Math 1501 E, Fall 2013

Exam #1

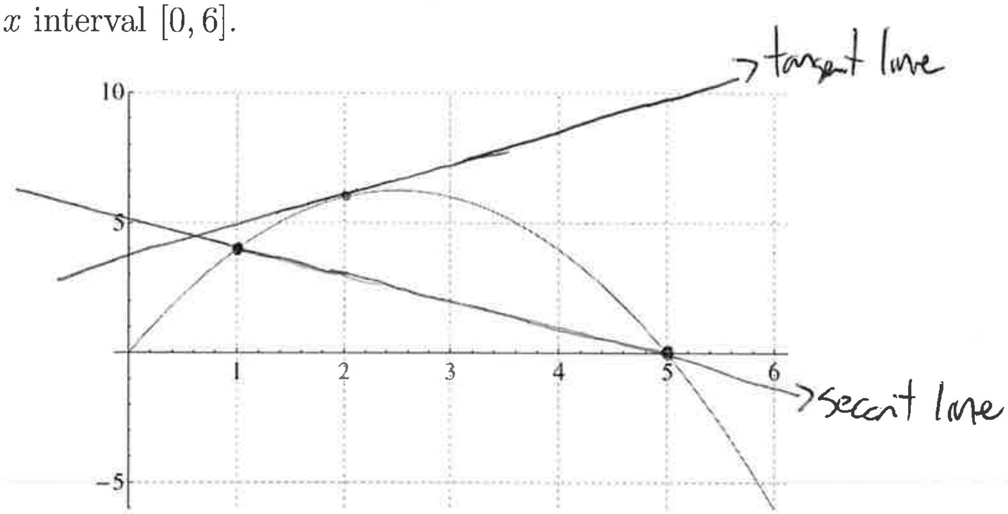
Name: Rubric

Section: ______

- You will have 50 minutes to complete the exam.
- No calculators, books, or notes allowed.
- Partial credit will be given. However, **no** credit will be given for a problem in which no work is shown, whether the answer is correct or not. Hence, show all applicable work.

Question:	1	2	3	4	5	Total
Points:	9	6	9	3	7	34
Score:						

1. Consider the function $f(x) = 5x - x^2$, the plot of which is shown below on the x interval $[0, 6]$.



(a) (4 points) Draw the secant line connecting the points at $x = 1$ and $x = 5$ on this curve. Then, find the average rate of change of $f(x)$ over the x interval $[1, 5]$, and explain how this quantity is related to the secant line you drew.

Average rate of change $= \frac{\Delta y}{\Delta x} = \frac{f(5)-f(1)}{5-1} = \frac{(5\cdot5-5^2)-(5\cdot1-1^2)}{4} = \boxed{-1}$

This average rate of change is the slope of the secant line.

(b) (5 points) Now, draw the tangent line to the function $f(x)$ at $x = 2$. Find the equation of this line. **Do not use techniques not presented in Chapter 2 in this class.**

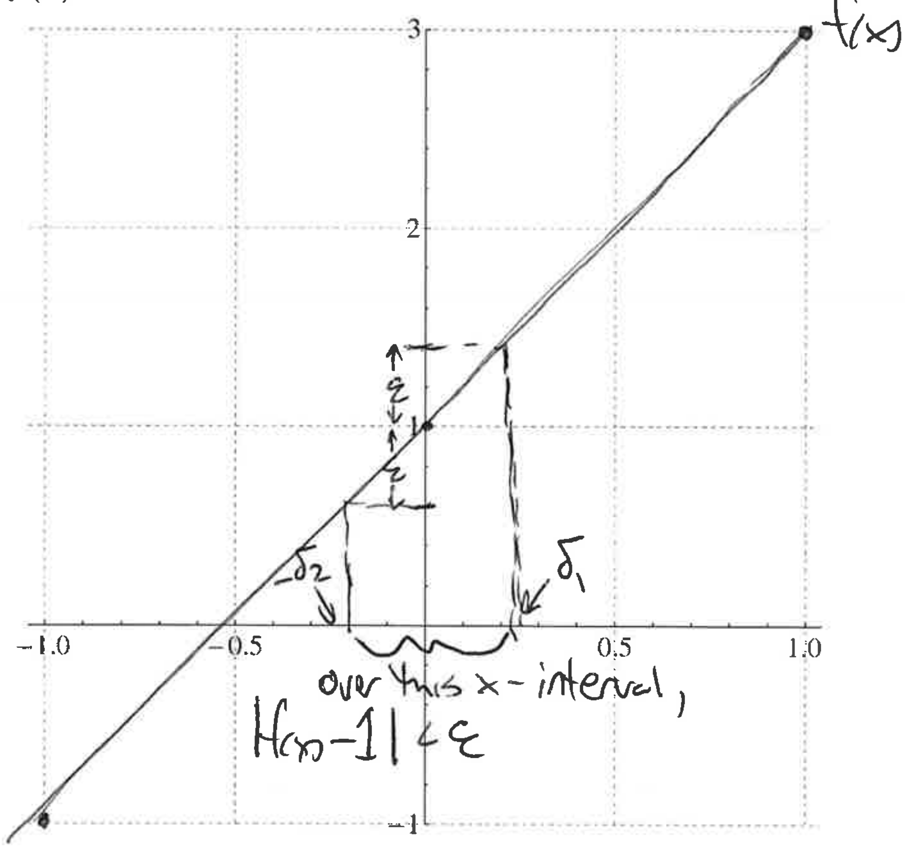
Slope of the tangent line is $\lim_{h\to0} \frac{f(2+h)-f(2)}{h} =$

$\lim_{h\to0} \frac{[5(2+h)-(2+h)^2]-[5\cdot2-2^2]}{h} = \lim_{h\to0} \frac{\cancel{10}+5h-\cancel{4}-4h-h^2-\cancel{10}+\cancel{4}}{h} =$

$\lim_{h\to0} \frac{h-h^2}{h} = \lim_{h\to0} 1-h = 1$. $y = 1x+b$ is the equation, and it has point $(2, 5\cdot2-2^2) = (2,6) \Rightarrow 6 = 2+b \Rightarrow b = 4$, $\boxed{y = x+4}$

2. Consider the function $y = f(x)$, where $f(x) = 2x + 1$. In this question, we will use the precise definition of a limit to prove that $\lim_{x \to 0} f(x) = 1$.

(a) (3 points) On the "graph paper" below, draw a plot of $f(x)$ over the interval $[-1, 1]$. Then, indicate graphically the range of x values for which $f(x)$ is within $\epsilon > 0$ of 1.



(b) (3 points) Now, algebraically determine an expression for $\delta > 0$ such that, if x is within δ of $x = 0$, then $f(x)$ is within $\epsilon > 0$ of 1.

Need $|f(x) - L| < \varepsilon \Rightarrow |2x + \cancel{1} - \cancel{1}| < \varepsilon \Rightarrow$

$-\varepsilon < 2x < \varepsilon \Rightarrow -\frac{\varepsilon}{2} < x < \frac{\varepsilon}{2}$. So, x needs to be within $\delta_1 = \delta_2 = \boxed{\delta = \frac{\varepsilon}{2}}$ of $x = 0$ for $f(x)$ to be within ε of 1.

3. Evaluate the following limits, or explain why they do not exist. **Do not use techniques we have not yet covered in this class.**

(a) (3 points) $\lim_{x\to 2} \frac{x^2+2x-8}{x^2-4}$

Rewrite $\frac{x^2+2x-8}{x^2-4} = \frac{(x+4)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$, cancel $x-2$, since

$x \neq 2 \Rightarrow \lim_{x\to 2} \frac{x+4}{x+2} = \frac{6}{4} = \boxed{\frac{3}{2}}$

(b) (3 points) $\lim_{\theta\to 0} \frac{1-\cos(2\theta)}{\theta^2}$. **Hint**: You may use $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$.

Rewrite: $\cos(2\theta) = \cos^2\theta - \sin^2\theta$; since $\cos^2\theta + \sin^2\theta = 1$,

$\cos(2\theta) = 1 - 2\sin^2\theta \Rightarrow \frac{1-\cos(2\theta)}{\theta^2} = \frac{2\sin^2\theta}{\theta^2} \Rightarrow$

$\lim_{\theta\to 0} \frac{2\sin^2\theta}{\theta^2} = 2\left(\cancel{\lim_{\theta\to 0}} \frac{\sin\theta}{\theta}\right)^2 = \boxed{2}$

(c) (3 points) $\lim_{x\to 0^+} x^3 \cos\left(\frac{2}{x}\right)$. **Hint**: Use the sandwich theorem, starting with the fact that $-1 \le \cos\left(\frac{2}{x}\right) \le 1$.

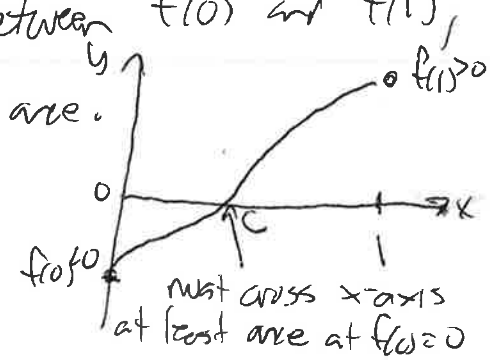
$-1 \le \cos\left(\frac{2}{x}\right) \le 1$. Since we are considering only positive x values (right hand limit as $x \to 0^+$), it is also true that $-x^3 \le x^3\cos\left(\frac{2}{x}\right) \le x^3$. $\lim_{x\to 0^+} (-x^3) = 0$; $\lim_{x\to 0^+} (x^3) = 0 \Rightarrow$

Since $x^3\cos\left(\frac{2}{x}\right)$ is sandwiched between $-x^3$ and x^3, and both of these have limit 0 as $x \to 0^+$, $\boxed{\lim_{x\to 0^+} x^3\cos\left(\frac{2}{x}\right) = 0}$, too

4. (3 points) A *continuous* function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Explain why the equation $f(x) = 0$ must have at least one solution (an x value for which the equation is true) between $x = 0$ and $x = 1$. Illustrate with a sketch.

The intermediate value theorem says that a continuous function $f(x)$ will take on all values between $f(a)$ and $f(b)$ on a finite interval $[a,b]$. In this case, $f(a) = f(0) < 0$ and $f(b) = f(1) > 0$, so, since 0 is between $f(0)$ and $f(1)$, $f(x)$ must equal zero in the $[0,1]$ interval at least once.

diagram $\longrightarrow$



5. (7 points) Find any vertical and horizontal asymptotes of the function $f(x) = \dfrac{2x+3}{x+5}$. Express these asymptotes as limits of the function.

Horizontal asymptotes: find $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{2x+3}{x+5} = \lim\limits_{x\to\infty} \dfrac{2+\frac{3}{x}}{1+\frac{5}{x}} = \boxed{2}$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{2+\frac{3}{x}}{1+\frac{5}{x}} = \boxed{2}$ also

So, horizontal asymptote at $\boxed{y=2}$

Vertical asymptotes: may occur when denominator is zero $\Rightarrow$ at $x=-5$

check by evaluating $\lim\limits_{x\to-5^+} f(x) = \lim\limits_{x\to-5^+} \dfrac{2x+3}{x+5} = \dfrac{-10+3}{0^+} = \dfrac{-7}{0^+} = \boxed{-\infty}$

$\lim\limits_{x\to-5^-} f(x) = \lim\limits_{x\to-5^-} \dfrac{-10+3}{0^-} = \dfrac{-7}{0^-} = \boxed{+\infty}$

So, yes, vertical asymptote at $\boxed{x=-5}$