MATH 2603, Fall 2015,Final Exam Sample: Closed book, no calculators.
Instructor: Esther Ezra.

Answer all questions **on this sheet**.

Name	GT IDnumber	Section Number

Problem 1. A $2 \times n$ checkerboard is to be tiled using two types of tiles. The first tile is a 1×1 square tile. The second tile is called an L-tile and is formed by removing the upperright 1×1 square from a 2×2 tile. The L-tiles can be used in any of the four ways they can be rotated, (that is, the "missing square" can be in any of four positions). Let $t(n)$ denote the number of tilings of the $2 \times n$ checkerboard using 1×1 tiles and L-tiles.

Write a recurrence equation for t_n, as well as the initial conditions for $n = 1, 2, 3$.

Initial conditions:

$$t(1) = 1, t(2) = 5, t(3) = 1 + 4 + 4 + 2 = 11.$$

Recurrence equation:

$$t_n = t_{n-1} + 4t(n-2) + 2t(n-3).$$

On the right hand-side, the first term corresponds to the case where the last column in the checkerboard consists of squares only. The second term corresponds to the case where the last two columns are tiles with one square and one L-shape, and the last term corresponds to the case where the last three columns are tiles with two L-shapes.

Problem 2.

a. How many different strings can be made with the letters in GATTACA?

$$\frac{7!}{3!2!}.$$

b. How many strings of 7 letters (A-Z) contain exactly three vowels (A,E,I,O,U)?

$$\binom{7}{3}21^4 5^3.$$

c. How many strings of 7 letters contain exactly three vowels if no letter can be repeated?

$$\binom{7}{3}\frac{21!}{17!}\frac{5!}{2!}$$

d. What is the coefficient of x^6y^9 in $(2x - 3y)^{15}$?

$$\binom{15}{6}2^6(-3)^9.$$

e. How many ways are there to distribute 10 identical balls to 3 children, if every child must get at least two balls?

This is like distributing 7 units to 3 children, where each gets at least one unit. Thus:

$$\binom{7-1}{2} = \binom{6}{2}.$$

f. How many integer-value solutions are there to $x_1 + x_2 + x_3 + x_4 = 90$, s.t. $x_i \geq -1$, for all $i = 1, \ldots, 4$.

Put a new variable $x_i' = x_i + 1$. Thus we have

$$x_1' + x_2' + x_3' + x_4' = 94,$$

and $x_i' \geq 0$, $i = 1, \ldots, 4$. Therefore number of solutions is

$$\binom{94+4-1}{3} = \binom{97}{3}.$$

Problem 3. Let $G = (V, E)$ be an (undirected) graph. Define the relation R as

$$R = \{(u, v) \in V \times V \mid \text{there is a path in } G \text{ from } u \text{ to } v\}.$$

a. Show that R is an equivalence relation.

Reflexivity: Each vertex is trivially connected to itself.

Symmetry: Assume there is a path from u to v, then since all edges are bidirectional the same path is also a path from v to u.

Transitivity: Assume there is a path π_1 from u to v, and a path π_2 from v to w. Then the concatenation $\pi_1\pi_2$ is a path from u to w.

b. What are the equivalence classes associated with R?

These are the connected components of the graph.

Problem 4. Use mathematical induction in order to show that, for any integer $n \geq 1$,

$$(n-1)^3 + n^3 + (n+1)^3$$

is divisible by 9. (Show the base case and then the inductive step.)

For the case $n = 1$ we have:

$$0^3 + 1^3 + 2^3 = 1 + 8 = 9,$$

which is divisible by 9.

We now assume that $(n-1)^3 + n^3 + (n+1)^3$ is divisible by 9, and show that $n^3 + (n+1)^3 + (n+2)^3$ is still divisible by 9. We write the latter as

$$(n-1)^3 + n^3 + (n+1)^3 + \big(n^3 + (n+1)^3 + (n+2)^3\big) - \big((n-1)^3 + n^3 + (n+1)^3\big)$$

$$= (n-1)^3 + n^3 + (n+1)^3 + (n+2)^3 - (n-1)^3.$$

We now claim that $(n+2)^3 - (n-1)^3$ is divisible by 9. Indeed, using the Binomial Theorem:

$$(n+2)^3 - (n-1)^3 = n^3 + 6n^2 + 12n + 8 - (n^3 - 3n^2 + 3n - 1) = 9n^2 + 9n + 9,$$

which is clearly divisible by 9. Thus $(n-1)^3 + n^3 + (n+1)^3$ is divisible by 9, which is consistent with our induction hypothesis.

Problem 5.

a. How many permutations of the letters ABCDEFGH contain neither of the strings ABC and EF as consecutive substrings? (permutation implies that no repetitions are allowed.)

This is an application of the Exclusion-inclusion principle. We thus have:

$$8! - 6! - 7! + 5!.$$

b. How many numbers between 1 and 900 (inclusive) are divisible by at least one of 2, 3, or 5?

This is an application of the Exclusion-inclusion principle. Since all pairs among $2, 3, 5$ **are relatively primes, we have:**

$$900 - \left(\frac{900}{2} + \frac{900}{3} + \frac{900}{5}\right) + \left(\frac{900}{6} + \frac{900}{10} + \frac{900}{15}\right) - \frac{900}{30}.$$

Problem 6. Solve the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} + 3^n,$$

with initial conditions $a_0 = 2, a_1 = 10$.

This is a non-homogeneous second order recurrence equation. Using Characteristic polynomial we have that the roots are 2 and -1. Thus the homogeneous solution is

$$C_1 2^n + C_2(-1)^n.$$

A particular solution to a_n is $a_n = (9/4) \cdot 3^n$. With the initial conditions we obtain:

$$a_n = 2^n - (5/4)(-1)^n + (1/4)3^{n+2}.$$

Problem 7. There are over 9.9 million people living in the state of Georgia. Prove that at least three of those people have the same birthday and the same last 4 digits in their social security numbers.

This is an application of the Pigeonhole principle.

There are $10^4 = 10000$ **possibilities for the last four digits of SSN, and there are** 366 **possible birthdays (**365 **is also okay), for a total of** $3,660,000$ **possible combinations. Since**

$$\left\lceil \frac{9,900,000}{3,660,000} \right\rceil = 3,$$

by the Pigeonhole Principle, there are at least three people with the same last four digits of SSN and the same birthday.

Problem 8. Find all solutions to the system of congruences

$$x \equiv 2 \mod 4$$

$$x \equiv 6 \mod 7.$$

Write $1 = (-1)(7) + (2)(4)$**. Since** $-7 \equiv 1 \mod 4$ **and** $8 \equiv 1 \mod 7$**,** $x = 2(-1)(7) + 6(2)(4) = 34$ **is the unique solution modulo** 28**. Thus all solutions are of the form** $6 + 28k$**, where** k **is an integer.**

Problem 9. Let $T = (V, E)$ be a tree defined on n vertices (that is, T is a connected graph with no circuits). Show that T contains exactly $n - 1$ edges. (You may assume that every tree contains at least one leaf.)

The proof proceeds by induction on n and is given in the lecture notes.

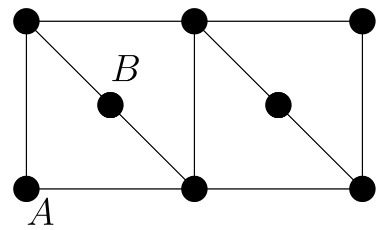

Figure 1:

Problem 10.

a. Is the following graph Hamiltonian?

No. Suppose for the sake of contradiction that the graph is Hamiltonian. Let C be the Hamiltonian cycle. Then it must contain the two edges adjacent to A, and the two adjacent to B. But then C contains a *smaller* cycle, which is impossible.

b. Which of the following graphs is Hamiltonian?

(i) $K_{3,3}$, (ii) K_{101}, (iii) $K_{2,3}$.

According to Dirac's Theorem $K_{3,3}$ and K_{101} are Hamiltonian. $k_{2,3}$ is not Hamiltonian: Need to have all 6 edges adjacent to each of the three vertices in one side of the graph.

c. Which of the graphs in (b) is Eulerian?

All these graphs are connected, then we will only consider the parity of their vertex degrees. $K_{3,3}$ and $K_{2,3}$ are not Eulerian. K_{101} is Eulerian.

Problem 11. The third matrix found in an application of the Floyd-Warshall algorithm is:

$$M_2 = \begin{bmatrix} 0 & \infty & 2 \\ \infty & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

What is the shortest distance between vertices 1 and 2?

The next iteration of the FW algorithm is the last one, and considers paths going through vertex 3**. Thus going from vertex** 1 **to vertex** 3 **and then to vertex** 2 **yields a total path length of** $2+1=3$**. Thus this is the improved path length (and the shortest one!).**

Problem 12.

a. Design a recursive algorithm to multiply two $n \times n$ matrices. You may assume, without loss of generality, that n is an integer power of 2.

Let A, B be the input matrices. We divide each of the two matrices into 4 blocks, each of which is a $(n/2) \times (n/2)$ submatrix. Let $A_{1,1}, A_{1,2}, A_{2,1}, A_{2,2}$ be the blocks of A, and $B_{1,1}, B_{1,2}, B_{2,1}, B_{2,2}$ be the blocks of B. We now observe that $A \times B$ can be obtained by

$$\begin{bmatrix} A_{1,1} \times B_{1,1} + A_{1,2}B_{2,1} & A_{1,1}B_{1,2} + A_{1,2}B_{2,2} \\ A_{2,1} \times B_{1,1} + A_{2,2}B_{2,1} & A_{2,1}B_{1,2} + A_{2,2}B_{2,2} \end{bmatrix}$$

Thus the computation of $A \times B$ involves 8 recursive calls, and $O(n^2)$ additional operations (to compute the addition of pairs of matrices).

b. What is the running time of your algorithm?

As stated in part a, we need to have 8 recursive calls, and $O(n^2)$ additional operations. Thus our running time $T(n)$ satisfies the recursive equation:

$$T(n) = 8T(n/2) + O(n^2).$$

The solution is $T(n) = O(n^3)$.

Problem 13. True-False. Mark in the left Margin.

1. **The following implication is correct:**

   $$\text{(For all } x \in \mathbb{R} \text{ there exists } y \in \mathbb{R}, \text{ s.t. } x = y^2) \to (2 + 2 = 5).$$

   T

2. **Let** a, b, x **be integers s.t.** $a \mid bx$**. If** a, b **are relatively primes then** $a \mid x$**. T**

3. $2^{1/3}$ **is a rational number. F**

4. **A graph on** n **vertices can have exactly one vertex of odd degree. F**

5. **At any step of the Prim's algorithm, the collection of edges just computed forms a connected subgraph. T**

6. **At any step of the Kruskal's algorithm, the collection of edges just computed forms a connected subgraph. F**

7. **The Fibonacci numbers** F_n **(with** $F_0 = 0, F_1 = 1$**) satisfy for each** $n \geq 1$**,** $F_{n-1}F_{n+1} = F_n^2 + (-1)^n$**. T**

8. **The number of spanning trees in a graph** G **on** n **vertices can be exponential in** n**. T**

9. **The number of shortest paths between two vertices** u, v **in a graph** G **can be exponential in the size of the graph. T**