**MATH 2602 K1-K3, Midterm 3 practice exercises, April 3**

**Problem 1 :** Let $a_1 = 4, a_2 = 7$ and $a_n = 10a_{n-1} - 21a_{n-2} + 7 + n$ for $n \geq 3$. Find $a_n$ for all $n$.

**Solution:** We search for a particular solution in a form $p_n = a + bn$. Then

$$a + bn = 10(a + b(n-1)) - 21(a + b(n-2)) + 7 + n.$$

After simplifying we get

$$a + bn = (-11a + 32b + 7) + (-11b + 1)n.$$

$$\begin{aligned} a &= -11a + 32b + 7 \\ b &= -11b + 1 \end{aligned}$$

So $12b = 1$, hence $b = 1/12$. And $12a = 32b + 7 = 32/12 + 7 = 8/3 + 7 = 29/3$, hence $a = 29/36$ and $p_n = 29/36 + (1/12)n$.

Now let's solve the corresponding homogeneous recurrence relation $a_n = 10a_{n-1} - 21a_{n-2}$. The characteristic polynomial is $x^2 = 10x - 21$, i.e. $x^2 - 10x + 21 = 0$. The characteristic roots are 3 and 7.

The solution to the homogeneous recurrence relation is

$$q_n = c_1 3^n + c_2 7^n.$$

Thus,

$$a_n = p_n + q_n = 29/36 + (1/12)n + c_1 3^n + c_2 7^n.$$

To find $c_1$ and $c_2$ we need to use $a_1 = 4, a_2 = 7$.

$$\begin{aligned} 4 &= 29/36 + 1/12 + 3c_1 + 7c_2 \\ 7 &= 29/36 + 2/12 + 9c_1 + 49c_2. \end{aligned}$$

**Problem 2:** How many integers between 1 and 200 (inclusive) are

(a) divisible by at least one of 3, 5, 7?

**Solution:** Let $A$ be the set of the integers between 1 and 200 dividable by 3, $B$ - divisible by 5 and $C$ divisible by 7.

$$|A| = \lfloor \frac{200}{3} \rfloor = \lfloor 66.666... \rfloor = 66.$$

$$|B| = \lfloor \frac{200}{5} \rfloor = \lfloor 40 \rfloor = 40.$$

$$|C| = \lfloor \frac{200}{7} \rfloor = \lfloor 28.57 \rfloor = 28.$$

The set $A \cap B$ are the integers between 1 and 200 divisible by 15, the set $A \cap C$ - divisible by 21, the set $B \cap C$ - divisible by 35 and the set $A \cap B \cap C$ - divisible by 105.

Using Inclusion-Exclusion we get

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

$$|A \cup B \cup C| = 66 + 40 + 28 - \lfloor \frac{200}{15} \rfloor - \lfloor \frac{200}{21} \rfloor - \lfloor \frac{200}{35} \rfloor + \lfloor \frac{200}{105} \rfloor.$$

$$|A \cup B \cup C| = 66 + 40 + 28 - 13 - 9 - 5 + 1 = 108.$$

(b) divisible by neither 5 nor 11?

**Solution:** Let $A$ be the set of the integers between 1 and 200 dividable by 5, $B$ - divisible by 11.

$$|A| = \lfloor \frac{200}{5} \rfloor = 40.$$

$$|B| = \lfloor \frac{200}{11} \rfloor = \lfloor 18.18... \rfloor = 18.$$

Let $U$ be all the integers between 1 and 200, $|U| = 200$. The set of integers between 1 and 200 that is divisible by neither 5 nor 11 is $U \setminus (A \cup B)$. Thus

$$|U \setminus (A \cup B)| = |U| - |A \cup B|.$$

$A \cap B$ is the set of integers between 1 and 200 that is divisible by 55, so $|A \cap B| = 3$. We get

$$|A \cup B| = |A| + |B| - |A \cap B| = 40 + 18 - 3 = 55.$$

So

$$|U \setminus (A \cup B)| = 200 - 55 = 145.$$

(c) divisible by exactly one of 6 or 8?

**Solution:** Let $A$ be the set of integers between 1 and 200 dividable by 6, $B$ - divisible by 8.

$$|A| = \lfloor \frac{200}{6} \rfloor = 33.$$

$$|B| = \lfloor \frac{200}{8} \rfloor = 25.$$

$A \cap B$ is the set integers between 1 and 200 dividable by 24.

$$|A \cap B| = \lfloor \frac{200}{24} \rfloor = 8.$$

The set of integers between 1 and 200 that are divisible by exactly one of 6 or 8 is $(A \cup B) \setminus (A \cap B)$. And

$$|(A \cup B) \setminus (A \cap B)| = |(A \cup B)| - |(A \cap B)| = |A| + |B| - 2|A \cap B| = 33 + 25 - 16 = 42.$$

(d) divisible by at least one of 3 and 5 but not by 10?

**Solution:** Let $A$ be the set of integers between 1 and 200 dividable by 3, $B$ - divisible by 5 and $C$ - divisible by 10.

Then $(A \cup B) \setminus C$ is the set of integers between 1 and 200 divisible by at least one of 3 and by 5 but not by 10. We have

$$|(A \cup B) \setminus C| = |A \cup B| - |(A \cup B) \cap C|.$$

$$|A \cup B| = |A| + |B| - |A \cap B| = \lfloor \frac{200}{3} \rfloor + \lfloor \frac{200}{5} \rfloor - \lfloor \frac{200}{15} \rfloor = 66 + 40 - 13 = 93.$$

We have

$$|(A \cup B) \cap C| = |(A \cap C) \cup (B \cap C| = |(A \cap C)| + |(B \cap C| - |(A \cap B \cap C|$$

$$= \lfloor \frac{200}{30} \rfloor + \lfloor \frac{200}{10} \rfloor - \lfloor \frac{200}{30} \rfloor = 20.$$

So

$$|(A \cup B) \setminus C| = 93 - 20 = 73.$$

(e) divisible by 5 but neither 3 nor 11?

Let $A$ be the set of integers between 1 and 200 dividable by 5, $B$ - divisible by 3 and $C$ - divisible by 11.

Then $A \setminus (B \cup C)$ is the set of integers between 1 and 200 divisible by 5 but neither 3 nor 11. And

$$|A \setminus (B \cup C)| = |A| - |A \cap (B \cup C)|.$$

The rest is similar as the example above.

**Problem 3:** Find the number of arrangements of the letters $A, B, C, D, E, F, G, H$ which contain.

(a) at least one of the patterns $ABC$ or $CED$.

(b) exactly one of the patterns $ABC$ or $CED$.

(c) contain neither $ABC$ nor $CED$.

**Solution:** We solved together yesterday. Answers a) $2(6!) - 4!$, b) $2(6!) - 2(4!)$ c) $8! - (2(6!) - 4!)$.

**Problem 4:** How many five-digit numbers can be formed

(a) So that all the digits are distinct?

**Solution:** We have 9 possibility to choose the first digit as we don't allow 0 at the beginning, after choosing the first digit there are 9 possibilities for the second digit as it can be any number between 0 - 9 except the first digit, for third digit we are left with 8 choices, for forth - 7 choices and for fifth 6 choices. Hence the number of five-digit numbers with all digits distinct is $9 \times 9 \times 8 \times 7 \times 6$.

(b) They have one or more repeated digits?

**Solution:** The five-digit numbers are $10000, 10000, ..., 99999$ in total 90000 of them. The number of five-digit numbers with distinct digits is $9 \times 9 \times 8 \times 7 \times 6$. Hence the number of five-digits number with one or more repeated digits is $90000 - 9 \times 9 \times 8 \times 7 \times 6$.

**Problem 5:** In a room where there are 51 people with ages between 1 and 100, show the following:

(a) Either two people have the same age or there are two people whose ages are consecutive integers.

**Solution:** We will group people based on their age group. Let's define 50 "boxes", in box 1 we "put" people with ages 1 or 2, in box 2 people with ages 3 or 4, ... , in box 50 people with ages 99 or 100. As we have 51 people and 50 boxes then by a pigeonhole principle there is a box with at least two people in it, so these people should have either the same age or their age should be consecutive integers.

(b) Either two people have the same age or one person's age is a multiple of another's.

**Solution:** We can write each number $n$ as $n = 2^k m$, where $m$ is an odd number. (e.g. $6 = 2^1 \times 3$, $5 = 2^0 \times 5$, $12 = 2^2 \times 3$). Let's take the boxes with odd number $1, 3, ..., 99$, in total 50 boxes. For each age $n$ we write $n = 2^k m$, and put that number in the box $m$. As we have more people than boxes one of the boxes will contain at least two number, so in a box $m$ we will have at least two numbers, i.e. $n_1 = 2^{k_1} m$ and $n_2 = 2^{k_2} m$. If $k_1 = k_2$ then $n_1 = n_2$ and we have two people with the same age, if $k_1 > k_2$ then $n_2$ divides $n_1$, and lastly if $k_2 > k_1$ then $n_1$ divides $n_2$.

(c) Some people shake hands. Show that among those who shook at lease one hand, two people shook the same number of hands.

**Solution:** Among the people that shook hands there is one (or maybe more) that shook the most hands. Let's say that the first person did the maximal number of handshakes, he did $n$ handshakes and all other people did no more than $n$ handshakes, as the first person did $n$ handshakes it means that there are $n$ people that shook hands with him. So in total there are at least $n+1$ who shook no more than $n$ hands, hence by the pigeonhole principal there are at least two people with the same number of handshakes.

**Problem 6:** An urn contains 10 red <u>numbered</u> balls and 6 white <u>numbered</u> balls. A sample of 8 balls is selected.

(a) How many samples contain exactly 3 red balls?

**Solution:** We need to select 3 red balls out of 10 and 5 while balls out of 6. Hence the number is

$$\binom{10}{3} \times \binom{6}{5}.$$

(b) How many samples contain at most 2 red balls?

As we have 6 white balls, then the sample should have exactly 2 red balls and 6 white balls. The number of such samples is

$$\binom{10}{2} \times \binom{6}{6} = \binom{10}{2}.$$

**Problem 7:** In how many ways can 10 identical stones be distributed among 14 (labelled) boxes. So that

(a) Each box contains no more than one stone.

**Solution:** We should choose 10 boxes to distribute the stones out of 14 boxes. Here there are $\binom{14}{10}$ ways to do it.

(b) The first box contains at least 3 stones and all the rest can contain any number of stones.

**Solution:** Let's first put 3 stones inside the first box. Then the remaining 7 identical stones should be distributed among 14 boxes and each box can get any number of stones. So the formula is $\binom{n+r-1}{r}$, where $r=7$ and $n=14$ so we get $\binom{14+7-1}{7} = \binom{20}{7}$

**Problem 8:** Find the coefficient of $x^{10}$ in the expansion of $(2x - \frac{1}{x})^{20}$.

**Solution:** The general term after expansion of $(2x - \frac{1}{x})^{20}$ will be

$$\binom{20}{k}(2x)^{20-k}\frac{1}{x^k} = \binom{20}{k}2^{20-k}x^{20-2k}.$$

We notice that $20 - 2k = 10$ for $k = 5$, so for $k = 5$ we get the term

$$\binom{20}{5}2^{15}x^{10}.$$

So the coefficient of $x^{10}$ is $\binom{20}{5}2^{15}$.