Name:	______________________________	**Section L__**
Signature:	____	

You will have **50 minutes** to complete this closed book,
no notes, no calculator exam.

Keep the exam booklet closed until
the beginning of the examination.

Make sure that your booklet has **6 pages** (including this one).
Write clear, complete, legible answers in the spaces provided.
Use the back of the page if needed, but clearly indicate when doing so.

Read each question carefully and completely.
Think about the problem being asked.

Good luck!

1	2	3	4	Bonus	Total
/ 10	/10	/15	/15	/4	/50+4B

1. Given the equation

$$y'' - 7y + 10y = 0$$

(a) Reduce it to a first order linear system.

(b) Choose the solutions of the system among the following vector functions

$$x_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{5t}, \quad x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}, \quad x_3 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t}$$

(c) Determine whether the solutions of the system are linearly independent.

(d) Write a general solution for the system.

(e) Write a general solution for the original second order equation.

Solution:

(a) (2 points) The linear system associated to this equation is

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) (2 points) x_1 and x_2 are solutions of the system.
x_3 is not a solution of the system.

(c) (2 points) We call Mr Wronskian :) :
We can compute the Wronskian in $t = 0$

$$W[x_1, x_2](0) = \det \begin{bmatrix} 1 & 1 \\ 5 & 2 \end{bmatrix} = -3 \neq 0$$

or we can compute the Wronskian for every t:

$$W[x_1, x_2](t) = \det \begin{bmatrix} e^{5t} & e^{2t} \\ 5e^{5t} & 2e^{2t} \end{bmatrix} = -3e^{7t} \neq 0$$

(d) (2 points)The general solution for the problem is a linear combination of the two solutions x_1 and x_2.

$$x = c_1 x_1 + c_2 x_2 = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$$

(e) (2 points) The general solution for the second order equation is

$$y = c_1 e^{5t} + c_2 e^{2t}$$

2. Given the equation

$$(5xy + 2y + 5)\,dx + 2x\,dy = 0$$

(a) Determine whether it is exact. If not, find an integrating factor.

(b) Solve the equation

Solution:

(a) (1 point) In order to check whether the equation is exact, one has to compare M_y and N_x.

$$M_y = 5x + 2 \neq 2 = N_x,$$

meaning that the equation is not exact .
(3 points) We find an integrating factor, solving the equation

$$\mu' = \frac{M_y - N_x}{N}\mu \quad \text{that is} \quad \mu' = \frac{5}{2}\mu \quad \text{and therefore} \quad \mu(x) = e^{\frac{5}{2}x}$$

(b) (1 point) The equation

$$e^{\frac{5}{2}x}(5xy + 2y + 5)\,dx + e^{\frac{5}{2}x}2x\,dy = 0$$

is exact.
(2 points) We compute

$$F(x, y) = \int e^{\frac{5}{2}x}2x\,dy + \psi(x) = 2xye^{\frac{5}{2}x} + \psi(x).$$

(1 point) We compare the partial derivative of F with respect to x with $e^{\frac{5}{2}x}(5xy+2y+5)$ and
(1 point) we obtain

$$\psi'(x) = 5e^{\frac{5}{2}x} \quad \text{and therefore} \quad \psi(x) = 2e^{\frac{5}{2}x} + k$$

(1 point) The implicit solution is given by

$$e^{\frac{5}{2}x}(2xy + 2) = c$$

3. Solve the following initial value problem

$$\begin{cases} y' + \frac{1}{x}y = xe^x \\ y(1) = 0 \end{cases}$$

Solution: (method of variation of parameters)
We look for a solution of the form $y = uy_1$ where y_1 is the solution of the complementary equation $y' + \frac{1}{x}y = 0$.
(4 points)

$$y_1 = \frac{1}{x}$$

(2 points) Plugging y into the equation gives

$$u'\frac{1}{x} - \frac{1}{x^2}u + \frac{1}{x^2}u = xe^x$$

(4 points) and therefore

$$u' = x^2e^x, \qquad \text{and integrating (by parts) we obtain} \qquad u = e^x(x^2 - 2x + 2) + c$$

(2 points) Thus

$$y = \frac{c}{x} + \frac{e^x(x^2 - 2x + 2)}{x}$$

(1 point) Setting $y(1) = 0$ we obtain $c = -e$.
(2 points) Therefore the solution of the initial value problem is

$$y = \frac{e^x(x^2 - 2x + 2)}{x} - \frac{e}{x}$$

4. Given the following autonomous equation

$$y' = y^2 + y - 2$$

(a) Sketch the graph of y' versus y.

(b) List and classify any critical points.

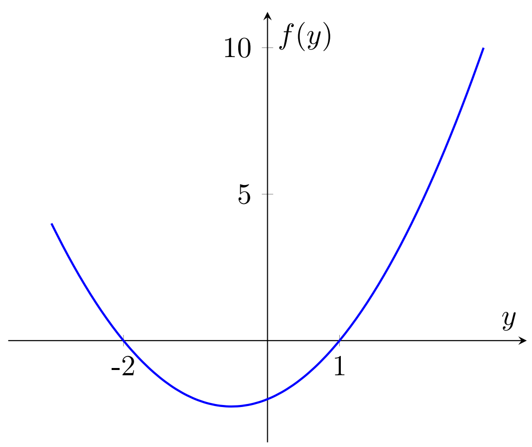
(c) Sketch the phase line.

(d) Determine points of inflection and study the concavity of the solution.

(e) Sketch the solutions in the y versus t plane.

Solution:

(a) (3 points)



(b) (3 points) $y = -2$ is asymptotically stable.
$y = 1$ is unstable.

(c) (3 points) See below.

(d) (3 points) We only have one point of inflection in $y = -\frac{1}{2}$.
The solution y is concave up in $(-2, -\frac{1}{2}) \cup (1, +\infty)$
and concave down in $(-\infty, -2) \cup (-\frac{1}{2}, 1)$.

(e) (3 points)

5. In the following, assume that all the functions are defined on a common interval (a, b). Prove that if y_1 and y_2 are solutions

$$y' + p(x)y = f_1(x), \qquad y' + p(x)y = f_2(x)$$

respectively, and c_1 and c_2 are constants, then

$$y = c_1 y_1 + c_2 y_2$$

is a solution of

$$y' + p(x)y = c_1 f_1(x) + c_2 f_2(x).$$

Solution:
(4 points) In order to prove this result (principle of superposition), one only needs to plug the solution into the equation:

$$\begin{aligned} y' + p(x)y &= (c_1 y_1 + c_2 y_2)' + p(x)(c_1 y_1 + c_2 y_2) \\ &= (c_1 y_1' + c_2 y_2') + p(x)(c_1 y_1 + c_2 y_2) \\ &= c_1(y_1' + p(x)y_1) + c_2(y_2' + p(x)y_2) \\ &= c_1 f_1(x) + c_2 f_2(x) \end{aligned}$$