Q1 TOOK 10 MINUTES (including writing it down)

TOTAL 58 MINUTES

① 9 | 9 | 22 | 32 | 32 | 33 | 39 | 39 | 42 | 49 | 52 | 58 | 65 | 70

ⓐ $n = 14$

$\boxed{\text{min} = 9}$ $\boxed{\text{max} = 70}$ $\boxed{\text{median} = \frac{39+39}{2} = 39}$

$$\text{mean} = \frac{9+9+22+32+32+33+39+39+42+49+52+58+65+70}{14}$$

Q1 MED Q3

9 9 22 32 32 33 39 | 39 42 49 52 58 65 70

ⓑ 1993 → 9

1994 → 9

$IQR = Q3 - Q1 = 20$

$$\underbrace{32}_{Q1} - 1.5 * \underbrace{20}_{IQR} = 2$$

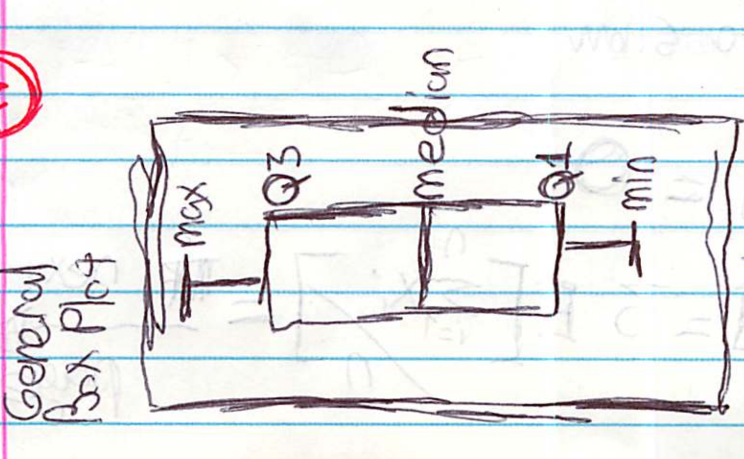
We should also calculate the VARIANCE but I do not have a calculator right now

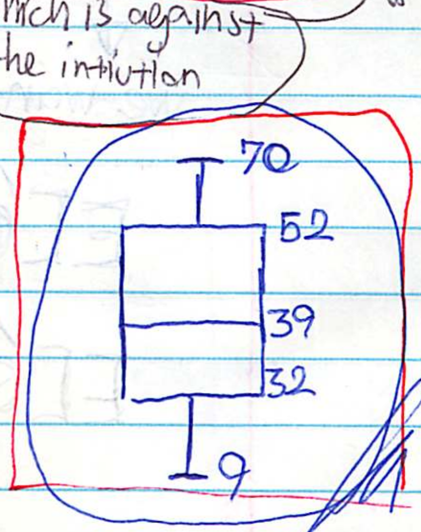
$9 \nless 2$

not an outliner

which is against the intuition

ⓒ





Q2 TOOK 26 MINUTES

Q2 $f(x|\theta) = c(1+\theta x) \quad -1 \leq x \leq 1$

a $\int_{-\infty}^{\infty} f(x|\theta)\,dx = 1 = \int_{-1}^{1} c(1+\theta x)\,dx$

$$\frac{1}{c} = x + \frac{\theta x^2}{2}\Big]_{-1}^{1} = \underbrace{\left(1+\theta/2\right)}_{x=1} - \underbrace{\left(-1+\theta/2\right)}_{x=-1}$$

$$= 2 \rightarrow \boxed{c = 1/2}$$

let's double check

$$\int_{-1}^{1} \frac{(1+\theta x)}{2}\,dx = \frac{x}{2} + \frac{\theta x^2}{4}\Big]_{-1}^{1} = 1/2 - (-1/2) = 1$$

b $\hat{\theta} = 3\bar{X} = 3\dfrac{\sum_{i=1}^{n} X_i}{n}$ is an unbiased estimator.

we want to show

$$E[\hat{\theta}] = \theta$$

$$E[\hat{\theta}] = 3\,E\left[\frac{\sum_{i=1}^{n} X_i}{n}\right] = \xrightarrow[\text{page}]{\text{next}}$$

Q2 (b) (continued)

$$E[\hat{\Theta}] = 3\, \tfrac{1}{n} \sum_{i=1}^{n} E[X_i]$$

$$= 3\, E[X_i]$$

$E[\hat{\Theta}] = 3E[X_i]$

$$E[X_i] = \int_{-1}^{1} x \left(\tfrac{1}{2} [1 + \Theta x] \right) dx$$

$$= \tfrac{1}{2} \int_{-1}^{1} (x + \Theta x^2)\, dx$$

$$= \tfrac{1}{2} \left(\frac{x^2}{2} + \frac{\Theta x^3}{3} \right) \Bigg]_{-1}^{1}$$

will vanish

$$= \tfrac{1}{2} \left(\Theta/3 - (-\Theta/3) \right)$$

$$= \Theta/3$$

$E[X_i] = \Theta/3$

unbiased

Then $\boxed{E[\hat{\Theta}] = 3E[X_i] = 3\,\Theta/3 = \Theta}$

Q.E.D.

ⓒ $V[\hat{\Theta}] = V[3\bar{X}]$

$= V\left[\frac{3}{n}\sum_{i=1}^{n} X_i\right]$

$= \left(\frac{3}{n}\right)^2 V\left[\sum_{i=1}^{n} X_i\right]$

I assume X_i's are independent

$= \left(\frac{3}{n}\right)^2 \left(V[X_1] + \dots V[X_n]\right)$

since X_i's are identically distributed

$= \frac{9}{n^2}\left(n V[X_i]\right)$

$= \frac{9}{n} V[X_i]$

$E(X_i^2) - E(X_i)^2$

by part ⓑ

$\left(\frac{\Theta}{3}\right)^2$

$E[X_i^2] = \int_{-1}^{1} x^2\left(\frac{1}{2}(1+\Theta x)\right)dx$

$= \frac{1}{2}\int_{-1}^{1}\left(x^2 + \Theta x^3\right)dx$

$= \frac{1}{2}\left(\frac{x^3}{3} + \frac{\Theta x^4}{4}\right)\Big|_{x=-1}^{x=1}$ $\longrightarrow$ next page

Q2 c (continued)

$$E[X_i^2] = \frac{1}{2}\left(\frac{1}{3} - \left(-\frac{1}{3}\right)\right) = \frac{1}{3}$$

Then

$$V[\hat{\theta}] = \left(\frac{9}{n}\right)\underbrace{V[X_i]}_{\frac{1}{3} - \left(\frac{\theta}{3}\right)^2 = \frac{3}{9} - \frac{\theta^2}{9}}$$

$$\boxed{V[\hat{\theta}] = \frac{3-\theta^2}{n}}$$

d $E[\hat{\theta}] = \theta \rightarrow$ unbiased

$$V[\hat{\theta}] = \frac{(3-\theta^2)}{n}$$

as $n \rightarrow \infty$ $V[\hat{\theta}] \rightarrow 0$ $\rightarrow$ consistent

e $MSE(\hat{\theta}) = V(\hat{\theta})$ since $\hat{\theta}$ is an unbiased estimator of θ

$$MSE(\hat{\theta}) = E(\hat{\theta}-\theta)^2 = V[\hat{\theta}] + \cancel{(E[\hat{\theta}]-\theta)^2}$$

0 since unbiased estimator

Q3 TOOK 22 MINUTES

(3) $f(x|\theta) = \theta x^{\theta-1} \quad 0 \le x \le 1$

(a) $E[X|\theta] = \int_{-\infty}^{\infty} x f_X(x|\theta)\,dx$

$= \int_0^1 x \theta x^{\theta-1}\,dx$

$= \int_0^1 \theta x^{\theta}\,dx$

$= \theta \left(\frac{x^{\theta+1}}{\theta+1} \right) \Bigg|_{x=0}^{x=1}$

$= \frac{\theta}{\theta+1}(1-0) = \frac{\theta}{\theta+1}$

$\int_0^1 \theta x^{\theta-1} = \theta \int_0^1 x^{\theta-1}\,dx = \theta \left[\frac{x^{\theta}}{\theta} \right]_0^1 = 1$

let $g(\theta) \equiv E[X|\theta] = \frac{\theta}{\theta+1}$

→ next page

③ ⓐ (continued)

set $\bar{x} \left(= \frac{\sum x_i}{n} \right) = g(\hat{\theta})$

$$\bar{x} = \frac{\hat{\theta}}{\hat{\theta}+1} = \frac{\hat{\theta}+1-1}{\hat{\theta}+1}$$

$$\bar{x} = 1 - \frac{1}{\hat{\theta}+1}$$

$$\frac{1}{\hat{\theta}+1} = 1 - \bar{x}$$

$$\frac{1}{1-\bar{x}} = \hat{\theta}+1$$

$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$

$$\hat{\theta} = \frac{1}{1-\bar{x}} - 1$$ or equivalently

$$\hat{\theta} = \frac{1-(1-\bar{x})}{1-\bar{x}} = \frac{\bar{x}}{1-\bar{x}}$$

3 b

$X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$ data

$$\text{lik}(\theta) = f(x_1, x_2, \ldots, x_n | \theta)$$

assuming independence

$$\text{lik}(\theta) = \prod_{i=1}^{n} f(x_i | \theta)$$

instead maximize logarithm function

$$\ell(\theta) = \sum_{i=1}^{n} \log\left(f(x_i|\theta)\right)$$

$$\theta x^{\theta - 1}$$

$$= \sum_{i=1}^{n} \left(\log(\theta) + \log(x_i^{\theta-1}) \right)$$

$$= \sum_{i=1}^{n} \left(\log(\theta) + (\theta - 1)\log(x_i) \right)$$

$$n\log(\theta) + (\theta - 1)\sum_{i=1}^{n}\log(x_i)$$

$$\left.\frac{d\,\ell(\theta)}{d\theta}\right|_{\theta=\theta^*} = 0 = \frac{n}{\theta^*} + \sum_{i=1}^{n}\log(x_i) = 0$$

maximizer

$$\frac{d^2\ell(\theta)}{d\theta^2} = -\frac{n}{\theta^2} < 0$$

$$\boxed{\theta^* = -\frac{n}{\sum_{i=1}^{n}\log(x_i)}}$$

with single point x_i instead of $\bar{x} = \frac{\sum x_i}{n}$

3) 6) $lik(\theta) = f(x|\theta) = \theta x^{\theta-1}$

$$\log(lik(\theta)) = \log(\theta x^{\theta-1})$$
$$= \log(\theta) + \log(x^{\theta-1})$$
$$= \log(\theta) + (\theta-1)\log(x)$$

$$\left.\frac{d \log(lik(\theta))}{d\theta}\right|_{\theta=\theta^*} = 0 = \frac{1}{\theta^*} + \log(x)$$

$$\theta^* = \cancel{} - \frac{1}{\log(x)}$$

$$\frac{d^2 \log(lik(\theta))}{d\theta^2} = -1/\theta^2 < 0$$

maximizer

if we will use a single point to estimate θ^*

$$\frac{n}{\theta} = -\sum_{i=1}^{n} \log(x_i)$$

$$\frac{n}{\sum_{i=1}^{n} \log(x_i)} = \theta^*$$