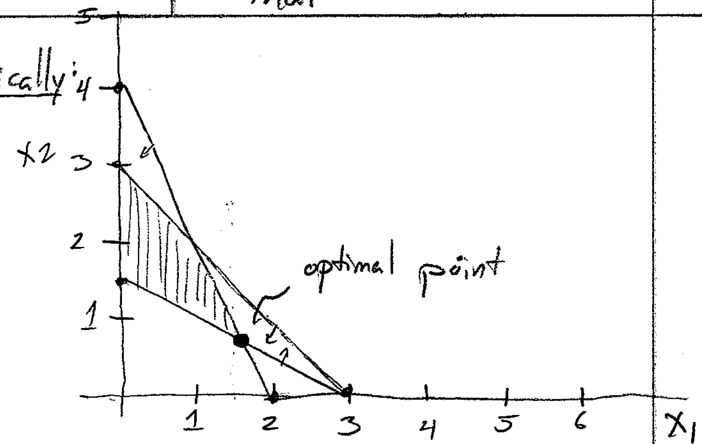
ISYE 3133 | Solutions | Final Exam

1) $\max\ z = 3x_1 - 2x_2$

a) s.t.
$$\begin{aligned} x_1 + 2x_2 &\geq 3 \\ 2x_1 + x_2 &\leq 4 \\ x_1 + x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Graphically:



Simplex:

$$\begin{array}{cccccc|c} & -3 & 2 & 0 & 0 & 0 & 0 \\ \hline & -1 & \boxed{-2} & 1 & 0 & 0 & -3 \\ & 2 & 1 & 0 & 1 & 0 & 4 \\ & 1 & 1 & 0 & 0 & 1 & 3 \end{array}$$

Dual simplex needs to occur to get a feasible solution. Pivot on row 1. Ratio of column 1 is 3, column 2 is $\frac{1}{\ }$, $1 < 3$, so pivot on row 1 column 2.

$$\begin{array}{ccccc|c} -4 & 0 & 1 & 0 & 0 & -3 \\ \hline 1/2 & 1 & -1/2 & 0 & 0 & 3/2 \\ \boxed{3/2} & 0 & 1/2 & 1 & 0 & 5/2 \\ 1/2 & 0 & 1/2 & 0 & 1 & 3/2 \end{array}$$

RHS > 0, feasible.
Row 0 coefficient of $x_1 < 0$, enter it in basis.
Row 1 ratio = 3, row 2 = $5/3$, row 3 = 3. Row 2

Pivot on column 1, row 2

$$\begin{array}{ccccc|c} 0 & 0 & 7/3 & 8/3 & 0 & 11/3 \\ \hline 0 & 1 & -2/3 & -1/3 & 0 & 2/3 \\ 1 & 0 & 1/3 & 2/3 & 0 & 5/3 \\ 0 & 0 & 1/2 & -1/3 & 1 & 2/3 \end{array}$$

Optimal Solution.

$x_1 = 5/3$, $x_2 = 2/3$, $z = 11/3$
$s_3 = 2/3$.

b) Write the dual:

$$\begin{aligned} \min\ & 3\pi_1 + 4\pi_2 + 3\pi_3 = w \\ \text{s.t.}\ & \pi_1 + 2\pi_2 + \pi_3 \geq 3 \\ & 2\pi_1 + \pi_2 + \pi_3 \geq -2 \\ & \pi_1 \leq 0,\ \pi_2, \pi_3 \geq 0 \end{aligned}$$

c) From Complementary slackness: If x, π optimal then
$x_1 e_1 = 0$, $x_2 e_2 = 0$, $\pi_1 s_1 = 0$, $\pi_2 s_2 = 0$, $\pi_3 s_3 = 0$.

Since $x_1, x_2, s_3 > 0$ in primal optimal, $e_1, e_2, \pi_3 = 0$ in dual optimal

$$\begin{aligned} \pi_1 + 2\pi_2 + \pi_3 - e_1 &= 3 \\ 2\pi_1 + \pi_2 + \pi_3 - e_2 &= -2 \end{aligned} \Rightarrow \begin{aligned} \pi_1 + 2\pi_2 &= 3 \\ 2\pi_1 + \pi_2 &= -2 \end{aligned} \Rightarrow \boxed{\begin{aligned} \pi_1 &= -7/3 \\ \pi_2 &= 8/3 \end{aligned}}$$

$(\pi_2 = -2 - 2\pi_1)$
$(\pi_1 - 4 - 4\pi_1 = 3)$

2) Given our optimal solution we know Basis is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Therefore $c_{BV} = \begin{bmatrix} 10 & 16 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 8 \\ 4 & 5 \end{bmatrix}$ $B^{-1} = \begin{bmatrix} -5/12 & 2/3 \\ 1/3 & -1/3 \end{bmatrix}$

a) Recall that our shadow price is just the values of π in the dual, and that these values $= c_{BV} B^{-1}$

$$\pi = c_{BV} B^{-1} = \begin{bmatrix} 10 & 16 \end{bmatrix} \begin{bmatrix} -5/12 & 2/3 \\ 1/3 & -1/3 \end{bmatrix} = \begin{bmatrix} 7/6 & 4/3 \end{bmatrix}$$

$\boxed{\text{Version 1: } 7/6 \text{, Version 2: } 4/3}$

Answer depended on which constraint was changed.

b) Need to check reduced costs of <u>ALL</u> variables because the change is in c_{BV} & c. B, B^{-1} do not change.

Version 1: New $c_{BV} = \begin{bmatrix} 10 & 15 \end{bmatrix}$ Version 2: $c_{BV} = \begin{bmatrix} 9 & 16 \end{bmatrix}$

Check $c_{BV} B^{-1} A - c$

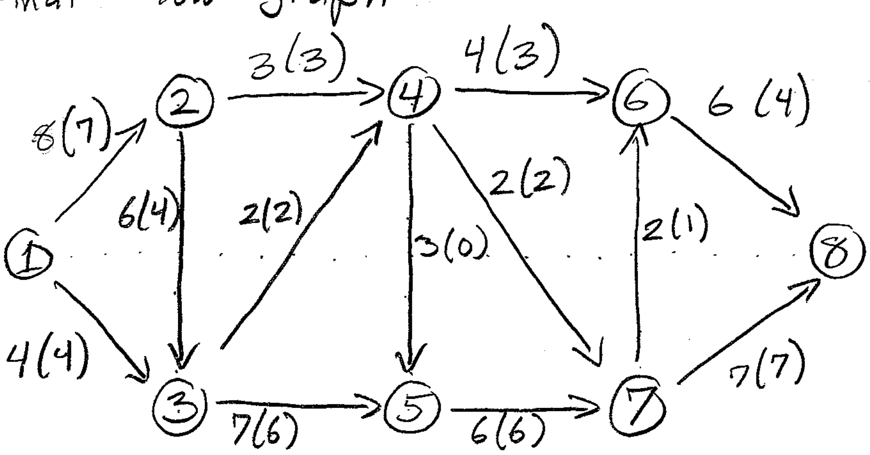
<u>Version 1</u>: $\begin{bmatrix} 10 & 15 & 21\tfrac{2}{3} & 5/6 & 5/3 \end{bmatrix} - \begin{bmatrix} 10 & 15 & 20 & 0 & 0 \end{bmatrix}$

All values will be positive. $\boxed{\text{No need to change basis.}}$

<u>Version 2</u>: $\begin{bmatrix} 9 & 16 & 18\tfrac{2}{3} & 1/12 & 2/3 \end{bmatrix} - \begin{bmatrix} 9 & 16 & 20 & 0 & 0 \end{bmatrix}$

$18\tfrac{2}{3} - 20 < 0$. $\boxed{x_3 \text{ enters basis, Yes}}$

3) Final Flow graph:



$\boxed{\text{Total Flow across} = 11}$

4) Label items across by number, both versions same (names of items change)

For linear knapsack problem, look @ value/weight ratio.

Ratios: 8 9 3 2 2.5 2.5 $3\frac{1}{3}$

		Total Value
a) Take item 2 first, $x_2=1$, leaves 14 pounds		54
		+40
Take item 1	$x_1=1$, leaves 9 pounds	94
		+10
Take item 7	$x_7=1$, leaves 6 pounds	104
		+15
Take item 3	$x_3=1$, leaves 1 pound	119
		+2.5
Either take $\frac{1}{8}$ of 5 or $\frac{1}{4}$ of 6, $x_5=\frac{1}{8}$	leaves 0 pounds $(\frac{1}{8})(20)=2.5$	$\boxed{121.5}$

b) If you cannot take both 1 & 2, you would take the one with the best ratio. (2).

$x_2=1$, leaves 14 pounds 54

$x_7=1$ leaves 11 pounds 64

$x_3=1$ leaves 6 pounds 79

$x_5=\frac{6}{8}$ or $x_6=1, x_5=\frac{2}{8}$ leaves 0 lbs. $\boxed{94}$

c) $x_1+x_2\le 1$

5) 1) Prune because you have a candidate solution with z value higher than z_1. (node 3)

2) Prune because node is infeasible

3) Prune. This is a candidate solution

4) Branch on $x_1\le 4$, $x_1\ge 5$.

Note: Having branch $x_1\le 4$, & $x_1\ge 4$ further up the tree is still feasible if $x_1=4$

6) There are several ways to look at this problem. Most common is to treat it as a machine scheduling problem (like HW 10).

Let $y_i = \begin{cases} 1 & \text{open warehouse } i \\ 0 & \text{o/w} \end{cases}$

$x_{ij} = \begin{cases} 1 & \text{serve factory } j \text{ from } i \\ 0 & \text{o/w} \end{cases}$

c_{ij} = cost of serving factory j from i

k_i = cost of opening warehouse i

a)

$$\min \sum_i \sum_j c_{ij} x_{ij} + \sum_i k_i y_i$$

s.t. $\sum_i x_{ij} = 1 \quad \forall j$ (1) (ensures every factory is supplied)

$x_{ij} \le y_i \quad \forall i, j$ (2) (ensures that if you serve j from i, i must be open)

$x_{ij} \in \{0,1\}, \ y_i \in \{0,1\}$

Note: In class/book (2) typically looked like $\sum x_{ij} \le 6 y_i \ \forall i$ (2') this is acceptable but the other version is better.

Consider the linear relaxation and let $x_{11} = 1$ & all other $x_{ij} = 0$. Using the (2) in my formulation, $y_1 = 1$ must be set.

In the other formulation, $y_1 = 1/6$ would be feasible in relaxation ($1 \le 6(1/6)$).

b) To the formulation listed, add new constraint

Version 1: $y_1 + y_4 \le 1$

Version 2: $y_3 + y_2 \le 1$

c) Let $y_i' = \begin{cases} 1 & \text{expand warehouse } i \\ 0 & \text{o/w} \end{cases}$

Add $\sum x_{ij} \le 2 + 3y_i'$ | or (2') to $\sum x_{ij} \le 2y_i + 3y_i'$

Also add constraint $y_i' \le y_i \ \forall i$ | (cannot expand an unopened warehouse)

Add objective $+ \sum_i \frac{k_i}{2} y_i'$ for both options.

For version 2, the 2 and 3 coefficients swap.

7) Let x_{ij} = amount supplied to j from i

$y_i = \begin{cases} 1 & \text{open warehouse } i \\ 0 & \text{o/w} \end{cases}$

c_{ij} = cost to supply one unit from i to j

k_i = cost to open i

d_j = demand @ j

b_i = capacity @ i

a) $\min \sum_j \sum_i c_{ij} x_{ij} + \sum_i k_i y_i$

s.t. $\sum_i x_{ij} \geq d_j \quad \forall j$ (demand must be met) (1)

$\sum_j x_{ij} \leq b_i \quad \forall i$ (capacity can't be exceeded) (2)

$x_{ij} \leq M_{ij} y_i \quad \forall i, j$ (must open to supply) (3)

$x_{ij} \geq 0$, int $\quad y_i \in \{0, 1\} \qquad M_{ij} = \min\{d_j, b_i\}$

Again, (2) & (3) can be replaced by (2') $\sum_j x_{ij} \leq b_i y_i$

b) Let $\bar{b}_i$ = # of times you expand capacity

Pay \$20 per time and get b extra $\left(\begin{array}{ll} \text{Version 1:} & b = 10 \\ \text{Version 2} & b = 20 \end{array}\right)$

To the objective you add $+ \sum_i 20 \bar{b}_i$

Constraint 2 becomes or (2') becomes

$\sum_j x_{ij} \leq b_i + b\bar{b}_i \qquad \sum_j x_{ij} \leq b_i y_i + b\bar{b}_i$

Add constraint

$b\bar{b}_i \leq b_i y_i$ (cannot increase total by more than b_i & only if you open)

Also need to change $M_{ij} = \min\{d_j, b_i + b\bar{b}_i\}$