MATH 2603, Fall 2015, Midterm Exam 1, Sep 24 2015: Closed book, no calculators.
Instructor: Esther Ezra.

Answer all questions **on this sheet**.

Name   GT IDnumber   Section Number

Problem 1. (15 points) Consider the statement $\mathcal{A}$:"If n is an integer then $\frac{n}{n+1}$ is not an integer".

a. (5 points) Is $\mathcal{A}$ true or false? Either prove true or give a counter example to prove false.

This is a false statement. Consider $n = 0$, we obtain $0/1 = 0$, which is an integer.

b. (10 points) Write down the converse, and the contrapositive of $\mathcal{A}$. Which of them is true? Which is false? Justify each answer with a proof or a counter example.

(Converse) If $n/(n+1)$ is not an integer then n is an integer. This is false: Consider $n = 1/2$, we have $n/(n+1) = 1/3$, which is not an integer.

(Contrapositive) If $n/n+1$ is integer then n is not an integer. This is false since this contrapositive statement is logically equivalent to $\mathcal{A}$ and this was proven to be false in part a.

Problem 2. (20 points) Define $\sim$ on the set of integers Z by $a \sim b$ if and only if $3a + b$ is a multiple of 4.

a. (10 points) Prove that $\sim$ defines an equivalence relation.

Reflexivity: $3a + a = 4a$**, which is clearly a multiple of** 4**.**

Symmetry: Suppose that $3a + b$ **is a multiple of** 4**. Consider now** $3b + a$**, then their sum is** $4a + 4b$**, which is obviously a multiple of** 4**, thus** $3b + a = (4a + 4b) - (3a + b)$ **must be a multiple of** 4**.**

Transitivity: Suppose that $3a + b$ **is a multiple of** 4 **and** $3b + c$ **is also a multiple of** 4**. We need to show** $3a + c$ **is a multiple of** 4**. Indeed, the sum of the first two is** $3a + b + 3b + c = 3a + 4b + c$**, and this must divide** 4 **by assumption. Since** $4b$ **is a multiple of** 4**, the remaining term** $3a + c$ **must divide** 4 **as well.**

b. (10 points) Find the equivalence classes of 0 and 2.

$$\overline{0} = \{a \in Z \mid a \equiv 0 \mod 4\}.$$
$$\overline{2} = \{a \in Z \mid a \equiv 2 \mod 4\}.$$

Problem 3. (15 points) Let $S = \{1, 2, \ldots, n\}$, where $n \geq 2$ is a fixed integer, and let $P(S)$ denote the power set of S.

a. (10 points) Prove that $(P(S), \subseteq)$ is a partially ordered set (that is, this is the collection of all pairs (S_1, S_2) of subsets of S, s.t. $S_1 \subseteq S_2$).

Reflexivity: Each subset contains itself.

Antisymmetry: If $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$ then we must have $S_1 = S_2$.

Transitivity: If $S_1 \subseteq S_2$ and $S_2 \subseteq S_3$ then we must have $S_1 \subseteq S_3$.

b. (5 points) Does $(P(S), \subseteq)$ have a maximum and a minimum elements? If so, what are these elements?

The maximum element is the entire set S, and the minimum element is $\emptyset$.

Problem 4. (15 points)

a. (5 points)

Find $gcd(82, 80)$ and $gcd(81, 79)$.

$$gcd(82, 80) = gcd(80, 2) = 2.$$
$$gcd(81, 79) = gcd(79, 2) = gcd(2, 1) = 1.$$

b. (10 points) Next, prove a more general property: For any natural number a, $gcd(a + 2, a)$ is 1 if a is odd, and 2 if a is even.

Assume first a is even, then $gcd(a + 2, a) = gcd(a, 2) = 2$ (since a is even it must divide 2).

Assume now that a is odd, then we have $gcd(a + 2, a) = gcd(a, 2) = 1$, as a does not divide 2, then they do not have any common divisors but 1.

Problem 5. (15 points)

a. (10 points) Describe the Sieve procedure to find all primes between 2 and n, where $n \geq 2$, is a natural number. Based on that, list all primes between 2 and 40.

The Sieve procedure:

- List all integers between 2 and n
- Circle 2 and cross out all multiples of 2 in the list
- Circle 3, the first number not yet crossed out or circled , and then cross out all multiples of 3
- Circle 5, the first number not yet crossed out or circle, and then cross out all multiples of 5
- At the general stage, circle the first number that is neither crossed out nor circled and then cross out all its multiples
- Continue until all numbers less than or equal to $\sqrt(n)$ have been circled or crossed out. When the procedure is complete, all the integers not crossed out are primes not exceeding n.

Thus by the Sieve procedure, the primes between 2 and 40 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37

b. (5 points) State (without a proof) the Fundamental Theorem of Arithmetic. Based on that, find the prime decomposition of 120.

The Fundamental Theorem of Arithmetic: Every natural number $n \geq 2$ can be written as $n = p_1 p_2 .... p_r$ for a unique set of primes $\{p_1, p_2, ... p_r\}$; equivalently, every integer $n \geq 2$ can be written $n = (q_1)^{\alpha_1} (q_2)^{\alpha_2} ... (q_s)^{\alpha_s}$ as the product of powers of distinct prime numbers $q_1, q_2, ..., q_s$. These primes and their exponents are unique.
Prime decomposition of 120: $120 = 2^3 \times 3 \times 5$.

Problem 6. (20 points) Answer true/false:

a. Let A, B, C, D be sets. Then $A \subseteq C$ and $B \subseteq D$ implies that $A \times B \subseteq C \times D$.
Answer: True.

b. Let A, B, C be sets. Then $A \nsubseteq B$ (A is not a subset of B), $B \subseteq C$ implies $A \nsubseteq C$.
Answer: False. Consider the case when $A = C - B$

c. If $a|b$ and $c|d$, then $ac|bd$.
Answer: True because given assumptions $b = ak$ and $d = ch$ for some h, k integers then $bd = akch$ so $ac|bd$

d. For any two integers, a, b, $gcd(a, b) \times lcm(a, b) = |a||b|$.
Answer: True, textbook page 110.

e. Let a, b be integers and let p be a prime number. If $p|a^5$ then $p|a$.
Answer: True.

f. There are only finitely many primes.
Answer: False, there are infinitely many primes as proven by Euclid, textbook pg 115.

g. The sum of two consecutive primes is never twice a prime.
Answers: True.

h. $\{n \in N \mid n > 2$ and $a^n + b^n = c^n,$ for some $a, b, c \in N\} \neq \emptyset$.
Answer: False by Fermat's last theorem.