Quiz 1 solution

1. Consider the differential equation
$$(\sec x)\frac{dy}{dx} = e^{y+\sin x}.$$

   (a) Find the general solution for $y(x)$. (8 points)

   (b) Find the particular solution that satisfies $y(0) = 0$. (2 points)

   Solution:

   (a) Separating variables
$$\begin{aligned}
&(\sec x)\frac{dy}{dx} = e^{y+\sin x} \\
\iff &\frac{1}{\cos x}\frac{dy}{dx} = e^{y}e^{\sin x} \\
\iff &\frac{1}{e^{y}}\,dy = e^{\sin x}\cos x\,dx \\
\iff &\int e^{-y}\,dy = \int e^{\sin x}\cos x\,dx \\
\iff &-e^{-y} = e^{\sin x} + C \quad \text{(substitution } u = \sin x\text{)} \\
\iff &e^{-y} = -e^{\sin x} - C \\
\iff &y(x) = -\ln(-e^{\sin x} - C)
\end{aligned}$$

   (b) If $y(0) = 0$, then $-\ln(-e^{\sin 0} - C) = 0$, so $-\ln(-1 - C) = 0 \Rightarrow -1 - C = 1 \Rightarrow C = -2$. Therefore the solution is given by
$$y(x) = -\ln(2 - e^{\sin x})$$

2. Solve the initial value problem given by (10 points)
$$\begin{cases} x\dfrac{dy}{dx} + 2y = 1 - \dfrac{1}{x}, \quad x > 0 \\ y(1) = \dfrac{1}{2}. \end{cases}$$

   Solution: Dividing by x we have
$$\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x} - \frac{1}{x^2}.$$

   The integrating factor is
$$\mu(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = e^{\ln x^2} = x^2.$$

   The solution is given by
$$y(x) = \frac{1}{x^2}\int x^2\left(\frac{1}{x} - \frac{1}{x^2}\right)dx = \frac{1}{x^2}\int (x-1)\,dx = \frac{1}{x^2}\left(\frac{x^2}{2} - x + C\right) = \frac{C}{x^2} - \frac{1}{x} + \frac{1}{2}.$$

   The initial condition $y(1) = \frac{1}{2}$ implies $\frac{1}{2} = C - 1 + \frac{1}{2}$, then $C = 1$. Therefore the solution is
$$y(x) = \frac{1}{x^2} - \frac{1}{x} + \frac{1}{2}.$$