TEST 1 Name Key

Math 2551 D Section ____ February 10, 2016

No books, notes, calculators, cell phones, or other electronic devices are allowed. Show your work and justify your answer to receive credit. Work neatly. There is a total of 100 points. Put your name and section number on each page of the test.

1. Consider the curve $\mathbf{r}(t) = t^2\,\mathbf{i} + (4t-2)\,\mathbf{j} + 3\cos(\pi t)\,\mathbf{k}$. Set up the appropriate integral (with limits) to compute the length of the curve between the points $P(1,2,-3)$ and $Q(16,14,3)$. [You do not need to complute the integral.]

$\underline{v}(t) = \underline{r}'(t) = 2t\,\underline{i} + 4\,\underline{j} - 3\pi\sin(\pi t)\,\underline{k}$ $\quad |\underline{v}(t)| = \sqrt{(2t)^2 + 4^2 + (-3\pi\sin\pi t)^2}$

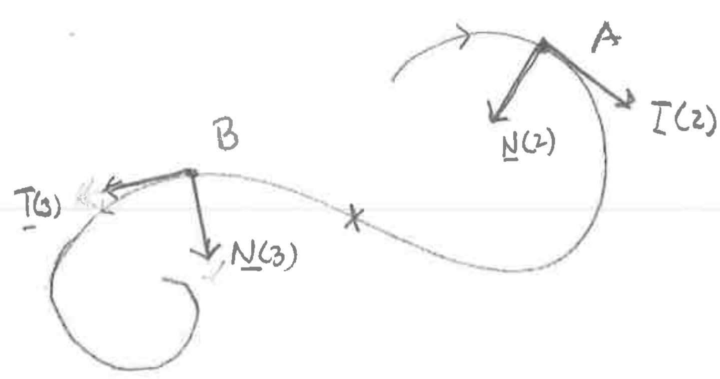
$= \sqrt{4t^2 + 16 + 9\pi^2\sin^2\pi t}$

$P(1,2,-3)$: $1 = t^2$; $2 = 4t-2 \Rightarrow t = 1$; $-3 = 3\cos\pi t$ $\quad t=1 \Rightarrow t^2 = 1$ ✓ $\quad 3\cos\pi t = 3(-1) = -3$ ✓ $\quad t = 1$ at P

$Q(16,14,3)$: $16 = t^2$; $14 = 4t-2 \Rightarrow 4t = 16$, $t = 4$; $3 = 3\cos\pi t$ $\quad t=4 \Rightarrow t^2 = 16$ ✓ $\quad 3\cos\pi t = 3\cos 4\pi = 3$ ✓ $\quad t = 4$ at Q

So length of curve between P and Q is $= \boxed{L = \int_1^4 \sqrt{4t^2 + 16 + 9\pi^2\sin^2\pi t}\, dt}$

2. Consider the sketch below of a plane curve traced (once) by a vector function $\mathbf{r}(t)$. The point A is $\mathbf{r}(2)$ and the point B is $\mathbf{r}(3)$.
   a. Sketch and label the unit tangent vectors $\mathbf{T}(2)$ and $\mathbf{T}(3)$.
   b. Sketch and label the unit normal vectors $\mathbf{N}(2)$ and $\mathbf{N}(3)$.
   c. Put an X at the point on the curve at which the curvature is a minimum.



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3. Consider the curve $\mathbf{r}(t) = (1+t^3)\,\mathbf{i} + (6-2t)\,\mathbf{j} + t^2\,\mathbf{k}$.

a. Find the parametric equations for the line that is tangent to the curve at the point when $t = 1$.

$\underline{v}(t)\ \underline{r}'(t) = 3t^2\underline{i} - 2\underline{j} + 2t\underline{k}$

$\underline{v}(1) = \underline{r}'(1) = 3\underline{i} - 2\underline{j} + 2\underline{k}$

$\underline{r}(1) = 2\underline{i} + 4\underline{j} + \underline{k}$

Tangent line to curve at pt where $t=1$.

$x = 2 + 3t$

$y = 4 + (-2)t$

$z = 1 + 2t$

$t \in \mathbb{R}$

b. Find the unit tangent vector $\mathbf{T}$ and the acceleration vector $\mathbf{a}$ at the point when $t = 1$.

$|\underline{v}(1)| = \sqrt{3^2 + (-2)^2 + 2^2} = \sqrt{9+4+4} = \sqrt{17}$

So $\underline{T}(1) = \dfrac{\underline{v}(1)}{|\underline{v}(1)|} = \dfrac{3\underline{i} - 2\underline{j} + 2\underline{k}}{\sqrt{17}} = \dfrac{3}{\sqrt{17}}\underline{i} - \dfrac{2}{\sqrt{17}}\underline{j} + \dfrac{2}{\sqrt{17}}\underline{k}$

$\underline{a}(t) = \underline{v}'(t) = 6\underline{i} + 0\underline{j} + 2\underline{k}$

$\underline{a}(1) = 6\underline{i} + 2\underline{k}$

c. Find the curvature κ at $t = 1$.

$\kappa = \dfrac{|\underline{v} \times \underline{a}|}{|\underline{v}|^3}$

$\underline{v} \times \underline{a} = \begin{vmatrix} i & j & k \\ 3 & -2 & 2 \\ 6 & 0 & 2 \end{vmatrix} = \underline{i}((-2)(2) - 0) - \underline{j}(3(2) - 6(2)) + \underline{k}(3(0) - 6(-2))$

$= -4\underline{i} + 6\underline{j} + 12\underline{k}$

So $|\underline{v} \times \underline{a}| = \sqrt{(-4)^2 + 6^2 + (12)^2} = \sqrt{16+36+144}$

$= \sqrt{196} = \sqrt{(49)4} = 7(2) = 14$

$= \dfrac{14}{(\sqrt{17})^3} = \dfrac{14}{17\sqrt{17}}$

d. Find the normal component of acceleration at $t = 1$.

$a_N = \sqrt{|a|^2 - (a_T)^2} \rightarrow$

$|\underline{a}| = |6\underline{i} + 2\underline{k}| = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$

$a_T = \frac{d}{dt}(|v(t)|) = \frac{d}{dt}(|3t^2\underline{i} - 2\underline{j} + 2t\underline{k}|)$

$= \frac{d}{dt}\left(\sqrt{9t^4 + 4 + 4t^2}\right) = \frac{1}{2}(9t^4 + 4 + 4t^2)^{-\frac{1}{2}}(36t^3 + 8t)$

$\frac{d}{dt}(|v(t)|)\Big|_{t=1} = \frac{1}{2}\dfrac{1}{\sqrt{9+4+4}}(44) = \dfrac{22}{\sqrt{17}}$

So $a_N = \sqrt{(2\sqrt{10})^2 - \left(\dfrac{22}{\sqrt{17}}\right)^2}$

$= \sqrt{40 - \dfrac{(22)^2}{17}} = \sqrt{\dfrac{196}{17}} = \dfrac{14}{\sqrt{17}}$

Another solution for a_N:

$a_N = \kappa|\underline{v}|^2 = \underset{\text{from part c}}{\dfrac{14}{17\sqrt{17}}}\ \underset{\text{from part b}}{(\sqrt{17})^2} = \dfrac{14}{17\sqrt{17}}(17) = \dfrac{14}{\sqrt{17}}$ $\boxed{a_N = \dfrac{14}{\sqrt{17}}}$

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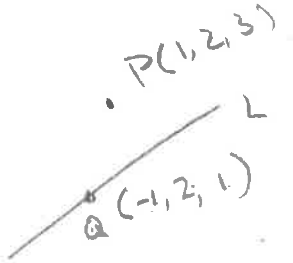
4. Consider the line

$$L : x = -1 + t, y = 2 + t, z = 1 - t, \infty < t < \infty$$

The point $P(1, 2, 3)$ is not on L. Find the equation of the plane that contains the line L and the point $P(1, 2, 3)$.

The vector $\underline{u} = (1)\underline{i} + (1)\underline{j} + (-1)\underline{k}$ is parallel to L
We'll form another vector (not parallel to $\underline{u}$) which is parallel to the plane by forming the vector $\overrightarrow{PQ}$ where Q is a point on L. Letting $t=0$, we have that $Q = (-1, 2, 1)$ is on L



so $\overrightarrow{PQ} = (-1-1)\underline{i} + (2-2)\underline{j} + (1-3)\underline{k} = -2\underline{i} - 2\underline{k}$ is parallel to the plane

$\underline{n} = \underline{u} \times \overrightarrow{PQ}$ will be normal to the plane

$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & -1 \\ -2 & 0 & -2 \end{vmatrix} = \underline{i}(-2-0) - \underline{j}(-2-2) + \underline{k}(0-(-2))$$

$= -2\underline{i} + 4\underline{j} + 2\underline{k}$ ← This vector is normal (orthogonal) to the plane. $P(1,2,3)$ is on the plane

$\boxed{\text{Then the plane is } -2(x-1) + 4(y-2) + 2(z-3) = 0 \quad -2x + 4y + 2z = 10}$

5. Let S be the surface described by $z = g(x, y) = \dfrac{y-3}{3x^2+1}$. Sketch and label the level curves of $g(x, y) = c$ for $c = 1$ and $c = -2$.

$g(x,y) = 1$

$\dfrac{y-3}{3x^2+1} = 1$

$y - 3 = 3x^2 + 1$

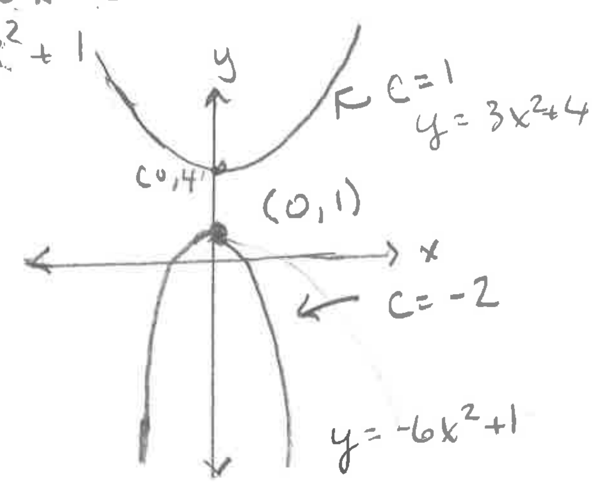
$y = 3x^2 + 4$

$g(x,y) = -2$

$\dfrac{y-3}{3x^2+1} = -2$

$y - 3 = -6x^2 - 2$

$y = -6x^2 + 1$



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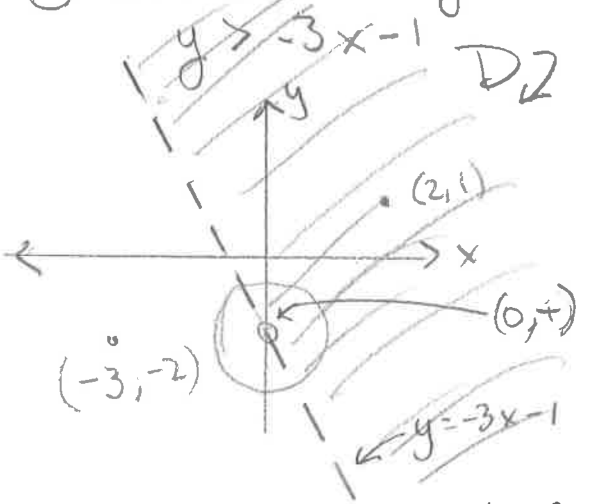
6. Let $f(x,y) = 2 + \ln(3x + y + 1)$. Let $D =$ domain of f.
   a. Find and sketch D, the domain of f.
   b. Is the domain a bounded set? Why or why not?
   c. Give an example of a boundary point of D.

ⓐ We need $3x+y+1 > 0$ since $\ln(t)$ is only defined for $t > 0$



Check a pt to see if we want that "side" of the plane

$(2,1)$: $-3(2)-1 = -7 < 1$ ✓

This pt satisfies $y > -3x-1$

$(-3,-2)$: $-3(-3)-1 = 8 \not< -2$ "Not less than"

So we don't want this "side" (the left side) of the plan

ⓑ The domain is not bounded since we cannot enclose the domain in a large circle centered at the origin. The domain is unbounded

ⓒ Any point on the dashed line is a bdy pt. For example $(0,-1)$ is a boundary pt of D

Any disk centered at $(0,-1)$ will contain point in D and points not in D. So $(0,-1)$ is a boundary pt of D.

7. Find $\lim_{(x,y)\to(0,0)} \dfrac{7x^4 - y^3}{(2x^2+5y)^2}$ or show that the limit does not exist. Justify your answer.

Consider

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{7x^4-y^3}{(2x^2+5y)^2} = \lim_{y\to 0} \frac{0-y^3}{(0+5y)^2}$$

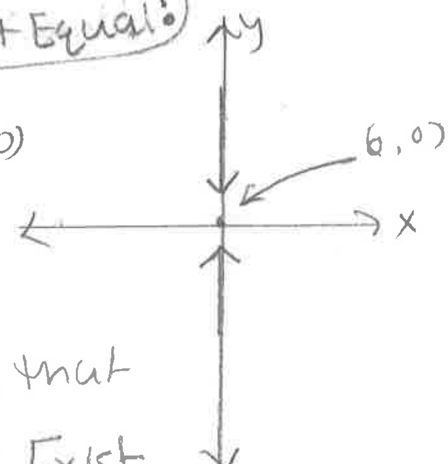
$$= \lim_{y\to 0} \frac{-y^3}{25y^2} = \lim_{y\to 0} \frac{-y}{25} = \boxed{0}$$

Also

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{7x^4-y^3}{(2x^2+5y)^2} = \lim_{x\to 0} \frac{7x^4}{(2x^2+0)^2}$$

$$= \lim_{x\to 0} \frac{7x^4}{4x^4} = \lim_{x\to 0} \frac{7}{4} = \boxed{\frac{7}{4}}$$

Not Equal!



Since the limit as we approach $(0,0)$ along the y-axis (where $x=0$) is 0 and the limit as we approach $(0,0)$ along the x-axis (where $y=0$) is $7/4$ and $0 \neq 7/4$, we can conclude that the $\lim_{(x,y)\to(0,0)} \dfrac{7x^4-y^3}{(2x^2+5y)^2}$ Does Not Exist