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ISyE 3044 — Spring 2013 — Test #1 Solutions

This test is 90 minutes. You're allowed one cheat sheet. **Just show your extremely neat answers.** All questions are 3 points, except #34, which is 1 point. Good luck!

1. Suppose I conduct a series of 4 independent experiments, each of which has a 20% chance of success. What's the probability that I'll see at least 2 successes?

   Solution: The number of successes $X \sim \text{Bin}(4, 0.2)$. Thus,

   $$\mathsf{Pr}(X \geq 2) \;=\; \sum_{x=2}^{4} \binom{4}{x} (0.2)^x (0.8)^{4-x} \;=\; 0.181. \quad \Box$$

2. Suppose $X_1, \ldots, X_n$ are i.i.d. from a $\text{Pois}(\lambda = 3)$ distribution. What is the *approximate* distribution of the sample mean $\bar{X}$ for large enough n?

   Solution: The CLT implies that

   $$\bar{X} \;\approx\; \text{Nor}\Big(\mu, \frac{\sigma^2}{n}\Big) \;\sim\; \text{Nor}\Big(3, \frac{3}{n}\Big). \quad \Box$$

3. Toss two dice and observe their sum. What is the expected number of tosses until you observe a sum of 3?

   Solution: Let $X \sim \text{Geom}(p)$ denote the number of tosses, where $p = \text{Pr}(\text{sum} = 3) = 2/36$. Thus, $\mathsf{E}[X] = 18$. $\Box$

4. If $X \sim \text{Bern}(0.5)$, find $\mathsf{E}[\ell\text{n}(X + 1)]$.

   Solution: By the Unconscious Statistician, $\mathsf{E}[\ell\text{n}(X+1)] = \frac{1}{2}\ell\text{n}(1) + \frac{1}{2}\ell\text{n}(2) = 0.347$. $\Box$

5. If X has a mean of -2 and a variance of 3, find $\mathsf{Var}(-X+1)$.

   Solution: $\mathsf{Var}(-X+1) = \mathsf{Var}(X) = 3$. $\square$

6. Suppose that X and Y are identically distributed with a mean of -2, a variance of 3, and $\mathsf{Cov}(X,Y) = 1$. Find $\mathsf{Corr}(X,Y)$.

   Solution: $\mathsf{Corr} = \mathsf{Cov}/\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)} = 1/3$. $\square$

7. Again suppose that X and Y are identically distributed with a mean of -2, a variance of 3, and $\mathsf{Cov}(X,Y) = 1$. Find $\mathsf{Var}(X+Y)$.

   Solution: $\mathsf{Var}(X) + \mathsf{Var}(Y) + 2\mathsf{Cov} = 8$. $\square$

8. Consider a Poisson process with rate $\lambda = 2$. What is the probability that the time between the 5th and 6th arrivals is less than 1?

   Solution: Denote the time between the arrivals by $X \sim \text{Exp}(2)$. Then $\Pr(X < 1) = 1 - e^{-\lambda x} = 1 - e^{-2} = 0.865$. $\square$

9. YES or NO? If X and Y are independent *geometric* random variables, are they necessarily uncorrelated?

   Solution: YES. $\square$

10. If X is Nor(3,4), find $\mathsf{Pr}(X < 1)$.

    Solution: $\Pr(X < 1) = \Pr(Z < \frac{1-3}{\sqrt{4}}) = \Pr(Z < -1) = 0.1587$. $\square$

11. If X and Y are i.i.d. standard normal random variables, find $\mathsf{Pr}(X - Y > 1)$.

    Solution: Note that $X - Y \sim \text{Nor}(0,2)$. Then $\Pr(X - Y > 1) = \Pr(Z > \frac{1-0}{\sqrt{2}}) \approx 0.24$. $\square$

12. Suppose X and Y are i.i.d. Exp(2). Find $\mathsf{Pr}(X+Y \le 1)$.

    Solution: Note that $X + Y \sim \text{Erlang}_{k=2}(\lambda = 2)$. Thus,

    $$\Pr(X+Y \le 1) \;=\; 1 - \sum_{i=0}^{k-1} \frac{e^{-\lambda x}(\lambda x)^i}{i!} \;=\; 1 - \sum_{i=0}^{1} \frac{e^{-2}(2)^i}{i!} \;=\; 1 - 3e^{-2} \;=\; 0.594. \quad \Box$$

13. Suppose U_1 and U_2 are i.i.d. Unif(0,1) random variables. What does $\lceil 6U_1 \rceil + \lceil 6U_2 \rceil$ do? (Recall that $\lceil x \rceil$ is the "ceiling" function.)

    Solution: This is just the sum of two dice. $\Box$

14. If U_1 and U_2 are i.i.d. Unif(0,1) random variables, what is the distribution of $-\frac{1}{5}\ell\text{n}\{U_1\} - \frac{1}{5}\ell\text{n}\{U_2\}$?

    Solution: $\text{Exp}(5) + \text{Exp}(5) = \text{Erlang}_2(5)$. $\Box$

15. What does FEL stand for?

    Solution: Future Event List. $\Box$

16. TRUE or FALSE? Arena is primarily a process-interaction discrete-event computer simulation language.

    Solution: TRUE. $\Box$

17. What Arena variable tells you how many people are in the queue `joey.queue`?

    Solution: `NQ(joey.queue)`. $\Box$

18. In Arena, is `TNOW` a variable or an attribute?

    Solution: It's a variable. $\Box$

19. TRUE or FALSE? In an Arena `PROCESS` block, it is possible to do a `SEIZE-DELAY` *without* an accompanying `RELEASE`.

    Solution: TRUE. $\Box$

20. TRUE or FALSE? An Arena `DECIDE` block can be used to probabilistically or conditionally route entities to more than 2 destinations.

    Solution: TRUE. $\Box$

21. TRUE or FALSE? In Arena, you can use the *same* server (resource) in multiple `PROCESS` blocks.

    Solution: TRUE. $\Box$

22. Calculate the integral $I = \frac{1}{\sqrt{2\pi}} \int_0^2 e^{-x^2/2}\, dx$.

    Solution: This is the integral of the Nor(0,1) p.d.f. Thus, $I = \Phi(2) - \Phi(0) = 0.4773$. $\Box$

23. Again consider the integral I from Question 22. Now use the following four $\text{Unif}(0,1)$ random numbers to compute a Monte Carlo estimate of I:

    $$0.78 \quad 0.15 \quad 0.33 \quad 0.84$$

    Solution: By class notes, we'll use

    $$\begin{aligned} \hat{I}_n &= \frac{b-a}{n} \sum_{i=1}^{n} f(a + (b-a)U_i) \\ &= \frac{2-0}{4} \sum_{i=1}^{4} f(2U_i) \\ &= \frac{1}{2} \sum_{i=1}^{4} \frac{1}{\sqrt{2\pi}} e^{-4U_i^2/2} \\ &= 0.459. \quad \Box \end{aligned}$$

24. Now re-do Question 24, except this time use the following "antithetic" Unif(0,1)'s (i.e., $1 - U$).

$$0.22 \quad 0.85 \quad 0.67 \quad 0.16$$

Solution: Using the U_i's from Question 23 and the same manipulations, except with $1 - U_i$, we have

$$\begin{aligned}\hat{I}'_n &= \frac{b-a}{n}\sum_{i=1}^{n} f(a+(b-a)(1-U_i)) \\ &= \frac{1}{2}\sum_{i=1}^{4}\frac{1}{\sqrt{2\pi}}e^{-4(1-U_i)^2/2} \\ &= 0.499. \quad \Box\end{aligned}$$

25. Finally, what is the difference between your "exact" answer from Question 22 and the *average* of your Monte Carlo answers from Questions 23 and 24?

Solution: The average of the previous two MC answers minus the exact answer is $0.479 - 0.477 = 0.002$, a remarkably small difference! The reason is that the antithetic run "balances out" the original run. $\Box$

26. Consider the ridiculous pseudo-random number generator $X_{i+1} = (5X_i + 1)\text{mod}(8)$. If $X_0 = 3$, calculate X_2.

Solution: $X_1 = (5X_0 + 1)\text{mod}(8) = 16\,\text{mod}(8) = 0$, and then we have $X_2 = (5X_1 + 1)\text{mod}(8) = 1\,\text{mod}(8) = 1$. $\Box$

27. Suppose I inscribe a circle of radius 1/2 in a unit square. Now I randomly toss 100 darts in the square and 76 happen to land in the circle. Use this sample to give me an estimate of π.

Solution: Let $\hat{p}_n = 0.76$ be the proportion of darts that hit the circle. Then we know that $\hat{\pi}_n = 4\hat{p}_n = 3.04$. $\Box$

28. If X and Y are i.i.d. Unif(0,1), draw a (rough) picture of the p.d.f. of $X/(X - Y)$.

Solution: The p.d.f. tails off at $\pm\infty$, and has asymptotes at 0 and 1; the p.d.f. is zero on [0,1]. See HW #2. $\Box$

29. Suppose that the probability that the Georgia Tech basketball team will win its first game of the season is 0.5. Also suppose that if the team wins game i ($i = 1, 2, \ldots$), then the team becomes very confident and will win game $i+1$ with probability 0.8. However, if the team loses game i, it becomes discouraged and will win game $i+1$ with probability of only 0.5. Name the probability distribution corresponding to the number of games the team will have to play before they get their first victory.

   Solution: The number of games is Geom(0.5). $\Box$

30. Consider the basketball set-up in Question 29. We will conduct Monte Carlo sampling to see how many games GT wins. To do so, suppose that I generously give you the following 10 Unif$(0,1)$ random numbers; call them $U_1, U_2, \ldots, U_{10}$:

   0.834 0.168 0.958 0.574 0.374 0.656 0.773 0.203 0.142 0.139

   Our simulation will declare that GT wins game i if $U_i < p_i$, where p_i is the conditional probability that GT wins game i (as discussed in Question 29). Using the above random numbers, determine how many games the team will have to play before they capture their 3rd victory.

   Solution: $p_1 = 0.5$, so $U_1 = 0.834$ corresponds to a loss.
   Then $p_2 = 0.5$, so $U_2 = 0.168$ corresponds to a win.
   Then $p_3 = 0.8$, so $U_3 = 0.958$ corresponds to a loss.
   Then $p_4 = 0.5$, so $U_4 = 0.574$ corresponds to a loss.
   Then $p_5 = 0.5$, so $U_5 = 0.374$ corresponds to a win.
   Then $p_6 = 0.8$, so $U_6 = 0.656$ corresponds to a win.

   Thus, it took 6 games. $\Box$

31. Suppose that two types of customers arrive at a single-server queue. Type-B (Georgia Tech) customers have priority over Type-A (UGA) customers (though nobody gets pre-empted if they're already being served.) Otherwise, service is FIFO within each type class. Assume the system starts out empty and idle.

Customer	Type	Interarrival time	Service time
1	A	3	13
2	A	6	8
3	B	5	4
4	B	3	6
5	B	11	2
6	A	6	8

When does the last customer leave the system?

Solution: Let's construct the following table, where we give priority to Type-B's.

Cust.	Type	Arrival time	Start Serv.	Service time	Depart	Time in Sys.
1	A	3	3	13	16	13
2	A	9	26	8	34	25
3	B	14	16	4	20	6
4	B	17	20	6	26	9
5	B	28	34	2	36	8
6	A	34	36	8	44	10

Thus, the last customer leaves at time 44. □

32. Continuing with Question 31, what is the average number of customers in the system by the time the last guy leaves?

    Solution: There are various ways to do this (e.g., you can draw the usual graph we do in class with all of the events), but the easiest way is just to add up all of the customer times-in-system and divide by the length of the simulation. Thus, the answer is $71/44 = 1.61$. □

33. Still continuing with Question 31, what is the average additional amount of time type-B's would have to spend in the system if they didn't have priority?

    Solution: Now let's see what happens when we don't have priorities.

Cust.	Type	Arrival time	Start Serv.	Service time	Depart	Time in Sys.
1	A	3	3	13	16	13
2	A	9	16	8	24	15
3	B	14	24	4	28	14
4	B	17	28	6	34	17
5	B	28	34	2	36	8
6	A	34	36	8	44	10

In the table from Problem 31 (where Type-B's have priority), we see that the average amount of time that the Type-B's spend in the system is $23/3 = 7.67$.

Similarly, from the table in this problem (where there is no priority), we see that the average amount of time that the Type-B's spend in the system is $39/3 = 13$.

So the difference is $16/3 = 5.33$. $\square$

34. What is the secret word?

Solution: Ramanujan. See the website
`http://en.wikipedia.org/wiki/Srinivasa_Ramanujan`. $\square$