Exam 1(50 mins), MATH 1501 Calculus 1A, 12 September 2013

Name: Key GT ID (not a number): Recitation:

Answers without substantiation do not count. You must show your work.
This exam is worth a total of 100 points, and the value of each question is listed with each question.
No calculators allowed, no books, no cheat sheet.

---

1. [10pts+15pts] Answer the following questions:

   (a) Find the limits:
   $$\lim_{x\to 0^-} \frac{\frac{1}{x-1}+\frac{1}{(x+1)^2}}{x(x-1)}$$

   (b) Consider two functions:
   $$f(x)=\sqrt{x-1}, \quad g(x)=\frac{3\pi}{1-e^x}.$$

   Write the formula for $f\circ g$ and evaluate $\lim_{x\to-\infty}(f\circ g)(x)$.

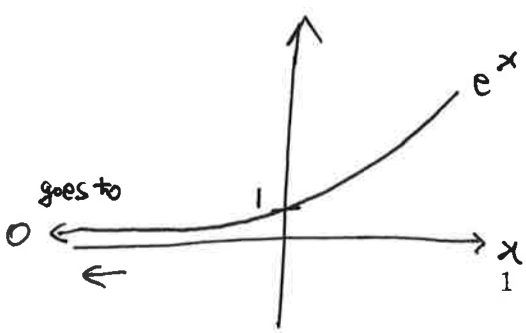
(a) $$\lim_{x\to 0^-}\left(\frac{\frac{1}{x-1}+\frac{1}{(x+1)^2}}{x(x-1)}\right)\cdot\frac{(x-1)(x+1)^2}{(x-1)(x+1)^2}$$

$$=\lim_{x\to 0^-}\frac{(x+1)^2+(x-1)}{x(x-1)^2(x+1)^2}=\lim_{x\to 0^-}\frac{x^2+2x+1+x-1}{x(x-1)^2(x+1)^2}$$

$$=\lim_{x\to 0^-}\frac{x^2+3x}{x(x-1)^2(x+1)^2}=\lim_{x\to 0^-}\frac{\cancel{x}(x+3)}{\cancel{x}(x-1)^2(x+1)^2}=3$$

(b) $$(f\circ g)(x)=\sqrt{\frac{3\pi}{1-e^x}-1}$$

$$\lim_{x\to-\infty}(f\circ g)(x)=\sqrt{\lim_{x\to-\infty}\frac{3\pi}{1-e^x}-1}=\sqrt{3\pi-1}$$



2. [5pts+15pts] Answer the following questions:

(a) State the definition of $\lim_{x \to x_0} f(x) = L$ using ϵ and δ.

(b) Use ϵ and δ to show that

$$\lim_{x \to 3} \left(-\frac{5}{2}x + 4 \right) = -\frac{7}{2}.$$

(a) For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

(b) For any $\varepsilon > 0$, we need to find $\delta > 0$ such that

$$0 < |x - 3| < \delta \Rightarrow \left| -\frac{5}{2}x + 4 + \frac{7}{2} \right| < \varepsilon$$

$$\left| -\frac{5}{2}x + \frac{15}{2} \right| < \varepsilon$$

$$|-5x + 15| < 2\varepsilon$$

$$|x - 3| < \frac{2\varepsilon}{5}$$

Therefore, we need to take $\delta = \frac{2\varepsilon}{5}$.

Name: Key GT ID (not a number): Recitation:

3. [10pts+10pts] Answer the following questions:

(a) What conditions must be satisfied by a function $f(x)$ if it is to be continuous at an interior point $x = c$ of its domain?

(b) Let

$$f(x) = \begin{cases} x^a \sin\left(\frac{1}{x}\right), & x > 0, \\ 0, & x \leqq 0. \end{cases}$$

Determine all real numbers a that make $f(x)$ is continuous at $x = 0$. Justify your answer. (Explain why your answer is correct.)

(a) $f(c)$ exists, $\lim_{x \to c} f(x)$ exists and

$\lim_{x \to c} f(x) = f(c)$

(b) $f(0) = 0$.

$\lim_{x \to 0^+} x^a \sin\left(\frac{1}{x}\right)$ must be 0.

Since $-x^a \leqq x^a \sin\left(\frac{1}{x}\right) \leqq x^a$,

by the Sandwich theorem,

$\lim_{x \to 0^+} x^a \sin\left(\frac{1}{x}\right) = 0$ only when $a > 0$.

If $a \leqq 0$, $\lim_{x \to 0^+} x^a \sin\left(\frac{1}{x}\right)$ does not exist.

Therefore, $a > 0$.

4. [10pts+10pts] Answer the following questions:

(a) State the Intermediate Value Theorem. Be sure to include all hypotheses and conclusions.

(b) Use the Intermediate Value Theorem to show that the equation $\sqrt{2x^2+1} = x + 3$ has a solution between -2 and -1.

(a) If f is continuous on $[a,b]$, and

if y_0 is between $f(a)$ and $f(b)$, then

$y_0 = f(c)$ for some c in $[a,b]$.

(b) $\underbrace{\sqrt{2x^2+1} - x - 3}_{\text{Let } = f(x)} = 0$ has a root between -2 and -1?

$f(-2) = \sqrt{9} + 2 - 3 = 2 > 0$

$f(-1) = \sqrt{3} + 1 - 3 = \sqrt{3} - 2 < 0$

Therefore, by the Intermediate Value Theorem, there is a solution between -2 and -1.

Name: Key GT ID (not a number): Recitation:

5. [10pts+5pts] Consider the following function

$$f(x) = x^2 - 5x + 1.$$

(a) Using *the difference quotient of f at a with increment h*, find the slope of the graph of $f(x)$ at the given point $P(a, a^2 - 5a + 1)$.

(b) Find the equation of the tangent line to the curve $y = f(x)$ at $(2, -5)$.

(a) $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{(a+h)^2 - 5(a+h) + 1 - (a^2 - 5a + 1)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 - \cancel{5a} - 5h \cancel{+1} \cancel{-a^2} + \cancel{5a} \cancel{-1}}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 2ah - 5h}{h}$$

$$= \lim_{h \to 0} h + 2a - 5$$

$$= 2a - 5$$

(b) $$f'(2) = -1$$

$$\therefore \quad y = -(x-2) - 5$$

$$= -x + 2 - 5$$

$$y = -x - 3.$$