Directions

1. You MUST justify all limits except for the following. But remember you must show that the " If " part is true.

$$\lim_{x \to \infty} \frac{A}{x^a} = 0 \text{ } \textit{where A is a constant} \textbf{ and } a > 0$$

If $f(x) \neq 0$ & $\lim_{x \to \infty} f(x) = \infty$ then $\lim_{x \to \infty} \frac{1}{f(x)} = 0$

If $f(x) \neq 0$ & $\lim_{x \to \infty} f(x) = 0$ then $\lim_{x \to \infty} \frac{1}{f(x)} = 0$

2. You can use $p-$ *integrals*: $\int_1^\infty \frac{dx}{x^p}$ Converges if $p > 1$ and Diverges if $p \leq 1$ (Just state that you are using a $p-$ *integral* to justify)

MATH 1552 - SPRING 2016
TEST 2 - SHOW YOUR WORK

NAME : ______________________________ TA : ______________________________

1. (20 points) Evaluate: $\int \boldsymbol{x} \, \mathbf{sin}(\boldsymbol{x}) \, \mathbf{cos}(\boldsymbol{x}) \, \boldsymbol{dx}$

$$\int \boldsymbol{x} \, \mathbf{sin}(\boldsymbol{x}) \, \mathbf{cos}(\boldsymbol{x}) \, \boldsymbol{dx} = \frac{1}{2} \, x \sin^2(x) - \frac{1}{2} \int \sin^2(x) \, dx \quad \textbf{(see (2) below)}$$

$$= \frac{\mathbf{1}}{\mathbf{2}} \, \boldsymbol{x} \, \mathbf{sin}^{\mathbf{2}}(\boldsymbol{x}) - \frac{\mathbf{1}}{\mathbf{4}} \boldsymbol{x} + \frac{\mathbf{1}}{\mathbf{8}} \mathbf{sin}(\mathbf{2} \, \boldsymbol{x}) + \boldsymbol{C}$$

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$u = x \qquad dv = \sin(x) \, \cos(x) \, dx$

$du = dx \qquad v = \dfrac{\mathbf{sin}^2(x)}{2}$ **(see (1) below)**

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$$\int \mathbf{sin}(\boldsymbol{x}) \, \mathbf{cos}(\boldsymbol{x}) \, \boldsymbol{dx} = \frac{\mathbf{sin}^{\mathbf{2}}(\boldsymbol{x})}{\mathbf{2}} \textbf{ . } \text{Use } u = \sin(x) \qquad \textbf{(1)}$$

$$\int \mathbf{sin}^{\mathbf{2}}(\boldsymbol{x}) \, \boldsymbol{dx} = \frac{\mathbf{1}}{\mathbf{2}} \, \boldsymbol{x} - \frac{\mathbf{1}}{\mathbf{4}} \, \mathbf{sin}(\mathbf{2} \, \boldsymbol{x}) \text{ . Use } \sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos(2 \, x) \qquad \textbf{(2)}$$

2. (20 points) Find the area under the graph of $y = x\,e^{-x}$ from $x = 0$ **to** $x = \mathit{infinity}$

a. $$\int_0^b \boldsymbol{x\,e^{-x}\,dx} = -\boldsymbol{e^{-x}}\,(\boldsymbol{x} + \mathbf{1}) = -\boldsymbol{e^{-b}}\,(\boldsymbol{b} + \mathbf{1}) + \mathbf{1}$$

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IBP

$$\int \boldsymbol{x\,e^{-x}\,dx} = -x\,e^{-x} + \int e^{-x}\,dx$$

$$= -x\,e^{-x} - e^{-x} = -\boldsymbol{e^{-x}}\,(\boldsymbol{x} + \mathbf{1})$$

$u = x \qquad dv = e^{-x}$

$du = dx \qquad v = -e^{-x}$
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b. $$\boldsymbol{Area} = \int_0^{\infty} \boldsymbol{x\,e^{x}\,dx} = \lim_{b \to \infty} \int_0^b \boldsymbol{x\,e^{-x}\,dx}$$

$$= \lim_{b \to \infty} \left(-\boldsymbol{e^{-b}}\,(\boldsymbol{b} + \mathbf{1}) + \mathbf{1} \right) = \mathbf{1} \quad \textbf{From a. above}$$

For this limit: as $b \to \infty$, $-e^{-b}(b+1) = -\dfrac{(b+1)}{e^b} \to 0 \quad \dfrac{\infty}{\infty}$ **use** *L'Hopital*

3. a. (10 points) **Does** $\displaystyle\int_{6}^{\infty} \frac{dx}{\sqrt{x^2 - 1}}$ **converge or diverge?** Use the **Direct Comparison Test** (DCT)

$$x^2 - 1 \le x^2 \;\Rightarrow\; \sqrt{x^2 - 1} \le x \;\Rightarrow\; \frac{1}{x} \le \frac{1}{\sqrt{x^2 - 1}}$$

$\displaystyle\int_{6}^{\infty} \frac{dx}{x}$ diverges because it is a p - *integral* with $p = 1$

$$\textbf{\textit{DCT}} \Rightarrow \int_{6}^{\infty} \frac{dx}{\sqrt{x^2 - 1}} \quad \textbf{Diverges}$$

b. (10 points) **Does** $\displaystyle\int_{0}^{\infty} \frac{dx}{e^x + e^{-x}}$ **Converge or diverge?** Use the **Limit Comparison Test** (LCT)

$$\int_{0}^{\infty} e^{-x}\,dx = \int_{0}^{\infty} \frac{dx}{e^x} \quad \text{converges}$$

$$\lim_{x\to\infty} \frac{\left(\dfrac{1}{e^x + e^{-x}}\right)}{e^{-x}} = \lim_{x\to\infty} \frac{e^x}{e^x + e^{-x}} \left(\frac{e^{-x}}{e^{-x}}\right) = \lim_{s\to\infty} \frac{1}{1 + e^{-2x}} = 1, \quad x \to \infty \Rightarrow e^{-2x} \to 0$$

$$\textbf{\textit{Since}} \int_{0}^{\infty} e^{-x}\,dx \;\; \textbf{\textit{converges, LCT}} \Rightarrow \int_{0}^{\infty} \frac{dx}{e^x + e^{-x}} \quad \textbf{\textit{converges}}$$

4. (20 points) Use the error estimates for the Trapezoid Rule and Simpson Rule. The integral is $\int_1^2 \frac{6}{x}\,dx$. DO NOT EVALUATE THIS INTEGRAL.

$$f(x) = \frac{6}{x} \Rightarrow f'(x) = -\frac{6}{x^2} \Rightarrow f''(x) = \frac{12}{x^3} \Rightarrow f^{(3)} = -\frac{36}{x^4} \Rightarrow f^{(4)}(x) = \frac{144}{x^5}$$

a. (10 points) Find the smallest value of n such that $|E_T| \leq 10^{-4}$

$$\max |f''(x)| = \max\left|\frac{12}{x^3}\right| \text{ over the interval } [1, 2] \text{ is } M = 12$$

$$\Rightarrow |E_T| \leq \frac{(12)(2-1)^3}{12\,n^2} = \frac{1}{n^2} \leq 10^{-4} \Rightarrow n^2 \geq 10^4 \Rightarrow n \geq 10^2 = 100$$

Smallest is $n = 100$

b. (10 points) Find the smallest value of n such that $|E_S| \leq 10^{-4}$. **Use $\sqrt[4]{0.8} \approx 0.94$**

$$\max|f^{(4)}(x)| = \max\left|\frac{144}{x^5}\right| \text{ over the interval } [1, 2] \text{ is } M = 144$$

$$\Rightarrow |E_S| \leq \frac{144\,(2-1)^5}{180\,n^4} \leq 10^{-4} \Rightarrow 0.8\left(\frac{1}{n^4}\right) \leq 10^{-4}$$

$$\Rightarrow n^4 \geq (0.8)\,10^4 \Rightarrow n \geq (0.94)\,(10) = 9.4$$

Smallest is $n = 10$

5. (20 points) Use L'Hopital Rule to evaluate the limits

a. $\lim_{x \to 1^+} x^{\frac{1}{x-1}}$ Indeterminant form 1^∞ because $\lim_{x \to 1^+} \frac{1}{x-1} = \infty$

** Evaluate $\lim_{x \to 1^+} \ln\left(x^{\frac{1}{x-1}}\right) = \lim_{x \to 1^+} \frac{\ln(x)}{x-1}$ $\frac{0}{0}$

$$= \lim_{x \to 1^+} \frac{\frac{1}{x}}{1} = 1 \Rightarrow \mathbf{\lim_{x \to 1^+} x^{\frac{1}{x-1}} = e^1 = e}$$

b. $\lim_{x \to 0^+} x \ln(x)$ Indeterminant form $0 \cdot \infty$. $x \ln(x) = \frac{\ln(x)}{\frac{1}{x}}$

$$\Rightarrow \lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} 2x = 0$$

$0 \cdot \infty$, then $\frac{\infty}{\infty}$ (use L'H), then simplify

$$\Rightarrow \mathbf{\lim_{x \to 0^+} x \ln(x) = 0}$$

c. $\lim_{x \to \infty} \frac{\left(3^{\frac{2}{x}} - 1\right)}{\frac{1}{x}}$ $\frac{0}{0}$

$$\mathbf{\lim_{x \to \infty} \frac{\left(3^{\frac{2}{x}} - 1\right)}{\frac{1}{x}}} = \lim_{x \to \infty} \frac{\ln(3)\, 3^{\frac{2}{x}} \left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x \to \infty} 2 \ln(3)\, 3^{\frac{2}{x}} = \mathbf{2 \ln(3)}$$