Quiz 5 Solution

1. Consider the vectors $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$, $\vec{\mathbf{v}}_2 = \begin{pmatrix} -3 \\ 7 \\ 6 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} 3 \\ -1 \\ h \end{pmatrix}$. For which value(s) of h is $\vec{\mathbf{b}}$ in $\mathrm{Span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$? For which value(s) of h are the vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{b}}$ linearly independent?.

   Solution: We need to check when the system $[\vec{\mathbf{v}}_1\ \vec{\mathbf{v}}_2]\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has solution.

$$\begin{pmatrix} 1 & -3 & 3 \\ -2 & 7 & -1 \\ -4 & 6 & h \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & 5 \\ 0 & -6 & h+12 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & h+42 \end{pmatrix}.$$

   The system $[\vec{\mathbf{v}}_1\ \vec{\mathbf{v}}_2]\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent if and only if $h = -42$. Now, the vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{b}}$ are linearly independent if and only if, the system $[\vec{\mathbf{v}}_1\ \vec{\mathbf{v}}_2\ \vec{\mathbf{b}}]\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a unique solution, and according to the reduced matrix, this happens if and only if $h \neq -42$. ■

2. Let $\vec{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{\mathbf{e}}_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{\mathbf{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, be the standard basis for $\mathbb{R}^3$. Let T be the linear transformation that satisfies $T(\vec{\mathbf{e}}_1) = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$, $T(\vec{\mathbf{e}}_2) = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$, and $T(\vec{\mathbf{e}}_3) = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$. Determine if the transformation is one-to-one and if it is onto. Justify.

   Solution: The standard matrix for T is

$$[T(\vec{\mathbf{e}}_1)\ T(\vec{\mathbf{e}}_2)\ T(\vec{\mathbf{e}}_3)] = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 3 \end{pmatrix}.$$

   Redicing the matrix we get

$$\begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & -3 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

   Since there is a pivot in every row and in every column, then the transformation is both one-to-one and onto. ■

3. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 3 \end{pmatrix}.$$

   Prove that it is invertible and compute its inverse.

   Solution: The matrix is the same as in the previous example, so, since the associated transformation is one-to-one and onto, then the matrix is invertible. To compute the inverse we reduce the matrix alongside the identity matrix

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 3 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 7 & -2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 7 & 3 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 15 & 6 & 2 \\ 0 & 1 & 0 & 17 & 7 & 2 \\ 0 & 0 & 1 & 7 & 3 & 1 \end{array}\right)$$

Therefore $A^{-1} = \begin{pmatrix} 15 & 6 & 2 \\ 17 & 7 & 2 \\ 7 & 3 & 1 \end{pmatrix}$. ■