MATH 2603, Fall 2015, Midterm Exam 2, Nov 5 2015: Closed book, no calculators.
Instructor: Esther Ezra.

Answer all questions **on this sheet**.

Name	GT IDnumber	Section Number

Problem 1. (15 points) We are given a $2 \times n$ board, which we can cover with either 1×2 or 2×1 tiles. That is, we can "break" the board to either horizontal or vertical tiles. Let t_n be the number of possibilities to break such a board.

a. Write a recurrence equation for t_n, as well as the initial conditions for $n = 1, 2$. What is your conclusion?

This is a Fibonacci recurrence:

$$t_n = t_{n-1} + t_{n-2},$$

with initial conditions $t_1 = 1$**,** $t_2 = 2$**.**

b. Solve the equation you formed in part a, in order to have an explicit form for t_n.

This is a shifted Fibonacci sequence, as the initial conditions are $t_0 = F_2$**,** $t_1 = F_3$**. Therefore**

$$t_n = F_{n+2} = \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2} \right)^{n+2} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+2} \right).$$

Problem 2. (20 points) Consider the alphabet consisting of the ten digits $\{0, 1, \ldots, 9\}$ and the five capital letters $\{'A', 'B', \ldots, 'E'\}$.

a. How many strings of length 9 can be generated if repetitions of symbols is *not* permitted?

$$\frac{15!}{6!}.$$

b. Same question when repetition of symbols is permitted.

$$15^9.$$

c. How many strings of length 9 can be formed using exactly two A's, three B's and four E's?

$$\frac{9!}{2!3!4!}.$$

d. How many strings of length 9 can be formed using exactly two 1's, three A's and four B's if the four B's are required to occur consecutively in the string?

Here the four consecutive B's are treated as a single character. Then we have 6 characters overall. Number of strings is thus:

$$\frac{6!}{2!3!}.$$

Problem 3. (20 points) How many integer-valued solutions are there to each of the following equations and inequalities?

a. $x_1 + x_2 + x_3 + x_4 = 40$, s.t. $x_i \geq 0$, for all $i = 1, \dots, 4$.

$$\binom{40+4-1}{3} = \binom{43}{3}.$$

b. $x_1 + x_2 + x_3 + x_4 = 40$, s.t. $x_i \geq 0$, for all $i = 1, \dots, 3$, and $x_4 = 1$.

The equation becomes $x_1 + x_2 + x_3 = 39$. Therefore, number of solutions is

$$\binom{39+3-1}{2} = \binom{41}{2}.$$

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c. $x_1 + x_2 + x_3 + x_4 = 40$, s.t. $x_i > 0$, for all $i = 1, \dots, 3$, $x_4 \geq -1$.

Define a new variable $x_4' = x_4 + 2 > 0$. We now have $x_1 + x_2 + x_3 + x_4' = 42$, and all variables are positive. Therefore number of solutions:

$$\binom{42-1}{3} = \binom{41}{3}.$$

d. $x_1 + x_2 + x_3 + x_4 \leq 40$, s.t. $x_i \geq 0$, for all $i = 1, \dots, 3$, and $x_4 \geq 3$.

Define a new variable $x_4' = x_4 - 3 \geq 0$. The equation becomes: $x_1 + x_2 + x_3 + x_4' \leq 37$, and all variables are non-negative. Therefore number of solutions:

$$\binom{37+4-1}{3} = \binom{40}{3}.$$

Problem 4. (10 points) Use induction in order to show that for all integers $n > 0$, $7^n - 3^n$ is divisible by 4.

Base case: $n = 1$. **Here we have** $7^1 - 3^1 = 4$**, which is divisible by 4. For** $n > 1$**, we assume by induction that** $7^n - 3^n$ **is divisible by 4, and show that** $7^{n+1} - 3^{n+1}$ **is also divisible by 4. Indeed,**

$$7^{n+1} - 3^{n+1} = (3+4)7^n - 3 \cdot 3^n = 3(7^n - 3^n) + 4 \cdot 7^n.$$

The first term is divisible by 4 due to the induction hypothesis, and the second term is a multiplication of 4, then it is obviously divisible by 4. Therefore $7^{n+1} - 3^{n+1}$ **is divisible by 4, as asserted.**

Problem 5. (10 points) We are given a universe U with $|U| = 100$, and three sets $A, B, B \subseteq U$ with $|A| = 30$, $|B| = 25$, $|C| = 30$. Each pair of sets among A, B, C have 15 elements in common, and there are 5 elements common to all these sets.

a. (5 points) Find $|A \cup B \cup C|$.

Using Exclusion-inclusion principle:

$$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C| = 30 + 25 + 30 - 3 \cdot 15 + 5 = 45.$$

b. (5 points) How many elements are contained only in C and not in A nor B?

This is the number of elements in $C \setminus (A \cup B)$, which is $C \setminus ((A \cup B) \cap C)$. Thus

$$|C \setminus ((A \cup B) \cap C)| = |C| - (|A \cap C| + |B \cap C| - |A \cap B \cap C|) = 30 - (15 + 15 - 5) = 5.$$

Problem 6. (10 points)

a. (5 points) Prove the Binomial Theorem, that is, for any x, y and any natural number n:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

$(x+y)^n = (x+y)\cdot(x+y)\cdots(x+y)$, **for** n **times. This product contains** n **closes, in each of which we choose either** x **or** y**. For any integer parameter** $0 \le k \le n$**, if we choose exactly** k x**'s, then we have** $n-k$ y**'s, resulting in the term** $x^k y^{n-k}$**, and the number of such possibilities is** $\binom{n}{k}$**. Therefore this contributes an overall factor of** $\binom{n}{k} x^k y^{n-k}$ **to the product. Running** k **from** 0 **to** n **yields the assertion of the Binomial Theorem.**

b. (5 points) Using part a, prove that the number of subsets chosen among a collection of n items is 2^n.

Here we need to choose $x = y = 1$**. In this case we obtain the identity:**

$$2^k = \sum_{k=0}^{n} \binom{n}{k}.$$

In the right hand side, $\binom{n}{k}$ **represents the number of subsets of size** k **among and** n**-element set, since we run** k **from** 0 **to** n**, we in fact obtain all possible subsets. The left hand side is** 2^n**, so the number of all subsets is** 2^n **as asserted.**

Problem 7. (15 points) True-False. Mark in the left Margin.

1. The following identity is always correct: $\binom{n}{k} = \sum_{i=0}^{n-k} \binom{n-1-i}{k-1}$, for any pair of integers $1 \leq k \leq n$. **T. (By the Hockey Stick Theorem on Pascal's triangle.)**
2. The system $x \equiv 3 \pmod 5$, $x \equiv 5 \pmod 8$ has a unique solution $\mod 40$. **T**
3. $n!$ and 3^n have the same order of growth. **F**
4. Multiplying two numbers of n digits each can be performed in $O(n)$ time. **F. (Best known algorithm is that of Karatsuba.)**
5. The Merge procedure on two sorted lists of n and m numbers, respectively, takes $n + m - 1$ comparisons in the worst case. **T**
6. Let f be a function from the Reals to the Integers, s.t. $f(x) = \lfloor x + 1 \rfloor$. Then f is onto the Integers. **T**