Math 1501 E, Fall 2013

Exam #3

Name: ______________________________

Section: ______________________________

- You will have 50 minutes to complete the exam.
- No calculators, books, or notes allowed.
- Partial credit will be given. However, **no** credit will be given for a problem in which no work is shown, whether the answer is correct or not. Hence, show all applicable work.
- $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Question:	1	2	3	Total
Points:	24	12	12	48
Score:				

1. Consider the function

$$f(x) = 12x - 9x^2 + 2x^3$$

on the interval $[0, 3]$.

(a) (6 points) Identify the critical point(s) of f on $[0, 3]$. Evaluate f at the critical point(s).

Critical points at $f'(x)=0$ or undefined:

$f'(x) = 12 - 18x + 6x^2 = 6(x^2 - 3x + 2) = 6(x-2)(x-1)$

$x=2$ $\quad f(2) = 24 - 36 + 16 = 4$

$x=1$ $\quad f(1) = 12 - 9 + 2 = 5$

(b) (5 points) Identify the region(s) within the interval $[0, 3]$ over which f is increasing or decreasing. Use this to classify the critical point(s) as either minimum(s) or maximum(s).

$[0,1]$: use $x = \frac{1}{2}$, $f'(\frac{1}{2}) = 12 - 9 + \frac{6}{4} > 0 \Rightarrow$ increasing

$[1,2]$: use $x = \frac{3}{2}$, $f'(\frac{3}{2}) = 12 - 27 + \frac{54}{4} < 0 \Rightarrow$ decreasing

$[2,3]$: use $x = 3$, $f'(3) = 12 - 54 + 54 > 0 \Rightarrow$ increasing

So, $x=1$ is a max (goes from ~~decreasing~~ increasing to ~~increasing~~ decreasing)

$x=2$ is a min (goes from decreasing to increasing)

(c) (3 points) Identify the inflection point(s) of f on $[0, 3]$.

Inflection points at $f''(x)=0$ or undefined

$f''(x) = -18 + 12x = 0$ at $x = \frac{18}{12} = \boxed{\frac{3}{2}}$

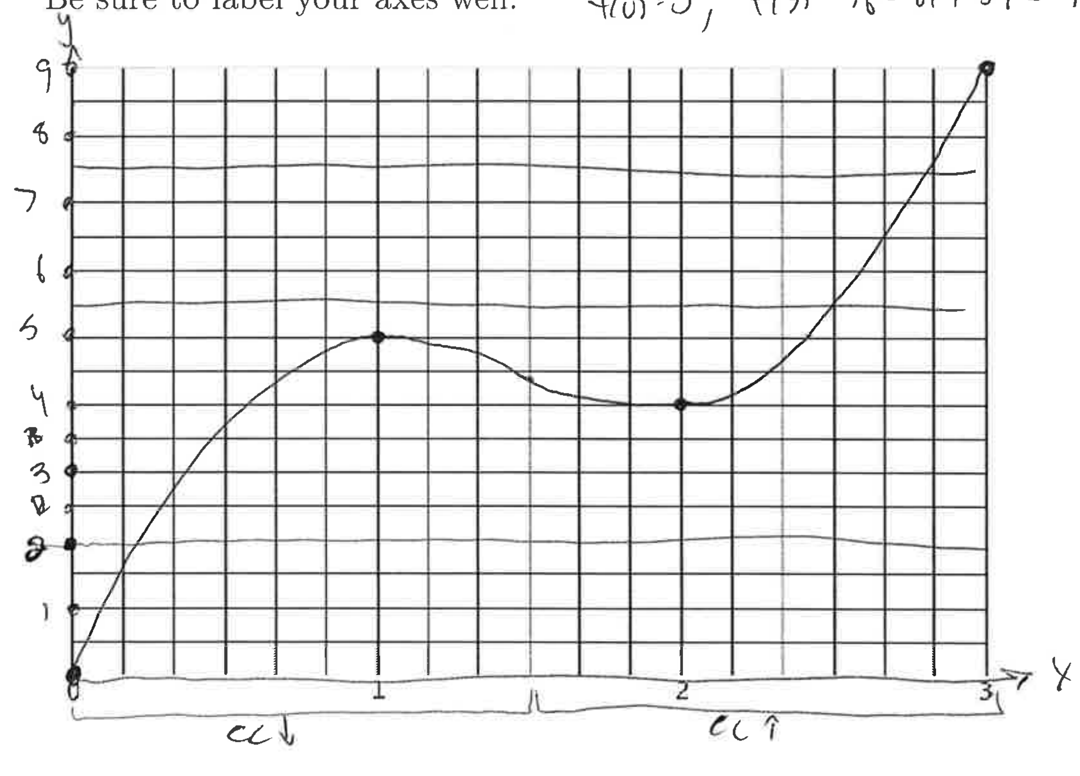
(d) (3 points) Identify the region(s) within the interval $[0, 3]$ over which f is concave up or concave down.

$[0, \frac{3}{2}]$: use $x = 1$, $f''(1) = -18 + 12 < 0 \Rightarrow$ Concave down

$[\frac{3}{2}, 3]$: use $x = 2$, $f''(2) = -18 + 24 > 0 \Rightarrow$ Concave up

(e) (5 points) Sketch a plot of f on $[0, 3]$ on the blank graph paper below. Be sure to label your axes well.

$f(0) = 0$, $f(3) = 36 - 81 + 54 = 9$

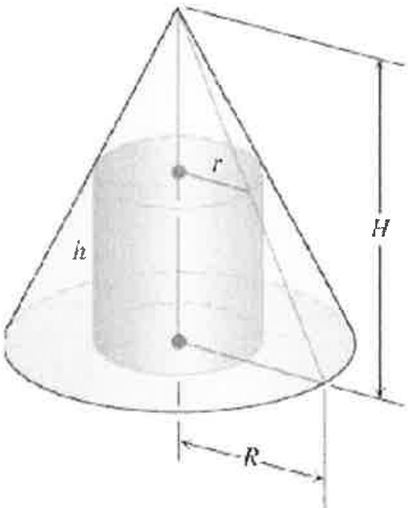


(f) (2 points) Identify the absolute minimum and maximum of f on $[0, 3]$.

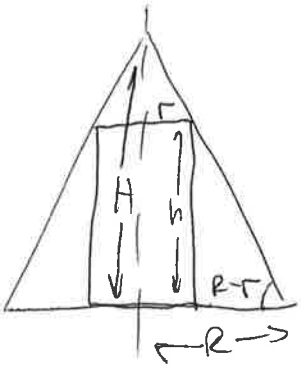
absolute min at $x = 0$ $(f(0) = 0)$

absolute max at $x = 3$ $(f(3) = 9)$

2. (12 points) Suppose that a right circular cylinder of height h and radius r is inscribed in a right circular cone of height H and radius R, as shown below. Find the value of r (in terms of R) that maximizes the total surface area of the cylinder (including the top and bottom) if $H = 3R$. **Extra Credit (3 points)**: what is the answer if $H = \frac{3}{2}R$?



$A = 2 \cdot \pi r^2 + 2\pi r h$



Similar triangles:

$\frac{h}{R-r} = \frac{H}{R} \Rightarrow$

$h = \frac{H}{R}(R-r)$

$A = 2\pi r^2 + 2\pi r \frac{H}{R}(R-r) = 2\pi r^2 + 2\pi r H - 2\pi \frac{H}{R} r^2$. Now, Maximize:

$\frac{dA}{dr} = 4\pi r + 2\pi H - 4\pi \frac{H}{R} r = 0 \Rightarrow 2\cancel{4\pi}\left(\frac{H}{R}-1\right) r = \cancel{8\pi} H$

$r = \frac{H}{2\left(\frac{H}{R}-1\right)}$. If $H = 3R$, we get $r = \frac{3R}{2(3-1)} = \frac{3R}{4}$.

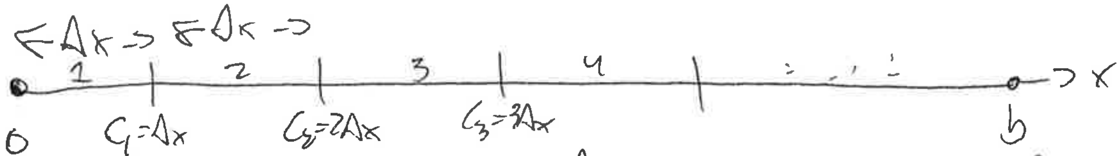
Now, r is on $[0, R]$. $\frac{d^2A}{dr^2} = 4\pi - 4\pi\frac{H}{R} = 4\pi(1-3) = -8\pi \Rightarrow$ $\frac{3R}{4}$, which is on $[0, R]$, is a max, and the absolute max since there are no other critical points.

E.C.: If $H = \frac{3R}{2}$, $r = \frac{3R}{2 \cdot 2(\frac{3}{2}-1)} = \frac{3R}{2} > R$, so it can't be the max. Instead, since $\frac{dA}{dr} > 0$ on $[0, R]$ in this case (at $r = R$, $\frac{dA}{dR} = 4\pi R + 3\pi R - 6\pi R > 0$), $r = R$ is the absolute max

3. (12 points) Evaluate the definite integral

$$\int_0^b 3x^2 - 4x^3 dx$$

by taking the limit of a Riemann sum as the norm of your partition goes to zero. In particular, choose a partition such that your subintervals are all of equal width, and choose c_k to be the right hand endpoint of the k^{th} subinterval. **Hint**: there are two useful formulas on the front cover of the exam.



$$\Delta x = \frac{b}{n}, \quad c_k = k\Delta x, \quad A = \sum_{k=1}^{n} \Delta x\, f(c_k) = \frac{b}{n}\sum_{k=1}^{n} 3(k\Delta x)^2 - 4(k\Delta x)^3$$

$$= \frac{b}{n}\sum_{k=1}^{n} 3\Delta x^2 k^2 - \frac{b}{n}\sum_{k=1}^{n} 4\Delta x^3 k^3 = 3\left(\frac{b}{n}\right)^3 \sum_{k=1}^{n} k^2 - 4\left(\frac{b}{n}\right)^4 \sum_{k=1}^{n} k^3$$

$$= 3\left(\frac{b}{n}\right)^3 \frac{n(n+1)(2n+1)}{6} - \cancel{4}\left(\frac{b}{n}\right)^4 \frac{n^2(n+1)^2}{\cancel{4}} = \frac{1}{2} b^3 \frac{\cancel{n^3}(1+\frac{1}{n})(2+\frac{1}{n})}{\cancel{n^3}} - b^4 \frac{\cancel{n^4}(1+\frac{1}{n})^2}{\cancel{n^4}}$$

$$\text{take } \lim_{n\to\infty} : \frac{1}{2} b^3 \underbrace{\lim_{n\to\infty} \frac{(1+\cancel{\frac{1}{n}}^{\,0})(2+\cancel{\frac{1}{n}}^{\,0})}{1}}_{2} - b^4 \underbrace{\lim_{n\to\infty} \frac{(1+\cancel{\frac{1}{n}}^{\,0})^2}{1}}_{1} = \boxed{b^3 - b^4}$$