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ISyE 3044 — Fall 2013 — Test #2

This test is 85 minutes. Everything is 3 points. You are allowed two cheat sheets. *Only turn in this answer page with neat, succinct answers.* Good Luck!

1. How would you simulate a Binomial$(3, 3/4)$ random variable in Arena?

   Solution: We have $\Pr(X = k) = \binom{3}{k}(3/4)^k(1/4)^{3-k}$, so that the Arena call is `DISC(0.0156, 0, 0.1562, 1, 0.5781, 2, 1, 3).` $\diamondsuit$

2. To get on a transporter in Arena, you `REQUEST` it. How do you leave the transporter?

   Solution: `FREE`. $\diamondsuit$

3. TRUE or FALSE? In Arena, it is possible to define a *set of sequences* (entity paths).

   Solution: TRUE. $\diamondsuit$

4. Consider an M/M/1/1 queue with arrival rate $\lambda = 2$/hr and service rate $\mu = 4$/hr. If the system is empty at time 0, what is the probability that there will be no people in the system at time 1 hr?

   Solution: At time $t = 1$, we have

$$P_0(t) \;=\; \frac{\mu}{\lambda+\mu} + \left[P_0(0) - \frac{\mu}{\lambda+\mu}\right]e^{-(\lambda+\mu)t} \;=\; \frac{4}{6} + \left[1 - \frac{4}{6}\right]e^{-6} \;=\; 0.6675. \quad \diamondsuit$$

5. Consider an M/M/1/1 queue with arrival rate $\lambda = 2$/hr and service rate $\mu = 4$/hr. What is the steady-state probability that there will be no people in the system?

   Solution: Take limit as $t \to \infty$ in the previous answer to get $P_0 = 2/3$. $\diamondsuit$

6. Consider an M/M/1 queue with arrival rate $\lambda = 2$/hr and service rate $\mu = 4$/hr. What is the steady-state probability that the system will not be empty?

   Solution: $\rho = 1/2$. $\diamondsuit$

7. State any form of Little's Law.

   Solution: $L = \lambda w$. $\diamondsuit$

8. Consider an M/M/4 queue with an arrival rate of 20 customers an hour. What is the smallest service rate that is required for this system to be stable?

   Solution: Must have $\rho = \lambda/(c\mu) = 5/\mu < 1$, so that $\mu > 5$. $\diamondsuit$

9. TRUE or FALSE? The M/D/1 queue is a special case of the M/G/1.

   Solution: TRUE. $\diamondsuit$

10. TRUE or FALSE? The effective arrival rate is always less than λ for an M/M/1/N queue.

    Solution: TRUE. $\diamondsuit$

11. Name Pollaczek's friend, i.e., the guy who also has his name associated with the famous M/G/1 steady-state equations.

    Solution: Khintchine. $\diamondsuit$

12. Customers arrive at Space Mountain in Disneyworld at the rate of 100/hr according to a Poisson process. They form one long FIFO line and are served according to an exponential distribution at the rate of 150/hr. Assume that interarrivals and services are all independent. Find the steady-state expected waiting time.

Solution: This is an M/M/1 system for which $\rho = \lambda/\mu = 2/3$. Therefore,

$$L_Q \;=\; \frac{\rho^2}{1-\rho} \;=\; 1.333.$$

Thus,

$$w_Q \;=\; L_Q/\lambda \;=\; 0.01333 \text{ hrs} \;=\; 48 \text{ sec.} \quad \diamondsuit$$

13. Continuing with Problem 12, now suppose that Disney has decided to use *two* parallel servers, each of whom can work at the rate of 75/hr. Again, assume services are exponential. Find the steady-state expected waiting time for this alternative system.

Solution: This is an M/M/2 with $\rho = \lambda/(2\mu) = 2/3$, as in the previous part of the problem. Then we have

$$P_0 \;=\; \left\{\left[\sum_{n=0}^{c-1}\frac{(c\rho)^n}{n!}\right] + \frac{(c\rho)^c}{(c!)(1-\rho)}\right\}^{-1} \;=\; \left\{1+\frac{4}{3}++\frac{(4/3)^2}{(2)(1/3)}\right\}^{-1} \;=\; 0.2,$$

so that

$$L_Q \;=\; \frac{(c\rho)^{c+1}P_0}{c(c!)(1-\rho)^2} \;=\; \frac{(4/3)^3(0.2)}{2(2)(1/3)^2} \;=\; \frac{16}{15} \;=\; 1.067.$$

Thus,

$$w_Q \;=\; L_Q/\lambda \;=\; 0.01067 \text{ hrs} \;\approx\; 38.41 \text{ sec.} \quad \diamondsuit$$

14. Consider the PRN generator $X_i = 16807\, X_{i-1} \bmod (2^{31}-1)$. If $X_0 = 444444$, find the PRN $R_1 = X_1/m$.

Solution: You can use my algorithm from class (or a very good calculator) to obtain $X_1 = 1027319367$, and hence $R_1 = 0.478$. $\diamondsuit$

15. Consider the following 24 PRN's.

0.06	0.16	0.36	0.69	0.76	0.85	0.92	0.47
0.10	0.18	0.29	0.38	0.91	0.62	0.41	0.30
0.11	0.45	0.72	0.28	0.31	0.48	0.57	0.72

How many runs up and down are there in this sequence?

Solution: We have

$$\begin{array}{cccccccc} & + & + & + & + & + & + & - \\ - & + & + & + & + & - & - & - \\ - & + & + & - & + & + & + & + \end{array}$$

This immediately yields $A = 7$ runs. $\diamondsuit$

16. Consider the set-up from Question 15. What is the approximate distribution of the number of runs up and down?

    Solution: We have $A \approx \text{Nor}(\frac{2n-1}{3}, \frac{16n-29}{90}) \sim \text{Nor}(15.67, 3.94)$. $\diamondsuit$

17. With your answers to Questions 15 and 16 in mind, do you *reject* or *fail to reject* the hypothesis that the PRN's are independent? Use level $\alpha = 0.05$.

    Solution: The test statistic is $Z_0 = (A - \mathsf{E}[A])/\sqrt{\mathsf{Var}(A)} = -4.37$. Since $|Z_0| > 1.96$, we reject. $\diamondsuit$

18. Suppose the random variable X has p.d.f. $f(x) = 3x^2/2$ for $-1 \le x \le 1$. Find the inverse of its c.d.f., i.e., $F^{-1}(U)$.

    Solution: The c.d.f. is

    $$F(x) = \int_{-1}^{x} \frac{3t^2}{2}\,dt = \frac{x^3+1}{2}.$$

    Setting $F(X) = (X^3 + 1)/2 = U$, we obtain $X = (2U - 1)^{1/3}$. $\diamondsuit$

19. If X is standard normal, use the inverse transform method with $U = 0.25$ to generate a realization of X.

    Solution: $X = \Phi^{-1}(0.25) = -0.675$. $\diamondsuit$

20. Suppose that $U_1 = 0.8$ and $U_2 = 0.5$ are realizations of two i.i.d. Unif(0,1)'s. Use the Box–Muller method to generate two i.i.d. standard normals.

Solution: $Z_1 = \sqrt{-2\ell\text{n}(U_1)}\cos(2\pi U_2) = -0.668$ and $Z_2 = \sqrt{-2\ell\text{n}(U_1)}\sin(2\pi U_2) = 0$. (Other answers are possible.) $\diamondsuit$

21. Use our desert island generator along with the first 12 PRN's from Question 15 to produce a Nor(0,1) realization.

    Solution: $\sum_{i=1}^{12} U_i - 6 = -0.78$. $\diamondsuit$

22. Let's play **Name That Distribution** (with parameters)! Assume that $U_1, U_2, \ldots$ are PRN's and $Z_1, Z_2, \ldots$ are i.i.d. standard normal deviates.

    (a) $-3U_1 + 2$.

    Solution: $\text{Unif}(-1, 2)$. $\diamondsuit$

    (b) $-3\ell\text{n}(U_1)$.

    Solution: Exp(1/3). $\diamondsuit$

    (c) $-3\ell\text{n}(U_1(1 - U_2))$.

    Solution: $\text{Erlang}_2(1/3)$. $\diamondsuit$

    (d) $\lceil \ell\text{n}(U_1)/\ell\text{n}(0.6) \rceil$.

    Solution: Geom(0.4). $\diamondsuit$

    (e) $U_1 + U_2$.

    Solution: Tria(0,1,2). $\diamondsuit$

    (f) $-3\Phi^{-1}(U_1) + 2$, where $\Phi(\cdot)$ is the standard normal c.d.f.

    Solution: $\text{Nor}(2, 9)$. $\diamondsuit$

(g) $\tan(2\pi U_1)$.

Solution: From class notes involving Box–Muller, we know that this is Cauchy (or Nor(0,1)/Nor(0,1) or $t(1)$, etc.). $\diamondsuit$

(h) $Z_1^2 + Z_2^2 + Z_3^2$.

Solution: $\chi^2(3)$. $\diamondsuit$

(i) Z_1/Z_2.

Solution: Cauchy. $\diamondsuit$

23. Suppose we have on hand PRN's $U_1 = 0.83$, $U_2 = 0.03$, $U_3 = 0.92$, $U_4 = 0.27$, and $U_5 = 0.06$. Generate a realization of $X \sim \text{Pois}(\lambda = 3.5)$ via the acceptance-rejection method we did in class. (You may not need to use all of the PRN's.)

    Solution: The algorithm says to continue sampling until $e^{-\lambda} = 0.0302 \geq \prod_{i=1}^{n+1} U_i$.

    For $n = 0$, we have $U_1 = 0.83$, so we don't stop.

    For $n = 1$, we have $U_1 U_2 = 0.0249 \leq 0.0302$, so we stop and take $X = 1$. $\diamondsuit$

24. If $X_1, X_2, \ldots$ come from a stationary MA(1) (order 1 moving average) process with coefficient $\theta = 0.9$, what is the covariance between X_4 and X_5?

    Solution: For the MA(1) with $k = 1$, we have $\mathsf{Cov}(X_i, X_{i+1}) = \theta = 0.9$. $\diamondsuit$

25. If $\mathcal{W}(t)$ is a Brownian motion process, find the *correlation* between $\mathcal{W}(4)$ and $\mathcal{W}(9)$.

    Solution:

$$\mathsf{Corr}(\mathcal{W}(4), \mathcal{W}(9)) = \frac{\mathsf{Cov}(\mathcal{W}(4), \mathcal{W}(9))}{\sqrt{\mathsf{Var}(\mathcal{W}(4))\mathsf{Var}(\mathcal{W}(9))}} = \frac{\min(4,9)}{\sqrt{(4)(9)}} = 2/3. \quad \diamondsuit$$

26. (1 point) What is the name of the lemma that establishes the fact that the empirical c.d.f. approaches the true c.d.f. as $n \to \infty$?

Solution: Glivenko–Cantelli. $\diamondsuit$