Instructions: *Print* your name, student ID number and recitation session in the spaces below.

Name: ____

Student ID:____________________________________

Recitation session:______________________________________

Exam 1, Calculus III (Math 2551)

09/24/2015 (Thursday)

Show your work clearly and completely!

No calculators are allowed.

You can bring a formula sheet of a one-side letter size paper.

Question	Points
1)	
2)	
3)	

Problem 1(30 points). Calculations.

(a) (5 pt)

$$\frac{d}{dt}[(e^t\mathbf{i} + t\mathbf{k}) \times (t\mathbf{j} + e^{-t}\mathbf{k})].$$

(b) (5 pt)

$$\frac{d}{dt}[(t^2\mathbf{i} - 2t\mathbf{j}) \cdot (t\mathbf{i} + t^3\mathbf{j})].$$

Solution:

(a)

$$\begin{aligned}\frac{d}{dt}[(e^t\mathbf{i} + t\mathbf{j}) \times (t\mathbf{i} + e^{-t}\mathbf{j})] &= \frac{d}{dt}\left(-t^2\mathbf{i} - \mathbf{j} + te^t\ \mathbf{k}\right) \\ &= -2t\mathbf{i} + (t+1)\,e^t\ \mathbf{k}\end{aligned}$$

(b)

$$\begin{aligned}\frac{d}{dt}[(t^2\mathbf{i} - 2t\mathbf{j}) \cdot (t\mathbf{i} + t^3\mathbf{j})] &= \frac{d}{dt}\left(t^3 - 2t^4\right) \\ &= 3t^2 - 8t^3.\end{aligned}$$

(c)(10 pt) Let $u(r, \theta, t) = \ln(x/y) - ye^{xz}$, calculate u_x, u_y and u_z.

(d)(10 pt) Set

$$f(x,y) = \begin{cases} \frac{x^3-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} .$$

Determine whether or not f has a limit at $(0,0)$.

Solution:

(c)

$$u_x = \frac{1}{x} - yze^{xz}, \; u_y = -\frac{1}{y} - e^{xz}, \; u_z = -xye^{xz}.$$

(d) When (x,y) approaches $(0,0)$ from x direction, that is, $y = 0$, we have

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{x^3-y^2}{x^2+y^2} &= \lim_{x\to 0} \frac{x^3}{x^2} = \lim_{x\to 0} x \\ &= 0. \end{aligned}$$

When (x,y) approaches $(0,0)$ from y direction, that is, $x = 0$, we have

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{x^3-y^2}{x^2+y^2} &= \lim_{y\to 0} \frac{-y^2}{y^2} \\ &= -1. \end{aligned}$$

The two limits are different, so $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Problem 2(32 pt) An object moves so that

$$\mathbf{r}(t) = \ln t\ \mathbf{i} + 2t\ \mathbf{j} + t^2\mathbf{k},\ \ t > 0.$$

(a)(6 pt) Compute the velocity, the acceleration and the speed of the object at an arbitrary time $t > 0$.

(b) (5 pt) Set up a definite integral equal to the length of the arc of the trajectory from $t = 1$ to $t = 4$.

(c) (5 points) Evaluate the integral in (b).

Solution:

(a) The velocity

$$\mathbf{v}\,(t) = \mathbf{r}'(t) = \frac{1}{t}\ \mathbf{i} + 2\ \mathbf{j} + 2t\mathbf{k},$$

the acceleration

$$\mathbf{a}\,(t) = \mathbf{v}'(t) = -\frac{1}{t^2}\ \mathbf{i} + 2\mathbf{k},$$

and the speed

$$v\,(t) = \|\mathbf{v}\,(t)\| = \sqrt{\frac{1}{t^2} + 4 + 4t^2} = \sqrt{\left(\frac{1}{t} + 2t\right)^2} = \frac{1}{t} + 2t.$$

(b) The arc length is

$$\int_1^4 \|\mathbf{v}\,(t)\|\,dt = \int_1^4 \left(\frac{1}{t} + 2t\right) dt.$$

(c)

$$\int_1^4 \left(\frac{1}{t} + 2t\right) dt = \left(\ln t + t^2\right)|_1^4 = \ln 4 + 15.$$

(d) (4 pt) Find the time $t_1 > 0$ and the coordinates of the point P where the object hits the yz plane.

(e) (6 pt) Find the equation of the line tangent to the trajectory at P.

(f) (6 pt) Find the curvature of the trajectory at P.

Solution:

(d) When it hits the yz plane, the x coordinate is zero. So $\ln t_1 = 0$, and $t_1 = 1$.

(e) The tangent vector is $\mathbf{v}(1) = \ \mathbf{i} + 2\ \mathbf{j} + 2\mathbf{k}$, and $\mathbf{r}(1) = \ 2\ \mathbf{j} + \mathbf{k}$. So the tangent line equation is

$$x = t,\ y = 2 + 2t,\ z = 1 + 2t.$$

(f) First, the unit tangent is

$$\begin{aligned} \mathbf{T}(t) &= \frac{\mathbf{v}(t)}{v(t)} = \frac{1}{\frac{1}{t} + 2t}\left(\frac{1}{t}\ \mathbf{i} + 2\ \mathbf{j} + 2t\mathbf{k}\right) \\ &= \frac{1}{1+2t^2}\mathbf{i} + \frac{2t}{1+2t^2}\mathbf{j} + \frac{2t^2}{1+2t^2}\mathbf{k}. \end{aligned}$$

So

$$T'(t) = -\frac{4t}{(1+2t^2)^2}\mathbf{i} + \frac{2-4t^2}{(1+2t^2)^2}\mathbf{j} + \frac{4t}{(1+2t^2)^2}\mathbf{k},$$

and the curvature at $t = 1$ is

$$\begin{aligned} \kappa &= \frac{\|T'(1)\|}{v(1)} = \frac{\left\|-\frac{4}{9}\mathbf{i} - \frac{2}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}\right\|}{3} \\ &= \frac{2\sqrt{21}}{27}. \end{aligned}$$

Problem 3(38 pt) At each point $P(x(t), y(t), z(t))$ of its motion, an object of mass m is subject to a force:

$$\mathbf{F}(t) = m(t\ \mathbf{i} + t^2\ \mathbf{j}).$$

Given that $\mathbf{v}(0) = \mathbf{k}$, and $\mathbf{r}(0) = \mathbf{i}$. Find the following:

(a) (8 pt) The velocity $\mathbf{v}(t)$.

(b) (4 pt) The speed $v(1)$.

Solution:

(a) The acceleration is

$$\mathbf{a}(t) = \frac{1}{m}\mathbf{F}(t) = t\ \mathbf{i} + t^2\ \mathbf{j}.$$

So

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{v}(0) + \int_0^t \mathbf{a}(s)\, ds = \mathbf{k} + \int_0^t \left(s\ \mathbf{i} + s^2\ \mathbf{j}\right) ds \\ &= \mathbf{k} + \left(\frac{1}{2}s^2\ \mathbf{i} + \frac{1}{3}s^3\ \mathbf{j}\right)\Big|_0^t = \frac{1}{2}t^2\ \mathbf{i} + \frac{1}{3}t^3\ \mathbf{j} + \mathbf{k}. \end{aligned}$$

(b) The speed

$$v(1) = \|\mathbf{v}(1)\| = \left\|\frac{1}{2}\ \mathbf{i} + \frac{1}{3}\ \mathbf{j} + \mathbf{k}\right\| = \sqrt{\frac{49}{36}} = \frac{7}{6}.$$

(c) (8 pt) The position function $\mathbf{r}(t)$.

(d) (9 pt) The tangential and normal components of the acceleration $\mathbf{a}(1)$.

(e) (9 pt) The osculating plane at $\mathbf{r}(1)$.

Solution:

(c) The position

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}(0)+\int_0^t \mathbf{v}(s)ds = \mathbf{i}+\int_0^t\left(\frac{1}{2}s^2\ \mathbf{i}+\frac{1}{3}s^3\ \mathbf{j}+\mathbf{k}\right)ds \\ &= \mathbf{i}+\left(\frac{1}{6}s^3\ \mathbf{i}+\frac{1}{12}s^4\ \mathbf{j}+s\mathbf{k}\right)\Big|_0^t \\ &= \left(1+\frac{1}{6}t^3\right)\mathbf{i}+\frac{1}{12}t^4\ \mathbf{j}+t\mathbf{k}.\end{aligned}$$

(d) The tangential component is

$$\begin{aligned}\mathbf{a}_T(1) &= \frac{\mathbf{a}(1)\cdot\mathbf{v}(1)}{v(1)} = \frac{(\mathbf{i}+\ \mathbf{j})\cdot\left(\frac{1}{2}\ \mathbf{i}+\frac{1}{3}\ \mathbf{j}+\mathbf{k}\right)}{\frac{7}{6}} \\ &= \frac{5}{7}.\end{aligned}$$

So the normal component is

$$\begin{aligned}\mathbf{a}_N(1) &= \sqrt{\|\mathbf{a}(1)\|^2-(\mathbf{a}_T(1))^2} = \sqrt{2-\left(\frac{5}{7}\right)^2} \\ &= \frac{2\sqrt{6}}{7}.\end{aligned}$$

(e) The normal vector of the osculating plane can be chosen to be

$$\begin{aligned}\mathbf{a}(1)\times\mathbf{v}(1) &= (\mathbf{i}+\ \mathbf{j})\times\left(\frac{1}{2}\ \mathbf{i}+\frac{1}{3}\ \mathbf{j}+\mathbf{k}\right) \\ &= \mathbf{i}-\ \mathbf{j}-\frac{1}{6}\mathbf{k}.\end{aligned}$$

Since $\mathbf{r}(1)=\frac{7}{6}\mathbf{i}+\frac{1}{12}\ \mathbf{j}+\mathbf{k}$, so the osculating plane at $\mathbf{r}(1)$ is

$$x-\frac{7}{6}-\left(y-\frac{1}{12}\right)-\frac{1}{6}(z-1)=0,$$

or

$$x-y-\frac{1}{6}z-\frac{11}{12}=0.$$