Math 2401, Fall 2014
Exam 1, (Sept 11, 2014)
Time Limit: 50 Minutes

Name : ______________________

GT Id : ______________________

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. Also, sign the Honor Code pledge at the bottom of this page, and follow the instructions below.

- On this exam you may **not** use your books, notes, or any electronic device other than a calculator.
- **Show all your work.** A correct answer not supported by calculations and/or explanation will receive no credit. An incorrect answer supported by substantially correct calculations and explanation may receive partial credit.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- If you need more space, use the back of the pages; clearly indicate when you have done so.

Problem	Points	Score
1	10	
2	9	
3	8	
4	11	
5	12	
Total:	50	

Honor Code Pledge: By signing below, you are verifying that you understand and uphold the Georgia Tech honor code.

Signature: ______________________

1. (a) (5 points) Find $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, where $\mathbf{a}=\mathbf{j}-\mathbf{k}$, $\mathbf{b}=2\mathbf{i}+\mathbf{k}$ and $\mathbf{c}=\mathbf{i}+2\mathbf{j}-3\mathbf{k}$.

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 0 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 2 & -3 \end{vmatrix} \quad \Big\} \text{(2 points)}$$

$$= 0 - (1)\begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} \quad \Big\} \text{(2 points)}$$

$$\left.\begin{aligned} &= -(-6-1) - (4-0) \\ &= 7-4 \\ &= 3 \end{aligned}\right\} \text{(1 point)}$$

(b) (5 points) Find the vector projection of $\mathbf{u}=\mathbf{i}+3\mathbf{j}$ onto $\mathbf{v}=\mathbf{i}+\mathbf{j}$.

$$\left.\begin{aligned} \text{proj}_{\vec{v}}\vec{u} &= \left(\vec{u}\cdot\frac{\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|} \\ &= \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} \\ &= \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} \end{aligned}\right\} \text{(2 points)}$$

$$= \frac{1(1)+3(1)}{1(1)+1(1)}(\vec{i}+\vec{j}) \quad \Big\} \text{(2 points)}$$

$$\left.\begin{aligned} &= \frac{4}{2}(\vec{i}+\vec{j}) \\ &= 2\vec{i}+2\vec{j} \end{aligned}\right\} \text{(1 point)}$$

2. (a) (5 points) Find the distance from the origin to the straight line $x = t$, $y = 1 - t$, $z = 2$.

The line passes through $P(0,1,2)$ and is parallel to $\vec{v} = \langle 1, -1, 0 \rangle$.

$\therefore$ Distance from $O(0,0,0)$ to the line $= \dfrac{|\vec{OP} \times \vec{v}|}{|\vec{v}|}$ } (2 points)

$\vec{OP} \times \vec{v} = \langle 0, 1, 2 \rangle \times \langle 1, -1, 0 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{vmatrix} = 2\vec{i} + 2\vec{j} - \vec{k}$ } (2 points)

$|\vec{OP} \times \vec{v}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$

$|\vec{v}| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$ } (1 point)

$\therefore$ The required distance is $\dfrac{3}{\sqrt{2}}$.

(b) (4 points) Find the point where the straight line through $A(1, 2, 0)$ and $B(2, 2, -1)$ intersects the yz-plane.

(2 points) { $\vec{AB} = \langle 2-1, 2-2, -1-0 \rangle = \langle 1, 0, -1 \rangle$

$\therefore$ Parametric equations for the straight line through A & B are: $\begin{cases} x = 1 + t(1) \\ y = 2 + t(0) \\ z = 0 + t(-1) \end{cases}$

i.e. $x = 1 + t$, $y = 2$, $z = -t$.

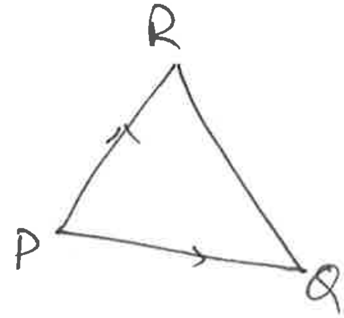
(1 point) { The equation of the yz-plane is $x = 0$.

(1 point) { So at the intersection point, $1 + t = 0$ i.e $t = -1$.

$\therefore$ The point of intersection is $(1+t, 2, -t) = (0, 2, 1)$

3. Let $P = (1,1,0)$, $Q = (1,0,1)$ and $R = (0,1,1)$.

(a) (6 points) Find an equation for the plane through P, Q, R.



(1 point) $\overrightarrow{PQ} = \langle 1-1,\ 0-1,\ 1-0 \rangle = \langle 0, -1, 1 \rangle$

$\overrightarrow{PR} = \langle 0-1,\ 1-1,\ 1-0 \rangle = \langle -1, 0, 1 \rangle$

2 pts. $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = -\vec{i} - \vec{j} - \vec{k}$,

which is normal to the plane through P, Q, R.

Using the point $P(1,1,0)$ & this normal vector, equation of the plane is

2 pts $(-1)(x-1) + (-1)(y-1) + (-1)(z-0) = 0$

or $-(x-1+y-1+z) = 0$

1 pt. i.e. $x+y+z-2=0$.

(b) (2 points) Find the area of ΔPQR.

area of $\Delta PQR = \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}|$ 1.5 pts

$= \frac{1}{2}\sqrt{(-1)^2+(-1)^2+(-1)^2}$ 0.5 pts

$= \frac{\sqrt{3}}{2}$.

4. A projectile is fired from a height of 1 m above the ground with an initial speed of $40\sqrt{2}\ m/sec$, making an angle of 45^o with the horizontal.

   (a) (7 points) Find the position vector for the path of the projectile. (Do not use the direct formula; take $g = 10\ m/sec^2$.)

1 We take $\vec{r}(0) = 0\vec{i} + (1)\vec{j} = \vec{j}$

2
$$\vec{v}(0) = 40\sqrt{2}\cos 45^\circ\,\vec{i} + 40\sqrt{2}\sin 45^\circ\,\vec{j}$$
$$= 40\sqrt{2}\cdot\frac{1}{\sqrt{2}}\vec{i} + 40\sqrt{2}\,\frac{1}{\sqrt{2}}\vec{j} = 40\vec{i} + 40\vec{j}$$

2 Integrating $\vec{a}(t) = -g\vec{j}$, we get
$$\vec{v}(t) = -gt\,\vec{j} + \vec{v}(0) = -gt\,\vec{j} + 40\vec{i} + 40\vec{j}$$
$$= 40\vec{i} + (40-10t)\vec{j}$$

2 Again by integration,
$$\vec{r}(t) = 40t\,\vec{i} + (40t - 5t^2)\,\vec{j} + \vec{r}(0)$$
$$= 40t\,\vec{i} + (40t - 5t^2)\vec{j} + \vec{j}$$
$$= (40t)\vec{i} + (1 + 40t - 5t^2)\vec{j}.$$

   (b) (4 points) What distance does it cover downrange when it reaches the maximum height?

2 At the maximum height, y-component of $\vec{v}(t) = 0$

i.e. $40 - 10t = 0$

$\therefore\ t = 4.$

2 In 4 seconds, the projectile covers $40t = 40(4) = 160$ downrange.

5. Let C be the curve $\mathbf{r}(t) = \sin 3t\mathbf{i} + \cos 3t\mathbf{j} - 4t\mathbf{k}$.

(a) (4 points) Find the unit tangent vector of the curve.

1 $\vec{r}\,'(t) = (3\cos 3t)\vec{i} - (3\sin 3t)\vec{j} - 4\vec{k}$

1 $|\vec{r}\,'(t)| = \sqrt{9\cos^2 3t + 9\sin^2 3t + 16} = \sqrt{9+16} = 5$

2 Unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \left(\frac{3}{5}\cos 3t\right)\vec{i} - \left(\frac{3}{5}\sin 3t\right)\vec{j} - \frac{4}{5}\vec{k}$

(b) (4 points) Find the length of the portion of the curve from $(0, 1, 0)$ to $(1, 0, -2\pi/3)$.

1 These two points correspond to $t = 0$ & $t = \pi/6$ respectively.

2 $\therefore$ length of the given portion of the curve $= \displaystyle\int_0^{\pi/6} |\vec{r}\,'(t)|\,dt$

1 $= \displaystyle\int_0^{\pi/6} 5\,dt = \Big[5t\Big]_0^{\pi/6}$

$= 5\pi/6$.

(c) (4 points) Find parametric equations for the line that is tangent to C at $t = 0$.

1 At $t=0$, $\vec{r}\,'(t) = \vec{r}\,'(0) = 3\vec{i} - (0)\vec{j} - 4\vec{k} = \langle 3, 0, -4\rangle$

1 Also $t=0$ corresponds to the point $(0, 1, 0)$ on the curve.

2 The tangent line to the curve at $t=0$ thus passes through $(0,1,0)$ & is parallel to the vector $\langle 3, 0, -4\rangle$. So its parametric equations are:

$$\begin{cases} x = 0 + t(3) \\ y = 1 + t(0) \\ z = 0 + t(-4) \end{cases} \quad \text{i.e.} \quad \begin{cases} x = 3t \\ y = 1 \\ z = -4t \end{cases}$$