ISyE 3044 — Practice Problems for Exam #2

1. Consider the following 10 pseudo-random numbers (read from left to right):

   0.98 0.51 0.66 0.97 0.31 0.08 0.27 0.42 0.16 0.68

   (a) Use the first two numbers to generate two observations from the $N(0,1)$ distribution.

   (b) Use the observations from part (a) to generate two observations from the normal distribution with mean 1 and variance 4.

   (c) Use *all* 10 numbers to generate an observation from the $N(0,1)$ distribution.

   (d) Use as many numbers as you need to generate an observation from the Poisson$(\lambda = 1)$ distribution.

   (e) Use as many numbers as you need to generate an observation from the geometric$(p = 0.9)$ distribution.

2. The random variable X has density function $f(x) = |x|$, $-1 \leq x \leq 1$.

   (a) Find the mean of X.

   (b) Apply the inverse-transform method to derive formula(s) for generating realizations of X.

   (c) Use the random number 0.75 to generate a realization of X.

3. Ten independent replications of a simulation of a bank (each run for 3 hours) gave the following estimates for the mean utilization of a teller:

Replication	1	2	3	4	5	6	7	8	9	10
Utilization	0.74	0.83	0.65	0.91	0.88	0.93	0.79	0.80	0.68	0.77

   Compute a 95% confidence interval for the mean teller utilization.

4. The scope of a simulation project was the estimation of the mean time, say μ, it takes to produce an item. We used 10 independent replications and the central limit theorem to compute the following 95% confidence interval for μ: $(4.8, 12.4)$.

   (a) What is the estimate of μ?

   (b) What is the relative error of this estimate?

   (c) Is the following statement correct? Yes No
   "The interval $(4.8, 12.4)$ contains the true mean with probability 0.95."

   (d) Compute a 90% confidence interval for μ.

(e) We wish to obtain a point estimate of μ such that $\Pr\{|\bar{\mu}_k - \mu| \leq 2\} \geq 0.95$. Compute an estimate of the additional number of replications that need to be made.

5. Consider the following 20 random numbers (read from left to right, and then down).

.594	.928	.515	.055	.507	.351	.262	.797	.788	.442
.097	.798	.227	.127	.474	.825	.007	.182	.929	.852

Conduct a chi-square test with four intervals for the hypothesis H_0: The numbers are $U(0,1)$. Use type I error $\alpha = 0.10$.

6. We are told that the following observations

0.201 1.075 0.656 2.282 0.992

are from the Weibull density

$$f(x) = 2\lambda^2 x e^{-(\lambda x)^2}, \quad x > 0.$$

(a) Compute the m.l.e. of λ.

(b) Use the Kolmogorov-Smirnov test to assess the goodness-of-fit of this distribution. Use type I error $\alpha = 0.05$ and use the test statistic for the case where all parameters are known.

(c) Alternatively, assume that the above five observations are from the gamma distribution with shape parameter equal to 2. Use the method of moments to compute an estimate for the scale parameter.

7. The following five observations are observed repair times for an airplane engine: 3.6, 23.3, 31.5, 17.9, 4.0.

The following questions can be answered independently of each other.

(a) Assume that the data come from the gamma distribution with *mean* equal to 10. Use the method of moments to estimate the shape and scale parameters. [Hint: Write $E(X^2)$ as a function of $E(X)$ and λ.]

(b) Use the Kolmogorov-Smirnov test with type I error 0.10 to assess the hypothesis "the data come from the gamma distribution with shape parameter equal to 2 and scale parameter equal to 0.2".

8. Study the following Examples from BCN&N (the first number indicates the chapter): 7.6, 7.7, 8.9, 8.10, 8.13, 9.12, 9.13, 9.14, 9.18, 11.9, 11.10, 11.12, 11.13, 11.15.

Critical Values c_α for Adjusted K-S Statistics

		α				
Case	Adjusted Test Statistic	0.15	0.10	0.05	0.025	0.01
All parameters known	$\left(\sqrt{n} + 0.12 + \frac{0.11}{\sqrt{n}}\right) D_n$	1.138	1.224	1.358	1.480	1.628
$\mathrm{Nor}(\bar{X}_n, S_n^2)$	$\left(\sqrt{n} - 0.01 + \frac{0.85}{\sqrt{n}}\right) D_n$	0.775	0.819	0.895	0.995	1.035
$\mathrm{Expo}(1/\bar{X}_n)$	$\left(D_n - \frac{0.2}{\sqrt{n}}\right)\left(\sqrt{n} - 0.01 + \frac{0.85}{\sqrt{n}}\right)$	0.926	0.990	1.094	1.190	1.308

Solutions

1. (a) $Z_1 = \sqrt{-2\ln(0.98)}\cos[2\pi(0.51)] = -0.2$ and $Z_2 = \sqrt{-2\ln(0.98)}\sin[2\pi(0.51)] = -0.01$.
   (b) $X_1 = 1 + 2Z_1 = 0.6$ and $X_2 = 1 + 2Z_2 = 0.98$.
   (c) $Z = \frac{\sum_{i=1}^{10} U_i - 10/2}{\sqrt{10/12}} = 0.044$.
   (d) $N = \min\{k \geq 1 : \prod_{i=1}^{k+1} U_i < e^{-1} = 0.368\} = 2$.
   (e) $X = \left\lceil \frac{\ln(1-0.98)}{\ln(1-0.9)} \right\rceil = 2$.

2. (a) $E(X) = 0$ by symmetry.
   (b) We have

$$F(x) = \begin{cases} \int_{-1}^{x}(-t)\,dt = 1/2 - x^2/2 & \text{if } -1 \leq x < 0 \\ F(0) + \int_0^x t\,dt = 1/2 + x^2/2 & \text{if } 0 \leq x \leq 1. \end{cases}$$

   Solving $F(X) = U$ for X we get

$$X = \begin{cases} -\sqrt{1-2U} & \text{if } 0 \leq U < 1/2 \\ \sqrt{2U-1} & \text{if } 1/2 \leq U \leq 1. \end{cases}$$

   (c) $X = \sqrt{2(0.75) - 1} = \sqrt{2}/2 = 0.707$.

3. $\bar{X}_{10} = 0.798$, $S_{10} = 0.09$, and $t_{9,0.025} = 2.26$. The confidence interval is $0.798 \pm 2.26\frac{0.09}{\sqrt{10}} = (0.734, 0.862)$.

4. (a) $\bar{\mu}_{10} = 8.6$, the midpoint of the interval.
   (b) The estimate of the relative error is

$$\frac{\text{halfwidth}}{\text{sample mean}} = 0.44.$$

   (c) No.
   (d) We have

$$\text{halwidth} = 3.8 = 2.26\frac{S_{10}}{\sqrt{10}} \Longrightarrow S_{10} = 5.31.$$

   The 90% confidence interval is

$$8.6 \pm 1.83\frac{5.31}{\sqrt{10}} = (5.31, 11.67).$$

   (e) We will use the standard normal quantile. Solving the inequality

$$z_{0.025}\frac{S_{10}}{\sqrt{k}} \leq 2$$

   for k, we have $k \geq (1.96 \times 5.31/2)^2 = 27.08$. Hence we should make about 18 more replications.

5. We have the following table

interval i	$[0, 0.25)$	$[0.25, 0.50)$	$[0.50, 0.75)$	$[0.75, 1)$
O_i	6	4	3	7
E_i	5	5	5	5

The test statistic is

$$\chi_0^2 = \frac{(6-5)^2}{5} + \frac{(4-5)^2}{5} + \frac{(3-5)^2}{5} + \frac{(7-5)^2}{5} = \frac{10}{5} = 2.$$

Since $\chi_0^2 \le \chi^2_{3,0.10} = 6.25$, we fail to reject H_0.

6. (a) $\hat{\lambda} = \left[(1/5)\sum_{i=1}^5 X_i^2\right]^{-1/2} = 0.8.$
(b) Using $\hat{F}(x) = 1 - e^{-(0.8x)^2}$ we compute the test statistic

$$D_5 = \max\left\{\max_{1\le i\le 5}\left[\frac{i}{5} - \hat{F}(X_{(i)})\right], \max_{1\le i\le 5}\left[\hat{F}(X_{(i)}) - \frac{i-1}{5}\right]\right\} = 0.277.$$

The adjusted test statistic is

$$\left(\sqrt{5} + 0.12 + \frac{0.11}{\sqrt{5}}\right) D_5 = 0.666.$$

Since this value is less than the critical value $c_{0.05} = 1.358$, we fail to reject H_0. (In the absence of a table for the Weibull distribution, we used the adjusted test statistic for the case where all parameters are known.)
(c) We have $\bar{X}_5 = 1.04$. Solving $\bar{X}_5 = 2/\lambda$ we get $\hat{\lambda} = 2/1.04 = 1.92$.

7. (a) The first two noncentral moments are $\bar{X}_5 = 16.06$ and $(1/5)\sum_{i=1}^5 X_i^2 = 376.9$. Solving the equations

$$\frac{\alpha}{\lambda} = 16.06 \quad \text{and} \quad \frac{\alpha}{\lambda^2} + \left(\frac{\alpha}{\lambda}\right)^2 = 376.9,$$

we get the method of moment estimates: $\tilde{\alpha} = 2.17$ and $\tilde{\lambda} = 0.135$.
(b) First sort the observations from smallest to largest: $X_{(1)} \le \cdots \le X_{(5)}$. Using $\hat{F}(x) = 1 - (1 + 0.2x)e^{-0.2x}$ we compute the test statistic

$$D_5 = \max\left\{\max_{1\le i\le 5}\left[\frac{i}{5} - \hat{F}(X_{(i)})\right], \max_{1\le i\le 5}\left[\hat{F}(X_{(i)}) - \frac{i-1}{5}\right]\right\} = \max\{0.209, 0.472\} = 0.472.$$

The adjusted test statistic is

$$\left(\sqrt{5} + 0.12 + \frac{0.11}{\sqrt{5}}\right) D_5 = 1.135.$$

Since this value is less than the critical value $c_{0.10} = 1.224$, we fail to reject H_0.