MATH 3012 A, Midterm 1

05/31/2013

Name: ______________________ GTID: ______________________

Key

Problem No.	Points
1	20
2	10
3	10
4	15
5	15
6	15
7	15
8	
9	
10	

TOTAL: 100

Please do show all your work including intermediate steps. Partial credit is available.

Problem 1 (20 points).

How many integer valued solutions to the following equations and inequalities:

5pts a) $x_1 + x_2 + x_3 + x_4 = 20$, all $x_i > 0$.

$\binom{19}{3}$ ✓

5pts b) $x_1 + x_2 + x_3 + x_4 = 20$, all $x_i \geq 0$.

$\binom{23}{3}$ ✓

5pts c) $x_1 + x_2 + x_3 + x_4 \leq 20$, all $x_i > 0$.

$\Leftrightarrow \quad x_1 + x_2 + x_3 + x_4 < 21$, all $x_i > 0$

$\Leftrightarrow \quad x_1 + x_2 + x_3 + x_4 + x_5 = 21$, all $x_i > 0$

$\binom{20}{4}$ ✓

5pts d) $x_1 + x_2 + x_3 + x_4 \leq 20$, all $x_i \geq 0$.

$\Leftrightarrow \quad x_1 + x_2 + x_3 + x_4 + x_5 = 20$, all $x_i \geq 0$

$\binom{20+5-1}{5-1} = \binom{24}{4}$ ✓

Turn over for more problems

Problem 2 (10 points).

A die is tossed ten times and the sequence of the outcomes is observed.

3 pts 1. How many different sequences are possible?

3 pts 2. How many of these sequences contain exactly two 1's?

4 pts 3. How many of these sequences contain at most two 1's?

1. 6^{10}

2. $\binom{10}{2} \cdot 5^8$

3. $\binom{10}{0} \cdot 5^{10} + \binom{10}{1} \cdot 5^9 + \binom{10}{2} \cdot 5^8$

Turn over for more problems

Problem 3 (10 points).

Find the coefficient of x^6 in the binomial expansion of

$$\left(4x + \frac{3}{x^2}\right)^{18}$$

$$\left(4x + \frac{3}{x^2}\right)^{18} = \sum_{k=0}^{18} \binom{18}{k} (4x)^k \left(\frac{3}{x^2}\right)^{18-k} \quad \cdots \text{ 5pts}$$

$$x^k \cdot (x^{-2})^{18-k} = x^6$$

$$\Rightarrow \quad 3k - 36 = 6$$

$$3k = 42$$

$$k = 14 \quad \text{----} \quad \text{3pts}$$

coefficient is $\binom{18}{14} 4^{14} \cdot 3^4$ --- 2pts

Turn over for more problems

Problem 4 (15 points).

a) Use the Euclidean algorithm to find $d = \gcd(85, 408)$. 10 pts

$408 = 85 \times 4 + 68$

$85 = 68 \times 1 + 17$

$68 = 17 \times 4$

$d = 17$

b) Use your work in (a) to find integers a and b so that $d = 85a + 408b$. Start by rewriting the results of the long division done previously (all but the last one): 5pts

by a)

$17 = 85 - 68$

$= 85 - (408 - 85 \times 4)$

$= 85 \times 5 - 408$

$\Rightarrow \quad a = 5 \quad b = -1$

Turn over for more problems

Problem 5 (15 points).
Jump induction is another induction scheme, which works in the following way: Given a statement $P(n)$, if

(a) $P(1)$ and $P(2)$ are both true and

(b) For any $k \geq 1$, $P(k)$ is true implies $P(k+2)$ is true

then $P(n)$ is true for all $n \geq 1$.

Use jump induction to show that for every integer $n \geq 1$

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}(1 + 2 + \cdots + n).$$

(a) $n=1$ $\quad 1^2 = (-1)^{1-1} \cdot 1$ --- 2pts

$n=2$ $\quad 1^2 - 2^2 = -3 = (-1)^{2-1}(1+2)$ --- 2pts

For any $k \geq 1$, Assume $P(k)$ is true.

i.e. $1^2 - 2^2 + 3^2 - \cdots + (-1)^{k-1} \cdot k^2 = (-1)^{k-1}(1+2+\cdots+k)$ --- 3pts

For $n = k+2$.

want to show

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{k-1}k^2 + (-1)^k(k+1)^2 + (-1)^{k+1}(k+2)^2$$

$$= (-1)^{k+1}(1+2+\cdots+k+(k+1)+(k+2)) \quad \text{--- 2pts}$$

we have $1^2 - 2^2 + 3^2 - \cdots + (-1)^{k-1}k^2 + (-1)^k(k+1)^2 + (-1)^{k+1}(k+2)^2$

$$\underset{\text{by I.H.}}{=} (-1)^{k-1}(1+2+\cdots+k) + (-1)^k(k+1)^2 + (-1)^{k+1}(k+2)^2$$

$$= (-1)^{k+1}(1+2+\cdots+k) - (-1)^{k+1}(k+1)^2 + (-1)^{k+1}(k+2)^2$$

$$= (-1)^{k+1}(1+2+\cdots+k) + (-1)^{k+1}(2k+3)$$

$$= (-1)^{k+1}(1+2+\cdots+k+(k+1)+(k+2)) \quad \text{--- 6pts}$$

Turn over for more problems

Problem 6 (15 points).

Suppose 51 numbers are chosen from the set $\{1, 2, 3, ..., 100\}$. Show that among those chosen numbers there are two numbers such that one is a multiple of the other.

Hint: Any natural number n can be written in the form $n = 2^k a$ with $k \geq 0$ and a odd.

For $n \in \{1, 2, \cdots, 100\}$

$n = 2^k \cdot a$

possible value for a are $1, 3, 5, 7, 9, \cdots, 97, 99$ ... 5pts

Those are "holes"

selected 51 numbers are "pigeons" ... 3pts

By pigeonhole-principle ... 3pts

$\exists\ n_1, n_2 \in \{1, 2, \cdots, 100\}$

s.t, $n_1 = 2^{k_1} \cdot a$, $n_2 = 2^{k_2} \cdot a$ ... 3pts

without loss of generality, we may assume $k_1 \leq k_2$.

then we have $n_1 | n_2$. ... 1pts

Turn over for more problems

Problem 6 (15 points).

Do ONE of the following two problems.

a)Prove the following identity using a combinatorial proof. Make sure to explain exactly what each side of the equation is counting. Let n and k be integers with $n \geq k \geq 2$. Then

$$k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}.$$

b)Show that decision version of the traveling salesman problem is in $\mathcal{NP}$. Given an input matrix of distances between n cities, the problem is to determine if there is a route visiting each city exactly once and returning to the origin city with total distance less than k.

a) Thinking of choosing a president and a vice president and k-2 members from a group of n people.

LHS : First choose a group of k people out of n : $\binom{n}{k}$

then choose a president from this group : k

then choose a v.p. from rest people in this group : $k-1$.

RHS : First choose president : n.

then choose v.p. : $n-1$

then choose k-2 members : $\binom{n-2}{k-2}$

Double counting. So LHS = RHS

b) Given a "Yes" certificate, i.e. a route visiting each city exactly once and returning to the origin city w/ total distance $\leq k$.

it takes $O(n)$ time to verify that the route passes through each city exactly once

it takes $O(n)$ time to verify total distance of ~~given~~ the route is less than or equal to k.

Hence the correctness can be verified in poly time of n.

Turn over for more problems

Hence it's in NP.

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