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Name Key

**TEST 1**

Math 1553 D Steinbart

September 17, 2015

Work neatly. Justify your answers and use proper notation. SHOW YOUR WORK TO RECEIVE CREDIT! No calculators or electronic devices are allowed.

There is a total of 100 points.

(15) 1. Solve the system of equations

$$\begin{aligned} x_1 - 3x_2 + x_3 &= -2 \\ -x_2 + x_3 &= -3 \\ 2x_1 + x_2 - x_3 &= 13 \end{aligned}$$

The augmented matrix for the system is

$$\begin{bmatrix} 1 & -3 & 1 & -2 \\ 0 & -1 & 1 & -3 \\ 2 & 1 & -1 & 13 \end{bmatrix} \xrightarrow[R_2 \to -R_2]{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 7 & -3 & 17 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 7R_2} \begin{bmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 4 & -4 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{4}R_3} \begin{bmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[R_1 \to R_1 - R_3]{R_2 \to R_2 + R_3} \begin{bmatrix} 1 & -3 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_1 \to R_1 - 3R_2$

$$\to \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

So

$x_1 = 5$

$x_2 = 2$

$x_3 = -1$

There is a solution to this system

The solution is unique.

(15) 2. Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, and $\mathbf{v_3} = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 5 \end{bmatrix}$. Are the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$ are linearly independent? Justify your answer. Be sure to show your work in a manner that can be followed.

Let $A = [\underline{v}_1 \ \underline{v}_2 \ \underline{v}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \\ 1 & 4 & 5 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_1]{R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \\ 0 & 3 & 4 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 3R_2]{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

This is in echelon form.

There is a pivot in every column. Every column of A is a pivot column. So the vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are linearly independent.

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(10) 3. Suppose $A$ is a $3 \times 5$ matrix and can be row reduced to $\begin{bmatrix} 1 & 0 & -2 & 4 & 6 \\ 0 & 1 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

a. Find the parametric vector form of the solutions set to $A\mathbf{x} = \mathbf{0}$

$$A \sim \begin{bmatrix} 1 & 0 & -2 & 4 & 6 \\ 0 & 1 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{0}$: $x_1 - 2x_3 + 4x_4 + 6x_5 = 0$, $x_2 + 3x_3 + 5x_4 = 0$

$x_1, x_2$ basic variables

$x_1 = 2x_3 - 4x_4 - 6x_5$

$x_2 = -3x_3 - 5x_4$

$x_3, x_4, x_5$ free variables

$$\text{So } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 - 4x_4 - 6x_5 \\ -3x_3 - 5x_4 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -4 \\ -5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -6 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$x_3, x_4, x_5 \in \mathbb{R}$

So the parametric vector form of the solution set to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3$ where $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ -5 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -6 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $t_1, t_2, t_3 \in \mathbb{R}$

b. Suppose that $A\begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Find the parametric vector form of the [redacted]

Since $\mathbf{p} = \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$, and knowing the solution set to $A\mathbf{x} = \mathbf{0}$ from above, we have the parametric vector form of the solution set to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{p} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3$ where $\mathbf{p} = \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are as in part (a), and $t_1, t_2, t_3 \in \mathbb{R}$.

(14) 4. $A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & -3 & 1 \end{bmatrix}$ Do the columns of $A$ span $\mathbb{R}^3$? Justify your answer.

$$\begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & -3 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_2]{} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Every row of $A$ has a pivot.

$A$ is $3 \times 4$. So the columns of $A$ span $\mathbb{R}^3$.

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3(b) $\mathbf{x} = \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} + t_1\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} -4 \\ -5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_3\begin{bmatrix} -6 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $t_1, t_2, t_3 \in \mathbb{R}$

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(18) 5. In parts (a), (b), and (c) below, $\cdot$ denotes a nonzero entry and $*$ denotes an entry that may be 0 or nonzero.

5. (a). $A$ is a $5 \times 4$ matrix. Suppose that $A$ can be row reduced to

$$\begin{bmatrix} \cdot & * & * & * \\ 0 & \cdot & * & * \\ 0 & 0 & \cdot & * \\ 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) Does $A\mathbf{x} = \mathbf{b}$ have a solution for all $\mathbf{b}$ in $\mathbb{R}^5$? No Why? There is not a pivot in each row of A

(ii) If $A\mathbf{x} = \mathbf{b}$ has a solution, is this solution unique? Yes Why?

There is a pivot in every column of A.
(or "Every column of A is a pivot column.")
This means that there are no free variables
So if a solution to $A\underline{x} = \underline{b}$ exists, that solution will be unique.

5. (b). $A$ is a $5 \times 6$ matrix. Suppose that $A$ can be row reduced to

$$\begin{bmatrix} \cdot & * & * & * & * & * \\ 0 & \cdot & * & * & * & * \\ 0 & 0 & \cdot & * & * & * \\ 0 & 0 & 0 & 0 & \cdot & * \\ 0 & 0 & 0 & 0 & 0 & \cdot \end{bmatrix}$$

to the equation $A\mathbf{x} = \mathbf{b}$.

(i) Does $A\mathbf{x} = \mathbf{b}$ have a solution for all $\mathbf{b}$ in $\mathbb{R}^5$? Yes Why? There is a pivot in every row of A.

(ii) If $A\mathbf{x} = \mathbf{b}$ has a solution, is this solution unique? No Why?

A has a column which is not a pivot column.
So there is a free variable.
So if $A\underline{x} = \underline{b}$ has a solution, then it has infinitely many solutions.

5. (c). $A$ is a $4 \times 5$ matrix. Suppose that $A$ can be row reduced to

$$\begin{bmatrix} \cdot & * & * & * & * \\ 0 & \cdot & * & * & * \\ 0 & 0 & \cdot & * & * \\ 0 & 0 & 0 & 0 & \cdot \end{bmatrix}.$$

(i) Does $A\mathbf{x} = \mathbf{b}$ have a solution for all $\mathbf{b}$ in $\mathbb{R}^5$? Yes Why? There is a pivot in every row of A.

(ii) If $A\mathbf{x} = \mathbf{b}$ has a solution, is this solution unique? No Why?

A has a column which is not a pivot column. So there is a free variable. So if $A\underline{x} = \underline{b}$ has a solution, it will have infinitely many solutions

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Note $\begin{bmatrix}-4\\4\end{bmatrix} = 4\begin{bmatrix}-1\\1\end{bmatrix}$ so $\begin{bmatrix}-4\\4\end{bmatrix}$, $\begin{bmatrix}-1\\1\end{bmatrix}$ are parallel and a vector param. form of L is the set of points $\underline{x}$ with $\underline{x} = \begin{bmatrix}3\\-2\end{bmatrix} + t\begin{bmatrix}-1\\1\end{bmatrix}$, $t \in \mathbb{R}$.

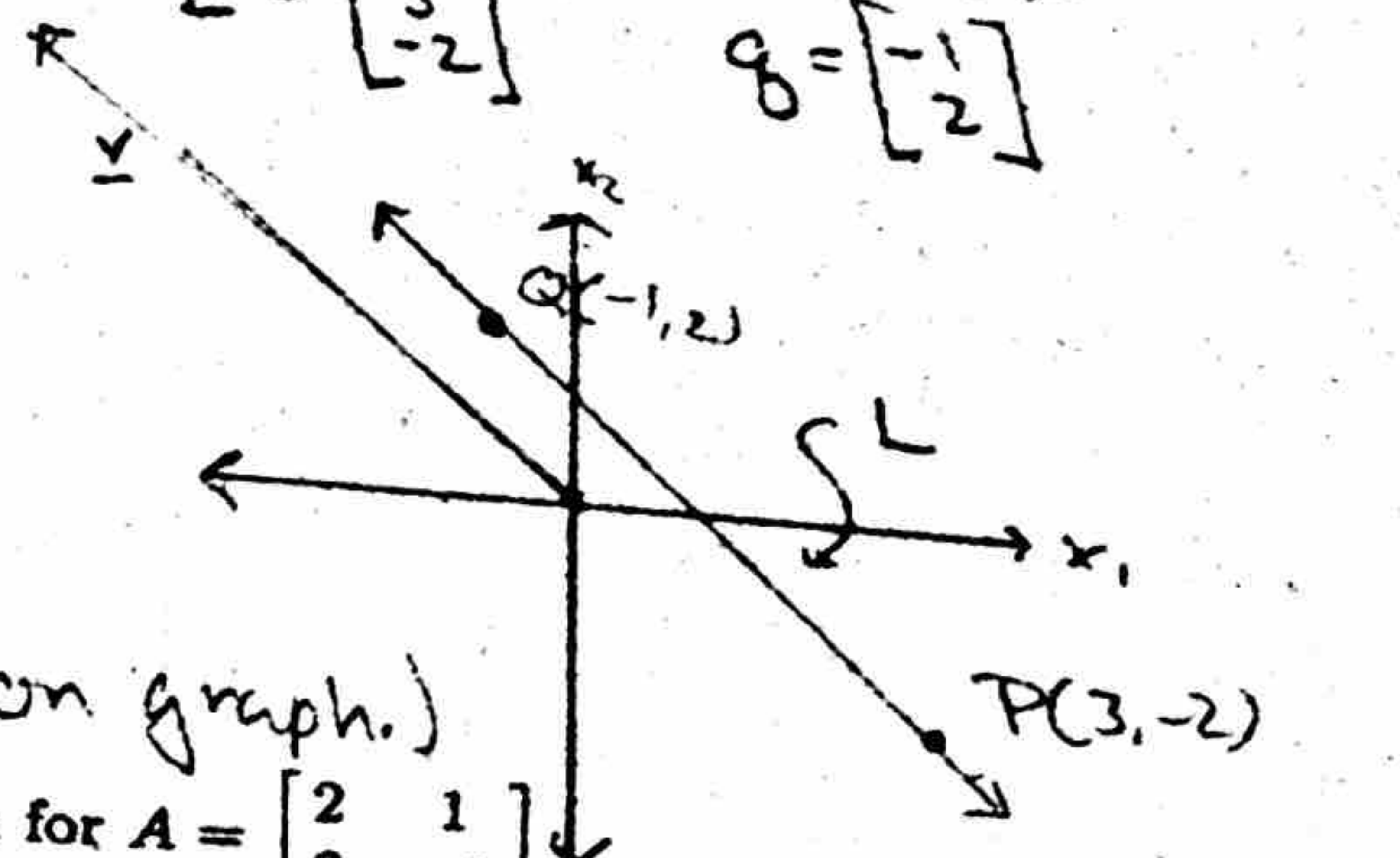
(12) 6. a. Find the vector parametric form of the line through the points $P(3,-2)$ and $Q(-1,2)$. Call this line $L$. Sketch $L$. Label $P$ and $Q$.

Let $\underline{v} = \overrightarrow{PQ} = \begin{bmatrix}-1\\2\end{bmatrix} - \begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}-4\\4\end{bmatrix}$   Let $\underline{p} = \begin{bmatrix}3\\-2\end{bmatrix}$   $\underline{q} = \begin{bmatrix}-1\\2\end{bmatrix}$

L is the set of points $\underline{x}$ with $\underline{x} = \underline{p} + t\underline{v}$ for all $t$ in $\mathbb{R}$

So $\underline{x} = \begin{bmatrix}3\\-2\end{bmatrix} + t\begin{bmatrix}-4\\4\end{bmatrix}$, $t \in \mathbb{R}$

(You did not need to include $\underline{v}$ on graph.)



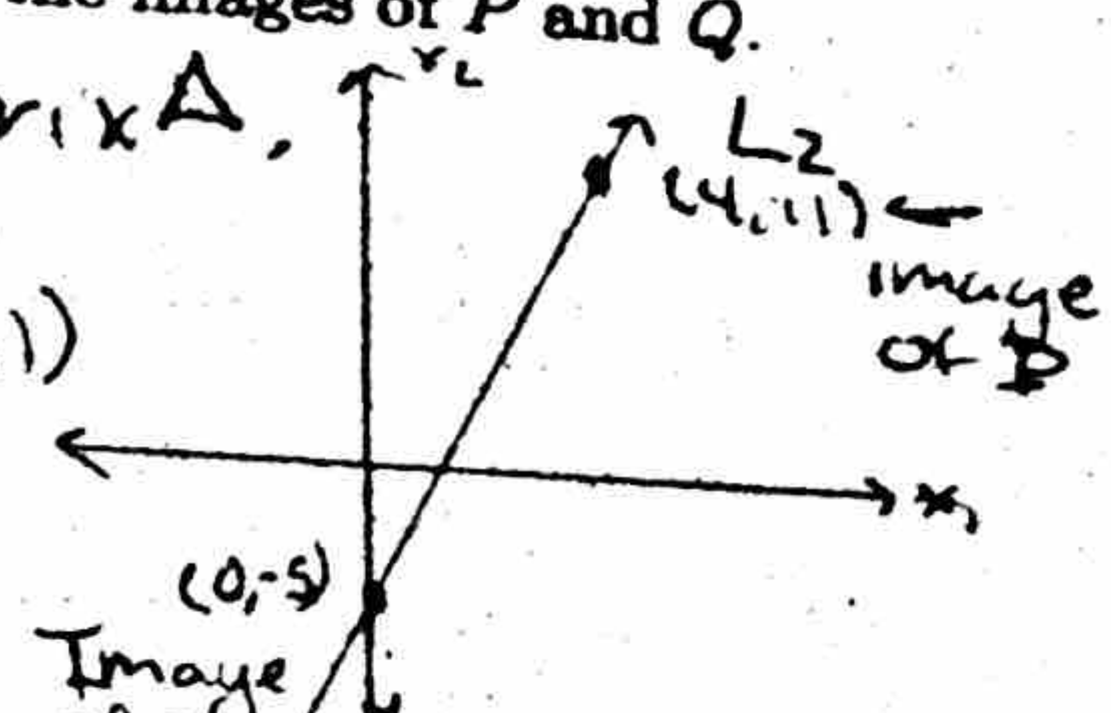

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation $T(\mathbf{x}) = A\mathbf{x}$ for $A = \begin{bmatrix}2 & 1\\3 & -1\end{bmatrix}$.

b. Let $L_2$ be the image under $T$ of the line $L$. Sketch $L_2$. Label the images of $P$ and $Q$.

T is linear since $T(\underline{x}) = A\underline{x}$ for a matrix $A$, so the image of a line is a line

$T(\underline{p}) = A\underline{p} = \begin{bmatrix}2 & 1\\3 & -1\end{bmatrix}\begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}6-2\\9+2\end{bmatrix} = \begin{bmatrix}4\\11\end{bmatrix} = \underline{p}' \to (4,11)$

$T(\underline{q}) = A\underline{q} = \begin{bmatrix}2 & 1\\3 & -1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}-2+2\\-3-2\end{bmatrix} = \begin{bmatrix}0\\-5\end{bmatrix} = \underline{q}' \to (0,-5)$




Note $T(\underline{x}) =$

For $\underline{x} \in L$, $T(\underline{x}) = T(\underline{p} + t\underline{v}) \overset{\text{since T is linear}}{=} T(\underline{p}) + tT(\underline{v}) = \begin{bmatrix}4\\11\end{bmatrix} + t\begin{bmatrix}-4\\-16\end{bmatrix}$, $t \in \mathbb{R}$

Aside. $T(\underline{v}) = A\underline{v} = \begin{bmatrix}2 & 1\\3 & -1\end{bmatrix}\begin{bmatrix}-4\\4\end{bmatrix} = \begin{bmatrix}-8+4\\-12-4\end{bmatrix} = \begin{bmatrix}-4\\-16\end{bmatrix}$

Since $\begin{bmatrix}-4\\-16\end{bmatrix} = -4\begin{bmatrix}1\\4\end{bmatrix}$, the vectors $\begin{bmatrix}-4\\-16\end{bmatrix}$ and $\begin{bmatrix}1\\4\end{bmatrix}$ are parallel. So a vector param. form of $L_2$ is $\underline{x} = \begin{bmatrix}4\\11\end{bmatrix} + t\begin{bmatrix}1\\4\end{bmatrix}$, $t \in \mathbb{R}$

Comment: It is not correct to write $L: \begin{bmatrix}3\\-2\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix}t$, $t \in \mathbb{R}$. A point is on L if and only if $\underline{x}$ (the vector associated with the point) can be written $\underline{x} = \begin{bmatrix}3\\-2\end{bmatrix} + \begin{bmatrix}-4\\4\end{bmatrix}t$, $t \in \mathbb{R}$ for some $t \in \mathbb{R}$.

(10) 7. Mark the following statements as TRUE or FALSE or fill in the blank. You do not need to justify your answer.

a. True If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are vectors in $\mathbb{R}^3$, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent. Since the vectors are in $\mathbb{R}^3$, they have 3 entries. There are 4 vectors.

b. False If $A$ is a $4 \times 3$ matrix, then the columns of $A$ span $\mathbb{R}^3$

c. True If $A$ is a $3 \times 5$ matrix, then the columns of $A$ are linearly dependent.

d. True $\mathbf{v}_1$ is not the zero vector $\mathbf{0}$. If the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent, then $\mathbf{v}_2 = c\mathbf{v}_1$ for some scalar $c$.

e. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $A$ be the standard matrix for $T$. $T$ is one-to-one if the columns of $A$ span $\mathbb{R}^m$. FALSE. See thm 1.9.

(b) A is 4×3. $A = [\underline{a}_1\ \underline{a}_2\ \underline{a}_3]$ where each $\underline{a}_i$ is in $\mathbb{R}^4$, so the columns of A are not in $\mathbb{R}^3$ and so cannot span $\mathbb{R}^3$

(e) T is 1-1 iff the columns of A are linearly independent.