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Homework 10

November 5, 2013

(Due: at the start of class on Tuesday/Wednesday, November 12/13)

1. In a study of the effect of advertising on brand shifting in the U.S. brewing industry, a Markov chain model was estimated to represent the chance of consumers shifting preferences among Anheuser-Busch (state 1), Miller (state 2) and "other " (state 3) beers from 1978 to 1979. The one-step transition matrix is

$$P = \begin{pmatrix} 0.9950 & 0.0046 & 0.0004 \\ 0.0000 & 0.9971 & 0.0029 \\ 0.0084 & 0.0522 & 0.9394 \end{pmatrix}$$

   Clearly brand loyalty is quite strong since the diagonal elements are all near 1. Suppose that there were 6 million beer drinkers in 1978, divided equally among the three brands. Answer the following questions:

   (a) How many consumers are expected to prefer Miller products in 1979?

   (b) If the transition matrix does not change over time (typically it does) and the number of beer drinkers is unchanged (certainly not true), how many consumers are expected to prefer Anheuser-Busch products in the long-run?

2. Consider the reflected random walk on state space $\{0, 1, 2, \dots\}$ with the following transition probabilities: $p_{00} = q + r$, $p_{01} = p$ and $p_{i,i-1} = q$, $p_{ii} = r$, $p_{i,i+1} = p$ for $i \geq 1$, where $p + q + r = 1$.

   (a) Is the Markov chain periodic or aperiodic. Explain and if it is periodic also give the period.

   (b) Is the Markov chain irreducible? Explain.

   (c) Find the stationary distribution when $p = 0.2$, $q = 0.4$, $r = 0.4$. Is the stationary distribution unique?

   (d) For the probabilities given in part (c), is the Markov chain positive recurrent? If so, why? If not, why not?

   (e) Is the Markov chain positive recurrent when $p = 0.5$, $q = 0.2$, $r = 0.3$? If so, why? If not, why not?

3. Let X be a Markov chain with state space $\{a, b, c, d, e, f\}$ and transition probabilities given by

$$P = \begin{pmatrix} .3 & .5 & 0 & 0 & 0 & .2 \\ 0 & .5 & 0 & .5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & .3 & 0 & 0 & 0 & .7 \\ .1 & 0 & .1 & 0 & .8 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   (a) Draw the transition diagram.

   (b) List the recurrent states.

   (c) List the irreducible set(s) and give their periods. (Hint: An irreducible set is a subset of state space where each state in the set communicates with each other.)

   (d) List the transient states.

4. For the Markov chain with state space $\{1, 2, 3, 4, 5\}$ and the following one-step transition matrix below,

$$P = \begin{pmatrix} 0.1 & 0.3 & 0.4 & 0.0 & 0.2 \\ 0.5 & 0.1 & 0.1 & 0.0 & 0.3 \\ 0.8 & 0.0 & 0.0 & 0.2 & 0.0 \\ 0.0 & 0.1 & 0.0 & 0.9 & 0.0 \\ 0.3 & 0.1 & 0.1 & 0.0 & 0.5 \end{pmatrix}$$

   (a) Classify as recurrent or transient the states of the Markov chains by first finding all of the irreducible subsets of states.

Extra (Littlefield Game 2): Suppose we agree to deliver an order in one day. The contract states that if we deliver the order within one day we receive \$1000. However, if the order is late, we lose money proportional to the tardiness until we receive nothing if the order is two days late. The length of time for us to complete the order is exponentially distributed with mean 0.7 days. For notation, let T be the length of time until delivery.

(a) What are the probabilities that we will deliver the order within one day and within two days?

(b) What is the expected tardiness?

(c) Notice that this contract is the one in the Littlefield simulation game 1. We now add two more contracts (which similarly have the rule of losing money proportional to tardiness):

   - price = \$750; quoted lead time = 7 days; maximum lead time = 14 days.
   - price = \$1250; quoted lead time = 0.5 day; maximum lead time = 1 days.

   Notice that in the \$1000 contract, quoted lead time = 1 day; maximum lead time = 2 days. What are the expected revenues for these three contracts? Which contract would be the most lucrative assuming the mean time to delivery is 0.7 days?

(d) Suppose the mean time to delivery is exponentially distributed with mean δ days. For what values of δ is the shortest \$1250 contract optimal, for what values is the medium length \$1000 contract optimal, and for what values is the longest \$750 contract optimal?