Math 2603-F **Exam 1** **Spring 2016**

Date: February 11, 2016 **Time: 12:05 to 13:25** **Duration of exam: 80 min**

Last Name (Print): ____________________ **First Name (Print):** ____________________

This exam contains 4 pages (including this cover page) and 8 questions. Total of points is 55.

Instructions

1. Please be sure your name appears correctly at the top of this page and that your initials appear at the top of the remaining pages.
2. Answer the questions in the space provided, using the backs of pages for overflow or rough work.
3. To obtain maximum marks show all your work, carefully justifying your answers.
4. The use of calculator, books or any other aids will not be allowed for the test.

Grade Table (for instructor use only)

Question	Points	Score
1	6	
2	4	
3	5	
4	9	
5	4	
6	8	
7	4	
8	15	
Total:	55	

1. (6 points) Write down the negation of each of the following statements in clear and concise English.

   (a) (2 points) $n > \pi$ or n is negative.

   Solution: $n \leq \pi$ and n is positive

   (b) (2 points) $\frac{1}{n}$ is not an integer for all natural numbers n.

   Solution: There exists a natural number n such that $\frac{1}{n}$ is an integer.

   (c) (2 points) There exists a function f that is differentiable and discontinuous.

   Solution: f is not differentiable or f is continuous for all functions f.

2. (4 points) Prove that if a is an irrational number and b a rational number then $a + b$ is an irrational number.

   Solution: Let us assume, for a contradiction, that $a+b$ is rational, say $a+b = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$, $q \neq 0$. Since b is rational, then $b = \frac{b}{m}$ for some $m, n \in \mathbb{Z}$, $m \neq 0$. Then

   $$a = \frac{a}{b} - \frac{p}{q} = \frac{pn - qm}{qn},$$

   which is a rational number (since $pn - qm, qn \in \mathbb{Z}$ with $qn \neq 0$). This contradicts the fact that a is irrational.

3. (5 points) Let A and B be sets. Prove that $(A \cap B)^{\mathsf{c}} = A^{\mathsf{c}} \cup B^{\mathsf{c}}$

   Solution: We have $x \in (A \cap B)^{\mathsf{c}}$ iff $x \notin A \cap B$. Now $x \notin A \cap B$ iff $x \notin A$ or $x \notin B$. Also note that $x \notin A$ or $x \notin B$ iff $x \in A^{\mathsf{c}}$ or $x \in B^{\mathsf{c}}$. Finally $x \in A^{\mathsf{c}}$ or $x \in B^{\mathsf{c}}$ iff $x \in A^{\mathsf{c}} \cup B^{\mathsf{c}}$.

4. (9 points) For $a, b \in \mathbb{R}$ consider the following relation:

   $$a \sim b \text{ if and only if } |a - 5| = |5 - b|.$$

   The aim of this exercise is to show that $\sim$ defines an equivalence relation over $\mathbb{R}$. Prove that the relation $\sim$ is:

   (a) (3 points) Reflexive.

   Solution: Let $a \in \mathbb{R}$. Then $|a - 5| = |5 - a|$, since $|x| = |-x|$. Thus $a \sim a$ for all $a \in \mathbb{R}$.

   (b) (3 points) Symmetric.

Solution: Suppose $a \sim b$. Then $|a-5| = |5-b|$ or $|5-a| = |b-5|$. Therefore $b \sim a$.

(c) (3 points) Transitive.

Solution: Suppose $a \sim b$ and $b \sim c$. Then $|a-5| = |5-b|$ and $|b-5| = |5-c|$. Since $|5-b| = |b-5|$ we conclude that $|a-5| = |5-c|$ or $a \sim c$.

5. (4 points) Consider the relation $\sim$ given in question 4 .

(a) (2 points) What is the equivalence class of 5?

Solution: $\overline{5} = \{x \in \mathbb{R} : |x-5| = 0\} = \{5\}$.

(b) (2 points) What is the equivalence class of -5?

Solution: $\overline{-5} = \{x \in \mathbb{R} : |x-5| = 10\} = \{-5, 15\}$.

(c) (Bonus 3 points) Define a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \# \text{ elements in } \overline{x},$$

that is each $x \in \mathbb{R}$ is mapped to the number of elements contained in its equivalence class.

6. (8 points) (a) (6 points) Compute $\gcd(282, 137)$.

Solution:

	a	b
282	1	0
137	0	1
8	1	-2
1	-17	35

(b) (2 points) Find integers m and n such that $282m + 137n = 3$.

Solution: From above we have that $-17 \cdot 282 + 35 \cdot 137 = 1$, therefore $(3 \cdot -17) \cdot 282 + (3 \cdot 35) \cdot 137 = 2$. So $m = -51$ and $n = 105$.

7. (4 points) Prove that if $k \in \mathbb{N}$, then $\gcd(3k+2, 5k+3) = 1$.

Solution: Let $g = \gcd(3k+2, 5k+3)$. Then $g|3k+2$ and $g|5k+3$. In other words, there exist $m, n \in \mathbb{Z}$ such that

$$3k + 2 = gn \tag{1}$$

and

$$5k + 3 = gm. \tag{2}$$

Multiplying equation (1) by 5 and equation (2) by 3 yields

$$15k + 10 = 5gn \tag{3}$$

and

$$15k + 9 = 3gm. \tag{4}$$

Now, subtracting equation (4) from (3) we obtain $(5n - 3m)g = 1$. Finally, since $g \geq 1$ and $g \in \mathbb{Z}$ it follows that $g = 1$.

8. (15 points) Decide whether each of the following statements is true or false. Justify your claim.

   (a) (3 points) If a and b are real numbers such that $a+b$ is a rational number, then a and b are rational numbers.

   Solution: False. Let $a = \sqrt{2}$ and $b = -\sqrt{2}$. Then $a + b = 0 \in \mathbb{Q}$ but $a, b \notin \mathbb{Q}$.

   (b) (3 points) If A and B are sets, then $A \cup (B \cap C) = (A \cup B) \cap C$.

   Solution: False. Let $A = \{1\}$, $B = \{2\}$ and $C = \emptyset$. Then

   $$A \cup (B \cap C) = \{1\} \neq \emptyset = (A \cup B) \cap C.$$

   (c) (3 points) Let A, B and C be sets. If $A \cap C = B \cap C$ then $A = B$.

   Solution: False. If $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{1\}$ then $A \cap C = \{1\} = B \cap C$ but $A \neq B$.

   (d) (3 points) For all nonzero integers a and b we have that $\gcd(a, b) = \gcd(-a, b)$.

   Solution: True. Let $g_1 = \gcd(a, b)$ and $g_2 = \gcd(-a, b)$. Let us show that $g_1 \leq g_2$ and $g_2 \geq g_1$. Since $g_1 | a$ we have that $g_1 | -a$, therefore $g_1 | -a$ and $g_1 | b$. Since g_2 is the gcd of $-a$ and b we have that $g_1 \leq g_2$. A similar argument shows that $g_2 \leq g_1$, thus $g_1 = g_2$.

   (e) (3 points) Let $A = \{a, b, c, d\}$ and $B = \{0, 2, 4, 8\}$. The set $f = \{(c, 4), (d, 0), (a, 0), (b, 2), (d, 4)\}$ defines a function from A to B.

   Solution: False. There are two pairs of the form (d, x) with $x \in B$.