MATH 2603, Fall 2015, Exam Sample 1: Closed book, no calculators.
Instructor: Esther Ezra.
This exam sample contains 8 questions, the actual exam will have only 6 questions.

Answer all questions **on this sheet**.

Name ____________________ GT IDnumber ____________________

Problem 1.

a. Prove that n is an even integer implies $n^2 + 3n$ is an even integer as well.

Solution:

$$n^2 + 3n = n(n+3).$$

Since n is an even integer the product $n(n+3)$ must be even as well.

b. What is the converse of the implication in (a)? Is this converse true or false?

Solution: The converse is "$n^2 + 3n$ even implies n is even". This statement is false as $n^2 + 3n = n(n+3)$ is always even. Indeed, if n is even then it follows from part a, and if n is odd we have $n + 3$ must be even, as in this case $n = 2k + 1$ for some integer k, and then $n + 3 = 2k + 1 + 3 = 2k + 4 = 2(k + 2)$. Thus the statement is incorrect as n can be odd. For example, choose $n = 3$, $n(n+3) = 18$.

Problem 2. Let A, B, C be subsets of some universal set U.

a. If $A \cap B \subseteq C$, and $A^C \cap B \subseteq C$, show that $B \subseteq C$. [Here A^C is the complement of A with respect to U, that is, this is $U \setminus A$).

Solution: Let x be an element in B, and let us consider $A \cap B$. Then either (i) x lies also in A, in which case $x \in A \cap B$, or (ii) x lies outside A, in which case $x \in A^C \cap B$. But since $A \cap B \subseteq C$ and $A^C \cap B \subseteq C$, it follows that in both cases (i) and (ii) x must lie in C. Therefore $x \in B$ implies $x \in C$ and thus $B \subseteq C$.

b. Given that $A \cap B = A \cap C$ and $A^C \cap B = A^C \cap C$, does it follow that $B = C$? Justify your answer.

solution: This statement is correct. Let us observe that $A^C \cap B = B \setminus A$, and $A^C \cap C = C \setminus A$ (you may draw the Venn diagram in order to see it). Next, let x be an element in B, then either (i) it lies outside A, in which case $x \in B \setminus A$, or (ii) it lies also inside A, in which case $x \in A \cap B$. In case (i), since $A^C \cap B = A^C \cap C$, we have $x \in C \setminus A$ as well. In case (ii), since $A \cap B = A \cap C$ we have $x \in A \cap C$. But this implies that $x \in C$ always! Thus $B \subseteq C$. Now, replacing the rules of B and C (as the proof is symmetric) we obtain $C \subseteq B$. Thus $B = C$.

Problem 3. Let A be the set of all natural numbers.

a. For any pair of natural numbers a, b, we say that $a \sim b$ iff ab is a perfect square (that is, the square of an integer). Show that $\sim$ defines an equivalence relation on A.

solution: We show that $\sim$ is reflexive, symmetric, and transitive.

Reflexive: $a \cdot a = a^2$ is definitely a perfect square. Thus $a \sim a$.

Symmetric: ab is a perfect square iff ba is a perfect square. Thus $a \sim b$ iff $b \sim a$.

Transitive: If ab is a perfect square and bc is a perfect square, we need to show ac is also a perfect square. Indeed, $ab \cdot bc$ is a perfect square. Let us rewrite is as:

$$ab \cdot bc = b^2(ac) = b^2(k^2),$$

for some integer k. But now

$$ac = \frac{ab \cdot bc}{b^2} = k^2,$$

then ac is also a perfect square. Thus $a \sim c$.

b. Repeat part a, when now $a \sim b$ iff a/b is a power of 2, that is, $a/b = 2^t$, for some integer t (positive, negative, or zero).

solution sketch:

Here, it is clear that $a/a = 1 = 2^0$, thus $\sim$ is reflexive. Given that $a/b = 2^t$, for some integer t, we have $b/a = 2^{-t}$, and thus $\sim$ is symmetric. Next, if $a/b = 2^t$, and $b/c = 2^s$, for two integers t, s, then

$$\frac{a}{c} = \frac{a}{b}\frac{b}{c} = 2^t 2^s = 2^{t+s},$$

and since $t + s$ is an integer, $a \sim c$.

Problem 4. **a.** Give an example of a partially ordered set that has a maximum and a minimum element, but is NOT totally ordered.

solution sketch: Let S be a non-empty set of at least two elements, say, $S = \{a, b\}$. Then such a poset is $(P(S), \subseteq)$, that is, the partially order set defined on the power set of S with the inclusion relation. Here the minimum element is the empty set $\emptyset$, and the maximum element is S. This poset is not totally ordered as the two elements $\{a\}, \{b\}$ are incomparable (non of them contains the other).

b. Define a binary relation R on the natural numbers N as follows:

$$R = \{(a, b) \mid a \text{ divides } b\}.$$

Show that R is a partial order.

solution:

Reflexive: For any $a \in N$, $a/a = 1$ and thus $a|a$.

Anti-symmetric: Let $a, b \in N$. Suppose $a|b$ and $b|a$. Then $b = q_1 a$, for some natural number q_1, and $a = q_2 b$, for some natural number q_2. Thus

$$a = q_2 b = q_2(q_1 a).$$

Since $a > 0$, we must have $q_1 q_2 = 1$, but then this implies that $q_1 = q_2 = 1$, as q_1, q_2 are natural numbers. Thus $a = b$.

Transitive: Let $a, b, c \in N$, and suppose $a|b$, and $b|c$. Then $b = q_1 a$ and $c = q_2 b$, where q_1, q_2 are natural numbers. Thus

$$c = q_2 b = q_2(q_1 a) = (q_1 q_2)a.$$

We thus have $a|c$, since $q_1 \cdot q_2$ is a natural number.

Problem 5. **a.** Exhibit integers x, y, such that $7x + 5y = 3$

solution: $x = 4, y = -5$.

b. If $7x + 5y = 3$, show that there exists an integer k s.t. $y = 9 + 7k$, and $x = -6 - 5k$.

solution: Let us put $y = 9 + 7k$, and $x = -6 - 5k$ and find the value of k. Since $7x + 5y = 3$ we have:

$$7(-6 + -5k) + 5(9 + 7k) = 3.$$

Thus

$$-42 - 35k + 45 + 35k = 3 + 0K = 3.$$

Thus we in fact got something stronger: the solution is $y = 9 + 7k$, and $x = -6 - 5k$, for any integer k.

Problem 6. **a.** Use the Euclidean algorithm in order to find $gcd(119, 93)$.

solution:

$gcd(119, 93) = gcd(93, 26) = gcd(26, 15) = gcd(15, 11) = gcd(11, 4) = gcd(4, 3) = gcd(3, 1) = 1.$

b. Now find $lcm(119, 93)$.

solution: Since we showed $gcd(119, 93) = 1$, we use the fact that

$$lcm(119, 93) = \frac{119 \cdot 93}{gcd(119, 93)} = 119 \cdot 93 = 11067.$$

c. If $k \in N$, prove that $gcd(5k + 3, 3k + 2) = 1$.

solution: By Euclidean Algorithm

$gcd(5k + 3, 3k + 2) = gcd(3k + 2, 2k + 1) = gcd(2k + 1, k + 1) = gcd(k + 1, k) = gcd(k, 1) = 1.$

Problem 7. Let a, b be integers and let p be a prime number. Answer true/false and explain:

a. If $p|a^2$ then $p|a$.

solution: True, as if p does not divide a, then since p is prime we must have, $gcd(p, a) = 1$ We now use the proposition that if m, b, x are integers s.t. $m|bx$, and m, b are relatively primes, we must have $m|x$. In our problem, $b, x = a$, and $m = p$. Then we must have that $p|a$.

b. If $p|a$ and $p|(a^2 + b^2)$, then $p|b$.

solution: True: if $p|a$ we must have $p|a^2$ as well. It is given that $p|(a^2 + b^2)$, so we must have $p|b^2$. Now we use part a in order to conclude $p|b$.

Problem 8. True or false:

$$\{n \in N \mid n > 2 \text{ and } a^n + b^n = c^n, \text{ for some } a, b, c \in N\} = \emptyset$$

solution: This is true according to the last Theorem of Fermat.