Instructions: *Print* your name, student ID number and recitation session in the spaces below.

Name: ____

Student ID:______________________________________

Recitation session:______________________________________

Exam 2, Calculus III (Math 2551)

10/29/2015 (Thursday)

Show your work clearly and completely!

No calculators are allowed.

You can bring a formula sheet of a one-side letter size paper.

Question	Points
1)	
2)	
3)	
4)	
5)	

Problem 1(20 points). Calculations.

(a) (5 pt) Find the directional derivative of

$$f(x,y,z) = x^2y + y^2z + z^2x$$

at $P(1,0,1)$ in the direction of $3\mathbf{j} - \mathbf{k}$.

(b) (5 pt) Find the rate of change of $f(x,y) = \ln\left(x^2 + y^2 + z^2\right)$ along the curve $\vec{r}(t) = \sin t\ \mathbf{i} + \cos t\ \mathbf{j} + \mathbf{e}^{2t}\mathbf{k}$.

Solution:

(a)

$$\nabla f = \left(z^2 + 2xy\right)\mathbf{i} + \left(x^2 + 2yz\right)\mathbf{j} + \left(y^2 + 2zx\right)\mathbf{k},$$
$$\nabla f(1,0,1) = \mathbf{i} + \mathbf{j} + 2\mathbf{k}, \quad \mathbf{u} = \frac{1}{\sqrt{10}}(3\mathbf{j} - \mathbf{k}),$$

so

$$f'_u(1,0,1) = \nabla f(1,0,1) \cdot \mathbf{u} = \frac{\sqrt{10}}{10}.$$

(b)

$$\nabla f = \frac{2}{x^2 + y^2 + z^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}),$$

$$\begin{aligned} \frac{df}{dt} &= \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \\ &= \frac{2}{1+e^{4t}}\left(\sin t\ \mathbf{i} + \cos t\ \mathbf{j} + \mathbf{e}^{2t}\mathbf{k}\right) \cdot \left(\cos t\ \mathbf{i} - \sin t\ \mathbf{j} + \mathbf{2e}^{2t}\mathbf{k}\right) \\ &= \frac{4e^{4t}}{1+e^{4t}}. \end{aligned}$$

(c)(5 pt) Find $\partial u/\partial t$ for $u = \sin(x-y) + \cos(x+y)\,,\ x = st, y = s^2 - t^2$.

(d)(5 pt) Find dy/dx if $xe^y + ye^x - 2x^2y = 0$.

Solution:

(c)

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \\ &= (\cos(x-y) - \sin(x+y))\, s + (-\cos(x-y) - \sin(x+y))\,(-2t) \\ &= (s+2t)\cos\left(st - s^2 + t^2\right) - (s-2t)\sin\left(st + s^2 - t^2\right).\end{aligned}$$

(d) Set $u = xe^y + ye^x - 2x^2y$,then

$$\begin{aligned}\frac{\partial u}{\partial x} &= e^y + ye^x - 4xy, \\ \frac{\partial u}{\partial y} &= xe^y + e^x - 2x^2,\end{aligned}$$

$$\frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} = -\frac{e^y + ye^x - 4xy}{xe^y + e^x - 2x^2}.$$

Problem 2(20 pt) Let $f(x, y) = \sin(x \cos y)$ and consider the graph surface $S : z = f(x, y)$.

(a) (7 points) Find the equation for the tangent plane to surface S at the point $P\left(1, \frac{\pi}{2}, 0\right)$, that is, $x = 1, y = \frac{\pi}{2}, z = 0$.

(b) (6 points) Find the equation for the normal line to S at $P\left(1, \frac{\pi}{2}, 0\right)$?

(d) (7 points) What is the direction for f to increase most rapidly at $\left(1, \frac{\pi}{2}\right)$? What is the maximal rate of increase at $\left(1, \frac{\pi}{2}\right)$?

Solution:

$$\nabla f = \cos y \cos(x \cos y)\,\mathbf{i} - x \sin y \cos(x \cos y)\,\mathbf{j},$$
$$\nabla f\left(1, \frac{\pi}{2}\right) = -\mathbf{j}.$$

(a) The tangent plane is $z = -\left(y - \frac{\pi}{2}\right)$.

(b) The normal line is $x = 1,\ y = \frac{\pi}{2} - t,\ z = -t$.

(c) The direction for f to increase most radidly at $\left(1, \frac{\pi}{2}\right)$ is $\nabla f\left(1, \frac{\pi}{2}\right) = -\mathbf{j}$. The maximal rate of increase is $\left\|\nabla f\left(1, \frac{\pi}{2}\right)\right\| = 1$.

Problem 3 (20 pt) Find the absolute extreme values taken on $f(x, y) = 3 + x - y + xy$ on the closed region enclosed by $y = x^2$ and $y = 4$.

Solution: The region is

$$D = \left\{(x, y) : -2 \le x \le 2,\ x^2 \le y \le 4\right\}.$$

First,

$$\nabla f = (y + 1)\,\mathbf{i} + (x - 1)\,\mathbf{j} = \vec{0}$$

at $(1, -1)$ which is not in the interior of D.

Next, we consider the boundary of D. On $y = x^2$ $(-2 \le x \le 2)$, $f = x^3 - x^2 + x + 3$, $df/dx = 3x^2 - 2x + 1 = 0$ has critical point. On $y = 4$ $(-2 \le x \le 2)$, $f = 5t - 1$ has no critical point. So the absolute extreme values can only be taken at the end points of the boundary, that is, $(-2, 4)$ and $(2, 4)$. The absolute maximum is -11 at $(-2, 4)$ and the absolute minimum is 9 at $(2, 4)$.

Problem 4 (20 points) A rectangular box has three of its faces on the coordinate planes and one vertex in the first octant on the paraboloid $z = 4 - x^2 - y^2$. Determine the maximum volume of the box.

Solution:

Let (x, y, z) be the vretex on the paraboloid. Then the volume is $f(x, y, z) = xyz$ and the side condition is $g(x, y, z) = x^2 + y^2 + z - 4 = 0$. We have $\nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$, $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$. By Lagrange multiplier method, we solve the equation $\nabla f = \lambda \nabla g$, that is

$$yz = 2\lambda x, \ xz = 2\lambda y, \ xy = \lambda.$$

Eliminating λ in above equations, we get $x^2 = y^2 = \frac{z}{2}$. From $g(x, y, z) = 0$, we get $4x^2 = 1$. So $x = y = 1, z = 2$. The maximal volume is 2.

Problem 5 (20 points)

(a) (10 points) (10 points) Find the volume of the solid bounded above by $z = x^3 y$ and below by the triangular region with vertices $(0,0)$, $(2,0)$ and $(0,1)$.

(b) (10 points) Use polar coordinates to evaluate the double integral $\int\int_D (x+y)\ dxdy$ where the region

$$D = \left\{1 \le x^2 + y^2 \le 4,\ x \ge 0, y \ge 0\right\}.$$

Solution:

(a) The triangular region is $D = \left\{0 \le x \le 2,\ 0 \le y \le 1 - \frac{1}{2}x\right\}$. So the volume is

$$\begin{aligned} & \int_0^2 \int_0^{1-\frac{1}{2}x} x^3 y\ dydx \\ = & \int_0^2 \frac{1}{2} x^3 y^2 |_0^{1-\frac{1}{2}x} dx = \frac{1}{2} \int_0^2 x^3 \left(1 - \frac{1}{2}x\right)^2 dx \\ = & \frac{1}{2} \int_0^2 \left(x^3 - x^4 + \frac{1}{4}x^5\right) dx \\ = & \frac{1}{2} \left(\frac{1}{4}x^4 - \frac{1}{5}x^5 + \frac{1}{24}x^6\right) |_0^2 = \frac{1}{2}\left(4 - \frac{32}{5} + \frac{64}{24}\right) \\ = & \frac{2}{15}. \end{aligned}$$

(b) In the polar coordinates, the region D becomes

$$\Gamma = \left\{1 \le r \le 2, 0 \le \theta \le \frac{\pi}{2}\right\}.$$

So the double integral is

$$\begin{aligned} & \int_0^2 \int_0^{\frac{\pi}{2}} (r\cos\theta + r\sin\theta)\, r d\theta dr \\ = & \int_0^2 r^2 dr \int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta)\, d\theta \\ = & \frac{1}{3} r^3 |_0^2\ (\sin\theta - \cos\theta)\, |_0^{\frac{\pi}{2}} \\ = & \frac{14}{3}. \end{aligned}$$