NAME →

ISyE 3044 — Spring 2013 — Test #2

This test is 90 minutes. You're allowed two cheat sheets. **Just show your extremely neat answers — any intermediate steps, multiple answers, or untidiness will be penalized.** All questions are 3 points, except #1 (1 point) and #33 (6 points). Good luck!

1. What is the secret word?

   Solution: Cramér. □

2. What is the Arena expression for a normal random variable with mean 4 and variance 9?

   Solution: `NORM(4,3)`. □

3. What was the capacity of the `QUEUE` block in our Area call center example?

   Solution: 0. □

4. How many sets of servers did we use in our Arena call center example?

   Solution: 3. □

5. TRUE or FALSE? You can use a single Arena `DECIDE` block to conditionally route customers to one of three possible destinations.

   Solution: TRUE. □

6. YES or NO? Is the linear congruential generator $X_{i+1} = (7X_i + 5)\text{mod}(8)$ full period?

   Solution: NO. If $X_0 = 1$, then we have $X_1 = 4$ and $X_2 = 1$, so it cycles after just 2 PRN's. $\Box$

7. Again consider the generator $X_{i+1} = (7X_i + 5)\text{mod}(8)$. Using $X_0 = 1$, calculate the PRN U_{123}.

   Solution: By the previous question, we have $X_0 = 1$, $X_1 = 4$, $X_2 = 1$, $X_3 = 4$, $\ldots$, $X_{123} = 4$. This implies that $U_{123} = 0.5$. $\Box$

8. Consider our desert island generator $X_{i+1} = 16807\, X_i \bmod(2^{31} - 1)$. If $X_0 = 76543$, find X_2.

   Solution: Using a very precise calculator (that keeps enough integer digits in storage), or using the algorithm from class, we find that

$$\begin{aligned} X_1 &= 16807\, X_0 \bmod(2^{31} - 1) \\ &= (16807)(76543) \bmod(2^{31} - 1) \\ &= 1286458201 \bmod(2^{31} - 1) \\ &= 1286458201. \end{aligned}$$

   Then, similarly,

$$X_2 = (16807)(1286458201) \bmod(2^{31} - 1) = 637626211. \quad \Box$$

9. Consider the following 20 PRN's.

0.56	0.26	0.36	0.59	0.71	0.85	0.99	0.77	0.63	0.60
0.50	0.38	0.29	0.38	0.91	0.62	0.41	0.30	0.11	0.45

   How many runs up and down do you get from this sequence?

   Solution: Letting $+/-$ denote an up / down move, respectively, we have

$$\begin{array}{cccccccccc} & - & + & + & + & + & + & - & - & - \\ - & - & - & + & + & - & - & - & - & + \end{array}$$

This translates to $A = 6$ runs. $\square$

10. Referring to Question 9, do a runs up and down test on this sequence of PRN's to decide whether or not they're independent. Use $\alpha = 0.10$. Do you accept or reject independence?

    Solution: By class notes, we have
$$\mathsf{E}[A] = \frac{2n-1}{3} = 13.0 \quad \text{and} \quad \mathsf{Var}(A) = \frac{16n-29}{90} = 3.23.$$
    Then by the previous answer, we have
$$Z_0 = \frac{A - \mathsf{E}[A]}{\sqrt{\mathsf{Var}(A)}} = \frac{6-13}{\sqrt{3.23}} = -3.89.$$
    Since $|Z_0| > z_{0.025} = 1.645$, we reject the null hypothesis of independence; and we conclude that the PRN's are dependent. $\square$

11. Again referring to the data set from Question 9, let's conduct a χ^2 goodness-of-fit test to test the hypothesis that the numbers are Unif(0,1). We'll use 4 equal-probability intervals and level $\alpha = 0.10$. What's the value of the g-o-f statistic, χ_0^2?

    Solution: The $k = 4$ intervals are $[0, 0.25]$, $(0.25, 0.5]$, $(0.5, 0.75]$, and $(0.75, 1]$, for which $E_1 = E_2 = E_3 = E_4 = 20/4 = 5$. We easily find $O_1 = 1$, $O_2 = 9$, $O_3 = 6$, and $O_4 = 4$. Thus,
$$\chi_0^2 \ = \ \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \ = \ \frac{1}{5}\sum_{i=1}^{k}(O_i - 5)^2 = 6.8. \quad \square$$

    Note: Some of you may have divided the subintervals as open on the right: $[0, 0.25)$, $[0.25, 0.5)$, $[0.5, 0.75)$, and $(0.75, 1]$. Amazingly, this changes the answer a little bit, because we then have $O_1 = 1$, $O_2 = 8$, $O_3 = 7$, and $O_4 = 4$, for which we obtain $\chi_0^2 = 6.0$. $\square$

12. Now referring to the instructions from Question 11, what's the appropriate χ^2 quantile value?

    Solution: $\chi^2_{\alpha,k-1} = \chi^2_{0.10,3} = 6.25$. $\square$

13. Again referring to the instructions from Question 11, do we accept or reject the null hypothesis of uniformity?

    Solution: Since $\chi_0^2 = 6.8 > 6.25$, we (barely) reject. $\square$

    Note: If you used the alternative subintervals (open on the right) when solving Question 11, you would have found that $\chi_0^2 = 6.0 < 6.25$, so that we (barely) accept! $\square$

14. What is 1 `XOR` 1?

    Solution: 0. $\square$

15. Consider a Tausworthe generator with $r = 2$, $q = 3$, $B_1 = 1$, $B_2 = 1$, and $B_3 = 0$. Find B_8.

    Solution: Using $B_i = (B_{i-r} + B_{i-q})\text{mod}(2) = (B_{i-2} + B_{i-3})\text{mod}(2)$, we quickly obtain

    $$B_1 = 1, \quad B_2 = 1, \quad B_3 = 0, \quad B_4 = 0, \quad B_5 = 1, \quad B_6 = 0, \quad B_7 = 1,$$

    and then things start to repeat. (In fact, this makes sense since the bits are indeed supposed to repeat every $2^q - 1 = 7$ iterations.) In any case, $B_8 = B_1 = 1$. $\square$

16. Suppose the random variable X has p.d.f. $f(x) = 3x^2/8$ for $0 \le x \le 2$. Find the inverse of its c.d.f., i.e., $F^{-1}(U)$.

    Solution: The c.d.f. is $F(x) = x^3/8$, for $0 < x < 2$. Set $F(X) = X^3/8 = U$ and solve for $F^{-1}(U) = X = 2U^{1/3}$. $\square$

17. If X is standard normal, use the inverse transform method with $U = 0.933$ to generate a realization of X.

    Solution: The c.d.f. is $X = \Phi^{-1}(U) = \Phi^{-1}(0.933) = 1.5$. $\square$

18. Suppose that X has the *Pareto* distribution with c.d.f. $F(x) = 1-(b/x)^a$, for $x > b$, where the constants $a > 1$ and $b > 0$. What is the distribution of the random variable $F(X)$?

    Solution: By the Inverse Transform Theorem, $F(X) \sim \text{Unif}(0, 1)$. $\Box$

19. Referring to Question 18, suppose that we are dealing with a Pareto distribution with $a = 2$ and $b = 1$. Show how to generate a realization of X via inverse transform.

    Solution: In this case, we use the ITT to set $F(X) = 1 - 1/X^2 = U$. Solving, we get $X = (1 - U)^{-1/2}$. $\Box$

20. The number of years until a certain critical component fails is geometrically distributed with a probability parameter of 0.01. Use the PRN $U = 0.15$ to generate a Geom(0.01) random variate via inverse transform.

    Solution: By the ITT method from class,

    $$X = \left\lceil \frac{\ell\text{n}(1-U)}{\ell\text{n}(1-p)} \right\rceil = \left\lceil \frac{\ell\text{n}(0.85)}{\ell\text{n}(0.99)} \right\rceil = 17. \quad \Box$$

    I would have also accepted $X = \lceil \ell\text{n}(U)/\ell\text{n}(1-p) \rceil = 189$. In fact, I would've also accepted 16 or 188 (if you used a slightly different definition of the Geom(p)).

21. Show how to use the `DISC` expression in Arena to generate a random variable that equals -3 with probability 0.4 and 7.2 w.p. 0.6.

    Solution: `DISC(0.4,-3,1.0,7.2)`. $\Box$

22. Suppose that $U_1 = 0.7$ and $U_2 = 0.1$ are realizations of two i.i.d. Unif(0,1)'s. Use the Box–Muller method to generate two i.i.d. standard normals.

Solution: We have

$$\begin{aligned} Z_1 &= \sqrt{-2\ell\text{n}(U_1)}\cos(2\pi U_2) = 0.683 \\ Z_2 &= \sqrt{-2\ell\text{n}(U_1)}\sin(2\pi U_2) = 0.496 \quad \square \end{aligned}$$

23. Use your answer from Question 22 to generate a Cauchy random variable.

Solution: $Z_1/Z_2 = 1.377$. Would have also accepted $Z_2/Z_1 = 0.726$. $\square$

24. If U_1 and U_2 are i.i.d. Unif(0,1), name the distribution (with parameters) of $3\sqrt{-2\ell\text{n}(U_1)}\cos(2\pi U_2) + 2$.

Solution: $3\,\text{Nor}(0,1) + 2 \sim \text{Nor}(2,9)$. $\square$

25. If U_1 and U_2 are i.i.d. Unif(0,1), name the distribution of $U_1 + U_2 - 1$.

Solution: $\text{Tria}(-1,0,1)$. $\square$

26. Suppose that $U_1, U_2, \ldots, U_{40}$ are i.i.d. Unif(0,1). Name the approximate distribution (with parameters) of $\sum_{i=1}^{40} U_i$.

Solution: By the usual properties of the Unif(0,1) distribution,

$$\mathsf{E}\left[\sum_{i=1}^{40} U_i\right] = \sum_{i=1}^{40} \mathsf{E}[U_i] = \frac{n}{2} = 20 \quad \text{and} \quad \mathsf{Var}\left(\sum_{i=1}^{40} U_i\right) = \sum_{i=1}^{40} \mathsf{Var}(U_i) = \frac{n}{12} = \frac{40}{12}.$$

Then by the CLT, $\sum_{i=1}^{24} U_i \approx \text{Nor}(20,\ 3.33)$. $\square$

27. Suppose that $U_1 = 0.65$, $U_2 = 0.45$, $U_3 = 0.82$, $U_4 = 0.09$, and $U_5 = 0.26$. Use our acceptance-rejection technique from class to generate a $\text{Pois}(\lambda = 2)$ random variate. (You may not need to use all of the uniforms.)

Solution: The procedure is to generate uniforms until $\prod_{i=1}^{n+1} U_i < e^{-\lambda} = 0.1353$.

$n = 0$: Since $U_1 = 0.65 > 0.1353$, we reject and continue.
$n = 1$: Since $U_1U_2 = 0.2925 > 0.1353$, we reject and continue.
$n = 2$: Since $U_1U_2U_3 = 0.2399 > 0.1353$, we reject and continue.
$n = 3$: Since $U_1U_2U_3U_4 = 0.0216 < 0.1353$, we stop with $n = 3$. □

28. Suppose that X_1, X_2, X_3, X_4 are i.i.d. Exp(3). Give an equation involving a *single* PRN U that you can use to generate a realization of $\max\{X_1, X_2, X_3, X_4\}$.

Solution: Let $Y = \max\{X_1, X_2, X_3, X_4\}$. As we did for a similar example in class, the c.d.f. of Y is

$$\begin{aligned} G(y) &= \Pr(Y \le y) = \Pr(\max \le y) = \Pr(\text{all } X_i\text{'s} \le y) \\ &= (\Pr(X_1 \le y))^n = (1 - e^{-\lambda y})^n \end{aligned}$$

By the Inverse Transform Theorem, set

$$G(Y) = (1 - e^{-\lambda Y})^n = U.$$

Solving, we eventually get

$$Y = -\frac{1}{\lambda}\ell\text{n}(1 - U^{1/n}) = -\frac{1}{3}\ell\text{n}(1 - U^{1/4}). \quad \Box$$

29. Suppose $X_i = \epsilon_i + \theta\epsilon_{i-1}$, where the ϵ_i's are i.i.d. standard normal for $i = 1, 2, \ldots$. What is the name of the $X_1, X_2, \ldots$ process?

Solution: First-order moving average process, or MA(1). □

30. Referring to Question 29, find the variance of the sample mean of two consecutive observations, i.e., $\mathsf{Var}((X_1 + X_2)/2)$.

Solution: By class notes, we know that the X_i's are all $\text{Nor}(0, 1 + \theta^2)$ with $\mathsf{Cov}(X_i, X_{i-1}) = \theta$. Thus,

$$\begin{aligned} \mathsf{Var}((X_1 + X_2)/2) &= \frac{1}{4}\left[\mathsf{Var}(X_1) + \mathsf{Var}(X_2) + 2\mathsf{Cov}(X_1, X_2)\right] \\ &= \frac{1}{2}\left[\mathsf{Var}(X_1) + \mathsf{Cov}(X_1, X_2)\right] \\ &= \frac{1}{2}\left[1 + \theta^2 + \theta\right]. \quad \Box \end{aligned}$$

31. TRUE or FALSE? The covariance function of an autoregressive process decays exponentially.

    Solution: TRUE. $\Box$

32. TRUE or FALSE? Consider an $M/M/1$ queue. Let I_{i+1} denote the interarrival time between the ith and $(i+1)$st customers; let S_i be the ith customer's service time; and let W_i denote the ith customer's waiting time before service. Then

$$W_{i+1} \ = \ \max\{W_i + S_i - I_{i+1},\ 0\}.$$

    Solution: TRUE. $\Box$

33. Two types of customers arrive to a post office. The first type stands in line and waits for service at the counter, and then leaves. The second type goes to check his/her post office box, and then either leaves or joins the counter line (each with a 50% chance). Draw a crude, high-level Arena block diagram to model this system.

Table 1: Standard normal values

z	$\Pr(Z \leq z)$
1	0.8413
1.28	0.9000
1.5	0.9332
1.645	0.9500
1.96	0.9750
2	0.9773

Table 2: $\chi^2_{\alpha,\nu}$ values

$\nu \setminus \alpha$	0.10	0.05	0.025
2	4.61	5.99	7.38
3	6.25	7.81	9.35
4	7.78	9.49	11.14
5	9.24	11.07	12.83
6	10.65	12.59	14.45

Table 3: $t_{\alpha,\nu}$ values

$\nu \setminus \alpha$	0.10	0.05	0.025
7	1.415	1.895	2.365
8	1.397	1.860	2.306
9	1.383	1.833	2.262
10	1.372	1.812	2.228

Table 4: $F_{0.025,m,n}$ values

$n \setminus m$	3	4	5
3	15.44	15.10	14.88
4	9.98	9.60	9.36
5	7.76	7.39	7.15