Print Your Name: Key-1

T.A. or Section Number: ____________

1. (18 points) Let $S : \Re^3 \to \Re^3$ be the transformation that dilates the vector by a factor of 4, then rotates the vector by an angle of $60°$ counterclockwise about the z-axis, and lastly reflects the vector about the yz-plane. If B is the matrix so that $S(\vec{x}) = B\vec{x}$, find the matrix B^{-1} for the INVERSE of S.

Let S_A : dilation by 4

S_B : rotate 60^0 cc about z-axis

S_C : reflect about yz-plane

Then $S = S_C \circ S_B \circ S_A$

So $S^{-1} = S_A^{-1} \circ S_B^{-1} \circ S_C^{-1}$

Then: S_A^{-1} = contract by $1/4$ $\Rightarrow [S_A^{-1}] = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$

S_B^{-1} = rotate by -60^0 $\Rightarrow [S_B^{-1}] = \begin{bmatrix} \cos(-60°) & -\sin(-60°) & 0 \\ \sin(-60°) & \cos(-60°) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

S_C^{-1} = reflect again $\Rightarrow [S_C^{-1}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then $B^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} -1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/8 & \sqrt{3}/8 & 0 \\ \sqrt{3}/8 & 1/8 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$

2. Let $T : \Re^2 \to \Re^3$ be the transformation given by the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 2x_1 + 2x_2 \\ -x_1 - x_2 \end{bmatrix}.$$

(a) (15 points) Prove that T is a linear transformation.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be two vectrs in $\mathbb{R}^2$, and let $a \in \mathbb{R}$. Then:

$$T(a\vec{x} + \vec{y}) = T\left(a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} ax_1 + y_1 \\ ax_2 + y_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} (ax_1+y_1) + (ax_2+y_2) \\ 2(ax_1+y_1) + 2(ax_2+y_2) \\ -(ax_1+y_1) - (ax_2+y_2) \end{bmatrix} = \begin{bmatrix} ax_1 + ax_2 \\ 2ax_1 + 2ax_2 \\ -ax_1 - ax_2 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ 2y_1 + 2y_2 \\ -y_1 - y_2 \end{bmatrix}$$

$$= a\begin{bmatrix} x_1 + x_2 \\ 2x_1 + 2x_2 \\ -x_1 - x_2 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ 2y_1 + 2y_2 \\ -y_1 - y_2 \end{bmatrix} = aT(\vec{x}) + T(\vec{y}),$$

so T is linear. qed

(b) (10 points) Find the matrix A so that $T(\vec{x}) = A\vec{x}$.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ -1 & -1 \end{bmatrix}$$

(c) (12 points) Is T one-to-one? Is T onto? Explain your answers mathematically.

Row reduce $A \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

only 1 pivotal column $\Rightarrow$ not 1-1

only 1 pivotal row $\Rightarrow$ not onto

Print Your Name: Key - 1

T.A. or Section Number: ____________

3. (a) (16 points) For the matrix A below, find elementary matrices E_1 and E_2 that would row reduce A to an upper trianglular matrix when multiplied on the left of A.

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

① $R_3 = R_3 - R_1$, so $\boxed{E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}$

$\Rightarrow E_1 A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 0 \\ 0 & -3 & -6 \end{bmatrix}$

② $R_3 = R_3 + R_2$, so $\boxed{E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}$

$\Rightarrow E_2 E_1 A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{bmatrix}$

(b) (14 points) Using your answer to part (a), find matrices L and U, where L is lower triangular and U is upper triangular, so that the matrix A can be written as $A = LU$.

From part (a), $\boxed{U = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{bmatrix}}$

and $\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}}$

(can find using "shortcut" or by multiplying $E_1^{-1} E_2^{-1}$)

4. (15 points) Given that $T : \Re^2 \to \Re^2$ is a linear transformation, $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix}$ and $T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\4\end{bmatrix}$, find $T\left(\begin{bmatrix}2\\1\end{bmatrix}\right)$.

If $T(\vec{x}) = A\vec{x}$, then $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$ and

$A\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$, so:

$$A\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} = \begin{bmatrix}2 & 3\\0 & 4\end{bmatrix} \Rightarrow A = \begin{bmatrix}2 & 3\\0 & 4\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}^{-1}$$

$$\Rightarrow A = \frac{1}{2}\begin{bmatrix}2 & 3\\0 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1 & 5\\-4 & 4\end{bmatrix} = \begin{bmatrix}-1/2 & 5/2\\-2 & 2\end{bmatrix}.$$

Then $T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}-1/2 & 5/2\\-2 & 2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \boxed{\begin{bmatrix}3/2\\-2\end{bmatrix}}$

BONUS: (5 points) Let A be an $n \times n$ matrix representing the linear transformation $T : \Re^n \to \Re^n$. If A is invertible, list five more equivalent statements from the Big Theorem of Linear Algebra.

Some possibilities:

(i) the RREF form of A is I_n

(ii) the columns of A form a linearly independent set

(iii) the range of T is $\mathbb{R}^n$

(iv) T is 1-1

(v) T is onto

(vi) A can be written as a product of elementary matrices

(vii) the columns of A span $\mathbb{R}^n$

(viii) $\det(A) \neq 0$

Print Your Name: Key-2

T.A. or Section Number: ______________

1. (15 points) Given that $T : \Re^2 \to \Re^2$ is a linear transformation, $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\0\end{bmatrix}$ and $T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}5\\2\end{bmatrix}$, find $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$.

Since $T(\vec{x}) = A\vec{x}$,

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix} \text{ and } A\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}5\\2\end{bmatrix}$$

$$\Rightarrow A\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} = \begin{bmatrix}3 & 5\\0 & 2\end{bmatrix}, \text{ so } A = \begin{bmatrix}3 & 5\\0 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}^{-1}$$

$$\Rightarrow A = \frac{1}{2}\begin{bmatrix}3 & 5\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}-2 & 8\\-2 & 2\end{bmatrix} = \begin{bmatrix}-1 & 4\\-1 & 1\end{bmatrix}$$

Then $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}-1 & 4\\-1 & 1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}$

$$= \boxed{\begin{bmatrix}7\\1\end{bmatrix}}$$

2. Let $T : \Re^2 \to \Re^3$ be the transformation given by the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \\ 2x_1 - 2x_2 \end{bmatrix}.$$

(a) (15 points) Prove that T is a linear transformation.

Let $\vec{x}, \vec{y} \in \mathbb{R}^2$ with $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, and $a \in \mathbb{R}$.

Then: $T(a\vec{x} + \vec{y}) = T\left(a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} ax_1 + y_1 \\ ax_2 + y_2 \end{bmatrix}\right)$

$$= \begin{bmatrix} (ax_1+y_1)-(ax_2+y_2) \\ -(ax_1+y_1)+(ax_2+y_2) \\ 2(ax_1+y_1)-2(ax_2+y_2) \end{bmatrix} = \begin{bmatrix} ax_1 - ax_2 \\ -ax_1 + ax_2 \\ 2ax_1 - 2ax_2 \end{bmatrix} + \begin{bmatrix} y_1 - y_2 \\ -y_1 + y_2 \\ 2y_1 - 2y_2 \end{bmatrix}$$

$$= a\begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \\ 2x_1 - 2x_2 \end{bmatrix} + \begin{bmatrix} y_1 - y_2 \\ -y_1 + y_2 \\ 2y_1 - 2y_2 \end{bmatrix} = aT(\vec{x}) + T(\vec{y}),$$ so T is linear. qed

(b) (10 points) Find the matrix A so that $T(\vec{x}) = A\vec{x}$.

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}$$

(c) (12 points) Is T one-to-one? Is T onto? Explain your answers mathematically.

Row reduce $A \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Only one pivotal column $\Rightarrow$ not 1-1

only one pivotal row $\Rightarrow$ not onto

Print Your Name: Key-2

T.A. or Section Number: ____________

3. (a) (16 points) For the matrix A below, find elementary matrices E_1 and E_2 that would row reduce A to an upper trianglular matrix when multiplied on the left of A.

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

① $R_3 = R_3 - R_1 \Rightarrow \boxed{E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}$

$$E_1 A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 2 & 0 \\ 0 & -2 & -4 \end{bmatrix}$$

② $R_3 = R_3 + R_2 \Rightarrow \boxed{E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}$

$$E_2 E_1 A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

(b) (14 points) Using your answer to part (a), find matrices L and U, where L is lower triangular and U is upper triangular, so that the matrix A can be written as $A = LU$.

From part (a), $\boxed{U = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}}$

Then $\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}}$

4. (18 points) Let $S : \Re^3 \to \Re^3$ be the transformation that dilates the vector by a factor of 2, then rotates the vector by an angle of 30° counterclockwise about the z-axis, and lastly reflects the vector about the xz-plane. If B is the matrix so that $S(\vec{x}) = B\vec{x}$, find the matrix B^{-1} for the INVERSE of S.

S_A: dilate by 2 $\Rightarrow$ S_A^{-1}: contract by 1/2

S_B: rotate 30° cc about z-axis $\Rightarrow$ S_B^{-1} = rotate by -30°

S_C: reflect about xz-plane $\Rightarrow$ S_C^{-1}: reflect again

So $B^{-1} = [S_A^{-1}][S_B^{-1}][S_C^{-1}]$

$$= \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} \cos(-30°) & -\sin(-30°) & 0 \\ \sin(-30°) & \cos(-30°) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ -1/2 & -\sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/4 & -1/4 & 0 \\ -1/4 & -\sqrt{3}/4 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

BONUS: (5 points) Let A be an $n \times n$ matrix representing the linear transformation $T : \Re^n \to \Re^n$. If A is invertible, list five more equivalent statements from the Big Theorem of Linear Algebra.

See Form 1.