# ISyE 3044 — Final Exam

**Fall 2013**

NAME: SOLUTIONS

| | Deductions |
|---|---|
| Problem 1: | |
| Problem 2: | |
| Problem 3: | |
| Problem 4: | |
| Problem 5: | |
| Problem 6: | |
| **SCORE:** | |

The test is designed for 90 minutes. No books and notes are allowed. You can use only the supplied formula sheet and tables, as well as a scientific calculator with single-variable statistical functions. *Show all your work to receive credit.* Signing your name implies strict compliance with the institute's honor code.

1. **[6 points]** Consider the density $f(x) = p(1-x)^{p-1}$, $0 < x < 1$, where $p > 0$ is an unknown (shape) parameter.

   (a) Derive the maximum likelihood estimator of $p$ based on a sample $X_1, X_2, \ldots, X_n$.

   ANSWER: $\hat{p} = -n \Big/ \sum_{i=1}^{n} \ln(1-X_i)$

$$L(p) = \prod_{i=1}^{n} p(1-x_i)^{p-1} = p^n \left[ \prod_{i=1}^{n} (1-x_i) \right]^{p-1}$$

$$\ell(p) = n \ln p + (p-1) \sum_{i=1}^{n} \ln(1-x_i)$$

$$\ell'(p) = \frac{n}{p} + \sum_{i=1}^{n} \ln(1-X_i) = 0 \iff p = -\frac{n}{\sum_{i=1}^{n} \ln(1-X_i)}$$

$$\ell''(p) = -\frac{n}{p^2} < 0.$$

(b) Consider the following observations: .03, .64 .34, .25, .53
Use the Kolmogorov-Smirnov test with type I error .15 to assess the fit of the model with $p = 2$.

ANSWER: Failure to reject hypothesis

$F(x) = 1-(1-x)^2$

| $X_{(i)}$ | .03 | .25 | .34 | .53 | .64 |
|---|---|---|---|---|---|
| $F(X_{(i)})$ | .059 | .438 | .564 | .779 | .870 |
| $\frac{i}{5}-F(X_{(i)})$ | .141 | — | .036 | .021 | .130 |
| $F(X_{(i)})-\frac{i-1}{5}$ | .059 | .238 | .164 | .179 | .070 |

$D_5 = \max\{.141, .238\} = .238$

Adjusted test statistic $= \left(\sqrt{5} + .12 + \frac{.11}{\sqrt{5}}\right)(.238) = .571$

Since the critical value at the 5% level is 1.138, we fail to reject the null hypothesis.

2. **[4 points]** The random variable $X$ has the density function

$$f(x) = \begin{cases} e^{2x} & x < 0 \\ e^{-2x} & x \geq 0. \end{cases}$$

(a) Compute the cumulative distribution function of $X$.

$$F(x) = \begin{cases} \frac{1}{2} e^{2x} & x < 0 \\ 1 - \frac{1}{2} e^{-2x} & x \geq 0 \end{cases}$$

ANSWER: ______________________________

(b) Use the inverse-transform method to obtain formulas for generating realizations of $X$.

$$X = \begin{cases} \frac{1}{2} \ln(2U) & 0 \leq U < \frac{1}{2} \\ -\frac{1}{2} \ln[2(1-U)] & \frac{1}{2} \leq U \leq 1 \end{cases}$$

ANSWER: ______________________________

Solve $F(X) = U$ for $X$.

4. **[5 points]** Consider the following list pseudo-random numbers (read from left to right): .13, .06, .25, .34, .89, .72, .61, .99, .58, .15, .64, .37

(a) Use the first two values to generate two independent realizations from the standard normal distribution. $Z_1 = [-2\ln(.13)]^{1/2} \cos[2\pi(.06)] = 1.88$

ANSWER: $Z_2 = [-2\ln(.13)]^{1/2} \sin[2\pi(.06)] = 0.74$

(b) Generate two realizations from the normal distribution with mean $-1$ and variance 4.

ANSWER: $-1 + 2(1.88) = 2.76$ ; $-1 + 2(0.74) = 0.48$

(c) Let's use the first two numbers from the list. Name the distribution of the realization

$$\left\lceil \frac{\ln(.87)}{\ln(.75)} \right\rceil + \left\lceil \frac{\ln(.94)}{\ln(.75)} \right\rceil$$

and specify the parameters.

ANSWER: negative binomial with $r = 2$ and $p = 0.25$

(d) Use the third number from the list at the top and an approximation method to generate a realization from the standard normal distribution.

ANSWER: $[.25^{.135} - .75^{.135}] / .1975 = -0.67$

(e) Use the standard normal sample from part (d) to generate a realization from the Poisson distribution with mean 36.

ANSWER: $\lceil 36 + (-0.67)\sqrt{36} - 0.5 \rceil = 32$

5. **[6 points]** A simulation project was used to compare two configurations for a production line by means of the mean monthly cost (in thousands of dollars). Ten independent replications for each configuration produced the following monthly averages ($Y_{ij}$ is the average from replication $j$ for configuration $i$):

| Run | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Config. 1 ($Y_{1j}$) | 52 | 42 | 50.1 | 49.9 | 42.5 | 49.6 | 45.5 | 46.9 | 55.5 | 44.5 |
| Config. 2 ($Y_{2j}$) | 51.3 | 35.3 | 48.2 | 47.8 | 36 | 47.4 | 40.8 | 43.1 | 56.8 | 39.2 |

(a) The experiments used common random numbers (CRN) in an attempt to induce (*circle the correct term*)

positive negative

correlation between the $Y_{1j}$ and $Y_{2j}$. Evaluate the effectiveness of CRN by computing the following estimate for this correlation:

$$\frac{\sum_{j=1}^{10}(Y_{1j}-\bar{Y}_1)(Y_{2j}-\bar{Y}_2)}{9S_1S_2} = \frac{\sum_{j=1}^{10} Y_{1j}Y_{2j} - 10\bar{Y}_1\bar{Y}_2}{9S_1S_2},$$

where $\bar{Y}_1$ and $\bar{Y}_2$ are the sample means of the $Y_{1j}$ and $Y_{2j}$, and $S_1$ and $S_2$ are the sample standard deviations of the $Y_{1j}$ and $Y_{2j}$, respectively.

ANSWER: estimate $= 0.99998 \approx 1$ $\rightarrow$ very effective!

(b) Conduct a paired-$t$ test with type I error .05 to assess the equality of the mean monthly costs under the two configurations.

ANSWER: There is strong evidence that $\mu_1 > \mu_2$.

$D_j = Y_{1j} - Y_{2j}$

$\bar{D} = 3.26$

$S = 2.59$

$t_{9,.025} = 2.26$

half length of 95% CI $= 2.26 \frac{2.59}{\sqrt{10}} = 1.85$

CI : $3.26 \pm 1.85 = (1.41, 5.11)$

6. **[5 points]** Short Questions.

(a) The "traditional" negative binomial distribution models the sum of $r$ i.i.d. geometric random variables; therefore $r$ is a nonnegative integer. There is a negative binomial distribution with a non-integer parameter $r$.

True False

(b) The Kolmogorov-Smirnov test can be used to assess the fit of discrete distributions.

True False

(c) Use the uniform random number .20 to generate a realization (sample) from the geometric distribution that counts the number of failures before the first success in sequence of Bernoulli($p = 0.75$) trials.

$$\left\lceil \frac{\ln(1-.20)}{\ln(1-.75)} \right\rceil - 1 = 0$$

ANSWER: 0

(d) If a simulation's run length is substantially long, Simio can report confidence intervals for a steady-state means (e.g., the mean size of a buffer). What method is it using?

ANSWER: batch means

(e) The sequential procedure of Kim and Nelson (2001), implemented in Simio, uses an indifference zone $\delta > 0$ and an error $\alpha \in (0, 1)$ to compare the means $\mu_i$ of $m$ systems. Suppose that smaller is better and that (unknown to us) $\mu_1 < \mu_2 < \cdots < \mu_m$. The procedure makes the following guarantee: the probability of selecting system 1 as the "best" when $\mu_2 - \mu_1 \geq \delta$ is $\geq 1 - \alpha$.

True False