1. (40 points) Random variables W, X, Y, Z are jointly independent. They have uniform distributions on the interval $[1, 4]$. Let $M = \max\{W, X, Y, Z\}$. Find the pdf, expected value, and standard deviation of M. You should know that the expected value is 3.4 to help check your work.

2. (15 points each) Items are stored on the perimeter of a circular carousel which rotates clockwise at a rate of one full revolution in 30 seconds. The picker stands at a fixed location. When the carousel has rotated the item to be in front of the picker, the carousel stops for 10 seconds. While the carousel is stopped, the picker removes the item. After the 10 second stop, the carousel starts to move again. 100 items are to be pulled from storage. If each item is in a random location (uniformly random) independent of other item locations, what is the expected amount of time required?

   You could buy a more expensive carousel which operates in the same way, but which can rotate either clockwise or counterclockwise. What would be the expected amount of time required to pull 100 items from storage?

   Instead of buying a more expensive carousel, you could pair the items (1st with 2nd, 3rd with 4th, etc.) and pick each pair in the best order. What would be the expected amount of time required?

   What would be the expected amount of time required if you used the pairing strategy with the more expensive carousel?

ISyE 2027C **Formulae for Quiz on Homework 8** **Spring 2015**

Formulas: $n!$ is the number of ways to arrange n items in a sequence. 1!=1 and $n! = n(n-1)!$. $\binom{n}{k}$ is the number of ways to pick k items out of n, when the order of the items does not matter. It equals $\frac{n!}{k!(n-k)!}$. A^C is the complement of A, the set of all things not in A. $P(A) + P(A^C) = 1$ for all A. ϕ denotes the empty set Ω^C. $(A \cup B)^C = A^C \cap B^C$. $(A \cap B)^C = A^C \cup B^C$.
If A and B are disjoint, $P(A \cup B) = P(A) + P(B)$.
If A and B are independent, $P(A \cap B) = P(A)P(B)$. In general $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
$0 \leq P(A) \leq 1$ for all A. $P(A|B) = P(A \cap B)/P(B) = P(B|A)P(A)/P(B)$ is the conditional probability of A given B.
The law of total probability is $P(A) = P(B)P(A|B) + P(B^C)P(A|B^C)$. More generally, if $E_1 \ldots E_n$ partition Ω then $P(A) = \sum_{i=1}^{n} P(E_i)P(A|E_i)$.
The law of total probability for expectation is $E[X] = P(A)E[X|A] + P(A^C)E[X|A^C]$. More generally, if $H_1 \ldots H_n$ partition Ω then $E[X] = \sum_{i=1}^{n} P(H_i)E[X|H_i]$.
If X is a discrete random variable, $E[X] = \sum_t tP(X = t)$, the weighted average of the values X can take.
If X is a continuous random variable with density function $f(t)$, then $\int_{-\infty}^{\infty} f(t)dt$ must equal 1. Then the cdf of X is $F(t) = \int_{-\infty}^{t} f(t)\ dt = P(X \leq t)$. Also, $E[X] = \int_{-\infty}^{\infty} tf(t)dt$ and LOTUS says that for any function g, $E[g(X)] = \int_{-\infty}^{\infty} g(t)f(t)dt$. LOTIS says $E[g(X)] = g(E[X])$ and is usually wrong.
Expectation is linear. This means that for any random variables X and Y and real number α, $E[\alpha X + Y] = \alpha E[X] + E[Y]$.
A Bernoulli variable with parameter p equals 1 w.p. p and equals 0 w.p. $1 - p$. If X is Bernoulli then $E[X] = p$.
The sum of n independent Bernoullis each with parameter p has binomial distribution $B(n, p)$. If $X \sim B(n, p)$ then $P(X = k) = \binom{n}{k}p^k(1-p)^{n-k}$ and $E[X] = np$.
Let Y be the number of times you flip a coin that has probability p of being heads, until you get your first head. Then Y has geometric distribution with parameter p. $P(Y = k) = p(1-p)^{k-1}$ and $E[Y] = 1/p$. The exponential distribution with mean $1/\lambda$ is defined as $P(\leq t) = 1 - e^{-\lambda t}$ for all $t \geq 0$. . These are the unique memoryless discrete and continuous distributions, respectively, meaning that $P(X \geq \alpha + \beta | X \geq \alpha) = P(X \geq \beta)$.
The variance $\sigma^2(X)$ of random variable X is defined to be $E[(X - E[X])^2]$. From linearity of expectation this simplifies to the more convenient $E[X^2] - (E[X])^2$. From the definition, $\sigma^2(\alpha X) = \alpha^2\sigma^2(X)$. The standard deviation of X is defined as $\sigma(X) = \sqrt{\sigma^2(X)}$. In general, variance is not additive. However, if X and Y are independent random variables, $\sigma^2(X + Y) = \sigma^2(X) + \sigma^2(Y)$. The variance of a Bernoulli variable with parameter p is $p(1-p)$. The variance of a B(n,p) distributed variable is $np(1-p)$. If X has uniform distribution on $[0, 1]$, $E[X] = .5$ and $\sigma^2(X) = 1/12$.
Chebyshev's inequality: $P(|X - E[X]| \geq k\sigma(X)) \leq 1/k^2$. The probability a random variable is k or more standard deviations from its mean is $\leq 1/k^2$. If X has a Poisson distribution with parameter λ then $P(X = k) = e^{-\lambda}\lambda^k/k!$ for all integers $k \geq 0$, and $E[X] = \lambda$. If X and Y are independent Poisson distributed variables then $X + Y$ has a Poisson distribution. A Poisson process with intensity rate r has interrarival times independently exponentially distributed each with parameter r. For any time interval of length t the number of arrivals has a Poisson distribution with parameter rt, and if time intervals are disjoint the corresponding Poisson variables are independent. If you lump together two independent Poisson processes with rates r_1 and r_2, you get a Poisson process with rate $r_1 + r_2$. If you split a Poisson process with rate r_1 by labelling each arrival red with independent probability p, and otherwise labeling it blue, you get two Poisson process with rates pr_1 and $(1-p)r_1$. The expected value of the kth smallest of n independent $U[0, 1]$ variables is $\frac{k}{n+1}$.