Feb. 27, 2014

Math 2401 M - Exam 2

First Name (Print): ______________ Last Name (Print): ___________ Signature: ________________

Please choose your section: ☐ M1 ☐ M2 ☐ M3

- There are **5** questions on **6** pages. The exam is worth 50 points in total.
- Answer the questions clearly and completely. You must provide work clearly justifying your solution.
- You can NOT write your work on the back of the page. Use it for scratch work if needed.
- You have 50 minutes to finish your work.

1. (3+4 points) Find the limit **OR** show the nonexistence of the limit of the following functions.

(a)

$$\lim_{(x,y)\to(-4,2),x\neq-4,y\neq y^2} \frac{x+4}{xy^2-xy+4y^2-4y}$$

Solution.

$$\begin{aligned}
&\lim_{(x,y)\to(-4,2),x\neq-4,y\neq y^2} \frac{x+4}{xy^2-xy+4y^2-4y}\\
&= \lim_{(x,y)\to(-4,2),x\neq-4,y\neq y^2} \frac{x+4}{(x+4)y(y-1)}\\
&= \lim_{(x,y)\to(-4,2),x\neq-4,y\neq y^2} \frac{1}{y(y-1)} = \frac{1}{2}.
\end{aligned}$$

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(b)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4-y^2}$$

Solution.

$$\lim_{(x,y)\to(0,0),y=kx^2,k^2\neq1} \frac{x^2y}{x^4-y^2} = \lim_{x\to0,y=kx^2,k^2\neq1} \frac{x^2kx^2}{x^4-(kx^2)^2} = \lim_{x\to0,y=kx^2,k^2\neq1} \frac{k}{1-k^2} = \frac{k}{1-k^2}$$

The limit changes with each value of k. By the two-path test, the limit does not exist.

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2. (6+2 points) Let

$$z = 4e^x \ln y, \quad x = \ln(u\cos v), \quad y = u\sin v.$$

(1) Express $\dfrac{\partial z}{\partial u}$ in terms of u and v by the Chain Rule;

(2) Evaluate $\dfrac{\partial z}{\partial u}$ at the point $(u, v) = (2, \frac{\pi}{4})$.

Solution.

(1)

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= 4e^x \ln y \frac{\cos v}{u\cos v} + 4e^x \frac{1}{y}\sin v \quad \text{(Substitute for } x \text{ and } y\text{, and simplify)} \\
&= 4(\cos v)\ln(u\sin v) + 4\cos v.
\end{aligned}$$

(2) $\dfrac{\partial z}{\partial u}(2, \dfrac{\pi}{4}) = \sqrt{2}\ln 2 + 2\sqrt{2}.$

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3. (6+4 points) Let $f(x, y, z) = \sin(xy) + e^{yz} + \ln(xz)$, and $P_0 = (1, 0, 1)$.

(a) Find directions in which $f(x, y, z)$ increases and decreases most rapidly at P_0, respectively.

Solution.

$$\begin{aligned} f_x(x, y, z) &= y\cos(xy) + \frac{1}{x} \quad \Longrightarrow \quad f_x(1, 0, 1) = 1, \\ f_y(x, y, z) &= x\cos(xy) + ze^{yz} \quad \Longrightarrow \quad f_y(1, 0, 1) = 2, \\ f_z(x, y, z) &= ye^{yz} + \frac{1}{z} \quad \Longrightarrow \quad f_z(1, 0, 1) = 1. \end{aligned}$$

Therefore, $\nabla f(1, 0, 1) = \vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} + \vec{\boldsymbol{k}}$.

$f(x, y, z)$ increases most rapidly in the direction of $\nabla f(1, 0, 1)$, which is $\frac{1}{\sqrt{6}}(\vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} + \vec{\boldsymbol{k}})$.

$f(x, y, z)$ decreases most rapidly in the direction of $-\nabla f(1, 0, 1)$, which is $-\frac{1}{\sqrt{6}}(\vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} + \vec{\boldsymbol{k}})$.

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(b) Find the derivative of $f(x, y, z)$ at P_0 in the direction of $\vec{\boldsymbol{v}} = 2\vec{\boldsymbol{i}} + \vec{\boldsymbol{j}} - 2\vec{\boldsymbol{k}}$.

Solution.

The direction of $\vec{\boldsymbol{v}}$ is $\vec{\boldsymbol{u}} = \frac{1}{3}(2\vec{\boldsymbol{i}} + \vec{\boldsymbol{j}} - 2\vec{\boldsymbol{k}})$, so

$$(D_{\vec{\boldsymbol{u}}})_{P_0} = \nabla f(1, 0, 1) \cdot \vec{\boldsymbol{u}} = (\vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} + \vec{\boldsymbol{k}}) \cdot \left[\frac{1}{3}(2\vec{\boldsymbol{i}} + \vec{\boldsymbol{j}} - 2\vec{\boldsymbol{k}})\right] = \frac{2}{3}.$$

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4. (15 points) Let

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

defined on the closed triangular region D in the first quadrant bounded by the lines $x = 0$, $y = 0$ and $x + 2y = 8$.

(a) Find the critical points of $f(x, y)$ in the interior of region D and classify them (local maxima, local minima or saddle points).

Solution.

Since $f(x, y)$ is differentiable, so the critical points can be founded by the first derivative test for local extreme values.

$$\begin{cases} f_x(x,y) = 2 - 2x, \\ f_y(x,y) = 2 - 2y. \end{cases} \Longrightarrow (x,y) = (1,1) \text{ is a critical point in interior of } D.$$

To classify the critical point $(1, 1)$, we calculate the second derivatives

$$f_{xx}(x,y) = -2 < 0, f_{xy}(x,y) = 0, f_{yy}(x,y) = -2.$$

The discriminant of $f(x, y)$ at $(1, 1)$ is

$$f_{xx}(1,1)f_{yy}(1,1) - (f_{xy}(1,1))^2 = 4 > 0,$$

so by the second derivative test for local extreme values, $f(x, y)$ has a local maximum value $f(1, 1) = 4$ at $(1, 1)$.

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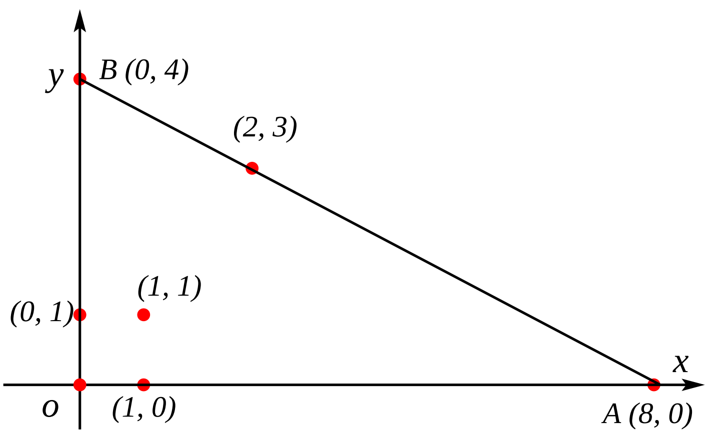
(b) Find the absolute maxima and minima of $f(x, y)$ on the region D.

Solution.



Interior points. We have found the critical points in the interior of D in part (a), and $f(x,y)$ has a local maximum value $f(1,1)=4$ at $(1,1)$.

Boundary points. $\partial D = OA \cup OB \cup AB$.

(i) On the segment OA, $y=0$, so

$$f(x,y)=f(x,0)=2+2x-x^2,\ \ 0\leq x\leq 8.$$

The extrema of $f(x,0)$ may occur at the interior point where $f'(x,0)=2-2x=0$, so $x=1$, and $f(1,0)=3$.

We also need to check the endpoints $x=0$ and $x=8$. Therefore, we have $f(0,0)=2$ and $f(8,0)=-46$.

(ii) On the segment OB, $x=0$, so

$$f(x,y)=f(0,y)=2+2y-y^2,\ \ 0\leq y\leq 4.$$

The extrema of $f(0,y)$ may occur at the interior point where $f'(0,y)=2-2y=0$, so $y=1$, and $f(0,1)=3$.

We also need to check the endpoints $y=0$ and $y=4$. Therefore, we have $f(0,0)=2$ and $f(0,4)=-6$.

(iii) On the segment AB, we have $y=4-\frac{x}{2}$, so

$$f(x,y)=f(x,4-\frac{x}{2})=2+2x+2(4-\frac{x}{2})-x^2-(4-\frac{x}{2})^2=-\frac{5}{4}x^2+5x-6,\ \ 0\leq x\leq 8.$$

Let $f'(x,4-\frac{x}{2})=-\frac{5}{2}x+5=0$, then $x=2$ and $y=4-\frac{2}{2}=3$, and $f(2,3)=-1$.

We have checked the endpoints $x=0$ and $x=8$ in (i) and (ii), and $f(0,4)=-6$ and $f(8,0)=-46$.

Look through the lists.

$f(1,1)=4$, $f(0,0)=2$, $f(1,0)=3$, $f(8,0)=-46$, $f(0,1)=3$, $f(0,4)=-6$, $f(2,3)=-1$.

Therefore $f(x,y)$ takes on absolute maximum value 4 at $(1,1)$, and absolute minimum value -46 at $(8,0)$.

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5. (10 points) Find the maximum and minimum distances from the point on the sphere $x^2 + y^2 + z^2 = \frac{3}{4}$ to the point $(1, -1, 1)$ by the **method of Lagrange multipliers**.

Solution.

We find the extreme values of

$$f(x, y, z) = (x-1)^2 + (y+1)^2 + (z-1)^2$$

$\big[$the square of the distance from (x, y, z) to the point $(1, -1, 1)\big]$ subject to the constraint

$$g(x, y, z) = x^2 + y^2 + z^2 - \frac{3}{4} = 0.$$

Solve the system

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z), && (0.1) \\ g(x, y, z) &= 0 && (0.2) \end{aligned}$$

for x, y, z, and λ. The gradient equation (0.1) gives

$$2(x-1)\vec{\boldsymbol{i}} + 2(y+1)\vec{\boldsymbol{j}} + 2(z-1)\vec{\boldsymbol{k}} = \lambda\big[2x\vec{\boldsymbol{i}} + 2y\vec{\boldsymbol{j}} + 2z\vec{\boldsymbol{k}}\big],$$

so

$$\begin{cases} 2(x-1) = 2\lambda x, \\ 2(y+1) = 2\lambda y, \\ 2(z-1) = 2\lambda z. \end{cases} \Longrightarrow \begin{cases} x = \frac{1}{1-\lambda}, \\ y = \frac{1}{\lambda-1}, \\ z = \frac{1}{1-\lambda}. \end{cases} . \tag{0.3}$$

Substitute (0.3) into (0.2), and we have

$$(\lambda - 1)^2 = 4 \Longrightarrow \lambda = -1 \text{ or } \lambda = 3.$$

If $\lambda = -1$, $(x, y, z) = (\dfrac{1}{2}, -\dfrac{1}{2}, \dfrac{1}{2})$ and $f(\dfrac{1}{2}, -\dfrac{1}{2}, \dfrac{1}{2}) = \dfrac{3}{4}$.

If $\lambda = 3$, $(x, y, z) = (-\dfrac{1}{2}, \dfrac{1}{2}, -\dfrac{1}{2})$ and $f(-\dfrac{1}{2}, \dfrac{1}{2}, -\dfrac{1}{2}) = \dfrac{27}{4}$.

Therefore, the minimum distance is $\frac{\sqrt{3}}{2}$, and the maximum distance is $\frac{3\sqrt{3}}{2}$.

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