ISYE 3232A Spring 2015 Test 2 Keys - A

I, ________________________, do swear that I abide by the Georgia Tech Honor Code. I understand that any honor code violations will result in a failure (an F).

Signature: ____________________

- You will have 1 hour 15 minutes.
- This quiz is closed book and closed notes. Calculators are not allowed. No scrap paper is allowed. Make sure that there is nothing on your desk except pens and erasers.
- If you need extra space, use the back of the page and indicate that you have done so.
- **Do not remove any page from the original staple**. Otherwise, there will be 5 points off.
- **Show your work on the test sheet**. If you do not show your work for a problem, we will give zero point for the problem even if your answer is correct.
- **We will not select among several answers**. Make sure it is clear what part of your work you want graded. If two answers are given, zero point will be given for the problem.
- Throughout, you will receive full credit (i) if the work is correct and (ii) if someone with no understanding of probability, set theory, and calculus could simplify your answer to obtain the correct numerical answer. **However, you must give a numerical answer where asked**.

1. $\underline{P}^{(n)} = \underline{P}^n$ and $\underline{a}^{(n)} = \underline{a}^{(0)}\underline{P}$.
2. Stationary distribution $\underline{\pi}$ is the solution to $\underline{\pi} = \underline{\pi}\,\underline{P}$ and $\sum_{i\in S}\pi_i = 1$.
3. Let X be an irreducible DTMC.
   (a) X is positive recurrent iff it has a (unique) stationary distribution $\underline{\pi}$.
   (b) If X is irreducible and positive, we know that it has a unique $\underline{\pi}$. Moreover, if X is aperiodic, then
      i. $\underline{P}^{(\infty)}$ exists; and
      ii. $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ for any $i \in S$.
   (c) If a unique $\underline{\pi}$ exists, then $\mathsf{E}[\#\text{ steps to come back to state } i \text{ for the 1st time} \mid X_0 = i] = \frac{1}{\pi_i}$.
4. Let X be an irreducible finite-state DTMC.
   (a) X is positive recurrent.
   (b) X has a unique $\underline{\pi}$.
   (c) If X is aperiodic, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ independent of starting state i.
   (d) If X is periodic with period $d \geq 2$,
   $$\pi_j = \lim_{n\to\infty} \frac{P_{ij}^{(n)} + P_{ij}^{(n+1)} + \ldots + P_{ij}^{(n+d-1)}}{d}.$$
5. Let X be an irreducible infinite-state DTMC. If a unique $\underline{\pi}$ exists, then it is positive recurrent. Otherwise, it is either recurrent or transient.

1. (15 points) A store operates from Monday to Friday. Inventory left at the end of each Friday can be used to satisfy the demand in the following week. Let $D_n \equiv$ iid demand in the nth period (weeks) and its distribution is given as follows:

d	0	1	2	3
$P(D_n = d)$	0.2	0.4	0.3	0.1

Currently there are 2 items in stock and an (s, S) inventory policy is employed with $s = 1$ and $S = 3$. So if the inventory at the end of each Friday is 1 or fewer, then an order is placed and the order arrives before the beginning of next Monday. Let X_n represent the inventory level at the end of Friday in the nth week.

X_n (at the end of week n)	inventory at the beginning of week $n+1$	D_{n+1}	X_{n+1} (at the end of week $n+1$)
3	3	0	3
		1	2
		2	1
		3	0
2	2	0	2
		1	1
		2	0
		3	0
1	3	0	3
		1	2
		2	1
		3	0
0	3	0	3
		1	2
		2	1
		3	0

(a) (5 points) Give state space S.

$$S = \{0, 1, 2, 3\}.$$

(b) (2 points) Give initial distribution $\underline{a}^{(0)}$.

$$\underline{a}^{(0)} = [0 \quad 0 \quad 1 \quad 0].$$

(c) (8 points) Give transition matrix $\underline{P}$.

$$\underline{P} = \begin{bmatrix} .1 & .3 & .4 & .2 \\ .1 & .3 & .4 & .2 \\ .4 & .4 & .2 & 0 \\ .1 & .3 & .4 & .2 \end{bmatrix}.$$

2. (15 points) Let $X = \{X_n : n = 0,1,2,\ldots\}$ be a discrete time Markov chain on state space $S = \{10,20,40\}$ with $\underline{a}^{(0)} = (.1,.2,.7)$ and transition matrix

$$\underline{P} = \begin{pmatrix} .2 & .5 & .3 \\ .5 & 0 & .5 \\ .8 & .2 & 0 \end{pmatrix}.$$

(a) (5 points) Find $\Pr\{X_2 = 40, X_3 = 20 \mid X_0 = 10\}$.

$$\begin{aligned} \Pr\{X_2 = 40, X_3 = 20 \mid X_0 = 10\} &= \Pr\{X_3 = 20 \mid X_2 = 40\} \Pr\{X_2 = 40 \mid X_0 = 10\} \\ &= (0.2) \times (.2 \times .3 + .5 \times .5 + .3 \times 0). \end{aligned}$$

(b) (5 points) Find $\Pr\{X_0 = 20 \mid X_1 = 10\}$.

$$\begin{aligned} \Pr\{X_0 = 20 \mid X_1 = 10\} &= \frac{\Pr\{X_1 = 10, X_0 = 20\}}{\Pr\{X_1 = 10\}} \\ &= \frac{\Pr\{X_1 = 10 \mid X_0 = 20\} \Pr\{X_0 = 20\}}{\Pr\{X_1 = 10\}} \\ &= \frac{.5 \times .2}{.1 \times .2 + .2 \times .5 + .7 \times .8}. \end{aligned}$$

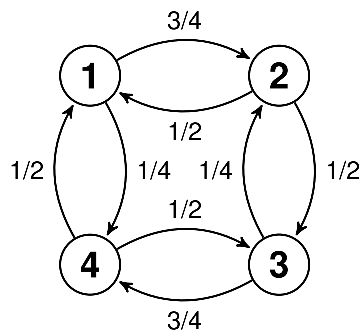
(c) (5 points) Set up equations to find stationary distributions $\underline{\pi}$. (Do not attempt to solve them. Just set them up.)

$$\begin{aligned} \pi_{10} &= .2\pi_{10} + .5\pi_{20} + .8\pi_{40} \\ \pi_{20} &= .5\pi_{10} + 0\pi_{20} + .2\pi_{40} \\ (\pi_{40} &= .3\pi_{10} + .5\pi_{20} + 0\pi_{40}) \\ \pi_{10} + \pi_{20} + \pi_{40} &= 1 \end{aligned}$$

3. (20 points, 5 points each) Let $X = \{X_n : n = 0, 1, 2, \ldots\}$ be a discrete time Markov chain on state space $S = \{1, 2, 3, 4\}$ with transition matrix

$$\underline{P} = \begin{pmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

(a) Draw a transition diagram.



(b) What is the period of each state?

Each state has period 2.

(c) Let $\underline{\pi} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Is $\underline{\pi}$ the unique stationary distribution of X? Explain your answer.

Yes. X is irreducible with a finite number of states. Thus X is irreducible and positive recurrent.

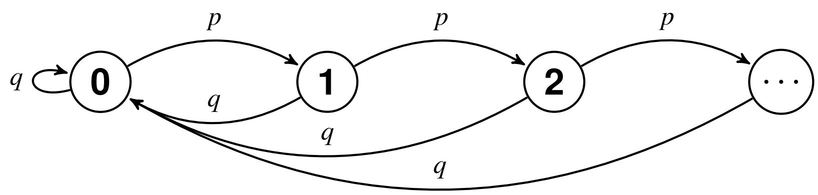
(d) Does $\lim_{n\to\infty} \Pr(X_n = 4 \mid X_0 = 2) = \frac{1}{4}$ hold? Explain your answer.

No. X is periodic. Thus the limiting distribution does not exist.

4. (5 points) Let $X = \{X_n : n = 0, 1, 2, \ldots\}$ be an irreducible discrete time Markov chain on state space $S = \{1, 2, 3, 4\}$. Let $\underline{\pi} = (\frac{1}{30}, \frac{5}{30}, \frac{9}{30}, \frac{15}{30})$. Assuming that we are currently in state 2, on average how many steps does it take to come back to state 2 for the first time in the long run? (If it takes you a long time to compute it, you are likely on a wrong track.)

$$\mathsf{E}[\tau_2 | X_0 = 2] = \frac{1}{\pi_2} = \frac{1}{5/30} = 6.$$

5. (15 points) Consider the following discrete time Markov chain.



(a) (10 points) Let $p = 0.99$ and $q = 0.01$. Calculate $\underline{\pi}$.

$$
\begin{aligned}
\pi_0 &= q\pi_0 + q\pi_1 + q\pi_2 + \cdots = q\sum_{i=0}^{\infty}\pi_i = q \\
\pi_1 &= p\pi_0 = pq \\
\pi_2 &= p\pi_1 = p^2 q \\
\pi_3 &= p\pi_2 = p^3 q \\
\vdots &\quad \vdots \quad \vdots \\
\pi_i &= p^i q
\end{aligned}
$$

Thus $\pi_i = (0.01) \times (0.99)^i$ for $i = 0, 1, 2, \ldots$.

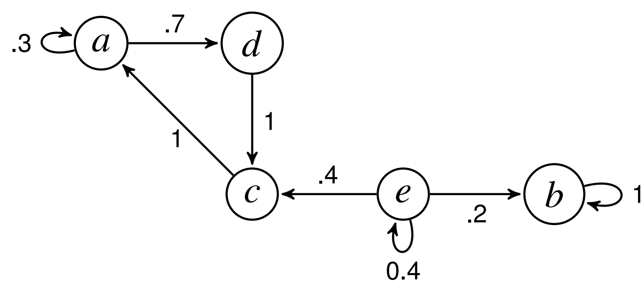
(b) (5 points) Is it positive recurrent? Why or why not?

Yes, because X is irreducible and the unique $\underline{\pi}$ exists.

6. (30 points, 5 points each) Consider a Markov chain with state space $\{a,b,c,d,e\}$ and transition matrix

$$\underline{P} = \begin{bmatrix} 0.3 & 0 & 0 & 0.7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0.2 & 0.4 & 0 & 0.4 \end{bmatrix}.$$

(a) (5 points) Draw a transition diagram.



(b) (5 points) List all the transient state(s) and irreducible set(s).
Sets $\{a,c,d\}$ and $\{b\}$ are recurrent sets while $\{e\}$ is transient.

(c) (5 points) What is the period for state c?
State c has period 1.

(d) (5 points) Find approximately $\Pr\{X_{100} = d \mid X_0 = b\}$.
It is impossible to move from b to d. Thus, the probability is 0.

(e) (5 points) Find approximately $\Pr\{X_{100} = d \mid X_0 = c\}$.
We need π_d. Note

$$\begin{aligned} \pi_a &= 0.3\pi_a + \pi_c, \\ \pi_d &= 0.7\pi_a, \\ \pi_c &= \pi_d = 0.7\pi_a, \end{aligned}$$

and

$$\pi_a + \pi_d + \pi_c = \pi_a + 0.7\pi_a + 0.7\pi_a = 2.4\pi_a = 1.$$

Thus

$$\underline{\pi} = [10/24 \quad 7/24 \quad 7/24]$$

and $\pi_d = 7/24 = 7/24$.

(f) (5 points) Find approximately $\Pr\{X_{100} = d \mid X_0 = e\}$.
First, the probability that we eventually joins $\{a,c,d\}$ from e is $\frac{0.4}{0.6} = 2/3$. Once we join set $\{a,c,d\}$, there is π_c probability that we stay in state a. Thus

$$\frac{2}{3} \times \pi_d = \frac{2}{3} \times \frac{7}{24} = \frac{7}{36}.$$

Turn to the next page for a bonus question. ⇒

Bonus: (5 points) Let X_n be the sum of the first n outcomes of tossing a six-sided die repeatedly in an independent fashion. We want to compute

$$\lim_{n\to\infty} \Pr\{X_n \text{ is divisible by } 4\}.$$

In order to compute the above probability, we will consider $Y_n = (X_n \bmod 4)$ where the mod function returns the remainder after a number is divided by a divisor. For example, $(7 \bmod 4) = 3$ and $(12 \bmod 4) = 0$. Find the transition matrix $\underline{P}$ of Y_n. (Just find $\underline{P}$. You don't need to calculate $\lim_{n\to\infty} \Pr\{X_n$ is divisible by $4\}$.)

Note that the mod function returns the remainder after a number is divided by divisor. Thus Y_n can take values from $\{0,1,2,3\}$.

Y_n	outcome of the $(n+1)$st roll	Y_{n+1}
0	1	1
	2	2
	3	3
	4	0
	5	1
	6	2
1	1	2
	2	3
	3	0
	4	1
	5	2
	6	3
2	1	3
	2	0
	3	1
	4	2
	5	3
	6	0
3	1	0
	2	1
	3	2
	4	3
	5	0
	6	1

$$\underline{P} = \begin{bmatrix} 1/6 & 2/6 & 2/6 & 1/6 \\ 1/6 & 1/6 & 2/6 & 2/6 \\ 2/6 & 1/6 & 1/6 & 2/6 \\ 2/6 & 2/6 & 1/6 & 1/6 \end{bmatrix}$$