Instructions: *Print* your name, student ID number and recitation session in the spaces below.

Name: ____

Student ID:____________________________________

Recitation session:_______________________________

Practice Exam 2, Calculus III (Math 2551)

Question	Points
1)	
2)	
3)	
4)	
5)	

Problem 1(20 points). Calculations.

(a) (5 pt) Find the directional derivative of

$$f(x,y,z) = xy + yz + zx$$

at $P(1,-1,1)$ in the direction of $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(b) (5 pt) Find the rate of change of $f(x,y) = xe^y + ye^{-x}$ along the curve $\vec{r}(t) = \ln t\ \mathbf{i} + t \ln t\ \mathbf{j}$.

Solution:

(a)Since

$$\nabla f = (y+z)\,\mathbf{i} + (x+z)\,\mathbf{j} + (y+x)\,k,$$

$$\nabla f(1,-1,1) = 2j,\ \mathbf{u} = \frac{\sqrt{6}}{6}(\mathbf{i} + 2\mathbf{j} + \mathbf{k}),$$

so

$$f'_u(1,-1,1) = \nabla f(1,-1,1) \cdot \mathbf{u} = \frac{2}{3}\sqrt{6}.$$

(b)

$$\nabla f = \left(e^y - ye^{-x}\right) i + \left(xe^y + e^{-x}\right) j,$$

$$\nabla f(\vec{r}(t)) = \left(t^t - \ln t\right) i + \left(t^t \ln t + \frac{1}{t}\right) j,$$

$$\frac{df}{dt} = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) = t^t\left(\frac{1}{t} + \ln t + (\ln t)^2\right) + \frac{1}{t}.$$

(c)(5 pt) Find $\partial u/\partial s$ for $u = x^2 - xy,\ x = s\cos t, y = t\sin s$.

(d)(5 pt) Find dy/dx if $x\cos(xy) + y\cos x = 2$.

Solution:

(c)

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \\ &= (2x - y)\cos t + (-x)\, t\cos s \\ &= 2s\cos^2 t - t\sin s\cos t - st\cos s\cos t.\end{aligned}$$

(d) Set $u = x\cos(xy) + y\cos x - 2$,then

$$\begin{aligned}\frac{\partial u}{\partial x} &= \cos(xy) - xy\sin(xy) - y\sin x, \\ \frac{\partial u}{\partial y} &= -x^2\sin(xy) + \cos x,\end{aligned}$$

$$\frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\cos(xy) - xy\sin(xy) - y\sin x}{x^2\sin(xy) - \cos x}.$$

Problem 2(20 pt) Consider the function $f(x,y,z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$.

(a) (6 points) Find the equation for the tangent plane to the level surface $f = 4$ at the point $P(1,4,1)$.

(b) (6 points) Find the equation for the normal line to $f = 4$ at $P(1,4,1)$.

(c) (8 points) Use differentials to estimate $f(0.9, 4.1, 1.1)$.

Solution:

(a)

$$\nabla f = \frac{1}{2\sqrt{x}}i + \frac{1}{2\sqrt{y}}j + \frac{1}{2\sqrt{z}}k, \quad \nabla f(1,4,1) = \frac{1}{2}i + \frac{1}{4}j + \frac{1}{2}k,$$

Tangent plane:

$$\frac{1}{2}(x-1) + \frac{1}{4}(y-4) + \frac{1}{2}(z-1) = 0.$$

(b) The normal line:

$$x = 1 + \frac{1}{2}t, \; y = 4 + \frac{1}{4}t, \; z = 1 + \frac{1}{2}t.$$

(c)

$$f(0.9, 4.1, 1.1) = f(1,4,1) + df,$$
$$df = \frac{1}{2}(-0.1) + \frac{1}{4}(0.1) + \frac{1}{2}(0.1) = 0.025.$$

Thus, the estimate of $f(0.9, 4.1, 1.1)$ is 4.025.

Problem 3 (20 pt) Find the area of the largest rectangle with edges parallel to the coordinate axes that can be inscribed in the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$.

Solution: Use Lagrangian multiplier method. Set the coordinates of the corner points of the rectangle to be $(x,y),(-x,y),(-x,-y),(x,-y)$. We need to maximize the area $f(x,y)=4xy$ with the side condition

$$g(x,y)=\frac{x^2}{9}+\frac{y^2}{4}-1=0.$$

Since

$$\nabla f=4y\mathbf{i}+4x\mathbf{j},\nabla g=\frac{2}{9}x\mathbf{i}+\frac{1}{2}y\mathbf{j},$$

by Lagrangian multiplier we solve the following system

$$4y=\lambda\frac{2}{9}x,\quad 4x=\lambda\frac{1}{2}y,\quad g(x,y)=0.$$

We have

$$\lambda=12,\ x=\frac{3}{2}\sqrt{2},\ y=\sqrt{2},$$

and the maximal area is 12.

Problem 4 (20 points) Find the absolute extreme values taken on $f(x,y) = -\frac{2y}{x^2+y^2+1}$ on the set $D = \left\{(x,y) : x^2+y^2 \leq 4\right\}$.

Solution: Critical points:

$$\nabla f = \frac{4xy}{(x^2+y^2+1)^2}\mathbf{i} + \frac{2y^2-2x^2-2}{(x^2+y^2+1)^2}\mathbf{j} = 0$$

at $P_1 = (0,1)$ and $P_2 = (0,-1)$ in D.

Next, we consider the boundary of D. We parametrize the boundary circle by

$$C : \vec{r}(t) = 2\cos t\ \mathbf{i} + 2\sin t\ \mathbf{j},\ \ t \in [0, 2\pi].$$

The value of f on the boundary is given by the function

$$F(t) = f(\vec{r}(t)) = -\frac{4}{5}\sin t.$$

$$F'(t) = -\frac{4}{5}\cos t = 0$$

at $t = \frac{1}{2}\pi$ and $t = \frac{3}{2}\pi$.Thus the critical points on the boundary C are $P_3 = \vec{r}\left(\frac{1}{2}\pi\right) = (0,2)$ and $P_4 = \vec{r}\left(\frac{3}{2}\pi\right) = (0,-2)$. Evaluating f at points $P_1, \cdots, P_4$:

$$f(0,1) = -1,\quad f(0,-1) = 1,\ \ f(0,2) = -\frac{4}{5},\quad f(0,-2) = \frac{4}{5}.$$

So f takes on its absolute maximum of 1 at $(0,-1)$ and its absolute minimum of -1 at $(0,1)$.

Problem 5 (20 points)

(a) (10 points) Find the area of the region enclosed by the parabolas $x = y^2$ and $x = 2y - y^2$.

(b) (10 points) Change the Cartesian integral

$$\int_0^1 \int_x^{\sqrt{2-x^2}} (x+2y)\ dydx$$

into an equivalent polar integral. Then evaluate the polar integral.

Solution:

(a) First, we find the intersection points by $y^2 = 2y - y^2$. The two intersection points are $(0,0)$ and $(1,1)$. So the enclosed region Ω is

$$\Omega = \left\{0 \le y \le 1,\ y^2 \le x \le 2y - y^2\right\}.$$

The area is

$$\begin{aligned}\int\int_\Omega dxdy &= \int_0^1 \int_{y^2}^{2y-y^2} dxdy = \int_0^1 \left(2y - 2y^2\right) dy \\ &= \frac{1}{3}.\end{aligned}$$

(b) The region is

$$\Omega = \left\{0 \le x \le 1,\ x \le y \le \sqrt{2 - x^2}\right\}.$$

In polar coordinates, it becomes

$$\Gamma = \left\{0 \le r \le \sqrt{2},\ \frac{\pi}{4} \le \theta \le \frac{\pi}{2}.\right\}$$

So

$$\begin{aligned}&\int_0^1 \int_x^{\sqrt{2-x^2}} (x+2y)\ dydx \\ =& \int_0^{\sqrt{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (r\cos\theta + 2r\sin\theta)\, rd\theta dr \\ =& \int_0^{\sqrt{2}} r^2 dr \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos\theta + 2\sin\theta)\, d\theta \\ =& \frac{1}{3} r^3|_0^{\sqrt{2}}\ (\sin\theta - 2\cos\theta)\,|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ =& \frac{2}{3}\left(\sqrt{2} + 2\right).\end{aligned}$$