ISyE 2027C Probability with Applications

Word Problems for class January 22 and homework due Thursday January 29, 2015

be ready for a quiz on THURSDAY January 29

Problems. Define the pertinent events. State the given information and the desired values in terms of probabilities involving those events.

1. You have two cards, one red on both sides, the second blue on one side and red on the other. While your eyes are closed, your friend picks one card at random and places it on a table without taking color into account. You open your eyes and see red. What is the probability that the other side of the card is red? E = the RR card is picked; F= you see red. The goal is to find $P(E|F)$. $P(E) = 1/2$; $P(F|E) = 1$ and $P(F|E^C) = .5$ obviously. $P(F) = 1 \cdot 1/2 + .5 \cdot 1/2 = .75$ from the law of total probability. Then $P(E|F) = P(F|E)P(E)/P(F) = 1 \cdot .5/.75 = 2/3$.

2. A game contestant chooses between a yellow urn and a purple urn. The yellow urn contains 10 white balls and 10 black balls. The purple urn contains 6 white balls and 6 black balls. The contestant draws two balls without replacement from the chosen urn without looking inside the urn. If the contestant draws two white balls, what the probability that the contestant chose the purple urn?

   You must assume that the urns are chosen randomly w.p. .5 each. Let U be the event that the purple urn is chosen, so U^C is the event that the yellow one is chosen and $P(U) = P(U^C) = .5$. Let W be the event that two white balls are drawn from the urn. The goal is to find $P(U|W)$. By counting, $P(W|U) = \binom{6}{2}/\binom{12}{2}$ or by conditional probabiltiy reasoning $P(W|U) = \frac{6}{12}\frac{5}{11}$. Similarly $P(W|U^C) = \frac{10}{20}\frac{9}{19}$. By the law of total probability,

   $$P(W) = P(U)P(W|U) + P(U^C)P(W|U^C) = .5\frac{5}{22} + .5\frac{9}{38}.$$

   Then $P(U|W) = P(W|U)P(U)/P(W) = \frac{5}{22}/(\frac{5}{22} + \frac{9}{38})$.

3. 60% of 70 year olds reach the age of 80. 35% of 70 year olds diagnosed with cancer reach 80. 68% of 70 year olds not so diagnosed reach 80. John is 70. What is the probability that he is diagnosed with cancer?

   Let E be the event that a random 70 year old, John, reaches 80. Let F be the event that a random 70 year old is diagnosed with cancer. The goal is to find $P(F)$. We are given $P(E) = .6; P(E|F) = .35; P(E|F^C) = .68$. Use the property that $P(F^C) = 1 - P(F)$ for any event F and solve the law of total probability for $P(F)$. $P(E) = P(F)P(E|F) + P(F^C)P(E|F^C)$.

4. Suppose the gender of a child is female w.p. .5 and male w.p. .5, independent of the genders of other children in the family. The Spohr family has 3 children. At least one of the children is a boy. What is the probability that the other two are girls?

   Let E be the event that the family has 1 boy and 2 girls. Let F be the event that the family has at least one boy. The goal is to find $P(E|F)$. Obviously $P(F|E) = 1$. $P(E) = 3/8$ because each of the 8 sequences is equally likely and E comprises three of them (MFF,FMF,FFM). $P(F) = 1 - P(F^C) = 1 - 1/8 = 7/8$. ($P(F^C) = 1/8$ because the probability of GGG is $1/2^3$.) Then $P(E|F) = P(F|E)P(E)/P(F) = 3/7$.

5. Suppose the gender of a child is female w.p. .5 and male w.p. .5, independent of the genders of other children in the family. The Hummel family has 3 children. The youngest is a boy. What is the probability that the other two are girls? Let E be the event that the family has 1 boy and 2 girls. Let F be the event that the youngest of the three children is a boy. The goal is to find $P(E|F)$. If you think of the kids as being born in reverse order, the independence of gender among the children tells you that the probability is $1/2^2 = 1/4$. To arrive at the answer more methodically, $P(F) = .5$. $P(E \cap F)$ is the probability that the genders, oldest to youngest, are FFM, which has probaibility 1/8. By definition, $P(E|F) = P(E \cap F)/P(F) = (1/8)/.5 = 1/4$.

6. Suppose that 50% of couples are physiologically predisposed such that each of their children is male with probability .6 and female w.p. .4, independent of each other. Suppose that the other couples are predisposed the other way, so that each of their children are female w.p. .6. The Czerny family has 3 children. At least one of the children is a boy. What is the probability that the other two are girls?

   Let E be the event that the family has 1 boy and 2 girls. Let F be the event that the family has at least one boy. Let M be the event that the Czerny couple is predisposed to have male children. The goal is to find $P(E|F)$. Obviously $P(F|E) = 1$. We are given that $P(M) = .5 \Rightarrow P(M^C) = 1 - .5 = .5$. Conditioned on M (respectively M^C), we know that each child is male w.p. .6 (respectively .4) independent of the other children. If we knew $P(E)$ and $P(F)$ we could compute $P(E|F)$. Using the law of total probability,

   $$P(F) = P(M)P(F|M)+P(M^C)P(F|M^C) = .5(1-.4^3)+.5(1-.6^3) = 1-.032-.108 = .86$$

   This is because conditioned on M the chance of all girls is $.4^3$ and conditioned on M^C it is $.6^3$. Similarly, using the law of total probability,

   $$P(E) = P(M)P(E|M) + P(M^C)P(E|M^C) = .5(3 \cdot .4^2 \cdot .6) + .5(3 \cdot .6^2 \cdot .4) = 1.5 \cdot .24 = .36$$

   The answer is therefore $P(E|F) = P(F|E)P(E)/P(F) = 1 \cdot .36/.86 = 18/43$.

   Suppose that 50% of couples are physiologically predisposed such that each of their children is male with probability .6 and female w.p. .4, independent of each other. Suppose that the other couples are predisposed the other way, so that each of their children are female w.p. .6. The Albinoni family has 3 children. The youngest is a boy. What is the probability that the other two are girls?

   Let E be the event that the family has 1 boy and 2 girls. Let F be the event that the youngest child is a boy. Let M be the event that the Czerny couple is predisposed to have male children. The goal is to find $P(E|F)$. From the previous problem we know $P(E) = .36$. Since the chance of each child being a boy is the same, $P(F|E) = 1/3$. In words, if exactly one child out of three is a boy, the chance that the boy is the youngest is one out of three. We need to know $P(F)$. By symmetry of the problem between boys and girls, $P(F) = P(F^c) \Rightarrow P(F) = .5$. If you aren't sure you see the symmetry, then by the law of total probability

   $$P(F) = P(M)P(F|M) + P(M^C)P(F|M^C) = .5 \cdot .6 + .5 \cdot .4 = .5$$

The answer is therefore $P(E|F) = P(F|E)P(E)/P(F) = .36\frac{1}{3}/.5 = .24$.

7. With your eyes closed, you draw 3 M&Ms without replacement from a bowl containing 10 orange, 15 green, 20 blue, and 25 yellow candies. You win \$20 if you get three oranges, and \$10 if you get one each of orange, green, and blue. What is the probability that you win money? If the first candy you draw is orange, what is the probablity that you win money?

   Let O be the event of getting 3 oranges and let F be the event of getting one each of o,g,b. The probability that you win money is $P(O \cup F)$. Since O and F are disjoint, this is simply

$$P(O) + P(F) = \frac{\binom{10}{3} + 10 \cdot 15 \cdot 20}{\binom{70}{3}} = \frac{3120}{\binom{70}{3}}$$

   Let G be the event that the first candy is orange. Then $P(O|G) = \frac{9}{69}\frac{8}{68}$ and $P(F|G) = \frac{15 \cdot 20}{\binom{69}{2}}$. Remember that O and F are disjoint. Then

$$P(O \cup F|G) = \frac{9}{69}\frac{8}{68} + \frac{300}{\binom{69}{2}} = \frac{72 + 600}{69 \cdot 68} = \frac{56}{391}$$

   .

8. You flip two coins and roll a 6-sided die. You win if the two coins come up the same, or if the die shows a 1. Given that you win, what is the probability that you rolled a 1? Given that you win, what is the probability that the first coin came up heads?

   Let M be the event that the two coins come up the same and let O be the event that the die shows a 1. These events are independent but not disjoint. Therefore the probability of winning is

$$P(M \cup O) = P(M) + P(O) - P(M \cap O) = .5 + 1/6 - 1/12 = 7/12.$$

   Let H be the event that the first coin comes up heads. The second question asks us to find $P(H|M \cup O)$. Your intuition should tell you that the answer is .5, because there are two equally likely ways to win, 2 heads and 2 tails. Regardless of whether or not you win, the first coin is equally likely to be heads as it is tails. To make this argument precise, consider the alternate game defined like this: you flip two coins and roll a 6-sided die. Your coins are foreign and you don't know which side to call "heads" and which to call "tails". After your first flip you decide to designate the side that came up as "heads". You win if the second coin comes up heads, or if the die shows a 1. Operationally, this is the same game as the original game. But it is also operationally equivalent to a game in which you only flip one coin and you win if the coin comes up heads or you roll a 1.

9. Todd is playing bridge, a 4-person game played with a standard 52-card deck. At the start of the game, each player is dealt 13 cards. If Todd's first card is an ace, what is the probability that he has two or more aces?

   Let A be the event that the first card is an ace and let T be the event that Todd has two or more aces. Then

$$P(T|A) = 1 - P(T^C|A). \tag{1}$$

T^C occurs, given that A occurs, if and only if the next 12 cards Todd gets are all not aces. There are 3 aces and 48 other cards left in the deck after A happens. Therefore

$$P(T^C|A) = \frac{\binom{48}{12}}{\binom{51}{12}} \tag{2}$$

Combine equations (1) and (2) for the answer.

If Todd's first card is the ace of spades, what is the probability that he has two or more aces?

Strange as it may seem, the answer is the same as that just given above. Here is a simple example. Todd gets 2 cards from a 4-card deck consisting of the aces and kings of spades and hearts. If his first card is the ace of spades, he gets two aces iff his second card is the ace of hearts. The probability is $1/3$ since there are 3 cards left. If his first card is an ace, there is one ace left in the deck out of 3 so the probability is still $1/3$. For the two questions below, let's see what happens. There are $\binom{4}{2} = 6$ possible pairs of cards Todd could get. Of those, one consists of two aces, one consists of two kings, and the other 4 consist of one king and one ace. Let E be the event that Todd has an ace (that means he has at least one ace) and let F be the event that Todd has 2 aces. $P(E) = 5/6$ and $P(F) = 1/6$. Obviously $P(E|F) = 1$. Hence $P(F|E) = 1 \cdot P(F)/P(E) = 1/5$. Let G be the event that Todd has the ace of spades. Obviously $P(G|F) = 1$. To get $P(F|G)$ we already know $P(F) = 1/6$ but we need to know $P(G)$. By symmetry, Todd is as likely to have the ace of spades as is the other player. Hence $P(G) = 1/2$. Or you can count the 3 ways Todd can have the ace of spades, AA,AKspades,AKhearts, out of 6 to get $3/6 = 1/2$. Hence

$$P(F|G) = 1/3 \neq 1/5.$$

Weird, isn't it?

If Todd has an ace, what is the probability that he has at least two aces?

Let E be the event that Todd has at least one ace. Let F be the event Todd has 2 or more aces. As above, $P(E|F) = 1$. We need $P(F|E)$. Again as above,

$$P(F|E) = P(E|F) \cdot P(F)/P(E) = P(F)/P(E) \tag{3}$$

We must calculate $P(F)$ and $P(E)$. The latter is simpler. $P(E) = 1 - P(E^C)$ and E^C means Todd has no aces. Hence

$$P(E^C) = \frac{\binom{48}{13}}{\binom{52}{13}} = 1 - P(E) \tag{4}$$

$P(F)$ is more complicated, like the more complicated keno calculations. Partition the event F into $F_2 \cup F_3 \cup F_4$ where F_n is the event that Todd has exactly n aces. Then

$$P(F) = P(F_4) + P(F_3) + P(F_2) = \frac{\binom{48}{9}}{\binom{52}{13}} + 4\frac{\binom{48}{10}}{\binom{52}{13}} + 6\frac{\binom{48}{11}}{\binom{52}{13}}. \tag{5}$$

The 6 in the equation above is $\binom{4}{2}$, the number of ways to pick two aces out of the four aces. The 4 is $\binom{4}{3}$, the number of ways to pick three out of the four aces. Combine the three equations to get the answer.

If Todd has the ace of spades, what is the probability that he has at least two aces?

Why is this different? One reason is that having at least two aces does not imply that Todd has the ace of spades. Define E to be the event that Todd has at least two aces. Define A to be the event that Todd has the ace of spades. The probability Todd has the ace of spades is simply $P(A) = 1/4$, by symmetry, since all 4 players are equally likely to have that card and the probabilities must sum to 1.

We want $P(E|A)$. It will be easier to calculate $P(E^C|A)$ and then use the formula

$$P(E|A) + P(E^C|A) = 1 \tag{6}$$

which is true for all events E and A. By definition, $P(E^C|A) = P(E^C \cap A)/P(A)$. What is the meaning of $E^C \cap A$? Todd has 0 or 1 aces, and Todd has the ace of spades. This means that Todd has exactly one ace, and that ace is the ace of spades. There are $\binom{48}{12}$ sets of 13 cards consisting of the ace of spades and no other spades. Therefore

$$P(E^C \cap A) = \frac{\binom{48}{12}}{\binom{52}{13}} \tag{7}$$

Combining the above we get

$$P(E|A) = 1 - P(E^C|A) = 1 - 4\frac{\binom{48}{12}}{\binom{52}{13}} \tag{8}$$

10. Todd is playing bridge again. His first two cards are hearts. What is the probability that he is void in diamonds (he has no cards in the diamond suit)?

    Each of his next 11 cards must not be a diamond. There are $\binom{37}{11}$ sets of 11 non-diamonds out of the 50 remaining cards, and $\binom{50}{11}$ sets of 11 out of the 50 remaining cards. The probability is therefore

$$\frac{\binom{37}{11}}{\binom{50}{11}}$$

Problems. Define the pertinent variables. State the given information in terms of your variables.

1. You are drunk, standing on the 2nd rung of a 3-rung ladder. No matter which rung you stand on, you step upwards with probability 1/3 and downwards with probability 2/3. If you try to step up from the 3rd (top) rung, you will fall off and break your arm. If you try to step down from the 1st rung, you will stumble away safely. What is the probability that you break your arm?

   Because what happens in the future depends only on where you are now and randomness, your chance of breaking your arm depends only on what rung you are on, and not on how you got there. Let X_i be the probability of breaking your arm if you start on rung i, for $i = 1, 2, 3$. The goal is to find X_2. The straightforward way to do this is to condition on stepping up or down. Let U be the event that you step up. $P(U) = 1/3, P(U^C) = 2/3$. By the law of total probability,

$$X_2 = P(U)X_3 + P(U^C)X_1 = \frac{1}{3}X_3 + \frac{2}{3}X_1.$$

Similarly, $X_3 = \frac{1}{3}1 + \frac{2}{3}X_2$ and $X_1 = \frac{1}{3}X_2 + \frac{2}{3}0$ because if you step down from rung 1 you are safe. Solve the 3 equations in 3 unknowns to get $X_3 = 7/15, X_2 = 1/5, X_1 = 1/15$.

2. You buy one unit of a mutual fund that tracks the S&P at a price of 1980. Each day the price goes up by 10, stays the same, or goes down by 10, each with probability 1/3. You give orders to sell if it reaches 2000, and to sell if it reaches 1950. What is the chance you will make a profit?

   The variables are $X_1 =$ the probability that the S&P reaches 2000 before it reaches 1950 if it starts at 1990; X_2 (respectively X_3, X_4) = the probability that the S&P reaches 2000 before it reaches 1950 if it starts at 1980 (resp. 1970,1960). The straightforward but long way to solve it is to write and solve the following 4 equations in 4 variables:

$$\begin{aligned} X_1 &= \frac{1}{3}X_1 + \frac{1}{3}1 + \frac{1}{3}X_2 \\ X_2 &= \frac{1}{3}X_2 + \frac{1}{3}X_1 + \frac{1}{3}X_3 \\ X_3 &= \frac{1}{3}X_3 + \frac{1}{3}X_2 + \frac{1}{3}X_4 \\ X_4 &= \frac{1}{3}X_4 + \frac{1}{3}X_3 + \frac{1}{3}0 \end{aligned}$$

   A shorter way to solve it is to see that starting at 1990 and aiming for 2000 is the same as starting at 1960 and aiming for 1950, since the chance of going up equals the chance of going down. Therefore $X_1 = 1 - X_4$. Similarly $X_2 = 1 - X_3$. This gives the solution $X_2 = 3/5$. X_1 equals 4/5. Once you see this answer you might guess an even simpler way to solve it. The average value of your mutual fund is the same as time progresses because it goes up by 10 with the same probability that it goes down by 10, and otherwise it does not move. Initially the value is 1980. Therefore

$$1980 = X_2 \cdot 2000 + (1 - X_2) \cdot 1950 \Rightarrow X_2 = 3/5$$

   . The technical name for this kind of argument is "martingale" but the topic is too advanced for our course.

Problems to solve completely.

1. You flip a fair coin until it comes up heads. What is the probability that you flip the coin an odd number of times?

   Let X be the number of flips. Let O be the event that X is odd. From class we know $P(X = n) = 1/2^n$ since there is exactly one way for $X = n$, namely, all tails except a head at the end. The straightforward solution is

$$\sum_{n \text{ odd}} P(X=n) = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \ldots = \frac{.5}{1-.25} = 2/3$$

. I'm using the formula $\frac{a}{1-r}$ for the sum of the infinite geometric progression $a, ar, ar^2, ar^3, \ldots$. The sneaky solution is to define $H =$ the event that the first flip is heads, and to see that

$$P(O|H^C) = P(O^C)$$

because if the first flip is not heads, then O happens if, starting from the 2nd flip, the number of flips is even. Therefore, by the law of total probability,

$$P(O) = P(H)P(O|H) + P(H^C)P(O|H^C) = .5 \cdot 1 + .5P(O^C) = .5 + .5(1 - P(O))$$

from which $1.5P(O) = 1 \Rightarrow P(O) = 2/3$.

2. You roll a 6-sided die repeatedly until you get the same number twice in a row. What is the probability that you will roll exactly twice? Exactly three times? Exactly 4 times?

   By symmetry assume the first roll is a 1. Then the chance of event E defined as rolling exactly twice equals the probability the 2nd roll is a 1, which is $P{*}E) = 1/6$. To roll 3 times, let F be the event that the 3rd roll equals the 2nd. Then $P(F) = 1/6$ whence $P(F^C) = 5/6$. The answer is $P(E^C \cap F) = (5/6)(1/6) = 5/36$ because E and F are independent. For 4 rolls, let G be the event that the 4th roll equals the 3rd. Then $P(G) = 1/6$ and by independence

   $$P(E^C \cap F^C \cap G) = \frac{5}{36}\frac{5}{36}\frac{1}{36} = \frac{25}{216}.$$

3. You roll a 4-sided die repeatedly until you have seen each number at least once. What is the probability that you will roll exactly 4 times? Exactly 5 times?

   There are 4! ways (counting the order) to roll exactly 4 times. The probability is therefore $4!/4^4 = 3/32$. To roll exactly 5 times, there must be two of one number, zero of another number, and 1 each of the other two numbers in the 1st 4 rolls. Then the 5th roll must be the missing number. There are 4 possibilities for the 5th roll. By symmetry, assume the 5th roll is a 1 and multiply the answer by 4. There are now 3 possibilities for the number that appears twice in the first 4 rolls. By symmetry, assume that is a 2 and multiply the answer by 3. The first 4 rolls therefore consist of one 4, one 3, and two 2's, in some order. There are 4!/2 such orders since the two 2's could be swapped without changing the order. The total number of ways to roll 5 times is therefore

   $$4 \cdot 3 \cdot 4!/2 = 6 \cdot 4! \Rightarrow \text{ the answer is } 6 \cdot 4!/4^5 = 9/64$$

4. You roll a 4-sided die repeatedly until you have seen the number 4 four times. What is the probablility that you will roll exactly 4 times? Exactly 5 times?

   There is only one way to roll 4 times. The probability is therefore $1/4^4$. To roll exactly 5 times, the last roll must be a 4. One of the first 4 rolls must not be a 4, and the rest must be 4's. The roll not equal to 4 can be 1st, 2nd, 3rd, or 4th, which makes 4 possibilities. The roll value can be 1, 2, or 3, making 3 possibilities. Therefore there are $4 \cdot 3 = 12$ ways to roll exactly 5 times. The probability is therefore $12/4^5 = 3/4^4$.