Full name: Math 2551, Section D___

# Quiz 7 — §15.8-16.1

Please ***clearly*** show all work. Scientific calculators are allowed, but no graphing calculators!

**(1)** Let $S$ be the region in the $xy$-plane bounded by the curves $y = x$, $y = 4x$, $y = 1/x$, and $y = 4/x$. Determine a change of variables $x = g(u, v)$, $y = h(u, v)$ which transforms $S$ into a rectangle $R$ in the $uv$-plane. Then express the integral

$$\iint_S (x^2 + y^2)\, dx\, dy$$

as an iterated integral in terms of your variables $u$ and $v$. **No need to evaluate!** [10 points]

One natural coordinate transformation to take is

$$\begin{cases} u = y/x \\ v = xy \end{cases}$$

so that the corresponding region $R$ in the $uv$-plane is the rectangle $1 \le u \le 4$, $1 \le v \le 4$. Solving for $x$ and $y$ in terms of $u$ and $v$ yields

$$\begin{cases} x = \sqrt{v/u} \\ y = \sqrt{uv} \end{cases}$$

from which we can compute the Jacobian determinant of the coordinate change:

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{pmatrix} -\frac{1}{2u}\sqrt{\frac{v}{u}} & \frac{1}{2}\frac{1}{\sqrt{uv}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{pmatrix} = -\frac{1}{2u}$$

The change of variables formula then gives

$$\iint_S (x^2 + y^2)\, dx\, dy = \iint_R \left(\frac{v}{u} + uv\right)\frac{1}{2u}\, du\, dv = \boxed{\int_1^4 \int_1^4 \left(\frac{v}{2u^2} + \frac{v}{2}\right) du\, dv}$$

**(2)** Let $C$ be the curve segment parameterized by $\mathbf{r}(t) = (1+t)\mathbf{i} + (t^2-1)\mathbf{j}$, where $-1 \le t \le 1$.
**(a)** Give an equation (in terms of $x$ and $y$) of the curve $C$. [4 points]

The curve is part of a parabola, namely the parabola $y = x^2 - 2x$. You could see this by noting that in the parameterization,

$$y = t^2 - 1 = (t+1)(t-1) = x(x-2) = x^2 - 2x$$

**(b)** [6 points] Use the parameterization given above to compute

$$\int_C (4x + \sqrt{y+1} - 4)\, ds$$

The speed of the parameterization is $|\mathbf{v}(t)| = \sqrt{1+4t^2}$, so

$$\begin{aligned}
\int_C (4x + \sqrt{y+1} - 4)\, ds &= \int_{-1}^{1} [4(1+t) + \sqrt{(t^2-1)+1} - 4]\sqrt{1+4t^2}\, dt \\
&= \int_{-1}^{1} (4t + |t|)\sqrt{1+4t^2}\, dt \\
&= \int_{-1}^{0} 3t\sqrt{1+4t^2}\, dt + \int_{0}^{1} 5t\sqrt{1+4t^2}\, dt \\
&= \boxed{\frac{1}{6}(5^{3/2} - 1)}
\end{aligned}$$