ISyE 2027C Probability with Applications
HINTS for Homework 7 for Quiz March 31 **March 24, 2015**

Simple Calculation Problems

1. Random variable X_j has exponential distribution with mean $4j$ for $j = 1, 2, 3$. The variables are jointly independent. Find $P(1 \le X_2 \le 12)$. Find $P((X_1 \ge 8) \cap (X_2 \le 4))$. Find $P(\min_j X_j \le 16)$.

   By definition, $F_{X_i}(t) = P(X_j \le t) = 1 - e^{-t/4j}$. Then $P(1 \le X_2 \le 12) = F_{X_2}(12) - F_{X_2}(2) = e^{-.25} - e^{-1.5}$. By independence the answer to the second question is $e^{-2}(1-e^{-5})$, and the third answer is $1 - e^{-4(1+\frac{1}{2}+\frac{1}{3})}$.

2. Random variable Y has cdf $F(t) \equiv P(Y \le t) = 1 - 1/t^3 : 1 \le t < \infty$. Find the pdf of Y. Find $E[Y]$. Find $\sigma^2(Y)$. $\frac{3}{4}$ $f_Y(t) = 0$ for $t < 1$. $f_Y(t) = 3t^{-4}$ for $t \ge 1$. $E[Y] = \int_1^\infty 3t^{-3}dt = 1.5$; $E[Y^2] = \int_1^\infty 3t^{-2}dt = 3$. Therefore $\sigma^2(Y) = 3 - (1.5)^2 = \frac{3}{4}$.

3. Random variables W_1, W_2, W_3 are jointly independent each uniformly distributed on $[0, 12]$. Let $W = \max_{1\le j\le 3} W_j$. Find $P(1 \le W \le 3)$. Find the pdf of W and find $E[W]$. $P(W \le t) = (t/12)^3$ by independence. $P(1 \le W \le 3) = (3^3 - 1)/12^3 = .01505$ The pdf is $t^2/576$ for $0 \le t \le 12$ and 0 elsewhere. $E[W] = \int_0^{12} t^3/576 = 12^4/(24^2 \cdot 4) = 9$. You can see the value 9 in a glance by randomly picking 4 points on a circle of circumference 12. One point is what you call 0 and 12. The other three points are $W_i : i = 1, 2, 3$.

Poisson Process Problems

1. 100 light bulbs in a chandelier have independent lifetimes each exponentially distributed with mean 2 months. 20 of the bulbs are blue; 30 are red; 50 are white. What is the probability that the first bulb to burn out is red? What is the probability that the first two bulbs to burn out are blue? What is the probability that the first 6 burnouts are red, white, blue, red, white, blue in that order? What is the expected number of working white bulbs after 1 month? Conditioned on there being 80 working bulbs after 2.5 months, what is the expected number of working blue bulbs?

   As explained in class, the answers are $.3; \frac{20\cdot 19}{100\cdot 99}$, $\frac{30\cdot 50\cdot 20\cdot 29\cdot 49\cdot 19}{100!/94!}$; $50(1-e^{-.5})$; $.8 \cdot 20 = 16$.

2. Consider the same chandelier as in the previous problem. What is the expected amount of time until there are only 99 bulbs working? 98? 1? 0? Leave your answers for 1 and 0 as summation formulae. If you number the bulbs from 1 to 100 now, what is the expected amount of time until bulb 55 burns out? What is the expected number of bulbs that will burn out more than 2 months but less than 3 months from now?

   As explained in class, the answers are the expected time until the 1st arrival of a Poisson process with rate 50 per month, .02 months; $.02 + \frac{2}{99}$ months;

   $$\sum_{j=2}^{100} \frac{2}{j} \text{ months };\sum_{j=1}^{100} \frac{2}{j} \text{ months };$$

   2 months; $100(e^{-1} - e^{-1.5}$.

3. Now consider two chandeliers. The first is identical to the one in the previous problems. The second is like the first, but each bulb has exponentially distributed lifetime with mean 4 months. What is the probability that the 1st bulb to burn out is from the first chandelier? The first chandelier initially behaves like a PP with rate 50 per month; the second like a PP with rate $100/4 = 25$ per month. $\frac{50}{50+25} = \frac{2}{3}$.

4. A mcg of U-238 emits particles according to a Poisson process at rate 2/sec. A mcg of U-235 emits particles according to a Poisson process at rate 10/sec. Ali H.K. receives a gift of 1 mcg uranium, which is U-238 with probability .5 and U-235 with probability .5.

   (a) Ali observes his gift for one second. What is the probability that no particles are emitted? Let 235 be the event that the gift is $U - 235$ and 235^C be the complementary event that it is $U - 238$. $P(235) = .5$. Let Z be the event that zero particles are emitted by the gift in one second. $P(Z|235^C) = e^{-2}$, which you can see from the probability a Poisson variable with mean 2 equals 0, or the probability an exponential variable with parameter 2 is greater than 1. Similarly $P(Z|235) = e^{-10}$. By the law of total probability for expected values, the answer is $P(Z) = .5e^{-2} + .5e^{-10}$.

(b) Suppose that the gift is U-238 and Ali carries it in his pocket for 11 hours. Write as a sum the probability that not more than 39,000 particles are emitted during that time. Use Chebyshev's inequality to bound that probability. Estimate square roots or use a calculator. The number of particles emitted by a PP of rate 2/sec over time 11 hours $= 11 \cdot 60^2$ seconds has Poisson distribution with mean (rate times time) $= 22 \cdot 60^2 \equiv \alpha$. Hence the answer is

$$e^{-alpha} \sum_{k=0}^{39000} \alpha^k/k!$$

. The expected number of particles is $79,200$. The standard deviation is therefore $\sqrt{79200} \approx 280$. $(79200 - 39000)/280 \approx 144 = 12^2$. So 39000 is about 12^2 standard deviations less than the mean, and the chance of that many or fewer particles is at most $1/12^4$. If you had to prove that the sum of those 39,001 terms was bounded by 12^{-4} without knowing about the Poisson distribution, you might have trouble.

(c) Ali observes that 15 particles are emitted from his mcg of uranium during a 3 second period. Conditioned on that observation, what is the probability that his uranium is U-235? Let A be the event of 15 particles arriving in 3 seconds. For $U - 238$ the number of arrivals has Poisson distribution with mean $2 \cdot 3 = 6$. For $U - 235$ the mean is 30. Then

$$P(235|A) = P(A|235)P(235)/P(A) = \frac{.5e^{-30}30^{15}/15!}{.5(e^{-30}30^{15} + e^{-6}6^{15})/15!} = \frac{e^{-30}30^{15}}{e^{-30}30^{15} + e^{-6}6^{15}}.$$

(d) The next day, Ali receives a second gift of 1 mcg uranium, of the type he didn't receive previously. He observes both gifts simultaneously. The first particle emitted by either one comes from the first. What is the probability that the first is U-235? If the first two particles emitted come from the first, what is the probability that the first is U-235? What is the probability that no particles are emitted (from either one) during the first 2 seconds?

Let A be the event that the first particle emitted comes from the 1st gift. We want $P(235|A) = P(A|235)P(235)/P(A) = \frac{10}{10+2}\frac{.5}{.5} = 5/6$. Let AA be the event that the first two particles emitted come from the 1st gift. We want $P(235|AA) = P(AA|235)P(235)/P(AA)$. Memoryless makes watching emissions like sampling with replacement. Regardless of where the first particle comes from, the next has a 5/6 chance of coming from the U-235. By the law of total probability, $P(AA) = P(235)P(AA|235) + P(235^C)P(AA|235^C) = .5((\frac{5}{6})^2 + (\frac{1}{6})^2) = 13/36$. So the answer is $\frac{25/36 \cdot .5}{13/36} = 25/26$. Can you see this in a glance? There are 25 realities in which both particles come from the U-235 and only 1 reality in which both come from the U-238.

Together the gifts make a PP with rate $10+2 = 12$. In 2 seconds the number of arrivals is Poisson distributed with mean 24. The answer is therefore e^{-24}.

5. Calls from Maryland arrive at a call center at a rate of 15/minute. Calls from Georgia arrive at the same center at a rate of 5/minute. What is the mean time between calls if both arrival processes are independent Poisson processes? Suppose each call from Maryland is silly with probability .3 and each call from Georgia is silly with probability .25. What is the mean time between silly calls? Suppose a silly call arrives at exactly 1:23. What is the expected number of seconds until the next non-silly call?

The total rate is 20/minute or one per 3 seconds. The answer is therefore 3 seconds. The silly call rate is $.3 \cdot 15 + .25 \cdot 5 = 5.75$ calls per minute. The mean time between is therefore $\frac{1}{5.75}$ minutes, or $\frac{4}{23}$ minutes. The non-silly call rate is $20 - 5.75 = 14.25$ calls per minute. By the memoryless property the expected number of seconds is $\frac{60}{14.25}$ until the next non-silly call. The answer would be the same if we supposed that a non-silly call arrives at 1:23. The answer would still be the same if we asked for the expected number of seconds between 1:23:00 and the next non-silly call. That is, it does not matter whether or not there is an arrival at 1:23:00.