MATH 2603, Fall 2015, Exam 2 Sample: Closed book, no calculators.
Instructor: Esther Ezra.

Answer all questions **on this sheet**.

Name	GT IDnumber	Section Number

Problem 1. Let t_n be the number of ternary strings in which we never have "20" occurring as a substring.

a. Write a recurrence equation for t_n, as well as the initial conditions for $n = 1, 2$.

Initial conditions:

$$t(1) = 3, t(2) = 8.$$

Note that we also have $t(0) = 1$, as when $n = 0$ we have just the "empty string".

Recurrence equation:

$$t_n = 2t_{n-1} + (t_{n-1} - t_{n-2}) = 3t_{n-1} - t_{n-2},$$

as when the string end with a '0', we must not have '2' at the preceding location. Then the number of strings in this case is $t_{n-1} - t_{n-2}$.

b. Solve the equation you formed in part a, in order to have an explicit form for t_n.

The characteristic polynomial is

$$x^2 = 3x - 1.$$

The roots are thus $x = (3 \pm \sqrt{5})/2$. Thus we have

$$t_n = c_1 \left(\frac{3+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{3-\sqrt{5}}{2}\right)^n.$$

Using the initial conditions we find c_1, c_2:

$$t_n = \frac{\sqrt{5}+3}{2\sqrt{5}} \left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}-3}{2\sqrt{5}} \left(\frac{3-\sqrt{5}}{2}\right)^n.$$

Problem 2. Consider the alphabet consisting of the ten digits $\{0, 1, \ldots, 9\}$ and the 26 capital letters $\{'A',' B', \ldots,' Z'\}$.

a. How many strings of length 10 can be generated if repetitions of symbols is permitted?

$$36^{10}.$$

b. Same question when repetition of symbols is not permitted.

$$P(36, 10).$$

c. Same question where we are now restricting the string to consist of only two 'A's, three 'D's, four '3's and one '0'.

$$\binom{10}{2, 3, 4, 1}.$$

d. Same question where we allow to have precisely four digits, and two of the remaining characters must be 'A's.

$$\binom{10}{4} 10^4 \binom{6}{4} 25^4.$$

Problem 3. How many integer-valued solutions are there to each of the following equations and inequalities?

a. $x_1 + x_2 + x_3 + x_4 = 50$, s.t. $x_i > 0$, for all $i = 1, \ldots, 4$.

$$\binom{49}{3}.$$

b. $x_1 + x_2 + x_3 + x_4 = 50$, s.t. $x_i \geq 0$, for all $i = 1, \ldots, 4$.

$$\binom{53}{3}.$$

c. $x_1 + x_2 + x_3 + x_4 \leq 50$, s.t. $x_i > 0$, for all $i = 1, \ldots, 4$.

$$\binom{50}{4}.$$

d. $x_1 + x_2 + x_3 + x_4 \leq 50$, s.t. $x_i \geq 0$, for all $i = 1, \ldots, 4$.

$$\binom{54}{4}.$$

e. $x_1 + x_2 + x_3 + x_4 = 50$, s.t. $x_i > 0$, for all $i = 1, \ldots, 3$, and $x_4 \geq 9$.

$$\binom{41}{3}.$$

Problem 4. Use induction in order to show:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(Show the base case and then the inductive step.)

For the case $n = 1$ we have:

$$1^2 = \frac{1 \cdot 2 \cdot 3}{6} = 1.$$

We now assume that the equation is correct for n, and we show that it still holds for $n+1$:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n(2n+1) + n + 1)}{6}$$

$$= \frac{(n+1)(2n^2 + 7n + 6}{6} = \frac{(n+1)((n+2)(2n+3))}{6},$$

and the last term is consistent with our induction hypothesis.

Problem 5. How many integers between 1 and 100 are divisible by 3, 4 or 5?

This is an application of the Exclusion-inclusion principle. Let A_i be the set of integers between 1 and 100 that are divisible by i. Then we have

$$|A_3| = \lfloor 100/3 \rfloor = 33, |A_4| = 100/4 = 25. |A_5| = 100/5 = 20.$$

Next let $A_{i,j}$ be the set of integers between 1 and 100 that are divisible by i and j, and $A_{i,j,k}$ be the set of integers between 1 and 100 that are divisible by i and j, and k. Since all pairs among $3, 4, 5$ are relatively primes, we have

$$|A_{3,4}| = \lfloor 100/12 \rfloor = 8, |A_{3,5}| = \lfloor 100/15 \rfloor = 6, |A_{4,5}| = 100/20 = 5,$$

and we also have $|A_{3,4,5}| = \lfloor 100/60 \rfloor = 1$. Thus by inclusion-exclusion principle, the number of integers between 1 and 100 that are divisible by 3, 4 or 5 is:

$$33 + 25 + 20 - (8 + 6 + 5) + 1 = 60.$$

Problem 6. Order the following terms in increasing "big-O"-order (if two functions are of the same order of growth, you should state this fact):

$$n^{4/3}, \quad \sqrt{n \log n}, \quad n \log \log n, \quad 2^{n/2}, \quad \log(n!), \quad n^{-2}, \quad 1, \quad n^{1/\log n}.$$

$$n^{-2} \ll 1 \approx n^{1/\log n} \ll \sqrt{n \log n} \ll n \log \log n \ll \log n! \ll n^{4/3} \ll 2^{n/2}.$$

Due to Stirling's approximation $\log(n!)$ behaves as:

$$\log\left(\sqrt{2\pi n}(n/e)^n\right) = n \log(n/e) + \log \sqrt{2\pi n}.$$

Since $e \approx 2.71$ is a constant and $\log \sqrt{2\pi n}$ is negligible, we have that $n \log n$ is the leading term, and so $\log(n!)$ behaves as $n \log n$.

Next,

$$n^{1/\log n} = \left(2^{\log n}\right)^{1/\log n} = 2.$$

Then $n^{1/\log n} = O(1)$.

Problem 7. Prove that for any natural number n, and any integer $1 \leq k \leq n$, we always have

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

This is the Pascal's triangle identity. The number of subsets of size k chosen among $\{1, 2, \ldots, n\}$ is $\binom{n}{k}$. We can classify each of these subsets as (i) the subset contains the (last) element n, (ii) the subset does not contain it.

The number of subsets of type (ii) is $\binom{n-1}{k}$ as we make the choice among $\{1, 2, \ldots, n-1\}$.

The number of subsets of type (i) is $\binom{n-1}{k-1}$ since we first isolate n and choose a subset of size $k-1$ from the remaining elements $\{1, 2, \ldots, n-1\}$. Then we append n to each of these subsets.

Problem 8. True-False. Mark in the left Margin.

1. The system $x \equiv 2(\mod 4)$, $x \equiv 6(\mod 7)$ has a unique solution $x \equiv 6(\mod 28)$. **T**
2. The function $f(x) = x^2 - 1$ is one-to-one and onto **F**
3. The number of lattice paths from $(0,0)$ to (n,n) is $\binom{2n}{n}$. **T**
4. In a regular n-gon one can draw at most $n-3$ diagonals that do not cross. **T**
5. The MergeSort algorithm divides first the input sequence into two equal-size subsequences, sorts each of them recursively, and then merges the two sorted subsequences to form the answer. **T**
6. The InsertionSort algorithm uses at most $O(n \log n)$ comparisons on a sequence of n numbers. **F**
7. The number of possibilities to distribute r distinct balls among n boxes, where every box can contain an arbitrary number of balls, is $\binom{n+r-1}{r}$. **F**
8. For all integers $n > 0$, $9^n - 5^n$ is divisible by 4. **T**