NAME →

ISyE 3044 — Fall 2012 — Test #1 Solutions

This test is 90 minutes. You're allowed one cheat sheet. **Just show your extremely neat answers.** Good luck!

1. Suppose X has p.d.f. $f(x) = 2x$, $0 < x < 1$. Find $\mathsf{E}[3X - 2]$.

   Solution: $\mathsf{E}[X] = \int_0^1 x\, 2x\, dx = 2/3$. Thus, $\mathsf{E}[3X - 2] = 0$. $\Box$

2. Suppose X has p.d.f. $f(x) = 2x$, $0 < x < 1$. Find $\mathsf{E}[\frac{2}{X} + X]$.

   Solution: $\mathsf{E}[\frac{2}{X} + X] = \int_0^1 (\frac{2}{x} + x)\, 2x\, dx = 14/3$. $\Box$

3. TRUE or FALSE? For any positive random variable X, we have $\mathsf{E}[X] \geq (\mathsf{E}[X^{1/2}])^2$.

   Solution: TRUE: $\mathsf{E}[X] - (\mathsf{E}[X^{1/2}])^2 = \mathsf{Var}(X^{1/2}) \geq 0$. $\Box$

4. TRUE or FALSE? Suppose X is a positive random variable with c.d.f. $F(x)$. Then $\mathsf{E}[X] = \int_0^\infty x(1 - F(x))\, dx$.

   Solution: FALSE: Not quite...

$$\begin{aligned}
\mathsf{E}[X] &= \int_0^\infty x f(x)\, dx \\
&= \int_0^\infty \int_0^x dy\, f(x)\, dx \\
&= \int_0^\infty \int_y^\infty f(x)\, dx\, dy \\
&= \int_0^\infty \Pr(X > y)\, dy \\
&= \int_0^\infty (1 - F(x))\, dx. \quad \Box
\end{aligned}$$

5. TRUE or FALSE? The number of times that the Atlanta Braves win in their next 4 games is well-represented by a Poisson process.

   Solution: FALSE: Even if the games were i.i.d. (which they probably are not), the number of wins would be binomial. $\square$

6. Suppose that machine breakdowns occur according to a Poisson process at the rate of 1/week. What's the probability that there will be exactly 1 breakdown during the next 2 weeks?

   Solution: Let X denote the number of breaks during the next 2 weeks. Then $X \sim \text{Pois}(2)$, so that $\Pr(X = 1) = e^{-2}2^1/1! = 0.2707$. $\square$

7. Again suppose that machine breakdowns occur according to a Poisson process at the rate of 1/week. What is the distribution of the time between the 1st and 5th breakdowns? (Name the distribution with any relevant parameters.)

   Solution: $\text{Erlang}_4(1)$. $\square$

8. Suppose that X has p.d.f. $f(x) = 2x$, $0 \le x \le 1$. What's the distribution of the random variable X^2? (Hint: Take a look at the c.d.f. $F(x)$.)

   Solution: Following the hint, we note that $F(X) = X^2$. And then the Inverse Transform Theorem immediately implies $X^2 \sim \text{Unif}(0,1)$. $\square$

   Alternatively, let $Y = X^2$. We have

   $$\Pr(Y \le y) = \Pr(X^2 \le y) = \Pr(X \le \sqrt{y}) = \int_0^{\sqrt{y}} 2x\,dx = y.$$

   This implies that $f_Y(y) = 1$, $0 \le y \le 1$, so that $Y \sim \text{Unif}(0,1)$. $\square$

9. Suppose that $U \sim \text{Unif}(0,1)$. Name the distribution of $-\frac{1}{4}\ell\text{n}(U)$. (Include any relevant parameters.)

   Solution: As mentioned numerous times in class, this is $\text{Exp}(4)$. $\square$

10. Suppose X is a normal random variable with mean 3 and variance 9. What is the probability that $X \geq 0$?

Solution:

$$\Pr(X \geq 0) = \Pr\left(Z \geq \frac{0-3}{\sqrt{9}}\right) = \Pr(Z \geq -1) = 1 - \Phi(-1) = \Phi(1) = 0.8413. \quad \square$$

11. Suppose the joint p.d.f. of X and Y is $f(x,y) = c$, $0 < x < y < 3$, for some appropriate value of c. Find c.

Solution: Set

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^3\int_0^y c\,dx\,dy = 9c/2.$$

This implies that $c = 2/9$. $\square$

12. Again suppose the joint p.d.f. of X and Y is $f(x,y) = c$, $0 < x < y < 3$. Find $\mathsf{Cov}(X,Y)$.

Solution: We have

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^3 (2/9)\,dy = \frac{2}{9}(3-x),\ 0 \leq x \leq 3,$$

so that $\mathsf{E}[X] = \int_0^3 x\,\frac{2}{9}(3-x)\,dx = 1$.

Similarly, $f_Y(y) = \frac{2}{9}y$, $0 \leq y \leq 3$, and $\mathsf{E}[Y] = 2$.

In addition,

$$\mathsf{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,f(x,y)\,dx\,dy = \int_0^3\int_0^y \frac{2}{9}xy\,dx\,dy = 9/4.$$

So finally, $\mathsf{Cov}(X,Y) = \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y] = \frac{9}{4} - (1)(2) = 1/4$. $\square$

13. Consider the random variables X, Y, Z, all of which are $\text{Nor}(0, 2)$, but with $\mathsf{Cov}(X, Y) = \mathsf{Cov}(X, Z) = \mathsf{Cov}(Y, Z) = 1$. Find $\mathsf{Var}(X - Y - Z)$.

    Solution:

    $$\mathsf{Var}(X) + \mathsf{Var}(Y) + \mathsf{Var}(Z) - 2\mathsf{Cov}(X, Y) - 2\mathsf{Cov}(X, Z) + 2\mathsf{Cov}(Y, Z) = 4. \quad \Box$$

14. Suppose that the joint p.d.f. of X and Y is $f(x, y) = c\,e^{-(x+y)}/(x^2 + y^2)$, for $1 < x < 2$, $0 < y < 1$, and some appropriate value of c. Are X and Y independent?

    Solution: No. You can't factor the joint p.d.f. $\Box$

15. Suppose that the times between buses are i.i.d. Exp(3/hr). Further suppose that I showed up at the bus stop at some random time and have already been waiting for an hour. What's the probability that the next bus will come within the next 20 minutes?

    Solution: Let $X \sim \text{Exp}(3)$ be the time that the next bus arrives. We want the probability that the next bus will show up by time 80 minutes (i.e., 4/3 hours), given that we've already been waiting an hour. By the memoryless property,

    $$\Pr(X \le 4/3 | X \ge 1) = \Pr(X \le 1/3) = 1 - e^{-1} = 0.632. \quad \Box$$

16. What distribution do you get if you add up two i.i.d. Nor(3,6) random variables?

    Solution: $\text{Nor}(6, 12)$. $\Box$

17. Suppose that $X_1, \ldots, X_{100}$ are i.i.d. with values 1 and -1, each with probability 0.5. (This is a simple random walk.) Find the approximate probability that the sample mean $\bar{X}_{100}$ will be between -0.1 and 0.1. (This corrects a typo in the original problem statement.)

Solution: Note that $\mathsf{E}[X_i] = 0$ and $\mathsf{Var}(X_i) = \mathsf{E}[X_i^2] - (\mathsf{E}[X_i])^2 = 1$. Then the Central Limit Theorem implies that the sample mean $\bar{X} \approx \text{Nor}(0, 1/100)$, and so

$$\begin{aligned}\Pr(-0.1 < \bar{X} < 0.1) &\approx \Pr\left(\frac{-0.1-0}{\sqrt{1/100}} < Z < \frac{0.1-0}{\sqrt{1/100}}\right) \\ &= \Pr(-1 < Z < 1) = 0.6826. \quad \Box\end{aligned}$$

18. Joey likes to play Dungeons and Dragons. Give him a simple algorithm to generate a 12-sided die toss. (If your algorithm isn't simple enough, a Medusa monster will kill him — so Joey is really counting on you to get this question correct.)

    Solution: If $U \sim \text{Unif}(0, 1)$, then you can use $\lceil 12U \rceil$. $\Box$

19. TRUE or FALSE? If you add up two i.i.d. Unif(0,1) random variables, you get a triangular distribution.

    Solution: TRUE. $\Box$

20. Suppose that I pick 20000 points randomly in a unit cube, into which I've inscribed a sphere with radius 1/2. (This sphere has volume $\frac{4}{3}\pi r^3 = \pi/6$.) Further suppose that 10452 of those random points fall inside the inscribed sphere. Using this information, give me an estimate of π.

    Solution: Let $p = \pi/6$ denote the probability that a random point will be in the sphere. Then an estimator for p is the sample proportion $\hat{p}_n = 0.5226$, and so the desired estimate is $\hat{\pi}_n = 6\hat{p}_n = 3.1356$. $\Box$

21. TRUE or FALSE? In this class, event-scheduling simulation programs usually take more coding effort than programs using the process-interaction approach.

    Solution: TRUE. $\Box$

22. What does "FEL" mean?

Solution: future events list. $\square$

23. What is the variance of a NORM(2,4) expression in Arena?

Solution: 16. $\square$

24. In Arena, what does NQ(Joe.Queue) do?

Solution: It indicates the number of customers waiting in the queue Joe.Queue. $\square$

25. What is the name of the Arena expression denoting the current time?

Solution: TNOW. $\square$

26. TRUE or FALSE? It is possible in the Arena PROCESS block to do a DELAY-RELEASE without a SEIZE.

Solution: TRUE. $\square$

27. TRUE or FALSE? In our call center example, some (but not all) of the servers could handle technical support on multiple product types.

Solution: TRUE. $\square$

28. Suppose that you want to evaluate the integral

$$I = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx.$$

What is the "exact" answer?

Solution: $\Phi(1) - \Phi(-1) = 0.6826$. $\square$

29. Suppose you didn't bring along your normal tables — darn! So now we'll estimate the above integral via Monte Carlo. The following numbers are a Unif(0,1) sample:

0.91 0.11 0.73 0.32

Use the Monte Carlo method from class to approximate the integral via the estimator $\hat{I}_4$.

Solution:

$$\begin{aligned}\hat{I}_4 &= \frac{b-a}{n}\sum_{i=1}^{n} f(a+(b-a)U_i) \\ &= \frac{2}{4}\sum_{i=1}^{4} f(-1+2U_i) \\ &= \frac{1}{2}\sum_{i=1}^{4} \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-(-1+2U_i)^2}{2}\right] \\ &= 0.656. \quad \text{(not bad!)} \quad \Box\end{aligned}$$

30. What is the expected value of the estimator $\hat{I}_4$?

Solution: By the answer to Question 28, $\mathsf{E}[\hat{I}_4] = I = 0.6826$. $\Box$

31. Joey works at an ice cream shop. Starting at time 0, five customer interarrival times are as follows (in minutes):

3 2 5 2 2

Customers are served in LIFO (last-in-first-out) fashion. The 5 customers order the following numbers of ice cream products, respectively:

6 2 3 1 4

Suppose it takes Joey 2 minutes to prepare each ice cream product. Further suppose that he charges \$4/ice cream. Sadly, the customers are unruly and each customer causes \$2 in damage for every minute the customer has to wait in line.

When does the first customer leave?

Solution:

cust	arrl time	start serv	serve time	depart	wait
1	3	3	12	15	0
2	5	31	4	35	26
3	10	25	6	31	15
4	12	23	2	25	11
5	14	15	8	23	1

Thus, the first customer leaves at time 15. □

32. Continuing the ice cream question, which customer is the second guy to be served?

    Solution: Customer 5. □

33. Still continuing the ice cream question, what is the average number of customers in the system during the first 15 minutes?

    Solution: Plot the number of customers in the system, $L(t)$, and compute $\frac{1}{15}\int_0^{15} L(t)\,dt = \frac{31}{15}$. □

34. Still still continuing, how much money will Joey make or lose with the above 5 customers?

    Solution: Profit $=$ 4(number of ice creams sold) $-$ 2(total wait times) $= 64 - 106 = -\$42$. □