1. The position of a moving object at time $t > 0$ is given by

$$\mathbf{r}(t) = (\cos t + t\sin t)\,\mathbf{i} + (\sin t - t\cos t)\,\mathbf{j} + \frac{\sqrt{3}}{2}t^2\,\mathbf{k}.$$

(a) (6 points) Find the curvature of the path.

1 $\vec{v}(t) = \vec{r}'(t) = \langle -\sin t + t\cos t + \sin t,\ \cos t + t\sin t - \cos t,\ \sqrt{3}\,t \rangle$
$= \langle t\cos t,\ t\sin t,\ \sqrt{3}\,t \rangle$

0.5 $|\vec{v}(t)| = \sqrt{t^2\cos^2 t + t^2\sin^2 t + 3t^2} = \sqrt{t^2 + 3t^2} = \sqrt{4t^2} = 2t$

1 $\vec{T}(t) = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \left\langle \dfrac{\cos t}{2},\ \dfrac{\sin t}{2},\ \dfrac{\sqrt{3}}{2} \right\rangle$

1 $\vec{T}'(t) = \left\langle -\dfrac{\sin t}{2},\ \dfrac{\cos t}{2},\ 0 \right\rangle$

0.5 $|\vec{T}'(t)| = \sqrt{\dfrac{\sin^2 t}{4} + \dfrac{\cos^2 t}{4} + 0} = \sqrt{\dfrac{1}{4}} = \dfrac{1}{2}$

2 $\kappa = \dfrac{|\vec{T}'(t)|}{|\vec{v}(t)|} = \dfrac{1/2}{2t} = \dfrac{1}{4t}$

(b) (6 points) Determine the tangential and normal components of acceleration.

$$\underbrace{a_T = \frac{d}{dt}|\vec{v}(t)|}_{2} = \underbrace{\frac{d}{dt}(2t) = 2}_{1}.$$

$$\underbrace{a_N = \kappa|\vec{v}(t)|^2}_{2} = \underbrace{\frac{1}{4t}(2t)^2 = \frac{4t^2}{4t} = t}_{1}.$$

2. (6 points) Find an equation for the osculating plane of the curve $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$ at $t = 0$.

$$\vec{v}(t) = \vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$|\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \left\langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

1

$$\vec{T}' = \left\langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0 \right\rangle$$

$$\vec{T}' = \sqrt{\frac{\cos^2 t}{2} + \frac{\sin^2 t}{2} + 0} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|} = \langle -\cos t, -\sin t, 0 \rangle$$

1

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \left(\frac{1}{\sqrt{2}}\sin t\right)\vec{i} - \left(\frac{1}{\sqrt{2}}\cos t\right)\vec{j} + \left(\frac{\sin^2 t}{\sqrt{2}} + \frac{\cos^2 t}{\sqrt{2}}\right)\vec{k}$$

$$= \left\langle \frac{1}{\sqrt{2}}\sin t, -\frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}} \right\rangle$$

2

At $t=0$, the binormal vector is $\vec{B} = \langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$, which is normal to the osculating plane (TN-plane) at that point. Also $t=0$ corresponds to the point $(1,0,0)$ on the curve. So the equation of the osculating plane is:

$$0(x-1) - \frac{1}{\sqrt{2}}(y-0) + \frac{1}{\sqrt{2}}(z-0) = 0$$

or, $$-\frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z = 0$$

i.e. $$y - z = 0.$$

2

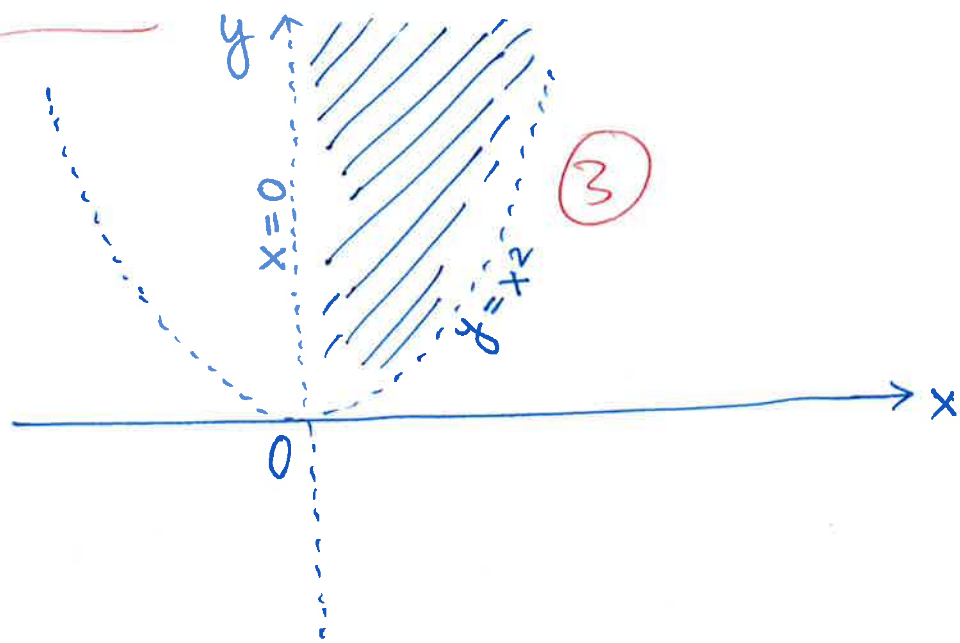
3. Let $f(x,y) = \frac{\ln x}{\sqrt{y-x^2}}$.

(a) (5 points) Find and sketch the domain of f.

Domain of f is determined by

$x > 0$ and $y - x^2 > 0$ i.e. $y > x^2$

2



3

(b) (1 point) State whether the domain of f is open, closed, both or neither.

1 Open

(c) (1 point) State whether the domain of f is bounded or unbounded.

1 Unbounded.

4. Find the limit or show that it does not exist.

(a) (6 points) $\displaystyle\lim_{\substack{(x,y)\to(-1,1)\\ y\neq 1}} \frac{y^2+xy}{y-1}$.

$$\lim_{\substack{(x,y)\to(-1,1)\\ y\neq 1\\ \text{along } x=-1}} \frac{y^2+xy}{y-1} = \lim_{y\to 1} \frac{y^2-y}{y-1} = \lim_{y\to 1}\frac{y\cancel{(y-1)}}{\cancel{(y-1)}} = 1$$ 2.5

$$\lim_{\substack{(x,y)\to(-1,1)\\ y\neq 1\\ \text{along } x=-y}} \frac{y^2+xy}{y-1} = \lim_{y\to 1} \frac{y^2+(-y)y}{y-1} = 0.$$ 2.5

Since the limits along two different paths are different, the given limit does not exist. 1

(b) (5 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2-xy}{\sqrt{x^2+y^2}}$.

In polar coordinates $$\frac{y^2-xy}{\sqrt{x^2+y^2}} = \frac{r^2\sin^2\theta - r\cos\theta\, r\sin\theta}{r}$$

$$= \frac{r^2(\sin^2\theta - \cos\theta\sin\theta)}{r}$$

$$= r(\sin^2\theta - \cos\theta\sin\theta) \quad \text{for } r\neq 0,$$ 3

and this approaches to '0' as $r\to 0$.
So the given limit is 0. 2

5. (a) (5 points) Find $f_{xz}(0,-1,1)$ if $f(x,y,z) = xyz\, e^{y/z}$.

1 $f_x = yz e^{y/z}$

3 $f_{xz} = yz\, e^{y/z}\left(\frac{-y}{z^2}\right) + y e^{y/z} = -\frac{y^2}{z} e^{y/z} + y e^{y/z}$

1 $f_{xz}(0,-1,1) = -\frac{(-1)^2}{1} e^{-1/1} + (-1) e^{-1/1} = -e^{-1} - e^{-1} = -2e^{-1}$

$= -\frac{2}{e}$.

(b) (4 points) Find the value of $\partial z/\partial x$ at $(1,1,1)$ if $4xy + z^3x - 4yz = 1$ defines z as a function of two independent variables x and y and the partial derivative exists.

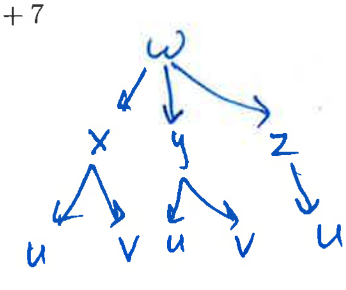
Let $F(x,y,z) = 4xy + z^3x - 4yz$

Then, $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{4y + z^3 - 0}{0 + 3z^2x - 4y} = -\frac{4y + z^3}{3z^2x - 4y}$ 3

$\therefore$ At $(1,1,1)$, $\frac{\partial z}{\partial x} = \frac{-(4+1)}{3-4} = -\frac{5}{-1} = 5$ 1

(c) (5 points) Express $\partial w/\partial v$ as a function of u and v if

$$w = 15 - z^2 + e^x \ln y, \quad x = \ln(u\cos v), \quad y = u\sin v \quad \text{and} \quad z = 2u^2 + 7$$



$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$ 2

$= (e^x \ln y)\frac{1}{u\cos v}(-u\sin v) + \frac{e^x}{y} u\cos v$ 2

$= -e^{\ln(u\cos v)} \ln(u\sin v) \cdot \frac{\sin v}{\cos v} + \frac{e^{\ln(u\cos v)}}{u\sin v} u\cos v$

$= -u\cos v \ln(u\sin v) \cdot \frac{\sin v}{\cos v} + \frac{u\cos v}{u\sin v} u\cos v$ 1

$= -u\sin v \ln(u\sin v) + \frac{u\cos^2 v}{\sin v}$.