{1}{2}$$

$$f^{(2)}(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \qquad \frac{f^{(2)}(0)}{2!} = -\frac{1}{8}$$

$$f^{(3)}(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}} \qquad \frac{f^{(3)}(0)}{3!} = \frac{1}{16}$$

$$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-\frac{7}{2}} \qquad \frac{f^{(4)}(0)}{4!} = -\frac{5}{128}$$

$$\Rightarrow \boldsymbol{P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} x^k = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4}$$

6. (20 points) Use the MacLaurin Series for $\ln(1+x)$ to find the MacLaurin Series for $\ln\left(\frac{1+x}{1-x}\right)$.

Simplify $\ln\left(\frac{1+x}{1-x}\right)$ first and then show all of your steps.

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1+x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \text{ .... (1)}$$

Replace $x \, with \, -x$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \text{ ....}$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} \text{ .... (2)}$$

Now add (1) and (2)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \text{ ....}$$

$$-\ (1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} \text{ ....}$$

$$\Rightarrow \mathbf{\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1+x)}$$

$$\mathbf{= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} \text{ ....} = \sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n+1}} \text{ (either is ok)}$$