MATH 1552 TEST 1, FALL 2015, GRODZINSKY

Print Your Name: Key - 1

T.A.: (circle one) Miheer Brandon Stephen Kabir

1. (12 points) A particle has an acceleration of $a(t) = e^{-3t}$ in m/s^2, with an initial velocity of $\frac{2}{3}$ m/s. Find the net distance traveled by the particle between $t = 0$ and $t = \ln 2$ seconds. Simplify as far as you can without a calculator.

$$v(t) = \int a(t)\,dt = \int e^{-3t}\,dt = -\frac{1}{3}e^{-3t} + C$$

Since $v(0) = \frac{2}{3}$, $-\frac{1}{3}e^{-3\cdot 0} + C = \frac{2}{3} \Rightarrow -\frac{1}{3} + C = \frac{2}{3}$,

so $C = 1$

We want to find: $\displaystyle\int_0^{\ln 2} v(t)\,dt = \int_0^{\ln 2} \left(-\frac{1}{3}e^{-3t} + 1\right)dt$

$$= \left(\frac{1}{9}e^{-3t} + t\right)\Big|_0^{\ln 2} = \left(\frac{1}{9}e^{-3\ln 2} + \ln 2\right) - \left(\frac{1}{9}e^0 + 0\right)$$

$$= \frac{1}{9}e^{\ln 2^{-3}} + \ln 2 - \frac{1}{9} = \frac{1}{9}\cdot\frac{1}{8} + \ln 2 - \frac{1}{9} = \boxed{\ln 2 - \frac{7}{72}}\ m$$

2. (12 points) For the function F given below, find $F'(1)$. Simplify as far as you can without a calculator.

$$F(x) = \int_{2x}^{\sin(\pi x)} \frac{1}{1+t^4}\,dt.$$

$$F'(x) = \frac{1}{1+(\sin \pi x)^4}(\pi\cos(\pi x)) - \frac{1}{1+(2x)^4}(2)$$

So:

$$F'(1) = \frac{1}{1+(\underbrace{\sin\pi}_{0})^4}(\pi\underbrace{\cos\pi}_{-1}) - \frac{1}{1+2^4}(2)$$

$$= \boxed{-\pi - \frac{2}{17}}$$

3. (10 points) Evaluate the integral:

$$\int \frac{x}{\sqrt{4-x^2}}dx.$$

$u = 4 - x^2$
$du = -2x\,dx$
$-\frac{1}{2}du = x\,dx$

$$= -\frac{1}{2}\int \frac{du}{\sqrt{u}}$$

$$= -\frac{1}{2}\cdot\frac{u^{1/2}}{1/2} + C$$

$$= \boxed{-\sqrt{4-x^2} + C}$$

4. (10 points) Evaluate the integral:

$$\int_e^{e^3} \frac{1}{x(\ln x)^4}dx.$$

$u = \ln x$
$du = \frac{1}{x}dx$

$$= \int_1^3 \frac{1}{u^4}du = -\frac{1}{3u^3}\Big|_1^3$$

$$= -\frac{1}{3\cdot 3^3} + \frac{1}{3\cdot 1^3} = -\frac{1}{81} + \frac{1}{3}$$

$$= \frac{26}{81}$$

5. (10 points) Evaluate the integral:

$$\int \frac{1}{x^2}\cot\left(\frac{4}{x}\right)dx.$$

$u = \frac{4}{x}$
$du = -\frac{4}{x^2}dx$
$-\frac{1}{4}du = \frac{1}{x^2}dx$

$$= -\frac{1}{4}\int \cot u\,du$$

$$= -\frac{1}{4}\ln|\sin u| + C$$

$$= -\frac{1}{4}\ln\left|\sin\left(\frac{4}{x}\right)\right| + C$$

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T.A.: (circle one) Miheer Brandon Stephen Kabir

6. The height in feet of a particular tree is modeled by the formula: $H(t) = t(t+2)$, where t is given in years.

(a) (14 points) Using the Trapezoidal rule with $n = 4$ subintervals, approximate the *average* height of the tree between $t = 0$ and $t = 2$ years. Then find the *maximum error* in your approximation. Recall: $|E_n^T| \leq \frac{(b-a)^3}{12n^2} \max |f''(c)|$. $\Delta t = \frac{1}{2}$

0 1/2 1 3/2 2

$$\text{Avg value} = \frac{1}{b-a}\int_a^b H(t)\,dt \approx \frac{1}{2}\cdot\frac{\Delta t}{2}\left[H(0) + 2H(1/2) + 2H(1) + 2H(3/2) + H(2)\right]$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\left[0(2) + 2\left(\frac{1}{2}\right)\left(\frac{5}{2}\right) + 2(1)(3) + 2\left(\frac{3}{2}\right)\left(\frac{7}{2}\right) + 2(4)\right]$$

$$= \frac{1}{8}\left[0 + \frac{5}{2} + 6 + \frac{21}{2} + 8\right] = \frac{27}{8} \text{ feet}$$

Since $H(t) = t^2 + 2t$
$H'(t) = 2t + 2$
$H''(t) = 2 \leftarrow$ max value

$$\Rightarrow |E_4^T| \leq \frac{2^3}{12\cdot 4^2}\cdot 2 = \frac{1}{12}$$

(b) (10 points) Approximate the *average* height of the tree between years $t = 0$ and $t = 2$ using a lower sum with $n = 4$ subintervals.

$$\text{Avg value} \approx \frac{1}{b-a}\cdot \Delta t\, L_H = \frac{1}{2}\cdot\frac{1}{2}\left[H(0) + H(1/2) + H(1) + H(3/2)\right]$$

$$= \frac{1}{4}\left[0 + \frac{5}{4} + 3 + \frac{21}{4}\right]$$

$$= \frac{1}{4}\cdot\frac{38}{4} = \frac{19}{8} \text{ feet}$$

(c) (10 points) Use the FTC to compute the actual *average* value of H on the interval $[0, 2]$.

$$Av = \frac{1}{2-0}\int_0^2 (t^2 + 2t)\,dt = \frac{1}{2}\left[\frac{1}{3}t^3 + t^2\right]\Big|_0^2$$

$$= \frac{1}{2}\left[\frac{1}{3}\cdot 8 + 4 - 0\right] = \frac{1}{2}\left[\frac{20}{3}\right] = \frac{10}{3} \text{ ft}$$

7. (12 points) Find the total area bounded between the curves $y = 6x$ and $y = \frac{6}{x^2}$ and the lines $x = \frac{1}{2}$, $x = 2$.

$$6x = \frac{6}{x^2} \Rightarrow x^3 = 1$$
$$x = 1$$

$6x < \frac{6}{x^2}$ $6x > 6/x^2$

$\frac{1}{2}$ 1 2

$$A = \int_{1/2}^{1} \left(\frac{6}{x^2} - 6x\right) dx + \int_{1}^{2} \left(6x - \frac{6}{x^2}\right) dx$$

$$= \left(-\frac{6}{x} - 3x^2\right)\Big|_{1/2}^{1} + \left(3x^2 + \frac{6}{x}\right)\Big|_{1}^{2}$$

$$= (-6-3) - \left(-12 - \frac{3}{4}\right) + (12+3) - (3+6)$$

$$= 3 + \frac{3}{4} + 6 = \boxed{9\tfrac{3}{4}}$$

BONUS: (5 points) Define $av(f)$ to be the average value of the function f on the interval $[a, b]$. Is this statement true or false: given two continuous functions f and g, $av(f + g) = av(f) + av(g)$. If it is true, prove it; if it is false, provide a counterexample.

The statement is true.

Note $av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

Then
$$av(f+g) = \frac{1}{b-a}\int_a^b (f+g)(x)\,dx$$
$$= \frac{1}{b-a}\int_a^b (f(x) + g(x))\,dx$$
$$= \frac{1}{b-a}\left[\int_a^b f(x)\,dx + \int_a^b g(x)\,dx\right]$$
$$= \frac{1}{b-a}\int_a^b f(x)\,dx + \frac{1}{b-a}\int_a^b g(x)\,dx$$
$$= av(f) + av(g). \blacksquare$$

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T.A.: (circle one) Miheer Brandon Stephen Kabir

1. (12 points) For the function F given below, find $F'(1)$. Simplify as far as you can without a calculator.

$$F(x) = \int_{3x}^{\cos\left(\frac{\pi x}{2}\right)} \frac{1}{1+t^4}\,dt.$$

$$F'(x) = \frac{1}{1+\left(\cos\frac{\pi x}{2}\right)^4}\left(-\frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right)\right) - \frac{1}{1+(3x)^4}(3)$$

So

$$F'(1) = \frac{1}{1+\underbrace{\left(\cos\frac{\pi}{2}\right)^4}_{=0}}\left(-\frac{\pi}{2}\overbrace{\sin\frac{\pi}{2}}^{=1}\right) - \frac{3}{1+3^4}$$

$$= \boxed{-\pi/2 - \frac{3}{82}}$$

2. (12 points) Find the total area bounded between the curves $y = 4x$ and $y = \frac{4}{x^2}$ and the lines $x = \frac{1}{2}$, $x = 2$.

$4x = \frac{4}{x^2} \Rightarrow x^3 = 1$, so $x = 1$

$\frac{4}{x^2} > 4x$ on $(\frac{1}{2}, 1)$, $4x > \frac{4}{x^2}$ on $(1, 2)$

$$A = \int_{1/2}^{1}\left(\frac{4}{x^2} - 4x\right)dx + \int_{1}^{2}\left(4x - \frac{4}{x^2}\right)dx$$

$$= \left(-\frac{4}{x} - 2x^2\right)\Big|_{1/2}^{1} + \left(2x^2 + \frac{4}{x}\right)\Big|_{1}^{2}$$

$$= (-4-2) - \left(-8 - \frac{1}{2}\right) + (8+2) - (2+4)$$

$$= 2 + \frac{1}{2} + 4 = \boxed{6.5}$$

3. The height in feet of a particular tree is modeled by the formula: $H(t) = t(t+3)$, where t is given in years.

(a) (14 points) Using the Trapezoidal rule with $n = 4$ subintervals, approximate the *average* height of the tree between $t = 0$ and $t = 2$ years. What is the maximum error in your approximation? Recall: $|E_n^T| \le \frac{(b-a)^3}{12n^2} \max |f''(c)|$. $\Delta t = \frac{1}{2}$, 0 1/2 1 3/2 2

$$AV = \frac{1}{b-a}\int_a^b f(x)\,dx \approx \frac{1}{b-a}\cdot\frac{1}{2}\Delta t\left[H(0) + 2H(1/2) + 2H(1) + 2H(3/2) + H(2)\right]$$

$$\approx \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\left[0 + 2\left(\frac{1}{2}\right)\left(\frac{7}{2}\right) + 2(1)(4) + 2\left(\frac{3}{2}\right)\left(\frac{9}{2}\right) + 10\right]$$

$$= \frac{1}{8}\left[\frac{7}{2} + 8 + \frac{27}{2} + 10\right] = \frac{1}{8}(18+17) = \boxed{\frac{35}{8}} \text{ feet}$$

Note: $H(t) = t^2 + 3t$

$H'(t) = 2t$

$H''(t) = 2 \leftarrow$ max value

So $|E_4^T| \le \frac{(2-0)^3}{12\cdot 4^2}(2) = \frac{2^4}{12\cdot 4^2} = \frac{1}{12}$

(b) (10 points) Approximate the *average* height of the tree between years $t = 0$ and $t = 2$ using a lower sum with $n = 4$ subintervals.

$$AV \approx \frac{1}{b-a}\Delta t\left[H(0) + H(1/2) + H(1) + H(3/2)\right]$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left[0 + \frac{7}{4} + 4 + \frac{27}{4}\right] = \frac{1}{4}\left[\frac{34}{4} + 4\right]$$

$$= \frac{25}{8} \text{ feet}$$

(c) (10 points) Use the FTC to compute the actual *average* value of H on the interval $[0, 2]$.

$$AV = \frac{1}{b-a}\int_a^b H(t)\,dt$$

$$= \frac{1}{2-0}\int_0^2 (t^2 + 3t)\,dt$$

$$= \frac{1}{2}\left[\frac{1}{3}t^3 + \frac{3}{2}t^2\right]\Big|_0^2$$

$$= \frac{1}{2}\left[\frac{8}{3} + 6\right] = \frac{26}{6} = \frac{13}{3} \text{ feet}$$

Key - 2

4. (10 points) Evaluate the integral:

$$\int \frac{1}{x^2} \tan\left(\frac{6}{x}\right) dx.$$

$u = 6/x$

$du = -6/x^2 \, dx$

$-\frac{1}{6} du = \frac{1}{x^2} dx$

$= \int -\frac{1}{6} \tan u \, du$

$= -\frac{1}{6} \ln|\sec u| + C$

$= \boxed{-\frac{1}{6} \ln|\sec(6/x)| + C}$

5. (10 points) Evaluate the integral:

$$\int \frac{x}{\sqrt{9-x^2}} dx.$$

$u = 9 - x^2$

$du = -2x \, dx$

$-\frac{1}{2} du = x \, dx$

$= -\frac{1}{2} \int \frac{du}{\sqrt{u}}$

$= -\frac{1}{2} \cdot 2\sqrt{u} + C$

$= \boxed{-\sqrt{9-x^2} + C}$

6. (10 points) Evaluate the integral:

$$\int_e^{e^3} \frac{1}{x(\ln x)^3} dx.$$

$u = \ln x$

$du = \frac{1}{x} dx$

$= \int_1^3 \frac{1}{u^3} dx = -\frac{1}{2u^2}\Big|_1^3$

$= -\frac{1}{2 \cdot 9} + \frac{1}{2 \cdot 1} = -\frac{1}{18} + \frac{1}{2}$

$= \frac{8}{18} = \boxed{\frac{4}{9}}$

7. (12 points) A particle has an acceleration of $a(t) = e^{-2t}$ in m/s^2, with an initial velocity of $\frac{1}{2}$ m/s. Find the net distance traveled by the particle between $t = 0$ and $t = \ln 3$ seconds. Simplify as far as you can without a calculator.

$$v(t) = \int a(t)\,dt = \int e^{-2t}\,dt = -\frac{1}{2}e^{-2t} + C$$

Since $v(0) = \frac{1}{2}$, $-\frac{1}{2}e^{0} + C = \frac{1}{2} \Rightarrow -\frac{1}{2} + C = \frac{1}{2}$

$\Rightarrow C = 1$

we want: $\int_0^{\ln 3} v(t)\,dt = \int_0^{\ln 3} \left(-\frac{1}{2}e^{-2t} + 1\right)dt$

$= \left(\frac{1}{4}e^{-2t} + t\right)\Big|_0^{\ln 3} = \frac{1}{4}e^{-2\ln 3} + \ln 3 - \frac{1}{4}$

$= \frac{1}{4} \cdot \frac{1}{3^2} + \ln 3 - \frac{1}{4} = \boxed{\ln 3 - \frac{2}{9}}$ m

BONUS: (5 points) Define $av(f)$ to be the average value of the function f on the interval $[a, b]$. Is this statement true or false: given two continuous functions f and g, $av(f+g) = av(f) + av(g)$. If it is true, prove it; if it is false, provide a counterexample.

See Form 1.

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Print Your Name: Key-3

T.A.: (circle one) Miheer Brandon Stephen Kabir

1. (12 points) For the function F given below, find $F'(1)$. Simplify as far as you can without a calculator.

$$F(x) = \int_{5x}^{\sin(\pi x)} \frac{1}{1+t^3} dt.$$

$$F'(x) = \frac{1}{1+(\sin\pi x)^3}(\pi\cos\pi x) - \frac{1}{1+(5x)^3}\cdot 5$$

So

$$F'(1) = \frac{1}{1+\underbrace{(\sin\pi)}_{=0}^3}(\pi\overbrace{\cos\pi}^{=-1}) - \frac{1}{1+5^3}\cdot 5$$

$$= \boxed{-\pi - \frac{5}{126}}$$

2. (12 points) A particle has an acceleration of $a(t) = e^{-4t}$ in m/s^2, with an initial velocity of $\frac{3}{4}$ m/s. Find the net distance traveled by the particle between $t = 0$ and $t = \ln 2$ seconds. Simplify as far as you can without a calculator.

$$v(t) = \int a(t)\,dt = \int e^{-4t}\,dt = -\frac{1}{4}e^{-4t} + C$$

Since $v(0) = \frac{3}{4}$, $-\frac{1}{4}e^{-4 \cdot 0} + C = \frac{3}{4} \Rightarrow -\frac{1}{4} + C = 3/4$,

so $C = 1$.

We want $\int_0^{\ln 2} v(t)\,dt = \int_0^{\ln 2}\left(-\frac{1}{4}e^{-4t} + 1\right)dt$

$$= \left(\frac{1}{16}e^{-4t} + t\right)\Big|_0^{\ln 2} = \frac{1}{16}e^{-4\ln 2} + \ln 2 - \frac{1}{16}$$

$$= \frac{1}{16}\cdot\frac{1}{2^4} + \ln 2 - \frac{1}{16} = \boxed{\ln 2 - \frac{15}{256}}$$

3. (10 points) Evaluate the integral:

$$\int_e^{e^2} \frac{1}{x(\ln x)^5}\,dx.$$

$u = \ln x$

$du = \frac{1}{x}dx$

$$= \int_1^2 \frac{du}{u^5} = -\frac{1}{4u^4}\Big|_1^2$$

$$= -\frac{1}{4 \cdot 16} + \frac{1}{4} = \boxed{\frac{15}{64}}$$

4. (10 points) Evaluate the integral:

$$\int \frac{1}{x^2}\cot\left(\frac{7}{x}\right)\,dx.$$

$u = \frac{7}{x}$

$du = -\frac{7}{x^2}dx$

$-\frac{1}{7}du = \frac{1}{x^2}dx$

$$= -\frac{1}{7}\int \cot u \, du$$

$$= -\frac{1}{7}\ln|\sin u| + C$$

$$= \boxed{-\frac{1}{7}\ln|\sin(7/x)| + C}$$

5. (10 points) Evaluate the integral:

$$\int \frac{x}{\sqrt{25-x^2}}\,dx.$$

$u = 25 - x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$$= -\frac{1}{2}\int \frac{du}{\sqrt{u}}$$

$$= -\frac{1}{2}\cdot 2\sqrt{u} + C$$

$$= \boxed{-\sqrt{25-x^2} + C}$$

Key - 3

6. The height in feet of a particular tree is modeled by the formula: $H(t) = t(t+4)$, where t is given in years.

(a) (14 points) Using the Trapezoidal rule with $n = 4$ subintervals, approximate the *average* height of the tree between $t = 0$ and $t = 2$ years. What is the maximum error in your approximation? Recall: $|E_n^T| \leq \frac{(b-a)^3}{12n^2} \max |f''(c)|$.

$$AV \approx \frac{1}{b-a} T_n \approx \frac{1}{2-0} \frac{1}{2} \frac{2-0}{4} \left[H(0) + 2H(1/2) + 2H(1) + 2H(3/2) + H(2) \right]$$

$$= \frac{1}{8}\left[0 + 2\left(\tfrac{1}{2}\right)\left(\tfrac{9}{2}\right) + 2(1)(5) + 2\left(\tfrac{3}{2}\right)\left(\tfrac{11}{2}\right) + (2)(6)\right]$$

$$= \frac{1}{8}\left[\frac{9}{2} + 10 + \frac{33}{2} + 12\right] = \frac{1}{8}\left[22 + 21\right] = \boxed{\frac{43}{8}} \text{ feet}$$

Since $H(t) = t^2 + 4t$ $\Rightarrow |E_4^T| \leq \frac{(2-0)^3}{12 \cdot 4^2} \cdot 2$

$H'(t) = 2t$

$H''(t) = 2 \rightarrow$ max value $= \boxed{\frac{1}{12}}$

(b) (10 points) Approximate the *average* height of the tree between years $t = 0$ and $t = 2$ using a lower sum with $n = 4$ subintervals.

$$AV \approx \frac{1}{b-a} \cdot \frac{b-a}{n} \left[H(0) + H(1/2) + H(1) + H(3/2) \right]$$

$$= \frac{1}{4}\left[0 + \frac{9}{4} + 5 + \frac{33}{4}\right] = \frac{1}{4}\left[\frac{42}{4} + 5\right]$$

$$= \frac{1}{4}\left[\frac{31}{2}\right] = \boxed{\frac{31}{8}} \text{ feet}$$

(c) (10 points) Use the FTC to compute the actual *average* value of H on the interval $[0, 2]$.

$$AV = \frac{1}{2-0} \int_0^2 (t^2 + 4t)\, dt$$

$$= \frac{1}{2}\left[\frac{1}{3}t^3 + 2t^2\right]\Big|_0^2$$

$$= \frac{1}{2}\left[\frac{8}{3} + 8\right] = \frac{32}{6} = \boxed{\frac{16}{3}} \text{ feet}$$

7. (12 points) Find the total area bounded between the curves $y = 2x$ and $y = \frac{2}{x^2}$ and the lines $x = \frac{1}{2}$, $x = 2$.

$\frac{2}{x^2}$ bigger, $2x$ bigger

$\frac{1}{2}$ 1 2

$$2x = \frac{2}{x^2} \Rightarrow x^3 = 1, \text{ so } x = 1$$

$$A = \int_{1/2}^{1} \left(\frac{2}{x^2} - 2x\right) dx + \int_{1}^{2} \left(2x - \frac{2}{x^2}\right) dx$$

$$= \left(-\frac{2}{x} - x^2\right)\Big|_{1/2}^{1} + \left(x^2 + \frac{2}{x}\right)\Big|_{1}^{2}$$

$$= (-2-1) - \left(-4 - \frac{1}{4}\right) + (4+1) - (1+2)$$

$$= 1 + \frac{1}{4} + 3 = \boxed{4.25}$$

BONUS: (5 points) Define $av(f)$ to be the average value of the function f on the interval $[a, b]$. Is this statement true or false: given two continuous functions f and g, $av(f+g) = av(f) + av(g)$. If it is true, prove it; if it is false, provide a counterexample.

See Form 1.

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T.A.: (circle one) Miheer Brandon Stephen Kabir

1. (12 points) Find the total area bounded between the curves $y = 8x$ and $y = \frac{8}{x^2}$ and the lines $x = \frac{1}{2}$, $x = 2$.

$8x = \frac{8}{x^2} \Rightarrow x^3 = 1, \quad x = 1$

$\frac{8}{x^2}$ larger, $8x$ larger on $\frac{1}{2}$, 1, 2

$$A = \int_{1/2}^{1} \left(\frac{8}{x^2} - 8x\right) dx + \int_{1}^{2} \left(8x - \frac{8}{x^2}\right) dx$$

$$= \left(-\frac{8}{x} - 4x^2\right)\Big|_{1/2}^{1} + \left(4x^2 + \frac{8}{x}\right)\Big|_{1}^{2}$$

$$= (-8-4) - (-16-1) + (16+4) - (4+8)$$

$$= 4 + 1 + 8 = \boxed{13}$$

2. (12 points) For the function F given below, find $F'(1)$. Simplify as far as you can without a calculator.

$$F(x) = \int_{4x}^{\cos\left(\frac{\pi x}{2}\right)} \frac{1}{1+t^3} dt.$$

$$F'(x) = \frac{1}{1 + \left(\cos\frac{\pi x}{2}\right)^3} \cdot \left(-\frac{\pi}{2}\sin\frac{\pi x}{2}\right) - \frac{1}{1+(4x)^3} \cdot 4$$

So

$$F'(1) = \frac{1}{1 + \left(\underbrace{\cos\frac{\pi}{2}}_{=0}\right)^3} \left(-\frac{\pi}{2}\overbrace{\sin\frac{\pi}{2}}^{=1}\right) - \frac{1}{1+4^3} \cdot 4$$

$$= \boxed{-\frac{\pi}{2} - \frac{4}{65}}$$

3. The height in feet of a particular tree is modeled by the formula: $H(t) = t(t+1)$, where t is given in years.

(a) (14 points) Using the Trapezoidal rule with $n = 4$ subintervals, approximate the *average* height of the tree between $t = 0$ and $t = 2$ years. What is the maximum error in your approximation? Recall: $|E_n^T| \le \frac{(b-a)^3}{12n^2} \max |f''(c)|$. $\Delta t = \frac{1}{2}$

$$AV = \frac{1}{b-a}\int_a^b H(t)\,dt \approx \frac{1}{2}\cdot\frac{1}{2}\cdot\Delta t\left[H(0)+2H(\tfrac{1}{2}) + 2H(1) + 2H(\tfrac{3}{2}) + H(2)\right]$$

$$= \frac{1}{8}\left[0 + 2(\tfrac{1}{2})(\tfrac{3}{2}) + 2(1)(2) + 2(\tfrac{3}{2})(\tfrac{5}{2}) + 2(3)\right]$$

$$= \frac{1}{8}\left[\frac{3}{2} + 4 + \frac{15}{2} + 6\right] = \frac{1}{8}[19] = \boxed{\frac{19}{8}} \text{ feet}$$

Since $H(t) = t^2 + t$, $H'(t) = 2t+1$, $H''(t) = 2$ $\Rightarrow \max|H''(t)| = 2$, so

$$|E| \le \frac{(2-0)^3}{12\cdot 4^2}\cdot 2 = \boxed{\frac{1}{12}}$$

(b) (10 points) Approximate the *average* height of the tree between years $t = 0$ and $t = 2$ using a lower sum with $n = 4$ subintervals.

$$AV \approx \frac{1}{b-a} L_H = \frac{1}{2}\cdot\Delta t\left[H(0) + H(\tfrac{1}{2}) + H(1) + H(\tfrac{3}{2})\right]$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left[0 + \frac{3}{4} + 2 + \frac{15}{4}\right] = \frac{1}{4}\left[2 + \frac{18}{4}\right]$$

$$= \frac{1}{4}\left[\frac{13}{2}\right] = \boxed{\frac{13}{8}} \text{ feet}$$

(c) (10 points) Use the FTC to compute the actual *average* value of H on the interval $[0, 2]$.

$$AV = \frac{1}{b-a}\int_a^b H(t)\,dt$$

$$= \frac{1}{2-0}\int_0^2 (t^2+t)\,dt = \frac{1}{2}\left[\frac{1}{3}t^3 + \frac{1}{2}t^2\right]\Big|_0^2$$

$$= \frac{1}{2}\left[\frac{8}{3} + 2\right] = \frac{1}{2}\left[\frac{14}{3}\right] = \boxed{\frac{7}{3}} \text{ feet}$$

Key-4

4. (10 points) Evaluate the integral:

$$\int \frac{1}{x^2} \tan\left(\frac{3}{x}\right) dx.$$

$u = 3/x$

$du = -3/x^2 \, dx$

$-\frac{1}{3} du = \frac{1}{x^2} dx$

$= -\frac{1}{3} \int \tan u \, du$

$= -\frac{1}{3} \ln|\sec u| + C$

$= \boxed{-\frac{1}{3} \ln|\sec(3/x)| + C}$

5. (10 points) Evaluate the integral:

$$\int_e^{e^2} \frac{1}{x(\ln x)^6} dx.$$

$u = \ln x$

$du = \frac{1}{x} dx$

$= \int_1^2 \frac{1}{u^6} du = -\frac{1}{5u^5}\Big|_1^2$

$= -\left(\frac{1}{5 \cdot 2^5} - \frac{1}{5 \cdot 1^5}\right) = \frac{1}{5} - \frac{1}{160}$

$= \boxed{\frac{31}{160}}$

6. (10 points) Evaluate the integral:

$$\int \frac{x}{\sqrt{16 - x^2}} dx.$$

$u = 16 - x^2$

$du = -2x \, dx$

$-\frac{1}{2} du = x \, dx$

$= -\frac{1}{2} \int \frac{du}{\sqrt{u}}$

$= -\frac{1}{2} \cdot 2\sqrt{u} + C$

$= \boxed{-\sqrt{16 - x^2} + C}$

7. (12 points) A particle has an acceleration of $a(t) = e^{-5t}$ in m/s^2, with an initial velocity of $\frac{4}{5}$ m/s. Find the net distance traveled by the particle between $t = 0$ and $t = \ln 2$ seconds. Simplify as far as you can without a calculator.

$$v(t) = \int a(t)\,dt = \int e^{-5t}\,dt = -\frac{1}{5}e^{-5t} + C$$

$$v(0) = \frac{4}{5} \Rightarrow -\frac{1}{5}e^{0} + C = \frac{4}{5},\quad -\frac{1}{5} + C = \frac{4}{5},\quad \boxed{C=1}$$

So we want:

$$\int_0^{\ln 2} v(t)\,dt = \int_0^{\ln 2}\left(-\frac{1}{5}e^{-5t} + 1\right)dt = \left(\frac{1}{25}e^{-5t} + t\right)\Big|_0^{\ln 2}$$

$$= \frac{1}{25}e^{-5\ln 2} + \ln 2 - \frac{1}{25} = \frac{1}{25}\cdot\frac{1}{2^5} + \ln 2 - \frac{1}{25}$$

$$= \frac{1}{25}\cdot\frac{1}{32} - \frac{1}{25} + \ln 2 = \boxed{\ln 2 - \frac{31}{800}}\ m$$

BONUS: (5 points) Define $av(f)$ to be the average value of the function f on the interval $[a, b]$. Is this statement true or false: given two continuous functions f and g, $av(f+g) = av(f) + av(g)$. If it is true, prove it; if it is false, provide a counterexample.

See Form 1.