NAME →

ISyE 3044 — Fall 2013 — Test #1 Solutions

Revised 10/4/13

This test runs for 85 minutes. You can use one cheat sheet. Good luck!

1. Probability problems.

   (a) Consider a Poisson process with rate $\lambda = 2$ arrivals per minute. Find the probability that the fifth interarrival time is more than one minute.

   Solution: $X \sim \text{Exp}(2)$. So $\Pr(X > 1) = e^{-\lambda x} = e^{-2} = 0.1353$. $\diamondsuit$

   (b) Suppose $X \sim \text{Exp}(\lambda)$. Find the median of X, i.e., the point m such that $P(X \le m) = 0.5$.

   Solution: By definition of the median (and the exponential c.d.f.), we have $\Pr(X \le m) = 1 - e^{\lambda m} = 0.5$. Solving gives $m = -\ell\text{n}(0.5)/\lambda = 0.693/\lambda$. $\diamondsuit$

   (c) If X is a Bernoulli(0.3) random variable, find $\mathsf{E}\left[\frac{X}{X-2}\right]$.

   Solution: By the Unconscious Statistician,

   $$\mathsf{E}\left[\frac{X}{X-2}\right] \;=\; \sum_x \frac{x}{x-2}\, f(x) \;=\; \frac{1(0.3)}{1-2} + \frac{0(0.7)}{0-2} \;=\; -0.3. \quad \diamondsuit$$

   (d) Let X, Y be i.i.d. Exponential with rate 0.5. Find $P\{X + Y \ge 1\}$.

   Solution:

   $$\Pr(X+Y \ge 1) \;=\; \Pr(\text{Erlang}_2(0.5) \ge 1) \;=\; \sum_{i=0}^{k-1} \frac{e^{-\lambda x}(\lambda x)^i}{i!} \;=\; \sum_{i=0}^{1} \frac{e^{-0.5}(0.5)^i}{i!} \;=\; 0.910. \quad \diamondsuit$$

(e) If $X \sim \text{Nor}(2,1)$ and $Y \sim \text{Nor}(3,1)$, and X and Y are independent, find $\Pr(X > Y)$.

Solution:

$$\begin{aligned}
\Pr(X > Y) &= \Pr(X - Y > 0) \\
&= \Pr(\text{Nor}(-1,2) > 0) \\
&= \Pr(\text{Nor}(0,1) > 1/\sqrt{2}) \\
&= 0.240. \quad \diamondsuit
\end{aligned}$$

(f) Suppose $X_1, X_2, \ldots, X_{100}$ are i.i.d. with mean 2 and variance 4. What is the approximate probability that the sample mean is between 1.8 and 2.2?

Solution: By the CLT, $\bar{X} \approx \text{Nor}(2,\, 0.04)$. Thus,

$$\begin{aligned}
\Pr\left(1.8 \le \bar{X} \le 2.2\right) &= \Pr\left(\frac{1.8 - 2.0}{\sqrt{0.04}} \le \frac{\bar{X} - 2.0}{\sqrt{0.04}} \le \frac{2.2 - 2.0}{\sqrt{0.04}}\right) \\
&\approx \Pr(-1 \le \text{Nor}(0,1) \le 1) \\
&= \Phi(1) - \Phi(-1) \;=\; 0.6826. \quad \diamondsuit
\end{aligned}$$

(g) Here's a fun fact (which you don't have to prove): If $X_1, X_2, \ldots, X_n$ are i.i.d. normal, then it turns out that the sample mean $\bar{X}$ and sample variance S^2 are independent. Are $\bar{X}$ and S^2 uncorrelated?

Solution: YES (since independence implies uncorrelated). $\diamondsuit$

(h) If $X \sim \text{Bern}(p)$, find $\mathsf{Corr}(X^2, X)$.

Solution: First of all,

$$\mathsf{E}[X^k] \;=\; \sum_x x^k f(x) \;=\; 1^k p + 0^k (1-p) \;=\; p \quad \text{for all } k.$$

This implies that

$$\mathsf{Corr}(X^2, X) \;=\; \frac{\mathsf{Cov}(X^2, X)}{\sqrt{\mathsf{Var}(X^2)\mathsf{Var}(X)}}$$

$$= \frac{\mathsf{E}[X^3] - \mathsf{E}[X^2]\mathsf{E}[X]}{\sqrt{(\mathsf{E}[X^4] - \mathsf{E}^2[X^2])(\mathsf{E}[X^2] - \mathsf{E}^2[X])}}$$

$$= \frac{p - p^2}{\sqrt{(p-p^2)(p-p^2)}} = 1. \quad \diamondsuit$$

In retrospect, the answer is pretty obvious (since X can only equal 0 or 1).

2. Short-answer ARENA / language questions.

   (a) Is Arena a Process-Interaction (P-I) or Event-Scheduling (E-S) language?

   Solution: P-I. $\diamondsuit$

   (b) TRUE or FALSE? Arena maintains a future event list.

   Solution: TRUE. $\diamondsuit$

   (c) TRUE or FALSE? You can use the CREATE block to generate negative interarrival times.

   Solution: FALSE. $\diamondsuit$

   (d) Suppose $X = i$ with probability $i/6$ for $i = 1, 2, 3$. How would you generate this random variable in Arena using the DISC function?

   Solution: DISC(0.1667,1, 0.5,2, 1.0,3). $\diamondsuit$

   (e) In Arena, how do you generate a normal random variable with mean -3 and variance 9?

   Solution: NORM(-3,3). $\diamondsuit$

   (f) YES or NO? Is it possible for an entity to seize more than one unit of a resource at the same time?

Solution: YES. $\diamondsuit$

(g) How would you determine how many customers are currently in the queue barber.queue?

Solution: NQ(barber.queue). $\diamondsuit$

(h) TRUE or FALSE? SEIZE – DELAY – RELEASE blocks can all be found in the Advanced Process template.

Solution: TRUE. $\diamondsuit$

(i) TRUE or FALSE? You can have customers from 4 different CREATE blocks feeding into the same PROCESS block.

Solution: TRUE. $\diamondsuit$

(j) TRUE or FALSE? Two different customers can have two different values for the same attribute.

Solution: TRUE. $\diamondsuit$

3. Random number / variate questions.

(a) Consider the linear congruential generator $X_{i+1} = (5X_i + 1)\text{mod}(16)$. Using $X_0 = 1$, calculate the first pseudo-random number U_1.

Solution: $X_1 = (5X_0 + 1)\text{mod}(16) = 6$, and this implies that $U_1 = X_1/m = 6/16 = 0.375$. $\diamondsuit$

(b) Consider the linear congruential generator $X_{i+1} = (5X_i + 1)\text{mod}(16)$. Using $X_0 = 1$, calculate the pseudo-random number U_{801}.

Solution: $X_0 = 1$, $X_1 = 6$, $X_2 = 15$, $\ldots$, $X_{15} = 0$, $X_{16} = 1$. Thus, this generator has a period of 16; and so $X_1 = X_{17} = \cdots = X_{801} = 6$. Finally, this implies that $U_{801} = X_{801}/m = 6/16 = 0.375$. $\diamondsuit$

(c) TRUE or FALSE? The midsquare uniform generator is pretty reasonable.

Solution: FALSE. $\diamondsuit$

(d) If Z is a standard normal random variable with c.d.f. $\Phi(z)$, find the distribution of $3\Phi(Z)+2$.

Solution: By the Inverse Transform Theorem, $\Phi(Z) \sim \text{Unif}(0,1)$, so that $3\Phi(Z)+2 \sim \text{Unif}(2,5)$. $\diamondsuit$

(e) If U is Unif(0,1), what's the distribution of $-3\,\ell\text{n}(U)$?

Solution: Exp(1/3). $\diamondsuit$

4. Suppose we're interested in evaluating the integral $I = \int_0^3 x^2\,dx$ using Monte Carlo integration.

(a) Use the following $n = 4$ Unif$(0,1)$ random numbers to come up with the usual estimator $\hat{I}_n$ for I:

0.25 0.73 0.98 0.12

Solution:

$$\begin{aligned} \hat{I}_n &= \frac{b-a}{n}\sum_{i=1}^{n} f(a+(b-a)U_i) \\ &= \frac{3}{4}\sum_{i=1}^{4} f(3U_i) \\ &= \frac{3}{4}\sum_{i=1}^{4} 9U_i^2 \\ &= 10.599. \quad \diamondsuit \end{aligned}$$

(b) What is the expected value of $\hat{I}_n$?

Solution: $I = 9$. $\diamondsuit$

(c) Re-do Question 4a using the alternative "antithetic" estimator,

$$\tilde{I}_n \equiv \frac{b-a}{n} \sum_{i=1}^{n} f(a+(b-a)(1-U_i)).$$

Solution: Now we have

$$\begin{aligned}\tilde{I}_n &= \frac{3}{4}\sum_{i=1}^{4} f(3(1-U_i)) \\ &= \frac{3}{4}\sum_{i=1}^{4} 9(1-U_i)^2 \\ &= 9.519. \quad \diamondsuit\end{aligned}$$

(d) Would you expect $\hat{I}_n$ and $\tilde{I}_n$ to positively or negatively correlated?

Solution: Negatively. $\diamondsuit$

(e) Combine your answers in Questions 4a and 4c in a smart way.

Solution: Use $\bar{I}_n \equiv \frac{\hat{I}_n+\tilde{I}_n}{2} = 10.059$. Actually, this answer happens to be inferior to the $\tilde{I}_n$ value, but it was just unlucky this time — usually, this combined estimator does better! $\diamondsuit$

5. Spectators at a Britney Spears concert have to get two things done in order to attend the concert:

(i) Go to the ticket window to purchase their tickets, and

(ii) Then go to the metal detection area to be checked for firearms.

Note that a concert-goer may have to stand in line at one or both of the stations. Suppose that the first concert-goer arrives at the ticket window at time 0. The next four concert-goers arrive at the ticket window with the following interarrival times:

22 48 9 94

The corresponding five service times at the ticket window are

52 31 18 49 64

The concert-goers immediately proceed to the firearms detection area. The five service times at this station are

72 51 13 59 24

(a) How many concert-goers experience a wait in either the ticket queue or firearms detection queue?

(b) When does the fifth concert-goer complete service at the firearms station?

(c) What is the average time-in-system for the five concert-goers?

Solution. Let's set up the usual type of table, except that I'll run the customers on the columns. . . .

customer	1	2	3	4	5
arrive time	0	22	70	79	173
start ticket	0	52	83	101	173
wait at ticket	0	30	13	22	0
ticket service time	52	31	18	49	64
leave ticket / arrive gun	52	83	101	150	237
start gun	52	124	175	188	247
wait at gun	0	41	74	38	10
gun service time	72	51	13	59	24
leave gun	124	175	188	247	271
time in system	124	153	118	168	98

The table immediately gives us the following solutions: (a) 4; (b) 271; (c) 132.2. ♢

6. *The Music Man* questions.

(a) How many trombones were at the big parade?

Solution: 76. See www.youtube.com/watch?v=ODu888i14-I. ♢

(b) How many cornets were close at hand?

Solution: 110. See www.youtube.com/watch?v=ODu888i14-I. $\diamondsuit$

(c) Multiple choice: Where is Prof. Harold Hill's "home sweet home"?

i. Louisiana
ii. Paris, France
iii. Gary, Indiana
iv. New York
v. Rome

Solution: Gary. See www.youtube.com/watch?v=acmSExn9O9g. Note that the other choices are explicitly mentioned in the song as *not* being correct! $\diamondsuit$