Math 2603, Fall 2015, Quiz 1 Solutions

September 11, 2015

1 Problem 1. (100 points)

For a, b integers, define $a \sim b$ if and only if $a^2 - b^2$ is divisible by 3.

(note: $a^2 - b^2$ is divisible by 3 if and only if $a^2 - b^2 = 3k$ for some integer k if and only if $a^2 - b^2 \equiv 0 \bmod 3$)

a. (50 points) *Prove that $\sim$ defines an equivalence relation on the integer numbers.*

Proof:
Reflexivity: Let $x \in \mathbb{Z}$. You have then that $x^2 - x^2 = 0 = 3 \cdot 0$, and so $x \sim x$. Since we did not specify anything about x beyond the fact that it was an integer, this must be true for all integers. Therefore $\sim$ is reflexive.

Symmetricity: Let x, y be integers and let $x \sim y$. You have then that $x^2 - y^2 = 3k$ for some integer k. By multiplying both sides of the expression by negative one, we have then that $y^2 - x^2 = -3k = 3(-k)$ is a multiple of three and therefore $y \sim x$. Therefore $\sim$ is symmetric.

Transitivity: Let x, y, z be integers such that $x \sim y$ and $y \sim z$. You have then that $x^2 - y^2 = 3k$ for some integer k and $y^2 - z^2 = 3l$ for some integer l (note, I had to use a different letter since they might be different). By adding these equations together, we have then $x^2 - y^2 + y^2 - z^2 = 3k + 3l$ and by simplifying, $x^2 - z^2 = 3(k + l)$ is a multiple of three and therefore $x \sim z$. Therefore $\sim$ is transitive.

Since $\sim$ is a relation that is reflexive, symmetric, and transitive, it is by definition then an equivalence relation.

b. (50 points) *What is $\overline{0}$ (the equivalence class of 0)? What is $\overline{1}$?*

By definition, $\overline{0} = \{a \mid a \sim 0\} = \{b \mid 0 \sim b\}$. We can however, simplify this a great deal.

$\overline{0} = \{a \mid a \sim 0\} = \{a \mid a^2 - 0^2 = 3k \text{ for some } k \in \mathbb{Z}\} = \{a \mid a^2 = 3k \text{ for some } k \in \mathbb{Z}\}$

This can still be simplified. Note that $n^2 = 3k$ for some integer k if and only if $n = 3l$ for some integer l. (even more generally, one can prove that for any prime p and any integer n, you have $n^2 \equiv 0 \bmod p$ if and only if $n \equiv 0 \bmod p$).
The proof of this fact (in the case $p = 3$): Suppose $n = 3k$. Then $n^2 = 3(3k)$ is a multiple of 3.
($\star$)Suppose $n = 3k + 1$. Then $n^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$ is not a multiple of 3.
Suppose $n = 3k + 2$. Then $n^2 = 9k^2 + 12k + 4 = (3k^2 + 4k + 1) + 1$ is not a multiple of 3.

Using this information then, $\overline{0} = \{a \mid a^2 = 3k \text{ for some } k \in \mathbb{Z}\} = \{a \mid a = 3k \text{ for some } k \in \mathbb{Z}\} = 3\mathbb{Z} = \{\dots, -6, -3, 0, 3, 6, \dots\}$.

For the equivalence class of 1, we can approach similarly.

$\overline{1} = \{a \mid a \sim 1\} = \{a \mid a^2 - 1 = 3k\} = \{a \mid a^2 = 3k + 1\}$

Note that for all integers a which are *not* multiples of three, you have $a^2 = 3k+1$ for some integer k (as shown above at $\star$). Therefore $\overline{1} = \mathbb{Z} \setminus 3\mathbb{Z} = (3\mathbb{Z} + 1) \cup (3\mathbb{Z} + 2) = \{\dots, -5, -4, -2, -1, 1, 2, 4, 5, \dots\}$

(*side note: $\star$ can be used as proof that if $n^2 = 3k + 2$ for some integer k, then n is not an integer*)