Quiz 5 **-** Name ____ Math 2403 - Section L ___

Good Luck!

This quiz has a back side! Don't forget about Question 3 and Bonus Question!

1. (5 points) Solve the following initial value problem $\begin{cases} y'' + 2y' + 5y = 0 \\ y(0) = 1;\ y'(0) = 1 \end{cases}$

   Solution: The associated characteristic polynomial is $p(r) = r^2 + 2r + 5$. Its roots are

   $$r_1 = -1 + 2i \quad \text{and} \quad r_2 = -1 - 2i.$$

   Therefore the fundamental set of solutions is given by

   $$\{e^{-t}\cos 2t, e^{-t}\sin 2t\}.$$

   The general solution of the problem is

   $$y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t.$$

   Setting the initial conditions we find $c_1 = 1$ and $c_2 = 2$, thus the solution of the IVP is

   $$y = e^{-t}\cos 2t + 2e^{-t}\sin 2t$$

2. (5 points) Find the general solution of the following differential equation: $y'' + 2y' + 5y = 3\sin 2t$

   Solution: The general solution of the problem is given by

   $$y = y_p + c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$$

   Using the method of undetermined coefficients, we guess a particular solution of the form

   $$y_p = A\sin 2t + B\cos 2t.$$

   Differentiating we have $y_p' = 2A\cos 2t - 2B\sin 2t$ and $4y_p'' = -4A\sin 2t - 4B\cos 2t$.
   Substituting into the equation we have:

   $$(A - 4B)\sin 2t + (4A + B)\cos 2t = 3\sin 2t$$

   The general solution of the problem is

   $$y = \frac{3}{17}\sin 2t - \frac{12}{17}\cos 2t + c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$$

3. (5 points) Given the equation $y'' + 4xy' + (4x^2 + 2)y = 8e^{-x(x+2)}$ and one of its solutions $y_1 = e^{-x^2}$,

(a) Given the solutions $y_1 = e^{-x^2}$ and $y_2 = xe^{-x^2}$, show that they form a fundamental set of solutions for the complementary equation.

(b) Given a solution of the form $y = ue^{-x^2}$, write the differential equation relative to the function u.

Solution:

(a) In order to show that $\{y_1, y_2\}$ is the fundamental set of solutions, we have to prove that y_1 and y_2 are two solutions of the problem and they are linearly independent. Plugging y_1 and y_2 into the equation we can easily verify that they are solutions. In order to prove that they are linearly independent we can either compute the Wronskian or observe that $\frac{y_2}{y_1}$ is not a constant.

(b) If $y = ue^{-x^2}$ then

$$y' = u'e^{-x^2} - 2xue^{-x^2} \quad \text{and} \quad y'' = u''e^{-x^2} - 4xu'e^{-x^2} - 2ue^{-x^2} + 4x^2ue^{-x^2}.$$

Therefore the equation becomes:

$$u''e^{-x^2} = 8e^{-x(x+2)}.$$

[Bonus] (2 points) Find the general solution of the equation $y'' + 4xy' + (4x^2 + 2)y = 8e^{-x(x+2)}$.

Solution: In order to find the general solution it is enough to solve the differential equation $u''e^{-x^2} = 8e^{-x(x+2)}$.
Multiplying by e^{x^2} we have

$$u'' = 8e^{-2x}$$

Therefore

$$u' - 4e^{-2x} + c_1$$

and

$$u = 2e^{-2x} + c_1 x + c_2$$

Therefore

$$y = ue^{-x^2} = e^{-x^2}(2e^{-2x} + c_1 x + c_2)$$

[Bonus+] (1 point) What is the Italian for recipe?

Solution: Ricetta