Math 2401

Quiz 8

Section B

1. Find a smooth parametrization for the curve $\mathcal{C}$ given by the intersection of the surfaces $x^2 + z^2 = 4$ and $y = 3$. Set up, but do not integrate, $\int_{\mathcal{C}} f\, ds$, with $f(x, y, z) = x^2 + y - 3$.

   Solution:

   $\vec{\mathbf{r}}(t) = (2\cos t)\vec{\mathbf{i}} + 3\vec{\mathbf{j}} + (2\sin t)\vec{\mathbf{k}}$, $0 \le t \le 2\pi$.
   $\vec{\mathbf{r}}'(t) = (-2\sin t)\vec{\mathbf{i}} + (2\cos t)\vec{\mathbf{k}}$, $\|\vec{\mathbf{r}}'(t)\| = \sqrt{(2\sin t)^2 + (2\cos t)^2} = \sqrt{4} = 2$.
   $ds = \|\vec{\mathbf{r}}'(t)\|\, dt = 2\, dt$, $f(\vec{\mathbf{r}}(t)) = (2\cos t)^2 + 3 - 3 = 4\cos^2 t$.
   $\int_{\mathcal{C}} f\, ds = \int_0^{2\pi} f(\vec{\mathbf{r}}(t))\, dt = \int_0^{2\pi} 8\cos^2 t\, dt.$

2. Find the work done by $\vec{\mathbf{F}}(x, y, z) = xy\vec{\mathbf{i}} + yz\vec{\mathbf{j}} + x\vec{\mathbf{k}}$, over the curve $\vec{\mathbf{r}}(t) = t\vec{\mathbf{i}} + t^2\vec{\mathbf{i}} + t^3\vec{\mathbf{k}}$, $0 \le t \le 2$ in the direction of increasing t.

   Solution:

   $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = t^3\vec{\mathbf{i}} + t^5\vec{\mathbf{j}} + t\vec{\mathbf{k}}, \qquad d\vec{\mathbf{r}} = \vec{\mathbf{r}}'(t)\, dt = (\vec{\mathbf{i}} + 2t\vec{\mathbf{j}} + 3t^2\vec{\mathbf{k}})\, dt.$
   $W = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^2 \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt = \int_0^2 (t^3\vec{\mathbf{i}} + t^5\vec{\mathbf{j}} + t\vec{\mathbf{k}}) \cdot (\vec{\mathbf{i}} + 2t\vec{\mathbf{j}} + 3t^2\vec{\mathbf{k}})\, dt = \int_0^2 2t^6 + 4t^3\, dt = \frac{368}{7}.$

3. Find a smooth parametrization of the curve $\mathcal{C}$ given by the intersection of the surfaces $2x^2 + y^2 + z^2 = 2$ and $y = z$. Find $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y, z) = -y\vec{\mathbf{i}} + z^2\vec{\mathbf{j}} + y^3\vec{\mathbf{k}}$.

   Solution:

   If $y = z$, then $2x^2 + y^2 + y^2 = 2 \Rightarrow 2x^2 + 2y^2 = 2 \Rightarrow x^2 + y^2 = 1$. A parametrization is $x = \cos t$, $y = \sin t$, $z = y = \sin t$, i.e., $\vec{\mathbf{r}}(t) = (\cos t)\vec{\mathbf{i}} + (\sin t)\vec{\mathbf{j}} + (\sin t)\vec{\mathbf{k}}$, $0 \le t \le 2\pi$.

   $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = (-\sin t)\vec{\mathbf{i}} + (\sin^2 t)\vec{\mathbf{j}} + (\sin^3 t)\vec{\mathbf{k}}, \qquad d\vec{\mathbf{r}} = \vec{\mathbf{r}}'(t)\, dt = ((-\sin t)\vec{\mathbf{i}} + (\cos t)\vec{\mathbf{j}} + (\cos t)\vec{\mathbf{k}})\, dt.$
   $\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt = ((-\sin t)\vec{\mathbf{i}} + (\sin^2 t)\vec{\mathbf{j}} + (\cos^3 t)\vec{\mathbf{k}}) \cdot ((-\sin t)\vec{\mathbf{i}} + (\cos t)\vec{\mathbf{j}} + (\cos t)\vec{\mathbf{k}})\, dt = (\sin^2 t + \sin^2 t \cos t + \sin^3 t \cos t)\, dt.$
   $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^{2\pi} (\sin^2 t + \sin^2 t \cos t + \sin^3 t \cos t)\, dt = \pi.$