Test 1
SOLUTIONS

1. **[5 points]** You run a candy plant and make three types of gum: Popalicious, Bubble Burst, and Fruity Chew. You can sell a carton of Popalicious for \$10, while a carton of Bubble Burst sells for \$20 and Fruity Chew sells for \$15. Popalicious requires 2 units of gum arabic to make. A carton of Bubble Burst also requires 2 units of gum arabic, but also requires 1 unit of fruit flavor. Fruity Chew requires 1 unit of gum arabic and 2 units of fruit flavor. Due to contractual obligations with your vendors, at least half of the total product you make has to be Fruity Chew. Model this as a Linear Program maximizing your revenue if your company has 14 units of gum arabic, and 20 units of fruit flavor.

   Solution: Let x_1: Popalicious, x_2: Bubble Burst, x_3: Fruity chew.

$$
\begin{aligned}
\max \quad & z = 10x_1 + 20x_2 + 15x_3 \\
\text{subject to} \quad & \\
& 2x_1 + 2x_2 + x_3 \leq 14 \quad \text{(Gum Arabic)} \\
& x_2 + 2x_3 \leq 20 \quad \text{(Fruit Flavor)} \\
& x_1 + x_2 - x_3 \leq 0 \quad \text{(Contract)} \\
& x_i \geq 0 \quad \forall i = 1,2,3
\end{aligned}
$$

2. **[10 points]** Solve the following LP using Simplex Tableaus, and Bland's Rule. (hint: you first have to find a basic feasible solution)

$$
\begin{aligned}
\max \quad & z = -2x_1 + 6x_2 - 4x_3 \\
\text{subject to} \quad & \\
& x_1 + 2x_2 \leq 5 \\
& x_1 + x_2 + x_3 \leq 6 \\
& 2x_1 + x_3 \geq 4 \\
& x_i \geq 0 \quad \forall i = 1,2,3
\end{aligned}
$$

   Solution: There are various ways to solve this after looking at the initial tableau. The solution presented here uses the 2 Phase Simplex method.

Initial Tableau:

2	-6	4	0	0	0	0
1	2	0	1	0	0	5
1	1	1	0	1	0	6
2	0	1	0	0	-1	4

The reason you cannot just make your pivot on column 2, row 1 is that you have an infeasible basis to start with. We therefore add artificial variable a3 to the problem and minimize its value in our new objective.

0	0	0	0	0	0	1	0
1	2	0	1	0	0	0	5
1	1	1	0	1	0	0	6
2	0	1	0	0	-1	1	4

Next we have to change Row 0 so that there is a 0 in the column associated with a3. We do this by subtracting row 3 from row 0.

-2	0	-1	0	0	1	0	-4
1	2	0	1	0	0	0	5
1	1	1	0	1	0	0	6
[2]	0	1	0	0	-1	1	4

We now have negative coefficients in row 0. We choose the pivot according to Bland's Rules.

0	0	0	0	0	0	1	0
0	2	-1/2	1	0	1/2	-1/2	3
0	1	1/2	0	1	1/2	-1/2	4
1	0	1/2	0	0	-1/2	1/2	2

This gives us the optimal tableau for phase 1. We have w=0, so we have a basic feasible solution to our original problem. We now remove the column for the artificial variable that is non-basic, and replace row 0 with our original objective function.

2	-6	4	0	0	0	0
0	2	-1/2	1	0	1/2	3
0	1	1/2	0	1	1/2	4
1	0	1/2	0	0	-1/2	2

Now we once again have to change row 0 to remove the 2 from column 1.

0	-6	2	0	0	1	-4
0	[2]	-1/2	1	0	1/2	3
0	1	1/2	0	1	1/2	4
1	0	1/2	0	0	-1/2	2

Now that we have a tableau we can work with we can pivot according to Bland's Rules. (Note: this is the tableau you get if you take our original and just pivot x_1 into the basis for constraint 3. If you pivoted in x_3 you would arrive at the above tableau after 2 pivot steps instead of 1.) Following this pivot we get:

0	0	3/2	3	0	5/2	5
0	1	-1/4	1/2	0	1/4	3/2
0	0	3/4	-1/2	1	1/4	5/2
1	0	1/2	0	0	-1/2	2

This is our optimal solution, all numbers in row 0 are non-negative.

3. [**2 points each**] Indicate whether or not the following statements are True or False.

   (a) Bland's rules for pivoting help us determine whether or not the problem has multiple optimal solutions.

   (b) If a constraint is considered to be "binding", the slack variable associated with that constraint must be one of the basic variables.

   (c) After running the first phase of the two phase simplex method, I am left with w=2. This means that my original problem is unbounded.

   Solutions

   (a) **False.** Bland's rules for pivoting are to prevent cycling, they have no bearing on multiple optimal solutions.

   (b) **False.** If a constraint is binding, then the slack variable for that constraint is 0. This means it can be a non basic variable, and more than likely is.

(c) **False.** If $w \neq 0$ then the original LP is infeasible. This is different then being unbounded.

4. **[3 points each]** Consider the following LP, with optimal basis $\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$.:

$$\begin{aligned} \max \quad z = & 10x_1 + 6x_2 + 7x_3 \\ \text{subject to} & \\ & 5x_1 + 2x_2 + 3x_3 \leq 100 \qquad \text{(Material)} \\ & 3x_1 + x_2 + x_3 \leq 45 \qquad \text{(Labor)} \\ & x_i \geq 0 \qquad \forall i = 1, 2, 3 \end{aligned}$$

(a) What values do x_1, x_2, x_3, z take in the optimal solution?

(b) What is the shadow price of Labor?

(c) What values of the cost c_1 keep the current basis optimal?

Solutions: Given our optimal basis we have that B= $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$. Which makes $B^{-1} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$.

(a) $x_1 = 0$ because is is non-basic in our optimal solution.
$\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 35 \\ 10 \end{bmatrix}$. $z = c_{BV}B^{-1}b = 280$.

(b) The shadow price of labor is the amount that the z value improves after increasing the right hand side of the labor constraint by 1. It is also the reduced cost of the variable x_4 in the optimal tableau. This can be calculated to be 4.

(c) In order for our current basis to remain optimal, $\bar{c}_1$ must be positive. Recall that $\bar{c}_1 = c_{BV}B^{-1}a_1 - c_1$.

$$\begin{aligned} c_{BV}B^{-1}a_1 - c_1 &\geq 0 \\ \begin{bmatrix} 6 & 7 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} - c_1 &\geq 0 \\ 17 - c_1 &\geq 0 \\ c_1 &\leq 17 \end{aligned}$$