ISyE 2027C Probability with Applications

TEST 2 **9 April 2015**

1. Poisson process A has arrival rate 20/minute. Poisson process B is independent and has arrival rate 30/minute. Find[1]:

   (a) The expected time between arrivals from A.

   (b) The expected time between now and the next arrival from A, given that the last arrival from A was exactly 1 second ago.

   (c) The expected time until the first arrival from A or B.

   (d) The probability that the next arrival is from B.

   (e) The expected time between the next two arrivals.

   (f) The probablity of exactly 19 arrivals from A and 31 arrivals from B during one minute.

   (g) The probability that there are no arrivals from A or B during the next 5 seconds.

   (h) The probability that there are no arrivals from B during the next 2 seconds.

   (i) The probability that there are no arrivals from A during the next 5 seconds and there are no arrivals from B during the next 7 seconds.

   (j) Explain the relationship between the answers to the previous three questions.

   (k) (5 points) The expected time until there has been at least one arrival from A and at least one arrival from B.

[1]Parts $a - -j$ are 2 points each. You should be able to answer them immediately.

2. (10 points) Jointly independent random variables $W_i : i = 1, 2, 3, 4$ have uniform distributions on $[-i, i]$, respectively. Let $W = \sum_{i=1}^{4} W_i$. Use Chebyshev's inequality to find a number α such that $P(|W| > \alpha) \leq 1/9$. Remember that the variance of a $U[0, 1]$ variable is $\frac{1}{12}$ and its mean is $\frac{1}{2}$.

3. (10 points) Random variable Y has cdf $F_Y(t) = 1 - 16/t^4 : 2 \leq t < \infty$. Write the integral that equals $E[Y^2]$.

4. (10 points) Random variables X and Y are independently uniformly distributed on $[0, 10]$ and $[0, 30]$ respectively. Find $E[\max\{X, Y)\}]$.

5. Parishioners arrive at church on Sunday morning according to a Poisson process at rate 2/minute starting at 9:00am until 11:00am. Each parishioner wears a hat with probability 1/3, independent of other parishioners, and brings an umbrella with probability 1/4, independent of whether she wears a hat and independent of other parishioners. The cloakroom has umbrella stands and baskets for hats.

   (a) (2 points) What is the probability that exactly 409 parishioners are present at 11:00am?

   (b) (2 points) What is the probability that the cloakroom has exactly 99 hats at 11:00am?

   (c) (5 points) If 299 parishioners arrive by 11:00am, what is the expected arrival time of the 100th parishioner? (Hint: consider the extreme case of 1 arrival if you are stuck.)

   (d) (5 points) What is the expected amount of time that elapses starting at 9:00am until there are at least 2 items in the cloakroom?

   5 points Extra Credit: At least 3 items?

   (e) (5 points) The sermon begins at 11:00am. Once the sermon begins, each parishioner falls asleep after an exponentially distributed amount of time with mean 12 minutes, independent of the others. The sermon ends when all parishioners are asleep. What is the expected number of sleeping parishioners (in the church) at 11:30am?

   (f) (5 points) What is the probability that at least one parishioner is (in church and) awake at noon?

   (g) (6 points) 300 parishioners arrive by 11:00am. What is the expected time at which the sermon ends?

   (h) (Extra credit 5 points) Make a probabilist's joke about the scenario of this problem.

6. You run a delivery service in a city whose streets form a rectangular grid. North-south streets are numbered 1 through 25 and are spaced 1000 feet apart; the 25 east-west streets are 200 feet apart and are named $A, B, C, \ldots, X, Y$. Delivery locations are uniformly distributed on the grid. Answer all questions using the very good approximation of continuous uniform distributions.

   (a) (2 points) Your location is at 1st and Y. What is the average distance your trucks travel to make a delivery and return?

   (b) (5 points) Your location is at 1st and Y. You send one truck to make three deliveries (to random locations) on S street. What is the expected distance your truck travels to make the deliveries and return?

   (c) (2 points) Your location is at 13th and M. (This is the center of the grid). What is the average distance your trucks travel to make a delivery and return?

   (d) (6 points) Your location is at 13th and M. You send one truck to make three deliveries (to random locations) on S street. What is the expected distance your truck travels to make the deliveries and return?