NAME →

ISyE 3044 — Fall 2012 — Test #2 Solutions

(Revised 12/11/12)

1. Short-answer Arena questions.

   (a) TRUE or FALSE? In Arena, it is possible to schedule 10 customers to show up *at the same time*.

   Solution: True. □

   (b) Which Arena template contains a `RELEASE` block (i.e., not part of a `PROCESS` block)?

   Solution: Advanced Process. (The Blocks template would also have been acceptable.) □

   (c) TRUE or FALSE? You can use a single Arena `DECIDE` block to probabilistically route customers to one of five possible destinations.

   Solution: True. □

   (d) TRUE or FALSE? In Arena, you can define a nonhomogeneous Poisson customer arrival process.

   Solution: True. □

   (e) TRUE or FALSE? The Arena manufacturing cell example that we studied in class used a set of sequences to move jobs around the cell.

   Solution: True. □

   (f) If the interarrival distribution for your Arena program is `NORM(2,2)`, what is the variance of an interarrival time?

   Solution: 4. □

2. Short-answer PRN questions.

(a) YES or NO? Is the linear congruential generator $X_{i+1} = (7X_i - 1)\text{mod}(8)$ full period?

Solution: No. For example, $X_0 = 1 \to X_1 = 6 \to X_2 = 1$, which shows that there's degeneration. $\Box$

(b) Again consider the generator $X_{i+1} = (7X_i - 1)\text{mod}(8)$. Using $X_0 = 1$, calculate the PRN U_{401}.

Solution: By the above problem, we see that $X_i = 6$ for all odd i. Thus, $U_{401} = X_{401}/m = 0.75$. $\Box$

(c) Consider the generator $X_{i+1} = 16807\, X_i \,\text{mod}(2^{31} - 1)$. If $X_0 = 543210$, find X_2.

Solution: Using a very precise calculator (that keeps enough integer digits in storage), or using the algorithm from class, we find that $X_1 = 539795882$ and then $X_2 = 1378463846$. $\Box$

(d) Consider three pseudo-random numbers R_1, R_2, R_3. Under the assumption of i.i.d. uniformity, what's the probability that $R_3 > R_1 > R_2 > 0.1$?

Solution: This is just a random permutation of the R_i's, each of which has been constrained to be > 0.1. Therefore, the desired probability is $(0.9)^3/3! = 0.1215$. $\Box$

(e) Consider the following 24 PRN's.

0.16	0.26	0.36	0.59	0.71	0.85	0.99	0.77
0.60	0.38	0.29	0.38	0.51	0.62	0.41	0.30
0.11	0.45	0.72	0.28	0.31	0.27	0.15	0.27

How many runs up and down do you get from this sequence?

Solution: Letting $+/-$ denote an up / down move, respectively, we have

$$\begin{array}{cccccccc}
 & + & + & + & + & + & + & - \\
- & - & - & + & + & + & - & - \\
- & + & + & - & + & - & - & +
\end{array}$$

This translates to $A = 9$ runs. $\square$

(f) Referring to Question 2e, do a runs up and down test on this sequence of PRN's to decide whether or not they're independent. Use $\alpha = 0.05$.

Solution: By class notes, we have

$$\mathsf{E}[A] = \frac{2n-1}{3} = 15.7 \quad \text{and} \quad \mathsf{Var}(A) = \frac{16n-29}{90} = 3.94.$$

Then by the previous answer, we have

$$Z_0 = \frac{A - \mathsf{E}[A]}{\sqrt{\mathsf{Var}(A)}} = \frac{9 - 15.7}{\sqrt{3.94}} = 3.38.$$

Since $|Z_0| > z_{0.025} = 1.96$, we reject the null hypothesis of independence; and we conclude that the PRN's are dependent. $\square$

(g) Referring to the data set from Question 2e, let's conduct a χ^2 goodness-of-fit test to test the hypothesis that the numbers are Unif(0,1). We'll use 3 equal-probability subintervals and level $\alpha = 0.10$. What's the value of the g-o-f statistic, χ_0^2?

Solution: The $k = 3$ intervals are $[0, 1/3]$, $(1/3, 2/3]$, and $(2/3, 1]$, for which $E_1 = E_2 = E_3 = 24/3 = 8$. We easily find $O_1 = 10$, $O_2 = 5$, and $O_3 = 9$. Thus, $\chi_0^2 = \sum_{i=1}^{k}(O_i - E_i)^2/E_i = 1.75$. $\square$

(h) Referring to the instructions from Question 2g, what's the appropriate χ^2 quantile value?

Solution: $\chi^2_{\alpha,k-1} = \chi^2_{0.10,2} = 4.61$. $\square$

(i) Again referring to the instructions from Question 2g, do we accept or reject the null hypothesis of uniformity?

Solution: Accept (er, well, fail to reject). $\square$

(j) What is 0 `XOR` 0?

Solution: 0. $\Box$

(k) Consider a Tausworthe generator with $r = 2$, $q = 3$, $B_1 = 1$, $B_2 = 0$, and $B_3 = 1$. Find B_{100}.

Solution: Using $B_i = (B_{i-r}+B_{i-q})\text{mod}(2) = (B_{i-2}+B_{i-3})\text{mod}(2)$, we quickly obtain

$$B_1 = 1, \quad B_2 = 0, \quad B_3 = 1, \quad B_4 = 1, \quad B_5 = 1, \quad B_6 = 0, \quad B_7 = 0,$$

and then things start to repeat. (In fact, this makes sense since the bits are indeed supposed to repeat every $2^q - 1 = 7$ iterations.) Thus, $B_{100} = B_{93} = \cdots = B_2 = 0$. $\Box$

3. Short-answer RV generation questions.

(a) Suppose the random variable X has p.d.f. $f(x) = x/8$ for $0 \le x \le 4$. Find the inverse of its c.d.f., i.e., $F^{-1}(U)$.

Solution: The c.d.f. is $F(x) = x^2/16$, for $0 < x < 4$. Set $F(X) = X^2/16 = U$ and solve for $F^{-1}(U) = X = 4\sqrt{U}$. $\Box$

(b) If X is standard normal, use the inverse transform method with $U = 0.10$ to generate a realization of X.

Solution: The c.d.f. is $X = \Phi^{-1}(U) = \Phi^{-1}(0.10) = -1.28$. $\Box$

(c) Suppose that X has the Weibull distribution with c.d.f. $F(x) = 1 - e^{-(\lambda x)^\alpha}$, for $x > 0$, where λ and α are positive constants. What is the distribution of the random variable $2F(X) - 1$?

Solution: By the Inverse Transform Theorem, $F(X) \sim \text{Unif}(0,1)$. Thus, $2F(X) - 1 \sim \text{Unif}(-1,1)$. $\Box$

(d) Use the PRN $U = 0.95$ to generate a Geom(0.2) random variate.

Solution: By the Inverse Transform Theorem method from class,

$$X = \left\lceil \frac{\ell\text{n}(1-U)}{\ell\text{n}(1-p)} \right\rceil = \left\lceil \frac{\ell\text{n}(0.05)}{\ell\text{n}(0.8)} \right\rceil = 14. \quad \Box$$

I would also have accepted $X = \lceil \ell\text{n}(U)/\ell\text{n}(1-p) \rceil = 1$.

(e) Suppose that $U_1 = 0.1$ and $U_2 = 0.9$ are realizations of two i.i.d. Unif(0,1)'s. Use the Box–Muller method to generate two i.i.d. standard normals.

Solution: We have

$$\begin{aligned} Z_1 &= \sqrt{-2\ell\text{n}(U_1)}\cos(2\pi U_2) = 1.736 \\ Z_2 &= \sqrt{-2\ell\text{n}(U_1)}\sin(2\pi U_2) = -1.262 \quad \Box \end{aligned}$$

(f) Use your answer from Question 3e to generate a $\chi^2(2)$ random variable.

Solution: $Z_1^2 + Z_2^2 = 4.606$. $\Box$

(g) If U is Unif(0,1), name the distribution of $\tan(2\pi U)$.

Solution: Cauchy. $\Box$

(h) If U_1, U_2, U_3 are i.i.d. Unif(0,1), name the distribution (with parameters) of $-4\{\ell\text{n}[U_1(1-U_2)U_3]\}$.

Solution: We have

$$\begin{aligned} -4\{\ell\text{n}[U_1(1-U_2)U_3]\} &= -4\,\ell\text{n}(U_1) - 4\,\ell\text{n}(1-U_2) - 4\,\ell\text{n}(U_3) \\ &\sim -4\,\ell\text{n}(U_1) - 4\,\ell\text{n}(U_2) - 4\,\ell\text{n}(U_3) \\ &\sim \text{Exp}(1/4) + \text{Exp}(1/4) + \text{Exp}(1/4) \\ &\sim \text{Erlang}_3(1/4). \quad \Box \end{aligned}$$

(i) If U_1 and U_2 are i.i.d. Unif(0,1), name the distribution of $U_1 + U_2$.

Solution: Triangular(0,1,2). $\Box$

(j) Suppose that $U_1, U_2, \ldots, U_{40}$ are i.i.d. Unif(0,1). Name the approximate distribution (with parameters) of $\sum_{i=1}^{40} U_i$.

Solution: By the usual properties of the Unif(0,1) distribution,

$$\mathsf{E}\Big[\sum_{i=1}^{40} U_i\Big] = \sum_{i=1}^{40} \mathsf{E}[U_i] = \frac{n}{2} = 20 \quad \text{and} \quad \mathsf{Var}\Big(\sum_{i=1}^{40} U_i\Big) = \sum_{i=1}^{40} \mathsf{Var}(U_i) = \frac{n}{12} = \frac{40}{12}.$$

Then by the CLT, $\sum_{i=1}^{40} U_i \approx \text{Nor}(20, 3.33)$. $\Box$

(k) Suppose that $U_1 = 0.45$, $U_2 = 0.15$, $U_3 = 0.92$, $U_4 = 0.09$, and $U_5 = 0.26$. Use our acceptance-rejection technique from class to generate a Pois($\lambda = 1.5$) random variate. (You may not need to use all of the uniforms.)

Solution: The procedure is to generate uniforms until $\prod_{i=1}^{n+1} U_i < e^{-\lambda} = 0.223$. Since $U_1 U_2 = 0.068 < 0.223$, we stop with $n+1 = 2$, i.e., $n = 1$. $\Box$

(l) Suppose that X_1, X_2, X_3, X_4 are i.i.d. Exp(1/3). Give an equation involving a *single* PRN U that you can use to generate a realization of $\min\{X_1, X_2, X_3, X_4\}$.

Solution: By class notes, we know that $\min\{X_1, X_2, X_3, X_4\} \sim \text{Exp}(n\lambda) \sim \text{Exp}(12) = -\frac{3}{4}\ell\text{n}(U)$. $\Box$

(m) Suppose Σ is the covariance matrix arising from a vector of random variables $(X_1, X_2, \ldots, X_n)$. It is known that you can find a matrix C such that $\Sigma = CC'$. Who is the C named after?

Solution: Prof. Cholesky. $\Box$

(n) Name 4 ways that you can generate a standard normal RV.

Solution: Inverse transform, rational approximation, convolution approximation, Box–Muller, etc., etc. $\Box$

4. Short-answer queueing questions.

(a) Suppose I run a small hospital experiencing Poisson arrivals at the rate of 10/day. Customers are served in FIFO style and the average time in the system is 1/4 day. (I didn't say the hospital was efficient.) What is the steady-state average number of people in the system?

Solution: By Little's Law, $L = \lambda w = 10(1/4) = 2.5$. $\Box$

(b) YES or NO? Is the expected steady-state time in system for an M/M/1 queue with arrival rate $\lambda = 1$ and service rate $\mu = 2$ shorter than the expected time in system for an M/M/2 with arrival rate $\lambda = 1$ and service rate $\mu = 1$ (more servers, but slower servers)?

Solution: Yes. Here's why. For the M/M/1, the traffic intensity is $\rho = \lambda/\mu = 0.5$. The expected time in system is

$$w = \frac{1}{\mu(1-\rho)} = \frac{1}{2(1-0.5)} = 1.$$

For the M/M/2, we need to do some preliminary calculations, where we use $c = 2$ servers, arrival rate $\lambda = 1$ and service rate $\mu = 1$.

$$\begin{aligned}
\rho &= \lambda/(c\mu) = 0.5 \\
P_0 &= \left\{ \left[\sum_{n=0}^{c-1} \frac{(c\rho)^n}{n!} \right] + \left[\frac{(c\rho)^c}{(c!)(1-\rho)} \right] \right\}^{-1} = 1/3 \\
L &= c\rho + \frac{(c\rho)^{c+1} P_0}{c(c!)(1-\rho)^2} = 4/3 \\
w &= L/\lambda = 4/3.
\end{aligned}$$

So the M/M/1's time in system is $<$ the M/M/2's. $\Box$

(c) YES or NO? Is the expected waiting time in an M/M/1/N less than that of an M/M/1 with the same λ and μ?

Solution: Yes (since there won't be as many customers entering the M/M/1/N). $\Box$