Problem 1(30 points). Calculations.

(a)(5 pt) $\frac{d}{dt}[(e^t\mathbf{i}+\sqrt{t}\mathbf{j})\bullet(e^{-t}\mathbf{i}-3\sqrt{t}\mathbf{j})]$.

(b) (5 pt) $\frac{d}{dt}[(t^2\mathbf{i}+\mathbf{j})\times(t^2\mathbf{i}-3t\mathbf{j})]$.

Solution:

(a)

$$\frac{d}{dt}[(e^t\mathbf{i}+\sqrt{t}\mathbf{j})\bullet(e^{-t}\mathbf{i}-3\sqrt{t}\mathbf{j})]=\frac{d}{dt}(1-3t)=-3.$$

(b)

$$\frac{d}{dt}[(t^2\mathbf{i}+\mathbf{j})\times(t^2\mathbf{i}-3t\mathbf{j})]=\frac{d}{dt}\left(t^4-3t\right)=4t^3-3.$$

(c)(10 pt) Let $h(r,\theta,t) = r^2e^{2t}sin(\theta - t)$, calculate h_r, h_t and h_{rt}.
(d)(10 pt) Set $f(x,y) = \frac{x-y^3}{x^3-y^3}$. Determine whether or not f has a limit at $(1,1)$.

Solution:

(c)

$$\begin{aligned}
h_r &= 2re^{2t}sin(\theta - t), \\
h_t &= 2r^2e^{2t}sin(\theta - t) - r^2e^{2t}\cos(\theta - t), \\
h_{rt} &= 4re^{2t}sin(\theta - t) - 2re^{2t}\cos(\theta - t).
\end{aligned}$$

(d) First, let (x,y) approaches $(1,1)$ from x direction, that is, $y = 1$. Then

$$\begin{aligned}
\lim_{(x,y)\to(1,1)} \frac{x-y^3}{x^3-y^3} &= \lim_{x\to 1} \frac{x-1}{x^3-1} = \lim_{x\to 1} \frac{1}{x^2+x+1} \\
&= \frac{1}{3}.
\end{aligned}$$

Then, let (x,y) approaches $(1,1)$ from y direction, that is, $x = 1$. Then

$$\begin{aligned}
\lim_{(x,y)\to(1,1)} \frac{x-y^3}{x^3-y^3} &= \lim_{y\to 1} \frac{1-y^3}{1-y^3} \\
&= 1.
\end{aligned}$$

Since the two limits are different, so the limit

$$\lim_{(x,y)\to(1,1)} \frac{x-y^3}{x^3-y^3}$$

does not exist.

Problem 2(30 pt) An object moves so that

$$\mathbf{r}(t) = 4\mathbf{i} + (1+3t)\mathbf{j} + (9-t^2)\mathbf{k},\ \ t \geq 0.$$

(a)(6 pt) Compute the velocity, the acceleration and the speed of the ball at an arbitrary time t.
(b) (6 pt) Find the time $t_1 > 0$ and the coordinates of the point P where the object hits the xy plane.
(c)(6 pt) Set up a definite integral equal to the length of the arc of the trajectory from $\mathbf{r}(0)$ to the point P. Do not evaluate the integral.

Solution:

(a) The velocity

$$\mathbf{v}\,(t) = \mathbf{r}'\,(t) = 3\mathbf{j} - 2t\mathbf{k}.$$

The acceleration

$$\mathbf{a}\,(t) = \mathbf{v}'\,(t) = -2\mathbf{k}.$$

The speed

$$v\,(t) = \|\mathbf{v}\,(t)\| = \sqrt{9+4t^2}.$$

(b) At time t_1 when it hits the xy plane, we have $9 - t_1^2 = 0$.So $t_1 = 3$.

(c) The arc length is

$$\int_0^3 \sqrt{9+4t^2}dt.$$

(d)(6 pt) Find the equation of the line tangent to the trajectory at P.
(e)(6 pt) Find the curvature of the trajectory at P.

Solution:

(d) At the point P, the tangent vector is $\mathbf{v}(3) = 3\mathbf{j} - 6\mathbf{k}$. Since $\mathbf{r}(3) = 4\mathbf{i} + 10\mathbf{j}$, so the tangent line is

$$x = 4,\ y = 10 + 3t,\ z = -6t.$$

(e) The unit tangent is

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)} = \frac{3\mathbf{j} - 2t\mathbf{k}}{\sqrt{9 + 4t^2}}.$$

So

$$\mathbf{T}'(t) = \frac{-2\mathbf{k}}{\sqrt{9 + 4t^2}} - 4t\left(9 + 4t^2\right)^{-\frac{3}{2}}(3\mathbf{j} - 2t\mathbf{k}).$$

At $t = 3$,

$$\mathbf{T}'(3) = \frac{1}{15\sqrt{5}}(-4\,\mathbf{j} - 2\,\mathbf{k})$$

and the curvature

$$\kappa = \frac{\|\mathbf{T}'(3)\|}{v(3)} = \frac{2}{45\sqrt{5}}.$$

Problem 3(40 pt) At each point $P(x(t), y(t), z(t))$ of its motion, an object of mass m is subject to a force:

$$\mathbf{F}(t) = -m(\sin t\ \mathbf{i} + \cos t\ \mathbf{j} + (\sin t + \cos t)\mathbf{k}).$$

Given that $\mathbf{v}(0) = \mathbf{i} + \mathbf{k}$, and $\mathbf{r}(0) = \mathbf{j} + 3\mathbf{k}$. Find the following:

(a) (8 pt) The velocity $\mathbf{v}(t)$.
(b) (4 pt) The speed $v(\pi)$.

Solution: (a) From $\mathbf{F} = m\mathbf{a}$, we have

$$\mathbf{a}(t) = -(\sin t\ \mathbf{i} + \cos t\ \mathbf{j} + (\sin t + \cos t)\mathbf{k}).$$

So

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{v}(0) + \int_0^t \mathbf{a}(t)\,dt \\
&= \mathbf{i} + \mathbf{k} - \int_0^t (\sin t\ \mathbf{i} + \cos t\ \mathbf{j} + (\sin t + \cos t)\mathbf{k})dt \\
&= \mathbf{i} + \mathbf{k} - (-\cos t\ \mathbf{i} + \sin t\ \mathbf{j} + (-\cos t + \sin t)\mathbf{k})\big|_0^t \\
&= \mathbf{i} + \mathbf{k} + (\cos t - 1)\,\mathbf{i} - \sin t\ \mathbf{j} + (\cos t - 1 - \sin t)\ \mathbf{k} \\
&= \cos t\ \mathbf{i} - \sin t\ \mathbf{j} + (\cos t - \sin t)\ \mathbf{k}.
\end{aligned}$$

(b) Ar $t = \pi$, the speed

$$v(\pi) = \|\mathbf{v}(\pi)\| = \sqrt{-\mathbf{i} - \mathbf{k}} = \sqrt{2}.$$

(c) (8 pt) The position function $\mathbf{r}(t)$.
(d) (10 pt) The tangential and normal components of the acceleration $\mathbf{a}(\pi)$.
(e) (10 pt) The osculating plane at $\mathbf{r}(\pi)$.
Solution:
(c) The position function

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{r}(0) + \int_0^t \mathbf{v}(t)\ dt \\
&= \mathbf{j} + 3\mathbf{k} + \int_0^t (\cos t\ \mathbf{i} - \sin t\ \mathbf{j} + (\cos t - \sin t)\ \mathbf{k})\ dt \\
&= \mathbf{j} + 3\mathbf{k} + (\sin t\ \mathbf{i} + \cos t\ \mathbf{j} + (\sin t + \cos t)\ \mathbf{k})|_0^t \\
&= \mathbf{j} + 3\mathbf{k} + \sin t\ \mathbf{i} + (\cos t - 1)\ \mathbf{j} + (\sin t + \cos t - 1)\,\mathbf{k} \\
&= \sin t\ \mathbf{i} + \cos t\ \mathbf{j} + (\sin t + \cos t + 2)\,\mathbf{k}.
\end{aligned}$$

(d) The tangential component

$$\begin{aligned}
\mathbf{a}_T(\pi) &= \frac{\mathbf{v}(\pi)\cdot\mathbf{a}(\pi)}{v(\pi)} = \frac{(-\mathbf{i}-\mathbf{k})\cdot(\mathbf{j}+\mathbf{k})}{\sqrt{2}} \\
&= -\frac{1}{\sqrt{2}}
\end{aligned}$$

and the normal component

$$\begin{aligned}
\mathbf{a}_N(\pi) &= \sqrt{\|\mathbf{a}(\pi)\|^2 - (\mathbf{a}_T(\pi))^2} \\
&= \sqrt{2-\frac{1}{2}} = \sqrt{\frac{3}{2}}.
\end{aligned}$$

(e) The unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\cos t\ \mathbf{i} - \sin t\ \mathbf{j} + (\cos t - \sin t)\ \mathbf{k}}{\sqrt{2-2\sin(2t)}}.$$

Thus

$$\begin{aligned}
\mathbf{T}'(t) &= \frac{-\sin t\ \mathbf{i} - \cos t\ \mathbf{j} + (-\sin t - \cos t)\ \mathbf{k}}{\sqrt{2-2\sin(2t)}} \\
&\quad + \frac{2\cos(2t)}{2-2\sin(2t)}(\cos t\ \mathbf{i} - \sin t\ \mathbf{j} + (\cos t - \sin t)\ \mathbf{k}).
\end{aligned}$$

The normal vector of the osculating plane is

$$\begin{aligned}
& \mathbf{T}(\pi)\times\mathbf{T}'(\pi) \\
=\ & \frac{\cos\pi\ \mathbf{i} - \sin\pi\ \mathbf{j} + (\cos\pi - \sin\pi)\ \mathbf{k}}{\sqrt{2-2\sin(2\pi)}} \times \frac{-\sin\pi\ \mathbf{i} - \cos\pi\mathbf{j} + (-\sin\pi - \cos\pi)\ \mathbf{k}}{\sqrt{2-2\sin(2\pi)}} \\
=\ & \frac{1}{\|\mathbf{v}(\pi)\|^2}\mathbf{v}(\pi)\times\mathbf{a}(\pi) = \frac{1}{2}(-\mathbf{i}-\mathbf{k})\times(\mathbf{j}+\mathbf{k}) \\
=\ & \frac{1}{2}(\mathbf{i}+\mathbf{j}-\mathbf{k}).
\end{aligned}$$

Since $\mathbf{r}(\pi) = -\mathbf{j} + \mathbf{k}$, so the osculating plane is

$$x + y + 1 - (z - 1) = 0,$$

or

$$x + y - z + 2 = 0.$$