Feb. 4, 2014

Math 2401 - Exam 1

First Name (Print): ______________ Last Name (Print): ___________ Signature: _________________

- There are **5** questions on **5** pages. The exam is worth 50 points in total.
- Answer the questions clearly and completely. You must provide work clearly justifying your solution.
- You can NOT write your work on the back of the page. Use it for scratch work if needed.
- You have 50 minutes to finish your work.

1. (6 points) Find an equation for the plane that passes through $P(1,1,-1)$, $Q(2,0,2)$ and $S(0,-2,1)$.

Ch. 12.5: # 23 on page 713

Solution.

$\overrightarrow{PQ} = \vec{\boldsymbol{i}} - \vec{\boldsymbol{j}} + 3\vec{\boldsymbol{k}}$, $\overrightarrow{PS} = -\vec{\boldsymbol{i}} - 3\vec{\boldsymbol{j}} + 2\vec{\boldsymbol{k}}$, and $\overrightarrow{PQ} \times \overrightarrow{PS}$ is normal to the plane.

$$\overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 7\vec{\boldsymbol{i}} - 5\vec{\boldsymbol{j}} - 4\vec{\boldsymbol{k}}.$$

The equation for the plane is

$$7(x-1) - 5(y-1) - 4(z+1) = 0,$$

or

$$7x - 5y - 4z = 6.$$

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2. (13 points) The parametrization of a smooth curve is given by

$$\vec{\boldsymbol{r}}(t) = (e^{-t})\vec{\boldsymbol{i}} + (\cos 2t)\vec{\boldsymbol{j}} + (\sin 2t)\vec{\boldsymbol{k}}.$$

Find the angle θ between the velocity vector and acceleration vector at time $t = 0$.

[Leaving θ as an inverse trigonometric function is acceptable.]

Ch. 13.1: # 14 on page 732(even simpler)

Solution.

$$\vec{\boldsymbol{v}}(t) = \frac{d\vec{\boldsymbol{r}}(t)}{dt} = (-e^{-t})\vec{\boldsymbol{i}} + (-2\sin 2t)\vec{\boldsymbol{j}} + (2\cos 2t)\vec{\boldsymbol{k}},$$
$$\vec{\boldsymbol{a}}(t) = \frac{d\vec{\boldsymbol{v}}(t)}{dt} = (e^{-t})\vec{\boldsymbol{i}} + (-4\cos 2t)\vec{\boldsymbol{j}} + (-4\sin 2t)\vec{\boldsymbol{k}},$$

$$\Rightarrow \vec{\boldsymbol{v}}(0) = -\vec{\boldsymbol{i}} + 0\vec{\boldsymbol{j}} + 2\vec{\boldsymbol{k}}, \quad \vec{\boldsymbol{a}}(0) = \vec{\boldsymbol{i}} - 4\vec{\boldsymbol{j}} + 0\vec{\boldsymbol{k}}.$$

Therefore,

$$\vec{\boldsymbol{v}}(0) \cdot \vec{\boldsymbol{a}}(0) = (-1)(1) + (0)(-4) + (2)(0) = -1,$$
$$|\vec{\boldsymbol{v}}(0)| = \sqrt{(-1)^2 + 0^2 + 2^2} = \sqrt{5},$$
$$|\vec{\boldsymbol{a}}(0)| = \sqrt{1^2 + (-4)^2 + 0^2} = \sqrt{17}.$$

Hence,

$$\theta = \cos^{-1}\left(\frac{\vec{\boldsymbol{v}}(0) \cdot \vec{\boldsymbol{a}}(0)}{|\vec{\boldsymbol{v}}(0)||\vec{\boldsymbol{a}}(0)|}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{85}}\right).$$

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3. (9 points) Solve the initial value problem for $\vec{\boldsymbol{r}}(t)$ as a vector function of t,

$$\begin{cases} \dfrac{d\vec{\boldsymbol{r}}}{dt} = \dfrac{3}{2}(t+1)^{\frac{1}{2}}\vec{\boldsymbol{i}} + e^{-t}\vec{\boldsymbol{j}} + \dfrac{1}{t+1}\vec{\boldsymbol{k}}, \\ \vec{\boldsymbol{r}}(0) = \vec{\boldsymbol{k}}. \end{cases}$$

Ch. 13.2: # 13 on page 739

Solution.

Integrating both sides of the differential equation with respect to t gives

$$\begin{aligned} \vec{\boldsymbol{r}}(t) &= \left(\int \frac{3}{2}(t+1)^{\frac{1}{2}}dt\right)\vec{\boldsymbol{i}} + \left(\int e^{-t}dt\right)\vec{\boldsymbol{j}} + \left(\int \frac{1}{t+1}dt\right)\vec{\boldsymbol{k}} \\ &= [(t+1)^{\frac{3}{2}}]\vec{\boldsymbol{i}} + (-e^{-t})\vec{\boldsymbol{j}} + [\ln(t+1)]\vec{\boldsymbol{k}} + \overrightarrow{\boldsymbol{C}}, \end{aligned} \tag{0.1}$$

where $\overrightarrow{\boldsymbol{C}}$ is a constant vector.

We determine the constant vector $\overrightarrow{\boldsymbol{C}}$ by the initial condition. From (0.1), we have

$$\vec{\boldsymbol{r}}(0) = \vec{\boldsymbol{i}} - \vec{\boldsymbol{j}} + \overrightarrow{\boldsymbol{C}}.$$

While $\vec{\boldsymbol{r}}(0) = \vec{\boldsymbol{k}}$, so $\overrightarrow{\boldsymbol{C}} = -\vec{\boldsymbol{i}} + \vec{\boldsymbol{j}} + \vec{\boldsymbol{k}}$. Therefore,

$$\vec{\boldsymbol{r}}(t) = [(t+1)^{\frac{3}{2}} - 1]\vec{\boldsymbol{i}} + (1 - e^{-t})\vec{\boldsymbol{j}} + [1 + \ln(t+1)]\vec{\boldsymbol{k}}.$$

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4. (10 points) The parametrization of a smooth curve is given by

$$\vec{\boldsymbol{r}}(t) = (t\sin t + \cos t)\vec{\boldsymbol{i}} + (t\cos t - \sin t)\vec{\boldsymbol{j}} + \frac{\sqrt{3}}{2}t^2\vec{\boldsymbol{k}}, \qquad t \in \mathbb{R}.$$

Ch. 13.3: # 8 and # 12 on page 745 (slightly changed)

(a) Find the arc length of the curve for $-2 \le t \le 0$.

Solution.

$$\vec{v}(t) = \vec{\boldsymbol{r}}'(t) = (t\cos t)\vec{\boldsymbol{i}} - (t\sin t)\vec{\boldsymbol{j}} + (\sqrt{3}t)\vec{\boldsymbol{k}}$$

$$\Rightarrow |\vec{v}(t)| = \sqrt{(t\cos t)^2 + (-t\sin t)^2 + (\sqrt{3}t)^2} = 2|t|.$$

$$L = \int_{-2}^{0} |\vec{v}(t)|dt = \int_{-2}^{0} 2|t|dt \underbrace{= \int_{-2}^{0} (-2t)dt}_{\text{since } t\in[-2,0]} = 4.$$

$\blacksquare$

(b) Find the arc length parameter s along the curve from the base point where $t_0 = 0$.

Solution.

$$s = s(t) = \int_0^t |\vec{v}(\tau)|d\tau = 2\int_0^t |\tau|d\tau = \begin{cases} t^2, & \text{if} \quad t \ge 0, \\ -t^2, & \text{if} \quad t < 0. \end{cases}$$

$\blacksquare$

5. (12 points) The parametrization of a helix is given by

$$\vec{\boldsymbol{r}}(t) = (a\cos t)\vec{\boldsymbol{i}} + (a\sin t)\vec{\boldsymbol{j}} + bt\vec{\boldsymbol{k}}, \qquad a, b \geq 0, \ \ a^2 + b^2 \neq 0.$$

(a) Find the curvature κ of the helix.

Ch. 13.4: an example given in the lecture!(example 5 on page 750)

Solution.

$$\vec{\boldsymbol{v}}(t) = \frac{d\vec{\boldsymbol{r}}(t)}{dt} = (-a\sin t)\vec{\boldsymbol{i}} + (a\cos t)\vec{\boldsymbol{j}} + b\vec{\boldsymbol{k}},$$
$$\vec{\boldsymbol{a}}(t) = \frac{d\vec{\boldsymbol{v}}(t)}{dt} = (-a\cos t)\vec{\boldsymbol{i}} + (-a\sin t)\vec{\boldsymbol{j}} + 0\vec{\boldsymbol{k}},$$

$$\vec{\boldsymbol{v}} \times \vec{\boldsymbol{a}} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = (ab\sin t)\vec{\boldsymbol{i}} + (-ab\cos t)\vec{\boldsymbol{j}} + a^2\vec{\boldsymbol{k}}.$$

$$|\vec{\boldsymbol{v}}| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2 + b^2}$$
$$|\vec{\boldsymbol{v}} \times \vec{\boldsymbol{a}}| = \sqrt{(ab\sin t)^2 + (-ab\cos t)^2 + (a^2)^2} = a\sqrt{a^2 + b^2}$$

$$\kappa = \frac{|\vec{\boldsymbol{v}} \times \vec{\boldsymbol{a}}|}{|\vec{\boldsymbol{v}}|^3} = \frac{a}{a^2 + b^2}.$$

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(b) Find the torsion τ of the helix

Ch. 13.5: an example given in the lecture!

Solution.

$$\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} = \begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix} = a^2 b.$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{\boldsymbol{v}} \times \vec{\boldsymbol{a}}|^2} = \frac{b}{a^2 + b^2}$$

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