Mar. 27, 2014

Math 2401 M - Exam 3

First Name (Print): ____________ Last Name (Print): __________ Signature: ______________

Please choose your section: ☐ M1 ☐ M2 ☐ M3

• There are **5** questions on **5** pages. The exam is worth 50 points in total.

• Answer the questions clearly and completely. You must provide work clearly justifying your solution. ***Sketch the region of the integration for Q1, Q2 and Q5.***

• You can NOT write your work on the back of the page. Use it for scratch work if needed.

• You have 50 minutes to finish your work.

Q1: #56 on page 866

Q2: #20 on page 876

Q3: #25 on page 884

Q4: An example shown in the lecture [Example 1 on page 894]

Q5: #37 and #38 mixed on page 903

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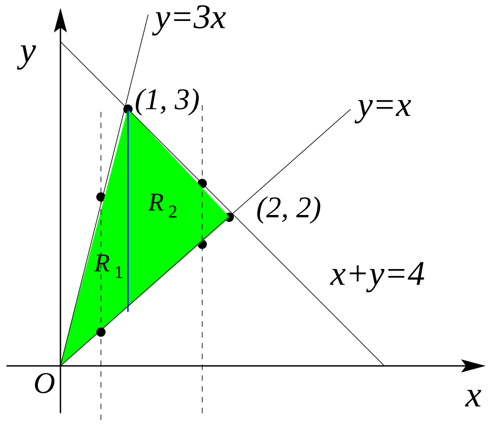
1. (10 points) Evaluate $\displaystyle\iint_R x\, dA$ where R is the region bounded by lines $y = x$, $y = 3x$ and $x + y = 4$.

Solution:

Sketch the region:



$$
\begin{aligned}
\iint_R x\, dA &= \iint_{R_1} x\, dA + \iint_{R_2} x\, dA \\
&= \int_{x=0}^{x=1} \int_{y=x}^{y=3x} x dy dx + \int_{x=1}^{x=2} \int_{y=x}^{y=4-x} x dy dx \\
&= \int_{x=0}^{x=1} xy\Big|_{y=x}^{y=3x} dx + \int_{x=1}^{x=2} xy\Big|_{y=x}^{y=4-x} dx \\
&= \int_{x=0}^{x=1} 2x^2 dx + \int_{x=1}^{x=2} (4x - 2x^2) dx \\
&= \frac{2}{3}x^3\Big|_{x=0}^{x=1} + (2x^2 - \frac{2}{3}x^3)\Big|_{x=1}^{x=2} = 2.
\end{aligned}
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Remark. You may calculate the double integral in the order $dxdy$.

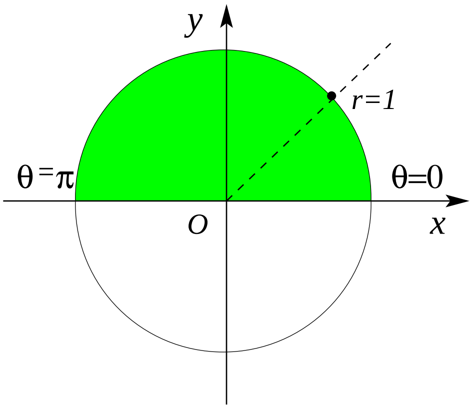
2. (10 points) Evaluate

$$\iint_R \ln(x^2+y^2+1)\,dA$$

where $R = \{(x,y) | -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}, 0 \le y \le 1\}$.

Solution:

Sketch the region:



Change the double integral into an equivalent polar integral.

$$\begin{aligned}
\iint_R \ln(x^2+y^2+1)\,dA &= \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1} \ln(r^2+1) r\,dr\,d\theta \quad \text{[\textit{Integration by parts}]} \\
&= \int_{\theta=0}^{\theta=\pi} \frac{1}{2}[\ln(r^2+1)(r^2+1)-(r^2+1)]\Big|_0^1 d\theta \\
&= \int_{\theta=0}^{\theta=\pi} (\ln 2 - \frac{1}{2})d\theta = (\ln 2 - \frac{1}{2})\pi.
\end{aligned}$$

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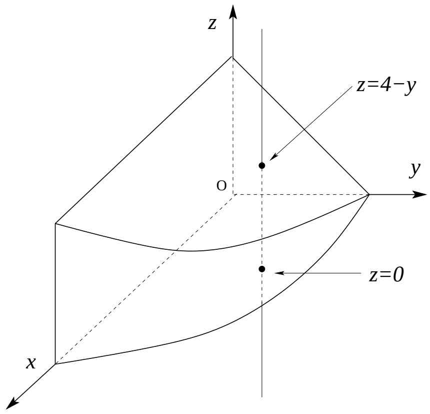
3. (10 points) Find the volume of the region D in the first octant bounded by the coordinate planes, the plane $y + z = 4$ and the cylinder $x = 16 - y^2$.



Solution:

$$\begin{aligned}
V = \iiint\limits_D dzdxdy &= \int_{y=0}^{y=4} \int_{x=0}^{x=16-y^2} \int_{z=0}^{z=4-y} dzdxdy \\
&= \int_{y=0}^{y=4} \int_{x=0}^{x=16-y^2} (4-y)dxdy \\
&= \int_{y=0}^{y=4} (4-y)(16-y^2)dy \\
&= 64y - 8y^2 - \frac{4}{3}y^3 + \frac{1}{4}y^4\Big|_{y=0}^{y=4} \\
&= \frac{320}{3}.
\end{aligned}$$

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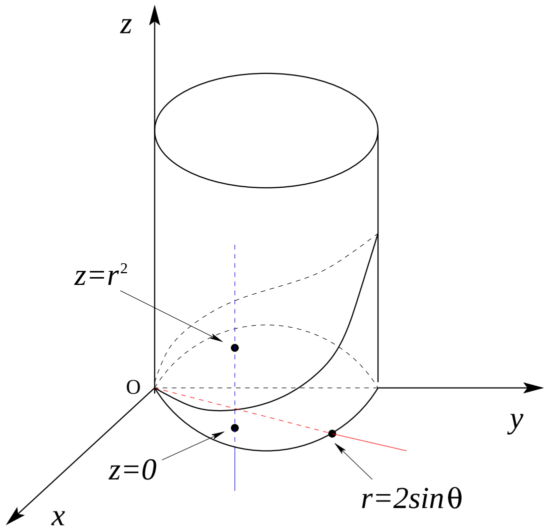
Remark. You may calculate the triple integral in other orders. However, a "good" order will significantly simplify the calculation. In order to choose an appropriate order, we need to look into the solid and its shadow on different coordinate planes.

4. (10 points) Find the volume of the solid D in space bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by $z = x^2 + y^2$.

Hint: Cylindrical coordinates.

Solution:

Sketch the region:



$x^2 + (y-1)^2 = 1 \iff (r\cos\theta)^2 + (r\sin\theta - 1)^2 = 1 \iff r = 2\sin\theta.$
$z = x^2 + y^2 \iff z = r^2.$

$$\begin{aligned} V = \iiint\limits_D r\,dz\,dr\,d\theta &= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\sin\theta} \int_{z=0}^{z=r^2} r\,dz\,dr\,d\theta \\ &= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\sin\theta} r^3\,dr\,d\theta \\ &= \int_{\theta=0}^{\theta=\pi} 4(\sin\theta)^4\,d\theta \quad [\textit{Double-angle identity on page 462}] \\ &= \frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\Big|_{\theta=0}^{\theta=\pi} \\ &= \frac{3}{2}\pi. \end{aligned}$$

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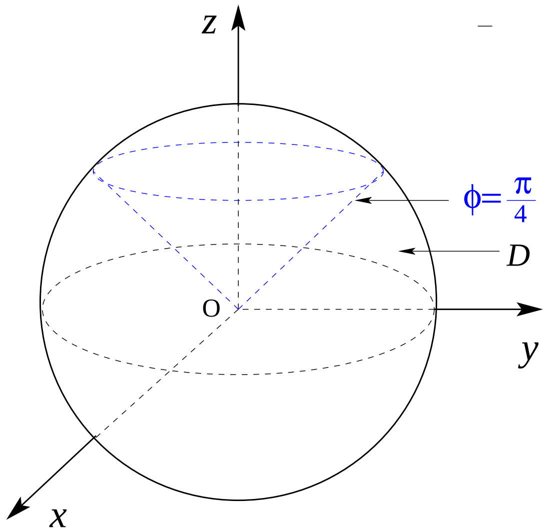
5. (10 points) Find the volume of the solid D bounded below by xy-plane, above by the cone $z=\sqrt{x^2+y^2}$, and on the sides by the sphere $\rho = 2$.

Solution:

Sketch the region:



$z=\sqrt{x^2+y^2} \iff \rho\cos\phi = \sqrt{(\rho\sin\phi)^2} \iff \rho\cos\phi = \rho\sin\phi \iff \rho = 0$ or $\phi = \frac{\pi}{4}$.
Since $\{\rho=0\} \subset \{\phi=\frac{\pi}{4}\}$, so $z=\sqrt{x^2+y^2} \iff \phi=\frac{\pi}{4}$.

$$\begin{aligned} V = \iiint_D \rho^2\sin\phi\, d\rho d\phi d\theta &= \int_{\theta=0}^{\theta=2\pi}\int_{\phi=\frac{\pi}{4}}^{\phi=\frac{\pi}{2}}\int_{\rho=0}^{\rho=2} \rho^2\sin\phi\, d\rho d\phi d\theta \\ &= \frac{8}{3}\int_{\theta=0}^{\theta=2\pi}\int_{\phi=\frac{\pi}{4}}^{\phi=\frac{\pi}{2}} \sin\phi\, d\phi d\theta \\ &= \frac{8}{3}\int_{\theta=0}^{\theta=2\pi} (-\cos\phi)\Big|_{\phi=\frac{\pi}{4}}^{\phi=\frac{\pi}{2}} d\theta \\ &= \frac{8}{3}\int_{\theta=0}^{\theta=2\pi} \frac{\sqrt{2}}{2} d\theta \\ &= \frac{8\sqrt{2}\pi}{3}. \end{aligned}$$

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