QUIZ 1 Name Key

Math 1553 D Steinbart August 27, 2015

Work neatly. Justify your answers and use proper notation. SHOW YOUR WORK TO RECEIVE CREDIT! Calculators, phones, or other electronic devices are not allowed. There is a total of 10 points.

1. (5 pts) There is a linear system of equations in the variables x_1, x_2, x_3, x_4. Suppose that the augmented matrix corresponding to this linear system has been row reduced to the following echelon form:

$$\begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ $\leftarrow$ I circled the pivots

a. Does there exist a solution to the linear system? Why or why not? $\boxed{2 \text{ pts}}$

b. If there is a solution, is it unique? Why or why not? $\leftarrow$ $\boxed{1 \text{ pt}}$

c. Find the parametric description of the solution set of the linear system. Be complete. $\boxed{2 \text{ pts}}$

(a). Yes the is a soln, because there is no pivot in final column of augmented matrix.

(b) No, the Solution is not unique. Because x_3 is a free variable - no pivots exist in third column.

(c) $$\rightarrow \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 9 & 0 & -3 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -3 - 9x_3$

$x_2 = -3 - 2x_3$

$x_4 = \frac{1}{2}$

x_3 free (any real #)

2. (4) Suppose that A is a 6×4 matrix with 4 pivot positions. and $\mathbf{b}$ is some vector in $\mathbb{R}^6$.

a. Does $A\mathbf{x} = \mathbf{b}$ always have a solution? Why or why not?

b. If $A\mathbf{x} = \mathbf{b}$ has a solution, is that solution unique? Why or why not?

(a) No, A has 6 rows but only 4 pivots. So there is a row of A (2 rows in fact) without a pivot, So there could be a pivot in the final column of the augmented matrix $[A \ \underline{b}]$ Thus $A\underline{x}=\underline{b}$ would have no soln.

(b) Yes. If a soln exists, it is unique. Since A has 4 columns and 4 pivots, every column of A is a pivot column, So there are no free variables

3. (1) Let $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}$ be vectors in $\mathbb{R}^5$. Complete the following sentence:

The vector $\begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ is in $Span\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}\}$ if... there are numbers $x_1, x_2, x_3,$ and x_4 so that

$$\begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + x_3 \underline{a}_3 + x_4 \underline{a}_4 .$$

QUIZ 2 Name Key

Math 1553 D Steinbart September 3, 2015

Work neatly. Justify your answers and use proper notation. SHOW YOUR WORK TO RECEIVE CREDIT! Calculators, phones, or other electronic devices are not allowed. There is a total of 10 points.

1. (3) A is a 5×4 matrix. Suppose that A can be row reduced to $\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Does $A\mathbf{x} = \mathbf{b}$ have a solution for all $\mathbf{b}$ in $\mathbb{R}^5$? Why or why not?

Matrix A does not have a pivot in every row. So the equation $A\underline{x} = \underline{b}$ does not have a soln for all $\underline{b}$ in $\mathbb{R}^5$. [There is a vector $\underline{b}_1$ in $\mathbb{R}^5$ so that $A\underline{x} = \underline{b}_1$ does not have a soln. In fact, there are infinitely many vectors $\underline{b}$ in $\mathbb{R}^5$ where $A\underline{x} = \underline{b}$ does not have a soln.]

2. (7) Matrix B can be row reduced to $\begin{bmatrix} 1 & -4 & 3 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

a. Find all solutions of $B\mathbf{x} = \mathbf{0}$. (That is, find the solutions set to the equation $B\mathbf{x} = \mathbf{0}$.) Express your answer in parametric vector form. Be complete.

$$B \xrightarrow{\text{row reduced}} \begin{bmatrix} 1 & -4 & 3 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & -4 & 0 & -8 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

← reduced echelon form of B.

Since we are solving the homogeneous equation $B\underline{x} = \underline{0}$ we deduce

$x_1 - 4x_2 - 8x_4 = 0$
$x_3 + 3x_4 = 0$

x_1, x_3 basic variable
x_2, x_4 free var.

So $x_1 = 4x_2 + 8x_4$
$x_3 = -3x_4$
x_2, x_4 free

So $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4x_2 + 8x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{bmatrix}$

$= \begin{bmatrix} 4x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 8x_4 \\ 0 \\ -3x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ 0 \\ -3 \\ 1 \end{bmatrix}$.

Answer: Solns of $B\underline{x} = \underline{0}$ are $\underline{x} = t_1 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 8 \\ 0 \\ -3 \\ 1 \end{bmatrix}$ where $t_1, t_2 \in \mathbb{R}$

This is parametric vector form

b. Suppose that we know that $\mathbf{p} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ is a solution to the equation $B\mathbf{x} = \begin{bmatrix} -1 \\ 5 \\ -7 \end{bmatrix}$.

Find all solutions of $B\mathbf{x} = \begin{bmatrix} -1 \\ 5 \\ -7 \end{bmatrix}$. Express your answer in parametric vector form. Be complete.

We know one soln to the nonhomogeneous equation $B\underline{x} = \begin{bmatrix} -1 \\ 5 \\ -7 \end{bmatrix}$, namely we know $B\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -7 \end{bmatrix}$. We know all solns to $B\underline{x} = \underline{0}$ (the homogeneous eq.) from part ⓐ. So we know all solns to the nonhomogeneous equation $B\underline{x} = \begin{bmatrix} -1 \\ 5 \\ -7 \end{bmatrix}$. The solns are $\underline{x} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} + t_1 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 8 \\ 0 \\ -3 \\ 1 \end{bmatrix}$ where t_1, t_2 are in $\mathbb{R}$.

parametric vector form

QUIZ 2 – Comments to solution Math 1553 D Steinbart

1. Comment: We could have answered this question just ~~know~~ knowing that A is a 5×4 matrix. A matrix cannot have moore pivots than columns. A has 4 columns so A has at most 4 pivots. We know A has 5 rows. So there cannot be a pivot in every row of A. Since A does not have a pivot in every row, the statement "$A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^5$" is FALSE. That means that there is at least one $\mathbf{b} \in \mathbb{R}^5$ so that $A\mathbf{x} = \mathbf{b}$ DOES NOT have a solution.

2a. Comment: We don't know what the 4×3 matrix B looks like. We do know that B can be row reduced to $\begin{bmatrix} 1 & -4 & 3 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ This is enough information to solve the homogeneous equation $B\mathbf{x} = \mathbf{0}$. Note that the *augmented matrix* corresponding to the equation $B\mathbf{x} = \mathbf{0}$ would be a 3×5 matrix and the right most column of this *augmented matrix* would be all 0's.

If we row reduced this *augmented matrix* to get to reduced echelon form, we would get $\begin{bmatrix} 1 & -4 & 0 & -8 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. So...

$x_1 - 4x_2 - 8x_4 = 0$

$x_3 + 3x_4 = 0$

etc...

2b. Comment: We can solved this problem without knowing B since we are told ONE solution to the nonhomogeneous equation.

FYI, the original B that I started with was $B = \begin{bmatrix} 1 & -4 & 3 & 1 \\ 0 & 0 & 1 & 3 \\ 2 & -8 & 5 & -1 \end{bmatrix}$.

Do you want to more practice ??

Practice problem: (Without using anwers from the quiz...) Find the solution set to $B\mathbf{x} = \begin{bmatrix} -1 \\ 5 \\ -7 \end{bmatrix}$ where $B = \begin{bmatrix} 1 & -4 & 3 & 1 \\ 0 & 0 & 1 & 3 \\ 2 & -8 & 5 & -1 \end{bmatrix}$. Express your answer in parametric vector form.

Note: Your solution will (probably) look different than the solution to the quiz problem. Your solution will have the same number of vectors, but one of the vectors will have differene numbers than appeared on the quiz solution. How can both answers be correct?

QUIZ 3 Name Key

Math 1553 D Steinbart September 10, 2015

Work neatly. Justify your answers and use proper notation. SHOW YOUR WORK TO RECEIVE CREDIT! Calculators, phones, or other electronic devices are not allowed. There is a total of 10 points.

6pts

1. Find the value(s) of h (if any) for which the vectors are linearly *dependent*.

$\begin{bmatrix}1\\-2\\-4\end{bmatrix}$, $\begin{bmatrix}-3\\7\\6\end{bmatrix}$, $\begin{bmatrix}2\\1\\h\end{bmatrix}$. Let $A=\begin{bmatrix}1&-3&2\\-2&7&1\\-4&6&h\end{bmatrix} \xmapsto[R_3\to R_3+4R_1]{R_2\to R_2+2R_1} \begin{bmatrix}1&-3&2\\0&1&5\\0&-6&h+8\end{bmatrix} \xrightarrow{R_3\to R_3+6R_2}$

$\begin{bmatrix}1&-3&2\\0&1&5\\0&0&h+38\end{bmatrix}$ If $h+38=0$ (so $h=-38$) there will be no pivot in the third column. This means that the equation $A\underline{x}=\underline{0}$ has a nontrivial solution. Thus there are numbers c_1, c_2, c_3 not all 0 so $A\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\underline{0}$.

So the vectors are linearly <u>dependent</u> if $h=-38$

$c_1\begin{bmatrix}1\\-2\\-4\end{bmatrix}+c_2\begin{bmatrix}-3\\7\\6\end{bmatrix}+c_3\begin{bmatrix}2\\1\\h\end{bmatrix}=\underline{0}$ if $h=-38$.

2pts

2. Let $A=\begin{bmatrix}1&2&3&2\\2&3&0&6\end{bmatrix}$, and define $T: \mathbb{R}^4 \to \mathbb{R}^2$ by $T(\mathbf{x})=A\mathbf{x}$. Let $\mathbf{u}=\begin{bmatrix}3\\1\\1\\-5\end{bmatrix}$ Find $T(\mathbf{u})$.

$T(\underline{u}) = A\underline{u} = A\begin{bmatrix}3\\1\\1\\-5\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}2\\3\end{bmatrix}+1\begin{bmatrix}3\\0\end{bmatrix}-5\begin{bmatrix}2\\6\end{bmatrix}$

$= \begin{bmatrix}3+2+3-10\\6+3+0+-30\end{bmatrix}$

$= \begin{bmatrix}-2\\-21\end{bmatrix}$

(OVER)

3. Let $A = \begin{bmatrix} 1 & -3 \\ -1 & 2 \\ 1 & 5 \end{bmatrix}$, and define $T : I\!R^2 \to I\!R^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Let $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix}$. Find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$. (That is, find a vector $\mathbf{x}$ so that $T(\mathbf{x}) = \mathbf{b}$.

2pt

$T(\underline{x}) = A\underline{x}$. So solve $A\underline{x} = \underline{b}$

$$\begin{bmatrix} 1 & -3 & -1 \\ -1 & 2 & 0 \\ 1 & 5 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & -1 \\ 0 & -1 & -1 \\ 0 & 8 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 0x_2 = 2$
$x_2 = 1$

So $\underline{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The image of $\underline{x}$ under T is $\underline{b}$

QUIZ 4 Name Key

Math 1553 D Steinbart October 1, 2015

Work neatly. Justify your answers and use proper notation. SHOW YOUR WORK TO RECEIVE CREDIT! Calculators, phones, or other electronic devices are not allowed. There is a total of 10 points.

(5) 1. For each matrix, determine if it has an inverse. If the matrix has an inverse, find the inverse. If it doesn't have an inverse, explain why not.

$$A = \begin{bmatrix} -5 & -15 \\ 4 & 12 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & -1 & -1 \\ 6 & 7 & 7 \\ 6 & 6 & 7 \end{bmatrix}$$

$\det A = (-5)(12)-(4)(-15) = -60+60 = 0$. Since $\det A = 0$, the matrix A is not invertible.

To see if B is invertible, we will row reduce B. We will augment the matrix B with the identity matrix I to keep track of our steps.

$$[B \mid I] = \left[\begin{array}{ccc|ccc} -1 & -1 & -1 & 1 & 0 & 0 \\ 6 & 7 & 7 & 0 & 1 & 0 \\ 6 & 6 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \mapsto R_3+6R_1]{R_2 \mapsto R_2+6R_1} \left[\begin{array}{ccc|ccc} -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 6 & 1 & 0 \\ 0 & 0 & 1 & 6 & 0 & 1 \end{array}\right] \xrightarrow[R_1+R_3]{R_2-R_3} \left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 7 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 6 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1+R_2} \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 7 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 6 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to -1R_1} \left[\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I} \middle| \underbrace{\begin{array}{ccc} -7 & -1 & 0 \\ 0 & 1 & -1 \\ 6 & 0 & 1 \end{array}}_{B^{-1}}\right].$$

We can row reduce B to the identity matrix. So B is invertible.

Further $B^{-1} = \begin{bmatrix} -7 & -1 & 0 \\ 0 & 1 & -1 \\ 6 & 0 & 1 \end{bmatrix}$

Check: $BB^{-1} = \begin{bmatrix} -1 & -1 & -1 \\ 6 & 7 & 7 \\ 6 & 6 & 7 \end{bmatrix}\begin{bmatrix} -7 & -1 & 0 \\ 0 & 1 & -1 \\ 6 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-6 & 1-1 & 1-1 \\ -42+42 & -6+7 & -7+7 \\ -42+42 & -6+6 & -6+7 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$ ✓

(OVER)

$b = \begin{bmatrix} 16 \\ -66 \\ 8 \end{bmatrix}$

(5) 2. $A = \begin{bmatrix} 2 & -6 & 2 \\ -10 & 28 & -6 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 & 2 \\ 0 & -2 & 4 \\ 0 & 0 & -2 \end{bmatrix}$. Let $\mathbf{b} = \begin{bmatrix} \cancel{8} \\ \cancel{8} \\ \cancel{-6} \end{bmatrix}$. Use the LU factorization method to solve $A\mathbf{x} = \mathbf{b}$.

Plan: ① Solve $L\underline{y} = \underline{b}$ for $\underline{y}$

② Solve $U\underline{x} = \underline{y}$ for $\underline{x}$

Then we'll have $A\underline{x} = (LU)\underline{x} = L(U\underline{x}) = L\underline{y} = \underline{b}$

① Solve $L\underline{y} = \underline{b}$: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 16 \\ -5 & 1 & 0 & -66 \\ 0 & 1 & 1 & 8 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 + 5R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 16 \\ 0 & 1 & 0 & 14 \\ 0 & 1 & 1 & 8 \end{array}\right] \xrightarrow{R_3 \mapsto R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 16 \\ 0 & 1 & 0 & 14 \\ 0 & 0 & 1 & -6 \end{array}\right]$

so $\underline{y} = \begin{bmatrix} 16 \\ 14 \\ -6 \end{bmatrix}$

② Solve $U\underline{x} = \underline{y}$: $\left[\begin{array}{ccc|c} 2 & -6 & 2 & 16 \\ 0 & -2 & 4 & 14 \\ 0 & 0 & -2 & -6 \end{array}\right] \xrightarrow[R_1 \mapsto R_1 + R_3]{R_2 \mapsto R_2 + 2R_3} \left[\begin{array}{ccc|c} 2 & -6 & 0 & 10 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & -6 \end{array}\right] \xrightarrow{R_1 \mapsto R_1 - 3R_2} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & -6 \end{array}\right]$

$\xrightarrow[R_3 \to -\frac{1}{2}R_3]{R_1 \mapsto \frac{1}{2}R_1,\ R_2 \mapsto -\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$ so $\underline{x} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

QUIZ 5 Name Key

Math 1553 D Steinbart October 22, 2015

Work neatly. Justify your answers and use proper notation. SHOW YOUR WORK TO RECEIVE CREDIT! Calculators, cell phones, and other electronic devices are not allowed on this quiz. There is a total of 10 points.

1. (3) Let $\mathbf{a_1} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{a_2} = \begin{bmatrix} 7 \\ 0 \\ -2 \end{bmatrix}$, and $\mathbf{a_3} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$. Determine if the set $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$ is linearly independent. Justify your answer.

We know that the columns of a 3x3 matrix are linearly independent if the determinant of the matrix is not 0. So let $A = [a_1\ a_2\ a_3]$

Then $\det A = \det\begin{bmatrix} -2 & 7 & 6 \\ 1 & 0 & 2 \\ 3 & -2 & 5 \end{bmatrix} = (-1)^{2+1}(1)\det A_{21} + (-1)^{2+2}(0)\det A_{22} + (-1)^{2+3}(2)\det A$

we'll expand by cofactors along the second row.

$= (-1)\det\begin{bmatrix} 7 & 6 \\ -2 & 5 \end{bmatrix} + 0 + (-1)(2)\det\begin{bmatrix} -2 & 7 \\ 3 & -2 \end{bmatrix} = -1\big(7(5) - (-2)(6)\big) - 2\big((-2)(-2) - 3(7)\big)$

$= -(35+12) - 2(4-21) = -47 + 34 = -13$. So $\det A \neq 0$. So the columns of A, namely a_1, a_2, a_3, are linearly independent.

2. (3) $B = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 0 \\ 2 & 0 & 1 & 5 \\ 0 & 1 & 8 & 0 \end{bmatrix}$. Find $det(B)$. Show all work in a readable manner.

There are two 0's in the fourth column. We'll find $\det B$ by expanding by cofactors down the fourth column.

$\det B = (-1)^{1+4}\, 6 \det B_{14} + (-1)^{2+4}\, 0 \det B_{24} + (-1)^{3+4}\, 5 \det B_{34} + (-1)^{4+4}(0)\det B_{44}$

$= -6\det B_{14} + (-5)\det B_{34} = -6(-35) - 5(9) = 210 - 45 = 165.$ $\boxed{\det B = 165}$

see Aside below

Aside: $\det B_{14} = \det\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 8 \end{bmatrix} = 1(0)8 + 2(1)(0) + (-1)(2)(1) - \big(\underbrace{0(0)(-1)}_{} + \underbrace{(1)(1)(1)}_{1} + \underbrace{8(2)(2)}_{32}\big)$

$= -2 - (33) = -35$

$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 8 \end{bmatrix}\begin{matrix} 1 & 2 \\ 2 & 0 \\ 0 & 1 \end{matrix}$

$B_{34} = \begin{bmatrix} 3 & 5 & -2 \\ 1 & 2 & -1 \\ 0 & 1 & 8 \end{bmatrix}$ $\begin{bmatrix} 3 & 5 & -2 \\ 1 & 2 & -1 \\ 0 & 1 & 8 \end{bmatrix}\begin{matrix} 3 & 5 \\ 1 & 2 \\ 0 & 1 \end{matrix}$

$\det B_{34} = \underbrace{3(2)8}_{48} + \underbrace{5(-1)0}_{0} + \underbrace{(-2)(1)(1)}_{-2} - \big(\underbrace{0(2)(-2)}_{0} + \underbrace{(1)(-1)3}_{3} + \underbrace{8(1)5}_{40}\big) = 48 - 2 - (0 - 3 + 40)$

$= 46 - 37 = 9$

3. (4) A is a 4×4 matrix with $detA = -3$. Find: $det(A^t)$ $det(A^4)$, $det(2A)$ $det(A^{-1})$.

$\det(A^t) = \det(A) = -3$

$\det(A^4) = \det(AAAA) = \det A \det A \det A \det A = (\det A)^4 = (-3)^4 = 81.$

$\det(2A) = 2^4 \det A = 16(-3) = -48$ since 2A means each of the 4 rows is multiplied by the factor of 2. There are 4 rows. So $\det(2A) = 2^4 \det A = -48$. ⊛

$\det A^{-1} = \frac{1}{\det A}$ if $\det A \neq 0$. So $\det A^{-1} = \frac{1}{-3} = -\frac{1}{3}$.

⊛ If B were a 3x3 matrix, then there are only 3 rows. So $\det 2B = 2^3 \det B$.

QUIZ 6 Name Key

Math 1553 D Steinbart October 29, 2015

Work neatly. Justify your answers and use proper notation. SHOW YOUR WORK TO RECEIVE CREDIT! Calculators, cell phones, and other electronic devices are not allowed on this quiz. There is a total of 10 points.

1. (3) Let $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$. a. Find the characteristic polynomial of ~~B~~ A. b. Find all eigenvalues of ~~B~~ A.

ⓐ The characteristic polynomial of B $= p_A(\lambda) = \det(A - \lambda I)$

$= \det\left(\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 10-\lambda & -9 \\ 4 & -2-\lambda \end{bmatrix}\right)$

$= (10-\lambda)(-2-\lambda) - ((-9)4) = \lambda^2 - 8\lambda - 20 + 36 = \lambda^2 - 8\lambda + 16 = (\lambda-4)^2$. So $p_A(\lambda) = (\lambda-4)^2$

ⓑ λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$

$p_A(\lambda) = 0$

$(\lambda-4)^2 = 0$

$\lambda = 4$

So the only eigenvalue of A is $\lambda = 4$

This eigenvalue has algebraic mult. 2.

2. (7) Let $B = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

a. Verify that $\lambda_1 = 2$ is an eigenvalue of B.

b. Find a basis for the eigenspace E_{λ_1}.

$\lambda_1 = 2$ is an eigenvalue of B if and only if $\det(B - 2I) = 0$

$\det(B - 2I) = \det\begin{bmatrix} 4-2 & 0 & 1 \\ -2 & 1-2 & 0 \\ -2 & 0 & 1-2 \end{bmatrix} = \det\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} = (2)(-1)(-1) + 0 + 1(-2)(-1) - [(-2)(-1)(1) + 0 + (-1)(-2)(1)]$

$= 2 + 2 - [2+2] = 0$

$\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{matrix} 2 & 1 \\ -2 & 0 \\ -2 & -1 \end{matrix}$

So $\lambda = 2$ is an eigen value of B.

The eigenspace $E_{\lambda_1} = \mathrm{Nul}(B - \lambda_1 I) = \mathrm{Nul}\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} = \{\underline{x} \mid (B - \lambda I)\underline{x} = \underline{0}\}$

So solve $(B - \lambda_1 I)\underline{x} = \underline{0}$:

$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$2x_1 + x_3 = 0$

$-x_2 + x_3 = 0$

augmented matrix

echelon form

x_3 is free variable

$2x_1 = -x_3$

$x_1 = -\frac{1}{2}x_3$

$-x_2 = -x_3$

$x_2 = x_3$

So $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$

So $\mathrm{Nul}(B - \lambda_1 I) = E_{\lambda_1} = \left\{ \underline{x} \mid \underline{x} = t\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}, t \in \mathbb{R} \right\}$

$\mathcal{B} = \left\{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis for E_{λ_1}.

Comment: The geometric mult. of λ_1 is 1 since $\dim E_{\lambda_1} = 1$.

The $\dim E_{\lambda_1} = 1$ since the basis for E_{λ_1} has One vector.

① Is the matrix a symmetric matrix

$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 0 & -2 \\ 5 & -2 & 8 \end{bmatrix}$

② Complete so A is symmetric

$A = \begin{bmatrix} 3 & \_ & \_ & 5 \\ -2 & 2 & \_ & \_ \\ 4 & 1 & 6 & 7 \\ \_ & 9 & \_ & 6 \end{bmatrix}$

③ Is $B = \begin{bmatrix} 5/\sqrt{29} & 2/\sqrt{29} \\ -2/\sqrt{29} & 5/\sqrt{29} \end{bmatrix}$ an orthogonal matrix?

Is $C = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$ an orthogonal matrix?

Find B^{-1}

Dot Products:

Recall if $\underline{u}$, $\underline{v}$ are vectors in $\mathbb{R}^n$, then

$$\underline{u} \cdot \underline{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n.$$

$$\|\underline{u}\| = \text{length (or norm) of } \underline{u} = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

Observe: $\underline{u} \cdot \underline{u} = \|\underline{u}\|^2$

If $\underline{v} \neq \underline{0}$, then $\underline{u} = \dfrac{\underline{v}}{\|\underline{v}\|} = \dfrac{1}{\|\underline{v}\|} \underline{v}$ is a vector of length 1.

Ex: Let $\underline{u} = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$, $\underline{w} = \begin{bmatrix} -9 \\ 2 \\ 3 \end{bmatrix}$

QUIZ 7 Name Key

Math 1553 D Steinbart November 5, 2015

Work neatly. Justify your answers and use proper notation. SHOW YOUR WORK TO RECEIVE CREDIT! Calculators, cell phones, and other electronic devices are not allowed on this quiz. There is a total of 10 points.

(4 pts) 1. Determine if the matrix is diagonalizable. Justify your answer. Show all work in a readable manner.

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}. \qquad C = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -3 & 0 \\ 1 & 5 & 0 \end{bmatrix}.$$

A is a symmetric matrix. So A is diagonalizable. (In fact, A is orthogonally diagonalizable.)

$P_B(\lambda) = \det(B - \lambda I) = \det\left(\begin{bmatrix} -2-\lambda & 12 \\ -1 & 5-\lambda \end{bmatrix}\right)$

$= (-2-\lambda)(5-\lambda) - ((-1)12)$

$= \lambda^2 - 3\lambda - 10 + 12$

$= \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$

So $P_B(\lambda) = (\lambda-1)(\lambda-2) = 0$ when $\lambda = 1, 2$

So the eigenvalues of the 2x2 matrix B are distinct: $\lambda_1 = 1$, $\lambda_2 = 2$, So B is diagonalizable.

C is lower triangular matrix. So the eigenvalues of C are the diagonal elements: $\lambda_1 = 2$, $\lambda_2 = -3$, $\lambda_3 = 0$

Since this 3x3 matrix C has 3 distinct eigenvalue, C is diagonalizable.

2. Complete this statements: The 5 × 5 matrix A is orthogonal if .... $A^{-1} = A^T$

3. Orthogonally diagonalize the matrix $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$. Show all work. Work neatly.

① Find eigenvalues of A: $P_A(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)^2 - 7^2$

$= \lambda^2 - 4\lambda + 4 - 49 = \lambda^2 - 4\lambda - 45 = (\lambda+5)(\lambda-9)$. λ is an eigenvalue when $P_A(\lambda) = 0$ So the eigenvalues of A are $\lambda_1 = -5$ and $\lambda_2 = 9$.

② Find eigenvectors: $\lambda_1 = -5$: Solve $(A - \lambda_1 I)\underline{x} = \underline{0}$

$A - \lambda_1 I = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} - (-5)I = \begin{bmatrix} 2+5 & 7 \\ 7 & 2+5 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 7 & 7 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 7 & 7 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{7}R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ x_1 basic variable, x_2 free variable

$x_1 + x_2 = 0$, $x_1 = -x_2$; $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ So $\underline{v}_1$ is an eigenvector of A corresponding to $\lambda_1 = -5$.

$\lambda_2 = 9$: Solve $(A - \lambda_2 I)\underline{x} = \underline{0}$. $A - \lambda_2 I = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} - 9I = \begin{bmatrix} 2-9 & 7 \\ 7 & 2-9 \end{bmatrix} = \begin{bmatrix} -7 & 7 \\ 7 & -7 \end{bmatrix}$

$\xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -7 & 7 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to -\frac{1}{7}R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ So x_1 is basic variable, x_2 is free variable. $x_1 - x_2 = 0$, $x_1 = x_2$; $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $\underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda_2 = 9$.

③ Normalize $\underline{v}_1$: $\|\underline{v}_1\| = \left\|\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$. So let $\underline{u}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|} = \frac{\underline{v}_1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\underline{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Normalize $\underline{v}_2$: $\|\underline{v}_2\| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Let $\underline{u}_2 = \frac{\underline{v}_2}{\|\underline{v}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

(Over)

④ Build P (an orthogonal matrix): Since $\underline{u}_1, \underline{u}_2$ are from distinct eigenvalues and A is symmetric we have $\underline{u}_1; \underline{u}_2$ are orthogonal (Note: $\underline{u}_1 \cdot \underline{u}_2 = 0$.) $P = [\underline{u}_1 \ \underline{u}_2] = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

⑤ Build D: D is the diagonal matrix built from the respective eigenvalues: $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & 9 \end{bmatrix}$

⑥ Put it together: So

$$A = PDP^{-1} = PDP^T \quad \text{for}$$

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad D = \begin{bmatrix} -5 & 0 \\ 0 & 9 \end{bmatrix} \quad P^{-1} = P^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Remark: In general, the matrix P is not a symmetric matrix. In this example, P happens to be symmetric.

Remark 2: Other solutions include (there are more!)

(i) $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad D = \begin{bmatrix} -5 & 0 \\ 0 & 9 \end{bmatrix}$

(ii) $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \quad D = \begin{bmatrix} 9 & 0 \\ 0 & -5 \end{bmatrix}$

$\begin{array}{cc} \uparrow & \uparrow \\ \underline{u}_2 & \underline{u}_2 \end{array} \qquad \begin{array}{cc} \uparrow & \uparrow \\ \lambda_2 & \lambda_1 \end{array}$

(iii) $P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, \quad D = \begin{bmatrix} -5 & 0 \\ 0 & 9 \end{bmatrix}$

$\begin{array}{cc} \uparrow & \uparrow \\ \underline{u}_1 & -\underline{u}_2 \end{array}$

Note: Since $\underline{u}_2$ is an eigenvector of A corresponding to $\lambda_2 = 9$ and $\underline{u}_2$ has length 1 ($\|\underline{u}_2\| = 1$) we have

$-\underline{u}_2$ is an eigenvector of A corresponding to $\lambda_2 = 9$ and $\|-\underline{u}_2\| = |(-1)| \, \|\underline{u}_2\| = 1(1) = 1$. So $-\underline{u}_2$ has length 1.