ISyE 3833 - HW 4 Due: Apr 6 / 7

1. Based on Winston Chapter 3, Review Question 7) Steelco manufactures two types of steel at three different steel mills. During a given month, each steel mill has 200 hours of blast furnace time available. Because of differences in the furnaces at each mill, the time and cost to produce a ton of steel differs for each mill. The time and cost for each mill are shown in the table below.

	Steel 1		Steel 2	
Mill	Cost ($ per ton)	Time (minutes per ton)	Cost ($ per ton)	Time (minutes per ton)
1	13	20	11	22
2	12	24	9	18
3	13	28	10	30

Each month, Steelco must manufacture at least 500 tons of Steel 1 and 600 tons of Steel 2.

An LP that minimized the cost of manufacturing the desired steel is shown below, where x_{ij} represents the tons of steel type i produced at mill j each month, for $i = 1, 2$ and $j = 1, 2, 3$:

$$
\begin{array}{llcccccccccccl}
\min & 13x_{11} & + & 11x_{21} & + & 12x_{12} & + & 9x_{22} & + & 13x_{13} & + & 10x_{23} & & \\
\text{s.t.} & x_{11} & & & + & x_{12} & & & + & x_{13} & & & \geq & 500 \\
 & & & x_{21} & & & + & x_{22} & & & + & x_{23} & \geq & 600 \\
 & 20x_{11} & + & 22x_{21} & & & & & & & & & \leq & 12000 \\
 & & & & & 24x_{12} & + & 18x_{22} & & & & & \leq & 12000 \\
 & & & & & & & & & 28x_{13} & + & 30x_{23} & \leq & 12000 \\
 & & & & & & & & \multicolumn{4}{r}{x_{11}, x_{21}, x_{21}, x_{22}, x_{13}, x_{23}} & \geq & 0.
\end{array}
$$

After converting the LP to standard maximization form, solution by the two-phase simplex algorithm, the following optimal dictionary is obtained:

$$
\begin{array}{rrcrcrcrcrcrcr}
x_{11} = & 450 & - & \frac{3}{4}x_{21} & - & x_{13} & - & \frac{3}{4}x_{23} & + & s_1 & + & \frac{3}{4}s_2 & + & \frac{1}{24}s_4 \\
x_{12} = & 50 & + & \frac{3}{4}x_{21} & & & + & \frac{3}{4}x_{23} & & & - & \frac{3}{4}s_2 & - & \frac{1}{24}s_4 \\
x_{22} = & 600 & - & x_{21} & & & - & x_{23} & & & + & s_2 & & \\
s_3 = & 3000 & - & 7x_{21} & + & 20x_{13} & + & 15x_{23} & - & 20s_1 & - & 15s_2 & - & \frac{5}{6}s_4 \\
s_5 = & 12000 & & & - & 28x_{13} & - & 30x_{23} & & & & & & \\
z = & -11850 & - & \frac{5}{4}x_{21} & & & - & \frac{1}{4}x_{23} & - & 13s_1 & - & \frac{39}{4}s_2 & - & \frac{1}{24}s_4,
\end{array}
$$

where s_1 and s_2 are the slack variables for the constraints requiring enough of Steel 1 and Steel 2 to be produced, respectively, and s_3, s_4 and s_5 are the slack variables in the constraints limiting the consumption of production time at each of the three mills, respectively. (Note that the definitions of s_1 and s_2 are

$$
\begin{aligned}
s_1 &= x_{11} + x_{12} + x_{13} - 500, \qquad \text{and} \\
s_2 &= x_{21} + x_{22} + x_{23} - 600.)
\end{aligned}
$$

Use this optimal dictionary to answer the following questions concerning sensitivity of the optimal solution and its value to changes in the data of the problem. In each case explain how you obtained your answer using the optimal dictionary.

(a) Suppose there is a proposal to change the number of hours per month that the blast furnace at Mill 2 can operate.

i. For what *range* of the number hours available at Mill 2 does the optimal basis stay feasible? We call this the *allowable range*.

Changing the number of hours per month that the blast furnace at Mill 2 can operate is equivalent to changing the slack variable for the corresponding constraint, s_4. In the optimal equations, s_4 has a negative coefficient in the second and fourth rows, so we need to check these rows to find out how far s_4 can increase without causing the current basis to become infeasible (a basic variable to become negative). From the second row, we see $50 - \frac{1}{24}s_4 \geq 0$ is needed, which implies $s_4 \leq 1200$. From the fourth row, we see $3000 - \frac{5}{6}s_4 \geq 0$ is needed, which implies $s_4 \leq 3600$. Thus, if the current basis is to stay feasible, s_4 can increase up to $\min\{50/\frac{1}{24}, 3000/\frac{5}{6}\} = 1200$. Similarly, by considering equations in which s_4 has a positive coefficient (the first row only), we see that $450 + \frac{1}{24}s_4 \geq 0$ is needed, and so s_4 can decrease to $-450/\frac{1}{24} = -10800$ without the optimal basis changing. Increasing s_4 corresponds to decreasing the time available at Mill 2, so the minimum this can be is $12000 - 1200 = 10800$ minutes, or 180 hours, and the maximum is $12000 + 10800 = 22800$ minutes, or 380 hours: the range is $[180, 380]$ hours per month. Within this range, which is often called the *allowable range*, the current basis is optimal, and the shadow price can be used to predict change in objective as a consequence of change in s_4.

ii. How will the optimal solution change as a function of change in the number of hours available at Mill 2, within the allowable range?

For number of hours available at Mill 2 within the allowable range, i.e., for s_4 in the range $[-10800, 1200]$, the current basis stays feasible (and optimal). Thus the tons of Steel 1 made at Mill 1 in the optimal solution will increase by 1 ton for each 24 minute decrease in the time Mill 2 is available, (and will decrease by 1 ton for each 24 minute increase in the time Mill 2 is available), since $x_{11} = 450 + \frac{1}{24}s_4$. Also, the tons of Steel 1 made at Mill 2 in the optimal solution will increase by 1 ton for each 24 minute increase in the time Mill 2 is available, (and will decrease by 1 ton for each 24 minute decrease in the time Mill 2 is available), since $x_{12} = 50 - \frac{1}{24}s_4$. The tons of Steel 2 made at Mill 2 in the optimal solution will not change with changes in the time Mill 2 is available, since $x_{22} = 600 + 0(s_4)$. As the optimal basis remains optimal, the nonbasic variables will remain nonbasic, and so the optimal solution will still make no steel at Mill 3, and make none of Steel 2 at Mill 1.

iii. What is the shadow price of the fourth constraint, dictating the maximum number of minutes that can be used at Mill 2, and, for changes within the allowable range, how will the monthly cost of the operation change per unit decrease in the number of hours at Mill 2? Will it increase or decrease, and by what rate?

The shadow price is the (negative of the) coefficient of s_4 in the objective equation, $\frac{1}{24}$, indicating a $\frac{1}{24}$ dollar reduction in cost per minute increase in the time available at Mill 2, i.e. about \$2.50 per hour. Since 190 hours is within the allowable range, and corresponds to s_4 increasing to $60 \times 10 = 600$ (minutes), the new optimal solution would be to make $450 + \frac{1}{24} \times 600 = 475$ tons of Steel 1 at Mill 1, $50 - \frac{1}{24} \times 600 = 25$ tons of Steel 1 at Mill 2 and 600 tons of Steel 2 at Mill 2, for a new optimal objective value of $-11850 - \frac{1}{24} \times 600 = -11,875$, so the cost has increased from \$11,850 to \$11,875 per month.

iv. If the blast furnace at Mill 2 reduced to be available for only 190 hours per month, what would the new optimal solution be, and what would the new optimal monthly cost be?

If 190 hours, which is 11,400 minutes, is available at Mill 2, then this corresponds to $s_4 = 600$, which is within the allowable range, the current basis stays feasible (and optimal). Thus the tons of Steel 1 made at Mill 1 will be $x_{11} = 450 + \frac{1}{24}(600) = 475$. Also, the tons of Steel 1 made at Mill 2 in the optimal solution will be $x_{12} = 50 - \frac{1}{24}(600) = 25$. The tons of Steel 2 made at Mill 2 in the optimal solution will remain $x_{22} = 600$. As the optimal basis remains optimal, the nonbasic variables will remain nonbasic, and so the optimal solution will still make no steel at Mill 3, and make none of Steel 2 at Mill 1. The new optimal monthly cost will be found from $z = -11850 - \frac{1}{24}(600) = -11875$, indicating that the monthly cost will increase by \$25 to \$11,875.

(b) What is the sensitivity of the optimal basis to changes in the number of hours per month that Mill 1's blast furnace is available? How much can these increase or decrease without affecting feasibility of the optimal basis, and how will increases or decreases affect the optimal solution and optimal value of the problem?

Since $s_3 > 0$ in the optimal solution, (s_3 is a basic variable, and its value is not zero), there is slack in the corresponding constraint: not all the available hours at Mill 1 are used up in the optimal solution. Since the hours haven't been completely used up, we can increase the number of Mill 1 hours available indefinitely without any effect on the optimal basis, or indeed the optimal solution. (Graphically, you can imagine that one of the constraint lines does not pass through the optimal solution, so moving it further away has no effect.)

Since $s_3 = 3000$ in the optimal solution, so the slack in the constraint is 3000 minutes, the Mill 1 blast furnace time available can decrease by 3000 minutes, or 50 hours, per month, without any change to the optimal solution. If the Mill 1 hours available was to decrease by more than 3000 minutes, the current optimal basis would become infeasible. (Graphically, you can imagine the constraint line moving towards the optimal solution, and once it passes it, the current optimal solution becomes infeasible, so the optimal basis would have to change.)

So the allowable range for Mill 1 hours available is $[150, \infty)$. Other than changing s_3, changes within this range will not change the optimal solution or its value.

(c) Consider changes to the number of tons of Steel 1 required.

i. For what range of the tons of Steel 1 required does the optimal basis stay feasible? (What is the allowable range for tons of Steel 1 required?)

Changing the tons of Steel 1 required is equivalent to changing the slack variable in that constraint, s_1. Using similar ideas, to those in part (a), we can determine that s_1 can increase by up to $3000/20 = 150$ and decrease by up to 450. Since the tons of Steel 1 produced is $500 + s_1$, the range of tons of Steel 1 required that keeps the current basis optimal is $[500 - 450, 500 + 150] = [50, 650]$. For changes within this range, the shadow price can be used to predict change in objective function as a consequence of change in the tons of Steel 1 required, and the optimal dictionary can be used to calculate the new production plans.

ii. How will the optimal solution change as a function of change in the tons of Steel 1 required, within the allowable range?

For number of tons of Steel 1 required in the allowable range, i.e., for s_1 in the range $[-450, 150]$, the current basis stays feasible (and optimal). Thus the tons of

Steel 1 made at Mill 1 in the optimal solution will increase by 1 ton for each ton increase in the number of tons of Steel 1 required, (and will decrease by 1 ton for each ton decrease in the number of tons of Steel 1 required), since $x_{11} = 450 + 1(s_1)$. Also, the tons of Steel 1 made at Mill 2 in the optimal solution will not change, since $x_{12} = 50 + 0(s_1)$, similarly, the tons of Steel 2 made at Mill 2 in the optimal solution will not change, $x_{22} = 600 + 0(s_1)$. As the optimal basis remains optimal, the nonbasic variables will remain nonbasic, and so the optimal solution will still make no steel at Mill 3, and make none of Steel 2 at Mill 1.

iii. What is the shadow price of the first constraint, dictating the minimum quantity of Steel 1 to be produced, and, for increases within the allowable range, how will the monthly cost of the operation change per unit increase in the tons of Steel 1 required? Will it increase or decrease, and by what rate?

The shadow price of the first constraint is 13, (the negative of the coefficient of s_1 in the z-row), indicating that the per unit change in the optimal objective value, per ton change in the tons of Steel 1 required, is \$13. Here increasing s_1 will decrease z, but z is the negative of the cost (conversion from a minimization to a maximization problem), so this means an increase in the cost of the operation, by a rate of \$13 per additional ton required.

iv. If the quantity of Steel 1 required decreased to 350 tons, what would the new optimal solution be, and what would the new optimal monthly cost be?

If the requirement for Steel 1 decreased to 350 tons, this corresponds to setting $s_1 = -150$, and so the new optimal solution would be to make $450 - 150 = 300$ tons of Steel 1 at Mill 1, 50 tons of Steel 1 at Mill 2 and 600 tons of Steel 2 at Mill 2, for a new optimal objective value of $-11850 - 13 \times (-150) = -9900$, so the cost has reduced from \$11,850 to \$9,900 per month.

v. If the quantity of Steel 1 required increased to 580 tons, what would the new optimal solution be, and what would the new optimal monthly cost be?

If the requirement for Steel 1 increased to 580 tons, this corresponds to setting $s_1 = 80$, and so the new optimal solution would be to make $450 + 80 = 530$ tons of Steel 1 at Mill 1, 50 tons of Steel 1 at Mill 2 and 600 tons of Steel 2 at Mill 2, for a new optimal objective value of $-11850 - 13 \times (80) = -12890$, so the cost has increased from \$11,850 to \$12,890 per month.

(d) Suppose there is some uncertainty about the cost per ton of Steel 1 produced at Mill 2.

i. By how much can this cost increase and by how much can it decrease from \$12 without affecting the optimal basis?

ii. If the cost increased by 50 cents per ton, what would the optimal solution be, and what would it cost?

iii. Can you deduce directly from the optimal dictionary what the optimal cost would be if the cost per ton of Steel 1 produced at Mill 2 decreased by 50 cents?

[Hint: be careful about the fact that the original problem is cost minimization, while z is expressed as maximization.]

If the objective coefficient of x_{12} changed by Δ, from -12 to $-12 + \Delta$, the z-row in the dictionary would become (taking the current z-row in the dictionary and adding on

Δ multiplied by the dictionary expression for x_{12}):

$$\begin{aligned} z &= -11850 - \frac{5}{4}x_{21} - \frac{1}{4}x_{23} - 13s_1 - \frac{39}{4}s_2 - \frac{1}{24}s_4 + \Delta(50 + \frac{3}{4}x_{21} + \frac{3}{4}x_{23} - \frac{3}{4}s_2 - \frac{1}{24}s_4), \\ &= -11850 + 50\Delta - (\frac{5}{4} - \frac{3}{4}\Delta)x_{21} - (\frac{1}{4} - \frac{3}{4}\Delta)x_{23} - 13s_1 - (\frac{39}{4} + \frac{3}{4}\Delta)s_2 - (\frac{1}{24} + \frac{1}{24}\Delta)s_4. \end{aligned}$$

And so to maintain optimality, we require

$$\begin{array}{rlcll} \frac{5}{4} - \frac{3}{4}\Delta & \geq 0 & \Rightarrow & \Delta & \leq \frac{5}{3} \\ \frac{1}{4} - \frac{3}{4}\Delta & \geq 0 & \Rightarrow & \Delta & \leq \frac{1}{3} \\ \frac{39}{4} + \frac{3}{4}\Delta & \geq 0 & \Rightarrow & \Delta & \geq -13, \quad \text{and} \\ \frac{1}{24} + \frac{1}{24}\Delta & \geq 0 & \Rightarrow & \Delta & \geq -1. \end{array}$$

Thus, we require $\Delta \leq \min\{\frac{5}{3}, \frac{1}{3}\} = \frac{1}{3}$ and $\Delta \geq \max\{-13, -1\} = -1$, i.e., we need

$$-1 \leq \Delta \leq \frac{1}{3}.$$

An alternative approach is to use the formula $\bar{c}_j = c_j + c_B^T \bar{a}_j$ for the coefficient of the jth nonbasic variable in the z-row, where c_j is its objective coefficient in the original, maximization, objective function, c_B is the vector of objective coefficients (in the original, maximization, objective function) of the basic variables, and $\bar{a}_j$ is the vector of coefficients of the variable in the other rows of the optimal dictionary. To ensure the current basis remains optimal, we require $\bar{c}_j \leq 0$ for all nonbasic variables, since the z-row is given by the expression $z = \bar{z} + \sum_{j \in \text{nonbasics}} \bar{c}_j x_j$. The original, maximization, objective function is

$$z = -13x_{11} - 11x_{21} - 12x_{12} - 9x_{22} - 13x_{13} - 10x_{23},$$

and the optimal basis is x_{11}, x_{12}, x_{22}, (and s_3 and s_5), so we have that $c_B^T = (-13, -12, -9, 0, 0)$, and if we are changing the objective of x_{12} to $-12 + \Delta$, this becomes $(-13, -12 + \Delta, -9, 0, 0)$. Now we calculate, for each nonbasic variable, its changed coefficient in the objective row:

$$\bar{c}_{x_{21}} = -11 + (-13, -12 + \Delta, -9, 0, 0) \begin{pmatrix} -\frac{3}{4} \\ \frac{3}{4} \\ -1 \\ -7 \\ 0 \end{pmatrix} = -\frac{5}{4} + \frac{3}{4}\Delta,$$

$$\bar{c}_{x_{13}} = -13 + (-13, -12 + \Delta, -9, 0, 0) \begin{pmatrix} -1 \\ 0 \\ 0 \\ 20 \\ -28 \end{pmatrix} = 0,$$

$$\bar{c}_{x_{23}} = -10 + (-13, -12 + \Delta, -9, 0, 0) \begin{pmatrix} -\frac{3}{4} \\ \frac{3}{4} \\ -1 \\ 15 \\ -30 \end{pmatrix} = -\frac{1}{4} + \frac{3}{4}\Delta,$$

$$\bar{c}_{s_1} = 0 + (-13, -12+\Delta, -9, 0, 0)\begin{pmatrix} 1 \\ 0 \\ 0 \\ -20 \\ 0 \end{pmatrix} = -13,$$

$$\bar{c}_{s_2} = 0 + (-13, -12+\Delta, -9, 0, 0)\begin{pmatrix} \frac{3}{4} \\ -\frac{3}{4} \\ 1 \\ -15 \\ 0 \end{pmatrix} = -\frac{39}{4} - \frac{3}{4}\Delta, \qquad \text{and}$$

$$\bar{c}_{s_4} = 0 + (-13, -12+\Delta, -9, 0, 0)\begin{pmatrix} \frac{1}{24} \\ -\frac{1}{24} \\ 0 \\ -\frac{5}{6} \\ 0 \end{pmatrix} = -\frac{1}{24} - \frac{1}{24}\Delta.$$

To ensure that the current basis stays optimal, we need

$$\begin{array}{rcccl}
\bar{c}_{x_{21}} \le 0 & \Leftrightarrow & -\frac{5}{4} + \frac{3}{4}\Delta \le 0 & \Leftrightarrow & \Delta \le \frac{5}{3} \\
\bar{c}_{x_{13}} \le 0 & \Leftrightarrow & 0 \le 0 & & \\
\bar{c}_{x_{23}} \le 0 & \Leftrightarrow & -\frac{1}{4} + \frac{3}{4}\Delta \le 0 & \Leftrightarrow & \Delta \le \frac{1}{3} \\
\bar{c}_{s_1} \le 0 & \Leftrightarrow & -13 \le 0 & & \\
\bar{c}_{s_2} \le 0 & \Leftrightarrow & -\frac{39}{4} - \frac{3}{4}\Delta \le 0 & \Leftrightarrow & \Delta \ge -13 \quad \text{and} \\
\bar{c}_{s_4} \le 0 & \Leftrightarrow & -\frac{1}{24} - \frac{1}{24} \le 0 & \Leftrightarrow & \Delta \ge -1.
\end{array}$$

As with the first approach, this shows us that we need

$$-1 \le \Delta \le \frac{1}{3}.$$

(The two approaches are equivalent.)

Since the objective coefficient of x_{12} in the maximization objective is -12, this means the coefficient can range between $-12 + (-1) = -13$ and $-12 + \frac{1}{3} = -11\frac{2}{3}$. Hence the cost per ton of Steel 1 produced at Mill 2 can range between $\$11\frac{2}{3}$ and $\$13$, i.e., it can decrease by up to $\approx\$0.33$ or increase by up to $\$1$ without affecting the optimal basis. In either case, the change in objective function is 50Δ, where the cost increase is $-\Delta$, since the optimal solution produces 50 tons of Steel 1 at Mill 2. So if the cost per ton increased by 50 cents, the optimal cost would increase by \$25. A cost per ton decrease of 50 cents is outside the range that ensures the current basis stays optimal, and so the change cannot be immediately deduced from the optimal dictionary.

Changing the objective coefficient of a variable within a small (allowable) range, as the range $[-13, -11\frac{2}{3}]$ is for the coefficient of x_{12}, does not change the optimal solution, only its value. So the optimal plan stays the same for a 50 cent increase in the cost per ton of Steel 1 at Mill 2. Again, since a decrease of 50 cents is outside the range, any change cannot be immediately deduced from the optimal dictionary.

(e) Suppose there is some uncertainty about the cost per ton of Steel 2 produced at Mill 1. By how much can this cost increase and by how much can it decrease from \$11 without affecting the optimal basis?

x_{21} is a nonbasic variable with coefficient $\overline{c}_{x_{21}} = -\frac{5}{4}$ in the optimal z-row. Since x_{21} is nonbasic, its value is zero in the current optimal solution: none of Steel 2 is made at Mill 1 in the optimal solution. Clearly increasing the cost of Steel 2 at Mill 1 cannot encourage more production! Thus it can increase indefinitely without affecting the optimal basis. (Its objective coefficient in the maximization problem, -11, can decrease indefinitely without affecting the optimal basis.)

If the cost of Steel 2 at Mill 1 decreases, on the other hand, the coefficient of x_{21} in the optimal objective equation will increase. If it increases above zero, the current basis would no longer be optimal. To see how far it can change, we use the formula $\overline{c}_{x_{21}} = c_{x_{21}} + c_B^T \overline{a}_{x_{21}}$ for the coefficient of x_{21} in the z-row, where $c_{x_{21}} = -11$ is the objective coefficient in the original, maximization, objective. Since x_{21} is nonbasic, c_B is unchanged. The only change is that Δ is added to $c_{x_{21}}$. We know that $c_{x_{21}} + c_B^T \overline{a}_{x_{21}} = -\frac{5}{4}$, since this is the coefficient we see for x_{21} in the current optimal objective row. Thus, after a change of Δ in the objective coefficient of x_{21}, we have

$$\overline{c}_{x_{21}} = (c_{x_{21}} + \Delta) + c_B^T \overline{a}_{x_{21}} = (c_{x_{21}} + c_B^T \overline{a}_{x_{21}}) + \Delta = -\frac{5}{4} + \Delta.$$

So for the current basis to stay optimal, we need

$$-\frac{5}{4} + \Delta \leq 0 \qquad \Leftrightarrow \qquad \Delta \leq \frac{5}{4}.$$

The objective coefficient is thus no greater than $-11 + \frac{5}{4} = -9.75$. In other words, the cost of Steel 2 and Mill 1 can decrease to $11 - \frac{5}{4} = 9.75$ dollars per ton without affecting the optimal basis.

2. Consider the problem faced by a small brewery, currently making a pale ale and an American beer, that require different proportions of scarce resources: corn, hops and malt. The quantities of these resources needed, per barrel of beer produced, is shown in the table below.

	Quantity per barrel beer produced			
Type of beer	Corn (lbs)	Hops (oz)	Malt (lbs)	Profit ($)
Pale ale	5	4	35	13
American beer	15	4	20	23
Available	480	160	1190	

The table also shows the profit made per barrel, and the quantity of each resource available in the coming production period. The brewery has formulated the following Linear Program to decide its production plan for the coming period, so as to maximize its total profit, where x_1 represents the number of barrels of pale ale to make, and x_2 represents the number of barrels of American beer:

$$
\begin{aligned}
\max z = \quad & 13x_1 + 23x_2 & \\
\text{s.t.} \quad & 5x_1 + 15x_2 \le 480 & (1)\\
& 4x_1 + 4x_2 \le 160 & (2)\\
& 35x_1 + 20x_2 \le 1190 & (3)\\
& x_1, x_2 \ge 0. &
\end{aligned}
$$

After the addition of slack variables, s_1, s_2 and s_3 for each of the three resource constraints, respectively, and application of the simplex method, the brewery determines the following optimal dictionary for the LP to be:

$$
\begin{array}{rcrcrcr}
x_1 & = & 12 & + & \frac{1}{10}s_1 & - & \frac{3}{8}s_2 \\
x_2 & = & 28 & - & \frac{1}{10}s_1 & + & \frac{1}{8}s_2 \\
s_3 & = & 210 & - & \frac{3}{2}s_1 & + & \frac{85}{8}s_2 \\
z & = & 800 & - & s_1 & - & 2s_2.
\end{array}
$$

(a) What is the basis matrix of the optimal solution, A_B?

$$A_B = \begin{bmatrix} 5 & 15 & 0 \\ 4 & 4 & 0 \\ 35 & 20 & 1 \end{bmatrix}$$

(b) What is the constraint coefficient matrix corresponding to the nonbasic variables, A_N?

$$A_N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(c) What is A_B^{-1}?

$$A_B^{-1} = \begin{bmatrix} -\frac{1}{10} & \frac{3}{8} & 0 \\ \frac{1}{10} & -\frac{1}{8} & 0 \\ \frac{3}{2} & -\frac{85}{8} & 1 \end{bmatrix}$$

(d) What is $A_B^{-1}b$, where b denotes the vector of available resource quantities?

$$A_B^{-1}b = \begin{bmatrix} 12 \\ 28 \\ 210 \end{bmatrix}$$

(e) What is c_B, where c_B denotes the vector of objective coefficients of basic variables? Now calculate $c_B^T A_B^{-1}$.

$$c_B = \begin{bmatrix} 13 \\ 23 \\ 0 \end{bmatrix}$$

and

$$c_B^T A_B^{-1} = [1, \quad 2, \quad 0] \,.$$

This can be verified by matrix multiplication, but can be observed directly from optimal dictionary: it is the vector of shadow prices of the slack variables (the negative of their coefficient in the z-row of the optimal dictionary).

(f) Suppose the brewery is thinking about introducing a new beer: a stout. One barrel of stout requires 6 oz of hops and 40 lbs of malt, and has a profit of \$16 per barrel. Stout does not use corn at all. Should the brewery make stout? Justify your answer.

We imagine introducing a new, nonbasic variable, x_3. Its coefficient in the optimal dictionary z-row, $\bar{c}_3$, is given by

$$\bar{c}_3 = c_3 - c_B^T A_B^{-1} a_{x_3}, \qquad \text{where} \qquad a_{x_3} = \begin{bmatrix} 0 \\ 6 \\ 40 \end{bmatrix}$$

and $c_3 = 16$. Using the fact that we have already deduced $c_B^T A_B^{-1}$, we simply calculate

$$\bar{c}_3 = c_3 - c_B^T A_B^{-1} a_{x_3} = 16 - [1, \quad 2, \quad 0] \begin{bmatrix} 0 \\ 6 \\ 40 \end{bmatrix} = 16 - 12 = 4.$$

Since this is positive, we would expect x_3 to enter the basis in the next simplex iteration, and hence expect that the brewery *should* make stout.

However, to be very certain that when x_3 enters the basis, it will do so at a positive (not zero) value, we may wish to note that, irrespective of the coefficients of x_3 in the dictionary for the current basis, since all current basic variables are strictly positive, some positive increase in x_3 must be possible, before any current basic variable drops to zero.

If we want to update the dictionary for the current basis, we need to introduce the column for x_3, in other words, we need to calculate

$$\bar{a}_{x_3} = A_B^{-1} a_{x_3} = \begin{bmatrix} -\frac{1}{10} & \frac{3}{8} & 0 \\ \frac{1}{10} & -\frac{1}{8} & 0 \\ \frac{3}{2} & -\frac{85}{8} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ 40 \end{bmatrix} = \begin{bmatrix} \frac{9}{4} \\ -\frac{3}{4} \\ -\frac{95}{4} \end{bmatrix}$$

The new dictionary would be

$$\begin{array}{rcrcrcrcr} x_1 & = & 12 & + & \frac{1}{10}s_1 & - & \frac{3}{8}s_2 & - & \frac{9}{4}x_3 \\ x_2 & = & 28 & - & \frac{1}{10}s_1 & + & \frac{1}{8}s_2 & + & \frac{3}{4}x_3 \\ s_3 & = & 210 & - & \frac{3}{2}s_1 & + & \frac{85}{8}s_2 & + & \frac{95}{4}x_3 \\ z & = & 800 & - & s_1 & - & 2s_2 & + & 4x_3. \end{array}$$

Note: your recitation class instructor may have used the notation B instead of A_B and N instead of A_N.

3. Write the linear programming dual of the following problems:

(a)

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ s.t. \quad & x_1 + x_2 \leq 2 \\ & 4x_1 + 2x_2 \leq 6 \\ & 2x_1 + 3x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & 2y_1 + 6y_2 + 4y_3 \\ s.t. \quad & y_1 + 4y_2 + 2y_3 \geq 3 \\ & y_1 + 2y_2 + 3y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

(b)

$$\begin{aligned} \min \quad & 2x_1 + x_2 + 4x_3 \\ s.t. \quad & x_1 - x_2 + x_3 = 2 \\ & 2x_1 + 3x_2 + 5x_3 \leq 12 \\ & -4x_1 + 2x_2 + 3x_3 \geq 1 \\ & x_1 \leq 0, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & 2y_1 + 12y_2 + y_3 \\ s.t. \quad & y_1 + 2y_2 - 4y_3 \geq 2 \\ & -y_1 + 3y_2 + 2y_3 \leq 1 \\ & y_1 + 5y_2 + 3y_3 = 4 \\ & y_1 \text{ unrestricted}, \; y_2 \leq 0, \; y_3 \geq 0 \end{aligned}$$

Note that there may be a number of equivalent forms for the dual LP. For example, an unrestricted variable may appear with the opposite sign in its coefficients throughout the LP. Or a nonpositive variable may be replaced by a nonnegative variable having the opposite sign in its coefficients throughout the LP. Constraints may be in the opposite direction, if multiplied through by -1. The main concern with changing forms of an LP in nonstandard form before taking its dual, is to be aware that if you convert a minimization to a maximization by multiplying by -1 before taking the dual, then strong and weak duality properties needs to be adjusted, for example, the dual LP optimal value will be equal to -1 times the primal LP optimal value.