ISyE 2027C Probability with Applications

TEST 2 SOLUTIONS **9 April 2015**

1. Poisson process A has arrival rate 20/minute. Poisson process B is independent and has arrival rate 30/minute. Find[1]:

   (a) The expected time between arrivals from A. $60/20 = 3$ seconds.

   (b) The expected time between now and the next arrival from A, given that the last arrival from A was exactly 1 second ago. 3 seconds (memoryless)

   (c) The expected time until the first arrival from A or B. $60/50 = 1.2$ seconds

   (d) The probability that the next arrival is from B. $30/(20 + 30) = .6$

   (e) The expected time between the next two arrivals. 1.2 seconds.

   (f) The probablity of exactly 19 arrivals from A and 31 arrivals from B during one minute.

   $$e^{-20}20^{19}/19! \cdot e^{-30}30^{31}/31!$$

   (g) The probability that there are no arrivals from A or B during the next 5 seconds.
   We expect 5 times 5/6 arrivals in 5 seconds. $e^{-25/6}$.

   (h) The probability that there are no arrivals from B during the next 2 seconds.
   We expect $2/2 = 1$ arrival. e^{-1}.

   (i) The probability that there are no arrivals from A during the next 5 seconds and there are no arrivals from B during the next 7 seconds.
   $e^{-5/3}e^{-7/2}$.

   (j) Explain the relationship between the answers to the previous three questions.
   No B arrivals in 5 seconds and no B arrivals in 2 more seconds by memorylessness has the same chance as none in 7 seconds. Hence the product of the first two answers equals the third.

   (k) (5 points) The expected time until there has been at least one arrival from A and at least one arrival from B.
   First arrival has expected time 1.2 seconds. Imagine the moment of the first arrival. Prob is .6 first arrival is B, in which case we expect to wait 3 seconds for the first A. Prob is .4 first arrival is A in which case we expect to wait 2 seconds for the first B.
   $1.2 + .6(3) + .4(2) = 3.8$ seconds.

[1] Parts $a - -j$ are 2 points each. You should be able to answer them immediately.

2. (10 points) Jointly independent random variables $W_i : i = 1, 2, 3, 4$ have uniform distributions on $[-i, i]$, respectively. Let $W = \sum_{i=1}^{4} W_i$. Use Chebyshev's inequality to find a number α such that $P(|W| > \alpha) \leq 1/9$. Remember that the variance of a $U[0, 1]$ variable is $\frac{1}{12}$ and its mean is $\frac{1}{2}$.

   To get the probability of 1/9 we need $1/k^2 = 1/9$ so $k = 3$ and $\alpha = 3\sigma(W)$. The variance of W_i is $\frac{1}{12}(2i)^2$. By independence, $\sigma^2(W) = \frac{1}{12}(4 + 16 + 36 + 64) = 10$. So $\alpha = 3\sqrt{10}$.

3. (10 points) Random variable Y has cdf $F_Y(t) = 1 - 16/t^4 : 2 \leq t < \infty$. Write the integral that equals $E[Y^2]$.

   The pdf is $f_Y(t) = 64t^{-5}$. The integral is

$$\int_2^\infty 64t^{-5}t^2 dt = \int_2^\infty 64t^{-3} dt$$

4. (10 points) Random variables X and Y are independently uniformly distributed on $[1, 2]$ and $[0, 3]$ respectively. Find $E[\max\{X, Y\}]$. $P(X \leq t) = 0$ for $0 \leq t < 1$; $t - 1$ for $1 \leq t \leq 2$; 1 for $2 \leq t \leq 3$. $P(Y \leq t) = t/3$ for $0 \leq t \leq 3$. $P(\max(X, Y) \leq t) = 0t/3 = 0$ for $0 \leq t < 1$; $(t - 1)t/3$ for $1 \leq t \leq 2$; $t/3$ for $2 \leq t \leq 3$. The corresponding pdfs are0, $(2t - 1)/3, 1/3$. The expected value is

$$\int_0^1 0t dt + \int_1^2 t(2t - 1)/3 dt + \int_2^3 t/3 dt = 0 + \frac{2}{9}7 - \frac{1}{6}3 + \frac{1}{6}5 = \frac{17}{9}$$

5. Parishioners arrive at church on Sunday morning according to a Poisson process at rate 2/minute starting at 9:00am until 11:00am. Each parishioner wears a hat with probability 1/3, independent of other parishioners, and brings an umbrella with probability 1/4, independent of whether she wears a hat and independent of other parishioners. The cloakroom has umbrella stands and baskets for hats.

   (a) (2 points) What is the probability that the cloakroom has exactly 99 hats at 11:00am? Hats arrive at rate 2/3 per minute for 120 minutes. The number of hats is Poisson with mean 80. $e^{-80}80^{99}/99!$.

   (b) (2 points) If 299 parishioners arrive by 11:00am, what is the probability that the cloakroom has exactly 99 hats at 11:00am? Each parishioner brings a hat with probability 1/3. $\binom{299}{99}(1/3)^{99}(2/3)^{200}$.

   (c) (5 points) If 299 parishioners arrive by 11:00am, what is the expected arrival time of the 100th parishioner? (Hint: consider the extreme case of 1 arrival if you are stuck.) There are 300 time intervals alll with equal expected value. The 100th parishioner arrives after the 100th interval out of 300, with expected time 9:00am plus $\frac{100}{300}120 = 40$ minutes = 9:40am.

   (d) (5 points) What is the expected amount of time that elapses starting at 9:00am until there are at least 2 items in the cloakroom? (Assume arrivals may continue past 11am until there are at least 2 items.) Prob a parishioner brings 2 items = 1/12. Prob no items = $(1-1/3)(1-1/4) = .5$. Parishioners with 1 or more items arrive as a Poisson process at rate 1/minute, and on average $(1/12)/(1/2) = 1/6$ of them have two items. Expected time until first arrival 1 minute. Prob 2 items then = 1/6. Prob must wait for another arrival $= 5/6$. Answer: $1 + \frac{1}{6}0 + \frac{5}{6}1 = \frac{11}{6}$ minutes or 1 minute and 50 seconds.

5 points Extra Credit: At least 3 items? People with items arrive at rate 1 per minute. The expected time until the first arrival is 1 minute. The probability is 1/6 that the first arrival brings 2 items. In that case the next arrival will bring the total to at least 3, in expected time 1 minute. With probability 5/6 the first arrival brings 1 item. This puts us in the situation of the previous problem. The answer is therefore $1 + \frac{1}{6}1 + \frac{5}{6}\frac{11}{6} = \frac{97}{36}$ minutes.

(e) (5 points) The sermon begins at 11:00am. Once the sermon begins, each parishioner falls asleep after an exponentially distributed amount of time with mean 12 minutes, independent of the others. The sermon ends when all parishioners are asleep. What is the expected number of sleeping parishioners (in the church) at 11:30am? Imagine each parishioner decides on the way to church to be asleep at 11:30am with probability $1 - e^{-30/12} = 1 - e^{-2.5}$. Sleep-inclined parishioners arrive as a PP at rate $2(1 - e^{-2.5})$ per minute for 120 minutes. The expected number is $240(1 - e^{-2.5})$.

(f) (5 points) What is the probability that at least one parishioner is (in church and) awake at noon? Wide-awake arrivals are a PP with rate $2e^{-60/12} = 2/e^5$ per minute. The expected number is $240/e^5$, so the probability of at least one arrival is $1 - P(\text{no arrivals}) = 1 - e^{-(240/e^5)}$.

(g) (6 points) 300 parishioners arrive by 11:00am. What is the expected time at which the sermon ends? The expected time until one is asleep is 12/300 minutes. The next on average takes an additional 12/299 minutes, etc.

$$\sum_{k=1}^{300} \frac{12}{k} \quad \text{minutes.}$$

(h) (Extra credit 5 points) Make a probabilist's joke about the scenario of this problem. I expected a joke about parishioners not remembering the sermon.

6. You run a delivery service in a city whose streets form a rectangular grid. North-south streets are numbered 1 through 25 and are spaced 1000 feet apart; the 25 east-west streets are 200 feet apart and are named $A, B, C, \ldots, X, Y$. Delivery locations are uniformly distributed on the grid. Answer all questions using the very good approximation of continuous uniform distributions.

   (a) (2 points) Your location is at 1st and Y. What is the average distance your trucks travel to make a delivery and return?

   The city is 24,000 feet "wide" east-west and 4800 feet "long" north-south. From the northwest corner (or any corner) the average round trip distance is $2(\frac{1}{2}24000 + \frac{1}{2}4800) = 28,800$ feet.

   (b) (5 points) Your location is at 1st and Y. You send one truck to make three deliveries (to random locations) on S street. What is the expected distance your truck travels to make the deliveries and return?] From Y to S and back is 2400 feet. From 1st and S you expect to go to the farthest of 3 $U[0, 24000]$ distances and back, with expected distance $\frac{3}{4}24000$ times two, which is 36000 feet. Answer: 38,400 feet.

   (c) (2 points) Your location is at 13th and M. (This is the center of the grid). What is the average distance your trucks travel to make a delivery and return?

   Your city in effect has half the width and half the length. $28800/2 = 14400$ feet.

   (d) (6 points) Your location is at 13th and M. You send one truck to make three deliveries (to random locations) on S street. What is the expected distance your truck travels to make the deliveries and return?

   The north-south travel from M to S and back is 2400 feet. Once at 13th and S, the probability is 1/4 that all three deliveries are to the east or all are to the west. The probability is 3/4 that one delivery is in one direction and the other two are in the other direction. In the former case you expect to travel $\frac{3}{4}12000 = 9000$ feet and back. In the latter case you expect to travel $\frac{1}{2}12000 + \frac{2}{3}12000$ and back. The expected distance is $2400 + \frac{1}{4}18000 + \frac{3}{4}28000 = 27900$ feet.