ISYE 3232A Spring 2015 Final - White

I, ________________________________, do swear that I abide by the Georgia Tech Honor Code. I understand that any honor code violations will result in an F.

Signature: ______________________

- Throughout, you will receive full credit if someone with no understanding of probability, set theory, and calculus could simplify your answer to obtain the correct numerical answer, provided that your work is correct.
- **Show your work**. If you do not show your work for a problem, we will give a zero point for the problem even if your answer is correct.
- You will have 2 hours.
- This exam is closed book and closed notes. Calculators are not allowed. No scrap paper is allowed. Make sure that there is nothing on your desk except pens and erasers.
- If you need extra space, use the back of the page and indicate that you have done so.
- **Do not remove any page from the original staple**. Otherwise, there will be 5 points off.
- **We will not select among several answers**. Make sure it is clear what part of your work you want graded. If two answers are given, zero point will be given for the problem.

1. E[Profit] = E[Sales] − {Fixed Cost + E[Ordering Cost] + E[Holding Cost]}.
2. Exponential with rate λ has mean $1/\lambda$, pdf $f(x) = \lambda e^{-\lambda x}$, and CDF $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$.
3. Suppose X_1 and X_2 are independent exponential with rate λ_1 and λ_2. Then $\Pr(X_1 < X_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$ and $\min(X_1, X_2) \sim$ expo with rate $(\lambda_1 + \lambda_2)$ and $\mathsf{E}[X_1 + X_2] = \mathsf{E}[\min(X_1, X_2)] + \mathsf{E}[\max(X_1, X_2)]$.
4. A Poisson process with rate λ has probability
$$\Pr(N(t) - N(s) = n) = \frac{e^{-\lambda(t-s)}(\lambda(t-s))^n}{n!} \text{ for } t > s,$$
and its inter-arrival times are iid exponentially distributed with rate λ. The nth customer's arrival time S_n is Erlang(n, λ) with
$$\Pr(S_n \leq x) = 1 - \sum_{i=0}^{n-1} \frac{e^{-\lambda x}(\lambda x)^n}{n!}.$$
5. A non-homogeneous Poisson process with rate function $\lambda(t)$ has probability
$$\Pr(N(t) - N(s) = n) = \frac{e^{-\int_s^t \lambda(w)dw}(\int_s^t \lambda(w)dw)^n}{n!} \text{ for } t > s.$$
6. A stationary distribution $\underline{\pi}$ should satisfy $\underline{\pi} = \underline{\pi P}$ and $\sum_{i\in S} \pi_i = 1$ for DTMC; and $\underline{0} = \underline{\pi G}$ and $\sum_{i\in S} \pi_i = 1$ for CTMC.
7. Little's Law: $L = \lambda_{\text{eff}} W$ and $L_q = \lambda_{\text{eff}} W_q$.
8. $\rho = \frac{\lambda_{\text{eff}}}{c\mu}$.

1. (15 points) A store operates from Monday to Friday. Inventory left at the end of each Friday can be used to satisfy the demand in the following week. Let $D_n \equiv$ iid demand in the nth period (weeks) and its distribution is given as follows:

d	0	1	2
$\Pr\{D = d\}$	0.2	0.5	0.3

The management decides to use the following inventory policy: whenever the inventory level in Friday evening is 0, an order is made and it will arrive by Monday morning. The order-up-to quantity is set to be 2. Each item sells at \$500 and each item costs \$300 to order. Unmet demand during a week is lost. Any unsold item in a week can be sold in the following week but each unsold item incurs a \$50 re-stocking cost over the weekend. Let X_n represent the inventory level at the beginning of Monday in the nth week.

$(s, S) = (0, 2)$

(a) (3 points) Give the state space of X_n and the transition matrix.

X_n	D_n	End of the week	X_{n+1}
2	0	2	2
	1	1	1
	2	0	2
1	0	1	1
	1	0	2
	2	0	2

$S = \{1, 2\}$

$$\underline{P} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} .2 & .8 \\ .5 & .5 \end{bmatrix} \end{array}$$

(b) (3 points) What is the stationary distribution of the inventory system? (A numerical answer is expected.)

$\pi_1 = .2\pi_1 + .5\pi_2 \longrightarrow .8\pi_1 = .5\pi_2 \longrightarrow \pi_2 = \frac{8}{5}\pi_1$

$\pi_1 + \pi_2 = 1 \longrightarrow \pi_1 + \frac{8}{5}\pi_1 = \frac{13}{5}\pi_1 = 1$

$$\boxed{\pi_1 = \frac{5}{13}, \quad \pi_2 = \frac{8}{13}.}$$

(c) (3 points) If this week starts with inventory level 1, what is the expected profit for the week, including any cost incurred over the weekend. (Hint: Don't forget the ordering cost.)

Order 2 items If left-over at the end of the week is 0.

$E[\text{profit} \mid X_n = 1] = 500\, E[1 \wedge D] - 50\, E[(1-D)^+] - \{2 \times 300 \times .8\}$

$= 500[0 \times 0.2 + 1 \times 0.5] - 50(1 \times 0.2) - 600 \times 0.8.$

(d) (3 points) If this week starts with inventory level 2, what is the expected profit for the week, including any cost incurred over the weekend. (Hint: Don't forget the ordering cost.)

$$E[\text{profit} \mid X_n = 2] = 500\,E[2 \wedge D] - 50\,E[(2-D)^+] - \{2 \times 300 \times 0.3\}$$

Order 2 items if leftover at the end of the week is 0.

$$= 500(0 \times 0.2 + 1 \times 0.5 + 2 \times 0.3) - 50(2 \times 0.2 + 1 \times 0.5) - 600 \times 0.3$$

(e) (3 points) Calculate the long-run average profit per week.

$$(\text{Answer in (c)}) \times \pi_1 + (\text{Answer in (d)}) \times \pi_2$$

$$= \frac{5}{13}\{\text{Answer in (c)}\} + \frac{8}{13}\{\text{Answer in (d)}\}$$

2. (10 points) A airline reservation system has two servers, A and B with one repair person. Server A's up time is iid exponential with mean 30 hours and server B's up time is iid exponential with mean 10 hours. It takes exponential time with mean 2 hours to fix server A or B. A repair person can fix only one machine at a time.

(a) (2.5 points) Suppose that the repair person is currently fixing server B. If the repair person cannot fix a server in 5 hours, the server is replaced with a new one. What is the probability that server B is replaced with a new one?

$$Pr(X > 5) = e^{-5 \cdot \frac{1}{2}} = e^{-2.5}$$

(b) (2.5 points) Server A has been up for 100 hours. Server B is new and just started to run. What is the probability that Server B breaks before Server A?

$$Pr(X_B < \text{server A's remaining up time}) = Pr(X_B < X_A) = \frac{\frac{1}{10}}{\frac{1}{10} + \frac{1}{30}}$$

(c) (5 points) Some change has been made to the reservation system, so the system will be up as long as at least one server is running. Suppose that both servers are new and start running at the same time. What is the expected time (*in hours*) until the reservation system is down?

$$E[\max(X_A, X_B)] = E[X_A] + E[X_B] - E[\min(X_A, X_B)]$$

$$= 30 + 10 - \frac{1}{\frac{1}{30} + \frac{1}{10}}$$

4/

$\lambda = 3/\min$

3. (15 points) Consider a small call center that opens 9 am and closes 4 pm. Calls arrive following a Poisson process with rate 3 calls per minute.

   (a) (3 points) What is the probability that exactly 2 calls arrive from 9:02 am to 9:04 am?

$$\Pr(N(4)-N(2)=2) = \frac{e^{-6}\, 6^2}{2!}$$

   (b) (4 points) Given that there is no arrival by 9:10am, what is the expected arrival *clock* time of the 6th call?

From 9:10 am, we need additional $\frac{1}{3} \times 6 = 2$ mins.

$$\boxed{\therefore\ 9{:}12\ \text{am}}$$

   (c) (4 points) It is known that 60% of calls are from female customers and 40% are from male customers. What is the expected inter-arrival time (*in minutes*) between calls from female customers?

$$PP(3) \xrightarrow{f} PP(1.8)$$

$$PP(3) \xrightarrow{m} PP(1.2)$$

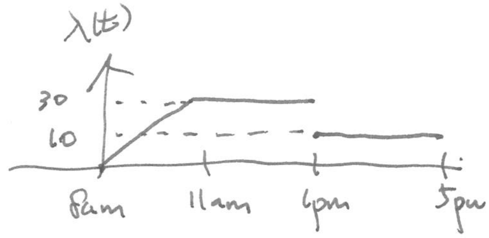
$$\frac{1}{\lambda_f} = \frac{1}{1.8} \text{ mins.}$$

   (d) (4 points) It is known that 60% of calls are from female customers and 40% are from male customers. What is the probability that exactly 5 calls from female customers and 3 calls from male customers arrive from 9:02 am to 9:04 am?

$$\Pr(N_f(4)-N_f(2)=5,\ N_m(4)-N_m(2)=3)$$

$$\overset{N_f(t) \perp\!\!\!\perp N_m(t)}{=\!=} \Pr(N_f(2)=5)\Pr(N_m(2)=3)$$

$$= \frac{e^{-3.6}(3.6)^5}{5!}\ \frac{e^{-2.4}(2.4)^3}{3!}.$$



4. (10 points) Siegbert runs a hot dog stand that opens at 8 am. From 8 am until 11 am customers seem to arrive, on the average, at a steadily increasing rate that starts with an initial rate of 0 customer per hour at 8 am and reaches a maximum of 30 customers per hour at 11 am. From 11 am until 1 pm the (average) rate seems to remain constant at 30 customers per hour. However, the (average) arrival rate then drops to constant 10 customers per hour from 1 pm until closing time at 5 pm. We assume that the numbers of customers arriving at Siegbert's stand during disjoint time periods are independent.

(a) (5 points) What is the probability that 100 customers arrive between 8 am and noon on Monday morning?

$$E[N(12pm) - N(8am)] = \frac{1}{2} \times 3 \times 30 + 30 = \frac{90}{2} + 30 = 45 + 30 = 75$$

$$Pr(N(12pm) - N(8am) = 100) = \frac{e^{-75}(75)^{100}}{100!}$$

(b) (5 points) Suppose that you starts to observe Siegbert's stand from noon (12 pm). Consider the second customer that you observe from noon (12 pm). What is the probability that this customer arrives by 1:30?

$$E[N(1{:}30) - N(12)] = 30 + \frac{1}{2} \times 10 = 35.$$

$$Pr(N(1{:}30) - N(12pm) \geq 2) = \sum_{n=2}^{\infty} \frac{e^{-35} 35^n}{n!}$$

$$= 1 - (e^{-35} + 35e^{-35}) = 1 - 36e^{-35} \,//$$

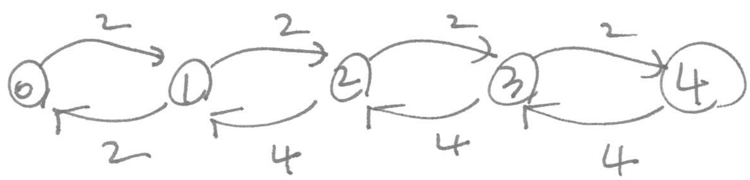
M/M/2/4

$\lambda = 2/hr$

$\mu = 2/hr$

5. (35 points) Calls arrive to DFK Ambulance Services according to a Poisson process with rate 2 per hour. DFK Ambulance Services operates 2 ambulances, and data suggests that the amount of time that an ambulance devotes to each call is exponentially distributed with a mean of 30 minutes. The service time begins when the ambulance is dispatched to the call and ends when the ambulance is available for dispatch to another call. DFK has at most 4 calls waiting or in service at any point in time; if there are already 4 calls in the system, additional calls are rerouted to another ambulance service. We will let $X(t)$ denote the number of calls in the system at time t. Thus, the state space is $S = \{0, 1, 2, 3, 4\}$. Then $X = \{X(t) : t \geq 0\}$ is a continuous time Markov chain.

(a) (3 points) Draw a rate diagram. Use rate in the units of *per hour*.



(b) (3 points) What is the expected holding time (*in hours*) for state 3?

$\frac{1}{6}$ hrs

(c) (8 points) Find the stationary distribution of the Markov chain. (A numerical answer is expected.)

$2\pi_0 = 2\pi_1 \Rightarrow \pi_1 = \pi_0$

$4\pi_1 = 2\pi_0 + 4\pi_2 \Rightarrow 2\pi_1 = 4\pi_2 \rightarrow \pi_2 = \frac{1}{2}\pi_1 = \frac{1}{2}\pi_0$

$(\cancel{4}+2)\pi_2 = \cancel{2\pi_1} + 4\pi_3 \Rightarrow \pi_3 = \frac{1}{2}\pi_2 = \frac{1}{4}\pi_0$

$(\cancel{4}+2)\pi_3 = \cancel{2\pi_2} + 4\pi_4 \Rightarrow \pi_4 = \frac{1}{2}\pi_3 = \frac{1}{8}\pi_0$

$\sum_{i=0}^{4} \pi_i = 1 \Rightarrow \pi_0\left[1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right] = 1$ 16+4+2+1

$$\boxed{\begin{array}{l} \pi_0 = \frac{8}{23} \\ \pi_1 = \frac{8}{23},\ \pi_2 = \frac{4}{23},\ \pi_3 = \frac{2}{23},\ \pi_4 = \frac{1}{23}. \end{array}}$$

If you could not get $\underline{\pi}$ from the previous problem, continue next problems assuming that $\underline{\pi} = (1/15, 2/15, 3/15, 4/15, 5/15)$. If you decide to use this $\underline{\pi}$, print your initial here:____________________

(d) (3 points) What is the steady state probability that an arriving call will be accepted by DFK? (A numerical answer is expected.)

$$1 - \pi_4 = \frac{22}{23}.$$

(e) (3 points) What is the effective arrival rate of calls (*per hour*) which enters the call center? (A numerical answer is expected.)

$$2(1-\pi_4) = 2\cdot\left(\frac{22}{23}\right)$$

W_q

(f) (5 points) Of those calls that are accepted, what is the steady state average length of time (*in hours*) the call waits in the queue until an ambulance is dispatched to that call? (A numerical answer is expected.)

$$W_q = \frac{l_q}{\lambda_{eff}} = \frac{1\cdot\pi_3 + 2\cdot\pi_4}{2\cdot\left(\frac{22}{23}\right)} = \frac{\frac{2}{23} + 2\cdot\frac{1}{23}}{2\cdot\frac{22}{23}}$$

$$= \frac{4}{44} = \frac{1}{11} \text{ hrs } (\approx 5.5 \text{ mins})$$

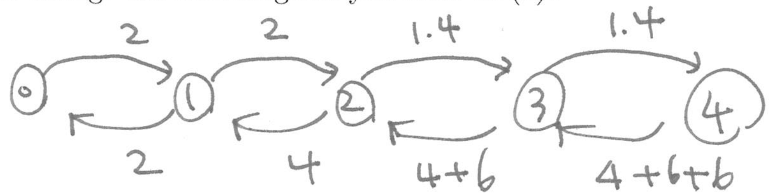
(g) (6 points) Suppose Caller X enters the system, and there are 5 calls already in the system. What is the probability that an ambulance will be dispatched to Caller X within 10 minutes? (Assume that sys capacity is now 6 not 4.)

~~Let S~~ 1. Note that Caller X's wait time is the sum of 4 iid exponential time with rate 4/hr (i.e., sum of four $\min(X_1, X_2) \sim \exp(4)$).

2. Thus, Caller X's wait time $\sim$ Erlang$(\overset{n}{4}, \overset{\lambda}{4})$

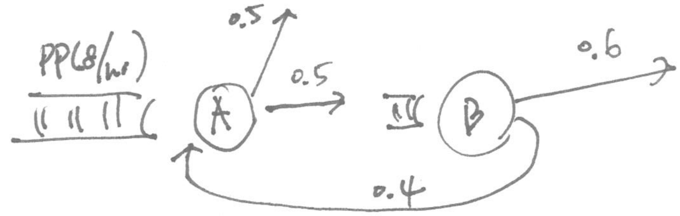
Then $\Pr($Caller X's wait time $< \frac{1}{6}) = 1 - \sum_{i=0}^{3} \frac{e^{-4/6}\left(\frac{4}{6}\right)^n}{n!}$ from the front page which gives CDF of Erlang.

(h) (4 points) Only 70% calls wait on hold if they know that they have to wait. Also, those calls already on hold have exponential patience with mean 10 minutes. How does it change the rate diagram you draw in (a)?



Alternatively, the # departure from the system $N(t) \sim PP(4/hr)$.

$$\therefore \Pr(N(\tfrac{1}{6}) \geq 4) = \sum_{n=4}^{\infty} \frac{e^{-\frac{4}{6}}(4/6)^n}{n!} = 1 - \sum_{n=0}^{3} \frac{e^{-4/6}(4/6)^n}{n!}$$



6. (15 points) A production line has two stations: A and B. Jobs arrive following a Poisson process with rate 8 per hour. All jobs are processed in Station A. 50% of jobs leave the production line but the other 50% moves to Station B. At station B, only 60% of inspected jobs pass inspection and leave the production line, but the other 40% are sent back to Station A for re-processing. Jobs can go through stations A and B multiple times.

   Station A's processing time is exponential with mean 10 minutes and B's processing time is exponential with mean 5 minutes. Each station has infinite queue capacity. $\mu_A = 6/hr$

   (a) (5 points) What is the arrival rate (*per hour*) to station A and B assuming the production line is stable? (A numerical answer is expected.) $\mu_B = 12/hr$

   $$\lambda_A = 8 + 0.4\lambda_B \rightarrow \lambda_A = 8 + 0.2\lambda_A \rightarrow 0.8\lambda_A = 8$$

   $$\lambda_B = 0.5\lambda_A$$

   $$\therefore \lambda_A = 10/hr, \quad \lambda_B = 5/hr$$

   If you could not get arrival rates from the previous problem, continue next problems assuming that arrival rates to A and B are 7 per hour and 9 per hour, respectively. If you decide to use these arrival rates, print your initial here:________________

   (b) (5 points) To make the production line stable, what is the minimum number of servers that we need? (A numerical answer is expected.)

   2 machines for A

   3 machines for B

   (c) (5 points) When we hire the numbers of machines you suggested in part (b), we observed that there are 5.5 jobs in Station A and 0.7 job in Station B on average. What is the long-run average system time (*in hours*)? (A numerical answer is expected.)

   $$L_{TOTAL} = 5.5 + 0.7 = 6.2$$

   $$W_{TOTAL} = \frac{L_{TOTAL}}{\lambda} = \frac{6.2}{8} \text{ hrs.}$$

Turn the page for a bonus question.

Bonus: (5 points) In class, we learned how to use QueueingCalculator.xlsx. A table below shows the list of queueing models the excel tool can handle and required parameters for each model.

Notation	MG1	MMc	MGc	MMcN	MMcKK
Parameters	lambda mu sigma2	lambda mu c	lambda mu c sigma2	lambda mu c N	lambda mu c K

For each situation below, choose the most adequate queueing notation and give values of the required parameters.

1. (2.5 points) A Drive-Thru of Bank of America has 2 ATM machines with a single queue line. Cars arrive with exponential interarrival time with mean 10 minutes and spend normal service time with mean 5 minutes and variance 4 minutes^2.

M/~~G~~/2 so use $\boxed{\text{MGc}}$

lambda $= \frac{1}{10}$/min (or 6/hr)

mu $= \frac{1}{5}$/min (or 12/hr)

c = 2

sigma2 = 4 (or $\frac{4}{3600}$ hrs^2)

2. (2.5 points) A small job shop has 10 identical machines and 2 identical repair persons. Each machine's up time is iid exponentially distributed with mean 20 days. Each repair person can fix a machine with exponential time with $\underbrace{\text{mean 12 hours}}_{=0.5\text{day}}$. Each repair person fixes one machine at a time.

MMcKK

lambda $= \frac{1}{20}$

mu = 2

c = 2

K = 10