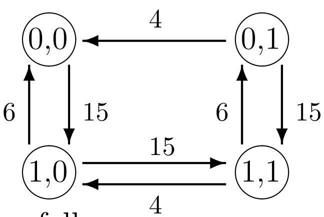
Question 1

(a) The continuous time Markov chain has four states. Represent the states as pairs (i, j), where $i = 1$ if server 1 is busy and $i = 0$ if server 1 is idle, and $j = 1$ if server 2 is busy and 0 if server 2 is idle. The rate transition diagram looks like:



The balance equations are as follows:

$$
\begin{aligned}
15\pi_{0,0} &= 6\pi_{1,0} + 4\pi_{0,1}\\
21\pi_{1,0} &= 15\pi_{0,0} + 4\pi_{1,1}\\
19\pi_{0,1} &= 6\pi_{1,1}\\
10\pi_{1,1} &= 15\pi_{1,0} + 15\pi_{0,1}
\end{aligned}
$$

Solving these equations with $\pi_{0,0} + \pi_{1,0} + \pi_{0,1} + \pi_{1,1} = 1$, we get that the stationary distribution is

$$\pi_{0,0} = \frac{64}{539}, \quad \pi_{1,0} = \frac{100}{539}, \quad \pi_{0,1} = \frac{90}{539}, \quad \pi_{1,1} = \frac{285}{539}.$$

(b)

$$
\begin{aligned}
N &= 0\pi_{0,0} + 1\pi_{1,0} + 1\pi_{0,1} + 2\pi_{1,1}\\
&= \frac{100}{539} + \frac{90}{539} + 2 \cdot \frac{285}{539}\\
&= \frac{760}{539}
\end{aligned}
$$

(c) We have by Little's law that

$$T = \frac{N}{\lambda_{ef}},$$

where T is the long-run average time spent in the system, and λ_{ef} is the effective arrival rate to the system. In this case, $\lambda_{ef} = \lambda(1 - \pi_{1,1})$, since $\pi_{1,1}$ is the proportion of arrivals that do not enter the system. Hence, we have

$$T = \frac{\frac{760}{539}}{15\left(1 - \frac{285}{539}\right)} = \frac{76}{381} \text{ hours.}$$

Question 2

(a) The continuous time Markov chain has 6 states, corresponding to the number of customers in the system (anywhere from 0 to 5). The rate transition diagram is given by:

$$\begin{tikzcd} 0 \arrow[r, shift left, "150"] & 1 \arrow[l, shift left, "15"] \arrow[r, shift left, "150"] & 2 \arrow[l, shift left, "30"] \arrow[r, shift left, "150"] & 3 \arrow[l, shift left, "45"] \arrow[r, shift left, "150"] & 4 \arrow[l, shift left, "52.5"] \arrow[r, shift left, "150"] & 5 \arrow[l, shift left, "60"] \end{tikzcd}$$

where rates are in terms of calls/hour. Then the balance equations are:

$$\begin{aligned} 150\pi_0 &= 15\pi_1 \\ 165\pi_1 &= 150\pi_0 + 30\pi_2 \\ 180\pi_2 &= 150\pi_1 + 45\pi_3 \\ 195\pi_3 &= 150\pi_2 + 52.5\pi_4 \\ 202.5\pi_4 &= 150\pi_3 + 60\pi_5 \\ 60\pi_5 &= 150\pi_4 \end{aligned}$$

Solving these (and enforcing that the total sum is 1) yields

$$\begin{aligned} &\pi_0 \approx 0.00052789 \quad \pi_1 \approx 0.0052789 \quad \pi_2 \approx 0.0263945 \\ &\pi_3 \approx 0.0879817 \quad \pi_4 \approx 0.251376 \quad \pi_5 \approx 0.628441. \end{aligned}$$

(b) An arriving call immediately talks to an operator if there are less than 3 customers in the system. So the answer is

$$\pi_0 + \pi_1 + \pi_2 \approx 0.00052789 + 0.0052789 + 0.0263945 = 0.0322013.$$

(c) A call does not get through if all five lines are busy. So we are looking for $\pi_5 \approx 0.628441$.

(d)

$$\begin{aligned} L &= \sum_{k=0}^{5} k\pi_k \\ &\approx 0(0.00052789) + 1(0.0052789) + 2(0.0263945)+ \\ &\quad 3(0.0879817) + 4(0.251376) + 5(0.628441) \\ &= 4.46972 \end{aligned}$$

$$N_q = 1\pi_4 + 2\pi_5 \approx 1(0.251376) + 2(0.628441) = 1.50826$$

(e) We have by Little's law that

$$T = \frac{N}{\lambda_{ef}},$$

where W is the long-run average time spent in the system, and λ_{ef} is the effective arrival rate to the system. In this case, $\lambda_{ef} = \lambda(1 - \pi_5)$, since π_5 is the proportion of calls that do not enter the system. Hence, we have

$$W \approx \frac{4.46972}{150(1 - .628441)} \approx 0.0801976 \text{ hours.}$$

Question 3

For system 1 we have 2 identical M/M/1 queues with parameters $\lambda/2$ and μ, hence using the formulas derived in class for each queue and adding them, we get

$$N_1 = 2\left(\frac{\lambda/2}{\mu - \lambda/2}\right) = \frac{2\lambda}{2\mu - \lambda} \quad N_{q1} = 2\left(\frac{(\lambda/2)^2}{\mu(\mu - \lambda/2)}\right) = \frac{\lambda^2}{\mu(2\mu - \lambda)}.$$

For system 2 we can use the formulas derived in class for an M/M/c queue, with parameters λ, μ and $c = 2$. We get

$$N_2 = \frac{\lambda}{2\mu - \lambda}\frac{4\mu}{2\mu + \lambda} = \frac{4\lambda\mu}{(2\mu + \lambda)(2\mu - \lambda)},$$

$$N_{q2} = \frac{\lambda^2}{\mu(2\mu - \lambda)}\frac{2\lambda}{2\mu + \lambda} = \frac{\lambda^3}{\mu(2\mu - \lambda)(2\mu + \lambda)}.$$

Now recall $2\mu - \lambda > 0$ so all of these quantities are positive. We will subtract the to compare:

$$N_1 - N_2 = \frac{2\lambda}{2\mu - \lambda} - \frac{4\lambda\mu}{(2\mu + \lambda)(2\mu - \lambda)} = \frac{2\lambda}{2\mu - \lambda}\left(1 - \frac{2\mu}{2\mu + \lambda}\right)$$

This amount is clearly always greater than zero if the parameters are positive so $N_1 > N_2$. Similarly

$$N_{q1} - N_{q2} = \frac{\lambda^2}{\mu(2\mu - \lambda)} - \frac{\lambda^3}{\mu(2\mu - \lambda)(2\mu + \lambda)} = \frac{\lambda^2}{\mu(2\mu - \lambda)}\left(1 - \frac{\lambda}{2\mu + \lambda}\right)$$

This amount is also clearly always greater than zero if the parameters are positive so $N_{q1} > N_{q2}$.

Since both systems have the same arrival rate, λ then by Little's Law it follows that $N_1 > N_2$ implies $T_1 > T_2$ and similarly, $N_{q1} > N_{q2}$ implies $T_{q1} > T_{q2}$.

We can therefore conclude that system 2 has lower waiting times, lower total times and fewer people in both the queue and the system as whole. So system 2 clearly is better than system 1. The intuition behind this is that in system 1 it is possible for a server to be idle in one queue while there are customers waiting in the other, while in system 2 there are never idle servers while customers are waiting.