Math 2602 K1-K3
Spring 2014
Midterm 1 practice
1/30/14
Time Limit: 80 Minutes

Name (Print): ____________________

Section ____________________

This exam contains 5 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, but you *can* use non symbolic calculator on this exam.

You are required to show your work on each problem on this exam.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

1. (10 points) Show that $n(n+1)(2n+1)$ is divisable by 3 for all integers n.

Proof by cases:

Case 1: $n = 3k$, k - an integer

$n(n+1)(2n+1) = 3 \cdot k(3k+1)(6k+1)$ is div. by 3.

Case 2: $n = 3k+1$, k is an integer

$n(n+1)\cdot(2n+1) = (3k+1)(3k+2)(6k+3) = 3(2k+1)(3k+1)(\ldots)$

$\vdots (3k+2)$ is divisable by 3.

Case 3 $n = 3k+2$: $n(n+1)(2n+1) = (3k+2)$

$\cdot (3k+3)\cdot(6k+5) = 3(k+1)(3k+2)(6k+5)$ is div by 3.

2. (10 points) Prove that if a and $a+b$ are rational numbers then b is rational.

As both a and $a+b$ are rationals we can write them as fractions.

$a = \frac{m}{n}$ and $a+b = \frac{k}{\ell}$ for integers n, m, k, ℓ.

Then $b = (a+b) - a = \frac{k}{\ell} - \frac{m}{n} = \frac{kn - \ell m}{\ell n}$

$kn - \ell m$ and ℓn are integers as well, so b is a rational number.

3. (10 points) Show that $\neg(p \to q) \leftrightarrow (p \wedge \neg q)$ is a tautology.

p	q	$\neg q$	$p \wedge \neg q$	$p \to q$	$\neg(p \to q)$	$\neg(p \to q) \leftrightarrow (p \wedge \neg q)$
T	T	F	F	T	F	T
T	F	T	T	F	T	T
F	T	F	F	T	F	T
F	F	T	F	T	F	T

So $\neg(p \to q) \leftrightarrow (p \wedge \neg q)$ is always true. ■

4. (10 points) Show the following logical equivalence $p \to (q \vee r) \Leftrightarrow (p \wedge \neg q) \to r$.

The only case when $p \to (q \vee r)$ is false is: $p = T$, $q \vee r = F$ i.e. $p = T$, $q = F$, $r = F$.

The only case that the statement $(p \wedge \neg q) \to r$ is false is: $p \wedge \neg q = T$, $r = F$ i.e. $p = T$, $q = F$, $r = F$

We got the same values for p, q, r.
Hence the two statements are logically equivalent. ■

PS. You could also do it with truth table with 8 rows.

PPS. Try to do it using the rules of logical equivalence.

5. (10 points) The binary relation $\mathcal{R}$ is defined by $\mathcal{R} = \{(x,y) \in \mathbb{R}^2 | \ x \le y\}$. Is $\mathcal{R}$ a) Reflexive, b) Symmetric, c)Antisymmetric, d)Transitive ?

Justify your answer.

a) For all $x \in \mathbb{R}$ $x \le x$ $\Rightarrow$ Reflexive

b) $x \le y$ does not imply that $y \le x$
e.g. $1 \le 2$ but $2 \nleq 1$. Not Symmetric.

c) $x \le y$ and $y \le x$ implies that $x = y$
hence it's Antisymmetric.

d) $x \le y$ and $y \le z$ implies that
$x \le z$, so it's Transitive.

6. (10 points) For integers a and b define $a \sim b$ if $a - b$ is divisible by 5.

a) Show that $\sim$ defines an equivalence relation on $\mathbb{Z}$.

b) What are the equivalence classes for $\sim$?

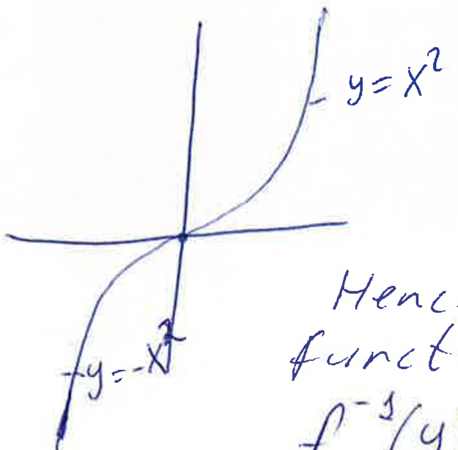
a) $a \sim a$ as $a - a = 0$ is divisible by 5.
$\sim$ is ~~symmetric~~ reflexive.
if $a - b$ is divisible by 5 then
so is $b - a$, hence $\sim$ is symmetric.
If $a \sim b$ and $b \sim c$ then
$a - b = 5k$, $b - c = 5m$ for integers k, m.
$a - c = 5k + 5m = 5(k-m)$ is divisible by 5.
hence $\sim$ is transitive.

b) $\bar{0} = 5\mathbb{Z}$, integers in $5k$ form $k \in \mathbb{Z}$
$\bar{1} = 5\mathbb{Z} + 1$ in form $5k+1$. $\bar{2} = 5\mathbb{Z} + 2$
$\bar{3} = 5\mathbb{Z} + 3$, $\bar{4} = 5\mathbb{Z} + 4$.

7. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x|x|$. Check if f is one-to-one and onto function. If so find the inverse function of f.

For $x \geq 0$ $f(x) = x^2$ increasing function

For $x < 0$ $f(x) = -x^2$ increasing function



Hence it's one-to-one and onto.

$y = x^2 \Rightarrow x = \sqrt{y}$ for $y \geq 0$

$y = -x^2 \Rightarrow x^2 = -y$, $x = \sqrt{-y}$ for $y < 0$

Hence $x = \sqrt{|y|}$ is the inverse function for all y's.

$f^{-1}(y) = \sqrt{|y|}$.

8. (10 points) Check if the sets have the same cardinality and justify your answer.

a) $\{\sqrt{n} \mid n \in \mathbb{N} \text{ and } n > 10\}$ and $\mathbb{N}$.

b) The intervals (a, b) and (c, ∞). Assume $a < b$.

a) $A = \{\sqrt{n} \mid \overset{n>10}{n \in \mathbb{N}}\} = \{\sqrt{11}, \sqrt{12}, \sqrt{13}, \dots\}$

$f : \mathbb{N} \to A$ $f(n) = \sqrt{n+10}$ is a one-to-one and onto from $\mathbb{N}$ to A. $f(1) = \sqrt{11}$, $f(2) = \sqrt{12}$, ...

Hence $|A| = |\mathbb{N}|$ - infinitely countable.

b) First let's show that $|(0,1)| = |(a,b)|$

$f : (0,1) \to (a,b)$, $f(x) = a + (b-a)x$

f is one-to-one and onto.

Now let's show that $|(0,1)| = |(c,\infty)|$

$g : (0,1) \to (c,\infty)$ $g(x) = c + \frac{1}{x}$ g is a bijection, hence $|(a,b)| = |(0,1)| = |(c,\infty)|$