ISyE 3044C — Final Exam
Fall 2014

Name (Last, First): ____________________, ____________________ **Score: ______**

Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	TOTAL

No books and notes are allowed. You can use only the supplied formula sheet and tables, as well as a scientific calculator with single-variable statistical functions. *This exam is in strict compliance with the Institute's Honor Code.*

1. **[6 points]** Consider the definite integral

$$\mu = \int_1^3 \exp\left[-\frac{(t-1)^2}{4}\right] dt.$$

(a) Use the Monte Carlo method with the following uniform(0,1) pseudo-random numbers to compute a point estimate of μ: .10, .23, .19, .98, .65, .45, .56, .74, .81, .03

Answer: ________________

(b) Compute an approximate 95% confidence interval for μ.

Answer: ________________

(c) We wish to compute an approximate 95% confidence interval for μ with a relative half-length (half-length over absolute value of point estimate) ≤ 0.1. Compute an estimate of the sample size that will be required to achieve this precision.

Answer: ________________

2. **[8 points]** Consider the density function $f(x) = bx^{b-1}, 0 < x < 1$, where $b > 0$ is an unknown parameter.

   (a) Compute the mean of this distribution.

   Answer: ________________

   (b) Use the method of moments to derive an estimator of b based on a sample $\{X_1, X_2, \ldots, X_n\}$.

   Answer: ________________

   (c) Derive the maximum likelihood estimator of b based on a sample $\{X_1, X_2, \ldots, X_n\}$.

   Answer: ________________

   (d) Consider the following data set: .99, .13, .24, .09, .18. Use the Kolmogorov-Smirnov test to assess the fit of the model with $b = 2$. Set the type I error to .05.

   Answer: ________________

3. **[5 points]** The random variable X has density function

$$f(x) = \begin{cases} 1 + x, & -1 \leq x < 0 \\ 1 - x, & 0 \leq x \leq 1. \end{cases}$$

   (a) Derive the cumulative distribution function of X.

   Answer: ___________________

   (b) Use the inverse-transform method to obtain a formula for generating realizations of X.

   Answer: ___________________

4. **[3 points]** Short questions.

   (a) Use the uniform(0,1) random number .13 to generate a realization from the geometric distribution with probability function $P(X = k) = (.25)^k \times .75, k = 0,1,\ldots$

   Answer: ___________________

   (b) Use the uniform(0,1) random number .73 and an approximation method to generate a realization from the standard normal distribution.

   Answer: ___________________

   (c) Use an appropriate random observation from part (b) to generate a realization from the Poisson distribution with mean 25.

   Answer: ___________________

5. **[8 points]** An enhancement of the Simio inventory model we studied was used to obtain "optimal" operational parameters for a single product. Each inventory replenishment requires a unit of a Resource called "Workers" during the lead time delay. In addition each replenished unit incurres a monetary revenue. The three controls were the capacity of the Workers resource, the reorder point, and the order-up-to level of the (s, S) policy. The three responses were the Average Daily Profit (revenue minus cost), the Average Fill Rate, and the combined response **Profit Times Fill Rate (PFR)**, defined as (Average Daily Profit)*(Average Fill Rate).

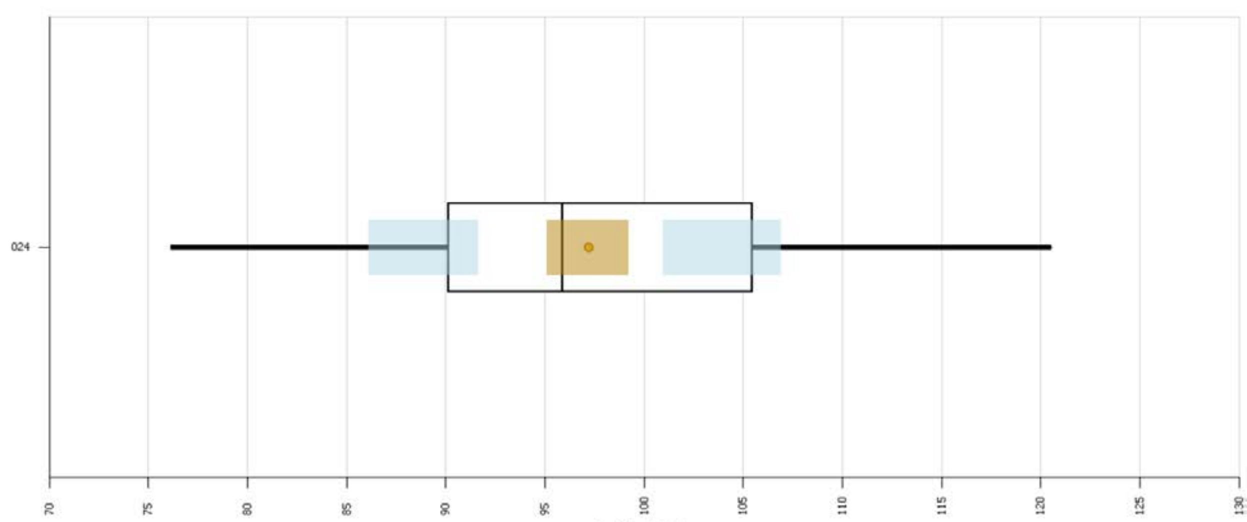
The Kim-Nelson (KN) Add-In was used to compare 8 scenarios with respect to the **PFR** response. The run length was 120 days, the confidence level was set at 0.95 and the Indifference Zone (or parameter) was set to 2. The screen capture below contains the outcome of the KN experiment.

Scenario			Replications		Controls	Inventory - Controls		Responses		
☐	Name	Status	Required	Completed	NumberOfWorkers	ReorderPoint	OrderUpToLevel	AverageDailyProfit	AverageFillRate	ProfitTimesFillRate
☑	024	Completed	23	23 of 23	1	70	120	117.289	0.804433	94.549
☐	012	Completed	23	23 of 23	2	70	120	107.285	0.839162	90.1142
☐	070	Completed	17	17 of 17	1	55	120	103.628	0.760948	79.006
☐	099	Completed	16	16 of 16	2	65	105	95.3207	0.797773	76.2816
☐	004	Completed	16	16 of 16	3	70	120	87.7702	0.84528	74.2791
☐	010	Completed	17	17 of 17	1	55	115	99.211	0.743249	73.9405
☐	075	Completed	16	16 of 16	2	60	120	89.87	0.773679	69.8102
☐	051	Completed	17	17 of 17	1	45	115	89.6893	0.717531	64.6919

(a) Scenario 24 with 1 worker, $s = 70$, and $S = 120$ yields the best *expected* PFR with probability $\geq$ 95% as long as the difference between the best and the second best configuration is ≥ 2.
True **False**

(b) If we decrease the indifference parameter, more replications may be needed until we declare a winner.
True **False**

(c) The following questions are based on the SMORE plot for Scenario 24 based on 100 replications with 95% confidence level, upper percentile 75%, and lower percentile 25%:



- Circle the 95% confidence interval for the *expected value* of PFR on the SMORE plot.

- The probability that scenario 24 induces a PFR less than 90 dollars is about 25%.
  True **False**

- The SMORE plot displays a confidence interval for the median of PFR.
  True **False**

- As we increase the number of replications, the distance between the point estimates for the 25- and 75-quartiles will shrink progressively.
  True **False**

- The probability that the PFR is between 90 and 106 dollars is approximately 50%.
  True **False**

ISyE 3044 — Important Formulas

Distributions

	Parameters	Density/p.m.f.	c.d.f.	Mean	Variance
Binomial	$n \geq 1, 0 < p < 1$	$\binom{n}{k} p^k (1-p)^{n-k}, k = 0, \ldots, n$		np	$np(1-p)$
Geometric	$0 < p < 1$	$(1-p)^{k-1} p, k = 1, 2, \ldots$	$\Pr\{X \leq k\} = 1 - (1-p)^k$	$1/p$	$(1-p)/p^2$
Negative Binomial	$r \geq 1, 0 < p < 1$	$\binom{k-1}{r-1} p^r (1-p)^{k-r}, k = r, r+1, \ldots$		r/p	$r(1-p)/p^2$
Poisson	$\lambda > 0$	$e^{-\lambda} \lambda^k / k!, k = 0, 1, \ldots$		λ	λ
Uniform	$-\infty < a < b < \infty$	$1/(b-a), a \leq x \leq b$	$(x-a)/(b-a)$	$(a+b)/2$	$(b-a)^2/12$
Normal	$-\infty < \mu < \infty, \sigma > 0$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$	$\Phi((x-\mu)/\sigma)$	μ	σ^2
Exponential	$\lambda > 0$	$\lambda e^{-\lambda x}, x > 0$	$1 - e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$
Gamma	$\lambda > 0, \beta > 0$	$\lambda(\beta x)^{\beta-1} e^{-\lambda x} / \Gamma(\beta)$		β/λ	β/λ^2
Erlang	Gamma with integer $\beta = k$		$1 - \sum_{i=0}^{k-1} e^{-\lambda x} (\lambda x)^i / i!$	k/λ	k/λ^2
Weibull	$\lambda > 0, \beta > 0$	$\beta\lambda(\lambda x)^{\beta-1} \exp[-(\lambda x)^\beta]$	$1 - \exp[-(\lambda x)^\beta], x > 0$	$\Gamma(1 + 1/\beta)/\lambda$	

Central Limit Theorem

$$\Pr\left\{ \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \leq z \right\} \to \Phi(z) \quad \text{as } n \to \infty.$$

Poisson Process

- The times between events are i.i.d. exponential with parameter λ;
- The number of events in an interval $[s, s+t]$ have the Poisson distribution with parameter λt;
- The numbers of events in nonoverlapping intervals are independent random variables.

Sums of Independent Random Variables

- If $X = \text{binomial}(n, p)$ and $Y = \text{binomial}(m, p)$ are independent, then $X + Y = \text{binomial}(n + m, p)$.
- If $X_1, \ldots, X_r$ are independent geometric(p) random variables, then $\sum_{i=1}^r X_i$ is a negative binomial(r, p) random variable.
- If $X = \text{Poisson}(\lambda)$ and $Y = \text{Poisson}(\mu)$ are independent, then $X + Y = \text{Poisson}(\lambda + \mu)$.
- If $X_1, \ldots, X_n$ are i.i.d. exponential(λ) random variables, then $\sum_{i=1}^n X_i = \text{Erlang}(n, \lambda)$.
- If $X_1 = N(\mu_1, \sigma_1^2), \ldots, X_n = N(\mu_n, \sigma_n^2)$ are independent, $\sum_{i=1}^n a_i X_i + b = N(\sum_{i=1}^n a_i \mu_i + b, \sum_{i=1}^n a_i^2 \sigma_i^2)$.

Queues $EA = 1/\lambda$; $cv_A^2 = \text{Var}(A)/(EA)^2$; $ES = 1/\mu$; $cv_S^2 = \text{Var}(S)/(ES)^2$; $\rho = \lambda/\mu$.

Little's Laws: $L = \lambda w$; $L_Q = \lambda w_Q$.

Kingman's formula for G/G/1 queues: $w_Q \approx \frac{\rho}{1-\rho} \cdot \frac{cv_A^2 + cv_S^2}{2} \cdot ES$ (holds as equality for M/G/1 queues).

Random Number Generation

Linear Congruential Generator $X_n = (aX_{n-1} + c) \mod m$

- For $m = 2^b$ and $c \neq 0$, the longest attainable period $p = m$ is achieved when $a = 4k + 1$ (for some integer k) and the largest common factor of c and m is 1.
- For $m = 2^b$ and $c = 0$, the longest attainable period $p = m/4$ is achieved when X_0 is odd and $a = 8k + 3$ or $a = 8k + 5$ (for some integer k).
- For m prime and $c = 0$, the longest attainable period $p = m - 1$ is achieved when the smallest integer k such that $a^k - 1$ is divisible by m is $k = m - 1$.

Random Variate Generation and Maximum Likelihood Estimation

Distribution	Generator	MLE's
Exponential	$-(1/\lambda)\ln R$	$\hat{\lambda} = 1/\bar{X}_n$
Erlang	$-(1/\lambda)\sum_{i=1}^{k} \ln R_i$	
Weibull	$(1/\lambda)[-\ln R]^{1/\beta}$	$\hat{\lambda} = \left[(1/n)\sum_{i=1}^{n} X_i^{\hat{\beta}}\right]^{-1/\hat{\beta}}$, no analytical expression for $\hat{\beta}$
$N(0,1)$	$Z_1 = (-2\ln R_1)^{1/2}\cos(2\pi R_2), Z_2 = (-2\ln R_1)^{1/2}\sin(2\pi R_2)$ approximation: $\frac{R^{0.135}-(1-R)^{0.135}}{0.1975}$	
$N(\mu,\sigma^2)$		$\hat{\mu} = \bar{X}_n, \widehat{\sigma^2} = \frac{n-1}{n} S_n^2$
Geometric	$\left\lceil \frac{\ln(1-R)}{\ln(1-p)} \right\rceil$ (ceiling function)	$\hat{p} = 1/\bar{X}_n$
Poisson	$\min\{n : \prod_{i=1}^{n+1} R_i < e^{-\lambda}\}$ approximation: $\lceil \lambda + Z\sqrt{\lambda} - 0.5 \rceil$	$\hat{\lambda} = \bar{X}_n$

Estimation Methods Data $X_1, \ldots, X_n$ from a distribution with unknown parameters $\theta = (\theta_1, \ldots, \theta_m)$

- Method of moments: Solve $E(X^k) = n^{-1}\sum_{i=1}^{n} X_i^k, k = 1, \ldots, m$, for $\theta_1, \ldots, \theta_m$.
- Maximum likelihood estimation: Maximize the likelihood $L(\theta) = \prod_{i=1}^{n} f(X_i; \theta)$ [or the log-likelihood $\ell(\theta) = \ln L(\theta)$].

Goodness-of-Fit Tests

- Chi-square: $\chi_0^2 = \sum_{i=1}^{k} (O_i - E_i)^2/E_i$ (with $E_i \geq 5$).
- Kolmogorov-Smirnov: $D_n = \max\left\{\max_{1\leq i\leq n}\left[\frac{i}{n} - \hat{F}(X_{(i)})\right], \max_{1\leq i\leq n}\left[\hat{F}(X_{(i)}) - \frac{i-1}{n}\right]\right\}$.

Critical Values for Adjusted K-S Statistics

		α				
Case	Adjusted Test Statistic	0.15	0.10	0.05	0.025	0.01
All parameters known	$\left(\sqrt{n} + 0.12 + \frac{0.11}{\sqrt{n}}\right) D_n$	1.138	1.224	1.358	1.480	1.628
Nor$(\bar{X}_n, S_n^2)$	$\left(\sqrt{n} - 0.01 + \frac{0.85}{\sqrt{n}}\right) D_n$	0.775	0.819	0.895	0.995	1.035
Expo$(1/\bar{X}_n)$	$\left(D_n - \frac{0.2}{\sqrt{n}}\right)\left(\sqrt{n} - 0.01 + \frac{0.85}{\sqrt{n}}\right)$	0.926	0.990	1.094	1.190	1.308

Table entry for p and C is the critical value t^* with probability p lying to its right and probability C lying between $-t^*$ and t^*.

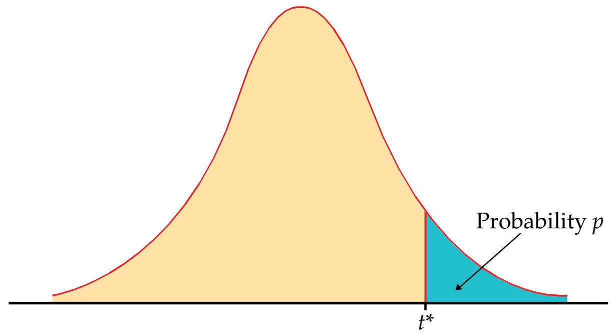


TABLE D

t distribution critical values

	Upper-tail probability p											
df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.000	1.376	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3	636.6
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.611	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
z^*	0.674	0.841	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.091	3.291
	50%	60%	70%	80%	90%	95%	96%	98%	99%	99.5%	99.8%	99.9%
	Confidence level C											