MATH1502 - Calculus II TEST 1 C Group - January 29, 2015

NAME: ____
STUDENT NUMBER:__________________________
GROUP (e.g. C1 or C3):__________________________
TEACHING ASSISTANT:________________________

Write your solutions to the questions on this testpaper - you may use both sides of each sheet of paper. There are 52 marks on this paper. Full marks (100%) is 50 marks. You may NOT use a calculator or any notes.

Question	Points	Ex
1		8
2		10
3		7
4		9
5		10
6		8
Total		52⇒50

Question 1
Calculate the following limit:

$$\lim_{x\to 0} \frac{e^{-3x} - e^{5x} + 8x}{1 - \cos 3x}. \qquad \text{(8 marks)}$$

Solution
Here

$$\begin{aligned} \lim_{x\to 0} \left(e^{-3x} - e^{5x} + 8x\right) &= 1 - 1 + 0 = 0; \\ \lim_{x\to 0} (1 - \cos 3x) &= 1 - 1 = 0, \end{aligned}$$

so we apply l'Hospital:

$$\begin{aligned} & \lim_{x\to 0} \frac{e^{-3x} - e^{5x} + 8x}{1 - \cos 3x} \\ = & \lim_{x\to 0} \frac{\frac{d}{dx}\left(e^{-3x} - e^{5x} + 8x\right)}{\frac{d}{dx}(1 - \cos 3x)} \\ = & \lim_{x\to 0} \frac{-3e^{-3x} - 5e^{5x} + 8}{3 \sin 3x}. \end{aligned}$$

Here

$$\begin{aligned} \lim_{x\to 0} \left(-3e^{-3x} - 5e^{5x} + 8\right) &= -3 - 5 + 8 = 0; \\ \lim_{x\to 0} (3 \sin 3x) &= 0. \end{aligned}$$

So apply l'Hospital again:

$$\begin{aligned} & \lim_{x\to 0} \frac{-3e^{-3x} - 5e^{5x} + 8}{3 \sin 3x} \\ = & \lim_{x\to 0} \frac{\frac{d}{dx}\left(-3e^{-3x} - 5e^{5x} + 8\right)}{\frac{d}{dx}(3 \sin 3x)} \\ = & \lim_{x\to 0} \frac{-3e^{-3x}(-3) - 5e^{5x}(5)}{9 \cos 3x} \\ = & \frac{9 - 25}{9} = -\frac{16}{9}. \end{aligned}$$

Question 2
Calculate

$$\lim_{x\to 0+}\left[\frac{1}{\ln(1+2x)}-\frac{1}{\sin 2x}\right]. \qquad \text{(10 marks)}$$

Solutions
Here

$$\lim_{x\to 0+}\frac{1}{\ln(1+2x)}=\infty=\lim_{x\to 0+}\frac{1}{\sin 2x},$$

so we have form $\infty-\infty$. We write

$$\begin{aligned}
&\lim_{x\to 0+}\left[\frac{1}{\ln(1+2x)}-\frac{1}{\sin 2x}\right]\\
=&\lim_{x\to 0+}\left[\frac{\sin 2x-\ln(1+2x)}{(\ln(1+2x))(\sin 2x)}\right].
\end{aligned}$$

Here the limit is of the form $\frac{0}{0}$, so we try l'Hospital:

$$\begin{aligned}
=&\lim_{x\to 0+}\left[\frac{\frac{d}{dx}\{\sin 2x-\ln(1+2x)\}}{\frac{d}{dx}\{(\ln(1+2x))(\sin 2x)\}}\right]\\
=&\lim_{x\to 0+}\frac{2\cos 2x-\frac{2}{1+2x}}{\left(\frac{2}{1+2x}\right)(\sin 2x)+(\ln(1+2x))(2\cos 2x)}.
\end{aligned}$$

Again the limit is of the form $\frac{0}{0}$, so we try more l'Hospital:

$$\begin{aligned}
=&\lim_{x\to 0+}\frac{\frac{d}{dx}\left\{2\cos 2x-\frac{2}{1+2x}\right\}}{\frac{d}{dx}\left\{\left(\frac{2}{1+2x}\right)(\sin 2x)+(\ln(1+2x))(2\cos 2x)\right\}}\\
=&\lim_{x\to 0+}\frac{-4\sin 2x+\frac{4}{(1+2x)^2}}{\left\{\begin{array}{c}\left(-\frac{4}{(1+2x)^2}\right)(\sin 2x)+\left(\frac{2}{1+2x}\right)(2\cos 2x)+\\ \left(\frac{2}{1+2x}\right)(2\cos 2x)+(\ln(1+2x))(-4\sin 2x)\end{array}\right\}}\\
=&\frac{0+4}{0+4+4+0}=\frac{1}{2}.
\end{aligned}$$

Question 3
Calculate the limit
$$\lim_{x\to\infty}\left(1+4e^{-x}\right)^{e^x}. \qquad \text{(7 marks)}$$

Solution
Here
$$\lim_{x\to\infty}\left(1+4e^{-x}\right)=1$$
and
$$\lim_{x\to\infty}e^x=\infty$$
so we have form 1^∞. So take log's:
$$\lim_{x\to\infty}\ln\left(1+4e^{-x}\right)^{e^x}=\lim_{x\to\infty}e^x\ln\left(1+4e^{-x}\right).$$
This has form $\infty\cdot 0$, so we rewrite as $\frac{0}{0}$ and then apply l"Hospital:
$$\begin{aligned}
&= \lim_{x\to\infty}\frac{\ln\left(1+4e^{-x}\right)}{e^{-x}}\\
&= \lim_{x\to\infty}\frac{\frac{d}{dx}\ln\left(1+4e^{-x}\right)}{\frac{d}{dx}e^{-x}}\\
&= \lim_{x\to\infty}\frac{\frac{1}{1+4e^{-x}}\left(-4e^{-x}\right)}{-e^{-x}}\\
&= \lim_{x\to 0+}\frac{4}{1+4e^{-x}}=\frac{4}{1+0}=4.
\end{aligned}$$
Then the original limit is
$$\lim_{x\to\infty}\left(1+4e^{-x}\right)^{e^x}=e^4.$$

Question 4

(a) For which $p > 0$ does

$$\int_1^\infty \frac{1}{(\ln(1+x))^p} \frac{1}{1+x} dx \qquad \text{(7 marks)}$$

converge? Hint: you may assume results proved in class about $\int_1^\infty \frac{1}{t^p} dt$.

(b) If it does converge, evaluate it.

(2 marks)

Solution

(a) First, note that $f(x) = \frac{1}{(\ln(1+x))^p} \frac{1}{1+x}$ is continuous in $[1, \infty)$. So we compute

$$\int_1^\infty \frac{1}{(\ln(1+x))^p} \frac{1}{1+x} dx = \lim_{b\to\infty} \int_1^b \frac{1}{(\ln(1+x))^p} \frac{1}{1+x} dx.$$

We make the substitution $t = \ln(1+x)$ (so $\frac{dt}{dx} = \frac{1}{1+x}$) and continue this as

$$\begin{aligned} &= \lim_{b\to\infty} \int_{\ln 2}^{\ln(1+b)} \frac{1}{t^p} dt \\ &= \int_{\ln 2}^\infty \frac{1}{t^p} dt. \end{aligned}$$

From class results, we know this converges iff $p > 1$.

(b) For $p > 1$, we can evaluate the integral as

$$\begin{aligned} \int_{\ln 2}^\infty \frac{1}{t^p} dt &= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{t^p} dt \\ &= \lim_{b\to\infty} \left[\frac{t^{-p+1}}{-p+1} \right]_{t=\ln 2}^{t=\ln b} \\ &= \lim_{b\to\infty} \left[\frac{(\ln b)^{-p+1} - (\ln 2)^{-p+1}}{-p+1} \right] = \frac{(\ln 2)^{-p+1}}{p-1}. \end{aligned}$$

Question 5

Does the following improper integral converge? (You may assume the comparison test and the result of Question 4).

$$\int_2^{\infty} \frac{1}{(\ln(2-\sin x + x))^3}\left(\frac{2+\sin(x^2)}{1+x}\right) dx. \qquad \text{(10 marks)}$$

Solution

We use the comparison test: we see that $\frac{1}{(\ln(2-\sin x+x))^3}\left(\frac{2+\sin(x^2)}{1+x}\right)$ is continuous in $[2,\infty)$ and for $x \geq 0$,

$$\frac{2+\sin(x^2)}{1+x} \leq \frac{3}{1+x}$$

while $2-\sin x \geq 1$, so

$$\ln(2-\sin x + x) \geq \ln(1+x),$$

so

$$\frac{1}{(\ln(2-\sin x + x))^3}\left(\frac{2+\sin(x^2)}{1+x}\right) \leq 3\frac{1}{(\ln(1+x))^3}\frac{1}{1+x}.$$

From the result of Question 4,

$$\int_2^{\infty} 3\frac{1}{(\ln(1+x))^3}\frac{1}{1+x} dx = 3\int_2^{\infty} \frac{1}{(\ln(1+x))^3}\frac{1}{1+x} dx$$

converges. By the comparison test,

$$\int_2^{\infty} \frac{1}{(\ln(2-\sin x + x))^3}\left(\frac{2+\sin(x^2)}{1+x}\right) dx \text{ also converges.}$$

Question 6
Find the sum of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + 9k + 20}. \qquad \text{(8 marks)}$$

Hint: use partial fractions.
Solution
We see that

$$k^2 + 9k + 20 = (k+4)(k+5)$$

so using partial fractions,

$$\frac{1}{k^2 + 9k + 20} = \frac{1}{(k+4)(k+5)} = \frac{1}{k+4} - \frac{1}{k+5}.$$

The nth partial sum is

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{k^2 + 9k + 20} \\ &= \sum_{k=1}^{n} \left[\frac{1}{k+4} - \frac{1}{k+5}\right] \\ &= \left[\frac{1}{5} - \frac{1}{6}\right] + \left[\frac{1}{6} - \frac{1}{7}\right] + ... + \left[\frac{1}{n+4} - \frac{1}{n+5}\right] \\ &= \frac{1}{5} - \frac{1}{n+5}. \end{aligned}$$

So

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{5} - \frac{1}{n+5}\right) = \frac{1}{5}.$$

Thus

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + 9k + 20} = \frac{1}{5}.$$