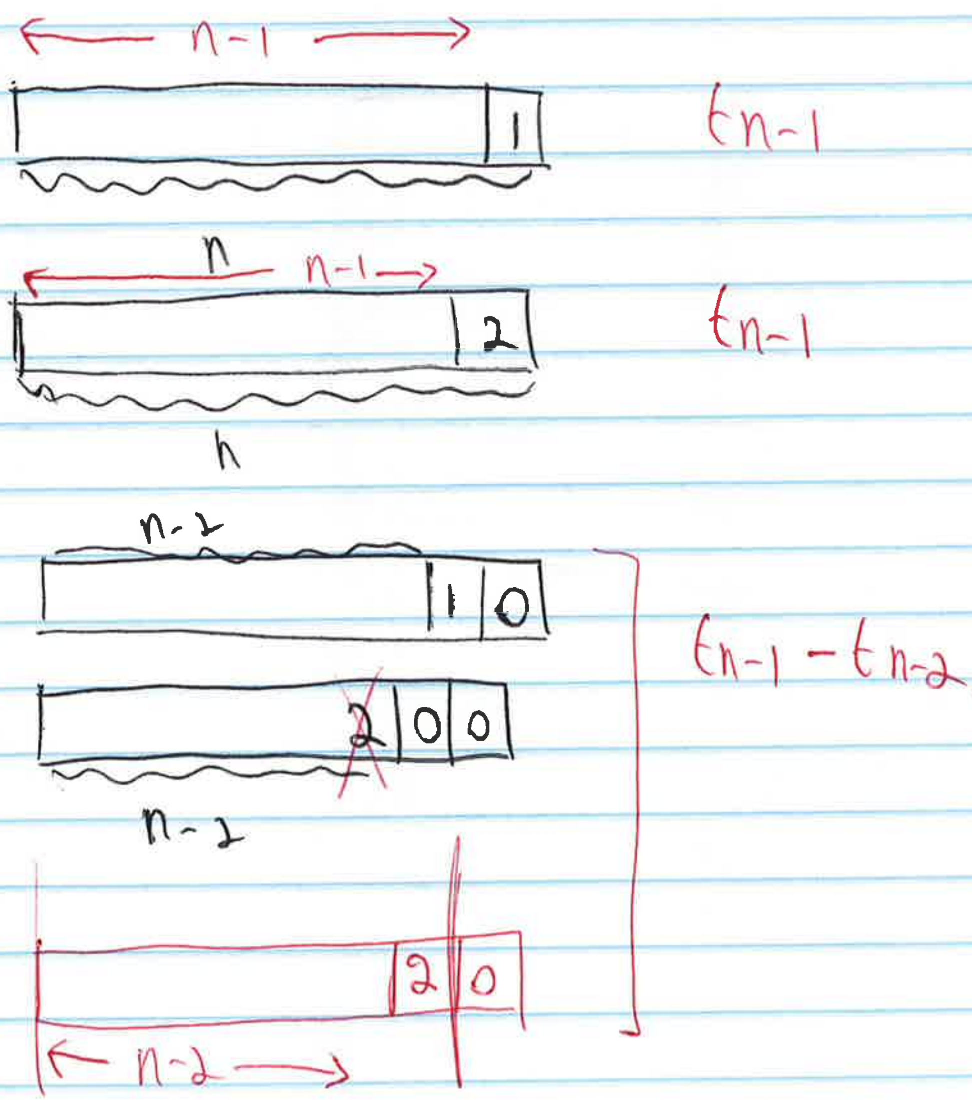
Midterm 2 exam sample

1. $n=1 \Rightarrow$ "1", "2", "0"

$n=2 \Rightarrow$ "00", "01", "02", "10", "11", "12"
"21", "22"

(n-2)th char must not be 2.



012 → 012 $\boxed{2}$

problem 2 :

36 possibilities

a. $\underset{1}{\sqcup}\ \underset{2}{\sqcup}\ \cdots\ \underset{10}{\sqcup}$

b. $36 \times 35 \times \ldots \times 27 = \dfrac{36!}{25!}$

c. (same question as b with restrictions)

$$\frac{10!}{2!\,3!\,4!\,1!}$$

Here, there are 6 letters, exactly two of which are 'A'. →

d. $\binom{10}{4}$ - choose 4 locations for digits.

10^4 - place 4 digits in these locations.

$\binom{6}{4}$ choose 4 locations for {"B", ..., "Z"}

25^4 - place these characters.

problem 3:

a. $x_1 + x_2 + x_3 + x_4 = 50$, $x_i > 0$, $\forall i = 1, .., 4$.

Analogy: folder-employee problem.

There are 49 gaps from which we need to choose 3.

b. $x_1 + x_2 + x_3 + x_4 = 50$. $x_i \geq 0$ $\forall i$.

Inflate the "folders" by 4

$\Rightarrow$ 54 folders, $\Rightarrow$ 53 gaps

$\Rightarrow \binom{53}{3}$

c. $x_1 + x_2 + x_3 + x_4 \leq 50$, $x_i > 0$ $\forall i$

Add a new variable x_5:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 50, \quad x_5 \geq 0$$

$\Rightarrow$ Inflate folders by 1.

$\Rightarrow$ $51 - 1 = 50$ gaps, choose 4 gaps.

d. $x_1 + x_2 + x_3 + x_4 \leq 50$, $x_i \geq 0$ $i = 1, .., 4$.

Add a new variable:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 50. \quad , x_5 \geq 0$$

=> Inflate folders by 5.

=> $55 - 1 = 54$ gaps choose 4 gaps.

e. $x_1 + x_2 + x_3 + x_4 = 50$, $x_i > 0$, $i = 1, 2, 3$
$x_4 \geq 9$.

Need to "deflate folders" by 8.

This brings us to $x_1 + x_2 + x_3 + x_4 = 42$,

$x_i > 0$, $i = 1, \ldots, 4$.

=> 41 gaps, choose 3 gaps.

problem 8:

1. This is the Chinese Remainder Theorem. 4, and 7 are relatively primes.

2. $f(x) = x^2 - 1$ is not one-to-one.

$$x_1^2 - 1 = x_2^2 - 1 \Rightarrow x_1^2 = x_2^2$$

$$\Rightarrow x_1 = \pm x_2,$$ e.g. $x = 3, -3$ are mapped to the same value.

3. Number of lattice paths is

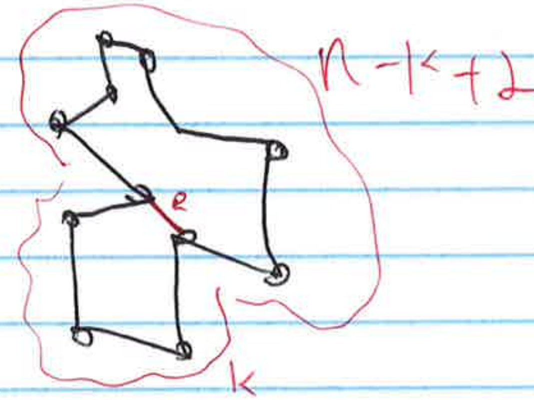
$$\binom{n+n}{n} = \binom{2n}{n}$$

[Number of binary strings of length $2n$ with n zeros and n ones].

ii.

$n=3$: 

diagonals $=0$.



$n>3$: use induction

we split the polygon into two polygons of sizes k and $(n-k+2)$

$\Rightarrow$ 1. Number of diagonals in first polygon is $k-3$.

2. Number of diagonal in second polygon is $n-k+2-3=n-k-1$

3. In addition, we need to add the edge e that we removed

$\Rightarrow (k-3)+(n-k-1)+1=n-3$.

5. This is exactly what Merge Sort does.

6. Insertion Sort uses $O(n^2)$ comparisons

7. The balls are distinct:

$$\underset{1}{\sqcup} \quad \underset{2}{\sqcup} \quad \dots \quad \underset{r}{\sqcup}$$

each ball has n possibilities to fall into a box $\Rightarrow n^r$.

8. $9^n - 5^n$ is divisible by 4, $n > 0$.

$n = 1 \Rightarrow 9 - 5 = 4$

$n > 1$: Use induction on n.

$$9^{n+1} - 5^{n+1} = 9 \cdot 9^n - 5 \cdot 5^n$$

$$= \underline{4} \cdot 9^n + 5\,[\underline{9^n - 5^n}].$$ [Both parts are divisible by 4].