MATH 2602 **Exam II** October 21, 2011

**Printed Name:** Solutions

**GT ID #:** ____________________

**Section (circle one):** (G1 - Geehoon Hong) (G2 - Ke Yin)

Instructions:

- There are 7 problems. Point values for each problem are as indicated.
- You may use scratch paper that I provide but ONLY THE WORK WRITTEN IN THIS BOOKLET WILL BE GRADED.
- On each question you must show all appropriate legible work to receive full credit.
- Box or circle your final answer.
- Calculators are not allowed.
- SCHOLASTIC DISHONESTY WILL NOT BE TOLERATED.

**Good Luck!**

| Total | /100 |
|---|---|

1. (6 points) How many ways can 8 people sit around a circular table if Alice and Bob won't sit next to each other?

# ways to choose a seat for Alice → 1

# ways to seat Bob → 5

# ways to seat the other 6 people → 6!

Answer = $\boxed{5 \cdot 6!}$

2. (14 points) How many nine-card hands from a standard deck of 52 cards contain exactly 2 kings, exactly 3 aces, and exactly 2 red cards? Note: At least one ace must be red!

Case 1:

2 black kings
2 black aces
1 red ace
1 other red card
3 other black cards

$\binom{2}{2} \cdot \binom{2}{2} \cdot \binom{2}{1} \cdot \binom{22}{1} \cdot \binom{22}{3}$

(the other cards should not be kings nor aces)

+

case 2

2 black kings
2 red aces
1 black ace
4 other black cards

$\binom{2}{2} \cdot \binom{2}{2} \cdot \binom{2}{1} \cdot \binom{22}{4}$

+

case 3

1 red king
1 black king
2 black aces
1 red ace
4 other black cards

$\binom{2}{1}\binom{2}{1}\binom{2}{2}\binom{2}{1}\binom{22}{4}$

3. You are performing an experiment consists of tossing a coin 7 times.

(a) (7 points) In how many ways can you get exactly 2 heads but not in two consecutive tosses?

exactly 2 H's $\rightarrow$ exactly 5 T's

_T_T_T_T_T_

the H's can't be consecutive $\rightarrow$ $\boxed{\binom{6}{2}}$ ways to place the H's among the T's

(b) (9 points) What is the probability that you will get at least 2 heads but no two of them in consecutive tosses if the coin is fair?

Similarly to (a), we can get exactly 3 H's (no 2 consecutive) in $\binom{5}{3}$ ways

_T_T_T_T_

" " 4 H's " in $\binom{4}{4}$ ways

_T_T_T_

Each outcome has probability $\left(\frac{1}{2}\right)^7$ So, Answer $= \boxed{\binom{6}{2}\cdot\frac{1}{2^7}+\binom{5}{3}\frac{1}{2^7}+\frac{1}{2^7}}$

(c) (9 points) What is the probability that you will get at least 2 heads but no two of them in consecutive tosses if the coin is biased with $P(H) = \frac{1}{3}$?

The count is the same as in (b), but the probabilities of the outcomes are different. $P(H)=\frac{1}{3} \rightarrow P(T)=\frac{2}{3}$

Answer $= \boxed{\binom{6}{2}\left(\frac{1}{3}\right)^2\cdot\left(\frac{2}{3}\right)^5+\binom{5}{3}\left(\frac{1}{3}\right)^3\cdot\left(\frac{2}{3}\right)^4+\binom{4}{4}\cdot\left(\frac{1}{3}\right)^4\cdot\left(\frac{2}{3}\right)^3}$

4. A new test for Alzheimer's Disease will detect the disease 95% of the time in a person who has Alzheimer's. In addition, the test will falsely detect the disease 15% of the time in a healthy person. If the test is given to a person selected at random from a group of people, 90 of whom are healthy and 10 of whom have Alzheimer's, what is the probability that

(a) (5 points) Alzheimer's will not be detected if the person has the disease?

A = event that a person has Alzheimer's

D = event that the test detects the disease.

Info given in the problem: $P(D|A)=0.95$, $P(D|A^c)=0.15$, $P(A^c)=0.9$

$P(A)=0.1$

$$P(D^c|A) = 1 - P(D|A) = 1-0.95 = \boxed{0.05}$$

(b) (13 points) the person has Alzheimer's if the test detects the disease?

$$P(A|D) = \frac{P(A)\cdot P(D|A)}{P(D)} = \frac{(0.1)\cdot(0.95)}{P(D\cap A)+P(D\cap A^c)} =$$

$$= \frac{(0.1)\cdot(0.95)}{P(A)\,P(D|A)+P(A^c)\cdot P(D|A^c)} = \boxed{\frac{(0.1)\cdot(0.95)}{(0.1)\cdot(0.95)+(0.9)\cdot(0.15)}}$$

5. (15 points) How many permutations of the 26 letters a-z contain at least one of the words *dog*, *gas*, or *left*?

A = set of all permutations of the 26 letters that contain the word "dog"

B = 〃 〃 〃 "gas"

C = 〃 〃 〃 "left"

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= \boxed{24! + 24! + 23! - 22! - 21! - 21! + 19!}$$

6. (11 points) Find the coefficient of $x^{-6}$ in $\left(3x^2 - \frac{1}{5x}\right)^{12}$.

$$\left(3x^2 - \frac{1}{5x}\right)^{12} = \sum_{k=0}^{12} \binom{12}{k} (3x^2)^{12-k} \left(-\frac{1}{5x}\right)^k = \sum_{k=0}^{12} \binom{12}{k} 3^{12-k} \left(-\frac{1}{5}\right)^k \cdot x^{2(12-k)-k}$$

We get $x^{-6}$ when

$$2(12-k)-k = -6$$
$$24 - 3k = -6$$
$$30 = 3k$$
$$k = 10$$

And the coeff. in front of $x^{-6}$ is $\boxed{\binom{12}{10} 3^2 \left(-\frac{1}{5}\right)^{10}}$

7. (11 points) In an RNA chain of 15 bases, there are 4 As, 6 Us, 4 Gs, and 1 C. If the chain begins with AU and ends with UG, how many chains are there?

$$A\ U\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}_{3 A's,\ 4 U's,\ 3 G's,\ 1 'C}\ U\ G$$

# ways to place the A's $\rightarrow \binom{11}{3}$

〃 〃 the U's $\rightarrow \binom{8}{4}$

〃 〃 〃 G's $\rightarrow \binom{4}{3}$

the C $\rightarrow \binom{1}{1}$

Answer = $\boxed{\binom{11}{3}\binom{8}{4}\cdot\binom{4}{3}}$

or simplified

$\boxed{\frac{11!}{3! \cdot 4! \cdot 3!}}$