Math 2401
Spring 2015
Exam 1
Jan. 29, 2015
Time Limit: 50 Minutes

Name : ____________________

GT Id : ____________________

TA: ____________________

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. Also, sign the Honor Code pledge at the bottom of this page, and follow the instructions below.

- On this exam you may **not** use books, notes, or any electronic devices other than a non-graphing calculator.
- **Show all your work.** A correct answer not supported by calculations and/or explanation will receive no credit. An incorrect answer supported by substantially correct calculations and explanation may receive partial credit.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- If you need more space, use the back of the pages; clearly indicate when you have done so.

Problem	Points	Score
1	10	
2	10	
3	11	
4	13	
5	6	
Total:	50	

Honor Code Pledge: By signing below, you are verifying that you understand and uphold the Georgia Tech honor code.

Signature: ____________________

1. (a) (5 points) Find the volume of the parallelepiped determined by the following vectors:

$$\mathbf{u} = -2\mathbf{j} + \mathbf{k},\ \mathbf{v} = \mathbf{i} + 3\mathbf{k} \ \text{ and } \ \mathbf{w} = \mathbf{i} - 4\mathbf{j}.$$

3

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} 0 & -2 & 1 \\ 1 & 0 & 3 \\ 1 & -4 & 0 \end{vmatrix} = 0 - (-2)\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & -4 \end{vmatrix}$$

$$= 2(0-3) + (-4-0)$$

$$= -6 - 4$$

$$= -10$$

$\therefore$ $\underbrace{\text{Volume of the parallelepiped} = |\vec{u}\cdot(\vec{v}\times\vec{w})|}_{2} = 10.$

(b) (5 points) Find the acute angle between the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

Let $\vec{a} = \vec{i} + \vec{j} = \langle 1, 1, 0 \rangle$

& $\vec{b} = \vec{j} + \vec{k} = \langle 0, 1, 1 \rangle$

If θ is an angle between $\vec{a}$ & $\vec{b}$,

$$\underbrace{\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}}_{1} = \underbrace{\frac{1(0) + 1(1) + 0(1)}{\sqrt{1^2+1^2+0^2}\ \sqrt{0^2+1^2+1^2}} = \frac{1}{\sqrt{2}\ \sqrt{2}} = \frac{1}{2}}_{3}$$

$\therefore$ $\underbrace{\theta = 60^\circ}_{1}$ (or $\frac{\pi}{3}$ radians).

2. (a) (5 points) Find the distance from the point $(1, -4, 0)$ to the plane $2x - y + 2z = 0$.

Given plane passes through $\underbrace{O(0,0,0)}_{1}$ 1 and

$\underbrace{\vec{n} = \langle 2, -1, 2 \rangle}_{1}$ is perpendicular to the plane.

$\therefore$ Distance 1 from $P(1, -4, 0)$ to the plane is

$$\left.\frac{|\overrightarrow{OP} \cdot \vec{n}|}{|\vec{n}|}\right\} 1$$

$$\left.\begin{aligned} &= \frac{|\langle 1, -4, 0\rangle \cdot \langle 2, -1, 2\rangle|}{\sqrt{2^2 + (-1)^2 + 2^2}} \\ &= \frac{|2+4+0|}{\sqrt{9}} = \frac{6}{3} = 2. \end{aligned}\right\} 2$$

(b) (5 points) For what value of α is the line $x = 2 - t, y = 2t - 1, z = 3t$ $(-\infty < t < \infty)$ perpendicular to the plane $2x + \alpha y - 6z = 13$?

Given line is parallel to $\overbrace{\langle -1, 2, 3 \rangle}^{1}$ and the given plane is perpendicular to $\underbrace{\langle 2, \alpha, -6 \rangle}_{1}$.

1 $\{$ So the line is perpendicular to the plane if $\langle -1, 2, 3\rangle$ & $\langle 2, \alpha, -6\rangle$ are parallel vectors.

2 $\{$ So, $\langle 2, \alpha, -6 \rangle = k \langle -1, 2, 3 \rangle$ for some scalar k.

Then, $2 = -k$, $\alpha = 2k$ & $-6 = 3k$

i.e. $k = -2$ and $\alpha = 2k = 2(-2) = -4$.

(OR use $\langle 2, \alpha, -6 \rangle \times \langle -1, 2, 3 \rangle = \langle 0, 0, 0 \rangle$).

3. A particle traveling in a straight line is located at the point $(1, 0, -2)$ at time $t = 0$. The particle has speed 6 at $(1, 0, -2)$ and is moving toward the point $(-1, 1, 0)$ with constant acceleration $-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

   (a) (5 points) Find the velocity of the particle at time $t = 0$.

Let $P=(1,0,-2)$ & $Q=(-1,1,0)$.

Then $\underbrace{\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle -1, 1, 0\rangle - \langle 1, 0, -2\rangle = \langle -2, 1, 2\rangle}_{1}$.

Now,

$$\left.\begin{aligned}\vec{v}(0) &= (\text{magnitude})\,(\text{direction})\\ &= 6\,\frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}\end{aligned}\right\}\ 2$$

$$2\ \left\{ = 6\,\frac{\langle -2,1,2\rangle}{\sqrt{(-2)^2+1^2+2^2}} = 6\,\frac{\langle -2,1,2\rangle}{3} \right. = 2\langle -2, 1, 2\rangle = \langle -4, 2, 4\rangle \quad (= -4\vec{i} + 2\vec{j} + 4\vec{k}).$$

   (b) (6 points) Find the velocity $\mathbf{v}(t)$ of the particle at time t.

1 $\left\{\begin{array}{l}\text{Integrating } \vec{a}(t) = -2\vec{i} + \vec{j} + 2\vec{k} = \langle -2, 1, 2\rangle,\\ \text{we get, } \cancel{\vec{v}(t) = \int \vec{a}(t)\,dt}\end{array}\right.$

$$\vec{v}(t) = \overbrace{\langle -2t, t, 2t\rangle}^{2} + \underbrace{\vec{v}(0)}_{1}$$

$$\left.\begin{aligned}&= \langle -2t, t, 2t\rangle + \langle -4, 2, 4\rangle\\ &= \langle -2t-4,\ t+2,\ 2t+4\rangle.\end{aligned}\right\}\ 1$$

$$\left(= (-2t-4)\vec{i} + (t+2)\vec{j} + (2t+4)\vec{k} \right).$$

4. A projectile fired from the origin over horizontal ground has position

$$\mathbf{r}(t) = 98\sqrt{3}\,t\mathbf{i} + (98t - 4.9t^2)\mathbf{j} \quad \text{at time } t.$$

(a) (3 points) Find the velocity of the projectile at time t.

$$\underbrace{\vec{v}(t) = \vec{r}\,'(t)}_{1} = \underbrace{98\sqrt{3}}_{1}\,\vec{i} + \underbrace{(98 - 9.8t)}_{1}\,\vec{j}\,.$$

(b) (6 points) What is the greatest height reached by the projectile?

At the greatest height,
vertical component of $\vec{v}(t) = 0$ 2

or, $98 - 9.8t = 0$
or $98 = 9.8t$ i.e. $t = 10$ 1

∴ The projectile reaches the greatest height at $t=10$,
and the greatest height is

$\cancel{98t - 4.9}$ $(98t - 4.9t^2)\Big|_{t=10}$ 2

$= 98(10) - 4.9(10)^2$
$= 980 - 490 = 490$ 1

(c) (4 points) What distance does the projectile cover downrange when it reaches the maximum height?

$$\overbrace{(98\sqrt{3}\,t)}^{2}\Big|_{\underbrace{t=10}_{1}}$$

$= 98\sqrt{3}\,(10)$
$= 980\sqrt{3}\,.$ 1

5. (6 points) Find parametric equations for the line that is tangent to the curve

$$\mathbf{r}(t) = \ln(t+1)\mathbf{i} + (2 - \sin t)\mathbf{j} + e^t\mathbf{k}$$

at the point $(0, 2, 1)$.

The point $(0,2,1)$ corresponds to $t=0$. 1

$\vec{r}'(t) = \left\langle \frac{1}{t+1},\ 0 - \cos t,\ e^t \right\rangle$ 1.5

$\therefore\ \vec{r}'(0) = \langle 1, -1, 1 \rangle$ 0.5

The parametric equations of the tangent line which passes through $(0,2,1)$ and is parallel to $\langle 1, -1, 1\rangle$ are:

3 $\left\{ \begin{array}{l} x = 0 + (1)t \\ y = 2 + (-1)t \\ z = 1 + (1)t \end{array} \right\}$ i.e. $x = t$, $y = 2 - t$, $z = 1 + t$, $-\infty < t < \infty$