MATH1502 - Calculus II TEST 2 C Group - February 17 - Spring 2015

Question	Score	Maximum
1		7
2		5
3		12
4		8
5		11
6		10
Total		53

Name ________________________________

Group (e.g. C1) _______________________

Student Number______________________

Teaching Assistant______________________

Answer all questions. There are 53 marks on the paper. 100% = 50 marks

No "cheatsheets" or calculators are allowed.

Question 1

Test the following series for convergence or divergence:

$$\sum_{k=10}^{\infty} \frac{\left(k^7 (\ln k) + k^6 \cos k\right)\left(1 + k^{-2}\right)}{\left(k^4 + (-1)^k k^2\right)\left(k^4 \ln k + \sqrt{k}\right)}. \qquad \text{(7 marks)}$$

Solution

We see that for very large k, the series terms behave roughly like

$$\frac{\left(k^7 \ln k\right)(1)}{\left(k^4\right)\left(k^4 \ln k\right)} = \frac{1}{k}.$$

Since $\sum \frac{1}{k}$ diverges ($p-$series with $p = 1$), this suggests we use the limit comparison test with

$$a_k = \frac{\left(k^7 (\ln k) + k^6 \cos k\right)\left(1 + k^{-2}\right)}{\left(k^4 + (-1)^k k^2\right)\left(k^4 \ln k + \sqrt{k}\right)}$$

and

$$b_k = \frac{1}{k}.$$

Now let us make this rigorous:

$$\begin{aligned}
& \lim_{k\to\infty} a_k/b_k \\
= & \lim_{k\to\infty} \left[\frac{\left(k^7 (\ln k) + k^6 \cos k\right)\left(1 + k^{-2}\right)}{\left(k^4 + (-1)^k k^2\right)\left(k^4 \ln k + \sqrt{k}\right)}\right] \Big/ \left[\frac{1}{k}\right] \\
= & \lim_{k\to\infty} \left[\frac{k^7 \ln k \left(1 + \frac{\cos k}{k \ln k}\right)\left(1 + k^{-2}\right)}{k^4\left(1 + (-1)^k k^{-2}\right) k^4 \ln k \left(1 + k^{-3.5}/\ln k\right)}\right] \frac{k}{1} \\
= & \lim_{k\to\infty} \frac{\left(1 + \frac{\cos k}{k \ln k}\right)\left(1 + k^{-2}\right)}{\left(1 + (-1)^k k^{-2}\right)\left(1 + k^{-3.5}/\ln k\right)} = 1,
\end{aligned}$$

finite and positive. Since $\sum \frac{1}{k}$ diverges ($p-$series with $p = 1$), by the limit comparison test,

$$\sum a_k = \sum \frac{\left(k^7 (\ln k) + k^6 \cos k\right)\left(1 + k^{-2}\right)}{\left(k^4 + (-1)^k k^2\right)\left(k^4 \ln k + \sqrt{k}\right)}$$

also diverges.

Question 2

Test the following series for convergence or divergence:

$$\sum_{n=2}^{\infty} \left(\frac{\sin^2 n}{2 + \ln n} \right)^n . \qquad \text{(5 marks)}$$

Solution

We use the root test, applied to

$$a_n = \left(\frac{\sin^2 n}{2 + \ln n} \right)^n .$$

We see that

$$\begin{aligned} \rho &= \lim_{n\to\infty} a_n^{1/n} \\ &= \lim_{n\to\infty} \left(\left(\frac{\sin^2 n}{2 + \ln n} \right)^n \right)^{1/n} \\ &= \lim_{n\to\infty} \frac{\sin^2 n}{2 + \ln n} \\ &= \lim_{n\to\infty} \frac{\sin^2 n}{(\ln n)\,(2/\ln n + 1)} = 0 < 1. \end{aligned}$$

Here we are using that $0 \leq \sin^2 n \leq 1$. (Graders, students do not need to justify that the limit is 0). By the root test, the series converges.

Question 3

Find the radius of convergence and interval of convergence of the following power series. Say which test(s) you are using. Also show your reasoning.

$$\sum_{k=1}^{\infty} \frac{3^k+1}{3^k-2} (-5x)^k . \qquad \text{(12 marks)}$$

Solution

The terms of the series are

$$a_k = \frac{3^k+1}{3^k-2} (-5x)^k .$$

We apply the root test to

$$|a_k| = \left| \frac{3^k+1}{3^k-2} (-5x)^k \right| = \frac{3^k+1}{3^k-2} (5|x|)^k .$$

(Of course, you can also use the ratio test). We compute

$$\begin{aligned}
&(\rho =) \lim_{k\to\infty} |a_k|^{1/k} \\
= &\lim_{k\to\infty} \left(\frac{3^k+1}{3^k-2} (5|x|)^k \right)^{1/k} \\
= &\lim_{k\to\infty} \left(\frac{3^k\left(1+1/3^k\right)}{3^k(1-2/3^k)} \right)^{1/k} (5|x|) \\
= &\lim_{k\to\infty} \frac{\left(1+1/3^k\right)^{1/k}}{(1-2/3^k)^{1/k}} (5|x|)
\end{aligned}$$

as $\lim_{k\to\infty} \left(1+1/3^k\right)^{1/k} = 1^0 = 1$ and also $\lim_{k\to\infty} \left(1-2/3^k\right)^{1/k} = 1^0 = 1$. By the root test, the series converges if

$$\rho = 5|x| < 1, \text{ that is, } |x| < \frac{1}{5}$$

and diverges if

$$\rho = 5|x| > 1, \text{ that is, } |x| > \frac{1}{5}.$$

So

$$\text{the radius of convergence is } \frac{1}{5} \qquad \text{(6 marks)}$$

Now test $x = \pm\frac{1}{5}$.
$\mathbf{x} = -\frac{1}{5}$
Here

$$\sum_{k=1}^{\infty} \frac{3^k + 1}{3^k - 2} (-5x)^k = \sum_{k=1}^{\infty} \frac{3^k + 1}{3^k - 2}.$$

Since

$$\lim_{k\to\infty} \frac{3^k + 1}{3^k - 2} = 1 \neq 0,$$

the nth term divergence test shows that the series diverges.

(3 marks)

$\mathbf{x} = \frac{1}{5}$
Here

$$\sum_{k=1}^{\infty} \frac{3^k + 1}{3^k - 2} (-5x)^k = \sum_{k=1}^{\infty} \frac{3^k + 1}{3^k - 2} (-1)^k.$$

Again, the series diverges by the nth term divergence test, since

$$\lim_{k\to\infty} \left| \frac{3^k + 1}{3^k - 2} (-1)^k \right| = 1 \neq 0.$$

(3 marks)

Summary
(a) The radius of convergence is $\frac{1}{5}$;
(b) The interval of convergence is

$$(-\frac{1}{5}, \frac{1}{5}).$$

Question 4

Find the radius of convergence of the following power series. Say which test you are using. Also show your reasoning.

$$\sum_{k=0}^{\infty} \frac{k!k^{2k}}{(3k)!} x^k. \qquad \text{(8 marks)}$$

You may assume that

$$\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e.$$

Solution

The kth term of our series is

$$a_k = \frac{k!k^{2k}}{(3k)!} x^k.$$

Because of the factorials, we assume $x \neq 0$, and apply the ratio test:

$$\begin{aligned}
& (\rho =) \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} \\
= & \lim_{k\to\infty} \left| \frac{(k+1)!\,(k+1)^{2k+2}}{(3k+3)!} x^{k+1} \right| \Big/ \left| \frac{k!k^{2k}}{(3k)!} x^k \right| \\
= & \lim_{k\to\infty} |x| \frac{(k+1)!}{k!} \frac{(3k)!}{(3k+3)!} \frac{(k+1)^{2k+2}}{k^{2k}} \\
= & \lim_{k\to\infty} |x| \frac{k+1}{1} \frac{1}{(3k+3)(3k+2)(3k+1)} \left(\frac{k+1}{k}\right)^{2k} (k+1)^2 \\
= & \lim_{k\to\infty} |x| \frac{(k+1)^3}{(3k+3)(3k+2)(3k+1)} \left(1+\frac{1}{k}\right)^{2k} \\
= & \lim_{k\to\infty} |x| \frac{k^3 (1+1/k)^3}{k^3 (3+3/k)(3+2/k)(3+1/k)} \left(\left(1+\frac{1}{k}\right)^k\right)^2 \\
= & \lim_{k\to\infty} |x| \frac{(1+1/k)^3}{(3+3/k)(3+2/k)(3+1/k)} \left(\left(1+\frac{1}{k}\right)^k\right)^2 \\
= & |x| \frac{1}{3(3)(3)} e^2 = \frac{|x|\,e^2}{27}.
\end{aligned}$$

So

$$\rho = \frac{|x|\, e^2}{27}.$$

By the ratio test, this converges if $\rho = \frac{|x|e^2}{27} < 1$, that is if

$$|x| < \frac{27}{e^2},$$

and diverges if $\rho = \frac{|x|e^2}{27} > 1$, that is if

$$|x| > \frac{27}{e^2}.$$

Hence the radius of convergence is $\frac{27}{e^2}$. (Our techniques do not tell us what is the interval of convergence).

Question 5

Let

$$f(x) = \ln(1 + e^x).$$

(i) Compute the 3rd degree Taylor polynomial $P_3(x)$ of f (about 0).

(7 marks)

(ii) Compute the Lagrange form of the remainder $R_3(x)$ and hence estimate the maximum error of $|R_3(x)| = |f(x) - P_3(x)|$ for x in $[0, 1]$.

(4 marks)

Solutions

(i)

$$f(x) = \ln(1 + e^x) \Rightarrow f(0) = \ln 2;$$

$$f'(x) = \frac{e^x}{1 + e^x} \Rightarrow f'(0) = \frac{1}{2}.$$

$$\begin{aligned} f''(x) &= \frac{(e^x)(1 + e^x) - (e^x)(e^x)}{(1 + e^x)^2} \\ &= \frac{e^x}{(1 + e^x)^2} \Rightarrow f''(0) = \frac{1}{4}. \end{aligned}$$

$$\begin{aligned} f'''(x) &= \frac{e^x (1 + e^x)^2 - e^x 2(1 + e^x) e^x}{(1 + e^x)^4} \\ &= \frac{e^x (1 + e^x) - e^x 2e^x}{(1 + e^x)^3} \\ &= \frac{e^x - e^{2x}}{(1 + e^x)^3} \Rightarrow f'''(0) = 0. \end{aligned}$$

Next,

$$\begin{aligned} f^{(4)}(x) &= \frac{(e^x - 2e^{2x})(1 + e^x)^3 - (e^x - e^{2x}) 3 (1 + e^x)^2 e^x}{(1 + e^x)^6} \\ &= \frac{(e^x - 2e^{2x})(1 + e^x) - (e^x - e^{2x}) 3e^x}{(1 + e^x)^4} \\ &= \frac{(e^x - 2e^{2x} + e^{2x} - 2e^{3x}) - (3e^{2x} - 3e^{3x})}{(1 + e^x)^4} \\ &= \frac{e^x - 4e^{2x} + e^{3x}}{(1 + e^x)^4} = \frac{e^x (1 - 4e^x + e^{2x})}{(1 + e^x)^4}. \end{aligned}$$

Then

$$\begin{aligned}
P_3(x) &= f(0) + f'(0)\,x + f''(0)\,\frac{x^2}{2!} + f'''(0)\,\frac{x^3}{3!} \\
&= \ln 2 + \frac{1}{2}x + \frac{1}{4}\frac{x^2}{2!} + 0\frac{x^3}{3!} \\
&= \ln 2 + \frac{1}{2}x + \frac{x^2}{8}. \qquad \text{(7 marks)}
\end{aligned}$$

Graders: please give up to 6 marks for differentation, and one mark for writing down the definition of the Taylor polynomial.
(ii) The Lagrange form of the remainder is

$$R_3(x) = \frac{1}{4!} f^{(4)}(c)\, x^4,$$

for some c between 0 and x. **(Graders, please give 1 mark for this).** Hence

$$R_3(x) = \frac{1}{4!} f^{(4)}(c)\, x^4 = \frac{1}{4!} \frac{e^c\left(1 - 4e^c + e^{2c}\right)}{\left(1 + e^c\right)^4} x^4.$$

Then for $x \in [0,1]$, we have $c \in [0,1]$, so $e^c \geq 1$ and then $1 + e^c \geq 2$. Hence

$$\begin{aligned}
|R_3(x)| &= \left| \frac{1}{4!} \frac{e^c\left(1 - 4e^c + e^{2c}\right)}{\left(1 + e^c\right)^4} x^4 \right| \\
&= \frac{1}{4!} \frac{e^c\left|1 - 4e^c + e^{2c}\right|}{\left(1 + e^c\right)^4} x^4 \\
&\leq \frac{1}{4!} \frac{e\left(1 + 4e + e^2\right)}{2^4} (1)^4 .
\end{aligned}$$

(4 marks)

Graders: students can use ANY correct estimation.

Question 6
Find the solution y of the differential equation

$$\frac{y'}{4x^3} + y = -\frac{1}{4x^2}e^{-x^4}$$

subject to

$$y(0) = 1. \qquad \text{(10 marks)}$$

Solution
We want the form $y' + p(x)\,y = q(x)$, so rewrite the above as

$$y' + 4x^3 y = -xe^{-x^4}.$$

So $p(x) = 4x^3$ and $q(x) = -xe^{-x^4}$.
Step 1 The integrating factor

$$H(x) = \int p(x)\,dx = \int 4x^3 dx = x^4.$$

The integrating factor is

$$e^{H(x)} = e^{x^4}.$$

Step 2 Multiply by the integrating factor

$$e^{x^4}y' + e^{x^4}4x^3 y = -x$$
$$\Rightarrow \frac{d}{dx}\left\{e^{x^4}y\right\} = -x.$$

Step 3 Integrate

$$\begin{aligned} e^{x^4}y &= -\int x\ dx + C \\ &= -\frac{x^2}{2} + C. \end{aligned}$$

Then

$$y(x) = e^{-x^4}\left(-\frac{x^2}{2} + C\right).$$

Step 4 Determine C using the initial condition
Given $y(0) = 1$ so

$$1 = y(0) = e^0(0 + C) = C$$

$$\Rightarrow C = 1.$$

So the solution is

$$y(x) = y(x) = e^{-x^4}\left(-\frac{x^2}{2} + 1\right).$$