Name: ______________________________ **Section L__**

Signature: ____

You will have **50 minutes** to complete this closed book,
no notes, no calculator exam.

Keep the exam booklet closed until
the beginning of the examination.

Make sure that your booklet has **6 pages** (including this one).
Write clear, complete, legible answers in the spaces provided.
Use the back of the page if needed, but clearly indicate when doing so.

Read each question carefully and completely.
Think about the problem being asked.

Good luck!

1	**2**	**3**	**4**	**Bonus**	**Total**
/ 15	**/15**	**/10**	**/10**	**/4**	**/50+4B**

1. Given the system of first order linear equations

$$y' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} y, \tag{1}$$

(a) Find two complex-valued solutions $y_1(t)$ and $y_2(t)$ of (1);

(b) Find two real-valued solutions $u_1(t)$ and $u_2(t)$ of (1);

(c) Assuming that u_1 and u_2 are linearly independent, find a fundamental matrix $U(t)$ for the system (1);

(d) Find the fundamental matrix e^{At} associated to (1).

(e) List and classify any critical points.

(a) Call $A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}$.
We want to find two solutions of the form $y_1(t) = e^{(\Re(\lambda)+i\Im(\lambda))t}(\Re(x) + i\Im(x))$ and $y_2(t) = e^{(\Re(\lambda)-i\Im(\lambda))t}(\Re(x) - i\Im(x))$ where λ is a complex eigenvalue of A and x is its associated eigenvector.
The complex conjugate eigenvalues of A are $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$. We can consider the associated eigenvectors $x_1 = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ 2+i \end{bmatrix}$.
Therefore the complex-valued solutions are:

$$y_1(t) = e^{\lambda_1 t}x_1 = e^{(-1+i)t}\begin{bmatrix} 1 \\ 2-i \end{bmatrix} \quad \text{and} \quad y_2(t) = e^{\lambda_2 t}x_2 = e^{(-1-i)t}\begin{bmatrix} 1 \\ 2+i \end{bmatrix}$$

(b) From y_1 we compute the real-valued solutions using Euler's formula:

$$y_1(t) = e^{-t}(\cos t + i\sin t)\begin{bmatrix} 1 \\ 2-i \end{bmatrix} = e^{-t}\begin{bmatrix} \cos t + i\sin t \\ -2\cos t - \sin t + i(\cos t - 2\sin t) \end{bmatrix}$$

Therefore,

$$u_1(t) = e^{-t}\begin{bmatrix} \cos t \\ -2\cos t - \sin t \end{bmatrix} \quad \text{and} \quad u_2(t) = e^{-t}\begin{bmatrix} \sin t \\ \cos t - 2\sin t \end{bmatrix}$$

(c)

$$U(t) = e^{-t}\begin{bmatrix} \cos t & \sin t \\ -2\cos t - \sin t & \cos t - 2\sin t \end{bmatrix}$$

(d) The fundamental matrix $e^{At} = U(t)U^{-1}(0)$.
Since $U(0) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, then $U^{-1}(0) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. Thus,

$$e^{At} = e^{-t}\begin{bmatrix} \cos t & \sin t \\ -2\cos t - \sin t & \cos t - 2\sin t \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = e^{-t}\begin{bmatrix} \cos t + 2\sin t & \sin t \\ -5\sin t & \cos t - 2\sin t \end{bmatrix}$$

(e) Since it is a homogeneous system, the critical point is the origin (the null vector $\tilde{y} = [0,0]^T$). It is a spiral point and it is asymptotically stable (since $\Re(\lambda) < 0$).

2. Given the following second order differential equation

$$y'' + 4y' + 3y = 2e^{-t} \tag{2}$$

(a) Classify the equation

(b) Find two solutions $y_1(t)$ and $y_2(t)$ of the complementary equation associated to (2);

(c) Verify that $\{y_1, y_2\}$ is a fundamental set of solutions for the problem (2).

(d) Find a particular solution $y_p(t)$ of (2);

(e) Find a general solution $y(t)$ of (2);

(a) Second order, linear, constant coefficients, nonhomogeneous.

(b) We look for a solution of the form $y = e^{rt}$, where r is the root of the characteristic polynomial associated to the equation, $p(r) = r^2 + 4r + 3$. Since the roots of p are real and distinct, we have

$$y_1(t) = e^{-t} \quad \text{and} \quad y_2(t) = e^{-3t}$$

(c) In order to check whether the solutions are linear independent, we verify that the Wronskian is nonzero:

$$W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det \begin{bmatrix} e^{-t} & e^{-3t} \\ -e^{-t} & -3e^{-3t} \end{bmatrix} = -3e^{-4t} + e^{-4t} \neq 0$$

(d) Since e^{-t} is already in the fundamental set of solutions, we look for a particular solution of the form $y_p = Ate^{-t}$. Differentiating, we have $y_p' = -Ate^{-t} + Ae^{-t}$, $y_p'' = Ate^{-t} - 2Ae^{-t}$. Plugging y_p and its derivatives into the equation we obtain $A = 1$ and therefore $y_p(t) = te^{-t}$.

(e) The general solution is

$$y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t) = te^{-t} + c_1e^{-t} + c_2e^{-3t}$$

3. Find a solution of the initial value problem

$$y'' + 4y' + 3y = 2e^{-t}, \quad y(0) = 2, \quad y'(0) = 1 \tag{3}$$

using the Laplace Transform. (Recall $\mathcal{L}(e^{at}) = \frac{1}{s-a}$). Assuming that $\mathcal{L}(y) = Y(s)$, and $\mathcal{L}(2e^{-t}) = \frac{2}{s+1}$, we have

$$p(s)Y(s) = \frac{2}{s+1} + (1+2s) + 8 = \frac{2s^2 + 11s + 11}{s+1}$$

that is

$$Y(s) = \frac{2s^2 + 11s + 11}{(s+1)^2(s+3)}$$

Using partial fractions

$$\frac{2s^2 + 11s + 11}{(s+1)^2(s+3)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3}$$

$$A(s^2 + 4s + 3) + B(s+3) + C(s^2 + 2s + 1) = 2s^2 + 11s + 11$$

that gives us $A = 3$, $B = 1$, and $C = -1$.
Therefore

$$Y(s) = \frac{2s^2 + 11s + 11}{(s+1)^2(s+3)} = \frac{3}{s+1} + \frac{1}{(s+1)^2} - \frac{1}{s+3}$$

and, taking the inverse Laplace Transform, we obtain

$$y(t) = 3e^{-t} + te^{-t} - e^{-3t}$$

4. Given the equation

$$y'' - 3y' + 2y = \frac{1}{1+e^{-x}} \tag{4}$$

and its solution $y_1 = e^{2x}$,

(a) Classify the equation

(b) Given a solution of the form $y = uy_1$, write the differential equation relative to the function u and reduce it to a first order equation.

(a) Second order, linear, constant coefficients, nonhomogeneous

(b) If $y = ue^{2x}$, differentiating we have $y' = u'e^{2x} + 2ue^{2x} = e^{2x}(u' + 2u)$ and $y'' = 2e^{2x}(u' + 2u) + e^{2x}(u'' + 2u') = e^{2x}(u'' + 4u' + 4u)$.
Therefore, substituting y, y' and y'' into the equation, we have

$$e^{2x}[(u' + 4u' + 4u - 3(u' + 2u) + 2u] = \frac{1}{1+e^{-x}}$$

that is

$$u'' + u' = \frac{e^{-2x}}{1+e^{-x}}$$

In order to reduce this equation to the first order, we consider $z = u'$ and

$$z' + z = \frac{e^{-2x}}{1+e^{-x}}$$

Bonus. Solve the problem (4).

The solution of the complementary equation is $z_1 = e^{-x}$. Using the method of variation of parameter, we have $z = ve^{-x}$ and plugging z and z' into the equation we obtain:

$$v'e^{-x} - ve^{-x} + ve^{-x} = \frac{e^{-2x}}{1 + e^{-x}}$$

Thus,

$$v = -\ln|1 + e^{-x}| + c_1$$

Recall $u' = z = -e^{-x}\ln|1 + e^{-x}| + e^{-x}c_1$. Integrating we have

$$u = (1 + e^{-x})\ln|1 + e^{-x}| - 1 - e^{-x} + e^{-x}c_1 + c_2$$

and

$$y = ue^{-x} = (e^{2x} + e^{x})\ln(1 + e^{-x}) + c_1e^{2x} + c_2e^{x}$$