Name:______________________________ **Section:**

ISyE 3833: Engineering Optimization
Final
April 30th, 2015
2:50pm to 5:00pm

Remarks.

- This is a closed book exam.
- There are 6 problems in total.
- Write your name on every sheet.

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Problem 1. (18 pts) Consider the linear program

$$\begin{aligned} \max z = &\quad\quad\;\; x_2 \\ \text{s.t.} \quad & x_1 - x_2 \le 1 \\ & x_1,\, x_2 \ge 0 \end{aligned}$$

(a) (3 pts) Give the extreme points of the feasible region.

(b) (4 pts) Give the first set of equations (dictionary) for solving this problem by the simplex method. Then use this dictionary to immediately observe that the problem is unbounded.

(c) (4 pts) Describe the unbounded solution revealed by the dictionary as the extreme point of the dictionary, together with a (half)-line that starts at this extreme point and goes off to infinity, as

$$(x_1, x_2) = (\quad\quad , \quad\quad) + t(\quad\quad , \quad\quad)$$

where t goes to infinity. (Fill in the blanks in the equation above.)

(d) (4 pts) Give the dual linear program.

(e) (3 pts) Give a simple argument to show that the dual is infeasible.

Problem 2. (12 pts) Suppose a given LP has an optimal solution and we modify it just by changing some right-hand side coefficients. (Note that this problem differs from the one in Midterm 2 because here we are changing a different set of coefficients.)

(a) (6 pts) Which of the following are possible for the modified LP? Check all that apply.

☐ A. It has an optimal solution with different objective value.

☐ B. It is unbounded.

☐ C. It is infeasible.

(b) (6 pts) Next, which of of the following are possible for the dual of the modified LP? Check all that apply.

☐ D. It has an optimal solution with different objective value.

☐ E. It is unbounded.

☐ F. It is infeasible.

Problem 3. (20 pts) Write linear constraints to model the following relationships. You can introduce extra variables (binary, integer or continuous) as needed. Clearly define any new variables you introduce. In the following, M is a given, positive constant.

(a) (4 pts) For binary variables x and y, $x = 1$ or $y = 0$ or both.

(b) (4 pts) Given non-negative integer variables x and y, we do not permit both variables to equal zero.

(c) (4 pts) For continuous variable x, with $0 \leq x \leq M$, we require that x is either 0, or at least a given positive number, k, where $k < M$.

(d) (4 pts) For continuous variables x_1 and x_2, with $0 \leq x_1 \leq 5$ and $0 \leq x_2 \leq 7$, we require that either $x_1 \leq 3$ or $x_2 \leq 4$ or both.

(e) (4 pts) Given binary variables x_1, x_2 and x_3, we require $x_3 = x_1 x_2$. First give the allowable values of (x_1, x_2, x_3) that satisfy this relationship and then give linear constraints so that only these values are satisfy (all of) the constraints.

Problem 4. (16 pts) A warehouse can store up to U units of a particular item. In each of the next N days, $n = 1, \ldots, N$, you need to determine the number of units to sell and then the number of units to buy. Selling in each day precedes buying, so the maximum amount that can be sold on a day is the initial inventory on hand at the beginning of the day. The maximum amount that can be bought in a day equals U minus the amount in inventory on hand after selling takes place. The inventory on hand after buying is held overnight in the warehouse, and equals the inventory on hand at the beginning of the next day. The number of units held overnight cannot exceed U. At the beginning of day 1, before selling takes place, there are s units of inventory in the warehouse. The unit selling and buying prices are p_n and c_n, $n = 1, \ldots, N$. In addition, if you choose to sell any units in period n, you must pay a fixed cost of f and if you choose to buy any units in period n, you must pay a fixed cost of g, where $f > 0$ and $g > 0$. Formulate an integer program to determine the number of units to be bought and sold in each period to maximize total profit. Carefully define your variables, and provide a brief description of each constraint.

Note: this is exactly the problem from Midterm 1 except for the addition of the fixed costs.

Problem 5. (16 pts) A baseball league has 8 teams. You, as the local expert in scheduling, have been asked to create a schedule for the league. Each of the 8 teams will play all 7 other teams exactly once. So the games are scheduled in 7 slots and in each slot there are 4 games, with every team playing one game in every slot. A schedule stipulates that team i plays team j in slot t, $i = 1, 2, \ldots, 7$, $j = i + 1, \ldots, 8$.

(a) (10 pts) Write down the constraints for an linear integer program that models the problem of finding a feasible schedule for the league (each team plays every other team exactly once, and every team has one game in every slot). Carefully define your variables and briefly describe each of your constraints.

(b) (6 pts) Many schedules are possible, but the teams have been ranked. Team 1 is predicted to be the best team, team 2 second best and so on. So it has been decided to grade possible schedules by awarding

- 3 points if team 1 plays team 2 in slot 7,
- 2 points for all games between teams 1,2,3 and 4 in slot 6 or 7 (excluding 1 vs. 2 in slot 7), and
- 1 point for all games between teams 1,2,3 and 4 in slots 4 or 5.

Add an objective function to your linear integer program from part (a) so as to model the problem of finding a feasible schedule that maximizes the total number of points.

Problem 6. (18 pts) Consider the integer program

$$\begin{aligned}
\max \quad & z = cx \\
\text{s.t.} \quad & Ax \leq b, \\
& x_j \text{ binary}, && j = 1, \dots, k \\
& x_j \geq 0, && j = k+1, \dots, n.
\end{aligned}$$

(a) (3 pts) Give the linear programming relaxation of the integer program.

(b) (3 pts) Let Z^* be the optimal value of the integer program and $Z_{LP}(0)$ be the optimal value of its linear programming relaxation. What is the relationship between these two values?

Let $Z_{LP}(i)$ be the optimal value of the linear programming relaxation at node i of the branch-and-bound tree.

(c) (3 pts) Suppose node 23 is a descendent of node 4 (e.g. child, grandchild, etc.) in the branch-and-bound tree. What is the relationship between $Z_{LP}(23)$ and $Z_{LP}(4)$?

Let Z_L be the value of the best feasible solution found so far.

(d) (6 pts) Suppose the LP relaxation is solved at node i of the tree.

i. In which of the following situations will node i have no child nodes (it is a leaf node or is pruned)? Check all that apply.

- ☐ The LP relaxation is infeasible.
- ☐ $Z_{LP}(i) \leq Z_L$.
- ☐ The LP relaxation solution has integral values for all of the binary variables.
- ☐ $Z_{LP}(i) \geq Z_L$.

ii. What conditions must be satisfied to update Z_L and what is the value it is updated to?

(e) (3 pts) When does the branch-and-bound search tree terminate, i.e., when can you be sure that the solution that gave the current value of Z_L is optimal?