Solutions to Homework 9

1) a) $\sum_{j=0}^{3}\sum_{i=0}^{3} c(i+j) = 1 \quad \Rightarrow 48c = 1$

$\Rightarrow c = \frac{1}{48}$

b) $P(X>Y) = P(X=1, Y=0) + P(X=2, Y=0) + P(X=3, Y=0) + P(X=2, Y=1) +$
$P(X=3, Y=1) + P(X=3, Y=2)$
$= \frac{1}{48}(1+2+3+3+4+5) = \frac{18}{48} = \frac{3}{8}$

2) a) $c\int_0^1\int_0^1 (x+y)\,dx\,dy = 1$

$c\int_0^1 \left(\frac{x^2}{2}+yx\right)\Big|_0^1 dy = 1$

$\Rightarrow c\int_0^1 \left(\frac{1}{2}+y\right)dy = 1 \Rightarrow c\left(\frac{1}{2}y+\frac{y^2}{2}\right)\Big|_0^1 = 1 \Rightarrow c = 1$

b) $f(x) = \int_0^1 (x+y)\,dy = xy + \frac{y^2}{2}\Big|_0^1 = x+\frac{1}{2} \qquad 0<x<1$

$P(X<1/2) = \int_0^{1/2}\left(x+\frac{1}{2}\right)dx = \frac{x^2}{2}+\frac{1}{2}x\Big|_0^{1/2} = \frac{1}{8}+\frac{1}{4} = \frac{3}{8}$

c) $F(a,b) = \begin{cases} 0 & \text{if } a<0 \text{ or } b<0 \\ \frac{a^2b}{2}+\frac{ab^2}{2} & \text{if } 0<a<1,\ 0<b<1 \\ \frac{b}{2}+\frac{b^2}{2} & \text{if } a>1,\ 0<b<1 \\ \frac{a^2}{2}+\frac{a}{2} & \text{if } 0<a<1,\ b>1 \end{cases}$

$F(a,b) = \int_0^a\int_0^b (x+y)\,dy\,dx = \int_0^a \left(xy+\frac{y^2}{2}\right)\Big|_0^b dx = \int_0^a \left(bx+\frac{b^2}{2}\right)dx = \frac{bx^2}{2}+\frac{b^2}{2}x\Big|_0^a = \frac{a^2b}{2}+\frac{ab^2}{2}$

3) a) $f(x,y) = xe^{-x(1+y)}$ cannot be factored into two functions such that one is just a function of x and the other is of y.

Thus, X and Y are dependent

b) $f(x,y) = x(2y+1)$ Hence, X and Y are independent

c) $f(x,y) = (x+y)^2 - (x-y)^2$

$= (x+y+x-y)(x+y-x+y)$

$= 2x \cdot 2y$

Hence, X and Y are independent