ISyE 3833 - HW 4 Due: Apr 6 / 7

1. Based on Winston Chapter 3, Review Question 7) Steelco manufactures two types of steel at three different steel mills. During a given month, each steel mill has 200 hours of blast furnace time available. Because of differences in the furnaces at each mill, the time and cost to produce a ton of steel differs for each mill. The time and cost for each mill are shown in the table below.

	Steel 1		Steel 2	
Mill	Cost ($ per ton)	Time (minutes per ton)	Cost ($ per ton)	Time (minutes per ton)
1	13	20	11	22
2	12	24	9	18
3	13	28	10	30

Each month, Steelco must manufacture at least 500 tons of Steel 1 and 600 tons of Steel 2.

An LP that minimized the cost of manufacturing the desired steel is shown below, where x_{ij} represents the tons of steel type i produced at mill j each month, for $i = 1, 2$ and $j = 1, 2, 3$:

$$
\begin{array}{llllllllllllll}
\min & 13x_{11} & + & 11x_{21} & + & 12x_{12} & + & 9x_{22} & + & 13x_{13} & + & 10x_{23} & & \\
\text{s.t.} & x_{11} & & & + & x_{12} & & & + & x_{13} & & & \geq & 500 \\
 & & & x_{21} & & & + & x_{22} & & & + & x_{23} & \geq & 600 \\
 & 20x_{11} & + & 22x_{21} & & & & & & & & & \leq & 12000 \\
 & & & & & 24x_{12} & + & 18x_{22} & & & & & \leq & 12000 \\
 & & & & & & & & & 28x_{13} & + & 30x_{23} & \leq & 12000 \\
 & & & & & & & \multicolumn{5}{r}{x_{11}, x_{21}, x_{21}, x_{22}, x_{13}, x_{23}} & \geq & 0.
\end{array}
$$

After converting the LP to standard maximization form, solution by the two-phase simplex algorithm, the following optimal dictionary is obtained:

$$
\begin{array}{rrlllllllllll}
x_{11} = & 450 & - & \frac{3}{4}x_{21} & - & x_{13} & - & \frac{3}{4}x_{23} & + & s_1 & + & \frac{3}{4}s_2 & + \ \frac{1}{24}s_4 \\
x_{12} = & 50 & + & \frac{3}{4}x_{21} & & & + & \frac{3}{4}x_{23} & & & - & \frac{3}{4}s_2 & - \ \frac{1}{24}s_4 \\
x_{22} = & 600 & - & x_{21} & & & - & x_{23} & & & + & s_2 & \\
s_3 = & 3000 & - & 7x_{21} & + & 20x_{13} & + & 15x_{23} & - & 20s_1 & - & 15s_2 & - \ \frac{5}{6}s_4 \\
s_5 = & 12000 & & & - & 28x_{13} & - & 30x_{23} & & & & & \\
z = & -11850 & - & \frac{5}{4}x_{21} & & & - & \frac{1}{4}x_{23} & - & 13s_1 & - & \frac{39}{4}s_2 & - \ \frac{1}{24}s_4,
\end{array}
$$

where s_1 and s_2 are the slack variables for the constraints requiring enough of Steel 1 and Steel 2 to be produced, respectively, and s_3, s_4 and s_5 are the slack variables in the constraints limiting the consumption of production time at each of the three mills, respectively. (Note that the definitions of s_1 and s_2 are

$$
\begin{aligned}
s_1 &= x_{11} + x_{12} + x_{13} - 500, \qquad \text{and} \\
s_2 &= x_{21} + x_{22} + x_{23} - 600.)
\end{aligned}
$$

Use this optimal dictionary to answer the following questions concerning sensitivity of the optimal solution and its value to changes in the data of the problem. In each case explain how you obtained your answer using the optimal dictionary.

(a) Suppose there is a proposal to change the number of hours per month that the blast furnace at Mill 2 can operate.

   i. For what *range* of the number hours available at Mill 2 does the optimal basis stay feasible? We call this the *allowable range*.

   ii. How will the optimal solution change as a function of change in the number of hours available at Mill 2, within the allowable range?

   iii. What is the shadow price of the fourth constraint, dictating the maximum number of minutes that can be used at Mill 2, and, for changes within the allowable range, how will the monthly cost of the operation change per unit decrease in the number of hours at Mill 2? Will it increase or decrease, and by what rate?

   iv. If the blast furnace at Mill 2 reduced to be available for only 190 hours per month, what would the new optimal solution be, and what would the new optimal monthly cost be?

(b) What is the sensitivity of the optimal basis to changes in the number of hours per month that Mill 1's blast furnace is available? How much can these increase or decrease without affecting feasibility of the optimal basis, and how will increases or decreases affect the optimal solution and optimal value of the problem?

(c) Consider changes to the number of tons of Steel 1 required.

   i. For what range of the tons of Steel 1 required does the optimal basis stay feasible? (What is the allowable range for tons of Steel 1 required?)

   ii. How will the optimal solution change as a function of change in the tons of Steel 1 required, within the allowable range?

   iii. What is the shadow price of the first constraint, dictating the minimum quantity of Steel 1 to be produced, and, for increases within the allowable range, how will the monthly cost of the operation change per unit increase in the tons of Steel 1 required? Will it increase or decrease, and by what rate?

   iv. If the quantity of Steel 1 required decreased to 350 tons, what would the new optimal solution be, and what would the new optimal monthly cost be?

   v. If the quantity of Steel 1 required increased to 580 tons, what would the new optimal solution be, and what would the new optimal monthly cost be?

(d) Suppose there is some uncertainty about the cost per ton of Steel 1 produced at Mill 2.

   i. By how much can this cost increase and by how much can it decrease from $12 without affecting the optimal basis?

   ii. If the cost increased by 50 cents per ton, what would the optimal solution be, and what would it cost?

   iii. Can you deduce directly from the optimal dictionary what the optimal cost would be if the cost per ton of Steel 1 produced at Mill 2 decreased by 50 cents?

   [Hint: be careful about the fact that the original problem is cost minimization, while z is expressed as maximization.]

(e) Suppose there is some uncertainty about the cost per ton of Steel 2 produced at Mill 1. By how much can this cost increase and by how much can it decrease from $11 without affecting the optimal basis?

2. Consider the problem faced by a small brewery, currently making a pale ale and an American beer, that require different proportions of scarce resources: corn, hops and malt. The quantities of these resources needed, per barrel of beer produced, is shown in the table below.

Type of beer	Quantity per barrel beer produced			
	Corn (lbs)	Hops (oz)	Malt (lbs)	Profit ($)
Pale ale	5	4	35	13
American beer	15	4	20	23
Available	480	160	1190	

The table also shows the profit made per barrel, and the quantity of each resource available in the coming production period. The brewery has formulated the following Linear Program to decide its production plan for the coming period, so as to maximize its total profit, where x_1 represents the number of barrels of pale ale to make, and x_2 represents the number of barrels of American beer:

$$
\begin{aligned}
\max z = \quad & 13x_1 + 23x_2 & \\
\text{s.t.} \quad & 5x_1 + 15x_2 \le 480 & (1)\\
& 4x_1 + \ 4x_2 \le 160 & (2)\\
& 35x_1 + 20x_2 \le 1190 & (3)\\
& x_1, x_2 \ge 0. &
\end{aligned}
$$

After the addition of slack variables, s_1, s_2 and s_3 for each of the three resource constraints, respectively, and application of the simplex method, the brewery determines the following optimal dictionary for the LP to be:

$$
\begin{array}{rcrcrcr}
x_1 & = & 12 & + & \frac{1}{10}s_1 & - & \frac{3}{8}s_2 \\
x_2 & = & 28 & - & \frac{1}{10}s_1 & + & \frac{1}{8}s_2 \\
s_3 & = & 210 & - & \frac{3}{2}s_1 & + & \frac{85}{8}s_2 \\
z & = & 800 & - & s_1 & - & 2s_2.
\end{array}
$$

(a) What is the basis matrix of the optimal solution, A_B?

(b) What is the constraint coefficient matrix corresponding to the nonbasic variables, A_N?

(c) What is A_B^{-1}?

(d) What is $A_B^{-1}b$, where b denotes the vector of available resource quantities?

(e) What is c_B, where c_B denotes the vector of objective coefficients of basic variables? Now calculate $c_B^T A_B^{-1}$.

(f) Suppose the brewery is thinking about introducing a new beer: a stout. One barrel of stout requires 6 oz of hops and 40 lbs of malt, and has a profit of $16 per barrel. Stout does not use corn at all. Should the brewery make stout? Justify your answer.

Note: your recitation class instructor may have used the notation B instead of A_B and N instead of A_N.

3. Write the linear programming dual of the following problems:

(a)

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ s.t. \quad & x_1 + x_2 \leq 2 \\ & 4x_1 + 2x_2 \leq 6 \\ & 2x_1 + 3x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(b)

$$\begin{aligned} \min \quad & 2x_1 + x_2 + 4x_3 \\ s.t. \quad & x_1 - x_2 + x_3 = 2 \\ & 2x_1 + 3x_2 + 5x_3 \leq 12 \\ & -4x_1 + 2x_2 + 3x_3 \geq 1 \\ & x_1 \leq 0, x_2 \geq 0 \end{aligned}$$