1. For the following power series, find the radius and interval of convergence. Give a complete justification for your solution. (20 points)

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{(n+1)3^{2n}}$$

Ratio test: $a_n = \frac{1}{(n+1)3^{2n}}$, $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+2)3^{2n+2}}}{\frac{1}{(n+1)3^{2n}}} = \frac{n+1}{n+2} \cdot \frac{1}{9}$ 4 points

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{9}$ 2 points

Then $R = 9$ (radius of convergence) 2 points

The series converges for $x \in (4-9, 4+9) = (-5, 13)$ 2 points

For $x = -5$, we have 2 points

$\sum_{n=1}^{\infty} \frac{(-9)^n}{(n+1)9^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$, is convergent 1 point

by the alternating series test. 1 point

For $x = 13$,

$\sum_{n=1}^{\infty} \frac{9^n}{(n+1)9^n} = \sum_{n=1}^{\infty} \frac{1}{n+1}$ 2 points

1 point

Since $\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1$, by limit comparison with the harmonic series, then $\sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$ 1 point

Interval of convergence: $[-5, 13)$ 2 points

2. (a) Compute the Taylor polynomial of order 3 generated by the function $f(x) = \sqrt{2}\sin x$, centered at $a = \frac{\pi}{4}$. (13 points)

$f'(x) = \sqrt{2}\cos x$, $f''(x) = -\sqrt{2}\sin x$, $f'''(x) = -\sqrt{2}\cos x$ 2 points

$f(\frac{\pi}{4}) = \sqrt{2}\cdot\frac{\sqrt{2}}{2} = 1$ $a_0 = 1$ 2 points

$f'(\frac{\pi}{4}) = \sqrt{2}\cdot\frac{\sqrt{2}}{2} = 1$ $a_1 = 1$ 2 points

$f''(\frac{\pi}{4}) = -\sqrt{2}\cdot\frac{\sqrt{2}}{2} = -1$ $a_2 = \frac{-1}{2}$ 2 points

$f'''(\frac{\pi}{4}) = -\sqrt{2}\cdot\frac{\sqrt{2}}{2} = -1$ $a_3 = \frac{-1}{3!} = \frac{-1}{6}$ 2 points

$P_3(x) = 1 + (x-\frac{\pi}{4}) - \frac{1}{2}(x-\frac{\pi}{4})^2 - \frac{1}{6}(x-\frac{\pi}{4})^3$ 3 points

(b) Find a bound for the error of the approximation for $x \in [\frac{\pi}{4}, \frac{\pi}{3}]$. (10 points)

$$R_3(x) = \frac{f^{(4)}(c)(x-\frac{\pi}{4})^4}{4!}$$ 2 points

$f^{(4)}(c) = \sqrt{2}\sin c$, then $|f^{(4)}(c)| \leq \sqrt{2}$ 2 points

2 points

Then

$$|R_3(x)| \leq \frac{|f^{(4)}(c)|\,|x-\frac{\pi}{4}|^4}{4!} \leq \frac{\sqrt{2}\,|\frac{\pi}{3}-\frac{\pi}{4}|^4}{4!}$$

$$= \frac{\sqrt{2}\cdot(\frac{\pi}{12})^4}{24} = \frac{\sqrt{2}\,\pi^4}{12^4\cdot 24}$$ 4 points

3. Find the Maclaurin series expansion for the following functions. Write them in sum notation.

(a) $f(x) = \frac{x^2}{3+2x}$. $= x^2 \cdot \frac{1}{3+2x} = x^2 \cdot \frac{1}{3(1+\frac{2}{3}x)}$ 2 points (11 points)

$= \frac{x^2}{3} \cdot \frac{1}{1-(\frac{-2}{3}x)} = \frac{x^2}{3} \sum_{n=0}^{\infty} \left(\frac{-2}{3}x\right)^n$ 3 points

2 points

$= \frac{x^2}{3} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^n} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^{n+1}} x^{n+2}$ 4 points

(b) $g(x) = \frac{1}{x}\ln(1+x^3)$. (11 points)

$= \frac{1}{x} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x^3)^n}{n}$ 5 points $\left(\text{Since } \ln(1+y) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} y^n\right)$

half wrong formula −3

$= \frac{1}{x} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{3n}$ 3 points

$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{3n-1}$ 3 points

wrong method, looks OK, wong ans +5

undef. / 3

4. Consider the following vectors and matrix

$$\vec{\mathbf{u}} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}; \quad \vec{\mathbf{v}} = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}; \quad A = \begin{pmatrix} -2 & -1 \\ 2 & 0 \\ 5 & 3 \end{pmatrix};$$

Compute the following

(a) $A^T\vec{\mathbf{u}}$. (8 points)

3 points

$$= \begin{pmatrix} -2 & 2 & 5 \\ -1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2\cdot 1 + 2\cdot 2 + 5\cdot(-1) \\ -1\cdot 1 + 0\cdot 2 + 3\cdot(-1) \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

3 points 2 points

(b) $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$. (7 points)

2 points

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -1 \\ -3 & 0 & 4 \end{vmatrix} = \hat{\imath}(2\cdot 4 - 0\cdot(-1)) - \hat{\jmath}(1\cdot 4 - (-3)(-1)) + \hat{k}(1\cdot 0 - (-3)\cdot 2)$$

2 points 2 points

$$= 8\hat{\imath} - \hat{\jmath} + 6\hat{k} = \begin{pmatrix} 8 \\ -1 \\ 6 \end{pmatrix}$$

1 point

(c) $\text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}})$ (Projection of $\vec{\mathbf{u}}$ onto $\vec{\mathbf{v}}$). (8 points)

$$= \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$$

1 points

$$\vec{u}\cdot\vec{v} = 1\cdot(-3) + 2\cdot 0 + (-1)\cdot 4 = -7$$ 2 points

$$\|\vec{v}\| = \sqrt{(-3)^2 + 0^2 + 4^2} = \sqrt{25} = 5$$ 2 points

$$\Rightarrow \text{proj}_{\vec{v}}(\vec{u}) = \frac{-7}{25}\cdot\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{21}{25} \\ 0 \\ \frac{-28}{25} \end{pmatrix}$$ 3 points

5. Compute the following determinant, by using row operations and properties of determinants to reduce it to an upper triangular matrix. (12 points)

$$\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}$$

$$\begin{matrix} r_3 \to r_3 + r_1 \\ = \\ r_4 \to r_4 - 3r_1 \end{matrix} \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 1 & 5 & 5 \\ 0 & 2 & 7 & 3 \end{vmatrix}$$ 4 points

$$\begin{matrix} r_3 \to r_3 - r_2 \\ = \\ r_4 \to r_4 - 2r_2 \end{matrix} \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & -5 \end{vmatrix}$$ 4 points

$$\begin{matrix} r_3 \leftrightarrow r_4 \\ = \end{matrix} - \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$ 3 points

$$= -1 \cdot 1 \cdot (-3) \cdot 1 = 3$$ 1 point

By cofactors

$$1 \cdot \begin{vmatrix} 1 & 5 & 4 \\ 2 & 8 & 5 \\ -1 & -2 & 3 \end{vmatrix} - (-1) \begin{vmatrix} 0 & 5 & 4 \\ -1 & 8 & 5 \\ 3 & -2 & 3 \end{vmatrix} - 3 \begin{vmatrix} 0 & 1 & 4 \\ -1 & 2 & 5 \\ 3 & -1 & 3 \end{vmatrix}$$

4 points 4 points 4 points

6. [**Bonus**] Use Taylor series to evaluate the limit (5 points)

$$\lim_{x\to 0}\frac{\ln(1+x^2)}{2-e^{x^2}-\cos x}$$

$$= \lim_{x\to 0}\frac{x^2-\frac{x^4}{2}+\frac{x^6}{3}-\frac{x^8}{4}+\cdots}{2-\left(1+x^2+\frac{x^4}{2}+\cdots\right)-\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\cdots\right)}$$

3 points (series)

$$= \lim_{x\to 0}\frac{x^2-\frac{x^4}{2}}{2-1-x^2-1+\frac{x^2}{2}}$$

$$= \lim_{x\to 0}\frac{x^2-\frac{x^4}{2}}{-x^2+\frac{x^2}{2}}$$

$$= \lim_{x\to 0}\frac{x^2\left(1-\frac{x^2}{2}\right)}{x^2\left(-1+\frac{1}{2}\right)}$$

1 point

$$= \lim_{x\to 0}\frac{1-\frac{x^2}{2}}{-\frac{1}{2}}$$

$$= \frac{1}{-1/2}$$

$$= -2.$$

1 point