Math 2401

Quiz 4

Section B

1. Show that the function $f(x,y) = \ln(x^2+y^2)$ satisfies the Laplace equation $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$.

$$f_x = \frac{2x}{x^2+y^2}, \quad f_{xx} = \frac{2}{x^2+y^2} - \frac{2x}{(x^2+y^2)^2}\cdot 2x = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2}$$

$$f_y = \frac{2y}{x^2+y^2}, \quad f_{yy} = \frac{2}{x^2+y^2} - \frac{2y}{(x^2+y^2)^2}\cdot 2y = \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2}$$

$$f_{xx}+f_{yy} = f_{xx} = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} = \frac{4}{x^2+y^2} - \frac{4(x^2+y^2)}{(x^2+y^2)^2} = 0$$

2. Find $\dfrac{\partial w}{\partial u}$ in terms of u and v, if $w = x^2e^z + \arctan(\frac{y}{x})$, $x = \dfrac{v^2}{u}$, $y = u+v$, $z = \cos u$.

$$\frac{\partial w}{\partial u} = w_x x_u + w_y y_u + w_z z_u$$

$$\frac{\partial w}{\partial x} = 2xe^z + \frac{1}{1+(\frac{y}{x})^2}\cdot\frac{-y}{x^2} = 2xe^z - \frac{y}{x^2+y^2}, \quad \frac{\partial w}{\partial y} = \frac{1}{1+(\frac{y}{x})^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}, \quad \frac{\partial w}{\partial z} = x^2e^z.$$

$$\frac{\partial x}{\partial u} = \frac{-v^2}{u^2}, \quad \frac{\partial y}{\partial u} = 1, \quad \frac{\partial z}{\partial u} = -\sin u.$$

$$\frac{\partial w}{\partial u} = \left(2xe^z - \frac{y}{x^2+y^2}\right)\cdot\frac{-v^2}{u^2} + \frac{x}{x^2+y^2}\cdot 1 + x^2e^z\cdot(-\sin u)$$

$$\frac{\partial w}{\partial u} = \left(2\frac{v^2}{u^2}e^{\cos u} - \frac{u+v}{(\frac{v^2}{u})^2+(u+v)^2}\right)\cdot\frac{-v^2}{u^2} + \frac{\frac{v^2}{u}}{(\frac{v^2}{u})^2+(u+v)^2} - (\tfrac{v^2}{u})^2e^{\cos u}\sin u$$

3. Consider the surface with equation $x^3+3x^2y^2+y^3+4xy-z^2=0$. Find the gradient, the equation of the normal line and the tangent plane at the point $(1,1,3)$.

$\nabla f(x,y,z) = (f_x, f_y, f_z) = (3x^2+6xy^2+4y, 6x^2y+3y^2+4x, -2z)$,

$\nabla f(1,1,3) = (13,13,-6)$,

Normal line: $\ell(t) = (1,1,3) + t(13,13,-6)$,

Tangent plane: $(13,13,-6)\cdot(x,y,z) = (13,13,-6)\cdot(1,1,3) \Rightarrow 13x+13y-6z = 8$.