Instructions: *Print* your name, student ID number and recitation session in the spaces below.

Name: ____

Student ID:____________________________________

Recitation session:_____________________________

Exam 1, Calculus II (Math 1502)

02/09/2015 (Monday)

Show your work clearly and completely!

No calculators are allowed.

You can bring a formula sheet of a one-side letter size paper.

Question	Points
1)	
2)	
3)	

Problem 1 (30 points):

(a) Evaluate the improper integral

$$\int_3^5 \frac{x}{\sqrt{x^2-9}}\,dx.$$

Solution:

$$\begin{aligned}\int_3^5 \frac{x}{\sqrt{x^2-9}}\,dx &= \lim_{a\to 3+}\int_a^5 \frac{x}{\sqrt{x^2-9}}\,dx\\ &= \lim_{a\to 3+}\sqrt{x^2-9}\,\Big|_a^5 = \lim_{a\to 3+}\sqrt{5^2-9}-\sqrt{a^2-9}\\ &= 4.\end{aligned}$$

(b) Which of the following improper integrals converge and which diverge? Why?

$$\int_1^\infty \frac{\sin(\pi x)}{x^2}dx, \qquad \int_2^\infty \frac{x\ dx}{\sqrt{x^3-5}}.$$

Solution: The first integral converges by comparison test since

$$\left|\frac{\sin(\pi x)}{x^2}\right| \le \frac{1}{x^2}$$

and $\int_1^\infty \frac{1}{x^2}dx$ converges. The second integral diverges by the limiting comparison test, since

$$\lim_{x\to\infty}\frac{x}{\sqrt{x^3-5}}\Big/\frac{x}{\sqrt{x^3}} = \lim_{x\to\infty}\frac{\sqrt{x^3}}{\sqrt{x^3-5}} = 1$$

and

$$\int_2^\infty \frac{x\ dx}{\sqrt{x^3}} = \int_2^\infty \frac{dx}{\sqrt{x}} \text{ diverges.}$$

(c) Use L'Hôpital's rule to find the limit

$$\lim_{x\to 0+} x^{\sin x}.$$

Solution:

$$\begin{aligned}\lim_{x\to 0+}\ln x^{\sin x} &= \lim_{x\to 0+}\sin x\ln x = \lim_{x\to 0+}\frac{\sin x}{x}\lim_{x\to 0+}\frac{\ln x}{\frac{1}{x}}\\ &= \lim_{x\to 0+}\frac{\cos x}{1}\lim_{x\to 0+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} \text{ (By L'Höpital)}\\ &= \lim_{x\to 0+}\cos x\lim_{x\to 0+} -x = 0,\end{aligned}$$

so

$$\lim_{x\to 0+} x^{\sin x} = e^0 = 1.$$

Problem 2 (30 points): Which of the following series converge, and which diverge? Use any method, and give reasons for your answers.

(a)

$$\sum_{k=1}^{\infty} \frac{k^k}{3^{k^2}}$$

Solution: Convergence by root test since

$$\lim_{k\to\infty} \left(\frac{k^k}{3^{k^2}}\right)^{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{3^k} = \lim_{k\to\infty} \frac{k}{3^k} = \lim_{k\to\infty} \frac{1}{3^k \ln 3} = 0 < 1.$$

(b)

$$\sum_{k=2}^{\infty} \frac{1}{k} \left(\frac{1}{\ln k}\right)^{\frac{3}{2}}$$

Solution: Convergence by the integral test, since

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x} \left(\frac{1}{\ln x}\right)^{\frac{3}{2}} dx &= \lim_{b\to\infty} \int_2^{b} \frac{1}{x} \left(\frac{1}{\ln x}\right)^{\frac{3}{2}} dx \\
&= \lim_{b\to\infty} -2 (\ln x)^{-\frac{1}{2}} |_2^b \\
&= \lim_{b\to\infty} -2 \left[(\ln b)^{-\frac{1}{2}} - (\ln 2)^{-\frac{1}{2}}\right] \\
&= 2 (\ln 2)^{-\frac{1}{2}} < \infty.
\end{aligned}$$

(c)

$$\sum_{k=1}^{\infty} (-1)^k \, k \sin(1/k) \, .$$

Solution: Diverges by the n-th term test since

$$\begin{aligned}
\lim_{k\to\infty} k \sin(1/k) &= \lim_{k\to\infty} \frac{\sin(1/k)}{1/k} = \lim_{k\to\infty} \frac{\cos(1/k)\left(-1/k^2\right)}{-1/k^2} \\
&= \lim_{k\to\infty} \cos(1/k) = 1
\end{aligned}$$

and thus the general term $a_k = (-1)^k \, k \sin(1/k)$ does not tend to zero when $k \to \infty$.

Problem 3 (30 points): Find the radius and interval of convergence of the following series. For what value of x does the series converges absolutely (conditionally)?

$$\sum_{k=1}^{\infty} \frac{\ln k}{k+2} (x+1)^k$$

Solution: Since

$$\begin{aligned} & \lim_{k\to\infty} \left| \frac{\ln(k+1)}{k+3} (x+1)^{k+1} / \frac{\ln k}{k+2} (x+1)^k \right| \\ = & \lim_{k\to\infty} \frac{\ln(k+1)}{\ln k} \frac{k+2}{k+3} |x+1| \\ = & |x+1|, \end{aligned}$$

by the ratio test, the power series converges absolutely when $|x+1| < 1$, that is, $-2 < x < 0$.

At $x = 0$, the series becomes

$$\sum_{k=1}^{\infty} \frac{\ln k}{k+2}$$

which diverges by the comparison test since

$$\frac{\ln k}{k+2} > \frac{1}{k+2} \text{ when } k > 3$$

and $\sum_{k=1}^{\infty} \frac{1}{k+2}$ diverges. At $x = -2$, the series becomes

$$\sum_{k=1}^{\infty} (-1)^k \frac{\ln k}{k+2}$$

which converges conditionally by the alternating series test, since the term $u_k = \frac{\ln k}{k+2}$is positive, tends to zero and decreasing when k is large. The decreasing property follows by the computation

$$\begin{aligned} \left(\frac{\ln x}{x+2}\right)' &= \frac{x+2-x\ln x}{x(x+2)^2} < \frac{x+2-2x}{x(x+2)^2} \\ &= \frac{2-x}{x(x+2)^2} < 0, \text{ when } x > e^2. \end{aligned}$$