Quiz 3 - Name ____ Math 2403 - Section L ___

Good Luck!

This quiz has a back side! Don't forget about Question 3 and Bonus Question!

1. (5 points) Given the following system of differential equations: $y' = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} y$,

   (a) Find the general solution. (b) Classify the equilibrium. (c) Sketch the phase portrait.

   Solution:

   (a) We solve the problem using the eigenvalue method.
   The matrix A has two real distinct eigenvalues, that are obtained by solving the following equation: $det(A - \lambda I) = (\lambda - 1)(\lambda + 3) = 0$.
   Therefore, we have $\lambda_1 = -3$ and $\lambda_2 = 1$.
   The corresponding eigenvectors are obtained by solving the homogeneous systems $(A - \lambda_i I)x_i = 0$ with $i = 1, 2$.
   With $\lambda_1 = -3$ we find $x_1 = (2, 1)^T$ and with $\lambda_2 = 1$ we find $x_2 = (-2, 1)^T$. The two independent solutions are

   $$y_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t} \qquad \text{and} \qquad y_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t}$$

   The general solution is given by

   $$y = c_1 y_1 + c_2 y_2 = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t}.$$

   (b) The equilibrium of the system is the point $(0, 0)$. Since the eigenvalues have opposite sign, it is a saddle point and therefore it is unstable.

   (c)

2. (5 points) Given the solution $y = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-\frac{t}{2}} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$ of the system $y' = \frac{1}{4} \begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix} y$,

   (a) Show that y is a general solution of the system. (b) Classify the equilibrium. (c) Sketch the phase portrait.

   Solution:

   (a) In order to show that y is a general solution, we have to prove that it is a solution and that the two solutions $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-\frac{t}{2}}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$ are linearly independent.
   Plugging y into the system we have

   $$-\frac{1}{2} c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-\frac{t}{2}} - 2c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} = \frac{1}{4} \begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-\frac{t}{2}} + \frac{1}{4} \begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

   which is an identity, meaning that y is a solution of the system.
   Moreover, we compute the Wonskian at $t = 0$

   $$W = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0,$$

   meaning that the solutions are linearly independent.
   Thus the linear combination of the two solutions is a general solution.

   (b) The equilibrium given by the origin is an asymptotically stable node, since the eigenvalues of the matrix are both real and negative.

   (c)

3. (5 points) Given the following system of differential equations $y' = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix} y$,

the characteristic polynomial associated to the matrix of the system is $p(\lambda) = -(\lambda - 2)(\lambda + 2)^2$. Let y_1 be the solution relative to the eigenvalue $\lambda_1 = 2$, find the solutions y_2 and y_3 associated to the repeated eigenvalues λ_2 and λ_3.

Solution: The repeated eigenvalues of the matrix are $\lambda_2 = \lambda_3 = -2$.
The solution relative to λ_2 is given by the eigenvalue method

$$y_2 = x_2 e^{\lambda_2 t} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-2t}.$$

Since the corresponding eigenspace has dimension one, we need to find a generalized eigenvector w corresponding to the eigenvalue -2, such that $(A + 2I)w = x_2$.
The vector which satisfies this system is given by $w = (1/2, 1/2, 0)^T$, and therefore the solution is

$$y_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{e^{-2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{-2t}.$$

[Bonus] (2 points) Find the solution y_1 of Question 3.

Solution: The solution associated to the eigenvalue $\lambda_1 = 2$ is given by

$$y_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{2t}.$$