Math 2401
Spring 2015
Exam 3
March. 12, 2014
Time Limit: 50 Minutes

Name : ______________________

GT Id : ______________________

TA: ______________________

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. Also, sign the Honor Code pledge at the bottom of this page, and follow the instructions below.

- On this exam you may **not** use your books, notes, or any electronic device other than a non-graphing calculator.
- **Show all your work.** A correct answer not supported by calculations and/or explanation will receive no credit. An incorrect answer supported by substantially correct calculations and explanation may receive partial credit.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- If you need more space, use the back of the pages; clearly indicate when you have done so.

Problem	Points	Score
1	12	
2	14	
3	12	
4	12	
Total:	50	

Honor Code Pledge: By signing below, you are verifying that you understand and uphold the Georgia Tech honor code.
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2. Let $f(x,y,z) = e^{xz} - x^2y$.

(a) (6 points) Find the derivative of f at P$(1,-1,0)$ in the direction of $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

$$\left.\begin{array}{l}\nabla f = \langle f_x, f_y, f_z \rangle = \langle z e^{xz} - 2xy, \; -x^2, \; x e^{xz} \rangle \\ \nabla f(P) = \langle 2, -1, 1 \rangle\end{array}\right\} 3$$

Unit vector in the direction of $\vec{u}$ is $\vec{v} = \dfrac{\vec{u}}{|\vec{u}|} = \dfrac{\langle 2,2,1\rangle}{\sqrt{4+4+1}}$ $\}$ 1

$$= \frac{1}{3}\langle 2,2,1\rangle.$$

$$\left.\begin{array}{l}\therefore D_{\vec{v}} f(P) = \nabla f(P) \cdot \vec{v} = \langle 2,-1,1\rangle \cdot \frac{1}{3}\langle 2,2,1\rangle \\ \qquad = \dfrac{4-2+1}{3} = 1.\end{array}\right\} 2$$

(b) (5 points) Find the linearization of $f(x,y,z)$ at $(1,-1,0)$.

$$3 \left\{\begin{array}{l} L(x,y,z) = f(1,-1,0) + f_x(1,-1,0)(x-1) + f_y(1,-1,0)(y+1) \\ \qquad + f_z(1,-1,0)(z-0)\end{array}\right.$$

$$2 \left\{\begin{array}{l} = (1+1) + 2(x-1) + (-1)(y+1) + 1(z) \\ = 2 + 2x - 2 - y - 1 + z \\ = 2x - y + z - 1.\end{array}\right.$$

(c) (3 points) Find the differential df.

$$df = f_x\,dx + f_y\,dy + f_z\,dz \quad \} \; 2$$

$$= (z e^{xz} - 2xy)\,dx + (-x^2)\,dy + x e^{xz}\,dz. \; \} \; 1$$

1. Let $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$.

(a) (4 points) Find the critical points for f.

$\nabla f = \langle f_x, f_y \rangle = \langle 12x - 6x^2 + 6y,\ 6y + 6x \rangle$. } 2

Setting $\nabla f = \langle 0, 0 \rangle$, we get $12x - 6x^2 + 6y = 0$ — ①
and $6y + 6x = 0$ — ②

From eq^n ②, $y = -x$. Using this in eq^n ①, we get

$12x - 6x^2 + 6(-x) = 0$

or, $6x - 6x^2 = 0$

or, $6x(1 - x) = 0 \Rightarrow x = 0, 1$.

$\therefore$ The critical points of f are $(0,0)$ & $(1,-1)$. 2

(b) (8 points) Choose any one of the critical points for f, and determine if f has a saddle point, local maximum or local minimum at that point.

We have, $f_{xx} = 12 - 12x$, $f_{xy} = 6$ and $f_{yy} = 6$. } 3

At $(0,0)$, $f_{xx} f_{yy} - (f_{xy})^2 = (12 - 12x)6 - (6)^2$ } 3
$= 12(6) - 36 = 36 > 0$,

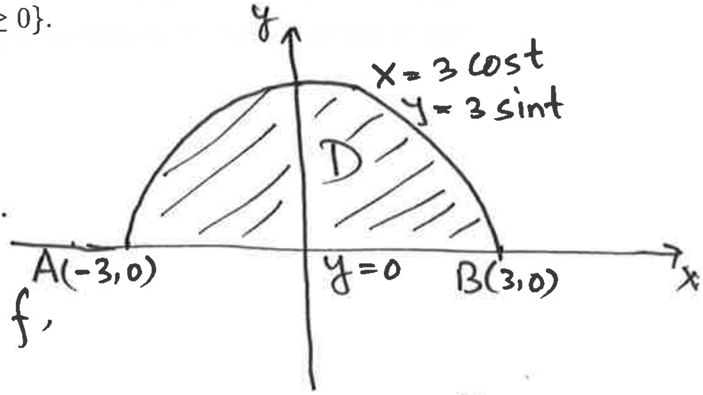
and since $f_{yy} = 6 > 0$, f has a local minimum } 2
at $(0,0)$.

OR

At $(1,-1)$: $f_{xx} f_{yy} - (f_{xy})^2 = (12 - 12(1))6 - 6^2 = -36 < 0$

So, f has a saddle point at $(1,-1)$.

3. (12 points) Find the absolute extreme values of the function $f(x, y) = x^2 + y^2 - 4x + 4y + 3$ on the closed semi-disk $D = \{(x, y) : x^2 + y^2 \le 9,\ y \ge 0\}$.



$\nabla f = \langle f_x, f_y \rangle = \langle 2x-4,\ 2y+4 \rangle$.

Setting $\nabla f = \langle 0,0 \rangle$, we get

(2) $2x-4=0,\quad 2y+4=0$ i.e. $x=2,\ y=-2$.

$\therefore (2,-2)$ is the only stationary point of f, and it is <u>not</u> in D.

On the line segment AB, f is given by

$$f(x,0) = x^2+0-4x+0+3 = x^2-4x+3 \quad (-3 \le x \le 3)$$
$$= g(x), \text{ say.}$$ (2)

$g'(x)=0 \Rightarrow 2x-4=0 \Rightarrow x=2.$

On the semicircular portion of the boundary, f is given by

$$f(3\cos t, 3\sin t) = 9\cos^2 t + 9\sin^2 t - 12\cos t + 12\sin t + 3$$
$$= 9 - 12\cos t + 12\sin t + 3$$
$$= 12 - 12\cos t + 12\sin t \quad (0 \le t \le \pi)$$
$$= h(t), \text{ say.}$$ (4)

$h'(t) = 0 \Rightarrow 12\sin t + 12\cos t = 0 \Rightarrow \sin t = -\cos t \Rightarrow t = \frac{3\pi}{4}.$

Now,

(1) $f(2,0) = 2^2 - 4(2) + 3 = -1$

$f\left(3\cos\frac{3\pi}{4}, 3\sin\frac{3\pi}{4}\right) = 12 - 12\cos\frac{3\pi}{4} + 12\sin\frac{3\pi}{4} = 12 - 12\left(\frac{-\sqrt{2}}{2}\right) + 12\frac{\sqrt{2}}{2}$
$= 12 + 6\sqrt{2} + 6\sqrt{2}$
$= 12 + 12\sqrt{2}$

(2) $f(-3,0) = 9+0+12+0+3 = 24$

$f(3,0) = 9+0-12+0+3 = 0$.

$\therefore f$ has abs. maximum value of $12+12\sqrt{2}$ at $\left(3\cos\frac{3\pi}{4}, 3\sin\frac{3\pi}{4}\right) = \left(\frac{-3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$, (1)

and abs. minimum value -1 at $(2,0)$.

4. (12 points) Use the method of Lagrange multipliers to find the maximum value of $f(x, y) = xy^2$ on the ellipse $2x^2 + y^2 = 6$.

Let $g(x,y) = 2x^2 + y^2 = 6$.

Lagrange Condition: $\nabla f = \lambda \nabla g$

$\Rightarrow \langle f_x, f_y \rangle = \lambda \langle g_x, g_y \rangle$

$\Rightarrow \langle y^2, 2xy \rangle = \lambda \langle 4x, 2y \rangle$

$\Rightarrow y^2 = 4\lambda x, \quad 2xy = 2\lambda y.$ (4)

$2xy = 2\lambda y \Rightarrow xy - \lambda y = 0 \Rightarrow y(x-\lambda) = 0$

$\Rightarrow y = 0$ or $x = \lambda$. (2)

If $y=0$, then $2x^2 + 0 = 6 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$. (2)

If $x = \lambda$, then using $y^2 = 4\lambda x$, we have $y^2 = 4x^2$. The given side condition then gives

$2x^2 + 4x^2 = 6$ or, $6x^2 = 6$ i.e. $x = \pm 1$. (2)

When $x = \pm 1$, $y^2 = 4(1) \Rightarrow y = \pm 2$.

$\therefore (\sqrt{3}, 0), (-\sqrt{3}, 0), (1, \pm 2), (-1, \pm 2)$ are the only points we need to consider for the maximum value of f.

Now, $f(\sqrt{3}, 0) = 0$, $f(-\sqrt{3}, 0) = 0$, $f(1, \pm 2) = 4$, $f(-1, \pm 2) = -4$

$\therefore f$ has maximum value of 4 at $(1, \pm 2)$. (2)