Math 2603, Fall 2015, Quiz 1 Solutions

September 11, 2015

1 Problem 1. (40 points)

a. Let a be an integer. Show that either a or $a+1$ is even.

Proof: Breaking into cases, either a is odd or it is even.
case 1: Suppose a is odd. Then $a = 2k+1$ for some $k \in \mathbb{Z}$, in which case $a+1 = 2k+2 = 2(k+1)$ is even by definition.
case 2: Suppose a is even. Then $a = 2k$ for some $k \in \mathbb{Z}$, in which case $a+1 = 2k+1$ is odd and therefore not even by definition.$\square$

(*note: the problem asked "either a is even or $a+1$ is even", so we need to also show or remind the reader that it cannot be both simultaneously*)

b. Show that n^2+n is even for any integer n

Proof: From the first part, we know that either n or $n+1$ is even.

case 1: Suppose n is even. So $n = 2k$ for some $k \in \mathbb{Z}$. We have then that $n^2+n = n(n+1) = 2k(n+1) = 2(kn+k)$ is even.
case 2: Suppose that $n+1$ is even. So $n+1 = 2k$ for some $k \in \mathbb{Z}$. We have then that $n^2+n = n(2k) = 2(kn)$ is even.$\square$

2 Problem 2. (60 points)

Which of the following conditions imply that $B = C$? In each case, either prove or give a counter example:

a.: $A \cup B = A \cup C$
(*note: the problem, reworded, is asking us to prove or disprove the implication "If $A \cup B = A \cup C$, then $B = C$"*)

This is false. There are several counterexamples. Here are two: $A = \{1\}, B =$

$\emptyset, C = \{1\}$. You have then $A \cup B = \{1\} = A \cup C$ but $B \neq C$.

$A = \{1,2\}, B = \{1,3,4\}, C = \{2,3,4\}$. You have that $A \cup B = \{1,2,3,4\} = A \cup C$ but $B \neq C$.

b.: $A \cap B = A \cap C$

This also is false. Again, there are several counterexamples. Here are two: $A = \emptyset, B = \emptyset, C = \{1\}$. You have then $A \cap B = \emptyset = A \cap C$ but $B \neq C$.

$A = \{1,2,3\}, B = \{1,4,5\}, C = \{1,6,7\}$. You have then that $A \cap B = \{1\} = A \cap C$ but $B \neq C$.

c.: $A \times B = A \times C$
note: In this problem, the intended interpretation is that A, B, C are still sets, and the $\times$ operation is for the Cartesian product. $A \times B = \{(a,b) \mid a \in A, b \in B\}$

This is technically false as well. In the case that $A = \emptyset$, you have $A \times B = \emptyset = A \times C$ for any choice of B and C, in particular when B and C are nonequal.

If you were however to add the condition that $A \neq \emptyset$ then this implication is in fact true.

Proof: Let A be nonempty. If B is empty, then $A \times B = \emptyset$ implying that $A \times C = \emptyset$, which could only happen if and only if C was also empty.

Suppose then that A, B, C are all nonempty.

Let $x \in B$. Then for every $a \in A$, you have $(a,x) \in A \times B$, implying that since $A \times B = A \times C$ you have $(a,x) \in A \times C = \{(a,c) \mid a \in A, c \in C\}$ which then implies that $x \in C$. Therefore $B \subseteq C$.

Similarly, letting $x \in C$, the same logic will imply that $x \in B$ and therefore $C \subseteq B$. Therefore $B = C$.