Math 2401
Exam 1

Name: Solutions & Key

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged:

Problem	Possible	Earned
1	5	
2	5	
3	10	
4	5	
5	5	
6	10	
7	10	
Total	50	

1. (5 pts) Determine the line through the point $\vec{p} = (3, -2, 1)$ that is parallel to the line $\vec{\ell}(t) = (1 + 2t, 2 - t, 3t)$.

$$\vec{\ell}(t) = (1,2,0) + t(2,-1,3)$$

$\Rightarrow \vec{\ell}(t)$ has direction $(2,-1,3) = \vec{v}$ 2 pts

Ans: $\vec{\ell}(t) = \vec{p} + t\vec{v} = \boxed{(3,-2,1) + t(2,-1,3)}$ 2 pts

$$= (3+2t, -2-t, 1+3t)$$

1 pt for correct answer

2. (5 pts) Compute the angle between the planes $x + y = 1$ and $y + z = 1$. You may leave your answer un-simplifed.

$x+y=1$ has normal $\vec{N}_1 = (1,1,0)$ 1 pt

$y+z=1$ has normal $\vec{N}_2 = (0,1,1)$ 1 pt

$$\vec{N}_1 \cdot \vec{N}_2 = (1,1,0)\cdot(0,1,1) = 1 \quad 1\text{ pt}$$

$$\left.\begin{aligned} |\vec{N}_1| &= \sqrt{1^2+1^2+0^2} = \sqrt{2} \\ |\vec{N}_2| &= \sqrt{0^2+1^2+1^2} = \sqrt{2} \end{aligned}\right\} \ 1\text{ pt}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1||\vec{N}_2|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}\cdot\sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

1 pt

3. (10 pts) Let $\vec{v}_1 = (1, 1, -1)$, $\vec{v}_2 = (2, 0, 2)$, and $\vec{v}_3 = (0, -2, 1)$.

(a) (3 pts) Compute $\vec{u}_1 = \vec{v}_1 - \vec{v}_3$ and $u_2 = \vec{v}_2 - \vec{v}_3$;

(b) (3 pts) Compute $\vec{u}_1 \times \vec{u}_2$;

(c) (4 pts) Determine the plane through the points $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

(a) $\vec{u}_1 = \vec{v}_1 - \vec{v}_3 = (1, 1, -1) - (0, -2, 1)$

$= (1, 3, -2)$ 1 pt

$\vec{u}_2 = \vec{v}_2 - \vec{v}_3 = (2, 0, 2) - (0, -2, 1) =$

$= (2, 2, 1)$ 1 pt

1 pt for no mistakes

(b) $\vec{u}_1 \times \vec{u}_2 = \underbrace{\det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ 2 & 2 & 1 \end{pmatrix}}_{1\text{ pt}} = \hat{i}(3 - (-4)) - \hat{j}(1 - (-4)) + \hat{k}(2 - 6)$ 1 pt

$= \underline{(7, -5, -4)}$ 1 pt

(c) Plane: $\underline{\vec{N} \cdot (\vec{x} - \vec{x}_0) = 0}$ 1 pt

$\vec{N}_3 = \underline{(7, -5, -4)}$ by (b) 1 pt

$\vec{x}_0 = \underline{(0, -2, 1)}$ 1 pt

$(7, -5, -4) \cdot (x - 0, y + 2, z - 1) = 0$

$\Leftrightarrow 7x - 5y - 10 - 4z + 4 = 0$

$7x - 5y - 4z = 6$ 1 pts

4. (5 pts) Evaluate the integral:

$$\int_0^1 \left[te^t\vec{i} + e^{-t}\vec{j} + \vec{k}\right] dt.$$

$$\int_0^1 e^{-t}\,dt = -e^{-t}\Big|_0^1 = -e^{-1} + 1 = 1 - e^{-1} \Big\} \ 1pt$$

$$\int_0^1 dt = 1 \Big\} \ 1pt$$

$$\int_0^1 te^t = (t-1)e^t\Big|_0^1 = 0 \cdot e^1 - (-1)e^0 = 1 \Big\} \ 2pts$$

$$\Rightarrow \int_0^1 \left[te^t\hat{i} + e^{-t}\vec{j} + \vec{k}\right] dt = 1\hat{i} + (1-e^{-1})\vec{j} + \vec{k}$$

$$= \left(1, \frac{e-1}{e}, 1\right) \Big\} \ 1pt$$

5. (5 pts) If $\vec{r}(t) = \left(\ln(t^2+1), \tan^{-1} t, \sqrt{t^2+1}\right)$ compute the velocity vector.

$$\frac{d}{dt}\left(\ln(t^2+1)\right) = \frac{1}{t^2+1} \cdot 2t = \frac{2t}{t^2+1} \Big\} \ 1pt$$

$$\frac{d}{dt}\left(\tan^{-1} t\right) = \frac{1}{t^2+1} = \frac{1}{t^2+1} \Big\} \ 1pt$$

$$\frac{d}{dt}\left(\sqrt{t^2+1}\right) = \frac{1}{2}(t^2+1)^{-1/2} \cdot 2t = \frac{t}{\sqrt{t^2+1}} \Big\} \ 1pt$$

$$\Rightarrow \vec{r}\,'(t) = \left(\frac{2t}{t^2+1}, \frac{1}{t^2+1}, \frac{t}{\sqrt{t^2+1}}\right) \Big\} \ 2pts$$

6. (10 pts) Find the length of the curve

$$\vec{r}(t) = e^t \cos t\vec{i} + e^t \sin t\vec{j} + e^t\vec{k}$$

from $\left(\frac{1}{4}\cos\left(\ln\frac{1}{4}\right), \frac{1}{4}\sin\left(\ln\frac{1}{4}\right), \frac{1}{4}\right)$ to $(1,0,1)$.

$$\vec{r}(t) = e^t(\cos t, \sin t, 1)$$

$$\vec{r}\,'(t) = e^t(-\sin t, \cos t, 0) + e^t(\cos t, \sin t, 1)$$

$$\Rightarrow \boxed{\vec{r}\,'(t) = e^t(\cos t - \sin t, \cos t + \sin t, 1)}$$ 2 pts

$$\Rightarrow |\vec{r}\,'(t)| = e^t\sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2 + 1}$$

$$= e^t\sqrt{\underbrace{\cos^2 t + \sin^2 t}_{1} - \cancel{2\sin t\cos t} + \underbrace{\cos^2 t + \sin^2 t}_{1} + \cancel{2\sin t\cos t} + 1}$$

$$= e^t\sqrt{3}$$

2 pts

$$S = \boxed{\int_a^b \sqrt{3}e^t\,dt = \sqrt{3}e^t\Big|_a^b = \sqrt{3}(e^b - e^a)}$$ 2 pts

$$\vec{r}(b) = (1,0,1) = (e^b\cos b, e^b\sin b, e^b) \Leftrightarrow \underline{b=0}$$ 1 pt

$$\vec{r}(a) = \left(\frac{1}{4}\cos\left(\ln\frac{1}{4}\right), \frac{1}{4}\sin\left(\ln\frac{1}{4}\right), \frac{1}{4}\right) = (e^a\cos a, e^a\sin a, e^a) \Leftrightarrow \underline{a = \ln\frac{1}{4}}$$ 1 pt

$$\text{Length}: \int_{\ln\frac{1}{4}}^{0} \sqrt{3}e^t\,dt = \sqrt{3}\left(e^0 - e^{\ln\frac{1}{4}}\right) = \sqrt{3}\left(1 - \frac{1}{4}\right) = \boxed{\frac{3\sqrt{3}}{4}}$$

2 pts

7. (10 pts) For the vector function:

$$\vec{r}(t) = \left(e^t \cos t, e^t \sin t, e^t\right).$$

Compute:

(a) (3 pts) The unit tangent vector $\vec{T}(t)$;

(b) (3 pts) The principal normal vector $\vec{N}(t)$;

(c) (4 pts) The binormal vector $\vec{B}(t)$.

$$\text{(a)}\quad \vec{r}\,'(t) = e^t\left(\cos t - \sin t, \cos t + \sin t, 1\right) \leftarrow 1\text{pt} \quad \text{by } \#6$$

$$|\vec{r}\,'(t)| = e^t\sqrt{3} \leftarrow 1\text{pt}$$

$$\Rightarrow \vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \boxed{\frac{1}{\sqrt{3}}\left(\cos t - \sin t, \cos t + \sin t, 1\right)} \quad 1\text{pt}$$

$$\text{(b)}\quad \vec{T}\,'(t) = \frac{1}{\sqrt{3}}\left(-\sin t - \cos t, -\sin t + \cos t, 0\right) \leftarrow 1\text{pt}$$

$$|\vec{T}\,'(t)| = \frac{1}{\sqrt{3}}\sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2 + 0^2}$$

1pt

$$= \frac{1}{\sqrt{3}}\sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1} + \cancel{2\sin t}\cos t + \underbrace{\cos^2 t - \cancel{2\sin t\cos t} + \sin^2 t}_{1}}$$

$$= \frac{\sqrt{2}}{\sqrt{3}}$$

$$\Rightarrow \vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = \boxed{\frac{1}{\sqrt{2}}\left(-\sin t - \cos t, \cos t - \sin t, 0\right)} \leftarrow 1\text{pt}$$

(c) $$\vec{B} = \underbrace{\vec{T} \times \vec{N}}_{1\ pt} = \underbrace{\det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{1}{\sqrt{3}}(\cos t - \sin t) & \frac{\cos t + \sin t}{\sqrt{3}} & 1/\sqrt{3} \\ \frac{-1}{\sqrt{2}}(\cos t + \sin t) & \frac{\cos t - \sin t}{\sqrt{2}} & 0 \end{pmatrix}}_{1\ pt}$$

$$= \hat{\imath}\left(-\frac{1}{\sqrt{6}}(\cos t - \sin t)\right)$$
$$- \hat{\jmath}\left(\frac{1}{\sqrt{6}}(\cos t + \sin t)\right)$$
$$+ \hat{k}\left(\frac{1}{\sqrt{6}}\left((\cos t - \sin t)^2 + (\cos t + \sin t)^2\right)\right)$$

1 pt

$$= \left(\frac{\sin t - \cos t}{\sqrt{6}}, \frac{\sin t + \cos t}{\sqrt{6}}, \frac{\cos^2 t + \sin^2 t - \cancel{2 \sin t \cos t} + \cos^2 t + \sin^2 t + \cancel{2 \sin t \cos t}}{\sqrt{6}} \right)$$

$$= \boxed{\left(\frac{\sin t - \cos t}{\sqrt{6}}, -\left(\frac{\sin t + \cos t}{\sqrt{6}}\right), \frac{2}{\sqrt{6}} \right) = \vec{B}(t)}$$

1 pt

Check: