Instructions: *Print* your name, student ID number and recitation session in the spaces below.

Name: ____

Student ID:____________________________________

Recitation session:______________________________

Exam 2, Calculus II (Math 1502)

03/11/2015 (Wednesday)

Show your work clearly and completely!

No calculators are allowed.

You can bring a formula sheet of a one-side letter size paper.

All problems have equal weigh!

Question	Points
1)	
2)	
3)	

Problem 1:

(a) Use Taylor polynomials to estimate $\sin 1$ within 0.01.

(b) Find the distance from the point $(0,0,12)$ to the line $x = 4t,\ y = -2t,\ z = 2t$.

Solution:

(a) Let $f(x) = \sin x$, then the Taylor polynomial

$$P_4(x) = P_3(x) = x - \frac{x^3}{3!}$$

and the error is at most $|x|^5/5!$ since $\left|f^{(n)}(x)\right| \leq 1$ for any n. So if we use

$$P_3(1) = 1 - \frac{1}{3!} = \frac{5}{6}$$

to approximate $\sin 1$, the error is at most $1/5! = 1/120$ which is within 0.01.

(b) The line passes through origin and with the direction vector $\vec{u} = [4,-2,2]$. Let the vector $\vec{v} = [0,0,12]$, then the distance is

$$\begin{aligned}
\|\vec{v} - proj_{\vec{u}}\vec{v}\| &= \left\| \vec{v} - \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{u} \right\| \\
& \left\| \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix} - \frac{24}{4^2+2^2+2^2} \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} 4 \\ -2 \\ 10 \end{bmatrix} \right\| = 2\sqrt{30}.
\end{aligned}$$

Problem 2: Find the general solution of the linear system

$$\begin{aligned} x_1 - x_2 - 3x_3 + x_4 - x_5 &= -2 \\ -2x_1 + 2x_2 + 6x_3 - 6x_5 &= -6 \\ 3x_1 - 2x_2 - 8x_3 + 3x_4 - 5x_5 &= -7. \end{aligned}$$

Solution:

The augmented matrix is

$$\begin{bmatrix} 1 & -1 & -3 & 1 & -1 & -2 \\ -2 & 2 & 6 & 0 & -6 & -6 \\ 3 & -2 & -8 & 3 & -5 & -7 \end{bmatrix}$$

and the reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & -4 & -5 \end{bmatrix}.$$

So x_1, x_2, x_4 are basic variables and x_3, x_5 are free variables. The general solution is

$$x_1 = 2x_3 - x_5 + 2,\ x_2 = 2x_5 - x_3 - 1,\ x_4 = 4x_5 - 5$$

and x_3 and x_5 are free parameters.

Problem 3:

(a) Determine whether the following sets are linearly dependent or linearly independent:

$$\begin{aligned} S_1 &= \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} \right\}, \\ S_2 &= \left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}, \\ S_3 &= \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\}. \end{aligned}$$

Solution:

S_1 is independent since the two vectors are not multiple of the other.

S_2 is dependent since the first two vectors are multiple of the other.

S_3 is dependent since there are four vectors in $\mathbf{R}^3$.

(b) Find the standard matrix of the transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ of reflection to the line $y = \sqrt{3}x$. (Hint: the line has an angle $\pi/3$ (i.e. $60°$) to the $x-$axis. Note that $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos \frac{\pi}{3} = \frac{1}{2}$.)

Solution:

The vector $T(\vec{e}_1)$ has the angle $120°$ $\left(\text{i.e. } \frac{2\pi}{3}\right)$ to the $x-$axis, so

$$T(\vec{e}_1) = \begin{bmatrix} \cos \frac{2\pi}{3} \\ \sin \frac{2\pi}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}.$$

The vector $T(\vec{e}_2)$ has the angle $30°$ $\left(\text{i.e. } \frac{\pi}{6}\right)$ to the $x-$axis, so

$$T(\vec{e}_1) = \begin{bmatrix} \cos \frac{\pi}{6} \\ \sin \frac{\pi}{6} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Thus the standard matrix is

$$A = [T(\vec{e}_1)\ \ T(\vec{e}_1)] = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$