MATH 1552 TEST 2, FALL 2015, GRODZINSKY

Print Your Name: Key-1

T.A.: (circle one) Miheer   Brandon   Stephen   Kabir

1. (16 points) Evaluate the integral:

$$\int \frac{2x}{x^2+4x+13}dx. = \int \frac{2x+4-4}{x^2+4x+13}dx$$

$$= \int \frac{2x+4}{x^2+4x+13}dx - \int \frac{4}{x^2+4x+4+9}dx$$

$$= \int \frac{2x+4}{x^2+4x+13}dx - \int \frac{4}{(x+2)^2+9}dx$$

$$= \int \frac{2x+4}{x^2+4x+13} - \frac{4}{9}\int \frac{dx}{\left(\frac{x+2}{3}\right)^2+1}$$

$$= \ln|x^2+4x+13| - \frac{4}{9}\cdot 3\tan^{-1}\left(\frac{x+2}{3}\right)+C$$

$$= \boxed{\ln|x^2+4x+13| - \frac{4}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+C}$$

2. (16 points) Evaluate the integral: $\int \sin^7(x)\cos^3(x)dx$.

$$= \int \sin^7 x \cos^2 x \cos x \, dx$$

$$= \int \sin^7 x(1-\sin^2 x)\cos x \, dx$$

$$= \int (\sin^7 x - \sin^9 x)\cos x \, dx$$

$u = \sin x$

$$= \int (u^7 - u^9)\,du = \frac{1}{8}u^8 - \frac{1}{10}u^{10} + C$$

$$= \boxed{\frac{1}{8}\sin^8(x) - \frac{1}{10}\sin^{10}(x) + C}$$

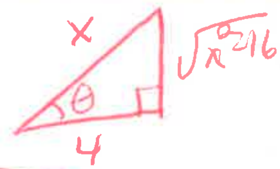
3. (16 points) Evaluate the integral: $\int \frac{1}{(x^2-16)^{3/2}}dx$.

Let $x = 4\sec\theta$, then $dx = 4\sec\theta\tan\theta\,d\theta$

$$\int \frac{1}{(x^2-16)^{3/2}}\,dx = \int \frac{4\sec\theta\tan\theta}{(16\sec^2\theta-16)^{3/2}}\,d\theta$$

$$= \frac{4}{64}\int \frac{\sec\theta\tan\theta}{\tan^3\theta}\,d\theta = \frac{1}{16}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \frac{1}{16}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \frac{1}{16}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$



$u = \sin\theta$
$du = \cos\theta\,d\theta$

$$= \frac{1}{16}\int \frac{du}{u^2} = -\frac{1}{16u} + C$$

$$= -\frac{1}{16}\cdot\csc\theta + C = \boxed{-\frac{1}{16}\cdot\frac{x}{\sqrt{x^2-16}} + C}$$

4. (16 points) Find the limit: $\lim_{x\to 0}\left[\ln(3x^2) - \ln(1-\cos(6x))\right]$.

$$= \lim_{x\to 0} \ln\left(\frac{3x^2}{1-\cos(6x)}\right)$$

$$= \ln\left(\lim_{x\to 0}\frac{3x^2}{1-\cos(6x)}\right) \quad \left[\frac{0}{0}\right]$$

$$\stackrel{L'H}{=} \ln\left(\lim_{x\to 0}\frac{6x}{+6\sin(6x)}\right) \quad \left[\frac{0}{0}\right]$$

$$\stackrel{L'H}{=} \ln\left(\lim_{x\to 0}\frac{6}{36\cos(6x)}\right) = \boxed{\ln\frac{1}{6}}$$

MATH 1552 TEST 2, FALL 2015, GRODZINSKY

Print Your Name: Key-1

T.A.: (circle one) Miheer Brandon Stephen Kabir

5. (20 points) Find a general solution to the differential equation:

$$(x^2 - 5x + 6)\frac{dy}{dx} = y(x+1).$$

$$\int \frac{1}{y}\,dy = \int \frac{x+1}{x^2-5x+6}\,dx$$

$$\frac{x+1}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2} \Rightarrow x+1 = A(x-2) + B(x-3)$$

$x=2$: $3 = B(-1)$, $B=-3$

$x=3$: $4 = A(1)$, $A=4$

$$\int \frac{1}{y}\,dy = \int \left[\frac{4}{x-3} + \frac{-3}{x-2}\right] dx$$

$$\ln|y| = 4\ln|x-3| - 3\ln|x-2| + C$$

$$\Rightarrow e^{\ln|y|} = e^{\ln\left|\frac{(x-3)^4}{(x-2)^3}\right| + C}$$

$$|y| = e^C \left|\frac{(x-3)^4}{(x-2)^3}\right|$$

Let $K = \pm e^C$, then $\boxed{y = K \cdot \frac{(x-3)^4}{(x-2)^3}}$

By parts 6. (16 points) Evaluate the integral: $\int e^x \sin(4x)dx$. $= I$

$u = \sin(4x) \qquad dv = e^x dx$

$du = 4\cos(4x)dx \qquad v = e^x$

So $I = e^x \sin(4x) - 4\int e^x \cos(4x)dx$

By parts again: $\quad u' = \cos(4x) \qquad dv' = e^x dx$

$du' = -4\sin(4x) \qquad v' = e^x$

So $I = e^x \sin(4x) - 4\left[e^x \cos(4x) + 4\int e^x \sin(4x)dx\right]$

$I = e^x \sin(4x) - 4e^x \cos(4x) - 16I$

$17I = e^x \sin(4x) - 4e^x \cos(4x)$

$I = \boxed{\frac{1}{17}e^x \sin(4x) - \frac{4}{17}e^x \cos(4x) + C}$

BONUS: (5 points) If a population grows/decays exponentially, the rate of growth is proportional to the population. Write and solve a differential equation to derive the generic formula for growth and decay.

We need to solve:

$$\frac{dy}{dt} = ky$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int k\,dt$$

$$\ln|y| = kt + c$$

$$e^{\ln|y|} = e^{kt+c} \qquad \text{Let } A_0 = \pm e^c$$

$$|y| = e^c e^{kt}$$

$$y = A_0 e^{kt}$$

MATH 1552 TEST 2, FALL 2015, GRODZINSKY

Print Your Name: Key-2

T.A.: (circle one) Miheer Brandon Stephen Kabir

1. (16 points) Evaluate the integral: $\int e^x \sin(7x)dx$. $= I$

By parts:

Let $u = \sin(7x)$ $\quad dv = e^x dx$

$du = 7\cos(7x)dx$ $\quad v = e^x$

Then:

$$I = e^x \sin(7x) - 7\int e^x \cos(7x)\,dx$$

By parts again:

$u' = \cos(7x)$ $\quad dv' = e^x dx$

$du' = -7\sin(7x)dx$ $\quad v' = e^x$

So:

$$I = e^x \sin(7x) - 7\left[e^x \cos(7x) + 7\int e^x \sin(7x)dx\right]$$

$$I = e^x \sin(7x) - 7e^x\cos(7x) - 49I$$

$$50I = e^x \sin(7x) - 7e^x \cos(7x)$$

$$I = \boxed{\frac{1}{50}e^x \sin(7x) - \frac{7}{50}e^x\cos(7x) + C}$$

2. (16 points) Find the limit: $\lim_{x\to 0}\left[\ln(4x^2) - \ln(1-\cos(9x))\right]$.

$$= \lim_{x\to 0} \ln\left(\frac{4x^2}{1-\cos(9x)}\right)$$

$$= \ln\left(\lim_{x\to 0}\left(\frac{4x^2}{1-\cos(9x)}\right)\right) \quad \left[\frac{0}{0}\right]$$

$$\overset{L'H}{=} \ln\left(\lim_{x\to 0}\frac{8x}{9\sin(9x)}\right) \quad \left[\frac{0}{0}\right]$$

$$\overset{L'H}{=} \ln\left(\lim_{x\to 0}\frac{8}{81\cos(9x)}\right) = \boxed{\ln\left(\frac{8}{81}\right)}$$

3. (16 points) Evaluate the integral: $\int \sin^5(x)\cos^3(x)dx$.

$$= \int \sin^5 x \cos^2 x \cos x \, dx$$

$$= \int \sin^5 x \left(1-\sin^2 x\right) \cos x \, dx$$

$$= \int \left(\sin^5 x - \sin^7 x\right) \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int \left(u^5 - u^7\right) du$$

$$= \frac{1}{6}u^6 - \frac{1}{8}u^8 + C$$

$$= \boxed{\frac{1}{6}\sin^6(x) - \frac{1}{8}\sin^8(x) + C}$$

MATH 1552 TEST 2, FALL 2015, GRODZINSKY

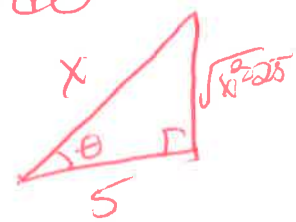
Print Your Name: Key-2

T.A.: (circle one) Miheer Brandon Stephen Kabir

4. (16 points) Evaluate the integral: $\int \frac{1}{(x^2-25)^{3/2}}\,dx$.

Let $x = 5\sec\theta$, then $dx = 5\sec\theta\tan\theta\,d\theta$

and $x^2 - 25 = 25\sec^2\theta - 25 = 25\tan^2\theta$



$$\int \frac{1}{(x^2-25)^{3/2}}\,dx = \int \frac{5\sec\theta\tan\theta}{(25\tan^2\theta)^{3/2}}\,d\theta$$

$$= \int \frac{5\sec\theta\tan\theta}{5^3\tan^3\theta}\,d\theta = \frac{1}{25}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{25}\int \frac{1/\cos\theta}{\sin^2\theta/\cos^2\theta}\,d\theta$$

$$= \frac{1}{25}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta \underset{du=\cos\theta\,d\theta}{\overset{u=\sin\theta}{=}} \frac{1}{25}\int \frac{1}{u^2}\,du = -\frac{1}{25u} + C$$

$$= \frac{-1}{25}\csc\theta + C = \boxed{-\frac{1}{25}\cdot\frac{x}{\sqrt{x^2-25}} + C}$$

5. (16 points) Evaluate the integral:

$$\int \frac{2x+6-6}{x^2+6x+13}\,dx.$$

$$= \int \frac{2x+6}{x^2+6x+13}\,dx - \int \frac{6}{x^2+6x+9+4}\,dx$$

$$= \ln|x^2+6x+13| - \int \frac{6}{(x+3)^2+4}\,dx$$

$$= \ln|x^2+6x+13| - \frac{3}{2}\int \frac{dx}{\left(\frac{x+3}{2}\right)^2+1}$$

$$= \ln|x^2+6x+13| - \frac{3}{2}\cdot 2\tan^{-1}\left(\frac{x+3}{2}\right) + C$$

$$= \boxed{\ln|x^2+6x+13| - 3\tan^{-1}\left(\frac{x+3}{2}\right) + C}$$

6. (20 points) Find a general solution to the differential equation:

$$(x^2 - 6x + 8)\frac{dy}{dx} = y(x + 2).$$

$$\int \frac{1}{y}\,dy = \int \frac{x+2}{x^2-6x+8}\,dx$$

$$\frac{x+2}{x^2-6x+8} = \frac{A}{x-4} + \frac{B}{x-2}$$

$$x+2 = A(x-2) + B(x-4)$$

$x=2$: $4 = B(-2)$, $B=-2$

$x=4$: $6 = A(2)$, $A=3$

$$\ln|y| = \int \left[\frac{3}{x-4} - \frac{2}{x-2}\right] dx$$

$$\ln|y| = 3\ln|x-4| - 2\ln|x-2| + C$$

$$e^{\ln|y|} = e^{3\ln|x-4| - 2\ln|x-2| + C}$$

Let $k = \pm e^C$

$$\boxed{y = k \cdot \frac{(x-4)^3}{(x-2)^2}}$$

BONUS: (5 points) If a population grows/decays exponentially, the rate of growth is proportional to the population. Write and solve a differential equation to derive the generic formula for growth and decay.

See Form 1.

MATH 1552 TEST 2, FALL 2015, GRODZINSKY

Print Your Name: Key-3

T.A.: (circle one) Miheer Brandon Stephen Kabir

1. (16 points) Evaluate the integral: $\int \sin^3(x)\cos^6(x)dx$.

$$= \int \sin^2 x \cos^6 x \cdot \sin x \, dx$$

$$= \int (1-\cos^2 x) \cos^6 x \sin x \, dx$$

$$= \int (\cos^6 x - \cos^8 x) \sin x \, dx$$

$u = \cos x$, $du = -\sin x \, dx$

$$= -\int (u^6 - u^8) du = \frac{1}{9}u^9 - \frac{1}{7}u^7 + C$$

$$= \boxed{\frac{1}{9}\cos^9 x - \frac{1}{7}\cos^7 x + C}$$

2. (16 points) Find the limit: $\lim_{x\to 0} \left[\ln(1-\cos(5x)) - \ln(2x^2)\right]$.

$$= \lim_{x\to 0} \ln\left(\frac{1-\cos(5x)}{2x^2}\right)$$

$$= \ln\left(\lim_{x\to 0} \frac{1-\cos(5x)}{2x^2}\right) \quad \left[\frac{0}{0}\right]$$

$$\overset{L'H}{=} \ln\left(\lim_{x\to 0} \frac{5\sin(5x)}{4x}\right) \quad \left[\frac{0}{0}\right]$$

$$\overset{L'H}{=} \ln\left(\lim_{x\to 0} \frac{25\cos(5x)}{4}\right) = \boxed{\ln\left(\frac{25}{4}\right)}$$

3. (16 points) Evaluate the integral:

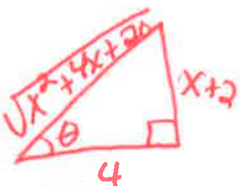
$$\int \frac{2x}{x^2+4x+20}\,dx. = \int \frac{2x}{x^2+4x+4+16}\,dx$$

Let $x+2 = 4\tan\theta$
$\Rightarrow x = 4\tan\theta - 2$
and $dx = 4\sec^2\theta\, d\theta$

$$= \int \frac{2x}{(x+2)^2+16}\,dx = \int \frac{2(4\tan\theta-2)}{16\tan^2\theta+16}\cdot 4\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int \frac{4\tan\theta - 2}{\sec^2\theta}\cdot \sec^2\theta\, d\theta = \frac{1}{2}\int (4\tan\theta - 2)\,d\theta$$

$$= \frac{1}{2}\left[-4\ln|\cos\theta| - 2\theta\right] + C = -2\ln|\cos\theta| - \theta + C$$



$\tan\theta = \frac{x+2}{4}$

$$= -2\ln\left|\frac{4}{\sqrt{x^2+4x+20}}\right| - \tan^{-1}\left(\frac{x+2}{4}\right) + C$$

$$= -2\ln 4 + \ln(x^2+4x+20) - \tan^{-1}\left(\frac{x+2}{4}\right) + C$$

4. (20 points) Find a general solution to the differential equation:

$$(x^2-5x+4)\frac{dy}{dx} = y(x+2).$$

$$\int \frac{1}{y}\,dy = \int \frac{x+2}{x^2-5x+4}\,dx$$

$\frac{x+2}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1}$
$x+2 = A(x-1) + B(x-4)$
$x=1$: $3 = B(-3)$, $B=-1$
$x=4$: $6 = A(3)$, $A=2$

$$\ln|y| = \int \left[\frac{2}{x-4} - \frac{1}{x-1}\right]dx$$

$$\ln|y| = 2\ln|x-4| - \ln|x-1| + C$$

$$|y| = e^{2\ln|x-4| - \ln|x-1| + C}$$

$$|y| = e^C \cdot \left|\frac{(x-4)^2}{x-1}\right|$$

Let $K = \pm e^C$

$$\boxed{y = K\frac{(x-4)^2}{x-1}}$$

MATH 1552 TEST 2, FALL 2015, GRODZINSKY

Print Your Name: Key-3

T.A.: (circle one) Miheer Brandon Stephen Kabir

5. (16 points) Evaluate the integral: $\int e^x \cos(3x)dx$. $= I$

By parts:

$u = \cos(3x) \qquad dv = e^x dx$

$du = -3\sin(3x)dx \qquad v = e^x$

So $I = e^x\cos(3x) + 3\int e^x \sin(3x)\,dx$

By parts again:

$u' = \sin(3x) \qquad dv' = e^x\,dx$

$du' = 3\cos(3x)\,dx \qquad v' = e^x$

So: $I = e^x\cos(3x) + 3\left[e^x\sin(3x) - 3\int e^x\cos(3x)dx\right]$

$I = e^x\cos(3x) + 3e^x\sin(3x) - 9I$

$10I = e^x\cos(3x) + 3e^x\sin(3x)$

$$\boxed{I = \frac{1}{10}e^x\cos(3x) + \frac{3}{10}e^x\sin(3x) + C}$$

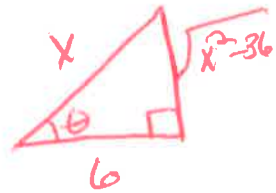
6. (16 points) Evaluate the integral: $\int \frac{1}{(x^2-36)^{3/2}}\,dx$.

Let $x = 6\sec\theta$, $dx = 6\sec\theta\tan\theta\, d\theta$

and $x^2-36 = 36\sec^2\theta - 36 = 36\tan^2\theta$

$$\int \frac{dx}{(x^2-36)^{3/2}} = \int \frac{6\sec\theta\tan\theta}{(36\tan^2\theta)^{3/2}}\,d\theta = \frac{6}{6^3}\int\frac{\sec\theta\tan\theta}{\tan^3\theta}\,d\theta$$

$$= \frac{1}{36}\int\frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{36}\int\frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \frac{1}{36}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta$$



$u=\sin\theta$, $du = \cos\theta\, d\theta$

$$=\frac{1}{36}\int\frac{du}{u^2} = -\frac{1}{36u}+C = -\frac{1}{36}\cdot\csc\theta + C$$

$$= \boxed{-\frac{1}{36}\cdot\frac{x}{\sqrt{x^2-36}}+C}$$

BONUS: (5 points) If a population grows/decays exponentially, the rate of growth is proportional to the population. Write and solve a differential equation to derive the generic formula for growth and decay.

See Form 1.

MATH 1552 TEST 2, FALL 2015, GRODZINSKY

Print Your Name: Key- 4

T.A.: (circle one) Miheer Brandon Stephen Kabir

1. (20 points) Find a general solution to the differential equation:

$$(x^2 - 6x + 5)\frac{dy}{dx} = y(x + 3).$$

$$\int \frac{1}{y}\,dy = \int \frac{x+3}{x^2-6x+5}\,dx$$

$$\frac{x+3}{x^2-6x+5} = \frac{A}{x-5} + \frac{B}{x-1}$$

$$x+3 = A(x-1) + B(x-5)$$

$x=1$: $4 = B(-4)$, $B=-1$

$x=5$: $8 = A(4)$, $A=2$

$$\ln|y| = \int \left[\frac{2}{x-5} - \frac{1}{x-1}\right] dx$$

$$\ln|y| = 2\ln|x-5| - \ln|x-1| + C$$

$$|y| = e^{2\ln|x-5| - \ln|x-1| + C}$$

$$|y| = e^C \cdot \left|\frac{(x-5)^2}{x-1}\right| \qquad \text{let } k = \pm e^C$$

$$\boxed{y = \frac{k(x-5)^2}{x-1}}$$

2. (16 points) Evaluate the integral: $\int \sin^3(x)\cos^4(x)dx$.

$$= \int \sin^2 x \cos^4 x \sin x \, dx$$

$$= \int (1-\cos^2 x)\cos^4 x \sin x \, dx$$

$$= \int (\cos^4 x - \cos^6 x) \sin x \, dx$$

$u = \cos x$, $du = -\sin x \, dx$

$$= -\int (u^4 - u^6)\, du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C$$

$$= \boxed{\frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C}$$

3. (16 points) Evaluate the integral:

$$\int \frac{2x}{x^2+2x+10}dx. = \int \frac{2x+2-2}{x^2+2x+1+9}dx$$

$$= \int \frac{2x+2}{x^2+2x+10}dx - \int \frac{2}{(x+1)^2+9}dx$$

$$= \int \frac{2x+2}{x^2+2x+10}dx - \frac{2}{9}\int \frac{dx}{\left(\frac{x+1}{3}\right)^2+1}$$

$$= \ln|x^2+2x+10| - \frac{2}{9}\cdot 3\tan^{-1}\left(\frac{x+1}{3}\right) + C$$

$$= \boxed{\ln|x^2+2x+10| - \frac{2}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + C}$$

MATH 1552 TEST 2, FALL 2015, GRODZINSKY

Print Your Name: Key-4

T.A.: (circle one) Miheer Brandon Stephen Kabir

4. (16 points) Find the limit: $\lim_{x\to 0}\left[\ln(1-\cos(8x)) - \ln(5x^2)\right]$.

$$= \lim_{x\to 0} \ln\left(\frac{1-\cos(8x)}{5x^2}\right)$$

$$= \ln\left(\lim_{x\to 0} \frac{1-\cos(8x)}{5x^2}\right) \quad \left[\frac{0}{0}\right]$$

$$\overset{L'H}{=} \ln\left(\lim_{x\to 0} \frac{8\sin(8x)}{10x}\right) \quad \left[\frac{0}{0}\right]$$

$$\overset{L'H}{=} \ln\left(\lim_{x\to 0} \frac{64\cos(8x)}{10}\right) = \boxed{\ln(6.4)}$$

5. (16 points) Evaluate the integral: $\int \frac{1}{(x^2-9)^{3/2}}\,dx$.

Let $x = 3\sec\theta$, then $dx = 3\sec\theta\tan\theta\,d\theta$

and $x^2-9 = 9\sec^2\theta - 9 = 9\tan^2\theta$.

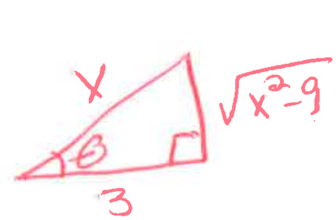
$$\int \frac{dx}{(x^2-9)^{3/2}} = \int \frac{3\sec\theta\tan\theta}{(9\tan^2\theta)^{3/2}}\,d\theta = \frac{3}{3^3}\int \frac{\sec\theta\tan\theta}{\tan^3\theta}\,d\theta$$

$$= \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \frac{1}{9}\int \frac{1}{u^2}\,du = -\frac{1}{9u} + C$$



$$= -\frac{1}{9}\csc\theta + C$$

$$= \boxed{-\frac{1}{9}\frac{x}{\sqrt{x^2-9}} + C}$$

6. (16 points) Evaluate the integral: $\int e^x \cos(6x)dx. = I$.

By parts: $u = \cos(6x)$ $\quad dv = e^x dx$

$du = -6\sin(6x)dx$ $\quad v = e^x$

Then: $I = e^x\cos(6x) + 6\int e^x \sin(6x)dx$

By parts again: $u = \sin(6x)$ $\quad dv = e^x dx$

$du = 6\cos(6x)dx$ $\quad v = e^x$

$$I = e^x\cos(6x) + 6\left(e^x\sin(6x) - 6\int e^x\cos(6x)\right)$$

$$I = e^x\cos(6x) + 6e^x\sin(6x) - 36I$$

$$37I = e^x\cos(6x) + 6e^x\sin(6x)$$

$$\boxed{I = \frac{1}{37}e^x\cos(6x) + \frac{6}{37}e^x\sin(6x) + C}$$

BONUS: (5 points) If a population grows/decays exponentially, the rate of growth is proportional to the population. Write and solve a differential equation to derive the generic formula for growth and decay.

See Form 1.