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TEST 3 Name Key

Math 2551 D Section ____ April 13, 2016

No books, notes, calculators, cell phones, or other electronic devices are allowed. Show your work and justify your answer to receive credit. Work neatly. There is a total of 100 points. Put your name and section number on each page of the test.

19 = 4+8+7

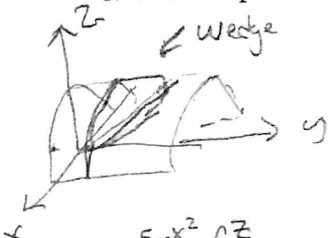
1. Consider the solid D bounded by the surfaces $z = 5 - x^2$, $z = y$, and $y = 1$.

(4) a. Sketch D.

Set up BUT DO NOT EVALUATE an iterated integral (with limits) to find the volume of D such that the integration

(8) b. with respect to z is performed first.

(7) c. with respect to y is performed first.



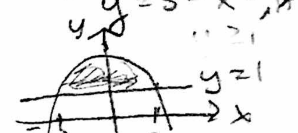
b. $y \le z \le 5-x^2$

$1 \le y \le 5-x^2$

$-2 \le x \le 2$

$z=y, \quad z=5-x^2$

$y=5-x^2$



$y=5-x^2$

$y=1$

$5-x^2=1$

$4=x^2$

$x=\pm 2$

(5) $V=\int_{-2}^{2}\int_{1}^{5-x^2}\int_{y}^{5-x^2} dz\,dy\,dx$

c. $1 \le y \le z$

$1 \le z \le 5-x^2$

$-2 \le x \le 2$

$y=1 \quad y=z$

$1=z$

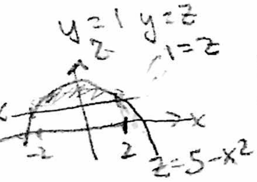
$1=z$

$5-x^2=z$

$1=5-x^2$

$x^2=4$

$x=\pm 2$

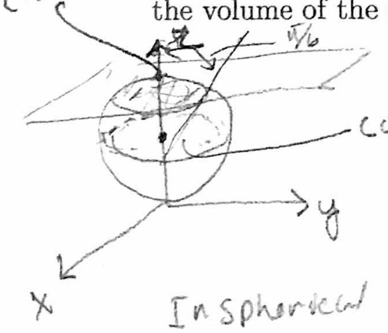


(5) $V=\int_{-2}^{2}\int_{1}^{5-x^2}\int_{1}^{z} dy\,dz\,dx$

(

12

2. The sphere tangent to the xy-plane and centered at $(0, 0, 4)$ can be described by the spherical coordinate equation $\rho = 8\cos\phi$. Let D be the solid that is bounded above by the sphere $\rho = 8\cos\phi$ and below by the plane $z = 6$. Set up BUT DO NOT EVALUATE an iterated integral in spherical coordinates (with limits) to compute the volume of the solid D.



The solid D is the "cap" part of the sphere

The sphere and plane intersect

$\rho\cos\phi = 6, \quad \rho = 8\cos\phi$

where $(8\cos\phi)\cos\phi = 6$

$8\cos^2\phi = 6$

$\cos^2\phi = \frac{6}{8} = \frac{3}{4}$

$\cos\phi = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$

But $\cos\phi > 0$ since $z=\rho\cos\phi = 6 > 0$ when the surfaces intersect

In spherical

$z=\rho\cos\phi$

$z=6$

$\rho\cos\phi=6$

$\rho=\frac{6}{\cos\phi}$

So $\frac{6}{\cos\phi} \le \rho \le 8\cos\phi$ (1)

So $\cos\phi = \frac{\sqrt{3}}{2}$, $0 \le \phi \le \pi$

then $\phi = \pi/6$ (2)

So $0 \le \phi \le \pi/6$

(3) $0 \le \theta \le 2\pi$

Putting (1), (2), (3) together

D: $\frac{6}{\cos\phi} \le \rho \le 8\cos\phi$

$0 \le \phi \le \pi/6$

$0 \le \theta \le 2\pi$

Take off 4 if they forget Jacobian
2 if they don't use $|J|$

Key

16

3. Let $I = \int\int_R (2x-3y)^{1/3}(2x+y)^4\, dx\, dy$ where R is the parallelogram bounded by $2x+y=4$, $2x+y=12$, $2x-3y=4$, and $2x-3y=-12$.

Make a change of variable as suggested below, writing I as an interated integral in terms of the variables u, v. Include the u and v limits. You do not need to compute the integral.

Note: $u = 2x+y, v = 2x-3y \Rightarrow x = \frac{3}{8}u + \frac{1}{8}v,\ y = \frac{1}{4}u - \frac{1}{4}v$.

$u = 2x+y \Rightarrow$ L_1 maps to $u=4$, L_2 maps to $u=12$

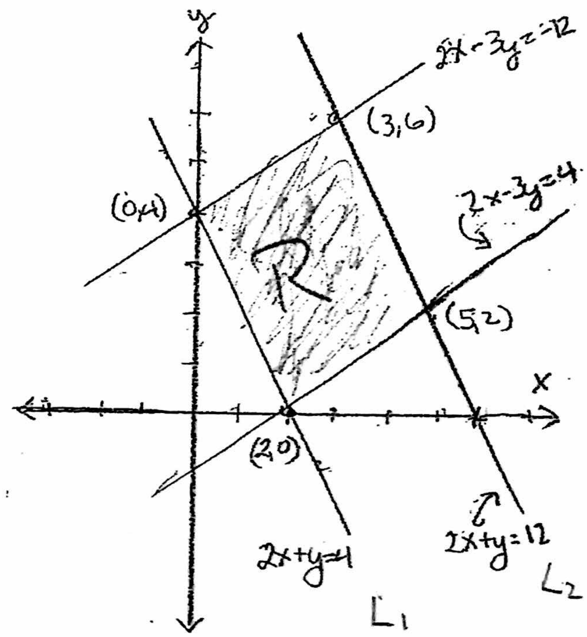
L_1: $2x+y=4$

L_2: $2x+y=12$

$v = 2x-3y \Rightarrow$ L_3 maps to $v=4$, L_4 maps to $v=-12$

L_3: $2x-3y=4$

L_3 $2x-3y=-12$



So R maps to the region S in the uv plane:

$4 \le u \le 12$

$-12 \le v \le 4$

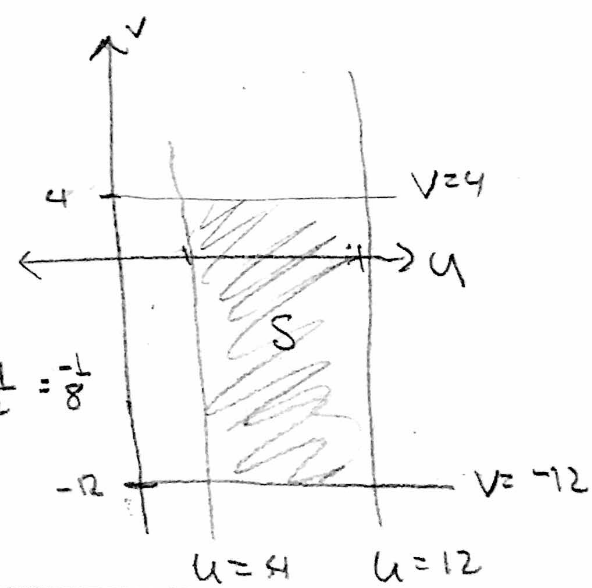
$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$x = \frac{3}{8}u + \frac{1}{8}v$

$y = \frac{1}{4}u - \frac{1}{4}v$

$$= \begin{vmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{4} & -\frac{1}{4} \end{vmatrix} = \left(\frac{3}{8}\right)\left(-\frac{1}{4}\right) - \left(\frac{1}{4}\right)\left(\frac{1}{8}\right)$$

$$= -\frac{3}{32} - \frac{1}{32} = \frac{-4}{32} = \frac{-1}{8}$$



So $I = \iint_R (\underbrace{2x-3y}_{v})^{1/3}(\underbrace{2x+y}_{u})^4\, dx\, dy$

$= \int_{-12}^{4}\int_{4}^{12} v^{1/3} u^4 \,|J|\, du\, dv$ ($J = -1/8$)

$$\boxed{= \int_{-12}^{4}\int_{4}^{12} v^{1/3} u^4 \left(\frac{1}{8}\right) du\, dv}$$

16 4. A wire with shape of the curve C has density $3+2x$ at the point (x,y). Set up the appropriate integral with limits to find the mass of the wire. $C: \mathbf{r}(t) = 2t\mathbf{i} + (5t+1)\mathbf{j}$, $0 \le t \le 2$.

$\delta = 3+2x$ $\quad \underline{r}(t) = 2t\underline{i} + (5t+1)\underline{j} \quad 0 \le t \le$

$\int_C \delta\, ds = \int_0^2 \delta(\underline{r}(t))|\underline{v}|dt$

so $x = 2t$

$\underline{r}'(t) = 2\underline{i} + 5\underline{j}$

$|\underline{v}| = |\underline{r}'(t)| = \sqrt{2^2+5^2} = \sqrt{29}$

$= \int_0^2 (3+2(2t)) \sqrt{29}\, dt$

$= \int_0^2 (3+4t)\sqrt{29}dt = \sqrt{29}\left(3t + 2t^2\right)\Big|_0^2 = \sqrt{29}(6+8-0)$

$= 14\sqrt{29}.$

19
=11+8 5. Consider the vector field $\mathbf{F} = y^2\mathbf{i} + (2xy-4)\mathbf{j} + 6z^2\mathbf{k}$.

11 a. Find f so that $\mathbf{F} = \nabla f$ (so $\mathbf{F}$ is a gradient field), or explain why no such f exists.

8 b. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the line segment from $(0,1,1)$ to $(3,2,1)$.

a. We can check if $\underline{F}$ is conservative. If $\underline{F}$ is not conservative then no such f exist.

$\underline{F} = y^2\underline{i} + (2xy-4)\underline{j} + 6z^2\underline{k} = M\underline{i} + N\underline{j} + P\underline{k}$ with

$M = y^2,\ N = 2xy-4,\ P = 6z^2$

Check: Are these all true?

$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$0 = 0$ ✓ $\qquad 0 = 0$ ✓ $\qquad 2y = 2y$ ✓

$\underline{F}$ is conservative

$\frac{df}{dx} = M \Rightarrow \frac{df}{dx} = y^2$. So $f = xy^2 + g(y,z)$

$\frac{df}{dy} = N = 2xy - 4 \quad \frac{\partial}{\partial x}(xy^2 + g(y,z)) = 2xy + \frac{\partial}{\partial y}(g(y,z)) = 2xy - 4$

So $\frac{\partial}{\partial y}(g(y,z)) = -4$. So $g(y,z) = -4y + h(z)$

& This gives $f = xy^2 + -4y + h(z)$

$P = \frac{\partial f}{\partial z} = 0 + 0 + h'(z)$. $P = 6z^2$

So $6z^2 = h'(z)$. We can take $h = 2z^3$

$\boxed{\text{So } f(x,y,z) = xy^2 - 4y + 2z^3 \text{ is a potential fn of } \underline{F}}$

b) $\int_C \underline{F} \cdot d\underline{r}$

$= f(3,2,1) - f(0,1,1)$

$= 3(2^2) - 4(2) + 2(1^3) - [0(1^2) - 4(1) + 2(1^3)]$

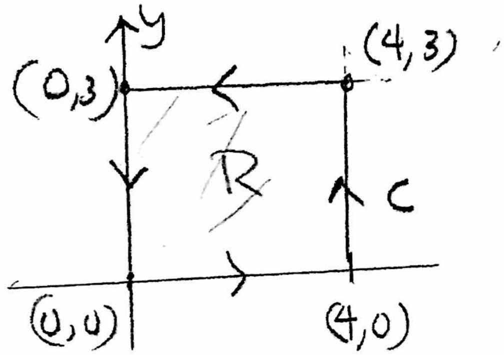
$= 12 - 8 + 2 - [-2]$

$= \boxed{8}$

18 6. Consider the vector field $\mathbf{F} = (2x + 5y + 1)\mathbf{i} + (3x^2 + 7y)\mathbf{j}$. The curve C is the given in the sketch.

9 a. Find the circulation of $\mathbf{F}$ around C.

9 b. Find the flux of $\mathbf{F}$ across C.



$F = (2x+5y+1)\,i + (3x^2+7y)\,j$

$= M\,i + N\,j$ for $M = (2x+5y+1)$

$N = (3x^2+7y)$

a. circ of $\underline{F}$ around $C = \oint_C F\cdot T\,ds = \oint_C \underset{\uparrow P}{M}\,dx + \underset{\uparrow Q}{N}\,dy$ $\quad 0 \le y \le 3$, $0 \le x \le 4$

Greens Thm. $\overset{?}{=} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ $\quad Q = N = 3x^2 + 7y \quad \frac{\partial Q}{\partial x} = 6x$ $\quad P = M = 2x + 5y + 1 \quad \frac{\partial P}{\partial y} = 5$

$= \iint_R (6x-5)\,dA = \int_0^3 \int_0^4 (6x-5)\,dx\,dy = \int_0^3 (3x^2 - 5x)\Big|_0^4 dy = \int_0^3 (48 - 20 - [0])\,dy$

$= \int_0^3 28\,dy = 28(3) = \boxed{84}$

b. Flux of F across $C = \oint_C F\cdot n\,ds = \oint_C M\,dy - N\,dx = \oint_C \underbrace{-N}_{P}\,dx + \underbrace{M}_{Q}\,dy$

$P = -N = -(3x^2+7y) \quad \frac{\partial P}{\partial y} = -7$

$Q = M = 2x + 5y + 1 \quad \frac{\partial Q}{\partial x} = 2$

$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$= \iint_R (2 - (-7))\,dA = \iint_R 9\,dA = 9(\text{area of } A) = 9(3)(4) = \boxed{108}$