MATH 1502 TEST 4, PAGE 1, FALL 2013, GRODZINSKY

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1. The augmented matrix for a system of equations is given by:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -7 & 5 \\ 0 & 1 & 0 & -1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

(a) (6 points) Is the matrix above in RREF? If not, use row operations to obtain a matrix in reduced row echelon form.

Not in RREF:

$$R_1 = R_1 + 7R_3 \quad \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

(b) (6 points) Write out the solution to the system (you may label the variables however you wish).

$$\begin{cases} x_1 = 5 \\ x_2 = 4 + t \\ x_3 = s \end{cases} \qquad \begin{cases} x_4 = t \\ x_5 = 0 \end{cases}$$

(c) (6 points) Describe the solution to part (b) geometrically (i.e., is it a point, line, plane, hyperplane, etc.).

Since there are two "free" variables, the solution is a plane in $\mathbb{R}^5$.

(d) (6 points) Which columns from the matrix form a linearly independent set? Explain how you obtained your answer.

Columns 1, 2, and 5 are pivotal, so they form a linearly independent set.

(e) (6 points) Describe the **span** of the columns geometrically (i.e., is the span a point, line, plane, hyperplane, etc.).

Since there are 3 pivotal columns and the columns are vectors in $\mathbb{R}^3$, the span is all of $\mathbb{R}^3$.

2. (10 points) Compute $A^T B$ for the matrices A and B below.

$$A = \begin{bmatrix} 6 & -6 & 1 \\ 1 & -5 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 5 & -5 & 4 \end{bmatrix}$$

$$A^T B = \begin{bmatrix} 6 & 1 \\ -6 & -5 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 5 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 11 & -5 & -2 \\ -31 & 25 & -14 \\ -19 & 20 & -17 \end{bmatrix}.$$

3. Let $\vec{v_1} = \begin{bmatrix} 4 \\ 1 \\ -6 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{v_3} = \begin{bmatrix} -20 \\ 1 \\ 15 \end{bmatrix}$.

(a) (10 points) Do the three vectors $\{v_1, v_2, v_3\}$ above form a linearly independent set? If so, explain why. If not, write $\vec{v_3}$ as a linear combination of $\vec{v_1}$ and $\vec{v_2}$.

$$\begin{bmatrix} 4 & -4 & -20 \\ 1 & 1 & 1 \\ -6 & 1 & 15 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 \\ 4 & -4 & -20 \\ -6 & 1 & 15 \end{bmatrix}$$

$$\xrightarrow[R_3+6R_1]{R_2-4R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -8 & -24 \\ 0 & 7 & 21 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since only 2 columns are pivotal, linearly dependent and $\vec{v_3} = -2\vec{v_1} + 3\vec{v_2}$

(b) (10 points) Describe the span of the vectors in part (a). Write a "formula" to find any vector in the span.

The span is all linear combinations of $\vec{v_1}$ and $\vec{v_2}$. So, if $\vec{w} \in \text{span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$,

then $\vec{w} = a\begin{bmatrix} 4 \\ 1 \\ -6 \end{bmatrix} + b\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 4a - 4b \\ a + b \\ -6a + b \end{bmatrix}, \quad a, b \in \mathbb{R}$$

This is a <u>plane</u> in $\mathbb{R}^3$.

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4. (15 points) (a) Solve the following system of equations using the Gauss-Jordan elimination method. If the solution is not unique, write your answer as a **vector equation**. You should continue your row operations until you obtain a matrix in reduced row-echelon form (RREF). (b) Then describe your solution geometrically (i.e., is it a line, plane, hyperplane, etc.).

$$\left\{\begin{array}{ll} 3x_1 + 3x_2 + 6x_3 & = 12 \\ -9x_1 - 9x_2 - 18x_3 & = -36 \\ -4x_2 + 8x_3 & = 8 \end{array}\right\}$$

$$\left[\begin{array}{ccc|c} 3 & 3 & 6 & 12 \\ -9 & -9 & -18 & -36 \\ 0 & -4 & 8 & 8 \end{array}\right] \xrightarrow[\substack{-\frac{1}{9}R_2 \\ -\frac{1}{4}R_3}]{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 1 & 1 & 2 & 4 \\ 0 & 1 & -2 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & -2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 6 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + 4x_3 = 6 \\ x_2 - 2x_3 = -2 \end{array} \Rightarrow \begin{array}{l} x_1 = 6 - 4x_3 \\ x_2 = -2 + 2x_3 \\ x_3 = x_3 \end{array}$$

So $\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix} x_3}$

This is a <u>line</u> in $\mathbb{R}^3$.

5. (5 points each) Given the points $P = (2, -1, -1)$ and $Q = (0, 2, 3)$ and the vector $\vec{v} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$ in $\Re^3$, calculate each of the following expressions, or explain why the expression is undefined.

(a) $P \cdot Q$ not defined - cannot dot points

(b) $\vec{PQ} \cdot \vec{v}$ $\vec{PQ} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$, So $\vec{PQ} \cdot \vec{v} = (-2)(-2) + (3)(0) + (4)(4)$

$= 4 + 16 = \boxed{20}$

(c) $||\vec{v}|| \cdot \vec{v}$ not defined - cannot dot a scalar with a vector

(d) $proj_{\vec{v}}\vec{PQ} = \dfrac{\vec{PQ} \cdot \vec{v}}{||\vec{v}||^2} \vec{v} = \dfrac{20}{4+16} \vec{v} = \vec{v} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$

(e) $\frac{\vec{v}}{||\vec{v}||} = \dfrac{1}{\sqrt{20}} \vec{v} = \begin{bmatrix} -2/\sqrt{20} \\ 0 \\ 4/\sqrt{20} \end{bmatrix}$

BONUS: (5 points) Prove the following vector property. Let $\vec{a}$, $\vec{b}$ be vectors in $\Re^n$ and let k be any real number. Then $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})$.

Let $\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$. Then:

$(k\vec{a}) \cdot \vec{b} = \begin{bmatrix} ka_1 \\ \vdots \\ ka_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = (ka_1)(b_1) + (ka_2)(b_2) + \dots + (ka_n)(b_n)$

$= ka_1b_1 + ka_2b_2 + \dots + ka_nb_n$

$= k(a_1b_1 + a_2b_2 + \dots + a_nb_n)$

$= k(\vec{a} \cdot \vec{b})$. qed

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1. (15 points) (a) Solve the following system of equations using the Gauss-Jordan elimination method. If the solution is not unique, write your answer as a **vector equation**. You should continue your row operations until you obtain a matrix in reduced row-echelon form (RREF). (b) Then describe your solution geometrically (i.e., is it a line, plane, hyperplane, etc.).

$$\left\{\begin{array}{ll} 2x_1 + 2x_2 + 4x_3 & = 8 \\ -6x_1 - 6x_2 - 12x_3 & = -24 \\ -5x_2 - 15x_3 & = 15 \end{array}\right\}$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 8 \\ -6 & -6 & -12 & -24 \\ 0 & -5 & -15 & 15 \end{array}\right] \xrightarrow[\substack{-1/6 R_2 \\ -1/5 R_3}]{1/2 R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & -3 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -3 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 7 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 - x_3 = 7 \\ x_2 + 3x_3 = -3 \end{array} \Rightarrow \begin{array}{l} x_1 = 7 + x_3 \\ x_2 = -3 - 3x_3 \\ x_3 = x_3 \end{array}$$

So
$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} x_3}$$

This is a <u>line</u> in $\mathbb{R}^3$.

2. Let $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$, $\vec{v_2} \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, and $\vec{v_3} = \begin{bmatrix} 2 \\ -16 \\ -3 \end{bmatrix}$.

(a) (10 points) Do the three vectors $\{v_1, v_2, v_3\}$ above form a linearly independent set? If so, explain why. If not, write $\vec{v_3}$ as a linear combination of $\vec{v_1}$ and $\vec{v_2}$.

$$\begin{bmatrix} 1 & -2 & 2 \\ 1 & 4 & -16 \\ -3 & 5 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 6 & -18 \\ 0 & -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -3 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

only 2 pivotal columns $\Rightarrow$ the set is linearly dependent and

$$\boxed{\vec{V_3} = -4\vec{V_1} - 3\vec{V_2}}$$

(b) (10 points) Describe the span of the vectors in part (a). Write a "formula" to find any vector in the span.

There are 2 pivotal columns $\Rightarrow$ span is a plane in $\mathbb{R}^3$.

If $\vec{w} \in \text{span}(\{\vec{v_1}, \vec{v_2}, \vec{v_3}\})$, then

$$\vec{w} = a\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + b\begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} a-2b \\ a+4b \\ -3a+5b \end{bmatrix}, \; a, b \in \mathbb{R}$$

3. (10 points) Compute A^TB for the matrices A and B below.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 6 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -1 & 0 \\ -8 & 0 & 3 \end{bmatrix}$$

$$A^TB = \begin{bmatrix} 2 & 0 \\ 1 & 6 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 6 & -1 & 0 \\ -8 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 12 & -2 & 0 \\ -42 & -1 & 18 \\ 16 & -4 & 3 \end{bmatrix}$$

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4. The augmented matrix for a system of equations is given by:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -4 & 3 \\ 0 & 1 & 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

(a) (6 points) Is the matrix above in RREF? If not, use row operations to obtain a matrix in reduced row echelon form.

Not yet in RREF:

$$R_1 = R_1 + 4R_3 \qquad \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

(b) (6 points) Write out the solution to the system (you may label the variables however you wish).

$$\begin{cases} x_1 = 3 \\ x_2 = 6 + t \\ x_3 = s \end{cases} \qquad \begin{cases} x_4 = t \\ x_5 = 0 \end{cases}$$

(c) (6 points) Describe the solution to part (b) geometrically (i.e., is it a point, line, plane, hyperplane, etc.).

Since there are two "free" variables this is a plane in $\mathbb{R}^5$.

(d) (6 points) Which columns from the matrix form a linearly independent set? Explain how you obtained your answer.

Columns 1, 2, and 5 are linearly independent because they are the pivotal columns.

(e) (6 points) Describe the **span** of the columns geometrically (i.e., is the span a point, line, plane, hyperplane, etc.).

There are 3 pivotal columns and each column is a vector in $\mathbb{R}^3$, so the span of the columns is all of $\mathbb{R}^3$.

5. (5 points each) Given the points $P = (1, -2, -4)$ and $Q = (4, 3, -1)$ and the vector $\vec{v} = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$ in $\Re^3$, calculate each of the following expressions, or explain why the expression is undefined.

(a) $P \cdot Q$ undefined (you cannot dot points)

(b) $\vec{PQ} \cdot \vec{v}$ $\vec{PQ} = \langle 3, 5, 3 \rangle$, so

$\vec{PQ} \cdot \vec{v} = (3)(3) + (5)(5) + (3)(0) = 9 + 25 = \boxed{34}$

(c) $||\vec{v}|| \cdot \vec{v}$ undefined (scalar · vector cannot occur)

(d) $proj_{\vec{v}}\vec{PQ}$ $= \dfrac{\vec{PQ} \cdot \vec{v}}{||\vec{v}||^2} \vec{v} = \dfrac{34}{9+25} \vec{v} = \vec{v} = \boxed{\begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}}$

(e) $\frac{\vec{v}}{||\vec{v}||}$ $= \dfrac{1}{\sqrt{34}} \vec{v} = \dfrac{1}{\sqrt{34}} \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{34} \\ 5/\sqrt{34} \\ 0 \end{bmatrix}$

BONUS: (5 points) Prove the following vector property. Let $\vec{a}$, $\vec{b}$ be vectors in $\Re^n$ and let k be any real number. Then $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})$.

See Form 1.