ISyE 2027C Probability with Applications

Homework 8 for Quiz April 7 and to prepare for Test 2 April 9 **2 April 2015**

1. Random variable X has cdf $\sin t$ for $0 \le t \le \pi/2$. Random variable Y is independent of X and has cdf $1 - \cos t$ for $0 \le t \le \pi/2$. Without doing any calculations, is $E[X]$ less than, equal to, or greater than $E[Y]$? Draw the two cdfs. The cdf of X is on or above the cdf of Y. It has higher probabilities of smaller values. Hence $E[X] < E[Y]$. This is called stochastic dominance. Is $\sigma^2(X)$ less than, equal to, or greater than $\sigma^2(Y)$? Find a symmetry. Calculate the mean and variance of X and of Y. Let $Z = \frac{1}{2}(X+Y)$. Calculate the pdf, mean, and variance of Z. $E[X] = \int_0^{\pi/2} tcostdt = \pi/2 - \int_0^{\pi/2} sintdt = \pi/2 - 1$. $E[X^2] = \int_0^{\pi/2} t^2costdt = \pi^2/4 - \int_0^{\pi/2} 2tsintdt = \pi^2/4 - 2$. $E[Y] = \int_0^{\pi/2} tsintdt = \pi/2 + 1$. $E[Y^2] = \int_0^{\pi/2} t^2sintdt = \pi - 2$. $E[Z] = \frac{1}{2}(E[X] + E[Y]$. $\sigma^2(Z) = \frac{1}{4}(\sigma^2(X) + \sigma^2(Y)$ by independence. The instantaneous probability $2Z = X + Y = t$ is $f_{2Z}(t) = \int_{s=0}^{t} P(X = s)P(Y = t - s)ds = \int_{s=0}^{t} (\cos s)(\sin t - s)ds$.

2. Components are placed one at a time on a square circuit board with side length 12 inches. The placement machine has a delicate plunger mechanism that holds the component with a negative air pressure nozzle, descends and releases the the component (e.g. a resistor) onto the board, ascends, and grabs the next component that will be placed. Each component has a specific location specified in x, y coordinates (each between 0 and 12) where it must be placed. This problem will guide you through some analysis of placement machines that you could be asked to do as an industrial engineer.

   Let n be the number of components to be placed. Let x_i, y_i be the coordinates of the ith such component. We are going to assume that x_i and y_i are each uniformly distributed on $[0, 12]$. This is a decent assumption, as is the assumption that x_i and y_i are pairwise independent. We are going to assume the stronger assumption that all $2n$ variables are jointly independent. Even if the first two assumptions were valid, the third one of joint independence could not be valid. Why not? Two components can't be in the same place. It is nonetheless a decent assumption which we will use for the rest of the problem.

   The plunger mechanism only moves vertically. The board is moved in the x and y dimensions so that the right location is directly under the plunger. x movement is provided by one motor and y movement is provided by another motor which runs simultaneously. Approximating the speed of each as constant, the time to move from one location to another is proportional to whichever coordinate difference is larger.

   (a) This is a word problem: Write the formula for the time to move from one location to another. $\max\{|x_i - x_{i+1}|, |y_i - y_{i+1}|\}$ divided by the speed.

   (b) Assuming a speed of 4 inches per second, what is the expected total travel time to place 100 components on a board? To be precise you will have to make an assumption or two. State your assumptions clearly. . Assume that the starting location of the board, when loaded on to the placement machine, is at $0, 0$. Assume that after the last placement the board has to move to $0, 0$ before being unloaded from the machine. There are other reasonable assumptions.

   The key formula we need is the cdf of $X = |x_1 - x_2|$. Once we have that we can find the distribution of the maximum of two variables with that cdf, which is the time to move from one location to another (when divided by 4). Scale so x_i has $U[0, 1]$ distribution. The instantaneous probability $f_X(t) = 2 - 2t$, which you can see from the integral

$$\int_0^1 dx_1 P(x_2 = x_1 - t) + P(x_2 = x_1 + t) = \int_0^1 dx(2 \text{ if } x - t \ge 0 \text{ and } x + t \le 1; 0 \text{ if } x - t < 0 \text{ and } x + t > 1; 1 \text{ o/w}$$

   or by reasoning as follows: for $0 \le x_1 \le 1 - t$ it is possible for $x_2 - x_1 = t$ with instantaneous probability 1 because x_2 has density 1 in the range $t \le x_2 \le 1$. The density of x_1 is also 1 for an instantaneous probability $\int_0^{1-t} 1dx_1 = 1 - t$. Symmetrically the instantaneous probability $x_1 - x_2 = t$ is also $1 - t$. Sum these to give $2 - 2t$. You can check this for plausibility. The integral should be 1 for any pdf. Sure enough, $\int_0^1 (2 - 2t)dt = 1$. Also it should be obvious to you that $E[X] = 1/3$ from the proof I gave in class that the expected distance between the kth smallest and $k + 1$st smallest out of n independent $U[0, 1]$ variables is $\frac{1}{n+1}$. (The proof was to generate

$n+1$ points randomly on a circle with circumference 1 and use symmetry.) Therefore it ought to be that $\int_0^1 2t - 2t^2 dt = \frac{1}{3}$, and that is true.

Now we compute $F_X(t) = \int_0^t 2 - 2x dx = 2t - t^2$. The maximum of two independent variables with that cdf has cdf $F(t) = (2t - t^2)^2 = t^2(2-t)^2$. The pdf is $4t^3 + 8t - 12t^2$ and the expected value is

$$\int_0^1 4t^4 + 8t^2 - 12t^3 dt = 4/5 + 8/3 - 3 = \frac{7}{15}.$$

It takes 3 seconds to move 12 inches. The expected time to move from one place to the next is therefore $3\frac{7}{15} = 1.4$ seconds. The expected travel time is approximately 140 seconds. More precisely, there are 99 moves with expected time 1.4 seconds each, and 2 moves with expected time $\frac{2}{3}3 = 2$ seconds each. (The time to move from 0,0 to the maximum of two $U[0,12]$ values at 4 inches per second is 2/3 of 3 seconds). The total is therefore $99(1.4) + 4$ seconds.

(c) What is the variance of the time to move from one location to another? Before scaling by 3 we were working with $U[0,1]$ variables. Returning to those variables, the expected value of the square is

$$\int_0^1 4t^5 + 8t^3 - 12t^4 dt = 4/6 + 8/4 - 12/5 = \frac{4}{15}.$$

The variance is therefore $\frac{4}{15} - \frac{49}{225} = \frac{11}{225}$. Scaling by 3 seconds gives a variance of $\frac{99}{225} seconds^2$.

(d) Why is it not simple to calculate the variance of the total travel time to place 100 components? . Because the travel time from component i to component $i+1$ is not independent of the time from $i+1$ to $i+2$. Both are influenced by x_{i+1} and y_{i+1}.

(e) This is not a probability question: Why is the total travel time not equal to the total process time for a board? Hint: there are at least two reasons. . Process time also includes the time to load the board and/or unload the board. If one board can be loaded at the same time another is unloaded you should count only one fo the two times for that board – o/w you double-count. After the board arrives at its location the plunger has to move down, drop the item, move up, and grab another item. If some of these activities must occur when the board is not moving, the time for those activitites must be added to the travel time when you count the process time. If some occur simultaneously you have to take the maximum of the time needed and the board move time needed.

(f) Suppose that the plunger does not begin to descend until the board in in the right location for the next placement. Suppose that it takes .2 seconds to descend and release the component. Once the component is released, the board may begin to move to prepare for the next placement. Suppose it takes the plunger .8 seconds to ascend and grab the next component. What is the expected time between the ith placement and the $i+1$st placement (for $i = 1$ to 99)?

The time is .2 seconds plus the maximum of .8 seconds and the travel time. Let Y denote the latter quantity. $P(Y \leq t) = 0$ for $t \leq .8$. For larger values of t, we already know $P(Y \leq 3t) = (2t - t^2)^2$ from a previous question (the $3t$ term comes from scaling by 3 seconds). Let $s = 3t$. Hence $P(Y \leq s) = (6s - s^2)^2/81$ for $s > .8$ and $P(Y = .8) = (4.8 - .64)^2/81 \equiv \beta$. So $E[Y] = .8\beta + \int_{0.8}^3 2s(6s - s^2)(6 - 2s)/81 ds$. The expected time is $.2 + E[Y]$.

(g) Demand for this product is very high. Your company can easily sell as many as you can make over the next year. Your company has one placement machine, which processes a certain number of boards per hour. As chief manufacturing engineer, you are considering buying 3 more placement machines, for a total of 4. How would you use them to process as many boards as possible per hour? One strategy would be to run each machine separately. Obviously that strategy would increase your production by a factor of 4. Another strategy would be to conceptually divide the board into 4 boards, each a 6 inch by 6 inch square, and run each board through all four machines. If loading a board onto the placement machine and unloading it took 0 time, and the plunger moved infinitely fast, how would the second strategy perform (i.e. by how much would it increase production)? Of course, your answer will be an estimate based on expected values.. Make the simplifying approximation that each 6 by 6 inch square has exactly 25 components. You should have figured out that each placement machine is responsible for one of the 6 by 6 squares. Each machine would take $25E[X/2]$ time per board because the scaling factor would be 6 inches / 4 inches per second rather than 12 inches / 4 inches per second. Hence production would increase by approximately a factor of 8.

(h) Repeat the previous question under the assumptions of 0 loading and unloading time, .2 second plunger descent and .8 second plunger ascent times as in an earlier question. The benefit will be smaller because the variable part of the process time is the maximum of .8 seconds and the travel time, but now the travel time ranges between 0 and 1.5 seconds rather than between 0 and 3 seconds. The travel time Y has cdf $P(Y \leq t) = 0$ for $t \leq .8$. The expected value of the variable part of the process time (using $s = 1.5t$ so that $P(Y \leq s) = (12s - 4s^2)^2/81$) is $.8(9.6 - 2.56)^2/81 + \int_{0.8}^{1.5} 2s(12s - 4s^2)(12 - 8s)/81ds$.

(i) (Continuation). A third strategy, which you could use whether you had 1 or 4 machines, would be to conceptually divide the board into 4 boards as previously, and re-sequence the 100 component placements into 4 sets of placements each on a 6 by 6 board. You would process a board on a single machine, but the order of its placements would be better planned so as to decrease the expected total travel time. What would be the impact of this strategy on the expected travel time per board? Each move, except for the three times you move from one 6 by 6 board to the next, will have half the expected travel time. Neglecting the slight increase in expected time for those three moves, this strategy would halve the expected travel time per board.

3. A rectangular warehouse 120 meters long (east-west) and 60 meters wide (north-south) could be filled with either 24 north-south aisles or 12 east-west aisles.

(a) Which choice do you intuitively prefer, or do you feel that the two choices are equally good? **FOR THE REST OF THIS QUESTION, ANSWER EACH PART TWICE, ONCE FOR EACH CHOICE.**

(b) If a single item is retrieved by a forklift in the southwest corner of the warehouse, and returned to the same corner, what is the expected distance traveled by the forklift if item locations are uniformly distributed in the warehouse? . 180 meters, 180 meters

(c) For the scenario of question 3b, what is the standard deviation? . $\sqrt{\frac{4}{12}(18000)}$ meters.

(d) If two items are retrieved in a single trip by a forklift in the southwest corner of the warehouse, and returned to the same corner, what is the expected distance traveled by the forklift if item locations are independently and uniformly distributed? You may neglect the case where the two items are in the same aisle, though it is better if you don't. Can you calculate the standard deviation? . In the simple case, $160 + 60 + 60$ meters, $80 + 120 + 120$ meters.

(e) Same as question 3d, but for three items. 360, 450 meters.

(f) Same as question 3b if the forklift starts and ends in the center of the south wall in the case of north-south aisles, and in the center of the west wall in the case of east-west aisles. 120, 150 meters

(g) Same as question 3f for 2 items. Hint: break the situation into two cases. In the first case both items are on the same side of the center of the wall. In the 2nd case there is one item on each side. Use the law of total probability for expectation. 220, 290 meters.

(h) Same as question 3f for 3 items. 307.5, 423.75 meters