Math 2401
Spring 2015
Exam 4
April 9, 2015
Time Limit: 50 Minutes

Name : ____________________

GT Id : ____________________

TA: ____________________

This exam contains 8 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. Also, sign the Honor Code pledge at the bottom of this page, and follow the instructions below.

- On this exam you may **not** use your books, notes, or any electronic devices other than a non-graphing calculator.
- **Show all your work.** A correct answer not supported by calculations and/or explanation will receive no credit. An incorrect answer supported by substantially correct calculations and explanation may receive partial credit.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- If you need more space, use the back of the pages; clearly indicate when you have done so.

Problem	Points	Score
1	15	
2	5	
3	16	
4	14	
Total:	50	

Honor Code Pledge: By signing below, you are verifying that you understand and uphold the Georgia Tech honor code.
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1. Evaluate the following integrals.

(a) (7 points) $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} (2+3\sqrt{x^2+y^2})\,dx\,dy$

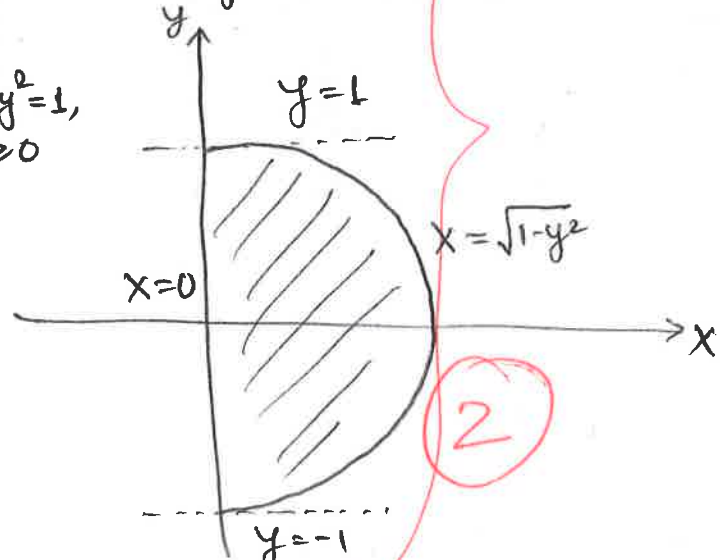
The region of integration (R) is bounded by

$x=0$

$x=\sqrt{1-y^2} \Rightarrow x^2 = 1-y^2,\ x\geq 0 \Rightarrow x^2+y^2=1,\ x\geq 0$

$y=-1$

& $y=1$.



2

In polar coordinates R is given by:

$0\leq r\leq 1,\quad -\pi/2 \leq \theta \leq \pi/2$

$\therefore$ The given integral can be written as

1 1

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{1} (2+3r)\,r\,dr\,d\theta$$

1

$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{1} (2r+3r^2)\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} (r^2+r^3)\Big]_{r=0}^{r=1}\,d\theta$$

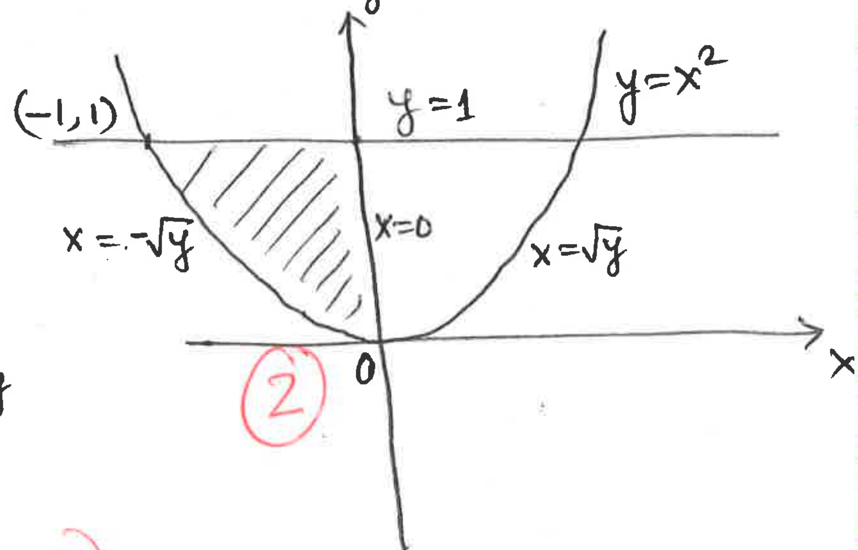
$$= \int_{-\pi/2}^{\pi/2} 2\,d\theta = 2\left(\pi/2+\pi/2\right) = 2\pi.$$

2

(b) (8 points) $\displaystyle\int_{-1}^{0}\int_{x^2}^{1}\int_{0}^{2x} \pi\cos\left(\frac{\pi y^2}{2}\right) dz\,dy\,dx.$

$$= \int_{-1}^{0}\int_{x^2}^{1} \pi\cos\left(\frac{\pi y^2}{2}\right)\cdot 2x\, dy\,dx$$ 1

Changing the order of integration, the above integral is equal to 1 2



$$\int_{0}^{1}\int_{-\sqrt{y}}^{0} 2\pi x\cos\left(\frac{\pi y^2}{2}\right) dx\,dy$$

$$= \int_{0}^{1} \pi\cos\left(\frac{\pi y^2}{2}\right) x^2\Big]_{x=-\sqrt{y}}^{0} dy$$

$$= \int_{0}^{1} \pi y\cos\left(\frac{\pi y^2}{2}\right) dy$$

$$= \sin\left(\frac{\pi y^2}{2}\right)\Big]_{0}^{1}$$

$$= \sin\pi/2 - \sin 0$$

$$= 1 - 0$$

$$= 1.$$

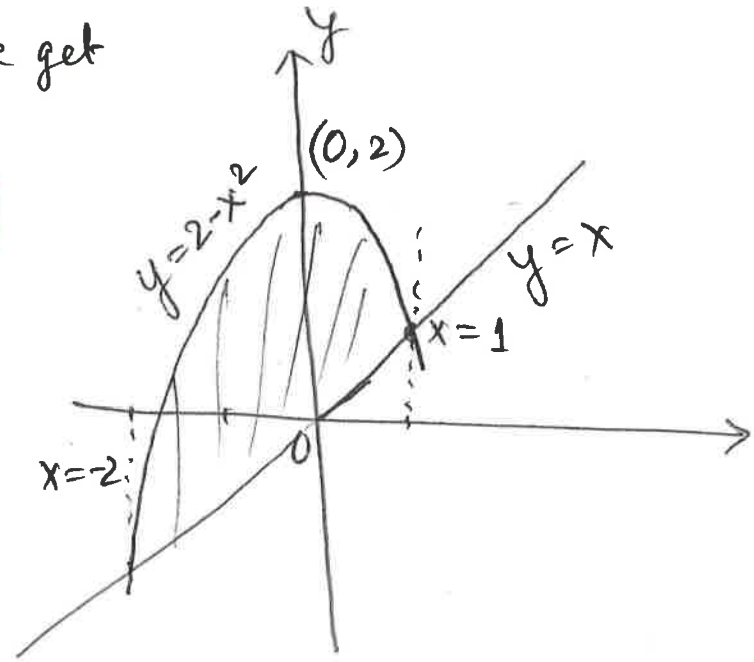
2

2. (5 points) Set up an iterated integral for the area of the region R in the xy-plane that is bounded by the parabola $y = 2 - x^2$ and the line $y = x$. (Do not evaluate.)

Solving $y=2-x^2$ & $y=x$, we get

$$x = 2-x^2$$
or $x^2+x-2=0$
or $(x+2)(x-1)=0$
i.e. $x=-2, 1$

2



$\therefore$ Area of $R = \iint_R dA$

$$= \int_{-2}^{1}\int_{x}^{2-x^2} dy\, dx$$

1

1 1

3. Let D be the region bounded below by the cone $z = \sqrt{x^2+y^2}$, and above by the sphere $x^2+y^2+(z-1)^2 = 1$.

(a) (2 points) Find cylindrical coordinate equations for the cone and the sphere.

1 $\quad z = \sqrt{x^2+y^2} \implies z = r$

1 $\quad x^2+y^2+(z-1)^2 = 1 \implies r^2 + (z-1)^2 = 1.$

(b) (6 points) Set up an iterated triple integral in **cylindrical coordinates** that gives the volume of D. (Do not evaluate.)

Solving $z = \sqrt{x^2+y^2}$ & $x^2+y^2+(z-1)^2 = 1$

we get $\quad z^2 + (z-1)^2 = 1$

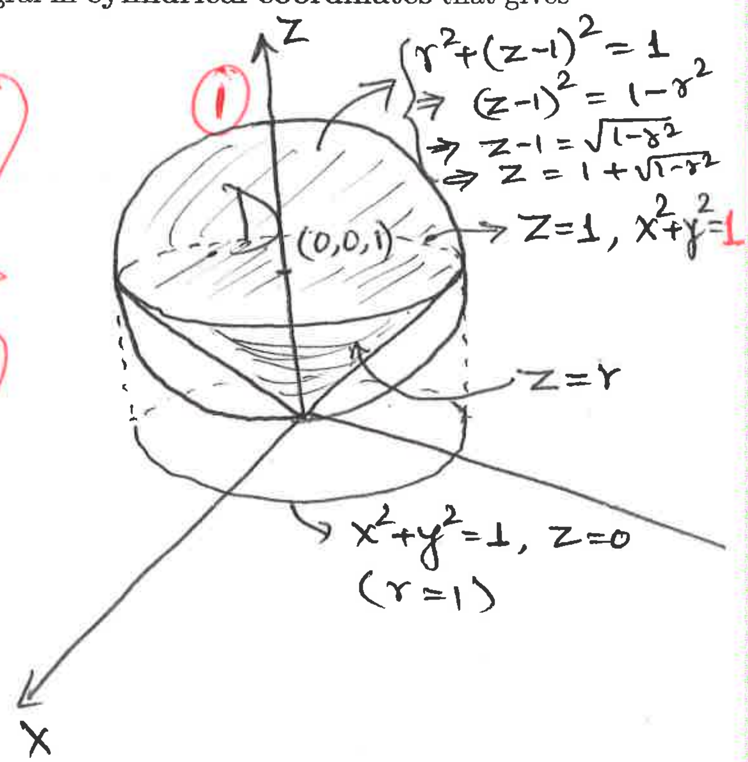
or $\quad z^2 + z^2 - 2z + 1 = 1$

or $\quad \cancel{2z^2 - 2z}\ 2z^2 - 2z = 0$

or, $\quad 2z(z-1) = 0$

i.e. $\quad z = 0,\ z = 1.$

2



Volume of D

$$= \underbrace{\int_0^{2\pi}\int_0^1}_{1} \int_{r}^{1+\sqrt{1-r^2}} \underbrace{dz\ r\,dr\,d\theta}_{1}$$

1

(c) (4 points) Find spherical coordinate equations for the cone and the sphere.

2 $$z = \sqrt{x^2+y^2} \Rightarrow z = r \Rightarrow \rho\cos\phi = \rho\sin\phi$$
$$\Rightarrow \cos\phi = \sin\phi \Rightarrow \phi = \frac{\pi}{4}$$

2 $$x^2+y^2+(z-1)^2 = 1 \Rightarrow x^2+y^2+z^2-2z+1 = 1$$
$$\Rightarrow \rho^2 - 2\rho\cos\phi = 0$$
$$\Rightarrow \rho = 2\cos\phi$$

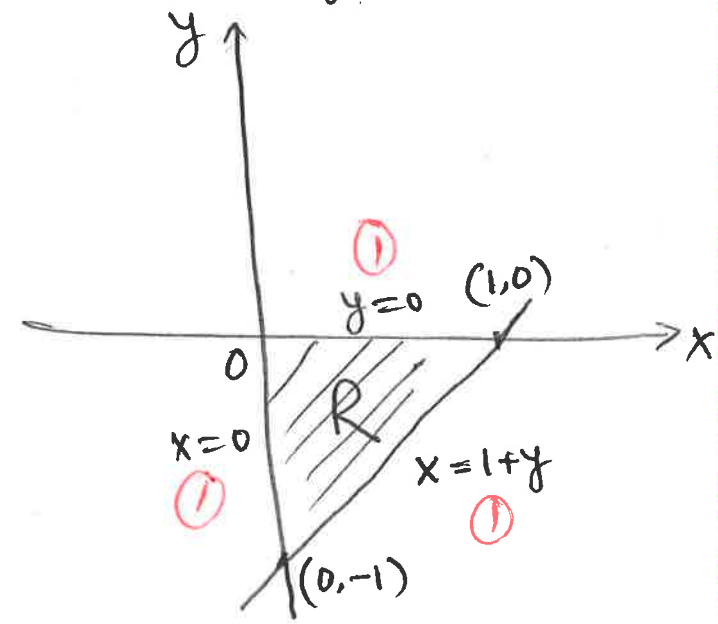
(d) (4 points) Set up an iterated triple integral in **spherical coordinates** that gives the volume of D. (Do not evaluate.)

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi} \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{1}$$

1 1 1

4. (a) (3 points) Sketch the region of integration (R) for $\int_{-1}^{0}\int_{0}^{1+y} \frac{6(x+2y)}{1+(x-y)^3}\,dx\,dy$.

The region of integration is bounded by $x=0$, $x=1+y$
$y=-1$ & $y=0$

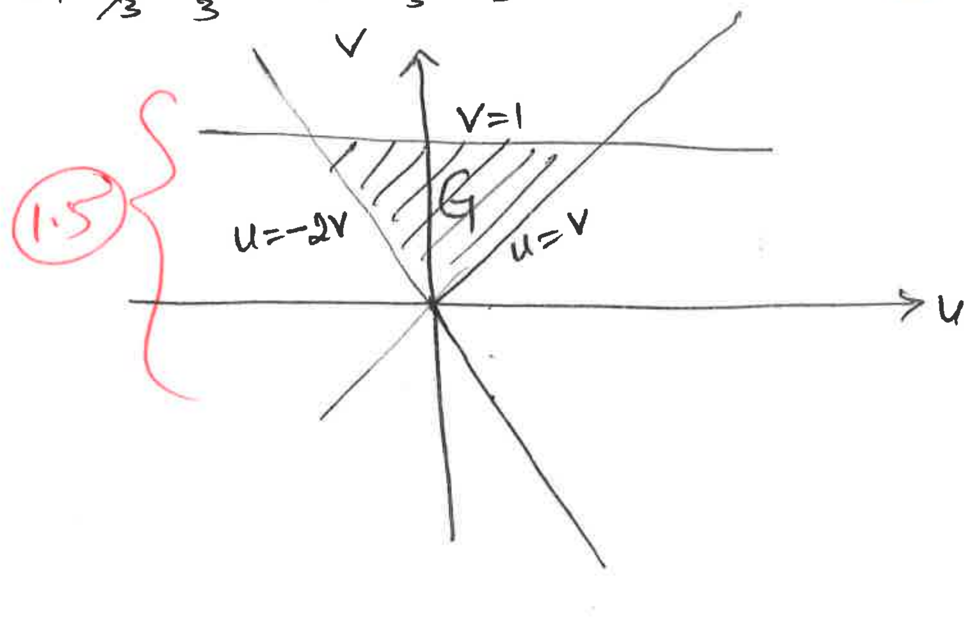


(b) (3 points) Find and sketch the region G in the $uv-$ plane that maps to R under the transformation $x = \frac{u}{3} + \frac{2v}{3}$, $y = \frac{u}{3} - \frac{v}{3}$.

$x=0 \Rightarrow \frac{u}{3} + \frac{2v}{3} = 0 \Rightarrow \frac{u}{3} = -\frac{2v}{3} \Rightarrow u = -2v$ → 0.5

$y=0 \Rightarrow \frac{u}{3} - \frac{v}{3} = 0 \Rightarrow \frac{u}{3} = \frac{v}{3} \Rightarrow u = v$ → 0.5

$x=1+y \Rightarrow \cancel{\frac{u}{3}} + \frac{2v}{3} = 1 + \cancel{\frac{u}{3}} - \frac{v}{3} \Rightarrow \frac{2v}{3} + \frac{v}{3} = 1 \Rightarrow v = 1$ → 0.5



(c) (2 points) Find the Jacobian of the above transformation.

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \text{1}$$

$$= \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix}$$

$$= -\frac{1}{9} - \frac{2}{9} = -\frac{3}{9} = -\frac{1}{3}. \quad \text{1}$$

(d) (6 points) Express the integral in part (*a*) as an integral over G and evaluate.

$$\iint_G \frac{6u}{1+v^3} |J(u,v)|\, du\, dv$$

$$x + 2y = \frac{u}{3} + \cancel{\frac{2v}{3}} + \frac{2u}{3} - \cancel{\frac{2v}{3}} = u$$

$$x - y = \frac{u}{3} + \frac{2v}{3} - \frac{u}{3} + \frac{v}{3} = v$$

1 1

$$= \int_0^1 \int_{-2v}^{v} \underbrace{\frac{6u}{1+v^3}}_{1} \underbrace{\left(\frac{1}{3}\right) du\, dv}_{1}$$

2

$$= \int_0^1 \int_{-2v}^{v} \frac{2u}{1+v^3}\, du\, dv = \int_0^1 \left. \frac{u^2}{1+v^3} \right]_{u=-2v}^{u=v} dv$$

$$= \int_0^1 \frac{1}{1+v^3}(v^2 - 4v^2)\, dv = \int_0^1 -\frac{3v^2}{1+v^3}\, dv$$

$$= -\ln(1+v^3)\Big]_0^1$$

$$= -\ln 2 + \ln 1 = -\ln 2$$