Problem 1 (24 points)**:**

a) Find $\frac{du}{dt}$, if

$$u(x,y) = 3xy^2 - x^2;\ x = t^2 + 2t,\ y = 3t.$$

b) Write an equation for the tangent plane of the surface

$$z^3 + xyz - 2 = 0$$

at the point $P(1,1,1)$.

c) Calculate the second-order partial derivatives of

$$g(x,y) = xy\sin(xy).$$

Answer:

a)

$$\begin{aligned}
\frac{du}{dt} &= u_x x_t + u_y y_t = \left(3y^2 - 2x\right)(2t+2) + 6xy(3) \\
&= \left(27t^2 - 2\left(t^2+2t\right)\right)(2t+2) + 54t\left(t^2+2t\right) \\
&= 2t\left(52t^2 + 75t - 4\right)
\end{aligned}$$

b) The normal vector is

$$\nabla\left(z^3 + xyz - 2\right) = \left(yz, xz, xy + 3z^2\right).$$

At $(1,1,1)$, the normal vector is $(1,1,4)$. So the tangent plane is

$$(x-1) + (y-1) + 4(z-1) = 0.$$

c)

$$\begin{aligned}
g_x &= y\sin(xy) + xy^2\cos(xy), \\
g_y &= x\sin(xy) + x^2y\cos(xy), \\
g_{xx} &= 2y^2\cos(xy) - xy^3\sin(xy), \\
g_{yy} &= 2x^2\cos(xy) - yx^3\sin(xy), \\
g_{xy} &= g_{yx} = \sin(xy) + 3xy\cos(xy) - x^2y^2\sin(xy).
\end{aligned}$$

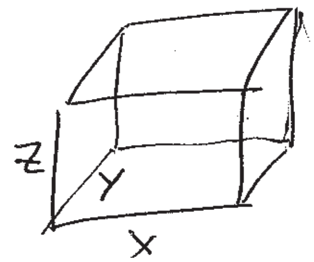
Problem 4 (12 points): Closed rectangular boxes 16 cubic feet in volume are to be constructed from three types of metal. The cost of the metal for the bottom of the box is \$0.50 per square foot, for the sides of the box \$0.25 per square foot, for the top \$0.10 per square foot. Find the dimensions that minimize cost of material.

Let the length, width and height be x, y, z.

Then the total cost is

$$f(x,y,z) = \frac{1}{2}xy + \frac{1}{4}(2xz + 2yz) + \frac{1}{10}xy$$

$$= \frac{3}{5}xy + \frac{1}{2}xz + \frac{1}{2}yz$$



with the condition

$$g(x,y,z) = xyz - 16 = 0$$

Note that we also require: $x > 0$, $y > 0$, $z > 0$.

$$\nabla f = \left(\frac{3}{5}y + \frac{1}{2}z\right)\vec{i} + \left(\frac{3}{5}x + \frac{1}{2}z\right)\vec{j} + \left(\frac{1}{2}y + \frac{1}{2}x\right)\vec{k}$$

$$\nabla g = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$$

by Lagrange multiplier method, the minimizer satisfies

$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} \frac{3}{5}y + \frac{1}{2}z = \lambda yz & (1) \\ \frac{3}{5}x + \frac{1}{2}z = \lambda xz & (2) \\ \frac{1}{2}x + \frac{1}{2}y = \lambda xy & (3) \end{cases}$$

$$x(1) - y(2) \Rightarrow \frac{1}{2}xz - \frac{1}{2}yz = 0 \Rightarrow x = y$$

$$y(2) - z(3) \Rightarrow \frac{3}{5}xy - \frac{1}{2}xz = 0 \Rightarrow z = \frac{6}{5}y$$

So from $xyz = 16$, it follows $\frac{6}{5}y^3 = 16$

$$\Rightarrow y = \sqrt[3]{\frac{40}{3}}, \quad x = \sqrt[3]{\frac{40}{3}}, \quad z = \frac{6}{5}\sqrt[3]{\frac{40}{3}}$$

Problem 3 (15 points)**:** Evaluate

$$\int\int_{\Omega} \cos\left(\frac{y-x}{y+x}\right)\ dxdy,$$

where Ω is the region in the first quadrant bounded by the lines $x+y=1$ and $x+y=2$.

(Hint: Use proper change of variables.)

Answer: Let $u=x+y, v=y-x$. Then $x=\frac{1}{2}(u-v)\,,\ y=\frac{1}{2}(u+v)$. The region Ω becomes

$$\Gamma=\left\{(u,v)\ \mid 1\le u\le 2,\ -u\le v\le u\right\}.$$

The Jacobian is

$$J=\left|\begin{array}{cc} x_u & x_v \\ y_u & y_v \end{array}\right| = \left|\begin{array}{cc} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array}\right| = \frac{1}{2}.$$

Then

$$\begin{aligned} & \int\int_{\Omega} \cos\left(\frac{y-x}{y+x}\right)\ dxdy \\ = & \int_1^2 \int_{-u}^{u} \cos\left(\frac{v}{u}\right) dvdu \\ = & 3\sin 1. \end{aligned}$$

Problem 7 (12 points): Evaluate

$$\int_C y\ dx + yz\ dy + z\,(x-1)\ dz$$

where C is the intersection of the sphere $x^2+y^2+z^2=4$ with the cylinder $(x-1)^2+y^2=1$ traversed from $(2,0,0)$ to $(0,0,2)$.

For any point (x,y,z) on C, Sinle (x,y) satisfies $(x-1)^2+y^2=1$ and (x,y) goes from $(2,0)$ to $(0,0)$,

So we let

$$\begin{cases} x-1=\cos u \\ y=\sin u \end{cases}, \quad u\in[0,\pi]$$

then $z=\sqrt{4-x^2-y^2}=\sqrt{4-(\cos u+1)^2-(\sin u)^2}$

$$=\sqrt{2(1-\cos u)}=\sqrt{4\sin^2\frac{u}{2}}=2\sin\frac{u}{2}$$

Thus C is $\vec{r}(u)=(\cos u+1)\vec{i}+\sin u\,\vec{j}+2\sin\frac{u}{2}\vec{k}$

with $u\in[0,\pi]$

$$\Rightarrow \int_C y\,dx+yz\,dy+z(x-1)\,dz$$

$$=\int_0^\pi\left[\sin u(-\sin u)+\sin u\left(2\sin\frac{u}{2}\right)\cos u+2\sin\frac{u}{2}\cos u\cos\frac{u}{2}\right]du$$

$$=\int_0^\pi\left[-\frac{1-\cos 2u}{2}+\sin(2u)\sin\frac{u}{2}+\sin u\cos u\right]du$$

$$=\int_0^\pi\left[-\frac{1}{2}+\frac{1}{2}\cos 2u+\frac{1}{2}\cos\frac{3u}{2}-\frac{1}{2}\cos\frac{5u}{2}+\frac{1}{2}\sin 2u\right]du$$

$$=\left[-\frac{1}{2}u+\frac{1}{4}\sin 2u+\frac{1}{3}\sin\frac{3u}{2}-\frac{1}{5}\sin\frac{5u}{2}-\frac{1}{4}\cos 2u\right]\Big|_0^\pi$$

$$=-\frac{1}{2}\pi-\frac{8}{15}$$

Problem 5 (15 points)**:** Find the area enclosed by the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1.$$

(Hint: Use Green's theorem.)

Answer:

We parametrize the curve C by

$$x\left(t\right) = \cos^3 t,\ y\left(t\right) = \sin^3 t, \quad 0 \le t \le 2\pi.$$

Then the enclosed area is

$$\begin{aligned} A &= \frac{1}{2}\oint_C xdy - ydx \\ &= \frac{1}{2}\int_0^{2\pi} \left(\cos^3 t\left(3\sin^2 t\cos t\right) - \sin^3 t\left(-3\cos^2 t\sin t\right)\right) dt \\ &= \frac{3}{2}\int_0^{2\pi} \sin^2 t\cos^2 t\ dt = \frac{3}{8}\pi. \end{aligned}$$

Problem 9 (12 points): Calculate the total flux of

$$\vec{v}(x,y,z) = 2x\ \mathbf{i} + xz\ \mathbf{j} + z^2\ \mathbf{k}$$

out of the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane.

Let T be the solid, then T is bounded by the base $\Omega = \{x^2+y^2 \le 9\}$ in xy-plane and the surface $z = 9-x^2-y^2$.
Let S be the surface enclosing T.

By divergence Theorem,

$$\iint_S (\vec{v}\cdot\vec{n})\,da = \iiint_T \nabla\cdot\vec{v}\ dx\,dy\,dz$$

$$= \iiint_T \left[\frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(z^2)\right] dx\,dy\,dz$$

$$= \iiint_T (2+2z)\,dx\,dy\,dz$$

$$= \iint_\Omega \left[\int_0^{9-x^2-y^2} (2+2z)\,dz\right] dx\,dy$$

$$= \iint_\Omega \left[2(9-x^2-y^2) + (9-x^2-y^2)^2\right] dx\,dy$$

$$= \int_0^3 \int_0^{2\pi} \left[2(9-r^2) + (9-r^2)^2\right] r\,d\theta\,dr$$

$$= 2\pi \int_0^3 (99r - 20r^3 + r^5)\,dr$$

$$= 2\pi \left(\frac{99}{2}r^2 - 5r^4 + \frac{1}{6}r^6\right)\Big|_{r=0}^{r=3}$$

$$= 324\pi$$