Feb. 18, 2014

Math 2401 M - Quiz 3

First Name (Print): ____________ Last Name (Print): ____________ Signature: ________________

- There are **3** questions on **2** pages. The quiz is worth 20 points in total.
- Answer the questions clearly and completely. You must provide work clearly justifying your solution.
- You can NOT write your work on the back of the page. Use it for scratch work if needed.
- You have 20 minutes to finish your work.

1. (2×2 points) Find the **domain** and **range** of functions below.

(a) $f(x,y) = \frac{1}{\ln(9-x^2-y^2)}$

Solution.

Domain: $D = \{(x,y) | x^2 + y^2 < 9, \text{and } \ x^2 + y^2 \neq 8\}$

Range: $\mathbb{R} \longleftarrow$ **WRONG**

Range: $(-\infty, 0) \cup [\frac{1}{\ln 9}, +\infty)$ ■

(b) $f(x,y) = \sin^{-1}(x-y)$

Solution.

Domain: $D = \{(x,y) | y - 1 \leq x \leq y + 1\}$

Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

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2. (4×2 points) Find the limit or show the nonexistence of the limit of functions.

(a)

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{(xy)^2}$$

Solution.

S1:

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{(xy)^2} \underbrace{=}_{\text{let } u=xy} \lim_{u\to 0} \frac{1-\cos(u)}{(u)^2} \underbrace{=}_{\text{L'Hopital's rule}} \lim_{u\to 0} \frac{\sin u}{2u} = \frac{1}{2}.$$

S2:

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{(xy)^2} = \lim_{(x,y)\to(0,0)} \frac{1}{2} \frac{\sin^2(\frac{xy}{2})}{(\frac{xy}{2})^2} \underbrace{=}_{\text{let } u=\frac{xy}{2}} \lim_{u\to 0} \frac{1}{2} \frac{\sin^2(u)}{u^2} = \frac{1}{2}.$$

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(b)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{(xy)^2}}$$

Solution.

$$\lim_{(x,y)\to(0,0)\ y=kx, k\neq 0} \frac{xy}{\sqrt{(xy)^2}} = \lim_{x\to 0} \frac{kx^2}{\sqrt{(kx^2)^2}} = \lim_{x\to 0} \frac{k}{|k|} = \frac{k}{|k|} = \begin{cases} 1, & \text{if} \quad k > 0, \\ -1, & \text{if} \quad k < 0. \end{cases}$$

Therefore, the limit does not exist by the two-path test.

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3. (4×2 points)

(a) Suppose that $f(x,y) = \tan^{-1}(\frac{x}{y})$. Find **ALL** the second-order derivatives of $f(x,y)$.

Solution.

$$\begin{aligned}
&\frac{\partial f}{\partial x}(x,y) = \frac{y}{x^2+y^2}, \quad \frac{\partial f}{\partial y}(x,y) = -\frac{x}{x^2+y^2}, \\
&\frac{\partial^2 f}{\partial x^2}(x,y) = -\frac{2xy}{(x^2+y^2)^2}, \\
&\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}, \\
&\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{2xy}{(x^2+y^2)^2}.
\end{aligned}$$

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(b) Assume that the equation $2x^2 - y^3 - xy = 0$ define a differentiable function $y = y(x)$. Find the value of $\frac{dy}{dx}$ at the point $(1,1)$.

Solution.

S1. Differentiate both sides of the equation with respect to x:

$$4x - 3y^2\frac{dy}{dx} - (y + x\frac{dy}{dx}) = 0 \Rightarrow \frac{dy}{dx} = \frac{4x-y}{x+3y^2} \Rightarrow \frac{dy}{dx}\Big|_{(x,y)=(1,1)} = \frac{3}{4}.$$

S2. Let $F(x,y) = 2x^2 - y^3 - xy$, then $F_x(x,y) = 4x - y$ and $F_y(x,y) = -3y^2 - x$.

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = \frac{4x-y}{x+3y^2} \Rightarrow \frac{dy}{dx}\Big|_{(x,y)=(1,1)} = \frac{3}{4}.$$

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