H.Ayhan and S.-H Kim

Solutions to Homework 8 Solutions

1. Let X_n be inventory at each Friday, thus we know $X_n \in S = \{0, 10, 20\}$. (Note that it's more convenient to model the chain without state 30, since it's transient and once left 30 it'll never go back to 30. We can know right now that stationary probability for 30, $\pi_{30} = 0$.)

   (a) Transition matrix
   $$P = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \\ 0.8 & 0.2 & 0 \\ 0.3 & 0.5 & 0.2 \end{bmatrix}$$
   Thus we have
   $$P^2 = \begin{bmatrix} 0.4 & 0.44 & 0.16 \\ 0.4 & 0.44 & 0.16 \\ 0.3 & 0.5 & 0.2 \\ 0.4 & 0.44 & 0.16 \end{bmatrix}$$
   The first probability can be found in row 3 column 2, $P(X_2 = 10|X_0 = 20) = P^2_{20,10} = 0.5$.

   (b) This problem intents you to use π_{10} to approximate $P(X_{100} = 10|X_0 = 20)$. Because 100 is a "large enough" time step ahead and the chain is irreducible and aperiodic, we know $\pi_{10} = P(X_{100} = 10|X_0 = 20)$. We'll calculate π_{10} later in the problem.
   If you have computational tool at hand and it's not forbidden from the question, we can apply following approach directly:
   $$P^{100} = \begin{bmatrix} 0.3833 & 0.4500 & 0.1667 \\ 0.3833 & 0.4500 & 0.1667 \\ 0.3833 & 0.4500 & 0.1667 \\ 0.3833 & 0.4500 & 0.1667 \end{bmatrix}.$$
   Therefor $P(X_{100} = 10|X_0 = 20) = 0.45$.
   NOTE: The P^{100} matrix has same values for each column and (by row) they sum up to one, it can be verified that those three value are $\pi_0, \pi_{10}, \pi_{20}$ respectively.

   (c) Digram is straight forward, so it's omitted here. Note that the transition matrix and the digram carries exactly the same information.

   (d) The chain is irreducible.

   (e) The chain is aperiodic. (Period is 1 for each state)

   (f) Let $f(i)$ denote the expected profit when on previous friday we have i inventory, and
   $$f(0) = -500 - 0 \times 20 - 30 \times 100 + 200 \times \mathbb{E}(\min(0, D)) = 700$$
   $$f(10) = -500 - 10 \times 20 - 20 \times 100 + 200 \times \mathbb{E}(\min(10, D)) = 1500$$
   $$f(20) = 20 \times 20 - 10 \times 100 + 200 \times \mathbb{E}(\min(20, D)) = 3200$$
   $$f(30) = 30 \times 20 - 0 \times 100 + 200 \times \mathbb{E}(\min(30, D)) = 3600$$
   Note that fixed cost only occurred when inventory is less or equal to 10, and D is the given demand.

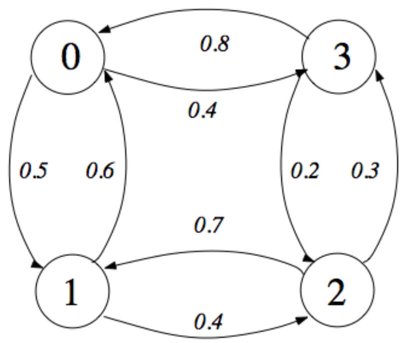
(g) Solving $\pi = (\pi_0, \pi_{10}, \pi_{20}) = \pi P, \sum \pi = 1, \pi_{30} = 0$, we can get

$$(\pi_0, \pi_{10}, \pi_{20}, \pi_{30}) = (0.3833, 0.4500, 0.1667, 0).$$

Therefore the expected profit per week is

$$f(0)\pi_0 + f(10)\pi_{10} + f(20)\pi_{20} + f(30)\pi_{30} = 1476$$

2. (a) Suppose the state space to be $\{0, 1, 2, 3, \}$, one can draw the transition diagram



(b)

$$P^{100} = \begin{bmatrix} 0.6875 & 0 & 0.3125 & 0 \\ 0 & 0.5625 & 0 & 0.4375 \\ 0.6875 & 0 & 0.3125 & 0 \\ 0 & 0.5625 & 0 & 0.4375 \end{bmatrix}.$$

(c)

$$P^{101} = \begin{bmatrix} 0 & 0.5625 & 0 & 0.4375 \\ 0.6875 & 0 & 0.3125 & 0 \\ 0 & 0.5625 & 0 & 0.4375 \\ 0.6875 & 0 & 0.3125 & 0 \end{bmatrix}.$$

(d) The chain is irreducible.

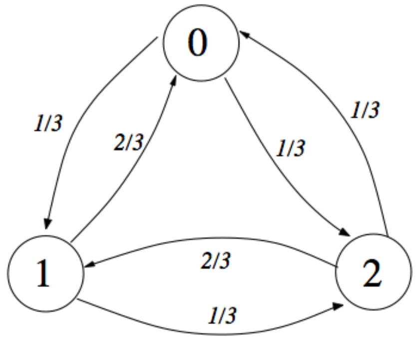
(e) It's periodic, and the period (of each state) is 2.

(f) Yes, it is. π satisfies the balance equation $\pi P = \pi$ and $\sum_{i=1}^{4} \pi_i = 1$. Since the Markov chain is irreducible, the stationary distribution is unique.

(g) $P_{11}^{100} = 11/16 \neq \pi_1$, $P_{11}^{101} = 0 \neq \pi_1$. Since the Markov chain has period 2,

$$\pi_1 = \frac{P_{11}^{100} + P_{11}^{101}}{2}.$$

3. (a) Assume the state space is $\{0, 1, 2\}$. One can draw the transition diagram as the following:



(1) Since any state commutes with each other, the Markov chain is irreducible.

(2) The stationary distribution is $\pi = (1/3, 1/3, 1/3)$, which is unique because the state space is irreducible.

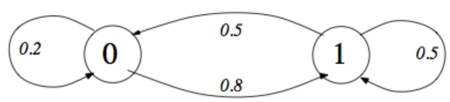
(3) The Markov chain is aperiodic.

(4) Each state can be visited again either in 2 steps (e.g. $0 \to 1 \to 0$) or 3 steps (e.g. $0 \to 2 \to 1 \to 0$). The period is the greatest common factor of 2 and 3, which is 1.

(5) Since Markov chain is irreducible, aperiodic, and has a finite state space, $\lim_{n\to\infty} P_{ij}^n = \pi_j$, where π is the unique stationary distribution obtained in (2). And $n = 100$ is long enough to "reach" the steady state, so

$$P^{100} \approx \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

(b) Assume the state space is $\{0, 1\}$. One can draw the transition diagram as the following:



(1) Since any state commutes with each other, the Markov chain is irreducible.

(2) The stationary distribution is $\pi = (5/13, 8/13)$, which is unique because the state space is irreducible.

(3) The Markov chain is aperiodic.

(4) Period of each state is 1.

(5) As discussed above,

$$P^{100} \approx \begin{bmatrix} 5/13 & 8/13 \\ 5/13 & 8/13 \end{bmatrix}.$$