ISYE 3232B Fall 2013 Final - White

I, ______________________________, do swear that I abide by the Georgia Tech Honor Code. I understand that any honor code violations will result in an F.

Signature: _________________________

- **Throughout, you will receive full credit if someone with no understanding of probability, set theory, and calculus could simplify your answer to obtain the correct numerical answer.**
- You will have 2 hours.
- This exam is closed book and closed notes. Calculators are not allowed. No scrap paper is allowed. Make sure that there is nothing on your desk except pens and erasers.
- If you need extra space, use the back of the page and indicate that you have done so.
- **Do not remove any page from the original staple**. Otherwise, there will be 3 points off.
- **Show your work on the test sheet**. If you do not show your work for a problem on your test sheet, we will give zero point for the problem even if your answer is correct.
- **We will not select among several answers**. Make sure it is clear what part of your work you want graded. If two answers are given, zero point will be given for the problem.

Poisson Process

- A Poisson arrival process $N(t)$ is a Poisson distributed with $\Pr\{N(t) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$.
- Nonhomogeneous (or non-stationary) Poisson process with $\lambda(t)$ has $\bar{\lambda} = \mathsf{E}[N(t+s) - N(s)] = \int_s^{t+s} \lambda(t)dt$ and $\Pr\{N(t+s) - N(s) = n\} = e^{-\bar{\lambda}}\left(\frac{\bar{\lambda}^n}{n!}\right)$.

Problem 1 (a), (b), (d), (d)
Problem 2 (a), (b), (c), (d)
Problem 3 (a), (b)
Problem 4 (a), (b)
Problem 5
Problem 6 (a), (b), (c), (d), (e), (f)
Bonus

$\lambda_J = \frac{1}{2}/min$ $\lambda_p = \frac{1}{4}/min$

J P

A B

C

1. (20 points, 5 points each) A small call center has three phone lines that are answered by two operators, John and Paul. That is, the call center can hold up to three calls (two in service and one in hold) and when the call center already has three calls, new calls will hear busy signals and have to call again to be connected. The call processing times by John are iid having exponential distribution with mean 2 minutes. The call processing times by Paul are iid having exponential distribution with mean 4 minutes. An arriving call to an empty system is always processed by John.

   Suppose that three calls (A, B, and C) arrive at an empty center at 8am with John taking call A, Paul taking call B, and call C waiting.

   (a) What is the probability that call A is still in service at 8:10am?

   $$Pr(X_A > 10) = e^{-\frac{10}{2}} = e^{-5}$$

   (b) What is the probability that all three calls still in the system at 8:10am?

   $$Pr(\min(X_A, X_B) > 10) = e^{-10(\lambda_A + \lambda_B)}$$

   $$= e^{-10(\lambda_J + \lambda_P)} = e^{-10(\frac{1}{2}+\frac{1}{4})}$$

   (c) Suppose that due to call B's special request, Paul had to talk to his supervisor which took exactly 4 minutes. (All other services are not influenced.) If the server came back at 8:04am and saw call A still in the service, what is the probability that call A is still in service at 8:10am?

   $$Pr(X_A > 10 \mid X_A > 4) = Pr(X_A > 6) = e^{-\frac{6}{2}} = e^{-3}.$$

   (d) What is the probability that call C leaves the system before call B?

   $$\underbrace{Pr(X_J < X_p)}_{\text{A finishes before B}} \underbrace{Pr(X_J < X_p)}_{\text{C finishes before B}} = \left(\frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{4}}\right)^2$$

2. (20 points, 5 points each) Suppose passengers arrive at a MARTA station between 10am - 5pm following a Poisson process with rate $\lambda = 60$ per hour. For notation, let $N(t)$ be the number of passengers arrived in the first t hours, $S_0 = 0$, S_n be the arrival time of the nth passenger, X_n be the interarrival time between the $(n-1)$st and nth passenger.

(a) What is the expected number of passengers arrived in the first two hours?

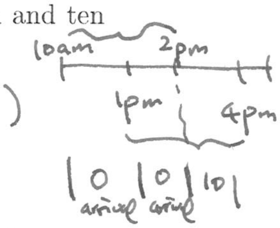
$$E[N(2)] = 60 \times 2 = 120$$

(b) What is the expected arrival time of the 45th passenger? Be specific. For example, your answer should be in a form of an exact clock time such as 11:11am or 4:56pm.

$$E[X_1 + \cdots + X_{45}] = 45 \times \frac{1}{60} = \frac{45}{60} \text{ hrs} = 45 \text{ mins}$$

$\therefore$ 10:45 am

(c) What is the probability that no passenger arrives between 10am and 2pm and ten passengers arrive between 1pm and 4pm?



$$\Pr(N(2pm) - N(10am) = 0,\ N(4pm) - N(1pm) = 10)$$

$$= \Pr(N(1pm) - N(10am) = 0) \cdot \Pr(N(2pm) - N(1pm) = 0) \cdot$$
$$\times \Pr(N(4pm) - N(2pm) = 10)$$

$$= \Pr(N(3) = 0) \cdot \Pr(N(1) = 0) \cdot \Pr(N(2) = 10)$$

$$= e^{-180} \times e^{-60} \times e^{-120}\frac{(120)^{10}}{10!}$$

(d) Suppose that 60% passengers are female and 40% are male. Given that 15 passengers arrived between 2 pm and 4 pm, what is the probability that 5 female passengers arrived between 2 pm and 4 pm?

$N(t) \sim PP(60)$

female $\to$ $N_f(t) \sim PP(36)$

male $\to$ $N_m(t) \sim PP(24)$

$$\Pr(N_f(4pm) - N_f(2pm) = 5 \mid N(4pm) - N(2pm) = 15)$$

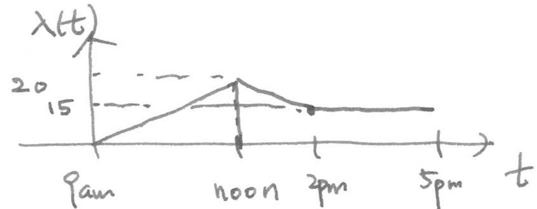
$$= \frac{\Pr(N_f(2) = 5,\ N_m(2) = 10)}{\Pr(N(2) = 15)}$$

$$= \frac{\dfrac{e^{-72}(72)^5}{5!} \cdot \dfrac{e^{-48}(48)^{10}}{10!}}{\dfrac{e^{-120}(120)^{15}}{15!}} = \underbrace{\binom{15}{5}(0.6)^5(0.4)^{10}}$$

This is the prob that 5 of 15 are female & 10 of 15 are male.

That is, # female given total = 15 follows Bin(15, 0.6).

$\boxed{\cancel{\#\text{ female} \mid \text{total} = 15 \sim \text{Bin}(15, 0.6)}}$



3. (10 points) Suppose customers arrive to a bank according to a Poisson process but the arrival rate fluctuates over time. From the opening time at 9 am until noon, the arrival rate increases linearly from 0 until it reaches 20 customers per hour. From noon to 2 pm, it decreases linearly to 15 customers per hour, and remains at 15 customers per hour until the bank closes at 5 pm. For notation, let $N(t)$ be the number of arrivals in the t hours since the bank opened and $\lambda(t)$ the arrival rate at t hours after opening.

   (a) (5 points) What is the expected number of customers per day?

   Area under $\lambda(t)$ from 9 am to 5 pm

   $$= \frac{1}{2} \times 3 \times 20 + \frac{1}{2} \times 2 \times 5 + 15 \times 7 = 30 + 5 + 105 = 140$$

   (b) (5 points) What is the probability of 6 arrivals between 9 am and noon?

   $$\Pr(N(\text{noon}) - N(9\text{am}) = 6) = \frac{e^{-30}\, 30^6}{6!}$$

4. (15 points) Two repair persons, Adam and Ben, are in charge of attending 3 similar machines. Each machine stays in working condition for a random amount of time before it breaks down. That random time is exponentially distributed with mean 10 hours. Adam's repair time is exponentially distributed with mean 2 hours and Ben's repair time is exponentially distributed with mean 5 hours. A repair person works on a broken machine one at a time. When both Adam and Ben are available, a broken machine is always sent to Adam as his mean repair time is shorter. We will model this system as a CTMC.

   (a) (5 points) Define states and clearly state what each state is.

   $S = \{0, 1A, 1B, 2, 3\}$

   0: no machine in repair

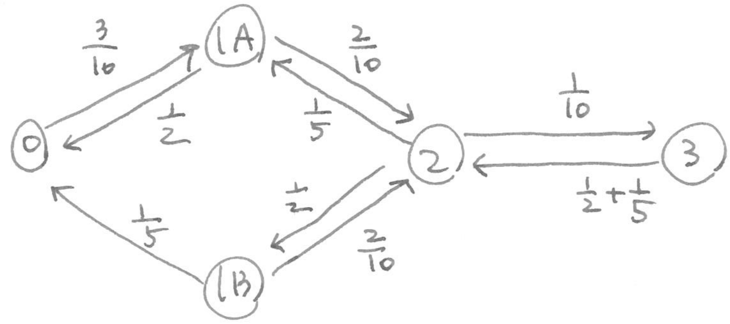
   1A: 1 machine in repair by Adam

   1B: 1 machine in repair by Ben

   2: 2 machines down & both Adam & Ben working

   3: 3 machines down and both Adam & Ben working

   (b) (10 points) Draw a transition rate diagram.



5. (5 points) Consider a CTMC $X = \{X(t), t \geq 0\}$ on $S = \{1, 2, 3\}$ with generator G given by

$$G = \begin{bmatrix} -12 & 8 & 4 \\ 2 & -2 & 0 \\ 5 & 1 & -6 \end{bmatrix}.$$

Set up all necessary equations to calculate stationary distribution $\underline{\pi}$.

$$12\pi_1 = 2\pi_2 + 5\pi_3$$

$$2\pi_2 = 8\pi_1 + \pi_3$$

$$\pi_1 + \pi_2 + \pi_3 = 0$$

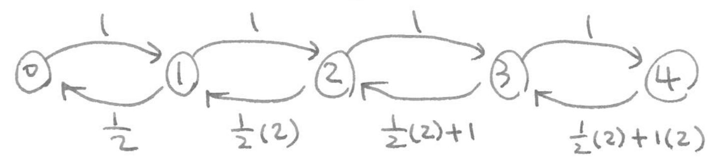
6. (30 points, 5 points each) Consider a call center that is staffed by 2 agents with four phone lines. Call arrivals follow a Poisson process with rate 1 per minute. An arrival call that finds all lines busy is lost. Call processing times are exponentially distributed with mean 2 minutes. Calls on hold have patience whose times are distributed iid exponential with mean 1 minute (i.e., a call on hold will hang up the phone without receiving a service if waiting time on hold exceeds their patience time). Let $X(t)$ represent the number of calls in the call center. The state space $S = \{0, 1, 2, 3, 4\}$ and $\underline{\pi} = (\frac{3}{19}, \frac{6}{19}, \frac{6}{19}, \frac{3}{19}, \frac{1}{19})$.

PP(1/min)

$\mu = \frac{1}{2}$/min

$\mu_p = 1$/min (patience)

(a) (5 points) Draw a transition rate diagram for the CTMC.



(b) (5 points) What is the probability that a call is rejected?

$$\pi_4 = \frac{1}{19}$$

(c) (5 points) Find throughput (Give the unit of your answer).

$$\lambda_{eff} = \lambda \Pr(\text{accepted}) = 1\left(1 - \frac{1}{19}\right) = \frac{18}{19}/\text{min}$$

(d) (5 points) When a call is randomly assigned to available servers, what is the utilization of an agent?

$$\underbrace{\frac{1}{2}\pi_1}_{\text{50\% chance that server A is picked}} + \underbrace{(\pi_2) + \pi_3 + \pi_4}_{\text{both are busy}} = \frac{1}{2}\cdot\frac{6}{19} + \frac{6}{19} + \frac{3}{19} + \frac{1}{19} = \frac{13}{19}$$

(e) (5 points) Find the long-run expected number of calls in the queue.

$$L_q = 1\cdot\pi_3 + 2\pi_4 = \frac{3}{19} + 2\cdot\frac{1}{19} = \frac{5}{19}$$

(f) (5 points) Find the long-run expected waiting time in the queue (Give the unit of your answer).

$$L_q = \lambda_{eff} W_q$$

$$W_q = \frac{L_q}{\lambda_{eff}} = \frac{\pi_3 + 2\pi_4}{18/19} = \frac{\frac{5}{19}}{\frac{18}{19}} = \underline{\underline{\frac{5}{18}\text{ mins}}}$$

Bonus (5 points - no partial points): A doctor has scheduled two appointments, one at 1pm and the other at 1:30pm. The amounts of time that appointments last are independent exponential random variables with mean 10 minutes. Assuming that both patients are on time, find the expected amount of time that the 1:30 appointment spends at the doctor's office.

If 1pm appt is done in 30 mins, then 1:30pm appt ~~waits~~ spends 10 mins. on avg.

If 1pm appt is still in the office when 1:30pm appt arrives, then the expected ~~wait~~ time in the office

$$= E[\text{1pm appt time}] + E[\text{1:30pm appt time}]$$

$$= 10 + 10 = 20 \text{ mins.}$$

$$\boxed{\therefore\ (1 - e^{-\frac{30}{10}})\,10 + e^{-\frac{30}{10}}(10+10)}\ \text{mins}$$