Full Name: Solutions

Section B

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

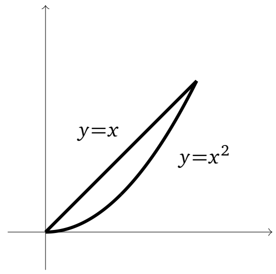
Signature: ______________________________

Math 2551 — Exam 4
November 18, 2015

Write your solutions clearly and legibly, showing all work. Use of notes, cheat sheets, the textbook, or any outside materials is not permitted. Only non-graphing, non-programmable calculators are permitted.

Problem	Points Possible	Points Earned
1	22	22
2	11	11
3	9	9
4	8	8
Total	50	50

(1) Let $\mathbf{F}(x,y)$ be the vector field $\mathbf{F}(x,y) = x^2\mathbf{i} + xy\mathbf{j}$, and let C be the simple closed curve drawn below.



(a) Compute the counterclockwise circulation of $\mathbf{F}$ along C without using Green's theorem, i.e., by computing a line integral instead of a double integral. [13 points]

Recall, the counterclockwise circulation is given by the line integral

$$\Gamma = \oint_C x^2\,dx + xy\,dy$$

To compute this, we must integrate over each edge of the loop separately, and then add the results together.

The parabolic edge: This edge is parameterized by $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$, where $0 \le t \le 1$. Thus the line integral over this edge is

$$\int x^2\,dx + xy\,dy = \int_0^1 (t^2)(dt) + (t)(t^2)(2t\,dt) = \int_0^1 (t^2 + 2t^4)\,dt = \frac{1}{3} + \frac{2}{5}$$

The straight edge: This edge is parameterized by $\mathbf{r}(t) = (1-t)\mathbf{i} + (1-t)\mathbf{j}$, where $0 \le t \le 1$. Thus the line integral over this edge is

$$\int x^2\,dx + xy\,dy = \int_0^1 (1-t)^2(-dt) + (1-t)(1-t)(-dt) = \int_0^1 -2(1-t)^2\,dt = -\frac{2}{3}$$

Adding the results over for these two edges together, we get that the circulation along C is

$$\Gamma = \frac{1}{3} + \frac{2}{5} - \frac{2}{3} = \frac{2}{5} - \frac{1}{3} = \boxed{\frac{1}{15}}$$

2 points for starting out with the correct (untranslated) line integral.
For each edge: 2 points for the parameterization, 2 points for the translation
Finally, 3 points for correctly evaluating

(b) For the same curve and vector field, now compute the outward flux of $\mathbf{F}$ across C, this time using Green's theorem, i.e., by computing a double integral instead of a line integral.
[9 points]

Green's theorem says that the outward flux of $\mathbf{F}$ across C is the double integral of the flux density of $\mathbf{F}$. The flux density of $\mathbf{F}$ is

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xy) = 3x$$

Thus, if R is the region enclosed by C, we get that the outward flux is

$$\iint_R 3x\,dA = \int_0^1 \int_{x^2}^{x} 3x\,dy\,dx = \int_0^1 (3x^2 - 3x^3)\,dx = \boxed{\frac{1}{4}}$$

6 points for setting up the correct iterated integral
3 points for evaluation

(2) Let S be the piece of the paraboloid $z = 2(x^2 + y^2)$, $z \leq 2$ which lies in the first octant. What is the surface area of S? [11 points]

I will parameterize S with respect to x and y (you could just as easily use r and θ):

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + 2(x^2 + y^2)\mathbf{k}$$

The domain R of the parameterization is the piece of the disk $x^2 + y^2 \leq 1$ that lies in the first quadrant. Expressing $d\sigma$ in terms of the parameters x and y yields

$$d\sigma = \left(1 + 16x^2 + 16y^2\right)^{1/2} dx\, dy$$

We therefore get that the surface area is

$$\iint_S d\sigma = \iint_R \left(1 + 16x^2 + 16y^2\right)^{1/2} dx\, dy \overset{(polar)}{=} \int_0^{\pi/2} \int_0^1 (1 + 16r^2)^{1/2}\, r\, dr\, d\theta = \boxed{\frac{\pi}{96}(17^{3/2} - 1)}$$

If you correctly parameterize S and correctly set up a double integral with respect to your parameters, that is worth 8 points.
If you incorrectly parameterize S and correctly set up a double integral with respect to your parameters, that is worth 4 points.

Finally, 3 points for evaluation.

(3) A spring C in space is parameterized by $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\mathbf{k}$, where $0 \le t \le 4\pi$. If the mass density of the spring is given by the function $\delta(x, y, z) = z$, what is the mass of C? [9 points]

The mass is the line integral of the density:

$$\text{Mass} = \int_C \delta(x, y, z)\,ds = \int_C z\,ds$$

Since the parameterization is given, we can immediately translate the integral into a t-integral. The velocity and speed are

$$\mathbf{v}(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \mathbf{k} \qquad\qquad |\mathbf{v}(t)| = \sqrt{5}$$

so that

$$\int_C z\,ds = \int_0^{4\pi} t\sqrt{5}\,dt = \boxed{8\sqrt{5}\pi^2}$$

7 points for correctly setting up a t-integral which computes the mass
2 points for evaluation

(4) Let $\mathbf{F}(x,y) = -xy\,\mathbf{i} + (xy - x^2)\mathbf{j}$. Does there exist a simple closed curve C in the plane for which both the counterclockwise circulation and the outward flux of $\mathbf{F}$ for C are > 0? If so, find one; if not, explain why not. [8 points]

The flux density of $\mathbf{F}$ is $x - y$ and the circulation density of $\mathbf{F}$ is $y - x$. Thus, if C is any simple closed curve in the plane enclosing a region R, the outward flux and counterclockwise circuation for C are by Green's theorem

$$\Phi = \iint_R (x-y)\,dA \qquad\qquad \Gamma = \iint_R (y-x)\,dA$$

Notice that Φ and Γ are negatives of each other! Thus it is impossible for *both* Φ and Γ to be strictly positive.

This problem is a little open-ended, but the rough breakdown should be the following: 2 points for all the computational work (i.e., computing the flux and circulation densities) and 6 points for understanding what to do with it (i.e., giving a reasonable answer). Here I decide what is reasonable.