MATH 1502 TEST 6, PAGE 1, FALL 2013, GRODZINSKY

Print Your Name: Key-1

T.A. or Section Number: ____________

1. For the matrix A given below, determine:
(a) (10 points) the rank and nullity of the matrix.
(b) (10 points) a basis for the column space of the matrix, and a geometric description of the column space (i.e., is it a line, plane, hyperplane, in $\Re^k$, etc.).
(c) (12 points) a basis for the nullspace of the matrix, and a geometric description of the nullspace.

$$A = \begin{bmatrix} 2 & 3 & 4 & -7 \\ -8 & -9 & -10 & 19 \\ 4 & -3 & -10 & 13 \end{bmatrix}$$

Row reduce to RREF:

$$\rightarrow \begin{bmatrix} 2 & 3 & 4 & -7 \\ 0 & 3 & 6 & -9 \\ 0 & -9 & -18 & 27 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -2 & 2 \\ 0 & 3 & 6 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) There are two pivotal columns, so $\boxed{\text{rank}(A) = 2}$ and $\boxed{\text{nullity}(A) = 4-2 = 2}$

(b) The first two columns are pivotal, so a basis for the column space is $\left\{ \begin{bmatrix} 2 \\ -8 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -9 \\ -3 \end{bmatrix} \right\}$.

This space is a $\boxed{\text{plane in } \mathbb{R}^3}$.

(c) Solving $A\vec{x} = \vec{0} \iff$
$x_1 = s - t$
$x_2 = -2s + 3t$
$x_3 = s$
$x_4 = t$

So a basis for the nullspace is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Nul (A) is a $\boxed{\text{plane in } \mathbb{R}^4}$.

2. (20 points) Find the determinant of the matrix A below.

$$A = \begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{bmatrix} -3 & -2 & 2 & 0 \\ -5 & 6 & 2 & 0 \\ -6 & 0 & 3 & 0 \\ 4 & 8 & 0 & -3 \end{bmatrix} \end{array} \begin{array}{c} - \\ + \\ - \\ + \end{array}$$

Expand on 4th column:

$$\det(A) = -3 \begin{vmatrix} -3 & -2 & 2 \\ -5 & 6 & 2 \\ -6 & 0 & 3 \end{vmatrix}$$ (expand on 3rd row)

$$= -3\left[(-6)\begin{vmatrix} -2 & 2 \\ 6 & 2 \end{vmatrix} + 3\begin{vmatrix} -3 & -2 \\ -5 & 6 \end{vmatrix}\right]$$

$$= (-3)\{(-6)(-4-12) + 3(-18-(10))\}$$

$$= (-3)\{-6(-16) + 3(-28)\}$$

$$= (-3)(96-84) = \boxed{-36}$$

3. (14 points) Given the vectors $\vec{a} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix}$, find a vector that is orthogonal to both $\vec{a}$ and $\vec{b}$.

The vector $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 5 \\ 3 & -4 & -1 \end{vmatrix}$$

$$= (3+20)\vec{i} - (-1-15)\vec{j} + (-4+9)\vec{k}$$

$$= 23\vec{i} + 16\vec{j} + 5\vec{k}, \text{ or } \begin{bmatrix} 23 \\ 16 \\ 5 \end{bmatrix}$$

(any multiple of this vector also works)

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4. (20 points) Find all eigenvalues and a basis for the eigenspace corresponding to each eigenvalue (i.e. all the eigenvectors) for the matrix B below.

$$B = \begin{bmatrix} -7 & -1 \\ 15 & 1 \end{bmatrix}$$

To find the eigenvalues, set $p(\lambda) = \det(B - \lambda I) = 0$:

$$p(\lambda) = \begin{vmatrix} -7-\lambda & -1 \\ 15 & 1-\lambda \end{vmatrix} = (-7-\lambda)(1-\lambda) + 15$$

$$= -7 + 6\lambda + \lambda^2 + 15 = \lambda^2 + 6\lambda + 8$$

$= (\lambda+4)(\lambda+2)$, so the eigenvalues are $\boxed{\lambda_1 = -4,\ \lambda_2 = -2}$

$\lambda_1 = -4$: $A - (-4)I = A + 4I = \begin{bmatrix} -3 & -1 \\ 15 & 5 \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}$ Solving $A\vec{x} = -4\vec{x}$ $\Leftrightarrow$ $x_1 = -\frac{1}{3}t$, $x_2 = t$,

so $\boxed{\left\{ \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} \right\}}$ is a basis for the eigenspace of $\lambda_1 = -4$

$\lambda_2 = -2$: $A - (-2I) = A + 2I = \begin{bmatrix} -5 & -1 \\ 15 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1/5 \\ 1 & 1/5 \end{bmatrix} \to \begin{bmatrix} 1 & 1/5 \\ 0 & 0 \end{bmatrix}$

Solving $A\vec{x} = -2\vec{x}$ $\Leftrightarrow$ $x_1 = -\frac{1}{5}t$, $x_2 = t$, so $\boxed{\left\{ \begin{bmatrix} -1/5 \\ 1 \end{bmatrix} \right\}}$ is a basis for the eigenspace of $\lambda_2 = -2$.

5. (14 points) Let S be the set of all vectors in $\Re^3$ that have the form $\begin{bmatrix} t \\ 2t \\ -t \end{bmatrix}$. Prove that S is a subspace of $\Re^3$. Recall: that means you must show that (i) S contains $\vec{0}$ and (ii) if $\vec{x}, \vec{y} \in S$ and $a \in \Re$, then $a\vec{x} + \vec{y} \in S$.

(i) If $t=0$, we have $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$, so $\vec{0} \in S$

(ii) Let $\vec{x}, \vec{y} \in S$, then $\vec{x} = \begin{bmatrix} t_1 \\ 2t_1 \\ -t_1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} t_2 \\ 2t_2 \\ -t_2 \end{bmatrix}$ for some $t_1, t_2 \in \mathbb{R}$. Let $a \in \mathbb{R}$. Then:

$$a\vec{x} + \vec{y} = \begin{bmatrix} at_1 + t_2 \\ 2at_1 + 2t_2 \\ -at_1 - t_2 \end{bmatrix} = \begin{bmatrix} at_1 + t_2 \\ 2(at_1 + t_2) \\ -(at_1 + t_2) \end{bmatrix}$$

$$\underline{\underline{t = at_1 + t_2}} \begin{bmatrix} t \\ 2t \\ -t \end{bmatrix}, \text{ so } a\vec{x} + \vec{y} \in S.$$

Thus, S is a subspace of $\mathbb{R}^3$ <u>qed</u>

BONUS: (5 points) Use the Big Theorem of Linear Algebra to explain why the eigenvalues λ of a matrix A must satisfy the equation $\det(A - \lambda I) = 0$.

If $\vec{x}$ is an eigenvector with associated eigenvalue λ, then $A\vec{x} = \lambda\vec{x}$ (by definition)

$\Longleftrightarrow (A - \lambda I)\vec{x} = \vec{0}$

Since $\vec{x} \neq \vec{0}$, this means there exists an $\vec{x} \neq 0$ with $(A - \lambda I)\vec{x} = \vec{0}$, so nullity $(A - \lambda I) \geq 1$

$\Longleftrightarrow$ the matrix $(A - \lambda I)$ is <u>not</u> invertible

$\Longleftrightarrow \det(A - \lambda I) = 0.$

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1. (20 points) Find all eigenvalues and a basis for the eigenspace corresponding to each eigenvalue (i.e. all the eigenvectors) for the matrix B below.

$$B = \begin{bmatrix} -8 & -1 \\ 14 & 1 \end{bmatrix}$$

First, find the eigenvalues by solving $p(\lambda) = \det(B - \lambda I) = 0$:

$$\det(B-\lambda I) = \begin{vmatrix} -8-\lambda & -1 \\ 14 & 1-\lambda \end{vmatrix} = (-8-\lambda)(1-\lambda) + 14$$

$$= -8 + 7\lambda + \lambda^2 + 14 = \lambda^2 + 7\lambda + 6$$

$$= (\lambda+6)(\lambda+1), \text{ so } \boxed{\lambda_1 = 6,\ \lambda_2 = -1}$$

$\lambda_1 = -6$: $A + 6I = \begin{bmatrix} -2 & -1 \\ 14 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 \\ 1 & 1/2 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ nullspace looks like $\begin{matrix} x_1 = -\frac{1}{2}t \\ x_2 = t \end{matrix}$ $\Rightarrow$ basis for the eigenspace is: $\boxed{\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}}$

$\lambda_2 = -1$: $A + I = \begin{bmatrix} -7 & -1 \\ 14 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1/7 \\ 1 & 1/7 \end{bmatrix} \to \begin{bmatrix} 1 & 1/7 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ solving $A\vec{x} = -\vec{x}$ gives $\begin{matrix} x_1 = -\frac{1}{7}t \\ x_2 = t \end{matrix}$ $\Rightarrow$ basis for the eigenspace is: $\boxed{\left\{ \begin{bmatrix} -1/7 \\ 1 \end{bmatrix} \right\}}$

2. (14 points) Given the vectors $\vec{a} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ -1 \\ -2 \end{bmatrix}$, find a vector that is orthogonal to both $\vec{a}$ and $\vec{b}$.

The cross product is orthogonal to both $\vec{a}$ and $\vec{b}$:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 4 & -3 \\ 5 & -1 & -2 \end{vmatrix} = (-8-3)\vec{\imath} - (-4+15)\vec{\jmath} + (-2-20)\vec{k}$$

$$= -11\vec{\imath} - 11\vec{\jmath} - 22\vec{k},$$

or $\begin{bmatrix} -11 \\ -11 \\ -22 \end{bmatrix}$ (any multiple of this vector also works)

3. (14 points) Let S be the set of all vectors in $\Re^3$ that have the form $\begin{bmatrix} 2t \\ -t \\ t \end{bmatrix}$. Prove that S is a subspace of $\Re^3$. Recall: that means you must show that (i) S contains $\vec{0}$ and (ii) if $\vec{x}, \vec{y} \in S$ and $a \in \Re$, then $a\vec{x} + \vec{y} \in S$.

(i) Letting $t=0$ gives the vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$, so $\vec{0} \in S$

(2) Let $\vec{x}, \vec{y} \in S$. Then $\vec{x} = \begin{bmatrix} 2t_1 \\ -t_1 \\ t_1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2t_2 \\ -t_2 \\ t_2 \end{bmatrix}$

for some $t_1, t_2 \in \mathbb{R}$. Let $a \in \mathbb{R}$. Then:

$$a\vec{x} + \vec{y} = \begin{bmatrix} 2at_1 + 2t_2 \\ -at_1 + (-t_2) \\ at_1 + t_2 \end{bmatrix} = \begin{bmatrix} 2(at_1 + t_2) \\ -(at_1 + t_2) \\ at_1 + t_2 \end{bmatrix}$$

$\underline{\underline{t = at_1 + t_2}}$ $\begin{bmatrix} 2t \\ -t \\ t \end{bmatrix}$, so $a\vec{x} + \vec{y} \in S$.

Thus, S is a subspace of $\mathbb{R}^3$.

qed

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4. For the matrix A given below, determine:
(a) (10 points) the rank and nullity of the matrix.
(b) (10 points) a basis for the column space of the matrix, and a geometric description of the column space (i.e., is it a line, plane, hyperplane, in $\Re^k$, etc.).
(c) (12 points) a basis for the nullspace of the matrix, and a geometric description of the nullspace.

$$A = \begin{bmatrix} 2 & 3 & 8 & -6 \\ -8 & -9 & -20 & 12 \\ 6 & 0 & -12 & 18 \end{bmatrix}$$

Row reduce to RREF:

$$\rightarrow \begin{bmatrix} 2 & 3 & 8 & -6 \\ 0 & 3 & 12 & -12 \\ 0 & -9 & -36 & 36 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -4 & 6 \\ 0 & 3 & 12 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 4 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) 2 pivotal and 2 non-pivotal columns

$\Rightarrow \boxed{\text{rank}(A) = \text{nullity}(A) = 2}$

(b) The pivotal columns form a basis for Col(A): $\left\{ \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ -9 \\ 0 \end{bmatrix} \right\}$

Col(A) is a $\boxed{\text{plane in } \mathbb{R}^3}$.

(c) Solve $A\vec{x} = \vec{0}$:

$$\begin{aligned} x_1 &= 2s - 3t \\ x_2 &= -4s + 4t \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

So a basis for Nul(A) is $\left\{ \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$

Nul(A) is $\boxed{\text{a plane in } \mathbb{R}^4}$.

5. (20 points) Find the determinant of the matrix A below.

$$A = \begin{array}{cc} \begin{matrix} + & - & + & - \end{matrix} & \\ \begin{bmatrix} 3 & 0 & 1 & 0 \\ 2 & -1 & 8 & 0 \\ 3 & -3 & 11 & 0 \\ 4 & -4 & 12 & 2 \end{bmatrix} & \begin{matrix} \\ + \\ - \\ + \end{matrix} \end{array}$$

Expand on column 4:

$$\det(A) = 2 \begin{vmatrix} 3 & 0 & 1 \\ 2 & -1 & 8 \\ 3 & -3 & 11 \end{vmatrix}$$

expand on row 1

$$= 2\left[3 \begin{vmatrix} -1 & 8 \\ -3 & 11 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix}\right]$$

$$= 2\left[3(-11+24) + (-6+3)\right]$$

$$= 2\left[3(13) - 3\right] = 2(36) = \boxed{72}$$

BONUS: (5 points) Use the Big Theorem of Linear Algebra to explain why the eigenvalues λ of a matrix A must satisfy the equation $\det(A - \lambda I) = 0$.

See Form 1.