MATH 2403 H1-H3. Differential Equations. First midterm exam. Feb 4, 2014.
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Student's Name: Sol   Section

Show all work to receive credit. Work on your own, without reference to notes or text. Use of calculator or any electronic device is not allowed. Answers should be as specific as possible and it should be evident how they were obtained. Work neatly.

1. Mark True or False. You do not need to justify your answers for this question.

   (a) (3 points) The Theorem of existence and uniqueness guarantees that the initial value problem

   $$y' = 2y + \tan(t), \quad y(0) = 2,$$

   has a unique solution defined for $0 < t < \infty$.

   — True X False

   (b) (3 points) The Theorem of existence and uniqueness guarantees that the initial value problem

   $$y' = 2t + \sin(y), \quad y(0) = 2,$$

   has a unique solution defined for $0 < t < \infty$.

   — True X False

   (c) (3 points) For the equation

   $$3x^2 - 2xy + 2 + (6y^2 - x^2 + 3)y' = 0,$$

   a function $\Psi(x, y)$ exists such that $\frac{\partial \Psi}{\partial x} = 3x^2 - 2xy + 2$, and $\frac{\partial \Psi}{\partial y} = 6y^2 - x^2 + 3$.

   X True — False

2. Consider

$$ty' + y = e^{3t}, \quad y(1) = 1.$$

(a) (10 points) Find the solution and determine $y(3)$.

$y' + \frac{y}{t} = \frac{1}{t}e^{3t}$

$\mu(t) = e^{\int \frac{1}{t}dt} = e^{\ln t} = t \Rightarrow t(y' + \frac{y}{t}) = e^{3t}$

$\Rightarrow \frac{d}{dt}(ty) = e^{3t} \Rightarrow ty = \int e^{3t}dt = \frac{1}{3}e^{3t} + C$

$\Rightarrow y = \frac{1}{3t}e^{3t} + \frac{C}{t}$

$y(1) = 1 \Rightarrow 1 = \frac{e^3}{3} + C \Rightarrow C = 1 - \frac{e^3}{3}$

$\therefore y = \frac{1}{3t}e^{3t} + \frac{1}{t} - \frac{e^3}{3t}$

$y(3) = \frac{e^9}{9} + \frac{1}{3} - \frac{e^3}{9}$.

(b) (10 points) Approximate the value of the solution at $t = 3$ using Euler's method with constant step size $h = 1$.

$f(t, y) = -\frac{y}{t} + \frac{1}{t}e^{3t}$

t_n	y_n
1	1
2	$1 + 1(-\frac{1}{1} + \frac{1}{1}e^3) = e^3$
3	$e^3 + 1(-\frac{e^3}{2} + \frac{1}{2}e^6) = \frac{e^3}{2} + \frac{e^6}{2}$

3. Consider the initial value problem

$$y' = \frac{2+2x}{3y^2-6y}, \quad y(0) = 1.$$

(a) (10 points) Find the solution.

$\int (3y^2-6y)\,dy = \int (2+2x)\,dx$

$y^3 - 3y^2 = 2x + x^2 + C$

$y(0) = 1 \Rightarrow \quad -2 = C \quad \therefore \quad y^3 - 3y^2 = x^2 + 2x - 2.$

(b) (6 points) Find the points (x, y) for which the solution has a vertical tangent, and use this information to provide the interval where the solution is defined. Note: you do not need to find the solution $y(t)$ explicitly.

There is a vertical tangent when

$3y^2 - 6y = 0$

$\Leftrightarrow 3y(y-2) = 0 \quad \Leftrightarrow y = 0 \text{ or } y = 2$

For $y = 0$: $\quad 0 = x^2 + 2x - 2 = x^2 + 2x + 1 - 3 =$

$= (x+1)^2 - 3$

$\Leftrightarrow \quad x = -1 \pm \sqrt{3}$

For $y = 2$: $\quad 2^3 - 3(2)^2 = x^2 + 2x - 2$

$\Leftrightarrow 8 - \cancel{12}^{\,3} + 2 = x^2 + 2x$

$\Leftrightarrow \quad -2 + 1 = (x+1)^2$

$\Leftrightarrow \quad -1 = (x+1)^2 \quad$ Not possible!!

The interval of the solution is $(-1-\sqrt{3},\ 1+\sqrt{3})$.

4. (10 points) Find the general solution to the following homogeneous equation

$$y' = \frac{x+3y}{x-y}.$$

Hint: Use $v = y/x$ to make the equation separable, and then use the change of variables $u = 1 + v$ for the integration.

$$y' = \frac{1+3\frac{y}{x}}{1-\frac{y}{x}} = \frac{1+3v}{1-v}$$

and $y' = (xv)' = xv' + v$

$$\Rightarrow \quad xv' + v = \frac{1+3v}{1-v} \quad \leftarrow \text{separable}$$

$$\Rightarrow \quad xv' = \frac{1+3v}{1-v} - v = \frac{1+3v-v+v^2}{1-v}$$

$$\Rightarrow \quad xv' = \frac{1+2v+v^2}{1-v} = \frac{(1+v)^2}{1-v}$$

$$\Rightarrow \int \frac{1-v}{(1+v)^2}\,dv = \int \frac{1}{x}\,dx = \ln|x| + C \quad (*)$$

$u = 1+v$
$du = dv$
$1-v = 1-(u-1) = 2-u$

$$\Rightarrow \int \frac{1-v}{(1+v)}\,dv = \int \frac{2-u}{u^2}\,du$$

$$= \int \left(\frac{2}{u^2} - \frac{1}{u}\right) du$$

$$= -\frac{2}{u} - \ln|u|$$

$$= -\frac{2}{1+v} - \ln|1+v|$$

$$= -\frac{2}{1+\frac{y}{x}} - \ln\left|1+\frac{y}{x}\right| = -\frac{2x}{x+y} - \ln\left|\frac{x+y}{x}\right|$$

From $(*)$ $\quad -\frac{2x}{x+y} - \ln\left|\frac{x+y}{x}\right| - \ln|x| = C$

$$\Leftrightarrow \quad -\frac{2x}{x+y} - \ln|x+y| = C.$$

5. Consider a population of mosquitoes.

(a) (10 points) In the the absence of predators the population grows at a rate proportional to the current population: $x' = rx$. If the size of the population doubles in a week, find r.

$x(t) = x_0 e^{rt}$, t in weeks.

$\Rightarrow x(1) = 2x_0 = x_0 e^{r(1)} = x_0 e^{r}$

$\Leftrightarrow 2 = e^{r} \Leftrightarrow r = \ln(2)$

(b) (10 points) Now, predators eat 70,000 mosquitoes in a week. Write the model that describes the action of predators on the mosquito population, and solve your model considering that initially there were 100,000 mosquitoes.

$x' = rx - 70{,}000$, $x(0) = 100{,}000$.

$\int \frac{dx}{rx - 70{,}000} = \int dt \Rightarrow \frac{1}{r}\ln|rx - 70{,}000| = t + C$

$\Rightarrow rx - 70{,}000 = ce^{rt}$

with $x(0) = 100{,}000 \Rightarrow r(100{,}000) - 70{,}000 = C$

$\therefore x = \frac{1}{r}\left(70{,}000 + (r \cdot 100{,}000 - 70{,}000)e^{rt}\right)$

$\Rightarrow x = \frac{70{,}000}{\ln 2} + \left(100{,}000 - \frac{70{,}000}{\ln 2}\right) e^{(\ln 2)t}$