MATH 1552 - SPRING 2016
TEST 1 - SHOW YOUR WORK

NAME: ___________________________ TA: __________________________

1. (15 points) Let h & b be positive constants and let $y = f(x) = h - \left(\frac{4h}{b^2}\right) x^2$, which is always a **parabola**. Find the area between the graph of $f(x)$ and the $x - axis$ between $x = -\frac{b}{2}$ & $x = \frac{b}{2}$ **in terms of h & b**. .

a. **Area =**

$$\mathbf{2 \int_0^{\frac{b}{2}} \left[h - \left(\frac{4h}{b^2}\right) x^2 \right] dx = h\,x - \frac{4}{3}\,\frac{h\,x^3}{b^2} \Bigg| = 2 \left(h\left(\frac{b}{2}\right) - \frac{4}{3}\,\frac{\left[h\left(\frac{b}{2}\right)^3\right]}{b^2} \right) = 2\left(\frac{h\,b}{2} - \frac{h\,b}{6}\right) = \frac{2}{3}\,h\,b}$$

2. (15 points) Let $f_1(x) = \frac{\log_c(x)}{x}$ & $f_2(x) = \frac{\log_{c^2}(x)}{x}$, *where* $a > 1$. **a.** (6 pts) Find the area between the graph of $f_1(x)$ and the $x - axis$ from $x = 1$ **to** $x = e$, call this area A_1. **b**. (6 pts) Find the area between the graph of $f_2(x)$ and the $x - axis$ from $x = 1$ **to** $x = e$, call this area A_2. **c**. (3 pts) Find the ratio $\frac{A_1}{A_2}$. **Recall:** $\mathbf{\log_d(x) = \frac{\ln(x)}{\ln(d)}}$.

SOLUTION: $\log_c(x) = \frac{\ln(x)}{\ln(c)}$ & $\log_{c^2}(x) = \frac{\ln(x)}{\ln\left(c^2\right)}$

a. $\mathbf{A_1 = \int_1^e \frac{\log_c(x)}{x}\, dx = \frac{1}{2}\,\frac{\ln^2(e)}{\ln(c)} = \frac{1}{2\ln(c)}}$

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$$\int \frac{\log_c(x)}{x}\, dx = \int \frac{\ln(x)}{\ln(c)\,x}\, dx = \frac{1}{\ln(c)} \int u\, du = \frac{\ln^2(x)}{2\,\ln(c)}$$

$u = \ln(x)$

$du = \frac{dx}{x}$

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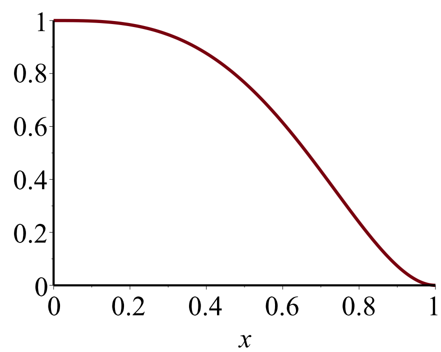
b. $A_2 = \int_1^e \frac{\log_{c^2}(x)}{x}\, dx = \frac{1}{4}\frac{\ln^2(e)}{\ln(c)} = \frac{1}{4\ln(c)}$ very similar to first integral

c. $\frac{A_1}{A_2} = \frac{\frac{1}{2\ln(c)}}{\frac{1}{4\ln(c)}} = 2$

3. (15 points) The graph below is the graph of the equation $x^3 + \sqrt{y} = 1$ in Quandrant 1. Find the area bounded by this graph and the *x* & *y axes*.



SOLUTION : Solve the equation for $y \Rightarrow y = (1 - x^3)^2$

$$Area = \int_0^1 (1 - x^3)^2\, dx = \int_0^1 (1 - 2x^3 + x^6)\, dx = \left(x - \frac{1}{2}x^4 + \frac{x^7}{7}\right)\Big| = 1 - \frac{1}{2} + \frac{1}{7} = \frac{9}{14}$$

4. (20 points) Evaluate the following integrals. Show your work

a. (6 pts) $\int_{2\pi}^{3\pi} 3\cos^2(x)\sin(x)\, dx = -\cos^3(x)\Big| = -\left(\cos^3(3\pi) - \cos^3(2\pi)\right) = -(-1-1) = 2$

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$\int 3\cos^2(x)\sin(x)\, dx = -3\int u^2\, du = -u^3 + C = -\cos^3(x) + C$

$u = \cos(x)$

$du = -\sin(x)\, dx$

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b. (7 pts) $\lim_{b \to 1} \int_0^b \frac{dx}{\sqrt{1-x^2}}$ Try evaluating the integral first. What is $\int \frac{dx}{\sqrt{1-x^2}}$?

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C \Rightarrow \int_0^b \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(b) - \sin^{-1}(0) = \sin^{-1}(b)$$

$$\Rightarrow \mathbf{\lim_{b \to 1} \int_0^b \frac{dx}{\sqrt{1-x^2}} = \lim_{b \to 1} \sin^{-1}(b) = \sin^{-1}(1) = \frac{\pi}{2}}$$

c. (7 pts) $\int (x+5)\sqrt[3]{x+1}\,dx = \int (u+4)\,u^{\frac{1}{3}}\,dx = \int \left(u^{\frac{4}{3}} + 4u^{\frac{1}{3}}\right)dx = \frac{3}{7}u^{\frac{7}{3}} + 3u^{\frac{4}{3}} + C$

$$= \mathbf{\frac{3}{7}(x+1)^{\frac{7}{3}} + 3(x+1)^{\frac{4}{3}} + C = \frac{3}{7}(x+1)^{\frac{4}{3}}(x+8) + C}$$

(either answer or any other equivalent answer)

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$u = x+1 \Rightarrow x = u-1 \Rightarrow x+5 = u+4$

$du = dx$
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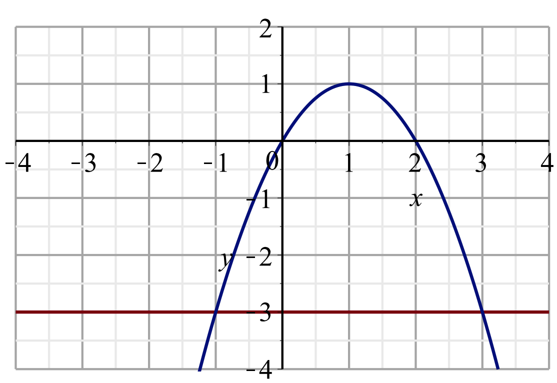
5. (15 points) a. (6 pts) Sketch the graphs of $y = -3$ & $y = 2x - x^2$ on the axis below. b. (3 pts) Shade in the region bounded above by the parabola and below by the line (3 pts). c.(6 pts) Find the area of the enclosed region.

a. Graph



c.

$$Area = \int_{-1}^{3} (-x^2 + 2x + 3)\, dx = \left(-\frac{x^3}{3} + x^2 + 3x \right)\Bigg| = \left(-\frac{27}{3} + 9 + 9 \right) - \left(\frac{1}{3} + 1 - 3 \right) = \mathbf{\frac{32}{3}}$$

6. (20 points) The figure below represents an electrical circuit with **resistence R (in ohms)** and **inductance L (in henries)**, both of which are positive **constants.** Also, $\boldsymbol{i = i(t)}$ is the current (in amperes) in the curcuit at time t after the circuit is closed , (the switch between a and b) connecting a closed circuit of **constant** positive **voltage** V. The circuit satisfies the IVP : $\boldsymbol{\frac{di}{dt} + \frac{R}{L} i = \frac{V}{L}}$ **with** $\boldsymbol{i(0) = 0}$. **Solve this IVP for the unknown function $\boldsymbol{i(t)}$.**

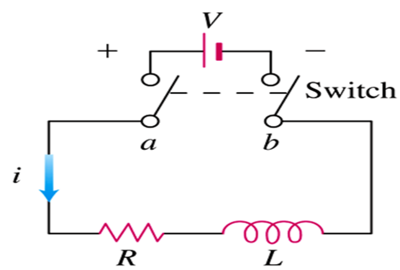


FIGURE 9.8 The *RL* circuit in Example 4.

SOLUTION: IVP : $\boldsymbol{\frac{di}{dt} + \frac{R}{L} i = \frac{V}{L}}$ **&** $\boldsymbol{i(0) = 0}$. This is a linear IVP with $P(t) = \frac{R}{L}$ & $Q(t) = \frac{V}{L}$

1. $\int P(t)\, dt = \int \frac{R}{L}\, dt = \frac{R}{L}\, t$

2. *The integrating factor is* $v(t) = e^{\frac{R}{L} t}$

3. Multiply both sides of the DE by the integrating factor $v(t) \Rightarrow e^{\frac{R}{L} t} \left(\frac{di}{dt} + \frac{R}{L} i \right) = \frac{V}{L} e^{\frac{R}{L} t}$

4. Now integrate both sides wrt t : (remember that $\int e^{\frac{R}{L} t} \left(\frac{di}{dt} + \frac{R}{L} i \right) dt = e^{\frac{R}{L} t} i(t)$).

5. Finally, multiply both sides by $e^{-\left(\frac{R}{L}\right) t}$ to get $\boldsymbol{i(t) = \frac{V}{R} + C e^{-\left(\frac{R}{L}\right) t}}$ Then use

$i(0) = 0 \Rightarrow 0 = \frac{V}{R} + C e^{0} \Rightarrow C = -\frac{V}{R}$. So the solution is :

$$i(t) = \frac{V}{R} - \frac{V}{R} e^{-\left(\frac{R}{L}\right)t}.$$