ISyE 3232A/B - Fall 2012
Homework 12 Solutions

Problem 1

a) $\lambda = 0.004$

b) The expected number of defects in the first spool is $E[N(100)] = 0.4$.

c) Recall that Poisson process has stationary increments. Then, $E[N(900) - N(800)] = E[N(100)] = 0.4$.

d)

$$P\{N(100) = 0\} = e^{-0.4}$$

e)

$$P\{N(1000) - N(900) = 1\} = P\{N(100) = 1\} = \frac{0.4^1 e^{-0.4}}{1!} = 0.4e^{-0.4}$$

f) Recall that Poisson process has independent increments.

$$P\{N(1000) - N(900) = 1 | N(900) = 0\} = P\{N(100) = 1\} = 0.4e^{-0.4}$$

g)

$$P\{N(1000) = 2\} = \frac{4^2 e^{-4}}{2!} = 8e^{-4}$$

h) Let T_1 be the time when the first defect occurs.

$$E[T_1] = \frac{1}{\lambda} = 250$$

i)

$$E[S_{10}] = \sum_{i=1}^{10} E[T_i] = 2500$$

j)

$$E[T_1 | T_1 > 100] = 100 + E[T_1] = 350$$

k) Let T_1' be the time when the first defect occurs in the second kilometer of wire. Since the Poisson process regenerates at all times, we have

$$E[T_1'] = 250$$

l) The number of major defects is also a Poisson process with rate $\lambda * 1/4 = 0.001$.

$$E[M(500)] = 0.5$$

m) Note that $M(t)$ and $L(t)$ are independent Poisson processes.

$$P\{M(1000) \geq 1 | L(1000) = 0\} = 1 - P\{M(1000) = 0\} = 1 - e^{-1}$$

n) Let A denote the event that the first 3 defects are minor. Then

$$P\{A\} = \left(\frac{3}{4}\right)^3$$

o) Let the expected cost due to defects of the first spool and of the first ten spools be denoted by E_1 and E_{10}, respectively. It follows that

$$E_1 = E[L(100)]P\{M(100) = 0\} + 10P\{M(100) > 0\} = (\frac{3}{4})(0.004)(100)e^{-0.1} + 10(1 - e^{-0.1})$$

$$E_{10} = 10E_1$$

Problem 2

(a) Let $N(t)$ denote the number of customer arrivals until time t. Since the arrival rate is time dependent, non-homogeneous Poisson process would be a good probability model for customer arrivals.

(b) The arrival rate is

$$\lambda(t) = \begin{cases} (15/3)t - 35 & 8 \leq t \leq 11 \\ 20 & 11 < t \leq 13 \\ -2t + 46 & 13 < t \leq 17 \end{cases}$$

(c) The probability of k arrivals between 8:30 and 9:30 is just

$$\begin{aligned} & \frac{(\int_{8.5}^{9.5} \lambda(s)ds)^k e^{-\int_{8.5}^{9.5} \lambda(s)ds}}{k!} \\ = & \frac{(\int_{8.5}^{9.5} (5s - 35)ds)^k e^{-\int_{8.5}^{9.5} (5s-35)ds}}{k!} \\ = & (10)^k e^{-10}/k!. \end{aligned}$$

Then $P(k = 0) \approx 0$

(d) Expected number of arrivals on a day is given by

$$\begin{aligned} E[N(17)] &= \int_8^{17} \lambda(s)ds \\ &= \int_8^{11} (5s-35)ds + 20(2) + \int_{13}^{17} (-2s+46)ds = 141.5 \approx 142. \end{aligned}$$

(e) To do this, note that (here minutes need to be changed into hours, e.g. $10mins = \frac{1}{6}$ hours)

$$P(T > \frac{1}{6}) = P(N(13+\frac{1}{6}) - N(13) = 0)$$

and

$$P(T > \frac{1}{3}) = P(N(13+\frac{1}{3}) - N(13) = 0).$$

Then

$$P(T > \frac{1}{6}) = e^{-\int_{13}^{13+\frac{1}{6}} (46-2t)dt} = e^{-3.323}.$$

Here T is not an exponential random variable, since it's not memoryless (try comparing $P(T > 15 \mid T > 5)$ with $P(T > 10)$).