ISyE 2027C **Quiz on Homework 7** **March 31, 2015**

1. Random variables X, Y, Z are jointly independent. They have uniform distributions on the intervals $[0,1], [0,2], [0,3]$ respectively. Let $W = \max\{X, Y, Z\}$. Find $E[W]$ and $\sigma^2(W)$.

2. Sherlock Holmes studies the daily personals section of the Times for a year. He observes that the number of items containing exactly 18 vowels and 25 consonants each week follows a Poisson distribution with mean 0.75 Whenever he spots such an item, he flips a fair coin, and treats himself and his colleague Dr. Watson each to a pint of bitters at the Baker Street pub. Watson keeps a careful record of the patients he is called to see at hospitals. The times between calls in which he treats a man who has an identical twin are independently exponentially distributed with a mean of 3 weeks. Whenever Watson returns from such a call, he treats himself and Holmes each to a pint of ale at the Baker Street pub. Suppose that these are the only two kinds of occasions on which they drink at that particular pub.
   While strolling on Baker Street, you peer into the pub and spot Holmes and Watson drinking. What is the probability that they are drinking bitters?

   Let X be the number of pints Watson drinks at the Baker Street pub from February 1 to March 21, 1899. What is $P(X = 4)$?.

   What is the probability that Holmes spends no money at the pub in February 1899 but Watson does spend money?

   Extra Credit: Starting on May 20, 1900, what is the expected number of weeks until Holmes has drunk at least one pint of ale and Watson has drunk at least one pint of bitters?

Formulas: $n!$ is the number of ways to arrange n items in a sequence. $1!=1$ and $n! = n(n-1)!$. $\binom{n}{k}$ is the number of ways to pick k items out of n, when the order of the items does not matter. It equals $\frac{n!}{k!(n-k)!}$. A^C is the complement of A, the set of all things not in A. $P(A) + P(A^C) = 1$ for all A. ϕ denotes the empty set Ω^C. $(A \cup B)^C = A^C \cap B^C$. $(A \cap B)^C = A^C \cup B^C$.
If A and B are disjoint, $P(A \cup B) = P(A) + P(B)$.
If A and B are independent, $P(A \cap B) = P(A)P(B)$. In general $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
$0 \leq P(A) \leq 1$ for all A. $P(A|B) = P(A \cap B)/P(B) = P(B|A)P(A)/P(B)$ is the conditional probability of A given B.
The law of total probability is $P(A) = P(B)P(A|B) + P(B^C)P(A|B^C)$. More generally, if $E_1 \dots E_n$ partition Ω then $P(A) = \sum_{i=1}^{n} P(E_i)P(A|E_i)$.
The law of total probability for expectation is $E[X] = P(A)E[X|A] + P(A^C)E[X|A^C]$. More generally, if $H_1 \dots H_n$ partition Ω then $E[X] = \sum_{i=1}^{n} P(H_i)E[X|H_i]$.
If X is a discrete random variable, $E[X] = \sum_t tP(X = t)$, the weighted average of the values X can take.
If X is a continuous random variable with density function $f(t)$, then $\int_{-\infty}^{\infty} f(t)dt$ must equal 1. Then the cdf of X is $F(t) = \int_{-\infty}^{t} f(t)\ dt = P(X \leq t)$. Also, $E[X] = \int_{-\infty}^{\infty} tf(t)dt$ and LOTUS says that for any function g, $E[g(X)] = \int_{-\infty}^{\infty} g(t)f(t)dt$. LOTIS says $E[g(X)] = g(E[X])$ and is usually wrong.
Expectation is linear. This means that for any random variables X and Y and real number α, $E[\alpha X + Y] = \alpha E[X] + E[Y]$.
A Bernoulli variable with parameter p equals 1 w.p. p and equals 0 w.p. $1 - p$. If X is Bernoulli then $E[X] = p$.
The sum of n independent Bernoullis each with parameter p has binomial distribution $B(n, p)$. If $X \sim B(n, p)$ then $P(X = k) = \binom{n}{k}p^k(1 - p)^{n-k}$ and $E[X] = np$.
Let Y be the number of times you flip a coin that has probability p of being heads, until you get your first head. Then Y has geometric distribution with parameter p. $P(Y = k) = p(1 - p)^{k-1}$ and $E[Y] = 1/p$. The exponential distribution with mean $1/\lambda$ is defined as $P(\leq t) = 1 - e^{-\lambda t}$ for all $t \geq 0$. . These are the unique memoryless discrete and continuous distributions, respectively, meaning that $P(X \geq \alpha + \beta | X \geq \alpha) = P(X \geq \beta)$.
The variance $\sigma^2(X)$ of random variable X is defined to be $E[(X - E[X])^2]$. From linearity of expectation this simplifies to the more convenient $E[X^2] - (E[X])^2$. From the definition, $\sigma^2(\alpha X) = \alpha^2\sigma^2(X)$. The standard deviation of X is defined as $\sigma(X) = \sqrt{\sigma^2(X)}$. In general, variance is not additive. However, if X and Y are independent random variables, $\sigma^2(X + Y) = \sigma^2(X) + \sigma^2(Y)$. The variance of a Bernoulli variable with parameter p is $p(1 - p)$. The variance of a B(n,p) distributed variable is $np(1 - p)$. If X has uniform distribution on $[0, 1]$, $E[X] = .5$ and $\sigma^2(X) = 1/12$.
Chebyshev's inequality: $P(|X - E[X]| \geq k\sigma(X)) \leq 1/k^2$. The probability a random variable is k or more standard deviations from its mean is $\leq 1/k^2$. If X has a Poisson distribution with parameter λ then $P(X = k) = e^{-\lambda}\lambda^k/k!$ for all integers $k \geq 0$, and $E[X] = \lambda$. If X and Y are independent Poisson distributed variables then $X + Y$ has a Poisson distribution. A Poisson process with intensity rate r has interrarival times independently exponentially distributed each with parameter r. For any time interval of length t the number of arrivals has a Poisson distribution with parameter rt, and if time intervals are disjoint the corresponding Poisson variables are independent. If you lump together two independent Poisson processes with rates r_1 and r_2, you get a Poisson process with rate $r_1 + r_2$. If you split a Poisson process with rate r_1 by labelling each arrival red with independent probability p, and otherwise labeling it blue, you get two Poisson process with rates pr_1 and $(1 - p)r_1$.