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Solutions to Homework 7

1. In the solutions below we use the following formulas: for two events A and B

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \bigcap B)}{\mathbb{P}(B)}$$

and for two random variables Y and Z and a constant k

$$\mathbb{E}[Y \mid Z = k] = \sum_{n=0}^{\infty} n\mathbb{P}\{Y = n \mid Z = k\}$$

and

$$\mathbb{E}[Y] = \sum_{k=0}^{\infty} \mathbb{E}[Y \mid Z = k]\mathbb{P}\{Z = k\}.$$

First, as computed in Problem 1(a) of Assignment 6,

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/6 & 0 & 0 & 5/6 \\ 3/6 & 1/6 & 0 & 2/6 \\ 2/6 & 3/6 & 1/6 & 0 \end{bmatrix} \qquad \Longrightarrow \qquad P^2 = \begin{bmatrix} 1/3 & 1/2 & 1/6 & 0 \\ 5/18 & 5/12 & 5/36 & 1/6 \\ 5/36 & 1/6 & 1/18 & 23/36 \\ 1/6 & 1/36 & 0 & 29/36 \end{bmatrix}.$$

(a) $\mathbb{P}\{X_2 = 6 \mid X_0 = 5\} = P^2_{3,4} = 23/36$,

(b) $\mathbb{P}\{X_2 = 5, X_3 = 4, X_5 = 6 \mid X_0 = 3\} = P^2_{1,3}P_{3,2}P^2_{2,4} = 1/216$,

(c)

$$\begin{aligned} \mathbb{E}[X_2 \mid X_0 = 6] &= 3\mathbb{P}\{X_2 = 3 \mid X_0 = 6\} + 4\mathbb{P}\{X_2 = 4 \mid X_0 = 6\} \\ &\quad +5\mathbb{P}\{X_2 = 5 \mid X_0 = 6\} + 6\mathbb{P}\{X_2 = 6 \mid X_0 = 6\} \\ &= 3P^2_{4,1} + 4P^2_{4,2} + 5P^2_{4,3} + 6P^2_{4,4} = 49/9, \end{aligned}$$

(d) If we know that $\alpha = (0, 0, 0.5, 0.5)$, then

$$\mathbb{P}\{X_2 = 6\} = \alpha_1 P^2_{1,4} + \alpha_2 P^2_{2,4} + \alpha_3 P^2_{3,4} + \alpha_4 P^2_{4,4} = 52/72 = 13/18,$$

(e)

$$\begin{aligned} \mathbb{P}\{X_4 = 3, X_1 = 5 \mid X_2 = 6\} &= \frac{\mathbb{P}\{X_4 = 3, X_1 = 5, X_2 = 6\}}{\mathbb{P}\{X_2 = 6\}} \\ &= \frac{\mathbb{P}\{X_4 = 3, X_2 = 6 \mid X_1 = 5\}\mathbb{P}\{X_1 = 5\}}{\mathbb{P}\{X_2 = 6\}} \\ &= \frac{P_{3,4}P^2_{4,1}\mathbb{P}\{X_1 = 5\}}{\mathbb{P}\{X_2 = 6\}} \end{aligned}$$

where

$$\mathbb{P}\{X_1 = 5\} = \alpha_1 P_{1,3} + \alpha_2 P_{2,3} + \alpha_3 P_{3,3} + \alpha_4 P_{4,3} = 1/12$$

an $\mathbb{P}\{X_2 = 6\}$ has been calculated in the previous problem. The finial ansewer is 1/156.

2. (a) In order to show that $\{X_n, n \geq 1\}$ is a discrete-time Markov chain it is enough to show that one step transition probabilities describe the evolution of the system.

First, the state space is $\mathcal{S} = \{1, 2, \ldots, 6\}$. (The state space has nothing to do with the fact that process is Markov or not.)

We have for $k_n > k_{n+1}$

$$\mathbb{P}\{X_{n+1} = k_{n+1} \mid X_n = k_n\} = 0$$

for all possible values of k_n.

If $k_n = k_{n+1} \in \mathcal{S} = \{1, 2, \ldots, 6\}$, then

$$\mathbb{P}\{X_{n+1} = k_{n+1} \mid X_n = k_n\} = k_n/6.$$

If $k_n < k_{n+1} \in \mathcal{S} = \{2, \ldots, 5, 6\}$, then

$$\mathbb{P}\{X_{n+1} = k_{n+1} \mid X_n = k_n\} = 1/6.$$

Since one step transition probabilities describe the evolution of the system, $\{X_n : n \geq 1\}$ is a Markov chain with the transition matrix

$$P = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{6}{6} \end{bmatrix}$$

Therefore, the pmf of X_1 is $\mathbb{P}\{X_1 = i\} = 1/6$ for $i \in \{1, 2, \ldots, 6\}$.

3. The state space is $\mathcal{S} = \{0, 1, 2, \ldots\}$. (The state space is unbounded, so has infinitely many elements.)

For $0 \leq k_n \leq n$,

$$\mathbb{P}\{Y_{n+1} = k_n + 1 \mid Y_n = k_n\} = 1/6 \text{ and}$$
$$\mathbb{P}\{Y_{n+1} = k_n \mid Y_n = k_n\} = 5/6.$$

Since one step transition probabilities describe the evolution of the system, $\{Y_n : n \geq 1\}$ is a Markov chain.

And we have $\mathbb{P}(Y_1 = 0) = 5/6$, $\mathbb{P}(Y_1 = 1) = 1/6$ and $\mathbb{P}(Y_1 = k) = 0$ for all other k values. Note: $\mathbb{P}(Y_1 = k)$ can be interpreted as $\mathbb{P}(Y_1 = k | Y_0 = 0)$ since before $1st$ roll, we have 0 sixes.