Quiz 3 solution

1. Consider the power series
$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{\sqrt{n}4^n}.$$
Find its interval of convergence. Show your work to justify your answer. (6 points)

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{\sqrt{n}4^n}} = \lim_{n\to\infty} \frac{1}{4\sqrt[n]{n^{1/2}}} = \frac{1}{4}.$$

Then, the radius of convergence is $\dfrac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}} = 4$. Therefore, the series converges absolutely on the interval $(-2-4, -2+4) = (-6, 2)$.

For $x = 2$, the series is
$$\sum_{n=0}^{\infty} \frac{(4)^n}{\sqrt{n}4^n} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}.$$
Which is divergent, since it is a p-series with $p = \frac{1}{2} < 1$.

For $x = -6$, the series becomes
$$\sum_{n=0}^{\infty} \frac{(-4)^n}{\sqrt{n}4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$
Which is convergent by the alternating series test. Therefore, the interval of convergence is $[-6, 2)$. ■

2. Find the Maclaurin series for the function $f(x) = \sin(2x^2)$ (write it in sum notation). Use this series expansion to obtain a power series for the indefinite integral (7 points)
$$\int \sin(2x^2)\, dx.$$

Since we have: $\sin y = y - \dfrac{y^3}{3!} + \dfrac{y^5}{5!} - \dfrac{y^7}{7!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!}$. Then,

$$\sin(2x^2) = (2x^2) - \frac{(2x^2)^3}{3!} + \frac{(2x^2)^5}{5!} - \frac{(2x^2)^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (2x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{4n+2}}{(2n+1)!}.$$

Integrating, we have
$$\int \sin(2x^2)\, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{4n+2}}{(2n+1)!}\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{4n+3}}{(4n+3)(2n+1)!}$$

3. Consider the function $f(x) = \dfrac{1}{\sqrt{3+x^2}}$. Find the Taylor polynomial of order 2, generated by f, about $x = 1$. (7 points)

The derivatives of $f(x)$ are

$$f'(x) = -\frac{x}{(x^2+3)^{3/2}}, \qquad f''(x) = \frac{3x^2}{(x^2+3)^{5/2}} - \frac{1}{(x^2+3)^{3/2}}.$$

At $x = 1$ we have: $f(1) = \frac{1}{2}$, $f'(1) = -\frac{1}{8}$, $f''(1) = -\frac{1}{32}$. Then, the coefficients for the Taylor polynomial are:

$$a_0 = f(1) = \frac{1}{2}, \quad a_1 = \frac{f'(1)}{1!} = -\frac{1}{8}, \quad a_2 = \frac{f''(1)}{2!} = -\frac{1}{64}.$$

Therefore, the order 2 Taylor polynomial is $p_2(x) = \frac{1}{2} - \frac{1}{8}x - \frac{1}{64}x^2$. ■