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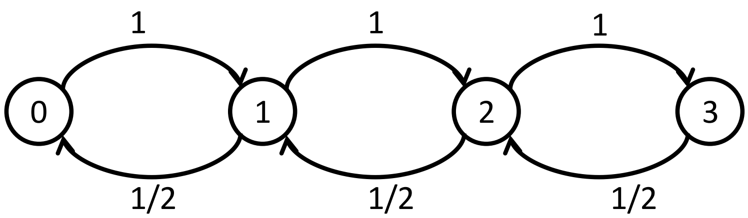
Solutions to Homework 13

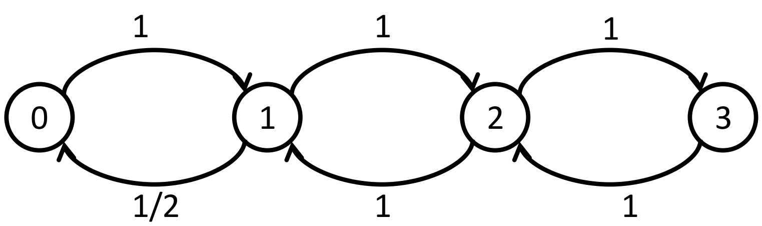
1. Let the state variable represent the number of customers in the system.

(a) The transition diagram is as below:



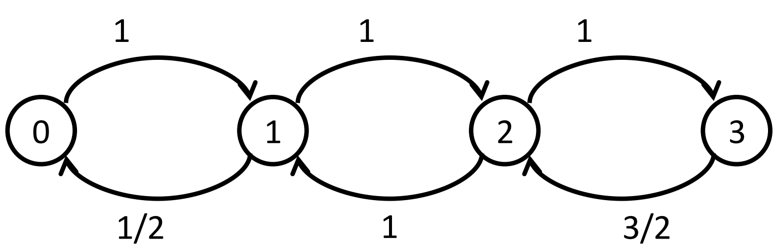

Solving for $\pi$ using the "cuts," we get $\pi = (1/15, 2/15, 4/15, 8/15)$. The throughput is $\lambda_{eff} = 1 \times (1 - \pi_3) = 7/15$.
The average queue length is $L = \pi_2 \times 1 + \pi_3 \times 2 = 4/3$. By Little's Law, $W = L/\lambda_{eff} = 20/7$.

(b) The transition diagram is as below:



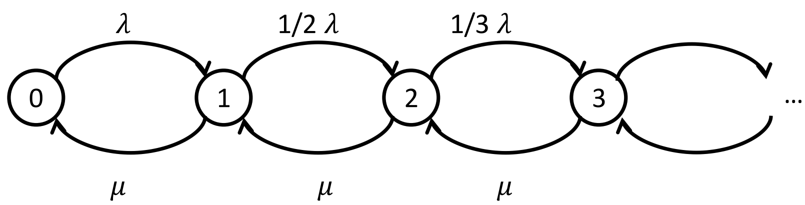

From this diagram, $\pi = (1/7, 2/7, 2/7, 2/7)$. The throughput is $\lambda_{eff} = 1 \times (1 - \pi_3) = 5/7$.
The average queue length is $L = \pi_3 \times 1 = 2/7$. By Little's Law, $W = L/\lambda_{eff} = 2/5$.

(c) The transition diagram is as below:



$\pi = (3/19, 6/19, 6/19, 4/19)$. The throughput is $\lambda_{eff} = 1 \times (1 - \pi_3) = 15/19$. The average queue length is $L = 0$. So the average waiting time is 0.

2. Let the state variable represent the number of customers in the system. The transtion diagram is as below:

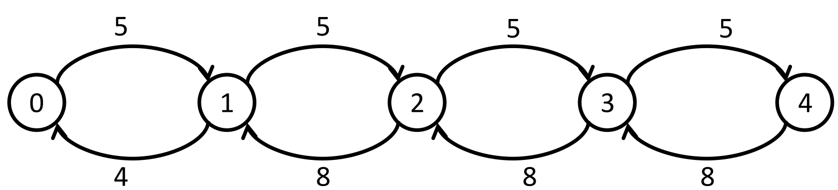



Using the cuts, we know: $\pi_i \cdot \frac{1}{1+i}\lambda = \pi_{i+1}\mu$ for $i = 0, 1, \cdots$. We also know, $\sum_{i=0}^{\infty} \pi_i = 1$. From these equations, we can obtain: $\pi_0 = \frac{1}{\sum_{i=0}^{\infty} \frac{1}{i!}(\frac{\lambda}{\mu})^i} = e^{-\lambda/\mu}$. The last equation comes from the fact $\sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$. For $i > 0$, $\pi_i = \frac{1}{i!}(\frac{\lambda}{\mu})^i \pi_0$.

The expected number of customers in the system is:

$$\sum_{i=1}^{\infty} i\pi_i = \sum_{i=1}^{\infty} \frac{1}{(i-1)!}(\frac{\lambda}{\mu})^i \pi_0 = \sum_{i=0}^{\infty} \frac{1}{i!}(\frac{\lambda}{\mu})^i \frac{\lambda}{\mu}\pi_0 = \frac{\lambda}{\mu}$$

3. The state represents the number of calls in the system. The transition diagram is as below:




(a) The generator is as below:

$$\begin{bmatrix} 5 & -5 & 0 & 0 & 0 \\ 4 & -9 & 5 & 0 & 0 \\ 0 & 8 & -13 & 5 & 0 \\ 0 & 0 & -13 & 5 & -8 \\ 0 & 0 & 0 & 8 & -8 \end{bmatrix}$$

(b) Using the cuts method, we know

$$\begin{aligned}\pi_0 \times 5 &= \pi_1 \times 4 \\ \pi_1 \times 5 &= \pi_2 \times 8 \\ \pi_2 \times 5 &= \pi_3 \times 8 \\ \pi_3 \times 5 &= \pi_4 \times 8 \\ 1 &= \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4\end{aligned}$$

The stationary distribution is $\pi = (.261, .327, .204, .128, .080)$.

(c) An arrival will be accepted if the system is not full (state 4). $\mathbb{P}(\text{accept}) = 1 - \pi_4 = .920$.

(d) An ambulance is dispatched directly if there is no call or one call in the system. $\mathbb{P}(\text{dispatch directly}) = \pi_0 + \pi_1 = .588$.

(e) There are 1 and 2 calls waiting in the system when the state is 3 and 4, respectively. No waiting in state 0, 1, or 2. Thus, the average waiting is: $1 \times \pi_3 + 2 \times \pi_4 = .288$.

(f) As in part (b), a call will be accepted with probability .920. There are three cases when a call being accepted:

i. When the call arrives and finds there are no call or one call in the system, his or her waiting time is zero.

ii. When the call arrives and finds there are 2 callers in the system, he or she needs to wait until 1 of the 2 finishes the service, which is the minimum of two exponential random variables. In this case, the average waiting time is 15/2. This happens with probability $\pi_2 = .204$.

iii. When the call arrives and finds there are 3 callers in the system, he or she needs to wait until 2 of the 3 finish the service, which is two times the minimum of two exponential random variables. The average waiting time is $2 \times 15/2 = 15$. This happens with probability $\pi_3 = .128$.

For those accepted to the system, the average waiting time is $\frac{.204 \times 15/2 + .128 \times 15}{.92} = 3.75$

(g) As argued in (f), the waiting time the sum of two $exp(1/7.5)$ random variables.

$$\mathbb{P}(W < 20) = \int_0^{20} 1/7.5 e^{-1/7.5x} \int_0^{20-x} 1/7.5 e^{-1/7.5y} dy dx = .191$$