ISyE 4803D – Spring 2014 – Final Exam – Solutions

Name (please print!)	**ID (please print!)**

- There are totally 5 problems, with total weight of 100 points + 10 bonus points for two non-obligatory parts of the problems.
  Note: each problem starts on a new page.
- All which is requested in all problems are answers (please write them as clearly as possible in the designated fields). You might add explanations; please use for your explanations clearly marked additional sheets.

Problem 1 [15 points] Pose optimization problem

$$\mathrm{Opt} = \min_{x \in \mathbf{R}^3} \left\{ \max[x_1+x_2, x_2+x_3, x_1+x_3] : \begin{array}{l} \max[x_1-5, |x_2-x_3|] \leq 1 \\ \min[x_1-2x_2, x_2+x_3] + \min[x_1,x_2,x_3] \geq -2 \\ x_1 - x_2 + x_3 = 1 \end{array} \right\}$$

as a Linear Optimization program.

Answer:

$$\mathrm{Opt} = \min_{x \in \mathbf{R}^3, t, v_1, v_2} \left\{ t : \begin{array}{l} t \geq x_1+x_2, t \geq x_2+x_3, t \geq x_1+x_3 \\ \hline x_1 - 5 \leq 1, x_2 - x_3 \leq 1, x_3 - x_2 \leq 1 \\ \hline x_1 - 2x_2 \geq v_1, x_2 + x_3 \geq v_1, x_1 \geq v_2, x_2 \geq v_2, x_3 \geq v_2, v_1 + v_2 \geq -2 \\ \hline x_1 - x_2 + x_3 = 1 \end{array} \right\}.$$

Problem 2 [20 points + 5 bonus points] Consider polyhedral set

$$X = \left\{ x \in \mathbf{R}^2 : \begin{array}{l} x_1 \le 0, x_2 \le 0, \\ -1 \le x_2 - x_1 \le 1 \end{array} \right\}$$

1. [3 points] Find the recessive cone of X and direction(s) of extreme ray(s) of this cone.
   Answer:

   $\mathrm{Rec}(X) =$

   Direction(s) of extreme ray(s) of $\mathrm{Rec}(X)$:

   Solution: The recessive cone of X is

   $$K = \{d = [d_1; d_2] : d_1 \le 0, d_2 \le 0, 0 \le d_2 - d_1 \le 0\} = \{[t; t] : t \le 0\}.$$

   The cone is its own extreme ray, and a direction of this ray is $[-1; -1]$

2. [2 points] Is X bounded?
   Y/N______
   Solution: No, X has a nontrivial recessive cone and thus is unbounded.

3. [10 points] List extreme points of X, if any
   Answer: Extreme points are

   Solution: at an extreme point v, 2 of the constraints with linearly independent vectors of coefficients among the constraints specifying X should become active. The options are:

   - $v_1 = 0, v_2 = 0 \Rightarrow$ extreme point $v^1 = [0; 0]$;
   - $v_1 = 0, v_2 - v_1 = -1 \Rightarrow$ extreme point $v^2 = [0; -1]$;
   - $v_1 = 0, v_2 - v_1 = 1 \Rightarrow v = [0; 1]$ – this point is not in X;
   - $v_2 = 0, v_2 - v_1 = -1 \Rightarrow v = [1; 0]$ – this point is not in X;
   - $v_2 = 0, v_2 - v_1 = 1 \Rightarrow$ extreme point $v^3 = [-1; 0]$.

4. [5 points] Represent X in the form $X = \mathrm{Conv}\{v^1, ..., v^M\} + \mathrm{Cone}\{r^1, ..., r^N\}$.
   Answer:

   $$X = \mathrm{Conv}\{[0; 0], [-1; 0], [0; -1]\} + \mathbf{R}_+ \cdot [-1; -1]$$

5. [non-obligatory; 5 bonus points]. What are the sets of those objectives $c = [c_1; c_2] \in \mathbf{R}^2$ for which problem

   $$\max_{x \in X} c^T x$$

- is solvable?
  Solution: these are c's which make nonpositive inner product with the only, up to positive factor, extreme direction $[-1;-1]$ of $\mathrm{Rec}(X)$, that is, c's with $c_1 + c_2 \geq 0$.
- has $[0;0]$ as an optimal solution?
  Solution: these are c's whicn make the problem solvable (i.e., with $c_1 + c_2 \geq 0$) and such that the maximum of values of the objective over the extreme points $[0;0]$, $[-1;0]$, $[0;-1]$ is attained at $[0;0]$, that is, with $-c_1 \leq 0$ and $-c_2 \leq 0$. Thus, the objectives in question are all nonnegative vectors c.

Note: An alternative way to get the above answers is just to draw a picture.

Problem 3 [15 points]. Fill the following table:

System of linear constraints	Feasibility [Y/N]	Certificate
$\begin{array}{rrrcl} 2x_1 & -x_2 & & \leq & 1 \\ & 2x_2 & -x_3 & \leq & 1 \\ -x_1 & & +2x_3 & \leq & 1 \\ x_1 & +x_2 & +x_3 & \leq & 1 \end{array}$	Y	$[0; 0; 0]$
$\begin{array}{rrrcl} 2x_1 & -x_2 & & \leq & 1 \\ & 2x_2 & -x_3 & \leq & 1 \\ -x_1 & & +2x_3 & \leq & 1 \\ x_1 & +x_2 & +x_3 & \geq & 3 \end{array}$	Y	$[1; 1; 1]$
$\begin{array}{rrrcl} 2x_1 & -x_2 & & \leq & 1 \\ & 2x_2 & -x_3 & \leq & 1 \\ -x_1 & & +2x_3 & \leq & 1 \\ x_1 & +x_2 & +x_3 & > & 3 \end{array}$	N	$[1; 1; 1; -1]$
$\begin{array}{rrrcl} 2x_1 & -x_2 & & \leq & 1 \\ & 2x_2 & -x_3 & \leq & 1 \\ -x_1 & & +2x_3 & \leq & 1 \\ x_1 & +x_2 & +x_3 & \geq & 4 \end{array}$	N	$[1; 1; 1; -1]$

Note: Certificate of feasibility of a system of constraints is a feasible solution to the system. Certificate of infeasibility is the vectors of weights of a legitimate aggregation of the constraints which yields a contradictory inequality.

Problem 4 [20 points]. You are given LO problem

$$\mathrm{Opt}(P) = \max_x \left\{ x_1 + x_2 + x_3 : \begin{array}{rrrcl} 2x_1 & -x_2 & & \leq & 1 \\ & 2x_2 & -x_3 & \leq & 1 \\ -x_1 & & +2x_3 & \leq & 1 \end{array} \right\} \qquad (P)$$

1. [5 points] Mark by "D" in the following list the problem dual to (P)

$A.$	$\mathrm{Opt}(D) = \max_\lambda \left\{ \lambda_1 + \lambda_2 + \lambda_3 : \begin{array}{rrrcl} 2\lambda_1 & & -\lambda_3 & = & 1 \\ -\lambda_1 & +2\lambda_2 & & = & 1 \\ & -\lambda_2 & +2\lambda_3 & = & 1 \\ \multicolumn{5}{c}{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0} \end{array} \right\}$	
$B.$	$\mathrm{Opt}(D) = \min_\lambda \left\{ \lambda_1 + \lambda_2 + \lambda_3 : \begin{array}{rrrcl} 2\lambda_1 & & -\lambda_3 & = & 1 \\ -\lambda_1 & +2\lambda_2 & & = & 1 \\ & -\lambda_2 & +2\lambda_3 & = & 1 \end{array} \right\}$	
$C.$	$\mathrm{Opt}(D) = \min_\lambda \left\{ \lambda_1 + \lambda_2 + \lambda_3 : \begin{array}{rrrcl} 2\lambda_1 & & -\lambda_3 & = & 1 \\ -\lambda_1 & +2\lambda_2 & & = & 1 \\ & -\lambda_2 & +2\lambda_3 & = & 1 \\ \multicolumn{5}{c}{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0} \end{array} \right\}$	
$D.$	$\mathrm{Opt}(D) = \min_\lambda \left\{ \lambda_1 + \lambda_2 + \lambda_3 : \begin{array}{rrrcl} 2\lambda_1 & & -\lambda_3 & \geq & 1 \\ -\lambda_1 & +2\lambda_2 & & \geq & 1 \\ & -\lambda_2 & +2\lambda_3 & \geq & 1 \\ \multicolumn{5}{c}{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0} \end{array} \right\}$	
$E.$	$\mathrm{Opt}(D) = \min_\lambda \left\{ \lambda_1 + \lambda_2 + \lambda_3 : \begin{array}{rrrcl} 2\lambda_1 & & -\lambda_3 & = & 1 \\ -\lambda_1 & +2\lambda_2 & & = & 1 \\ & -\lambda_2 & +2\lambda_3 & = & 1 \\ \multicolumn{5}{c}{\lambda_1 \leq 0, \lambda_2 \leq 0, \lambda_3 \leq 0} \end{array} \right\}$	

Solution: The dual problem is problem C.

2. [5 points] Is (P) feasible? If yes, certify its feasibility, otherwise certify infeasibility.
Answer:

Solution: (P) is feasible, a certificate being $x = [1; 1; 1]$.

3. [5 points] Is the dual problem feasible? If yes, certify it feasibility, otherwise certify infeasibility.
Answer:

Solution: The dual problem is feasible, a certificate being $\lambda = [1; 1; 1]$.

4. [5 points] Are the primal and the dual problems solvable? If yes, point out primal and dual optimal solutions and optimal values.
Answer:

Solution: The primal and the dual problems are solvable with optimal solutions $x^* = [1;1;1]$, $\lambda^* = [1;1;1]$. Indeed, these solutions are feasible for the respective problems, and the corresponding duality gap is

$$\sum_{i=1}^{3} \lambda_i^* - \sum_{i=1}^{2} x_i^* = 0,$$

which is a certificate of optimality for both solutions. Equivalent certificate is complementary slackness: products of λ_i^* and the residuals in the constraints of (P) as evaluated at our x^* are zeros. The optimal values in the problems are equal to 3.
The simplest way to solve the problems is to note that the dual problem has 3 equality constraints on 3 variables, and the resulting system of linear equations has a unique solution $\lambda^* = [1;1;1]$. By complementary slackness, the system of primal constraints should at every primal optimal solutuions to be satisfied as a system of equations, which immediately yields x^*.

Problem 5 [30 points + 5 bonus points]

1. [10 points] Mark by "C" those of the functions below which are convex on the indicated domains (think of the functions as equal to $+\infty$ outside of these domains):

#	Function and domain	Convexity [Y/N]
$A.$	$f(x) = \frac{1}{x_1} + \frac{1}{x_2} : \{x_1 > 0, x_2 > 0\} \to \mathbf{R}$	
$B.$	$f(x) = \frac{1}{x_1} + \frac{1}{x_2} : \{x_1 > 0, x_2 < 0\} \to \mathbf{R}$	
$C.$	$f(x) = \ln(e^{x_1+x_2} + e^{x_1-x_2}) : \mathbf{R}^2 \to \mathbf{R}$	
$D.$	$f(x) = e^{-x^2/2} : \mathbf{R} \to \mathbf{R}$	
$E.$	$f(x) = e^{-x^2/2} : \{x \geq 1\} \to \mathbf{R}$	
$F.$	$f(x) = e^{-x^2/2} : \{\lvert x\rvert \geq 1\} \to \mathbf{R}$	

The convex functions are A, C, E. Nonconvexity of B follows from considering restriction of this function on the ray $\{x_1 = 1, x_2 < 0\}$ of the domain – on this ray, the function is concave, but not convex. Now, for the function $\phi(x) = e^{-x^2/2}$ we have $\phi''(x) = [x^2 - 1]\,e^{-x^2/2}$, that is, the second order derivative is nonnegative when $|x| \geq 1$. This explains convexity of E and nonconvexity of D; as about F, there is no such thing as "convex function with nonconvex domain."

2. [10 points] Fill the following table

Inequality	Domain in $\mathbf{R}^3$	The inequality is valid everywhere in the domain [Yes/No]
$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \geq 6 - x_1 - x_2 - x_3$	$x > 0$	Y
$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \geq 6 - x_1 - x_2 - x_3$	$x_1 \neq 0, x_2 \neq 0, x_3 \neq 0$	N
$\frac{1}{x_1x_2x_3} \geq 4 - x_1 - x_2 - x_3$	$x > 0$	Y
[non-obligatory; 5 bonus points] $\frac{1}{x_1x_2x_3} \geq e^{3-x_1-x_2-x_3}$	$x > 0$	Y

Solution: The valid inequalities are the first, the third, and the fourth. The first two of the valid inequalities are Gradient Inequalities

$$\forall x \in X : f(x) \geq f(\bar{x}) + (x - \bar{x})^T \nabla f(\bar{x})$$

for convex functions $f(x)$, convex domains X and points $\bar{x} \in X$.
In the first inequality,

$$f(x) = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, X = \{x > 0\}, \bar{x} = [1; 1; 1]$$

In the third inequality,

$$f(x) = \frac{1}{x_1x_2x_3}, X = \{x > 0\}, \bar{x} = [1; 1; 1]$$

The fourth inequality is equivalent form of the Gradient Inequality written down for

$$f(x) = \ln(\frac{1}{x_1x_2x_3}), X = \{x > 0\}, \bar{x} = [1;1;1],$$

that is, the inequality

$$\forall(x > 0) : \ln(\frac{1}{x_1x_2x_3}) \geq -[x_1 - 1] - [x_2 - 1] - [x_3 - 1] := 3 - x_1 - x_2 - x_3$$

(take exponents of both sides).

The second inequality can be violated on the indicated domain, e.g., at the point $x = [-1; -1; -1]$.

3. [9 points] $a_1, ..., a_{10}$ are given positive reals. Find the optimal value and an optimal solution in the optimization problem

$$\mathrm{Opt} = \min\left\{\sum_{i=1}^{10} \frac{a_i}{x_i^2} : x > 0, \sum_{i=1}^{10} x_i^3 \leq 1\right\} \qquad (*)$$

Answer: The optimal value is
An optimal solution is
<u>Solution:</u> The problem is convex, so that a KKT point is an optimal solution. The KKT optimality conditions read

$$\begin{array}{ll}
(a) & \nabla_x \left[\sum_{i=1}^{10} \frac{a_i}{x_i^2} + \lambda\left[\sum_{i=1}^{10} x_i^3 - 1\right]\right] = 0\\
(b) & \lambda[\sum_{i=1}^{10} x_i^3 - 1] = 0\\
(c) & x > 0, \sum_i x_i^3 \leq 1, \lambda \geq 0.
\end{array}$$

From (a),

$$\lambda = \frac{2a_i}{3x_i^5}, \ i = 1, ..., 10$$

which combines with (b) to imply $\sum_{i=1}^{10} x_i^3 = 1$. Thus,

$$x_i = \mu a_i^{1/5}, i = 1, ..., 10 \qquad [\mu = \left(\tfrac{2}{3\lambda}\right)^{1/5}]$$

and

$$\mu^3 \sum_{i=1}^{10} a_i^{3/5} = 1 \Rightarrow \mu = \frac{1}{[\sum_{j=1}^{10} a_j^{3/5}]^{1/3}} \Rightarrow x_i = \frac{a_i^{1/5}}{\left[\sum_{j=1}^{10} a_j^{3/5}\right]^{1/3}}.$$

The vector x and the real μ we have found give rise to a KKT point $(x; \lambda = 2/(3\mu^5))$. Thus, an optimal solution and the optimal value are

$$x_i = \frac{a_i^{1/5}}{\left[\sum_{j=1}^{10} a_j^{3/5}\right]^{1/3}}, \ 1 \leq i \leq 10, \mathrm{Opt} = \left[\sum_{j=1}^{10} a_j^{3/5}\right]^{5/3}.$$

4. [1 point, follow-up to the previous item] Is the optimal solution to $(*)$ you have found the unique optimal solution to the problem?
Y/N_____
Solution: The optimal solution is unique due to strong convexity of the objective in our convex problem.

5. [non-obligatory, 5 bonus points] Given positive reals $a_1, ..., a_{10}$, consider the optimization problem

$$\text{Opt} = \min_x \left\{ \frac{1}{2} \sum_{i=1}^{10} \frac{a_i}{x_i^2} : x > 0, \sum_{i=1}^{10} \frac{1}{x_i} \geq 1 \right\} \qquad (!)$$

(a) [1 point] Is the problem convex?
Y/N_____
Solution: No – the problem includes a nonconvex constraint $\sum_j 1/x_j \geq 1$.

(b) [4 points] Find the optiml value and an optimal solution to the problem.
Answer: The optimal value is
An optimal solution is
Solution: Passing to the variables $y_i = 1/x_i$, the problem becomes the convex problem

$$\min_y \left\{ \frac{1}{2} \sum_{i=1}^{n} a_i y_i^2 : y > 0, 1 - \sum_i y_i \leq 0 \right\}$$

Finding KKT point, we get the system of relations

$$\begin{array}{l} a_i y_i - \lambda = 0,\ i = 1, ..., 10 \\ \lambda \left[1 - \sum_{i=1}^{10} \right] = 0 \\ y > 0, 1 - \sum_i y_i \leq 0, \lambda \geq 0 \end{array}$$

whence $\lambda > 0$, $y_i = \lambda/a_i$ for all i and $\sum_i y_i = 1$, that is, $\lambda = \frac{1}{\sum_i 1/a_i}$. Consequently, an optimal solution to the y-problem is

$$y_i = \frac{1}{a_i [\sum_{j=1}^{10} 1/a_j]}\ \forall i$$

and an optimal solution to the x-problem is

$$x_i = 1/y_i = a_i [\sum_j 1/a_j],$$

and the optimal value in the original problem is

$$\text{Opt} = \frac{1}{2 \sum_j 1/a_j}.$$