Math 1501 E, Fall 2013

Exam #3

Name: Rubric

Section: ______

- You will have 50 minutes to complete the exam.
- No calculators, books, or notes allowed.
- Partial credit will be given. However, **no** credit will be given for a problem in which no work is shown, whether the answer is correct or not. Hence, show all applicable work.

Question:	1	2	3	Total
Points:	12	12	32	56
Score:				

1. Suppose that Newton, when deriving his Law of Cooling, had thought that the differential equation describing the rate at which the temperature $T(t)$ of an object changes in time was

$$\frac{dT}{dt} = -k\,(T - T_a)^3\ ,$$

where $k > 0$ and $T_a > 0$ is the ambient temperature.

(a) (8 points) Solve this separable differential equation using the initial condition $T(0) = 3T_a$.

$$\frac{dT}{(T-T_a)^3} = -k\,dt \Rightarrow \int \frac{dT}{(T-T_a)^3} = -k\int dt \Rightarrow$$

$$-\frac{1}{2}(T-T_a)^{-2} = -kt + C, \quad (T-T_a)^{-2} = 2kt + C \Rightarrow$$

$$T - T_a = (2kt+C)^{-\frac{1}{2}}, \quad \boxed{T = T_a + (2kt+C)^{-1/2}}$$

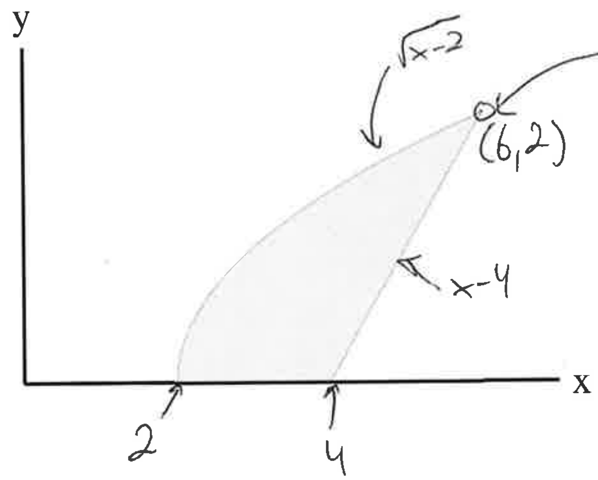
$$T(0) = \cancel{T_a} + (2k\cancel{0}+C)^{-1/2} = \overset{3}{\cancel{8}}T_a \Rightarrow 2T_a = (2k\cancel{0}+C)^{-1/2} \Rightarrow \boxed{C = (2T_a)^{-2}}$$

(b) (4 points) Starting from $T(0) = 3T_a$, how long will it take the object to reach a temperature of $2T_a$?

$$2T_a = \cancel{T_a} + \left[2kt + (2T_a)^{-2}\right]^{-1/2} \Rightarrow T_a^{-2} = 2kt + (2T_a)^{-2}$$

$$2kt = \frac{3}{\cancel{4}}T_a^{-2} \Rightarrow \boxed{t = \frac{3}{8k}T_a^{-2}}$$

2. (12 points) Find the area between the curves $y = \sqrt{x-2}$ and $y = x - 4$ illustrated in the (mostly) unlabeled plot below. **Hint**: there are two ways to do this, one being a bit easier than the other.



Here, $\sqrt{x-2} = x-4 \Rightarrow$

$x - 2 = (x^2 - 8x + 16)$

$x^2 - 9x + 18 = 0$

$(x-6)(x-3) = 0$

$\boxed{x = 6}$ $\boxed{y = 6 - 4 = 2}$

"Hard" way

$$A = \int_2^6 \sqrt{x-2}\,dx - \int_4^6 x - 4\,dx = \frac{2}{3}(x-2)^{3/2}\Big|_2^6 - \left[\frac{x^2}{2} - 4x\right]_4^6$$

$$= \frac{2}{3}(4)^{3/2} - 0 - \left[\frac{36}{2} - 24 - \frac{16}{2} + 16\right] = \frac{16}{3} - \frac{6}{3} = \boxed{\frac{10}{3}}$$

$\frac{2}{3} 8 = \frac{16}{3}$ $\quad 18 - 24 - 8 + 16 = 2$

"Easy" way

$x = y + 4, \quad x = y^2 + 2 \Rightarrow$

$$A = \int_0^2 [y + 4 - (y^2 + 2)]\,dy = \int_0^2 (y + 2 - y^2)\,dy = -\frac{y^3}{3} + 2y + \frac{y^2}{2}\Big|_0^2 = -\frac{8}{3} + 4 + 2$$

$$= -\frac{8}{3} + \frac{18}{3} = \boxed{\frac{10}{3}}$$

3. Perform the following calculations:

(a) (8 points) $\int_0^{\pi/2} \sin^5 x \cos x dx$

let $u = \sin x \Rightarrow du = \cos x dx \Rightarrow$

$$\int_0^{\pi/2} \sin^5 x \cos x dx = \int_{\sin 0}^{\sin(\pi/2)} u^5 du = \frac{u^6}{6}\Big|_0^1 = \boxed{\frac{1}{6}}$$

(b) (8 points) $\int \sqrt{1-x^2} dx$

let $x = \sin\theta \Rightarrow 1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$,

$dx = \cos\theta d\theta \Rightarrow$

$$\int \sqrt{1-x^2} dx = \int \sqrt{\cos^2\theta} \cos\theta d\theta = \int \cos^2\theta d\theta$$

$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$, so we have

$$\frac{1}{2}\int 1 + \cos(2\theta) d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\underbrace{\sin(2\theta)}_{2\sin\theta\cos\theta}\right] = \frac{1}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$

$\theta = \sin^{-1}(x)$, $\sin\theta = x$, $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2} \Rightarrow \boxed{\frac{1}{2}\left[\sin^{-1}(x) + x\sqrt{1-x^2}\right] + C}$

(c) (8 points) $\int \frac{5x+11}{x^2+6x-7}dx$

$x^2+6x-7 = (x+\cancel{7})(x-1) \Rightarrow$

$\frac{5x+11}{(x+7)(x-1)} = \frac{A}{(x+7)} + \frac{B}{(x-1)} \Rightarrow 5x+11 = A(x-1)+B(x+7)$

evalute at $x=1$: $5+11 = B(8) \Rightarrow B=2$

evalute at $x=-7$: $\underbrace{-35+11}_{-24} = A(-8) \Rightarrow A=3$

$\int \frac{5x+11}{x^2+6x-7}dx = \int \frac{3}{x+7}dx + \int \frac{2}{(x-1)}dx = \boxed{3\ln|x+7| + 2\ln|x-1| + C}$

(d) (8 points) $\int x\sin(3x)dx$

let $u=x$, $v'=\sin(3x)$

$\Downarrow$

$u'=1$, $v=-\frac{1}{3}\cos(3x)$

$\int x\sin(3x)dx = -\frac{x}{3}\cos(3x) + \frac{1}{3}\underbrace{\int \cos(3x)\,dx}_{\frac{1}{3}\sin(3x)}$

$= \boxed{-\frac{x}{3}\cos(3x) + \frac{1}{9}\sin(3x) + C}$