Full Name: Solutions Section B___

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Signature: ______________________________

Math 2551 — Exam 1
September 9, 2015

Write your solutions clearly and legibly, showing all work. Use of notes, cheat sheets, the textbook, or any outside materials is not permitted. Only non-graphing, non-programmable calculators are permitted.

Problem	Points Possible	Points Earned
1	13	13
2	12	12
3	11	11
4	14	14
Total	50	50

(1) A particle is moving on a surface S and has position vector

$$\mathbf{r}(t) = \sin t \cos t\,\mathbf{i} + \sin^2 t\,\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k}.$$

(a) If the particle starts moving at time $t_0 = 0$, how many units of arc length has it traveled at time $t > 0$? [8 points]

Though not necessary, we may use the trig identity $\sin 2t = 2\sin t \cos t$ to rewrite the position vector as $\mathbf{r}(t) = \frac{1}{2}\sin 2t\,\mathbf{i} + \sin^2 t\,\mathbf{k} + \frac{2}{3}t^{3/2}\mathbf{k}$. The velocity is therefore

$$\mathbf{v}(t) = \cos 2t\,\mathbf{i} + 2\sin t \cos t\,\mathbf{j} + t^{1/2}\mathbf{k} = \cos 2t\,\mathbf{i} + \sin 2t\,\mathbf{j} + t^{1/2}\mathbf{k}$$

$$\implies \qquad |\mathbf{v}(t)| = \sqrt{\cos^2 2t + \sin^2 2t + t} = \sqrt{1+t}$$

We then get the arc length as a function of time by integrating speed:

$$s(t) = \int_0^t \sqrt{1+\tau}\,d\tau = \left[\frac{2}{3}(1+\tau)^{3/2}\right]_{\tau=0}^{\tau=t} = \boxed{\frac{2}{3}(1+t)^{3/2} - \frac{2}{3}}$$

3 points for computing $\mathbf{v}(t)$, and additional 2 points for computing $|\mathbf{v}(t)|$, and the final 3 points for computing $s(t)$.

(b) Which of the following is a possible equation for S? You must justify your answer to receive credit. [5 points]

(I) $x^2 + y^2 = z^2$ (II) $x^2 + y^2 = y$ (III) $x^2 + y^2 = 1$

All three equations start with $x^2 + y^2$, so it makes sense to compute $x^2 + y^2$ in terms of t:

$$\begin{aligned} x^2 + y^2 &= (\sin t \cos t)^2 + (\sin^2 t)^2 \\ &= \sin^2 t \cos^2 t + \sin^4 t \\ &= \sin^2 t\left(\cos^2 t + \sin^2 t\right) \\ &= \sin^2 t \\ &= y \end{aligned}$$

This shows that $x^2 + y^2 = y$ is a possible equation for S, so the answer is (II)

Partial credit is available for reasonable answers (I make the decision what counts as reasonable).

(2) Let $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Which of the following two vectors has the largest magnitude? You must justify your answer to receive credit. [12 points]

(I) $\mathbf{v} \times \mathbf{w}$ (II) $\mathrm{proj}_{\mathbf{v}}\mathbf{w}$

First compute the cross product:

$$\mathbf{v} \times \mathbf{w} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -1 \\ 2 & 2 & -1 \end{pmatrix} = 4\mathbf{i} + 8\mathbf{k},$$

so $|\mathbf{v} \times \mathbf{w}| = \sqrt{16 + 64} = \sqrt{80}$.

The projection is given by

$$\mathrm{proj}_{\mathbf{v}}\mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|^2}\mathbf{v} = \frac{1}{9}(2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = \frac{2}{9}\mathbf{i} - \frac{2}{9}\mathbf{j} - \frac{1}{9}\mathbf{k}$$

The magnitude of this vector is therefore

$$|\mathrm{proj}_{\mathbf{v}}\mathbf{w}| = \sqrt{(2/9)^2 + (2/9)^2 + (1/9)^2} = \sqrt{1/9} = 1/3.$$

Clearly $\mathbf{v} \times \mathbf{w}$ has larger magnitude.

Alternatively, you could have avoided computing the projection explicitly and instead just computed its magnitude:

$$|\mathrm{proj}_{\mathbf{v}}\mathbf{w}| = \frac{|\mathbf{v} \cdot \mathbf{w}|}{|\mathbf{v}|} = \frac{1}{3}.$$

Either way is fine.

3 points for computation of $\mathbf{v} \times \mathbf{w}$ with an additional 2 points for computing its magnitude. Similarly 3 points for computation of $\mathrm{proj}_{\mathbf{v}}\mathbf{w}$ and 2 points for computing its magnitude. Finally, the remaining 2 points for choosing the right answer.

(3) An object is moving in space with a constant acceleration $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. At time $t = 0$ the object is at the point $(1, 1, 1)$ and has velocity $\mathbf{k}$. How far is the object from the plane H defined by $x + y + z = 1$ at time t? [11 points]

The position vector of the object is

$$\mathbf{r}(t) = \mathbf{r}(0) + \mathbf{v}(0)t + \frac{1}{2}\mathbf{a}t^2 = \left(1 + \frac{1}{2}t^2\right)\mathbf{i} + (1 - t^2)\mathbf{j} + \left(1 + t + \frac{1}{2}t^2\right)\mathbf{k}.$$

Thus at time t, the object is at the point $Q(t) = (1 + \frac{1}{2}t^2, 1 - t^2, 1 + t + \frac{1}{2}t^2)$.

We next choose a point P_0 on the plane — any point on the plane will do. We will take $P_0 = (1, 0, 0)$. We also let $\mathbf{n}$ be a normal vector to the plane, $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. The distance formula says that at time t the object is at a distance

$$d(t) = \frac{|\overrightarrow{P_0Q}(t) \cdot \mathbf{n}|}{|\mathbf{n}|}$$

from H. Since

$$\overrightarrow{P_0Q}(t) = \frac{1}{2}t^2\mathbf{i} + (1 - t^2)\mathbf{j} + \left(1 + t + \frac{1}{2}t^2\right)\mathbf{k}$$

the numerator in the equation for d is

$$|\overrightarrow{P_0Q}(t) \cdot \mathbf{n}| = \left|\frac{1}{2}t^2 + 1 - t^2 + 1 + t + \frac{1}{2}t^2\right| = |2 + t|.$$

We are therefore left with

$$d(t) = \frac{|2 + t|}{|\mathbf{n}|} = \boxed{\frac{|2 + t|}{\sqrt{3}}}$$

6 points for correctly computing $\mathbf{r}(t)$, 2 points for computing $\overrightarrow{P_0Q}$, and then 3 points for completing the computation.

(4) Suppose an object has position vector $\mathbf{r}(t) = \cos(t^2)\mathbf{i} + \sin(t^2)\mathbf{j}$. Compute the tangential and normal components a_T and a_N of the object's acceleration at time $t > 0$. [14 points]

Computing a_T is a straightforward computation:

$$\mathbf{v}(t) = -2t\sin t^2\mathbf{i} + 2t\cos t^2\mathbf{j} \quad \Longrightarrow \quad |\mathbf{v}(t)| = \sqrt{4t^2\sin^2(t^2) + 4t^2\cos^2(t^2)} = 2t$$

$$a_T = \frac{d|\mathbf{v}|}{dt} = \boxed{2}$$

To compute a_N, there are two different methods, namely using one of the following two identities:

$$a_N = |\mathbf{v}|\left|\frac{d\mathbf{T}}{dt}\right| \qquad \text{or} \qquad a_N = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Method 1: We must first compute $\mathbf{T}$ and its derivative:

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -\sin t^2\mathbf{i} + \cos t^2\mathbf{j} \quad \Longrightarrow \quad \frac{d\mathbf{T}}{dt} = -2t\cos t^2\mathbf{i} - 2t\sin t^2\mathbf{j} \quad \Longrightarrow \quad \left|\frac{d\mathbf{T}}{dt}\right| = 2t$$

Therefore

$$a_N = |\mathbf{v}|\left|\frac{d\mathbf{T}}{dt}\right| = \boxed{4t^2}$$

Method 2: We must first compute $\mathbf{a}$ and its magnitude.

$$\mathbf{a}(t) = \mathbf{v}'(t) = \left(-2\sin t^2 - 4t^2\cos t^2\right)\mathbf{i} + \left(2\cos t^2 - 4t^2\sin t^2\right)\mathbf{j} \quad \Longrightarrow \quad |\mathbf{a}(t)| = \sqrt{4 + 16t^4}$$

Therefore

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{(4 + 16t^4) - 4} = \boxed{4t^2}$$

2 points for computing $\mathbf{v}$ and 2 more points for computing $|\mathbf{v}(t)|$. Then 2 points for computing a_T.

If you used method 1, then you get 2 points each for correctly computing $\mathbf{T}$, $\mathbf{T}'$, $|\mathbf{T}'|$, and a_N, for a total of 8 points.

If you used method 2, you get 2 points for correctly computing $\mathbf{a}$, 4 points for computing $|\mathbf{a}|$, and 2 points for a_N, for a total of 8 points.