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Solutions to Homework 11

1. Let S_1 and S_2 be independent and exponentially distributed random variables with means $\frac{1}{\mu_1} = 2$ and $\frac{1}{\mu_2} = 5$. They corresponds to the service time of the first and the second teller.

(a) Since A starts service immediately after arrival, $\mathbb{P}(T_A \geq 5) = \mathbb{P}(S_1 \geq 5) = e^{-\frac{1}{2}5} = 0.0821$.

(b) By the same reason, $\mathbb{E}(T_A) = \mathbb{E}(S_1) = 2$.

(c) This amount, in mathematical expression, will be

$$\begin{aligned}\mathbb{E}(T_A \mid T_A > 5) &= \mathbb{E}(S_1 \mid S_1 > 5)\\ &= 5 + \mathbb{E}(S_1) \text{ // by memoryless property of exponential random variables}\\ &= 7.\end{aligned}$$

(d)

$$\mathbb{P}(T_A < T_B) = \mathbb{P}(S_1 < S_2) = \frac{\mu_1}{\mu_1 + \mu_2} = 5/7.$$

(e) The time from noon till a customer leaves is $\min\{T_A, T_B\}$. Its expectation will be the same as

$$\mathbb{E}(\min\{S_1, S_2\}) = \frac{1}{\mu_1 + \mu_2} = 10/7.$$

(f) If a customer leaves, one of the tellers will be able to serve C. So it is the same as time till one departure, which is computed in (e).

(g) The total time C spends in the system will be the summation of the waiting time $\min\{T_A, T_B\}$ and its service time. The service time will depend on at which teller the customer gets service. So by conditioning, we have

$$\begin{aligned}\mathbb{E}(\text{service time of C }) &= \mathbb{E}(S_1 \mid T_A < T_B)\mathbb{P}(T_A < T_B) + \mathbb{E}(S_2 \mid T_B < T_A)\mathbb{P}(T_B < T_A)\\ &= 2 \times 5/7 + 5 \times 2/7 \text{ by problem (e)}\\ &= 20/7.\end{aligned}$$

Thus, the expected time C in the system is $10/7 + 20/7 = 30/7$.

(h) The expected time from noon till C start service is $\mathbb{E}(\min\{T_A, T_B\})$, which has already been studied in (e). Let's now study the expected time from C enter the service till the system is empty. No matter which teller C goes to, both of the teller are busy upon this time. By memoryless property, the two tellers can be imagined just start service together at this time. The expected time left for both of them to finish is still $\mathbb{E}(\max\{S_1, S_2\})$. So

$$\begin{aligned}\mathbb{E}(\text{time until system is empty}) &= \mathbb{E}(\min\{T_A, T_B\}) + \mathbb{E}(\max\{S_1, S_2\})\\ &= 2\mathbb{E}(\min\{S_1, S_2\}) + \mathbb{E}(S_1) + \mathbb{E}(S_2) - \mathbb{E}(\min\{S_1, S_2\})\\ &= \mathbb{E}(S_1) + \mathbb{E}(S_2) = 7.\end{aligned}$$

(i) B leaves before A means that C start service at teller 2.

$$\begin{aligned}\mathbb{P}(T_C < T_A \mid T_B < T_A) &= \mathbb{P}(S_2 < S_1) \text{ // again by memoryless property}\\ &= \frac{\mu_2}{\mu_1 + \mu_2} = 2/7.\end{aligned}$$

(j) The probability that A leaves last is

$$\begin{aligned}\mathbb{P}(T_B < T_A, T_C < T_A) &= \mathbb{P}(T_C < T_A \mid T_B < T_A)\mathbb{P}(T_B < T_A) \\ &= 2/7 \times 2/7 = 4/49.\end{aligned}$$

The probability that B leaves last is

$$\begin{aligned}\mathbb{P}(T_A < T_B, T_C < T_B) &= \mathbb{P}(T_C < T_B \mid T_A < T_B)\mathbb{P}(T_A < T_B) \\ &= \mathbb{P}(T_C < T_B \mid T_A < T_B) \times 5/7 \\ &= 5/7 \times 5/7 = 25/49. \text{ // computation of the first } 5/7 \text{ is similar as in (i)}\end{aligned}$$

The probability that C leaves last is

$$\begin{aligned}\mathbb{P}(T_A < T_C, T_B < T_C) &= \mathbb{P}(T_A < T_C, T_B < T_A) + \mathbb{P}(T_B < T_C, T_A < T_B) \\ &= \mathbb{P}(T_A < T_C \mid T_B < T_A)\mathbb{P}(T_B < T_A) + \mathbb{P}(T_B < T_C \mid T_A < T_B)\mathbb{P}(T_A < T_B) \\ &= 5/7 \times 2/7 + 2/7 \times 5/7 = 20/49.\end{aligned}$$

(k) By memoryless property, we can imagine that everything just starts at 12:10. In order that D enter service, two customers must leave. The average time from 12:10 till one departure is same as in (e), which is 10/7. The average time from then to next departure will be the same, 10/7, due to memoryless property. Now, it is left to compute the expected service time of D. Using a similar conditioning in (g)

$$\begin{aligned}\mathbb{E}(\text{service time of D }) &= \mathbb{E}(S_1 \mid T_A < T_B, T_C < T_B)\mathbb{P}(T_A < T_B, T_C < T_B) \\ &\quad + \mathbb{E}(S_1 \mid T_B < T_A, T_A < T_C)\mathbb{P}(T_B < T_A, T_A < T_C) \\ &\quad + \mathbb{E}(S_2 \mid T_B < T_A, T_C < T_A)\mathbb{P}(T_B < T_A, T_C < T_A) \\ &\quad + \mathbb{E}(S_2 \mid T_A < T_B, T_B < T_C)\mathbb{P}(T_A < T_B, T_B < T_C) \\ &= 2 \times 5/7 \times 5/7 + 2 \times 2/7 \times 5/7 + 5 \times 2/7 \times 2/7 + 5 \times 5/7 \times 2/7 \\ &= 20/7.\end{aligned}$$

So the total time D spends in the system is $10/7 + 10/7 + 20/7 = 40/7$.