MATH 1552 TEST 3, FALL 2015, GRODZINSKY

Print Your Name: Key 1 (white)

T.A.: (circle one) Miheer Brandon Stephen Kabir

5. (6 points each) Determine if each series below converges or diverges. Give a short (one-line) justification for your answer.

(a) $\sum_{k=3}^{\infty} \frac{1}{k-2}$ Let $j=k-2$, then we have:

$= \sum_{j=1}^{\infty} \frac{1}{j}$, which is the harmonic series, so it $\boxed{\text{diverges}}$.

(b) $\sum_{k=2}^{\infty} \left[\frac{1}{k-1} - \frac{1}{k+3}\right]$ This series is telescoping, so it $\boxed{\text{converges}}$

$= 1 - \cancel{\frac{1}{5}} + \frac{1}{2} - \cancel{\frac{1}{6}} + \frac{1}{3} - \cancel{\frac{1}{7}} + \frac{1}{4} - \frac{1}{8} + \cancel{\frac{1}{5}} - \frac{1}{9} + \cdots$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$

(c) $\sum_{k=0}^{\infty} \frac{3^k}{4^k+2}$ Note that $4^k+2 > 4^k$, so $\frac{3^k}{4^k+2} < \frac{3^k}{4^k} = \left(\frac{3}{4}\right)^k$.

Since $\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k$ converges (geometric with $r=\frac{3}{4}<1$), this series $\boxed{\text{converges}}$ by Basic Comparison.

(d) $\sum_{n=1}^{\infty} \left(1 - \frac{5}{n}\right)^n$

Note $\lim_{n\to\infty} \left(1-\frac{5}{n}\right)^n = e^{-5} \neq 0$, so the series $\boxed{\text{diverges}}$ by the n^{th} term / divergence test.

6. (14 points) Determine if the series below converges or diverges. JUSTIFY YOUR ANSWER FULLY using the convergence tests from class. The justification will count for the majority of the points.

$$\sum_{n=0}^{\infty} \frac{5^n n!}{(2n)!}$$

Using the Ratio Test:

$$\lim_{n\to\infty} \frac{5^{n+1}(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{5^n n!} = \lim_{n\to\infty} \frac{\cancel{5^n}\cdot 5\cdot (n+1)\,\cancel{n!}\cdot \cancel{(2n)!}}{\underbrace{(2n+2)}_{2}(2n+1)\cancel{(2n)!}\cdot \cancel{5^n n!}}$$

$$= \lim_{n\to\infty} \frac{5}{4n+2} = 0 < 1, \text{ so the series}$$

$\boxed{\text{Converges}}$.

7. (14 points) Determine if the series below converges or diverges. JUSTIFY YOUR ANSWER FULLY using the convergence tests from class. The justification will count for the majority of the points.

$$\sum_{k=1}^{\infty} \frac{4k}{(k^7+5)^{1/3}}$$

Using Basic Comparison:

Compare to $\sum_{k=1}^{\infty} \frac{1}{k^{4/3}}$, which converges (p-series, $p = 4/3 > 1$).

Since $(k^7+5)^{1/3} > (k^7)^{1/3}$, then $\frac{1}{(k^7+5)^{1/3}} < \frac{1}{k^{7/3}}$, so

$\frac{4k}{(k^7+5)^{1/3}} < \frac{4k}{k^{7/3}} = 4 \cdot \frac{1}{k^{4/3}}$. $\sum_{k=1}^{\infty} 4 \cdot \frac{1}{k^{4/3}}$ converges, so our series also $\boxed{\text{converges}}$ by basic comparison.

BONUS: (5 points) Suppose that $\sum_k a_k$ and $\sum_k b_k$ are both convergent series, and $b_k > 0$ for all values of k. Does $\sum_k \frac{a_k}{b_k}$ converge? If so, prove it. If not, provide a counterexample.

Not necessarily! For example, let $\sum_k a_k = \sum_k \frac{1}{k^3}$ and $\sum_k b_k = \sum_k \frac{1}{k^2}$.

Then $\sum_k \frac{a_k}{b_k} = \sum_k \frac{1}{k}$, which diverges, even though $\sum a_k$ and $\sum b_k$ converge.

MATH 1552 TEST 3, FALL 2015, GRODZINSKY

Print Your Name: Key 2 (blue)

T.A.: (circle one) Miheer Brandon Stephen Kabir

1. (12 points) Determine if the sequence given below converges or diverges. If it converges, find its limit and specify if the limit is the least upper bound (l.u.b.) or the greatest lower bound (g.l.b.) of the terms.

$$\left\{\left(\frac{n-1}{n+1}\right)^{2n}\right\}$$

Let $y=\left(\frac{n-1}{n+1}\right)^{2n}$.

Then $\ln y = 2n\ln\left(\frac{n-1}{n+1}\right) = \frac{2\ln\left(\frac{n-1}{n+1}\right)}{1/n}$. $\Rightarrow \lim_{n\to\infty}\ln y = \lim_{n\to\infty}\frac{2\ln\left(\frac{n-1}{n+1}\right)}{1/n}$ $\left[\frac{0}{0}\right]$

$\overset{L'H}{=} \lim_{n\to\infty}\frac{2\left[\frac{1}{n-1}-\frac{1}{n+1}\right]}{-1/n^2}$ ($\frac{1}{n-1}-\frac{1}{n+1} = \frac{2}{n^2-1}$) $= \lim_{n\to\infty}\frac{-2n^2(2)}{n^2-1} = -4$ so

$\lim_{n\to\infty} y = \boxed{e^{-4}}$ and the sequence $\boxed{\text{converges}}$.

As the terms are increasing, $\boxed{\text{l.u.b.} = e^{-4}}$.

2. (12 points) Sum the series:

$$\sum_{k=1}^{\infty}\frac{4^{k+1}-1}{9^k}$$

$$=\sum_{k=1}^{\infty}\frac{4^{k+1}}{9^k}-\sum_{k=1}^{\infty}\frac{1}{9^k}$$

$$=4\sum_{k=1}^{\infty}\left(\frac{4}{9}\right)^k-\sum_{k=1}^{\infty}\left(\frac{1}{9}\right)^k$$

$$=4\cdot\frac{4/9}{1-4/9}-\frac{1/9}{1-1/9}=4\cdot\frac{4}{9}\cdot\frac{9}{5}-\frac{1}{9}\cdot\frac{9}{8}$$

$$=\frac{16}{5}-\frac{1}{8}=\boxed{\frac{123}{40}}$$

3. (12 points) Evaluate the integral:

$$\int_0^4 \frac{1}{(x-3)^2}\,dx.$$

$$= \int_0^3 \frac{dx}{(x-3)^2} + \int_3^4 \frac{dx}{(x-3)^2} = \lim_{b\to 3^-} \int_0^b \frac{dx}{(x-3)^2} + \lim_{a\to 3^+} \int_a^4 \frac{dx}{(x-3)^2}$$

$$= \lim_{b\to 3^-} -\frac{1}{x-3}\Big|_0^b + \lim_{a\to 3^+} -\frac{1}{x-3}\Big|_a^4$$

$$= \lim_{b\to 3^-} \Big[\underbrace{-\frac{1}{b-3}}_{\to -(-\infty)} + \frac{1}{-3}\Big] + \lim_{a\to 3^+} \Big[-\frac{1}{1} + \underbrace{\frac{1}{a-3}}_{\to\infty}\Big]$$

$= \infty + \infty \Rightarrow$ both integrals diverge

$\Rightarrow$ the integral $\boxed{\text{diverges}}$.

4. (12 points) Determine if the series $\sum_{k=2}^{\infty} \frac{\ln(k^4)}{k^2}$ converges or diverges. JUSTIFY YOUR ANSWER FULLY using convergence tests from class. The justification will count for the majority of the points.

Note that $\ln(k^4) = 4\ln k$. We'll use the integral test.

If $f(x) = \frac{\ln(x^4)}{x^2}$, then $f'(x) = \frac{4(1-2\ln x)}{x^3}$ and $f'(x) < 0$ when $x > e^{1/2} \approx 0.648$, so f is decreasing for $x \geq 2$.

Then: $\int_2^\infty \frac{\ln(x^4)}{x^2}\,dx = \lim_{N\to\infty} 4\int_2^N \frac{\ln x}{x^2}\,dx$

By parts:
$u = \ln x \quad dv = \frac{1}{x^2}dx$
$du = \frac{1}{x}dx \quad v = -\frac{1}{x}$

$$= 4\lim_{N\to\infty} \Big[-\frac{1}{x}\ln x + \int \frac{1}{x^2}dx\Big]\Big|_2^N$$

$$= 4\lim_{N\to\infty} \Big[-\frac{1}{x}\ln x - \frac{1}{x}\Big]\Big|_2^N = 4\lim_{N\to\infty} \Big[\underbrace{-\frac{\ln N}{N}}_{\frac{1/N}{1}\to 0} - \underbrace{\frac{1}{N}}_{\to 0} + \frac{\ln 2}{2} + \frac{1}{2}\Big]$$

$= 4\Big[\frac{\ln 2}{2} + \frac{1}{2}\Big]$ which is finite $\Rightarrow$ the integral converges,
so the series also converges

Key 2 (blue)

5. (6 points each) Determine if each series below converges or diverges. Give a short (one-line) justification for your answer.

(a) $\sum_{k=0}^{\infty} \frac{6^k}{7^k+3}$ Compare to $\sum \left(\frac{6}{7}\right)^k$, which converges (geometric with $r = 6/7 < 1$):

$7^k+3 > 7^k$, so $\frac{1}{7^k+3} < \frac{1}{7^k} \Rightarrow \frac{6^k}{7^k+3} < \frac{6^k}{7^k} = \left(\frac{6}{7}\right)^k$,

so the series also converges.

(b) $\sum_{k=4}^{\infty} \frac{1}{k-3}$

Note that if we let $j = k-3$, we obtain:

$\sum_{j=1}^{\infty} \frac{1}{j}$, which is the harmonic series, so it diverges.

(c) $\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^n$

$\lim_{n\to\infty} \left(1-\frac{3}{n}\right)^n = e^{-3} \neq 0$, so the series diverges

(d) $\sum_{k=3}^{\infty} \left[\frac{1}{k-2} - \frac{1}{k+4}\right]$

This series is telescoping so it converges

$= \left(\frac{1}{1} - \cancel{\frac{1}{7}}\right) + \left(\frac{1}{2} - \frac{1}{8}\right) + \left(\frac{1}{3} - \frac{1}{9}\right) + \left(\frac{1}{4} - \frac{1}{10}\right)$

$+ \left(\frac{1}{5} - \frac{1}{11}\right) + \left(\frac{1}{6} - \frac{1}{12}\right) + \left(\cancel{\frac{1}{7}} - \frac{1}{13}\right) + \cdots$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$

6. (14 points) Determine if the series below converges or diverges. JUSTIFY YOUR ANSWER FULLY using the convergence tests from class. The justification will count for the majority of the points.

$$\sum_{k=1}^{\infty} \frac{7k}{(k^9+5)^{1/4}}$$

~~Limit Comparison Test~~ with $\sum \frac{1}{K^{5/4}}$, which converges (p-series with $p = 5/4 > 1$)

$$\lim_{n\to\infty} \frac{7n}{(n^9+5)^{1/4}} \cdot \frac{n^{5/4}}{1} = 7 \lim_{n\to\infty} \frac{n^{4/4} \cdot n^{5/4}}{(n^9+5)^{1/4}} = 7 \lim_{n\to\infty} \left(\frac{n^9}{n^9+5}\right)^{1/4}$$

$$= 7\left(\lim_{n\to\infty} \frac{n^9}{n^9+5}\right)^{1/4} = 7 \cdot 1^{1/4} = 7.$$ Since $0 < 7 < \infty$,

our series also converges.

7. (14 points) Determine if the series below converges or diverges. JUSTIFY YOUR ANSWER FULLY using the convergence tests from class. The justification will count for the majority of the points.

$$\sum_{n=0}^{\infty} \frac{3^n n!}{(2n)!}$$

Ratio Test:

$$\lim_{n\to\infty} \frac{3^{n+1}(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{3^n n!} = \lim_{n\to\infty} \frac{\cancel{3^n} \cdot 3 (\cancel{n+1}) \cancel{n!} \cdot \cancel{(2n)!}}{\underbrace{(\cancel{2n+2})}_{2} (2n+1) \cancel{(2n)!} \cancel{3^n} \cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{3}{4n+2} = 0 < 1,$$

so the series converges

BONUS: (5 points) Suppose that $\sum_k a_k$ and $\sum_k b_k$ are both convergent series, and $b_k > 0$ for all values of k. Does $\sum_k \frac{a_k}{b_k}$ converge? If so, prove it. If not, provide a counterexample.

See Form 1.

MATH 1552 TEST 3, FALL 2015, GRODZINSKY

Print Your Name: Key-3 (green)

T.A.: (circle one) Miheer Brandon Stephen Kabir

1. (6 points each) Determine if each series below converges or diverges. Give a short (one-line) justification for your answer.

(a) $\sum_{n=1}^{\infty} (1 - \frac{7}{n})^n$

Note $\lim_{n\to\infty} (1-\frac{7}{n})^n = e^{-7} \neq 0$,

so the series diverges.

(b) $\sum_{k=5}^{\infty} \frac{1}{k-4}$ Letting $j = k-4$, this series becomes $\sum_{j=1}^{\infty} \frac{1}{j}$, the harmonic series, so it diverges.

(c) $\sum_{k=4}^{\infty} \left[\frac{1}{k-3} - \frac{1}{k+2}\right]$ This is telescoping, so it converges.

$= (1-\cancel{\frac{1}{6}}) + (\frac{1}{2} - \cancel{\frac{1}{7}}) + (\frac{1}{3} - \frac{1}{8}) + (\frac{1}{4} - \frac{1}{9}) + (\frac{1}{5} - \frac{1}{10})$
$+ (\cancel{\frac{1}{6}} - \frac{1}{11}) + (\cancel{\frac{1}{7}} - \frac{1}{12}) + \cdots$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

(d) $\sum_{k=0}^{\infty} \frac{2^k}{5^k+4}$

Compare to $\sum (\frac{2}{5})^k$, which converges (geometric with $r = 2/5 < 1$):

$\frac{2^k}{5^k+4} < \frac{2^k}{5^k} = (\frac{2}{5})^k$, so this series converges.

2. (12 points) Sum the series:

$$\sum_{k=1}^{\infty} \frac{6^{k+1}-1}{7^k}$$

$$= \sum_{k=1}^{\infty} \frac{6^{k+1}}{7^k} - \sum_{k=1}^{\infty} \frac{1}{7^k}$$

$$= 6\sum_{k=1}^{\infty} \left(\frac{6}{7}\right)^k - \sum_{k=1}^{\infty} \left(\frac{1}{7}\right)^k = 6 \cdot \frac{6/7}{1-6/7} - \frac{1/7}{1-1/7}$$

$$= 6 \cdot \frac{6}{\cancel{7}} \cdot \frac{\cancel{7}}{1} - \frac{1}{\cancel{7}} \cdot \frac{\cancel{7}}{6} = 36 - \frac{1}{6} = \boxed{35\tfrac{1}{5}}$$

3. (12 points) Evaluate the integral:

$$\int_0^5 \frac{1}{(x-4)^2}\,dx.$$

$$= \int_0^4 \frac{dx}{(x-4)^2} + \int_4^5 \frac{dx}{(x-4)^2}$$

$$= \lim_{b\to 4^-} \int_0^b \frac{dx}{(x-4)^2} + \lim_{a \to 4^+} \int_a^5 \frac{dx}{(x-4)^2}$$

$$= \lim_{b\to 4^-} -\frac{1}{x-4}\Big|_0^b + \lim_{a\to 4^+} -\frac{1}{x-4}\Big|_a^5$$

$$= \lim_{b\to 4^-} \Big[\underbrace{-\frac{1}{b-4}}_{\to\infty} + \frac{1}{-4}\Big] + \lim_{a\to 4^+}\Big[-\frac{1}{1} + \underbrace{\frac{1}{a-4}}_{\infty}\Big]$$

$$= \infty + \infty \quad \Rightarrow \text{both integrals diverge}$$

$$\Rightarrow \text{the integral } \boxed{\text{diverges}}$$

MATH 1552 TEST 3, FALL 2015, GRODZINSKY

Print Your Name: Key 3 (green)

T.A.: (circle one) Miheer Brandon Stephen Kabir

4. (12 points) Determine if the series $\sum_{k=2}^{\infty} \frac{\ln(k^4)}{k^2}$ converges or diverges. JUSTIFY YOUR ANSWER FULLY using convergence tests from class. The justification will count for the majority of the points.

See Form 1.

5. (14 points) Determine if the series below converges or diverges. JUSTIFY YOUR ANSWER FULLY using the convergence tests from class. The justification will count for the majority of the points.

$$\sum_{n=0}^{\infty} \frac{6^n n!}{(2n)!}$$

Ratio Test

$$\lim_{n\to\infty} \frac{6^{n+1}(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{6^n n!}$$

$$= \lim_{n\to\infty} \frac{\cancel{6^n} \cdot 6\,\cancel{(n+1)}\cancel{n!} \cdot \cancel{(2n)!}}{\underbrace{\cancel{(2n+2)}}_{2}(2n+1)\cancel{(2n)!}\,\cancel{6^n n!}}$$

$$= \lim_{n\to\infty} \frac{6}{4n+2} = 0 < 1,$$

so the series Converges

6. (12 points) Determine if the sequence given below converges or diverges. If it converges, find its limit and specify if the limit is the least upper bound (l.u.b.) or the greatest lower bound (g.l.b.) of the terms.

$$\left\{ \left(\frac{n-1}{n+1}\right)^{2n} \right\}$$

Let $y = \left(\frac{n-1}{n+1}\right)^{2n}$

Then $\ln y = 2n \ln\left(\frac{n-1}{n+1}\right) \Rightarrow \ln y = \frac{2\ln\left(\frac{n-1}{n+1}\right)}{1/n}$.

So $\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{2\ln\left(\frac{n-1}{n+1}\right)}{1/n}$ $\left[\frac{0}{0}\right]$

$\overset{L'H}{=} \lim_{n\to\infty} \frac{2\left[\frac{1}{n-1} - \frac{1}{n+1}\right]}{-1/n^2} = \lim_{n\to\infty} -\frac{2n^2(2)}{n^2-1} = -4$,

So $\lim_{n\to\infty} y = e^{-4}$ and the sequence converges

As the terms are increasing, $\boxed{\text{l.u.b.} = e^{-4}}$

7. (14 points) Determine if the series below converges or diverges. JUSTIFY YOUR ANSWER FULLY using the convergence tests from class. The justification will count for the majority of the points.

$$\sum_{k=1}^{\infty} \frac{6k}{(k^8+5)^{1/3}}$$

Compare to $\sum \frac{1}{k^{5/3}}$, which converges (p-series with $p = 5/3 > 1$):

$k^8 + 5 > k^8 \Rightarrow (k^8+5)^{1/3} > k^{8/3}$

$\Rightarrow \frac{6k}{(k^8+5)^{1/3}} < \frac{6k}{k^{8/3}} = 6 \cdot \frac{1}{k^{5/3}}$.

Since $\sum \frac{6}{k^{5/3}}$ converges, our series also converges.

BONUS: (5 points) Suppose that $\sum_k a_k$ and $\sum_k b_k$ are both convergent series, and $b_k > 0$ for all values of k. Does $\sum_k \frac{a_k}{b_k}$ converge? If so, prove it. If not, provide a counterexample.

See Form 1.

MATH 1552 TEST 3, FALL 2015, GRODZINSKY

Print Your Name: Key - 4 (gold)

T.A.: (circle one) Miheer Brandon Stephen Kabir

1. (12 points) Determine if the sequence given below converges or diverges. If it converges, find its limit and specify if the limit is the least upper bound (l.u.b.) or the greatest lower bound (g.l.b.) of the terms.

$$\left\{\left(\frac{n-1}{n+1}\right)^{3n}\right\}$$

Let $y = \left(\frac{n-1}{n+1}\right)^{3n}$

$\Rightarrow \ln y = 3n \ln\left(\frac{n-1}{n+1}\right) \Rightarrow \ln y = \dfrac{3\ln\left(\frac{n-1}{n+1}\right)}{1/n}$

So $\lim\limits_{n\to\infty} \ln y = \lim\limits_{n\to\infty} \dfrac{3\ln\left(\frac{n-1}{n+1}\right)}{1/n} \left[\frac{0}{0}\right] \stackrel{L'H}{=} \lim\limits_{n\to\infty} \dfrac{3\left[\frac{1}{n-1} - \frac{1}{n+1}\right]}{-1/n^2}$

$= \lim\limits_{n\to\infty} \dfrac{3\left(\frac{2}{n^2-1}\right)}{-1/n^2} = \lim\limits_{n\to\infty} \dfrac{-6n^2}{n^2-1} = -6$, so

$\lim\limits_{n\to\infty} y = \boxed{e^{-6}}$ and the sequence $\boxed{\text{converges}}$. As the terms are increasing, $\boxed{\text{l.u.b.} = e^{-6}}$.

2. (12 points) Evaluate the integral:

$$\int_0^2 \frac{1}{(x-1)^2}\,dx.$$

$= \displaystyle\int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^2} + \lim_{a\to 1^+} \int_a^2 \frac{dx}{(x-1)^2}$

$= \lim\limits_{b\to 1^-} -\dfrac{1}{x-1}\Big|_0^b + \lim\limits_{a\to 1^+} -\dfrac{1}{x-1}\Big|_a^2$

$= \lim\limits_{b\to 1^-} \left[-\underbrace{\frac{1}{b-1}}_{-(-\infty)} + \frac{1}{-1}\right] + \lim\limits_{a\to 1^+} \left[-\frac{1}{1} + \underbrace{\frac{1}{a-1}}_{\to\infty}\right]$

$= \infty + \infty \Rightarrow$ both integrals diverge

$\Rightarrow$ the integral $\boxed{\text{diverges}}$

3. (6 points each) Determine if each series below converges or diverges. Give a short (one-line) justification for your answer.

(a) $\sum_{k=5}^{\infty} \left[\frac{1}{k-4} - \frac{1}{k+1}\right]$ This series is telescoping, so it converges.

$= (1 - \cancel{\tfrac{1}{6}}) + (\tfrac{1}{2} - \tfrac{1}{7}) + (\tfrac{1}{3} - \tfrac{1}{8}) + (\tfrac{1}{4} - \tfrac{1}{9}) + (\tfrac{1}{5} - \tfrac{1}{10}) + (\cancel{\tfrac{1}{6}} - \tfrac{1}{11}) + \cdots$

$= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5}$

(b) $\sum_{n=1}^{\infty} \left(1 - \frac{4}{n}\right)^n$

$\lim_{n\to\infty} \left(1 - \frac{4}{n}\right)^n = e^{-4} \neq 0$, so the series diverges

(c) $\sum_{k=6}^{\infty} \frac{1}{k-5}$ Letting $j = k-5$, we have $\sum_{j=1}^{\infty} \frac{1}{j}$, the harmonic series, which diverges.

(d) $\sum_{k=0}^{\infty} \frac{5^k}{8^k+6}$

Comparing to $\sum \left(\frac{5}{8}\right)^k$, which converges (geometric with $r = 5/8 < 1$):

$8^k + 6 > 8^k$, so $\frac{5^k}{8^k+6} < \frac{5^k}{8^k} = \left(\frac{5}{8}\right)^k$,

so the series converges.

MATH 1552 TEST 3, FALL 2015, GRODZINSKY

Print Your Name: Key - 4 (gold)

T.A.: (circle one) Miheer Brandon Stephen Kabir

4. (12 points) Determine if the series $\sum_{k=2}^{\infty} \frac{\ln(k^4)}{k^2}$ converges or diverges. JUSTIFY YOUR ANSWER FULLY using convergence tests from class. The justification will count for the majority of the points.

See Form 1.

5. (14 points) Determine if the series below converges or diverges. JUSTIFY YOUR ANSWER FULLY using the convergence tests from class. The justification will count for the majority of the points.

$$\sum_{n=0}^{\infty} \frac{4^n n!}{(2n)!}$$

Ratio Test:

$$\lim_{n\to\infty} \frac{4^{n+1}(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{4^n n!}$$

$$= \lim_{n\to\infty} \frac{\cancel{4^n} \cdot 4 \cdot \cancel{(n+1)} \cancel{n!} \cdot \cancel{(2n)!}}{\underbrace{(2n+2)}_{2}(2n+1)\cancel{(2n)!} \cdot \cancel{4^n} \cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{4}{4n+2} = 0 < 1,$$ so the series $\boxed{\text{converges}}$.

6. (12 points) Sum the series:

$$\sum_{k=1}^{\infty} \frac{5^{k+1}-1}{6^k}$$

$$= \sum_{K=1}^{\infty} \frac{5^{K+1}}{6^K} - \sum_{K=1}^{\infty} \frac{1}{6^K}$$

$$= 5\sum_{K=1}^{\infty} \left(\frac{5}{6}\right)^K - \sum_{K=1}^{\infty} \left(\frac{1}{6}\right)^K = 5 \cdot \frac{5/6}{1-5/6} - \frac{1/6}{1-1/6}$$

$$= 5 \cdot \frac{5}{6} \cdot \frac{6}{1} - \frac{1}{6} \cdot \frac{6}{5} = 25 - \frac{1}{5} = \boxed{24\,4/5}$$

7. (14 points) Determine if the series below converges or diverges. JUSTIFY YOUR ANSWER FULLY using the convergence tests from class. The justification will count for the majority of the points.

$$\sum_{k=1}^{\infty} \frac{3k}{(k^{11}+5)^{1/5}}$$

Compare to

$\sum \frac{1}{K^{6/5}}$, which converges (p-series with $p = 6/5 > 1$):

$$\lim_{n\to\infty} \frac{3n}{(n^{11}+5)^{1/5}} \cdot \frac{n^{6/5}}{1} = 3 \lim_{n\to\infty} \frac{n^{11/5}}{(n^{11}+5)^{1/5}}$$

$$= 3\left(\lim_{n\to\infty} \frac{n^{11}}{n^{11}+5}\right)^{1/5} = 3 \cdot 1^{1/5} = 3.$$

Since $0 < 3 < \infty$, the series also converges by the Limit Comparison Test.

BONUS: (5 points) Suppose that $\sum_k a_k$ and $\sum_k b_k$ are both convergent series, and $b_k > 0$ for all values of k. Does $\sum_k \frac{a_k}{b_k}$ converge? If so, prove it. If not, provide a counterexample.

See Form 1.