Question 1

$$\max \quad 2x_1 + x_2 - 3x_3 + 5x_4$$

$$\begin{aligned} \text{s.t.} \quad & x_1 + 2x_2 + 2x_3 + 4x_4 + x_5 = 40 \\ & 2x_1 - x_2 + x_3 + 2x_4 + x_6 = 8 \\ & 4x_1 - 2x_2 + x_3 - x_4 + x_7 = 10 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \end{aligned}$$

$\mathcal{B} = \{x_5, x_6, x_7\}$, $\mathcal{N} = \{x_1, x_2, x_3, x_4\}$.

$$\begin{aligned} z &= 0 + 2x_1 + x_2 - 3x_3 + 5x_4 \\ x_5 &= 40 - x_1 - 2x_2 - 2x_3 - 4x_4 \qquad \min\left\{\frac{40}{4}, \frac{8}{2}\right\} = 4 \\ \leftarrow x_6 &= 8 - 2x_1 + x_2 - x_3 - 2x_4 \\ x_7 &= 10 - 4x_1 + 2x_2 - x_3 + x_4 \end{aligned}$$

$\mathcal{B} = \{x_5, x_4, x_7\}$, $\mathcal{N} = \{x_1, x_2, x_3, x_6\}$.

$$\begin{aligned} z &= 20 - 3x_1 + \tfrac{7}{2}x_2 - \tfrac{11}{2}x_3 - \tfrac{5}{2}x_6 \\ \leftarrow x_5 &= 24 + 3x_1 - 4x_2 + 2x_6 \\ x_4 &= 4 - x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 - \tfrac{1}{2}x_6 \\ x_7 &= 14 - 5x_1 + \tfrac{5}{2}x_2 - \tfrac{3}{2}x_3 - \tfrac{1}{2}x_6 \end{aligned}$$

$B = \{x_2, x_4, x_7\}$ $\qquad N = \{x_1, x_5, x_3, x_6\}$

$$z = 41 \underbrace{-3/8}\, x_1 \underbrace{-7/8}\, x_5 \underbrace{-1/2}\, x_3 \underbrace{-3/4}\, x_6$$
$$x_2 = 6 + 3/4\, x_1 - 1/4\, x_5 + 1/2\, x_6$$
$$x_4 = 7 - 5/8\, x_1 - 1/8\, x_5 - 1/2\, x_3 - 1/4\, x_6$$
$$x_7 = 23 - 31/8\, x_1 - 3/8\, x_5 - 3/2\, x_3 + 1/4\, x_6$$

optimal solution $= (0, 6, 0, 7)$
optimal value $= 41$

2. In standard equality form

$$\begin{cases} \max & z = x_1 + x_2 \\ \text{s.t.} & -x_1 + x_2 + s_1 = 2 \\ & x_1 - x_2 + s_2 = 2 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{cases}$$

the slack variables form a basic feasible solution so no Phase 1 is required.

$$\left.\begin{aligned} s_1 &= 2 + x_1 - x_2 \\ s_2 &= 2 - x_1 + x_2 \\ z &= x_1 + x_2 \end{aligned}\right\} \text{Initial dictionary}$$

Choose x_1 to enter the basis (arbitrarily).
s_2 must leave the basis.

$$\begin{aligned} s_1 &= 4 - s_2 \\ x_1 &= 2 - s_2 + x_2 \\ z &= 2 - s_2 + 2x_2 \end{aligned}$$

As x_2 increases from zero, z increases, improving the objective value, while s_1 and x_1 remain nonnegative, and hence the solution remains feasible. Indeed, taking $x_2 = t$, $x_1 = 2+t$ is feasible for any $t \geq 0$, with objective $z = 2 + 2t \to \infty$ as $t \to \infty$.

$\therefore$ The LP is unbounded.

3(a) Phase 1 LP

$$\begin{array}{ll}\max & w = -a_1 - a_2 \\ \text{s.t.} & x_1 + x_2 + x_3 + s_1 = 4 \\ & 2x_1 + x_2 - x_3 - s_2 + a_1 = 1 \\ & -x_2 + x_3 - s_3 + a_2 = 1 \\ & x_1, x_2, x_3, s_1, s_2, s_3, a_1, a_2 \geq 0 \end{array}$$

Initial Phase 1 dictionary

$$\begin{aligned} s_1 &= 4 - x_1 - x_2 - x_3 \\ a_1 &= 1 - 2x_1 - x_2 + x_3 + s_2 \\ a_2 &= 1 + x_2 - x_3 + s_3 \\ w &= -2 + 2x_1 - s_2 - s_3 \end{aligned}$$

x_1 enters, a_1 leaves since $\frac{1}{2} < 4$

$$\begin{aligned} s_1 &= \tfrac{7}{2} + \tfrac{1}{2}a_1 - \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 - \tfrac{1}{2}s_2 \\ x_1 &= \tfrac{1}{2} - \tfrac{1}{2}a_1 - \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 + \tfrac{1}{2}s_2 \\ a_2 &= 1 + x_2 - x_3 + s_3 \\ w &= -1 - a_1 - x_2 + x_3 - s_3 \end{aligned}$$

x_3 enters, a_2 leaves

$$\begin{aligned} s_1 &= 2 + \tfrac{1}{2}a_1 - 2x_2 + \tfrac{3}{2}a_2 - \tfrac{1}{2}s_2 - \tfrac{3}{2}s_3 \\ x_1 &= 1 - \tfrac{1}{2}a_1 - \tfrac{1}{2}a_2 + \tfrac{1}{2}s_2 + \tfrac{1}{2}s_3 \\ x_3 &= 1 + x_2 - a_2 + s_3 \\ w &= 0 - a_1 - a_2 \end{aligned}$$

This is optimal with $w=0$ $\therefore$ Initial feasible basis is $\{x_1, x_3, s_1\}$, and basic feasible solution is $x = (1,0,1)^T$.

3(b) $z = 5 + x_2 + \frac{3}{2}s_2 + \frac{7}{2}s_3$

s_3 enters, s_1 leaves

$s_3 = \frac{4}{3} - \frac{4}{3}x_2 - \frac{1}{3}s_2 - \frac{2}{3}s_1$

$x_3 = \frac{7}{3} - \frac{1}{3}x_2 - \frac{1}{3}s_2 - \frac{2}{3}s_1$

$x_1 = \frac{5}{3} - \frac{2}{3}x_2 + \frac{1}{3}s_2 - \frac{1}{3}s_1$

$z = \frac{29}{3} - \frac{11}{3}x_2 + \frac{1}{3}s_2 - \frac{7}{3}s_1$

s_2 enters, $\min\{\frac{4}{3}/\frac{1}{3}, \frac{7}{3}/\frac{1}{3}\} = 4 \Rightarrow s_3$ leaves

$s_2 = 4 - 4x_2 - 3s_3 - 2s_1$

$x_3 = 1 + x_2 + s_3$

$x_1 = 3 - 2x_2 - s_3 - s_1$

$z = 11 - 5x_2 - s_3 - 3s_1$

$\therefore$ Optimal solution: $x = (3, 0, 1)^T$

Optimal value: $z = 11$

Question 4

$\min\ 3x_1$

s.t. $2x_1 + x_2 + x_3 - x_4 = 6$

$3x_1 + 2x_2 + x_3 = 4$

$x_1, x_2, x_3, x_4 \geq 0$

Phase 1 LP:

$\max\ W = -x_5 - x_6$

s.t. $2x_1 + x_2 + x_3 - x_4 + x_5 = 6$

$3x_1 + 2x_2 + x_3 + x_6 = 4$

$\mathcal{B} = \{x_5, x_6\}, \quad \mathcal{N} = \{x_1, x_2, x_3, x_4\}$.

$W = -10 + 5x_1 + 3x_2 + 2x_3 - x_4$

$x_5 = 6 - 2x_1 - x_2 - x_3 + x_4 \qquad \min\{\overset{x_5}{\tfrac{6}{2}}, \overset{x_6}{\tfrac{4}{3}}\} = 4/3$

$\leftarrow x_6 = 4 - 3x_1 - 2x_2 - x_3$

$\mathcal{B} = \{x_5, x_1\} \quad \mathcal{N} = \{x_6, x_2, x_3, x_4\}$.

$W = -10/3 - 5/3\,x_6 - 1/3\,x_2 + 1/3\,x_3 - x_4$

$x_5 = 10/3 + 2/3\,x_6 + 1/3\,x_2 - 1/3\,x_3 + x_4 \qquad \min\{\overset{x_5}{\tfrac{10/3}{1/3}}, \overset{x_1}{\tfrac{4/3}{1/3}}\} = 4$

$\leftarrow x_1 = 4/3 - 1/3\,x_6 - 2/3\,x_2 - 1/3\,x_3$

$\mathcal{B} = \{x_5, x_3\}$ $\quad \mathcal{N} = \{x_6, x_2, x_1, x_4\}$.

$W = -2 - 2x_6 - x_2 - x_1 - x_4$

$x_5 = 2 + x_6 + x_2 + x_1 + x_4$

$x_3 = 4 - x_6 - 2x_2 - 3x_1$

Optimal solution is reached.

Since optimal value $\neq 0$, that means that $x_5 = x_6 = 0$ is not a feasible solution for Phase 1 LP.

Therefore the original LP is infeasible.