ISyE 2027C Probability with Applications **TEST I** **February 19, 2015**

PART 1: See-it-in-a-glance problems, 2 points each. Circle the correct answer.

1. You flip a fair coin 5 times. Given that the coin came up the same all 5 times, is the probability that the 3rd flip was a head: $< .5$, $\overline{= .5}$, or $> .5$?

   HHHHH and TTTTT both have probability $.5^5$ so by symmetry the answer is .5.

2. You flip an unfair coin 5 times. On each flip the probablity of heads is 0.6. Given that the coin came up the same all 5 times, is the probability that the 3rd flip was a head: $< .6$, $= .6$, or $\overline{> .6}$?

   Consider a simple extreme case of 2 flips and P(heads) = .9. $P(HH = .81), P(TT = .01)$ so $P(HH|$ flips the same$) = .81/.82 > .9$.

3. You roll a (6-sided) die and receive $2k$ dollars if you roll a k. What is the expected amount you will get: $< \$7$, $\overline{= \$7}$, or $> \$7$?

   The expected roll value is 3.5. By linearity the answer is \$2 times 3.5 = \$7.

4. You roll a (6-sided) die and receive $4k^2$ dollars if you roll a k. Is the expected amount you will get: $< \$49$, $= \$49$, or $\overline{> \$49}$?

   Since $f(k) = k^2$ is convex it should be more than \$49. Or consider the simple case of a two-sided die that comes up 6 half the time and 1 half the time. The expected amount would be more than \$72 which is more than \$49.

5. Jesse is dealt 13 cards, one at a time, for a bridge game. If Jesse gets no clubs, what is the probability that she has the king of hearts?: $1/13$, $\binom{39}{12}/\binom{39}{13}$, $\binom{38}{12}/\binom{52}{13}$, $1/12$, $1/4$,$\overline{1/3}$. Jesse gets a random $13/39$ of the non-clubs.

PART 2: Simple Problems, 4 points each

6. You roll a 20-sided die (with faces numbered from 1 to 20) until you roll a 17. What is the expected number of times that you roll? 20. (mean of geometric distribution).

7. The discrete random variable X has the following distribution: $P(X = 3) = .4; P(X = 9) = .1; P(X = 10) = .5$. Find $E[X], E[10X - 333]$, and $E[X^2]$

   $3 \cdot .4 + 9 \cdot .1 + 10 \cdot .5 = 7.1$; $71 - 333 = -262$; $9 \cdot .4 + 81 \cdot .1 + 100 \cdot .5 = 61.7$

8. The continuous random variable Y has uniform distribution in the range $[1, 3]$. Find $E[Y]$ and $E[\sqrt{Y}]$.

   The density of Y is $\frac{1}{2}$.

$$\int_1^3 \frac{1}{2} y \, dy = 3^2/4 - 1^2/4 = 2$$

$$\int_1^3 \frac{1}{2}\sqrt{y} dy = \frac{1}{2}\frac{2}{3}(3^{1.5} - 1^{1.5}) = \sqrt{3} - 1/3$$

9. *Answer this question without calculating any integrals.* The continuous random variable Z has uniform distribution in the range $[100, 300]$. Find $E[Z]$ and $E[\sqrt{Z}]$. $100 \cdot 2 = 200$; $\sqrt{Z} = \sqrt{100Y} = 10\sqrt{Y}$ hence $10(\sqrt{3} - 1/3)$. (Use linearity of expectation.)

10. You flip a fair coin 2 times. Given that you got at least one head, what is the probability that the first flip was a head?

    2/3 Let F be the even of at least one head and let $H1$ be the event that the 1st flip was a head. $P(H1|F) = P(H1 \cap F)/P(F) = .5/.75$.

11. You flip a fair coin 9 times. Given that you got at least one head, what is the expected number of heads? $\frac{9}{2}/(1 - 2^{-9})$. Let H be the event that you get at least one head. Let X be the number of heads. $E[X] = 9/2$; $P(H) = 1 - 2^{-9}$. $X = 0$ when H^C occurs so $E[X|H^C] = 0$. Plug into $E[X] = P(H)E[X|H] + P(H^C)E[X|H^C]$.

    PART 2: Problems, 10 points each unless specified otherwise. Half credit on the first two for defining events, and stating the given probabilities and the value in question in terms of your events.

12. You have three cards, one red on both sides, the second blue on one side and red on the other, the third blue on both sides. While your eyes are closed, your friend picks one card at random and places it on a table without taking color into account. You open your eyes and see blue. What is the probability that the other side of the card is red? 1/3 (Same as homework problem since the red-red card is ruled out.)

13. A kitten gambols on a ledge between a bathtub and her mother, taking 4 inch steps. The ledge is 12 inches wide. The kitten starts in the center of the ledge. When the kitten is in the center, she steps towards her mother with probability .5 and towards the bathtub with probability .5. When the kitten is close to her mother, she steps towards her mother with probability 0.25 and otherwise steps to the center. When the kitten is close to the bathtub, she steps towards the tub with probability 1/3 and otherwise steps to the center. The bathtub is filled iwth water. If the kitten lands on her mother, the mother will playfully bite her.

    (a) (10 points) Define variables and write equations which, if solved, would tell you the probability that the kitten lands in the bathtub (rather than getting bitten).

    X_1 = prob kitten ends in tub starting next to tub; X_2 = same starting in middle of ledge; X_3=same starting on ledge next to mother. $X_1 = 1/3 + (2/3)X_2$; $X_2 = .5X_1 + .5X_3$; $X_3 = .25 + .75X_2$.

    (b) (10 points) Define variables and write equations which, if solved, would tell you the expected number of steps that the kitten will take (until she gets wet or bitten).

    X_i = expected number of steps kitten takes starting at $i = 1$ next to tub; $i = 2$ in middle of ledge; $i = 3$ near mother. $X_1 = 1/3 + (2/3)(1 + X_2)$; $X_3 = .25 + .75(1 + X_2)$; $X_2 = 1 + .5X_1 + .5X_2$.

(c) (5 points extra credit) Explain why this problem is unrealistic. Only humorous answers will receive credit. Cats hate water but love catnip.

14. With your eyes closed, you draw 2 M&Ms without replacement from a bowl containing 10 orange, 15 green, 20 blue, and 25 yellow candies. You win \$20 if you get two oranges, \$15 if you get one each of orange and green, and \$10 if you get one each of orange and blue. Conditioned on the event that you win money, what is the probability that your first candy was orange?

    Let M be the event that you win money and O be the event that the first candy is orange. $P(O) = 10/70 = 1/7$. $P(M) = \frac{1}{7}\frac{9}{69} + \frac{10 \cdot 15}{\binom{70}{2}} + \frac{10 \cdot 20}{\binom{70}{2}}$. $P(M|O) = (9 + 15 + 20)/69 = 44/69$. Then $P(O|M) = P(M|O)P(O)/P(M)$.

15. (10 points) You gamble at the Casino Cockayne, where the odds actually are in your favor. Each time you bet \$100, you win the bet with probability 0.4 and lose with probabilty 0.6. So why are the odds in your favor? When you lose a bet, you receive 0, for a net loss of \$100. When you win a bet, you receive \$400, for a net gain of \$300. Write an equation that, if solved, would give the probability that you will go broke if you start with \$100.

    Let p be the probability in question. $p = .6 + .4p^4$.

    (5 points extra credit) Write an equation that, if solved, would give the probability that you will go broke if you start with \$1000. If q is the desired probability then $q = p^{10}$. Substitute $p = q^{1/10}$ into the previous answer to get $q^{1/10} = .6 + .4q^{2/5}$.

PART 3: A more complicated problem. 20 points.

16. You work for the USDA and receive an anonymous tip that one of the three "Sir K" meat processing plants has an *E Coli* contamination. The three plants produce 1000, 2000, and 3000 kilos of ground beef per hour, respectively. Based on your past experience with whistleblowers, you assess the probability to be 0.75 that the tip is correct, and 0.25 that it is a fraudulent tip sent in by "Gawain Farms," a competing meat processor. If the tip is correct, each of the three processing plants is equally likely to be the one that is contaminated.

    Each hour, you can test a random kilo out of the 6000 produced. If the plant from which the beef came is contaminated, the probability is 2/3 that the test will report contamination. If the beef comes from a plant that is not contaminated, the probability is .01 that the test will incorrectly report contamination. If the test reports contamination, you can require Sir K to tell you which of the three plants produced the kilo that you tested.

    Suppose you test for 24 hours, and every test reports no contamination. What is the probability that the tip was fraudulent? Be sure that your events are clearly defined. Remember that there is no need to write the answer with a single formula.

    Let T be the event that the tip is correct. Let $C1, C2, C3$ be the events that plant 1,2,3 respectively is contaminated.

    As a warmup, let $B1, B2, B3$ be the event that the kilo tested in the first hour is from plant 1,2,3 respectively. We are given $P(C1|T) = P(C2|T) = P(C3|T) = 1/3$ and these events are disjoint. Also $P(Ci|T^C) = 0$ for $i = 1, 2, 3$. Given $P(B1) = 1/6$, $P(B2) = 1/3$, $P(B3) = 3000/6000 = 1/2$. Let R be the event that the test in the first hour reports contamination. Then $P(R|T) = \frac{1}{3}(P(R|T \cap C1) + P(R|T \cap C2) + P(R|T \cap C3)) = \frac{1}{3}\frac{2}{3}(P(B1) + P(B2) + P(B3)) = 2/9$. Hence $P(R^C|T) = 7/9$.

    Now for the actual problem, let R^C24 be the event that all 24 tests report no contamination. If there is no contamination, which is equivalent to T^C, $P(R^C24|T^C) = (.99)^{24}$. If there is contamination and plant 1 is contaminated, $P(R|C1) = \frac{1}{6}\frac{2}{3} + \frac{5}{6}.01 \equiv \alpha_1$. Similarly, $P(R|C2) = \frac{1}{3}\frac{2}{3} + \frac{2}{3}.01 \equiv \alpha_2$ and $P(R|C3) = \frac{1}{2}\frac{2}{3} + \frac{1}{2}.01 \equiv \alpha_3$. Then $P(R^C24|Ci) = (1 - \alpha_i)^{24}$ for $i = 1, 2, 3$. Hence

    $$P(R^C24|T) = \sum_{i=1}^{3} \frac{1}{3}(1 - \alpha_i)^{24}$$

    We seek

    $$P(T^C|R^C24) = P(R^C24|T^C)P(T^C)/P(R^C24) = .25P(R^C24|T^C)/(.25P(R^C24|T^C)+.75P(R^C24|T)).$$