MATH 2602 **Exam I** September 16, 2011

Printed Name: Solutions

Signature: ______________________

Section (circle one): **G1** **G2**

Instructions:

- There are 5 questions. Point values for each problem are as indicated.
- On each question you must show all appropriate legible work to receive full credit.
- Calculators are not allowed.
- SCHOLASTIC DISHONESTY WILL NOT BE TOLERATED.

Good Luck!

1. (20 points) Use induction to prove that for all $n \geq 1$,

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3}.$$

Base step: $n=1$: $\left.\begin{array}{l} LHS = 1^2 = 1 \\ RHS = \frac{1\cdot(4-1)}{3} = 1 \end{array}\right\}$ equal ✓

Induction Hypothesis: Suppose that $1^2+3^2+\cdots+(2k-1)^2 = \frac{k(4k^2-1)}{3}$ for some $k \geq 1$.

Induction Step: We will show that $1^2+3^2+5^2+\cdots+(2k+1)^2 = \frac{(k+1)(4(k+1)^2-1)}{3}$

Indeed, $1^2+3^2+5^2+\cdots+(2k+1)^2 \underset{\uparrow \text{ I.H.}}{=} \frac{k(4k^2-1)}{3} + (2k+1)^2 =$

$= \frac{4k^3+12k^2+11k+3}{3}$ $\cancel{(k+1)(4k^2+8k}$ and also, $\frac{(k+1)(4(k+1)^2-1)}{3} = \frac{4k^3+12k^2+11k+3}{3}$

2. (12 points) Express the generating function of the sequence

$$a_0 = 1, a_1 = 1, a_2 = 2, a_n = -a_{n-1} + 2a_{n-3}, n \geq 3$$

as a polynomial or as a quotient of polynomials.

$$+\left\{\begin{array}{l} f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\ x f(x) = \quad a_0 x + a_1 x^2 + a_2 x^3 + \cdots + a_{n-1} x^n + \cdots \\ -2x^3 f(x) = \qquad -2a_0 x^3 - \cdots - 2a_{n-3} x^n - \cdots \end{array}\right.$$

$$(1+x-2x^3) f(x) = a_0 + (a_1+a_0)x + (a_2+a_1)x^2 + \underbrace{(a_3+a_2-2a_0)}_{=0} x^3 + \cdots + \overbrace{(a_n+a_{n-1}-2a_{n-3})}^{=0} x^n + \cdots$$

$$(1+x-2x^3) f(x) = 1 + (1+1)x + (2+1)x^2 = 1 + 2x + 3x^2$$

$$\boxed{f(x) = \frac{1+2x+3x^2}{1+x-2x^3}}$$

3. (20 points) Solve the recurrence $a_0 = 0$, $a_1 = 0$, $a_{n+1} = 4a_n + 12a_{n-1} + 3^{n-1}$ for $n \geq 1$ by completing the following steps:

(a) Find a particular solution to the given recurrence.

Guess $p_n = A \cdot 3^n$

$$A \cdot 3^{n+1} = 4 \cdot A \cdot 3^n + 12 A 3^{n-1} + 3^{n-1}$$

$$3^{n-1}(9A) = 3^{n-1}(12A + 12A + 1)$$

$$9A = 12A + 12A + 1$$

$$15A = -1$$

$$A = -\frac{1}{15}$$

$$\boxed{p_n = -\frac{1}{15} \cdot 3^n}$$

(b) Find the general solution to the corresponding homogeneous recurrence.

Characteristic equation:

$$x^2 - 4x - 12 = 0$$

$$(x-6)(x+2) = 0$$

$$\rightarrow x_1 = 6, \; x_2 = -2$$

$$\boxed{q_n = c_1 \cdot 6^n + c_2 \cdot (-2)^n}$$

(c) Find a_n.

$$a_n = p_n + q_n = -\frac{1}{15} \cdot 3^n + c_1 \cdot 6^n + c_2 (-2)^n$$

$n = 0$: $n = 1$:

$$\begin{cases} 0 = -\frac{1}{15} + c_1 + c_2 \\ 0 = -\frac{1}{5} + 6c_1 - 2c_2 \end{cases} \rightarrow \begin{cases} c_1 + c_2 = \frac{1}{15} \\ 6c_1 - 2c_2 = \frac{1}{5} \end{cases} \rightarrow \begin{cases} c_1 = \frac{1}{24} \\ c_2 = \frac{1}{40} \end{cases}$$

$$\boxed{a_n = -\frac{1}{15} \cdot 3^n + \frac{1}{24} \cdot 6^n + \frac{1}{40} \cdot (-2)^n}$$

4. (a) (15 points) For the functions $f(n) = n^2 \ln n$ and $g(n) = n(\ln n)^2 + 7n$ decide whether $f \prec g$, $g \prec f$, or $f \asymp g$. Justify your answer.

$n \cdot (\ln n)^2 \succ 7n$ so $g \asymp n(\ln n)^2$

Compare f and $n(\ln n)^2$:

$$\lim_{n\to\infty} \frac{n^2 \ln n}{n(\ln n)^2} = \lim_{n\to\infty} \frac{n}{\ln n} \underset{\substack{\uparrow \\ \text{L'Hospital's} \\ \text{Rule}}}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \infty$$

So, $f \succ n(\ln n)^2$ and therefore $\boxed{f \succ g}$

(b) (15 points) For the functions $f(n) = 2(n!)^2$ and $g(n) = (2n)^n$ decide whether $f \prec g$, $g \prec f$, or $f \asymp g$. Justify your answer.

$$\lim_{n\to\infty} \frac{2(n!)^2}{(2n)^n} = \lim_{n\to\infty} \frac{2\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2}{(2n)^n} = \lim_{n\to\infty} \frac{2 \cdot 2 \cdot \pi \cdot n \cdot \left(\frac{n}{e}\right)^{2n}}{(2n)^n}$$

$$= \lim_{n\to\infty} 4\pi n \left(\frac{n^2}{2n \cdot e^2}\right)^n = \lim_{n\to\infty} \underbrace{4\pi \cdot n}_{\downarrow \, \infty} \left(\frac{n}{2e^2}\right)^n = \infty$$

$\left(\frac{n}{2e^2}\right) \to \infty$

So, $\boxed{g \prec f}$

5. (18 points) Which sequence is associated with each of the following generating functions?

(a) $3x^2 - 5x^4 + x^5$

$$\boxed{0, 0, 3, 0, -5, 1, 0, 0, \ldots}$$

(b) $\frac{5}{(1-2x)(1+3x)}$

$$\frac{5}{(1-2x)(1+3x)} = \frac{2}{1-2x} + \frac{3}{1+3x} = 2 \cdot \sum_{n=0}^{\infty} (+2)^n x^n + 3 \cdot \sum_{n=0}^{\infty} (-3)^n x^n$$

$$= \sum_{n=0}^{\infty} 2^{n+1} x^n + \sum_{n=0}^{\infty} 3(-3)^n x^n$$

$$= \sum_{n=0}^{\infty} \left(2^{n+1} - (-3)^{n+1}\right) x^n$$

$$\boxed{\begin{gathered} a_n = 2^{n+1} - (-3)^{n+1} \\ \text{for } n = 0, 1, 2, \ldots \end{gathered}}$$

(c) $\frac{1}{(1+x)^2}$

We know that $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

So, $\frac{1}{(1+x)^2} = 1 + 2(-x) + 3(-x)^2 + 4(-x)^3 + \cdots$

$$= 1 - 2x + 3x^2 - 4x^3 + \cdots$$

So, the sequence is $\boxed{1, -2, 3, -4, 5, -6, \ldots}$