Quiz 4 - Name ____ Math 2403 - Section L ___

Good Luck!

This quiz has a back side! Don't forget about Question 3 and Bonus Question!

1. (5 points) Given the following system of differential equations $y' = \begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix} y$,

   (a) Find the real general solution. (b) Classify the equilibrium.

   (a) The characteristic polynomial is $p(\lambda) = (\lambda - 2)^2 + 9$, therefore the matrix has two complex conjugate eigenvalues $\lambda_1 = 2+3i$ and $\lambda_2 = 2-3i$. It is enough to find one eigenvector, and then take the conjugate as a second one. Solving the homogeneous linear system $(A - \lambda_1 I)x_1 = 0$ gives $x_1 = (1+i, 1)^T$. Therefore we have

$$e^{2t}(\cos 3t + i \sin 3t) \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

   and taking the real and imaginary parts of it yields

$$y = c_1 e^{2t} \begin{bmatrix} \cos 3t - \sin 3t \\ \cos 3t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} \sin 3t + \cos 3t \\ \sin 3t \end{bmatrix}$$

   (b) The equilibrium (the origin) is an unstable spiral point.

2. (5 points) Given the following system of differential equations $y' = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} y$,

   (a) Find the general solution. (b) Classify the equilibrium. (c) Sketch the phase portrait.

   (a) The characteristic polynomial is $p(\lambda) = (\lambda+1)^2$. Therefore, there are two repeated eigenvalues $\lambda_1 = \lambda_2 = -1$. The associated eigenspace is not complete, in particular, solving the system $(A - \lambda I)x = 0$ gives us $x = x_1 = x_2 = (1, 1)^T$. We can write the first solution $y_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$. For a second solution we need to find a vector w such that $(A - \lambda I)w = x$. Solving this system yields $w = (1, 0)^T$. Therefore the second solution is given by $y_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t}$. The general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} \right)$$

   (b) The equilibrium (the origin) is an asymptotically stable improper node.

   (c)

3. (5 points) Given the following system of differential equations: $y' = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} y + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

(a) Find the equilibrium solution. (b) Classify the equilibrium. (c) Find the fundamental matrix associated to the homogeneous system.

(a) The equilibrium is given by solving the system $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$, therefore

$$y_{eq} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

(b) y_{eq} is an unstable node.

(c) Since the matrix is triangular, its eigenvalues are the diagonal entries: $\lambda_1 = 1$ and $\lambda_2 = 2$ and the associated eigenvectors are $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The fundamental matrix associates with the homogeneous system is

$$X(t) = \begin{bmatrix} e^t & 0 \\ -e^t & e^{2t} \end{bmatrix}$$

Its determinant is nonzero.

[Bonus] (2 points) Find the general solution of the nonhomogeneous system.

The general solution of the homogeneous problem is $\tilde{y} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}$ and in order to find the general solution of the nonhomogeneous problem, it is enough to shift this solution:

$$y = y_{eq} + \tilde{y} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}$$