MATH 1552 QUIZ 4, FALL 2015, GRODZINSKY

Print Your Name: Key-1

T.A.: (circle one) Miheer Brandon Stephen Kabir

1. (a) (10 points) Find the third degree Taylor polynomial for the function $f(x) = \cos x$ about $x = \frac{\pi}{3}$.

$a = \pi/3, \quad n = 3$

K	$f^{(k)}(x)$	$f^{(k)}(\frac{\pi}{3})$	$\frac{f^{(k)}(\frac{\pi}{3})}{k!}(x-\pi/3)^k$
0	$\cos x$	$1/2$	$1/2$
1	$-\sin x$	$-\sqrt{3}/2$	$-\sqrt{3}/2\,(x-\pi/3)$
2	$-\cos x$	$-1/2$	$-1/4\,(x-\pi/3)^2$
3	$\sin x$	$\sqrt{3}/2$	$\sqrt{3}/12\,(x-\pi/3)^3$

So $\boxed{P_3(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}(x-\frac{\pi}{3}) - \frac{1}{4}(x-\frac{\pi}{3})^2 + \frac{\sqrt{3}}{12}(x-\frac{\pi}{3})^3}$

(b) (10 points) Determine the maximum error in your approximation if we try to approximate $f(\frac{\pi}{2})$ using your polynomial in part (a) (you do NOT need to calculate the approximate function value). Recall: $|R_n(x)| \leq \max|f^{(n+1)}(c)|\frac{|x-a|^{n+1}}{(n+1)!}$. Simplify as far as you can without a calculator.

Here, $n=3$, $a=\frac{\pi}{3}$, and $x=\frac{\pi}{2}$, so:

$$|R_3(\pi/2)| \leq \max|f^{(4)}(c)|\,\frac{|\frac{\pi}{2}-\frac{\pi}{3}|^4}{4!}$$

Note that $f^{(4)}(x) = \cos x$ is <u>decreasing</u> on $[\pi/3, \pi/2]$,

so $|f^{(4)}(c)| \leq \cos\frac{\pi}{3} = \frac{1}{2}$

$$\Rightarrow |R_3(\pi/2)| \leq \frac{1}{2}\cdot\frac{(\pi/6)^4}{24}$$

$$= \boxed{\frac{1}{48}\left(\frac{\pi}{6}\right)^4}$$ max error

2. (15 points) Determine if the alternating series below converges absolutely, converges conditionally, or diverges. JUSTIFY YOUR ANSWER fully using the convergence tests from class. The justification will count for the majority of the points.

$$\sum_{k=3}^{\infty}(-1)^k\frac{1}{(k-2)^{1/3}}$$

Note that $\sum_{K=3}^{\infty}\frac{1}{(K-2)^{1/3}}$ diverges by Basic Comparison:

$(K-2)^{1/3} < K^{1/3}$, so $\frac{1}{(K-2)^{1/3}} > \frac{1}{K^{1/3}}$. Since $\sum_{K=3}^{\infty}\frac{1}{K^{1/3}}$

diverges (p-series, $p=\frac{1}{3}<1$), $\sum_{K=3}^{\infty}\frac{1}{(K-2)^{1/3}}$ also diverges.

But since our series is:

alternating

and $\lim_{n\to\infty}\frac{1}{(n-2)^{1/3}}=0$

and $a_{n+1}=\frac{1}{(K-1)^{1/3}} < \frac{1}{(K-2)^{1/3}}=a_n \Rightarrow$ decreasing

$\Rightarrow$ $\boxed{\text{series converges conditionally}}$

3. (15 points) Find the radius and interval of convergence for the power series: $\sum_{k=1}^{\infty}\frac{5^k}{k^3}(x-2)^k$.

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{5^{n+1}}{(n+1)^3}\cdot\frac{n^3}{5^n}\right|$$

$$=\lim_{n\to\infty}\frac{5n^3}{(n+1)^3}=5, \quad \text{so } \boxed{R=\frac{1}{5}}$$

The series converges absolutely when

$|x-2|<\frac{1}{5} \Rightarrow \frac{9}{5}<x<\frac{11}{5}$.

Checking endpoints:

$x=\frac{11}{5}$: $\sum\frac{5^k}{K^3}\left(\frac{1}{5}\right)^K=\sum\frac{1}{K^3}$, which converges ($p$-series, $p=3>1$)

$x=\frac{9}{5}$: $\sum\frac{5^K}{K^3}\left(-\frac{1}{5}\right)^K=\sum\frac{(-1)^K}{K^3}$, which converges absolutely

$\Rightarrow \boxed{I.C.=\left[\frac{9}{5},\frac{11}{5}\right]}$

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1. (a) (10 points) Find the third degree Taylor polynomial for the function $f(x) = \cos x$ about $x = \frac{\pi}{6}$.

k	$f^{(k)}(x)$	$f^{(k)}(\pi/6)$	$\frac{f^{(k)}(\pi/6)}{k!}(x-\pi/6)^k$
0	$\cos x$	$\sqrt{3}/2$	$\sqrt{3}/2$
1	$-\sin x$	$-1/2$	$-1/2\,(x-\pi/6)$
2	$-\cos x$	$-\sqrt{3}/2$	$-\sqrt{3}/4\,(x-\pi/6)^2$
3	$\sin x$	$1/2$	$1/2 \cdot \frac{1}{3!}(x-\pi/6)^3 = \frac{1}{12}(x-\pi/6)^3$

So $\boxed{P_3(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x-\frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x-\pi/6\right)^3}$

(b) (10 points) Determine the maximum error in your approximation if we try to approximate $f(\frac{\pi}{2})$ using your polynomial in part (a) (you do NOT need to calculate the approximate function value). Recall: $|R_n(x)| \leq \max|f^{(n+1)}(c)|\frac{|x-a|^{n+1}}{(n+1)!}$. Simplify as far as you can without a calculator.

Here, $a = \frac{\pi}{6}$, $x = \frac{\pi}{2}$, and $n = 3$, so:

$$|R_3(\pi/2)| \leq \max|f^{(4)}(c)|\,\frac{\left|\frac{\pi}{2}-\frac{\pi}{6}\right|^4}{4!}$$

Note that $f^{(4)}(x) = \cos x$ is decreasing on $[\pi/6, \pi/2]$,

so $\max|f^{(4)}(c)| \leq \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

Then: $|R_3(\frac{\pi}{2})| \leq \frac{\sqrt{3}}{2} \cdot \frac{(\pi/3)^4}{24}$

So $= \boxed{\frac{\sqrt{3}}{48}\left(\frac{\pi}{3}\right)^4}$ max error

2. (15 points) Find the radius and interval of convergence for the power series: $\sum_{k=1}^{\infty} \frac{6^k}{k^4}(x-3)^k$.

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{6^{n+1}}{(n+1)^4} \cdot \frac{n^4}{6^n}\right|$

$= \lim_{n\to\infty} \frac{6n^4}{(n+1)^4} = 6$, so $\boxed{R = \frac{1}{6}}$

The series converges absolutely for $|x-3| < \frac{1}{6} \Rightarrow 17/6 < x < 19/6$.

Checking endpoints:

$x = \frac{19}{6}$: $\sum \frac{6^k}{k^4}\left(\frac{1}{6}\right)^k = \sum \frac{1}{k^4}$ converges (p-series, $p = 4 > 1$)

$x = \frac{17}{6}$: $\sum \frac{6^k}{k^4}\left(-\frac{1}{6}\right)^k = \sum \frac{(-1)^k}{k^4}$ converges absolutely

so $\boxed{I.C. = \left[\frac{17}{6}, \frac{19}{6}\right]}$

3. (15 points) Determine if the alternating series below converges absolutely, converges conditionally, or diverges. JUSTIFY YOUR ANSWER fully using the convergence tests from class. The justification will count for the majority of the points.

$$\sum_{k=4}^{\infty} (-1)^k \frac{1}{(k-3)^{1/4}}$$

Basic comparison:

Note that $(k-3)^{1/4} < k^{1/4}$, so $\frac{1}{(k-3)^{1/4}} > \frac{1}{k^{1/4}}$.

As $\sum \frac{1}{k^{1/4}}$ diverges (p-series, $p = 1/4 < 1$),

the series $\sum \frac{1}{(k-3)^{1/4}}$ also diverges.

But as the series is alternating and:

- $\lim_{n\to\infty} \frac{1}{(n-3)^{1/4}} = 0$
- $a_{n+1} = \frac{1}{(k-2)^{1/4}} < \frac{1}{(k-3)^{1/4}} = a_n$ terms are decreasing

$\Rightarrow$ the series $\boxed{\text{converges conditionally}}$