MATH 2401 QUIZ 9

Problem 1. Explicitly state and use the component test for conservative fields, then compute the scalar potential function for:

$$F(x, y, z) = \pi yz \cos \pi x\, \mathbf{i} + z \sin \pi x\, \mathbf{j} + y \sin \pi x\, \mathbf{k}$$

Solution:

$$\frac{\partial P}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial z} \qquad \frac{\partial M}{\partial z} \stackrel{?}{=} \frac{\partial P}{\partial x} \qquad \frac{\partial N}{\partial x} \stackrel{?}{=} \frac{\partial M}{\partial y}$$

$$(a) \quad \frac{\partial P}{\partial y} = \sin \pi x = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \pi y \cos \pi x = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \pi z \cos \pi x = \frac{\partial M}{\partial y}$$

hence conservative.

$$(b) \quad \frac{\partial f}{\partial x} = \pi yz \cos \pi x \quad \Rightarrow \quad f(x, y, z) = \int \pi yz \cos \pi x dx = yz \sin \pi x + \varphi(y, z)$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = z \sin \pi x + \frac{\partial \varphi}{\partial y} \quad \Rightarrow \quad \frac{\partial \varphi}{\partial y} = 0 \quad \Rightarrow \varphi(y, z) = \varphi(z),$$

$$\frac{\partial f}{\partial z} = y \sin \pi x + \frac{\partial \varphi}{\partial z} \quad \Rightarrow \quad \frac{\partial \varphi}{\partial z} = 0 \quad \Rightarrow \varphi(z) = C \quad (\mathit{true\ const}),$$

$$f(x, y, z) = yz \sin \pi x + C$$

Problem 2. Evaluate:

$$\int_c \left(\left(2xy + z^2\right)\ \mathbf{i} + x^2\, \mathbf{j} + (2xz)\ \mathbf{k} \right) \cdot d\mathbf{r}$$

over the curve $C: \quad \mathbf{r}(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + (2\pi t - t^2)\ \mathbf{k} \quad 0 \leq t \leq 2\pi.$

Solution: First observe that

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = 2x = \frac{\partial M}{\partial y}$$

This is a conservative vector field so that the fundamental theorem for line integrals applies. Next note that $C: \quad \mathbf{r}(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + (2\pi t - t^2)\ \mathbf{k} \quad 0 \leq t \leq 2\pi$ is a closed curve, thus we immediately have $\int_c ((2xy + z^2)\ \mathbf{i} + x^2\, \mathbf{j} + (2xz)\ \mathbf{k}) \cdot d\mathbf{r} = 0$.

Alternative solution: $\mathbf{r}'(t) = -\sin t\, \mathbf{i} + \cos t\, \mathbf{j} + (2\pi - 2t)\ \mathbf{k}$, and

$$\int_c \left(\left(2xy + z^2\right)\ \mathbf{i} + x^2\, \mathbf{j} + (2xz)\ \mathbf{k} \right) \cdot d\mathbf{r} =$$

$$\int_0^{2\pi} \left(\left(2 \cos t \sin t + \left(2\pi t - t^2\right)^2 \right)\ \mathbf{i} + (\cos t)^2\ \mathbf{j} + \left(2 \cos t \left(2\pi t - t^2\right)\right)\ \mathbf{k} \right)$$

$$\cdot \left(-\sin t\, \mathbf{i} + \cos t\, \mathbf{j} + (2\pi - 2t)\ \mathbf{k}\right) dt$$

pretty much doomed here with 20 minutes, but we know diligence will yield zero.

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Problem 3. Use Green's theorem to evaluate:

$$\oint_c e^x \sin y dx + e^x \cos y dy$$

where $C: \ (x-a)^2 + (y-b)^2 = r^2$

Solution:

$$\oint_c e^x \sin y dx + e^x \cos y dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_R \left(e^x \cos y - e^x \cos y \right) dA$$
$$= 0$$