Print Your Name: Key - 1

T.A. or Section Number: ____________

1. (14 points) Use the convergence tests from class to determine whether the series converges or diverges. JUSTIFY YOUR ANSWER in a complete argument as we did in class. The justification counts for the majority of the points.

$$\sum_{k=1}^{\infty} \frac{8^k(2k)!}{(k!)(k!)}$$

Use the Ratio Test:

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{8^{n+1}(2(n+1))!}{(n+1)!\,(n+1)!} \cdot \frac{n!\,n!}{8^n(2n)!}$$

$$= \lim_{n\to\infty} \frac{8\cdot \cancel{8^n}(2n+2)(2n+1)\cancel{(2n)!}}{(n+1)\cancel{n!}\,(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}\,\cancel{n!}}{\cancel{8^n}\cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{8\cdot(2n+2)(2n+1)}{(n+1)(n+1)} \text{ (same degrees) } = \frac{8\cdot 4}{1} = 32$$

Since $L = 32 > 1$, by the Ratio Test, this series $\boxed{\text{diverges.}}$

2. (14 points) Use the convergence tests from class to determine whether the series converges or diverges. JUSTIFY YOUR ANSWER in a complete argument as we did in class. The justification counts for the majority of the points.

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^5}$$

Use the Integral Test:

$$\int_2^{\infty} \frac{1}{x(\ln x)^5}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^5}\,dx$$

$u = \ln x$

$du = \frac{1}{x}dx$

$$\lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u^5}\,du = \lim_{b\to\infty} -\frac{1}{4u^4}\Big|_{\ln 2}^{\ln b}$$

$$= \lim_{b\to\infty} \left[\underbrace{-\frac{1}{4(\ln b)^4}}_{\to 0} + \frac{1}{4(\ln 2)^4}\right] = \frac{1}{4(\ln 2)^4} < \infty,$$ so the integral & series both $\boxed{\text{converge}}$

3. (14 points) Express the repeating decimal 0.13131313.... as an infinite series. Then, sum this series to write the repeating decimal as a fraction.

$$0.131313\ldots = 0.13 + 0.0013 + .000013 + \cdots$$

$$= 0.13\left(1 + \frac{1}{100} + \frac{1}{10000} + \cdots\right)$$

$$= 0.13 \sum_{k=0}^{\infty} \left(\frac{1}{100}\right)^k$$

$$= 0.13\left(\frac{1}{1 - 1/100}\right)$$

$$= \frac{13}{100} \cdot \frac{100}{99} = \boxed{\frac{13}{99}}$$

4. (15 points) Determine if the following alternating series converges absolutely, converges conditionally, or diverges. JUSTIFY YOUR ANSWER in a complete argument as we did in class. The justification counts for the majority of the points.

$$\sum_{k=0}^{\infty} (-1)^k \left(1 - \frac{3}{k}\right)^{k^2}$$

Use the Root Test: (on absolute value)

$$R = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \sqrt[n]{\left|\left(1 - \frac{3}{n}\right)^{n^2} (-1)^n\right|}$$

$$= \lim_{n \to \infty} \left(1 - \frac{3}{n}\right)^n = e^{-3} = \frac{1}{e^3}$$

Since $R = \frac{1}{e^3} < 1$, by the Root Test, the absolute value series $\boxed{\text{converges}}$.

Therefore, the alternating series $\boxed{\text{converges absolutely}}$.

Print Your Name: Key-1

T.A. or Section Number: ____________

5. (a) (14 points) Use a convergence test from class to show that the series below is convergent. JUSTIFY your answer in a complete argument.

$$\sum_{k=2}^{\infty} \frac{1}{k^2+6k+5}$$

<u>Comparison Test</u>

Note that for $k \geq 2$, $k^2+6k+5 > k^2$, so $\frac{1}{k^2+6k+5} < \frac{1}{k^2}$. The series $\sum_{k=2}^{\infty} \frac{1}{k^2}$ is a convergent series (p-series with $p=2>1$); thus, by direct comparison, $\sum_{k=2}^{\infty} \frac{1}{k^2+6k+5}$ also <u>converges</u>.

<u>Note</u>: Limit Comparison and Integral tests would also work.

(b) (14 points) The sum in part (a) can be evaluated. Find the sum.

$$\frac{1}{(k^2+6k+5)} = \frac{1}{(k+5)(k+1)} = \frac{A}{k+5} + \frac{B}{k+1}$$

$$\Rightarrow 1 = A(k+1) + B(k+5)$$

$k=-5$: $1 = A(-4)$, $A = -1/4$ $\quad$ $k=-1$: $1 = B(4)$, $B = 1/4$

Thus: $$\sum_{k=2}^{\infty} \frac{1}{k^2+6k+5} = \frac{1}{4}\sum_{k=2}^{\infty}\left[\frac{1}{k+1} - \frac{1}{k+5}\right]$$

$$= \frac{1}{4}\left[\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{4}-\frac{1}{8}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{6}-\frac{1}{10}\right) + \left(\frac{1}{7}-\frac{1}{11}\right) + \cdots\right]$$

$$= \frac{1}{4}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right) = \frac{1}{4}\left(\frac{20+15+12+10}{60}\right) = \frac{1}{4}\left(\frac{57}{60}\right) = \boxed{\frac{57}{240}}$$

6. (15 points) Determine if the following alternating series converges absolutely, converges conditionally, or diverges. JUSTIFY YOUR ANSWER in a complete argument as we did in class. The justification counts for the majority of the points.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{3k+\sqrt{k}}$$

Consider the absolute value: $\sum_{K=1}^{\infty} \frac{1}{3K+\sqrt{K}}$

Note that $3K+\sqrt{K}$ has degree 1.

Limit Comparison with $\sum_{K=1}^{\infty} \frac{1}{K}$, which diverges (harmonic series):

$$\lim_{n\to\infty} \frac{1}{3n+\sqrt{n}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{3n+\sqrt{n}} = \frac{1}{3}$$

Since $0 < \frac{1}{3} < \infty$, both series diverge.

Check for conditional convergence:

(1) Since $3(K+1)+\sqrt{K+1} > 3K+\sqrt{K}$,

$\frac{1}{3(K+1)+\sqrt{K+1}} < \frac{1}{3K+\sqrt{K}} \Rightarrow a_{K+1} < a_K$ Decreasing ✓

(2) $\lim_{n\to\infty} \frac{1}{3n+\sqrt{n}} = 0$ ✓

So the series Converges conditionally

BONUS: (5 points) TRUE OR FALSE: Suppose that $\sum_k a_k = A$ and $\sum_k b_k = B$, where the two series are not identical, $A \neq 0$, $B \neq 0$, and $b_n > 0$ for all n. Suppose that $\sum_k \frac{a_k}{b_k}$ converges. Then

$$\sum_k \frac{a_k}{b_k} = \frac{A}{B}.$$

If the statement is true, prove it in general. If the statement is false, provide a counterexample to show that it fails.

This statement is false.

A counterexample:

Let $\sum_K a_K = \sum_{K=0}^{\infty} \left(\frac{1}{3}\right)^K$, then $\sum a_K = \frac{1}{1-1/3} = \frac{3}{2}$.

$\sum_K b_K = \sum_{K=0}^{\infty} \left(\frac{3}{4}\right)^K$, " $\sum b_K = \frac{1}{1-3/4} = 4$.

But $\sum_K \frac{a_K}{b_K} = \sum_{K=0}^{\infty} \left(\frac{4}{9}\right)^K = \frac{1}{1-4/9} = \frac{9}{5} \neq \frac{3/2}{4}$.

Print Your Name: Key-2

T.A. or Section Number: ____________

1. (15 points) Determine if the following alternating series converges absolutely, converges conditionally, or diverges. JUSTIFY YOUR ANSWER in a complete argument as we did in class. The justification counts for the majority of the points.

$$\sum_{k=1}^{\infty}(-1)^k\left(1-\frac{7}{k}\right)^{k^2}$$

Root test on $\sum_{K=1}^{\infty}\left|(-1)^k\left(1-\frac{7}{k}\right)^{k^2}\right| = \sum_{K=1}^{\infty}\left(1-\frac{7}{k}\right)^{k^2}$:

$R = \lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\left(\left(1-\frac{7}{n}\right)^{n^2}\right)^{1/n} = \lim_{n\to\infty}\left(1-\frac{7}{n}\right)^n = e^{-7}$.

Since $e^{-7}<1$, the absolute value series converges.

Thus, the alternating series converges absolutely.

2. (14 points) Express the repeating decimal 0.27272727.... as an infinite series. Then, sum this series to write the repeating decimal as a fraction.

$$0.2727\ldots = \frac{27}{100}+\frac{27}{10^4}+\frac{27}{10^6}+\cdots$$

$$= \frac{27}{100}\left[1+\frac{1}{10^2}+\frac{1}{10^4}+\cdots\right]$$

$$= \frac{27}{100}\sum_{K=0}^{\infty}\left(\frac{1}{100}\right)^k$$

$$= \frac{27}{100}\cdot\frac{1}{1-1/100} = \frac{27}{100}\cdot\frac{100}{99} = \boxed{\frac{27}{99}}$$

3. (14 points) Use the convergence tests from class to determine whether the series converges or diverges. JUSTIFY YOUR ANSWER in a complete argument as we did in class. The justification counts for the majority of the points.

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$$

Integral test: $\int_2^\infty \frac{dx}{x(\ln x)^3} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^3}$

$u = \ln x$, $du = \frac{1}{x}dx$: $\lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u^3}\,du = \lim_{b\to\infty} -\frac{1}{2u^2}\Big|_{\ln 2}^{\ln b}$

$= \lim_{b\to\infty} \Big[\underbrace{-\frac{1}{2(\ln b)^2}}_{\to 0} + \frac{1}{2(\ln 2)^2}\Big] = \frac{1}{2(\ln 2)^2} < \infty$

Since the integral converges, the series also converges.

4. (15 points) Determine if the following alternating series converges absolutely, converges conditionally, or diverges. JUSTIFY YOUR ANSWER in a complete argument as we did in class. The justification counts for the majority of the points.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{5k+\sqrt{k}}$$

Check for absolute convergence:

$\sum_{k=1}^{\infty} \left|\frac{(-1)^k}{5k+\sqrt{k}}\right| = \sum_{k=1}^{\infty} \frac{1}{5k+\sqrt{k}}$

Note that $\sqrt{k} < k$, so

$5k+\sqrt{k} < 5k+k = 6k$

and $\frac{1}{5k+\sqrt{k}} > \frac{1}{6k} = \frac{1}{6}\cdot\frac{1}{k}$

The series $\frac{1}{6}\sum_{k=1}^{\infty}\frac{1}{k}$ diverges (multiple of harmonic series), so $\sum_{k=1}^{\infty} \frac{1}{5k+\sqrt{k}}$ diverges by the Comparison Test.

Check for conditional convergence:

(1) $\lim_{n\to\infty} \frac{1}{5n+\sqrt{n}} = 0$ ✓

(2) Since $k < k+1$, then

$5k+\sqrt{k} < 5(k+1)+\sqrt{k+1}$

so $\frac{1}{5k+\sqrt{k}} > \frac{1}{5(k+1)+\sqrt{k+1}}$

$\Rightarrow a_k > a_{k-1}$, so the terms are decreasing ✓

$\Rightarrow$ the series converges conditionally

Print Your Name: Key -2

T.A. or Section Number: ____________

5. (a) (14 points) Use a convergence test from class to show that the series below is convergent. JUSTIFY your answer in a complete argument.

$$\sum_{k=2}^{\infty} \frac{1}{k^2+5k+4}$$

Limit Comparison with $\sum_{K=2}^{\infty} \frac{1}{K^2}$, which converges (p-series with $p=2>1$):

$\lim_{n\to\infty} \frac{1}{n^2+5n+4} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+5n+4} = 1$

Since $0<1<\infty$, both series converge

(b) (14 points) The sum in part (a) can be evaluated. Find the sum.

$$\frac{1}{K^2+5K+4} = \frac{1}{(K+1)(K+4)} = \frac{A}{K+1} + \frac{B}{K+4}$$

$\Rightarrow 1 = A(K+4) + B(K+1)$

$K=-4: 1 = B(-3), \quad B=-1/3$

$K=-1: 1 = A(3), \quad A = 1/3$

So $\sum_{K=2}^{\infty} \frac{1}{K^2+5K+4} = \frac{1}{3} \sum_{K=2}^{\infty} \left[\frac{1}{K+1} - \frac{1}{K+4}\right]$

$= \frac{1}{3}\left[\left(\frac{1}{3}-\frac{1}{6}\right) + \left(\frac{1}{4}-\frac{1}{7}\right) + \left(\frac{1}{5}-\frac{1}{8}\right) + \left(\frac{1}{6}-\frac{1}{9}\right) + \cdots\right]$

$= \frac{1}{3}\left[\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right] = \frac{1}{3}\left[\frac{20+15+12}{60}\right] = \frac{1}{3}\left[\frac{37}{60}\right] = \boxed{\frac{47}{180}}$

6. (14 points) Use the convergence tests from class to determine whether the series converges or diverges. JUSTIFY YOUR ANSWER in a complete argument as we did in class. The justification counts for the majority of the points.

$$\sum_{k=1}^{\infty} \frac{6^k (2k)!}{(k!)(k!)}$$

Ratio test

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{6^{n+1}(2n+2)!}{(n+1)!(n+1)!} \cdot \frac{n!\,n!}{6^n(2n)!}$$

$$= \lim_{n\to\infty} \frac{6\cdot\cancel{6^n}(2n+2)(2n+1)\cancel{(2n)!}}{(n+1)\cancel{n!}(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}\,\cancel{n!}}{\cancel{6^n}\cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{6\cdot(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{6\cdot 2\cdot 2}{1} \text{ (since degrees are equal)}$$

$= 24 > 1$, so the series diverges

BONUS: (5 points) TRUE OR FALSE: Suppose that $\sum_k a_k = A$ and $\sum_k b_k = B$, where the two series are not identical, $A \neq 0$, $B \neq 0$, and $b_n > 0$ for all n. Suppose that $\sum_k \frac{a_k}{b_k}$ converges. Then

$$\sum_k \frac{a_k}{b_k} = \frac{A}{B}.$$

If the statement is true, prove it in general. If the statement is false, provide a counterexample to show that it fails.

See Form 1.