ISyE 2027 Exam # 3
Fall 2014

Name KEY

Please be neat and show all your work so that I can give you partial credit.
GOOD LUCK.

Question 1
Question 2
Question 3
Question 4
Total

(25) 1. Suppose X is a uniform random variable in the interval $(0,1)$ (i.e. $f(x) = 1$ for $0 < x < 1$ and 0 otherwise). Define $Y = 2X^2$ Compute

(a) (15) the cumulative distribution function of Y.

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

$$P(Y \leq y) = P(2X^2 \leq y) = P\left(X^2 \leq \frac{y}{2}\right) = P\left(X \leq \sqrt{\frac{y}{2}}\right) = \sqrt{\frac{y}{2}}$$

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \sqrt{\frac{y}{2}} & 0 < y < 2 \\ 1 & \text{if } y \geq 2 \end{cases}$$

(b) (10) the probability density function of Y.

$$f_y(y) = \begin{cases} \frac{1}{2\sqrt{2}} y^{-\frac{1}{2}} & 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(25) **2.** (a) (15) Suppose X and Y have the joint density function $f(x, y) = x + 2y^3$ for $0 < x < 1$ and $0 < y < 1$ and 0 otherwise. Compute the probability density function of X.

$$f_X(x) = \int_0^1 (x+2y^3)\,dy = xy + \frac{2y^4}{4}\Big|_0^1 = x + \frac{1}{2}$$

$$f_X(x) = \begin{cases} x+\frac{1}{2} & \text{if } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

(b) (10) Compute $E[X]$

$$E[X] = \int_0^1 x\left(x+\frac{1}{2}\right)dx = \frac{x^3}{3} + \frac{x^2}{4}\Big|_0^1$$

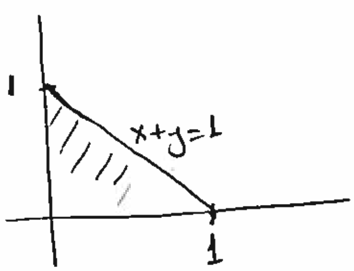
$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}$$

(25) **3.** (a) (10) Suppose X and Y have the joint density function $f(x, y) = 6xy^2$ for $0 < x < 1$ and $0 < y < 1$ and 0 otherwise. Are X and Y independent? Justify your answer.

Yes. X and Y are independent since the joint density function can be factored into two functions of x and y.

(b) (15) Compute $P\{X + Y < 1\}$.



$$P\{X+Y<1\} = \int_0^1 \int_0^{1-y} 6xy^2 \, dx \, dy$$

$$= \int_0^1 3x^2y^2 \Big|_0^{1-y} dy = 3\int_0^1 y^2(1-y)^2 \, dy$$

$$= 3\int_0^1 (y^2 - 2y^3 + y^4) \, dy$$

$$= 3\left(\frac{y^3}{3} - \frac{2y^4}{4} + \frac{y^5}{5}\right)\Big|_0^1$$

$$= 3\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right)$$

$$= 3\left(\frac{10-15+6}{30}\right) = \frac{1}{10}$$

(25) **4.** (15) (a) Suppose X and Y are independent random variables and take values 1, 2, 3, and 4 with probabilities 0.1, 0.2, 0.3, and 0.4. Compute the probability mass function of $X + Y$

$P\{X+Y=2\} = P\{X=1\}P\{Y=1\} = 0.01$

$P\{X+Y=3\} = P\{X=1\}P\{X=2\} + P\{X=2\}P\{X=1\} = 0.04$

$P\{X+Y=4\} = P\{X=1\}P\{Y=3\} + P\{X=3\}P\{Y=1\} + P\{X=2\}P\{Y=2\} = 0.1$

$P\{X+Y=5\} = P\{X=1\}P\{Y=4\} + P\{X=4\}P\{Y=1\} + P\{X=2\}P\{Y=3\} + P\{X=3\}P\{Y=2\} = 0.2$

$P\{X+Y=6\} = P\{X=2\}P\{Y=4\} + P\{X=4\}P\{Y=2\} + P\{X=3\}P\{Y=3\} = 0.25$

$P\{X+Y=7\} = P\{X=3\}P\{Y=4\} + P\{X=4\}P\{Y=3\} = 0.24$

$P\{X+Y=8\} = P\{X=4\}P\{Y=4\} = 0.16$

(b) (10) Compute $E[X + Y]$.

$E[X+Y] = 2\times 0.01 + 3\times 0.04 + 4\times 0.1 + 5\times 0.2 + 6\times 0.25 + 7\times 0.24 + 8\times 0.16$

$= 0.02 + 0.12 + 0.4 + 1 + 1.5 + 1.68 + 1.28$

$= 6.0$