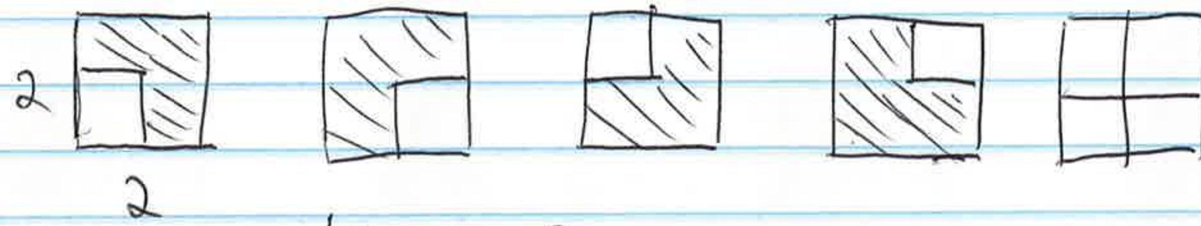
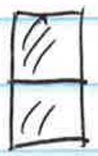
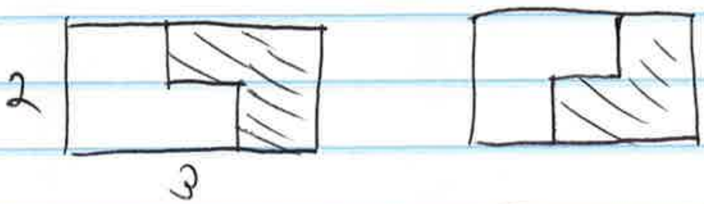
Practice Exam

problem 1

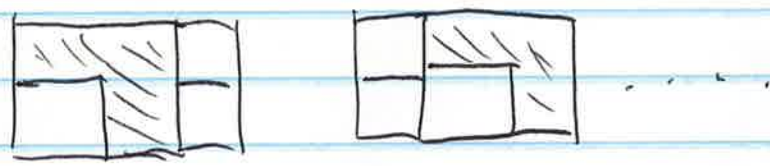




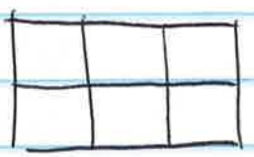
$t(2) = \cancel{4}\ 5.$



→ two case when $n = 3$

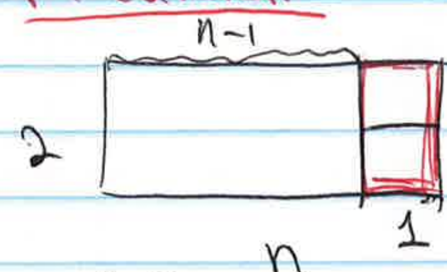


. . . . 8 cases when $n = 3$.

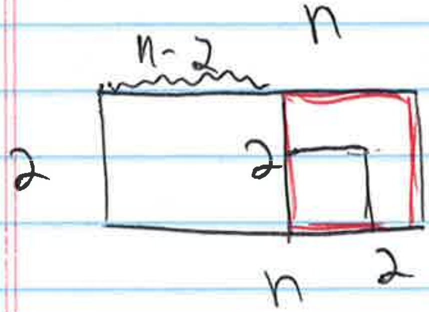


One more case when $n = 3$.

Recursion:



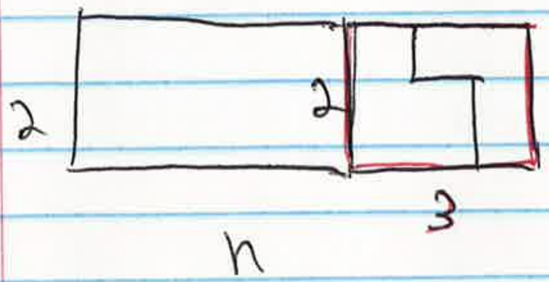
$t(n-1)$



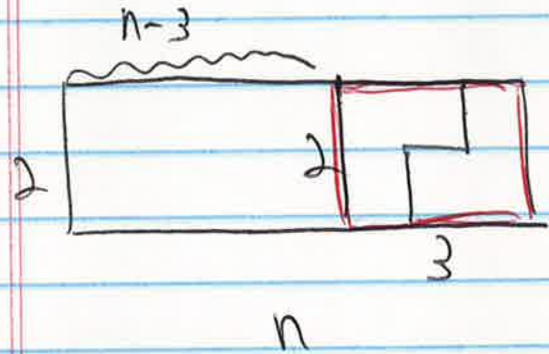
→ tile the right square 2x2 with each of the 4 possibilities

$4 \cdot t(n-2)$

(continue)



tile the right rectangle 2×3, with each of the two possibilities



$2t(n-3)$

$\Downarrow$

$$t(n) = t(n-1) + 4t(n-2) + 2t(n-3).$$

d. $(2x-3y)^{15}$.

The coefficient of x^6y^9 is where

$$\binom{15}{k}(2x)^{k}(-3y)^{15-k}, \quad k=6$$

$$\Downarrow$$

$$\binom{15}{6}2^6(-3)^9.$$

e. This is like

$$x_1+x_2+x_3=10, \quad \text{s.t. } x_i \geq 2, \; i=1,2,3.$$

Define a new variable:

$$x_i' = x_i - 1 \geq 1.$$

$$\Rightarrow \quad x_1'+x_2'+x_3' = 10-3 = 7$$

Then this is the question of distributing 7 units among 3 variables, each of which $\geq 1 \Rightarrow \binom{7-1}{3-1} = \binom{6}{2}$.

f. Full solution is given in the form.

problem 2

a. This is a MISSISSIPPI Question.

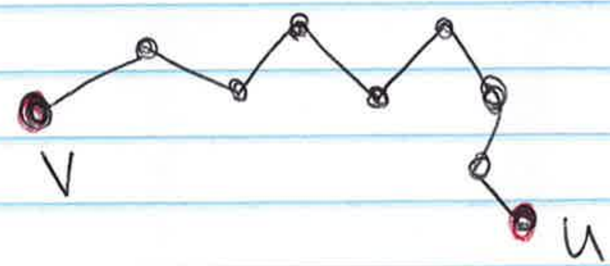
We get the multinomial coefficient

$$\frac{7!}{3!\,2!}.$$

b. Choose first 3 locations for vowels (and 4 for the rest). Then there are 5^3 possibilities for vowels at these locations, and 21^4 possibilities for the remaining letters at the 4 other locations.

c. Similarly to b, but now there are $\frac{5!}{2!}$ possibilities for vowels, and $\frac{21!}{17!}$ for the rest.

problem 3 Connectivity is an equivalence relation.

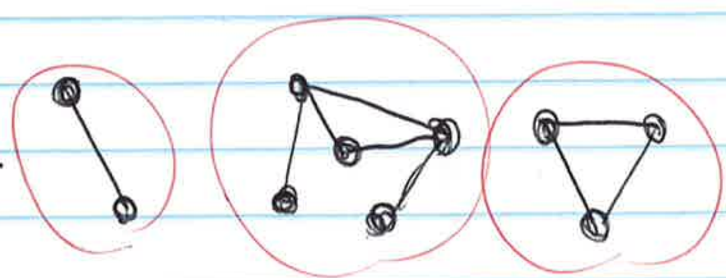


symmetry: V is reachable from u iff u is reachable from V.

Transitivity:

concatenate the two paths $V \rightsquigarrow u$, $u \rightsquigarrow W \Rightarrow V \rightsquigarrow W$.

Connected components:



problem 5

A B C D E F G H.

a.

Here, we refer to ABC as a single character. We do the same with EF.

- Number of permutations that contain "ABC" (consecutive) is $6!$, as we have 6 chars: $\underbrace{ABC}$, D, E, F, G, H.
- Number of permutations containing "EF" is $7!$
- Number of permutations containing both: $5!$ (as we have 5 chars).

$\Downarrow$

By Exclusion-Inclusion:

$$8! - 7! - 6! + 5!$$

6. A_i = Numbers in $[1, 900]$ that are divisible by i.

$A_{i,j}$ = Numbers in $[1, 900]$ divisible by i and j.

$A_{i,j,k}$ = Numbers in $[1, 900]$ divisible by i, j, and k.

we have:

$$|A_2| = \frac{900}{2}, \quad |A_3| = \frac{900}{3}, \quad |A_5| = \frac{900}{5}.$$

since 2, 3, 5 are relatively primes:

$$|A_{2,3}| = \frac{900}{2 \cdot 3}, \quad |A_{2,5}| = \frac{900}{2 \cdot 5}$$

$$|A_{3,5}| = \frac{900}{15}, \quad |A_{2,3,5}| = \frac{900}{2 \cdot 3 \cdot 5}$$

$$\Downarrow$$

$$900 - \left(\frac{900}{2} + \frac{900}{3} + \frac{900}{5}\right) + 900\left(\frac{1}{6} + \frac{1}{10} + \frac{1}{15}\right) - 900/30.$$

problem 6: $a_n = a_{n-1} + 2a_{n-2} + 3^n$,

$a_0 = 2$, $a_1 = 10$.

First, find an homogeneous solution:

Characteristic polynomial:

$$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0 \Rightarrow x = +2, -1.$$

$\Rightarrow$ Homogeneous solution:

$$c_1(2^n) + c_2(-1)^n.$$

Next, find a particular solution to a_n:

Guess: $c \cdot 3^n$

Check: $c \cdot 3^n = c \cdot 3^{n-1} + 2c \cdot 3^{n-2} + 3^n$

$$\Rightarrow \quad 9c = 3c + 2c + 9$$

$$\Rightarrow \quad 4c = 9$$

$$\Rightarrow \quad c = 9/4$$

(continue): $a_n = \frac{9}{4} 3^n = \frac{3^{n+2}}{4}$

we thus have:

$$\underline{a_n = c_1 2^n + c_2 (-1)^n + \frac{3^{n+2}}{4}}$$

Applying initial conditions:

$$a_0 = 2 = c_1 \cdot 1 + c_2 + \frac{9}{4}$$

$$a_1 = 10 = c_1 \cdot 2 - c_2 + \frac{27}{4}$$

$$\Downarrow$$

$$12 = 3c_1 + 9 \Rightarrow \underline{c_1 = 1}$$

$$\Rightarrow \underline{c_2} = 2 - 1 - \frac{9}{4} = \underline{-\frac{5}{4}}$$

$$\Downarrow$$

$$a_n = 2^n - \frac{5}{4}(-1)^n + \frac{3^{n+2}}{4}.$$

problem: 7, : Full solution is given in the online form.

Problem 8

Since $\gcd(4,7)=1$, there are s, t, with

$$1 = s\cdot 7 + t\cdot 4,$$

we have: $s=-1$, $t=2$.

We next claim that the solution x is:

$$x = 2(-1)(7) + 6\cdot(2)(4) = 34 \equiv 6 \mod 28.$$

This is because $(-1)(7) \equiv 1 \mod 4$, and $(2)\cdot(4) \equiv 1 \mod 7$.

$$\Downarrow$$

$$x = 2(-1)(7) + 6(2)(4) \equiv 2 \mod 4$$

$$x = \quad \downarrow \quad \equiv 6 \mod 7.$$

$\Rightarrow$ solution is consistent

$\Rightarrow$ $x \equiv 6 \mod 28$

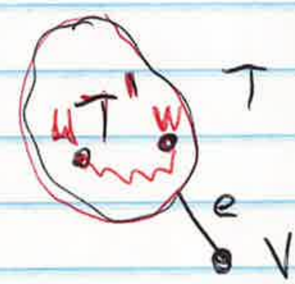
problem 9: T is connected w/o circuits.

Proof proceeds by induction on n.

Base case: $n=1 \Rightarrow$ graph consists of only one vertex, and 0 edges [Trivial].

$n>1$: T must contain a leaf, V.

Let us remove V and its adjacent edge e from T.



Let T' be the remaining graph.

We now claim that T is connected with no circuits:

- T has no circuits $\Rightarrow$ T' has no circuits.

(Continue):

- T is connected, the removal of v and e does not violate connectivity, as any pair of vertices u, w in T' are still connected via a path in T'.

$\Rightarrow$ Apply induction hypothesis on T' $\Rightarrow$ T' has $(n-1)$ vertices

$\Rightarrow$ T' has $(n-2)$ edges.

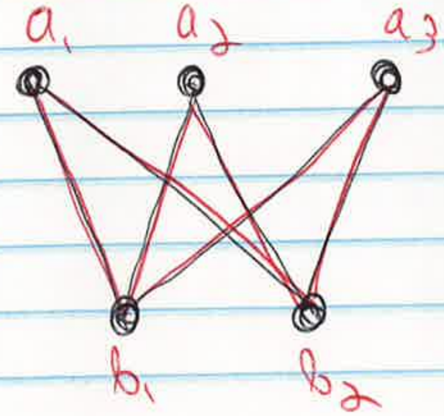
Next, return e to T, we obtain:

$(n-2)+1 = n-1$ edges,

$\Rightarrow$ our induction hypothesis is consistent. $\blacksquare$

Problem 10:

$K_{2,3}$

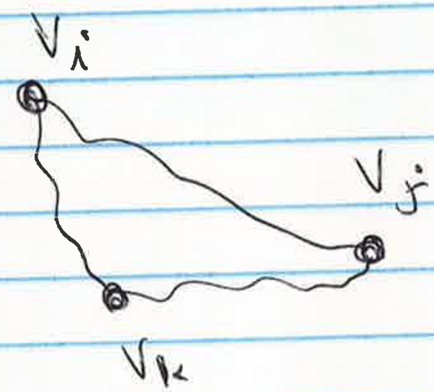


If $K_{2,3}$ has a Hamiltonian cycle, this cycle must contain, for each vertex a_i, both of its adjacent edges $\Rightarrow$ but then cycle contains all 6 edges of $K_{2,3}$, which is impossible. Because there are only 5 vertices.

Problem 11:

Recall that in the FW-algorithm we define $M_k(i,j)$ to be the shortest path length from i to j, which uses only the vertices $V_1, V_2, \ldots, V_k$ as intermediate vertices.

$$M_k(i,j) = \min \begin{cases} M_{k-1}(i,j), \\ M_{k-1}(i,k) + M_{k-1}(k,j) \end{cases}$$



Thus:

$$M_5(1,3) = \min\{\underbrace{-1}_{M_4(1,3)}, \underbrace{-4+1}_{M_4(1,5)+M_4(5,3)}\} = -3$$

problem 13

1. "For all $x \in \mathbb{R}$, $\exists y \in \mathbb{R}$, s.t. $x = y^2$"

is false. Thus it can imply anything, even another false statement as "$2+2=5$".

2. Claim: If a, b, x are integers, with $a \mid bx$, then if a, b are relatively primes then $a \mid x$.

Proof: $\gcd(a, b) = 1$ by definition.

$\Rightarrow$ there are integers m, n s.t.:

$$1 = ma + nb$$

Multiplying by x:

$$x = xma + xnb.$$

clearly, $a \mid (xma)$.

(continue).

since $a \mid bx$ we also have: $a \mid (x n b)$

$$\Rightarrow \quad a \mid \underbrace{(x m a) + (x n b)}_{x}$$

$$\Rightarrow \quad a \mid x.$$

3. $2^{1/3}$ is irrational.

proof: In fact, $2^{\frac{1}{n}}$ is irrational for any integer ≥ 3.

Assume by contradiction that $2^{1/n}$ is rational. $\Rightarrow$ There are two integers p, q s.t.: $2^{\frac{1}{n}} = p/q$

$$\Downarrow$$

$$2 = \frac{p^n}{q^n} \Rightarrow p^n = 2q^n$$

there are no integers a, b, c, w/ $a^n + b^n = c^n$, $n \geq 3$.

$\Rightarrow p^n = q^n + q^n$, $n \geq 3$. But this is impossible according to Fermat's last Theorem. $\Rightarrow$ contradiction.

4. A graph on n vertices <u>cannot</u> have exactly one vertex of odd degree.

<u>proof:</u> sum of degrees is $2|E|$

$$\sum \deg(v) = 2|E|$$

$$\Downarrow$$

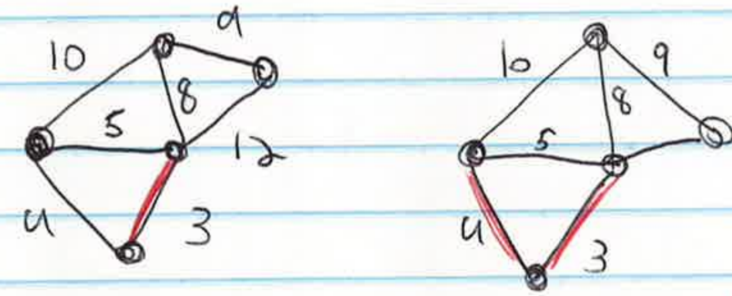
$$\sum_{\deg(v) \text{ is even}} \deg(v) + \sum_{\deg(v) \text{ is odd}} \deg(v) = 2|E|$$

$$\Downarrow$$

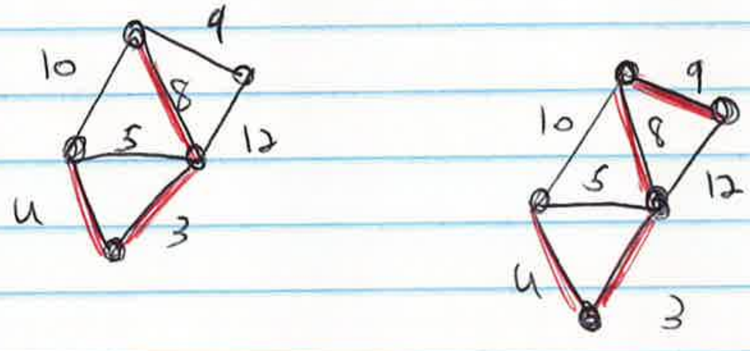
$$\sum_{\deg(v) \text{ is odd}} \deg(v) = 2|E| - \sum_{\deg(v) \text{ is even}} \deg(v)$$

RHS is <u>even</u> $\Rightarrow$ number of vertices of odd degree is even.

5. Prim's algorithm always connects the current component (containing a tree) with a vertex outside this component, so the edges just computed form a connected subgraph.



subgraph is connected at any step.



6. Kruskal's Algorithm only adds edges that do not close cycles, but they can form a disconnected subgraph

7 prove that: $F_{n-1} \cdot F_{n+1} = F_n^2 + (-1)^n$.

Base case: $n=1$, we have:

$$F_0 \cdot F_2 = 0 \cdot 1 = 0$$

$$F_1^2 + (-1)^1 = 1^2 - 1 = 0 \quad //$$

$n>1$: Assume this holds for n (use induction) and show that holds for $n+1$: Assume that:

$$F_{n-1} \cdot F_{n+1} = F_n^2 + (-1)^n.$$

Next:

$$F_n \cdot F_{n+2} = F_n(F_n + F_{n+1})$$

since $F_{n+2} = F_n + F_{n+1}$

$$= F^2_n + F_n \cdot F_{n+1}$$

$$= \underbrace{F_{n-1} \cdot F_{n+1} - (-1)^n}_{= F_n^2} + F_n \cdot F_{n+1}$$

use induction

$$= F_{n+1}[F_{n-1} + F_n] + (-1)^{n+1}$$

(continue):

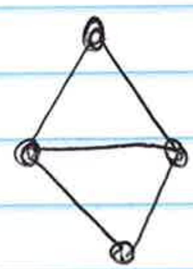
since $F_{n+1} = F_{n-1} + F_n$;

$$= F_{n+1} \cdot F_{n+1} + (-1)^{n+1}$$

$$= F_{n+1}^2 + (-1)^{n+1}$$

//

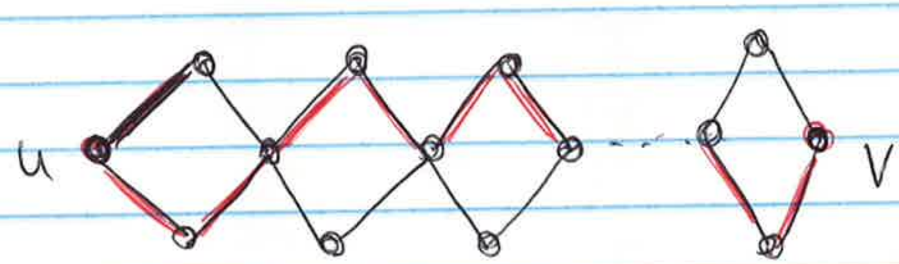
8. Number of spanning trees is exponential in n:

$|V| = 4$ 

This graph admits $8 = 2^{4-1}$ spanning trees.

This can also be derived by Kirchhoff's Theorem.

9. Number of shortest paths between two vertices u, v can be exponential in $|V|$.



we have n gadgets 

Number of vertices: $4+3(n-1) = 3n+3$.

at each gadget we have two possibilities of going upward or downward

$\Rightarrow$ Number of paths: 2^n.

Number of circuits: 2^n.

(as we go along a path from u to v, and then return on the unused edges.