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Solutions to Homework 6

1. (a) Observe that the state space of X_n is

$$\mathcal{S} = \{3, 4, 5, 6\},$$

since if the inventory drops below 3 we order up to 6. So at the beginning of a day the minimum number of items is equal to 3.
The initial state is deterministic and the initial distribution is given by

$$P(X_0 = 5) = 1.$$

Next, we find the transition matrix. Note that

$$P(X_{n+1} = 3|X_n = 3) = P(X_{n+1} = 4|X_n = 3) = P(X_{n+1} = 5|X_n = 3) = 0,$$

since whenever the inventory level goes below 3 we order up to 6. Therefore,

$$P(X_{n+1} = 6|X_n = 3) = 1.$$

Now,

$$P(X_{n+1} = 3|X_n = 4) = 1/6,$$

since the demand is equal to 1 with probability 1/6 and if the demand during day n is 1 we end up with 3 items and do not order. Otherwise we order, hence

$$P(X_{n+1} = 6|X_n = 4) = 5/6.$$

Going in this fashion, the transition matrix can be shown to be

$$\mathbb{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/6 & 0 & 0 & 5/6 \\ 3/6 & 1/6 & 0 & 2/6 \\ 2/6 & 3/6 & 1/6 & 0 \end{bmatrix},$$

where $\mathbb{P}_{ij} = P(X_{n+1} = j + 2|X_n = i + 2)$

(b) Let Y_n be the amount in stock at the end of day n. The inventory at the end of a day can be any value from 0 to 5. It cannot be 6 because at the beginning of a day the maximum number of items we can have in the inventory is 6 and the demand is strictly greater than zero with probability 1. So the state space in this case is

$$\mathcal{S} = \{0, 1, 2, 3, 4, 5\}.$$

If you think 6 must be included in the state space include 6 in the transition matrix and see what happens.
The initial state is deterministic and the initial distribution is given by

$$P(Y_0 = 2) = 1.$$

Observe that

$$P(Y_{n+1} = 5 | Y_n = 0) = P(D_{n+1} = 1) = 1/6,$$

where D_{n+1} is the demand during day $n+1$. Similarly

$$\begin{aligned} P(Y_{n+1} = 4 | Y_n = 0) = P(D_{n+1} = 2) = 3/6 \text{ and} \\ P(Y_{n+1} = 3 | Y_n = 0) = P(D_{n+1} = 3) = 2/6. \end{aligned}$$

Going in this fashion, we come up with the following transition matrix

$$\mathbb{P} = \begin{bmatrix} 0 & 0 & 0 & 2/6 & 3/6 & 1/6 \\ 0 & 0 & 0 & 2/6 & 3/6 & 1/6 \\ 0 & 0 & 0 & 2/6 & 3/6 & 1/6 \\ 2/6 & 3/6 & 1/6 & 0 & 0 & 0 \\ 0 & 2/6 & 3/6 & 1/6 & 0 & 0 \\ 0 & 0 & 2/6 & 3/6 & 1/6 & 0 \end{bmatrix},$$

where $\mathbb{P}_{ij} = P(Y_{n+1} = j-1 | Y_n = i-1)$.

2. Observe that the state space of X_n is

$$S = \{0, 1, 2, ...\}$$

since we count the days in a row without any injuries.

The initial distribution is given by

$$P(X_0 = 0) = 1$$

Next we find the transition matrix. Note that

$$\begin{aligned} P(X_{n+1} = j+1 | X_n = j) = 0.99 \text{ and} \\ P(X_{n+1} = 0 | X_n = j) = 0.01. \end{aligned}$$

Therefore the transition probability matrix is

$$\mathbb{P} = \begin{bmatrix} 0.01 & 0.99 & 0 & 0 & 0 & \dots \\ 0.01 & 0 & 0.99 & 0 & 0 & \dots \\ 0.01 & 0 & 0 & 0.99 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \dots \end{bmatrix}.$$

3. Since $X_{n+1} = \max\{X_n, U_{n+1}\}$ where $\{U_n : n \geq 1\}$ is an i.i.d. sequence of uniform distribution on $\{1, 2, 3, 4, 5, 6\}$, $\{X_n : n \geq 1\}$ is a Markov chain with state space $\mathcal{S} = \{1, 2, \dots, 6\}$ and transition probability

$$\begin{aligned} &\mathbb{P}\{X_{n+1} = i \mid X_n = i\} = \mathbb{P}\{U_{n+1} \leq i\} = i/6, \\ &\mathbb{P}\{X_{n+1} = j \mid X_n = i\} = 0, && \forall j < i \\ &\mathbb{P}\{X_{n+1} = j \mid X_n = i\} = \mathbb{P}\{U_{n+1} = j\} = 1/6, && \forall j > i \end{aligned}$$

The transition matrix is

$$P = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{6}{6} \end{bmatrix}$$

4. $\{Y_n : n \geq 1\}$ is a Markov chain, because $Y_{n+1} = Y_n + U_{n+1}$ where $U_n = 1$ when the n-th roll is 6 and $U_n = 0$ when the n-th roll is not 6 and obviously $\{U_n : n \geq 1\}$ is an i.i.d. sequence.

   The state space is $\mathcal{S} = \{0, 1, 2, \ldots\}$. (The state space is the set of nonnegative integers, which has infinitely many (countably many) elements.)

   The transition probabilities are given as follows: for all $i \in \mathcal{S}$

$$\begin{aligned} &\mathbb{P}\{Y_{n+1} = i + 1 \mid Y_n = i\} = \mathbb{P}\{(n+1)\text{-th roll is } 6\} = 1/6, \\ &\mathbb{P}\{Y_{n+1} = i \mid Y_n = i\} = \mathbb{P}\{(n+1)\text{-th roll is not } 6\} = 5/6, \\ &\mathbb{P}\{Y_{n+1} = j \mid Y_n = i\} = 0 \quad \text{for } j \neq i \text{ and } j \neq i + 1. \end{aligned}$$