1. Find the general solution for the equation (14 points)

$$y\frac{dy}{dx} = 8x^3 + 2x^3y^2, \qquad y > 0.$$

$$y\frac{dy}{dx} = 2x^3(4+y^2)$$

$$\frac{y}{4+y^2}\,dy = 2x^3\,dx$$

$$\int \frac{y}{4+y^2}\,dy = \int 2x^3\,dx$$

5 points (separation and set up)

$$\frac{1}{2}\ln(4+y^2) = \frac{x^4}{2} + C$$

5 points (integration)

$$\ln(4+y^2) = x^4 + 2C$$

$$4+y^2 = e^{x^4+2C}$$

$$y^2 = e^{x^4+2C} - 4$$

$$y = \sqrt{e^{x^4+2C} - 4}$$

4 points (solve for y)

2. Find the solution for the following initial value problem (15 points)

$$\begin{cases} xy' + 2y = 3xe^{x^3}, \\ y(1) = 0. \end{cases}$$

$$y' + \frac{2}{x}y = 3e^{x^3}$$

1 point

$$\mu(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

3 points (integrating factor)

$$y(x) = \frac{1}{\mu(x)} \int \mu(x) q(x)\,dx$$

$$= \frac{1}{x^2} \int x^2 \cdot 3e^{x^3}\,dx$$

$$= \frac{1}{x^2}\left(e^{x^3} + C\right)$$

6 points (solution, either by formula or multiplying by $\mu(x)$)

$$y(1) = 0 \implies \frac{1}{1^2}\left(e^{1^3} + C\right) = 0$$

$$\implies e + C = 0$$

$$\implies C = -e$$

4 points (finding C)

$$\therefore y(x) = \frac{1}{x^2}\left(e^{x^3} - e\right)$$

1 point (final answer)

3. Compute the following limit (14 points)

$$\lim_{x\to\infty}(1+2x)^{1/(2\ln x)}$$

$$= \lim_{x\to\infty} e^{\ln\left[(1+2x)^{\frac{1}{2\ln x}}\right]} = \lim_{x\to\infty} \exp\left(\frac{1}{2\ln x}\cdot \ln(1+2x)\right)$$

2 points (exponential notation)

$$= \exp\left(\lim_{x\to\infty} \frac{\ln(1+2x)}{2\ln x}\right)$$

3 points (algebraic manipulation to obtain a "L'Hôpitalizable" form)

$$= \exp\left(\lim_{x\to\infty} \frac{\frac{2}{1+2x}}{\frac{2}{x}}\right)$$

L'Hôpital (form "$\frac{\infty}{\infty}$")

1 point (justification)

4 points (use of L'Hôpital)

$$= \exp\left(\lim_{x\to\infty} \frac{2x}{2+4x}\right)$$

$$= \exp\left(\tfrac{1}{2}\right) = e^{\frac{1}{2}} = \sqrt{e}$$

4 points (computation of limit and final answer)

4. Determine the values of λ for which the integral $\int_0^\infty e^{\lambda x}\,dx$ is convergent. In the cases where it is convergent, evaluate the integral. (14 points)

For $\lambda = 0$, $e^{\lambda x} = 1$, and $\int_0^\infty 1\,dx = \infty$ 2 points (case $\lambda = 0$)

For $\lambda \neq 0$
1 point (definition of integral)

$$\int_0^\infty e^{\lambda x}\,dx = \lim_{b\to\infty} \int_0^b e^{\lambda x}\,dx = \lim_{b\to\infty} \left.\frac{1}{\lambda} e^{\lambda x}\right|_0^b$$

2 points (integration)

$$= \lim_{b\to\infty} \frac{1}{\lambda}\left(e^{\lambda b} - 1\right)$$

1 point (correct notation)

$$\lim_{b\to\infty} e^{\lambda b} = \begin{cases} \infty & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda < 0 \end{cases}$$

Then, $\displaystyle \lim_{b\to\infty} \frac{1}{\lambda}\left(e^{\lambda b} - 1\right) = \begin{cases} \infty, & \text{if } \lambda > 0, \\ -\frac{1}{\lambda}, & \text{if } \lambda < 0. \end{cases}$ 2 points (case $\lambda > 0$) 2 points (case $\lambda < 0$) (computations)

So, $\int_0^\infty e^{\lambda x}\,dx$ is convergent if and only if $\lambda < 0$,

and in this case $\displaystyle \int_0^\infty e^{\lambda x}\,dx = -\frac{1}{\lambda}$

2 points (final answer)

2 points (value for $\lambda < 0$)

5. Determine if the following integral is convergent or divergent, using a convergence test and providing justification. (14 points)

$$\int_1^\infty \frac{2x+3}{\sqrt{x^6+x+2}}\,dx$$

By comparison test, for $x \geq 1$

$$0 \leq \frac{2x+3}{\sqrt{x^6+x+2}} \leq \frac{2x+3}{\sqrt{x^6}} = \frac{2x+3}{x^3} = \frac{2}{x^2} + \frac{3}{x^3}$$

5 points (comparison, selection of the function, justification)

Since $\int_1^\infty \frac{2}{x^2} + \frac{3}{x^3}\,dx = \int_1^\infty \frac{2}{x^2}\,dx + \int_1^\infty \frac{3}{x^3}\,dx$

3 points (statement about convergence or diverg. of "test" function)

are both convergent. (p-integral, $p>1$).

2 points (justification, convergence of $g(x)$)

then $\int_1^\infty \frac{2x+3}{\sqrt{x^2+x+2}}\,dx$ is convergent.

4 points (conclusion and final answer) (correct use of test)

By limit comparison test, with $f(x) = \frac{2x+3}{\sqrt{x^6+x+2}}$, $g(x) = \frac{1}{x^2}$

4 points (comparison, $g(x)$)

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{2x+3}{\sqrt{x^6+x+2}}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2x^3+3x^2}{\sqrt{x^6+x+2}}$$

$$= \lim_{x\to\infty} \frac{x^3\left(2+\frac{3}{x}\right)}{\sqrt{x^6\left(1+\frac{1}{x^5}+\frac{2}{x^6}\right)}} = \lim_{x\to\infty} \frac{x^3\left(2+\frac{3}{x}\right)}{x^3\sqrt{1+\frac{1}{x^5}+\frac{2}{x^6}}}$$

$$= \lim_{x\to\infty} \frac{2+\frac{3}{x}}{\sqrt{1+\frac{1}{x^5}+\frac{2}{x^6}}} = 2$$

4 points (limit computation)

Since $0 < 2 < \infty$, by limit comparison test, $\int_1^\infty \frac{2x+3}{\sqrt{x^6+x+2}}\,dx$ converges, because $\int_1^\infty \frac{1}{x^2}\,dx$ converges (p-integral, $p>1$)

4 points (correct use of test)

2 points (justification of convergence of $g(x)$)

6. Consider the series (15 points)

$$\sum_{n=1}^{\infty} \frac{3}{n^2 + 7n + 12}$$

First, use a convergence test to prove that it is convergent, **then** compute its sum.

By direct comparison 1 point (test)

$$\frac{3}{n^2+7n+12} \leq \frac{3}{n^2}$$ 1 point (selection of function)

Since $\sum_{n=1}^{\infty} \frac{3}{n^2} = 3 \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p>1$)

2 points (statement) 1 point (justification)

then $\sum_{n=1}^{\infty} \frac{3}{n^2+7n+12}$ converges. 2 points (correct use of test)

$\frac{27}{28}$

Now, by partial fractions decomposition

$$\frac{3}{n^2+7n+12} = \frac{3}{n+3} - \frac{3}{n+4} = \frac{3}{n+3} - \frac{3}{(n+1)+3}$$

3 points (partial fractions)

then

$$\sum_{n=1}^{\infty} \frac{3}{n^2+7n+12} = \sum_{n=1}^{\infty} \left(\frac{3}{n+3} - \frac{3}{(n+1)+3} \right)$$

$$= \frac{3}{1+3} - \lim_{n\to\infty} \frac{3}{(n+1)+3}$$ 3 points (algebra)

$$= \frac{3}{4}$$ 2 points (answer)

7. Determine if the following series is convergent or divergent using a convergence test and providing justification. (14 points)

$$\sum_{n=1}^{\infty} \frac{(2n+1)!}{5^n (n!)^2}$$

Ratio test

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\dfrac{(2(n+1)+1)!}{5^{n+1}(n+1)!(n+1)!}}{\dfrac{(2n+1)!}{5^n n! n!}} = \frac{5^n (2n+3)! \, n! \, n!}{5^{n+1}(2n+1)!(n+1)!(n+1)!}$$

4 points (correct expression)

$$= \frac{(2n+2)(2n+3)}{5(n+1)(n+1)}$$

3 points (simplification)

Then, $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{4}{5} < 1$

3 points (limit)

Therefore, the series is convergent by the ratio test.

4 points (correct use of test)

Bonus question: Consider the series

$$\sum_{n=1}^{\infty} \frac{(n!)^n + n^{(n^2)}}{3^n n^{(n^2)}}.$$

Determine, giving a complete justification, if the series is convergent or divergent. (5 points)

$$\sum_{n=1}^{\infty} \frac{(n!)^n + n^{(n^2)}}{3^n n^{(n^2)}} = \underbrace{\sum_{n=1}^{\infty} \frac{(n!)^n}{3^n n^{(n^2)}}}_{S_1} + \underbrace{\sum_{n=1}^{\infty} \frac{n^{(n^2)}}{3^n n^{(n^2)}}}_{S_2}$$

$S_2 = \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a geometric series with $r = \frac{1}{3} < 1$, hence, convergent.

For S_1, applying the root test

$$\sqrt[n]{|a_n|} = \frac{n!}{3\, n^n} = \frac{1}{3} \cdot \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} < \frac{1}{3}$$

then, $\lim_{n\to\infty} \sqrt[n]{|a_n|} < \frac{1}{3} < 1$

By the root test, S_1 is convergent.

Therefore, $\sum_{n=1}^{\infty} \frac{(n!)^n + n^{(n^2)}}{3^n n^{(n^2)}}$ is convergent.

Note: $0 < \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n}{n} < \frac{1}{n}$,

by "Sandwich" Theorem $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.