MATH 2403 H1-H3 Differential Equations. Second Midterm Exam. March 11, 2014.
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Student's Name: Sol Section

Show all work to receive credit Work on your own, without reference to notes or text. Use of calculator or any electronic device is not allowed. Answers should be as specific as possible and it should be evident how they were obtained. Work neatly. To receive credit, you must show your work.

1. Mark True or False. You do not need to justify your answers for this question.

   (a) (3 points) The Theorem of existence and uniqueness guarantees that

   $$\overline{\mathbf{x}}' = \begin{pmatrix} 1 & -5 \\ 3 & -2 \end{pmatrix} \overline{\mathbf{x}}.$$

   has ~~a~~ unique solution defined for $-\infty < t < \infty$.
   ✓ True — False

   (b) (3 points) If $\overline{\mathbf{x}}_1$ and $\overline{\mathbf{x}}_2$ are linearly independent solutions of a given 2x2 system $\overline{\mathbf{x}}' = A\overline{\mathbf{x}}$, then there exists a t_0 such that the Wronskian satisfies

   $$W(\overline{\mathbf{x}}_1, \overline{\mathbf{x}}_2)(t_0) = 0.$$

   — True X False

   (c) (3 points) For a 4-dimensional linear system with constant coefficients, the eigenvalues are $\lambda_1 = -3$, $\lambda_2 = -0.2$, $\lambda_3 = -2 + 0.7i$, $\lambda_4 = -2 - 0.7i$. The origin is a stable equilibrium point.
   ✓ True — False

2. (15 points) For the equation

$$dx + \left(\frac{x}{y} - 4y^2\right) dy = 0$$

find an integrating factor that depends only on y that makes the equation exact and solve the equation. Hint: Calculate

$$Q(y) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

and show that it depends only on y. Calculuate the integrating factor $\mu(y)$ from $\frac{d\mu}{dy} = Q(y)\mu$.

$M = 1, \quad N = \frac{x}{y} - 4y^2$

$Q(y) = \frac{\frac{1}{y} - 0}{1} = \frac{1}{y}$

Then $\frac{d\mu}{dy} = \frac{1}{y}\mu \iff \frac{d\mu}{\mu} = \frac{dy}{y} \iff \ln\mu = \ln y$

$\Rightarrow \mu = y$.

The new equation:

$$y\,dx + (x - 4y^3)\,dy = 0$$

There exists ψ such that

$$\frac{\partial \psi}{\partial x} = y, \quad \frac{\partial \psi}{\partial y} = x - 4y^3$$

$\Rightarrow \psi = xy + h(y), \quad \frac{\partial \psi}{\partial y} = x + h'(y)$

$\Rightarrow h'(y) = -4y^3 \Rightarrow h(y) = -y^4$

$\therefore \psi = xy - y^4 = C$

The solution is $xy - y^4 = C$.

3. (15 points) Obtain a bound for the local truncation error for the Euler's method in terms of h for the solution of

$$y' = -3y - 2, \quad y(0) = -1,$$

in the interval $0 \le t \le 4$. Use this bound to determine the step size h required to obtain a local error of at most 10^{-5}. Recall, if $y = \phi(t)$ is the exact solution, then $|e_n| \le Mh^2/2$, where $|\phi''(t)| \le M$ on the interval of interest.

$y' + 3y = -2$ ← linear

$\mu = e^{\int 3dt} = e^{3t}$

$\Rightarrow ye^{3t} = \int -2e^{3t}\,dt = -\frac{2}{3}e^{3t} + C$

$\Rightarrow y = -\frac{2}{3} + Ce^{-3t}$

$y(0) = -1 \Rightarrow \quad -1 = -\frac{2}{3} + C \quad \Rightarrow C = -\frac{1}{3}$

$\therefore\ y = -\frac{2}{3} - \frac{1}{3}e^{-3t} \quad \equiv \phi(t)$

$\phi'(t) = e^{-3t}$

$\phi''(t) = -3e^{-3t} \quad \Rightarrow \quad |\phi''(t)| < 3$ for $0 \le t \le 4$.

Therefore $|e_n| \le \frac{3}{2}h^2$

To get an error of at most 10^{-5}, we need h as

$\frac{3}{2}h^2 \le 10^{-5} \quad \Rightarrow \quad h \le \sqrt{\frac{2}{3} \cdot 10^{-5}}$

4. (8 points) Find the general solution of the system and draw the phase portrait.

$$\overline{\mathbf{x}}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \overline{\mathbf{x}}.$$

$$p(\lambda) = \det\begin{pmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{pmatrix} = (-2-\lambda)(4-\lambda) + 5$$

$$= \lambda^2 - 2\lambda - 8 + 5 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$$

$$\Rightarrow \lambda_1 = 3, \ \lambda_2 = -1.$$

For $\lambda = 3$: $\begin{pmatrix} -5 & 1 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -5v_1 + v_2 = 0 \Rightarrow v_2 = 5v_1$

$$\bar{v} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

For $\lambda = -1$: $\begin{pmatrix} -1 & 1 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = v_2$

$$\bar{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \quad \bar{x} = c_1 e^{3t}\begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Saddle

5. Consider the system

$$\overline{\mathbf{x}}' = \begin{pmatrix} 1 & \alpha \\ -\alpha & 3 \end{pmatrix} \overline{\mathbf{x}}.$$

(a) (3 points) Calculuate the eigenvalues depending on α.

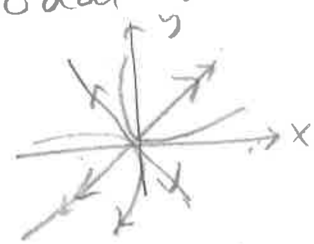
(b) (10 points) Find the critical value(s) of α for which the qualitative nature of the phase portrait changes. Draw representative phase portraits.

a) $p(\lambda) = (1-\lambda)(3-\lambda) + \alpha^2 = \lambda^2 - 4\lambda + 3 + \alpha^2$

$\Rightarrow \lambda = \dfrac{4 \pm \sqrt{16 - 4(3+\alpha^2)}}{2} = \dfrac{4 \pm \sqrt{4 - 4\alpha^2}}{2}$

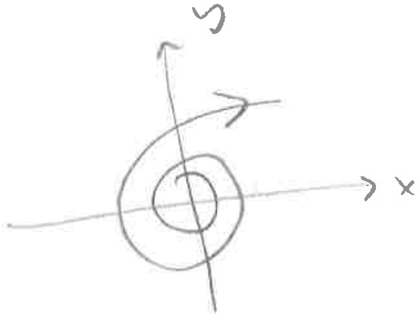
$= 2 \pm \sqrt{1-\alpha^2}$

b) If $1-\alpha^2 > 0 \Rightarrow$ eigenvalues are real, positive $\Rightarrow$ nodal source



This is for $-1 < \alpha < 1$

If $1-\alpha^2 < 0 \Rightarrow$ eigenvalues are complex with positive real part $\Rightarrow$ unstable spiral



This is for $\alpha < -1$ or $\alpha > 1$.

The critical values are $\alpha = 1$ and $\alpha = -1$.

6. (15 points) Write the real general solution for the system and draw the phase portrait.

$$\overline{x}' = \begin{pmatrix} -3 & 1 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \overline{x}.$$

$$p(\lambda) = \det \begin{pmatrix} -3-\lambda & 1 & 0 \\ -1 & -3-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} =$$

$$= (2-\lambda)\left((-3-\lambda)^2 + 1 \right) = 0 \iff \lambda = 2 \text{ or}$$

$$-3-\lambda = \pm i \iff \lambda = 2, -3+i, -3-i.$$

For $\lambda = 2$: $\begin{pmatrix} -5 & 1 & 0 \\ -1 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{matrix} v_1 = 0 \\ v_2 = 0 \\ v_3 \text{ free} \end{matrix}$

$$\therefore \overline{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda = -3+i$: $\begin{pmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & 5-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} v_3 = 0 \\ v_1 = i v_2 \end{matrix}$

$$\therefore \overline{v} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$$

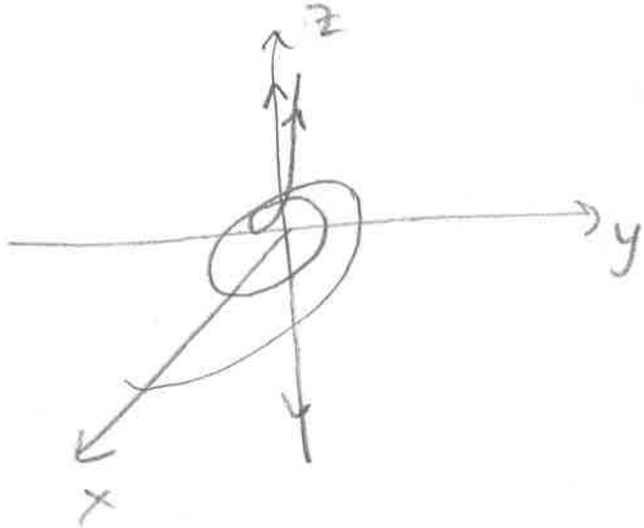
Complex solution:

$$\overline{x}_1 = e^{(-3+i)t}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} = e^{-3t}(\cos t + i \sin t)\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$$

$$= e^{-3t}\left[\begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} + i \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \right]$$

General solution:

$$\overline{x} = c_1 e^{2t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} + c_3 e^{-3t}\begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}$$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