Math 2401
Exam 2
Section K
Number

Name: Solution

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged:____________________

Problem	Possible	Earned
1	4	
2	6	
3	5	
4	5	
5	10	
6	10	
7	10	
Total	50	

1. (4 pts) Determine if the following limits exist and if so compute the value:

(a) (2 pts) $\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$;

(b) (2 pts) $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$.

(a) $x^3+y^3 = (x+y)(x^2-xy+y^2)$

$\Rightarrow \dfrac{x^3+y^3}{x+y} = x^2-xy+y^2$ 1

$\Rightarrow \lim_{(x,y)\to(1,-1)} \dfrac{x^3+y^3}{x+y} = \lim_{(x,y)\to(1,-1)} x^2-xy+y^2 = 1-(1)(-1)+1$
$= 3$ 1

(b) Test along path $y=kx^2$

$\left.\dfrac{x^4}{x^4+y^2}\right|_{y=kx^2} = \dfrac{x^4}{x^4+k^2x^4} = \dfrac{1}{1+k^2}$ if $x \neq 0$

$\lim_{\substack{(x,y)\to(0,0)\\ y=kx^2}} \dfrac{x^4}{x^4+y^2} = \lim_{x\to 0} \dfrac{1}{1+k^2} = \dfrac{1}{1+k^2}$ 1

Different choice of k gives different limit. 1
Hence limit does not exist

2. (6 pts) For the function $w(x,y) = x^2 + y^2$ and $x(r,s) = r - s$ and $y(r,s) = r + s$ compute:

(a) (1 pt) $\frac{\partial x}{\partial r}$ and $\frac{\partial x}{\partial s}$;

(b) (1 pt) $\frac{\partial y}{\partial r}$ and $\frac{\partial y}{\partial s}$;

(c) (2 pts) $\frac{\partial w}{\partial r}$;

(d) (2 pts) $\frac{\partial w}{\partial s}$.

Your answer to parts (c) and (d) should be expressed in terms of the variables r and s.

(c) $\frac{\partial x}{\partial r} = 1 \qquad \frac{\partial x}{\partial s} = -1$

(b) $\frac{\partial y}{\partial r} = 1 \qquad \frac{\partial y}{\partial s} = 1$

(c) $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = 2x \cdot 1 + 2y \cdot 1$

1

$= 2(r-s) + 2(\cancel{y}r+s)$

$= 4r$

1

(d) $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = 2x(-1) + 2y(1)$

1

$= 2(y-x) =$

$= 2(r+s-r+s) = 4s$

1

3. (5 pts) For the function $f(x, y) = xy$:

(a) (1 pt) Compute the gradient of $f(x, y)$.

(b) (2 pts) What is the largest value that the directional derivative can have at the point $(1, 1)$?

(c) (2 pts) Find the directions so that the directional derivative of f in the direction $\vec{u}$ is 0.

(a) $\nabla f(x,y) = (y, x)$ 1 pt.

(b) $\nabla f(1,1) = (1,1)$

∇f is maximum in direction of ∇f 1 pt.

$|\nabla f| = \sqrt{2}$ $\Rightarrow$ $\vec{u} = \frac{(1,1)}{\sqrt{2}}$ 1 pt.

Maximum value is $\nabla f(1,1) \cdot \frac{(1,1)}{\sqrt{2}} = (1,1) \cdot \frac{(1,1)}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$.

(c) $D_{\vec{u}} f = 0 \Leftrightarrow \nabla f \cdot \vec{u} = 0 \Leftrightarrow (1,1) \cdot (u_1, u_2) = 0$

1 pt ① $\Leftrightarrow u_1 + u_2 = 0$

② $u_1^2 + u_2^2 = 1 \Rightarrow 2u_1^2 = 1 \Rightarrow u_1 = \pm \frac{1}{\sqrt{2}}$

$\vec{u} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ or $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ are the directions

1 pt.

So $\nabla f(1,1) \cdot \vec{u} = 0$.

4. (5 pts) For the equation $x^2 + y^2 + z = 4$ and the point $(1, 1, 2)$ find:

(a) the tangent plane to the level surface at the point;

(b) the normal line through the point.

(a) $f(x, y, z) = x^2 + y^2 + z - 4$

$\nabla f(x, y, z) = (2x, 2y, 1) \rightarrow$ 1 pt

$\nabla f(1,1,2) = (2, 2, 1) \longrightarrow$ 1 pt

Tangent Plane : 1 pt

$\boxed{\nabla f(1,1,2) \cdot ((x, y, z) - (1,1,2)) = 0} \iff$

$(2,2,1) \cdot (x-1, y-1, z-2) = 0 \iff$

$2x - 2 + 2y - 2 + z - 2 = 0 \iff \boxed{2x + 2y + z = 6}$

$\hookrightarrow$ 1 pt

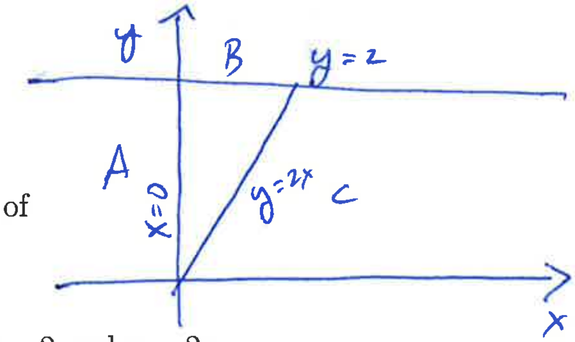
(b) $\ell(t) = (1,1,2) + t\nabla f(1,1,2)$

$= (1,1,2) + t(2,2,1) = \underbrace{(1+2t, 1+2t, 2+t)}_{\text{1 pt}}$

5. (10 pts) Find the absolute maximum and minimum values of

$$g(x, y) = 2x^2 - 4x + y^2 - 4y + 1$$

on the set in the first quadrant bounded by the lines $x = 0$, $y = 2$ and $y = 2x$.



2 pts

$\nabla g(x,y) = (4x-4, 2y-4)$

$\nabla g = \vec{0} \iff (4x-4, 2y-4) = (0,0) \iff (1,2) = (x,y)$

$g(1,2) = 2-4+4-8+1 = 3-8 = \boxed{-5 = g(1,2)} \rightarrow$ Minimum value

Three boundary components: (2 pts per bdy component)

1 pt for Max/Min

$\underline{A}$: $x=0$, $0 \le y \le 2$ $\quad g|_A = y^2 - 4y + 1 = h_A(y)$

$h_A'(y) = 2y - 4 \Rightarrow$ Critical pt on bdy are $y = 0, 2$

$\boxed{g(0,0) = 1} \longrightarrow$ Maximum value

$\boxed{g(0,2) = 4-8+1 = -3}$

$\underline{B}$: $y=2$ (2) $0 \le x \le 1$ $\quad g|_B = 2x^2 - 4x + 4 - 8 + 1 = 2x^2 - 4x - 3 = h_B(x)$

$h_B'(x) = 4x - 4 \Rightarrow$ Critical points on B are $x = 0, 1$

$h_B(0) = \boxed{-3 = g(0,2)}$

$h_B(1) = \boxed{g(1,2) = -5}$

$\underline{C}$: $y = 2x$ $\quad 0 \le x \le 1$ (2) $\quad g|_C = 2x^2 - 4x + 4x^2 - 8x + 1 = 6x^2 - 12x + 1 = h_C(x)$

$h_C'(x) = 12x - 12 \Rightarrow$ Critical Points are $x = 0, 1$

$\boxed{g(0,0) = h_C(0) = 1}$ $\quad \boxed{g(1,2) = -5}$

6. (10 pts) Use the method of Lagrange multipliers to find the maximum and minimum values of

$$f(x, y, z) = x - y + z$$

subject to the constraint that

$$x^2 + y^2 + z^2 = 1.$$

3 pts

$\nabla f = (1, -1, 1)$ $\quad$ $\nabla f = \lambda \nabla g \iff 1 = 2\lambda x, \ -1 = 2\lambda y, \ 1 = 2\lambda z$

$\nabla g = (2x, 2y, 2z)$

1 pt

$\lambda \neq 0$ esle get $1 = 0, \ -1 = 0, \ 1 = 0 \ \rightarrow\leftarrow$

$\Rightarrow x = \frac{1}{2\lambda} \quad y = -\frac{1}{2\lambda} \quad z = \frac{1}{2\lambda}$

2 pts

$x^2 + y^2 + z^2 = \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = \frac{3}{4\lambda^2} = 1$

$\Rightarrow 4\lambda^2 = 3 \quad \Rightarrow 2\lambda = \pm\sqrt{3} \quad \Rightarrow \frac{1}{2\lambda} = \frac{\pm 1}{\sqrt{3}}$

So the candidate points are

$x = \frac{1}{\sqrt{3}}, \ y = -\frac{1}{\sqrt{3}}, \ z = \frac{1}{\sqrt{3}}$

$x = -\frac{1}{\sqrt{3}}, \ y = \frac{1}{\sqrt{3}}, \ z = -\frac{1}{\sqrt{3}}$

2 pts

2 pts

$f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} - \left(-\frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3} \rightarrow$ Max

$f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} = -\frac{3}{\sqrt{3}} = -\sqrt{3} \rightarrow$ Min

7. (10 pts) For the function $f(x, y) = e^{-y}(x^2 + y^2)$:

(a) Compute the critical points;

(b) For each critical point found from part (a) determine if the function is a local maximum, local minimum, or a saddle point by using the second derivative test.

$$\nabla f = \left(2xe^{-y},\ -e^{-y}(x^2+y^2) + e^{-y}\cdot 2y\right) \Big\} \ 2\text{pts}$$

$$\nabla f = \vec{0} \iff 2xe^{-y} = 0 \quad e^{-y}(2y - x^2 - y^2) = 0$$

$$\iff x = 0 \quad 2y - x^2 - y^2 = 0$$

$$\iff x = 0, \quad y(2-y) = 0$$

$(0,0)$ and $(0,2)$ are the critical points $\Big\}$ 2pts

(b) $$\text{Hess}\, f(x,y) = \begin{pmatrix} 2e^{-y} & -2xe^{-y} \\ -2xe^{-y} & -e^{-y}(2y-x^2-y^2) + e^{-y}(2-2y) \end{pmatrix} \Big\} \ 2\text{pts}$$

$(0,0)$ $$\text{Hess}\, f(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & -0+2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$\det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} > 0$, $f_{xx} > 0$ $\Rightarrow$ local min $\Big\}$ 2pts

$(0,2)$ $$\text{Hess}\, f(x,y) \begin{pmatrix} 2e^{-2} & 0 \\ 0 & -e^{-2}(4-4) + e^{-2}(2-4) \end{pmatrix} = \begin{pmatrix} 2e^{-2} & 0 \\ 0 & -2e^{-2} \end{pmatrix}$$

$\det < 0 \Rightarrow$ Saddle point $\Big\}$ 2pts