FINAL EXAM - MATH 2304

Sections: D1 - D2 - D3

Name: ____________________________ **Section D__**

Signature: ______________________________________

You will have **2 hours and 50 minutes** to complete this closed book, no notes, no calculator exam.

Keep the exam booklet closed until the beginning of the examination.

Make sure that your booklet has **12 pages** (including this one).
Write clear, complete, legible answers in the spaces provided.
Use the back of the page if needed, but clearly indicate when doing so.

Read each question carefully and completely.
Think about the problem being asked.

Good luck!

1	2	3	4	5	6	7	8	9	10	Total
/10	/10	/10	/10	/10	/10	/10	/10	/10	/10	/100

1. Given the following first order equation

$$(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)dx + (3x^2y^2 + 4y)dy = 0$$

   (a) Show that the equation is not exact.

   (b) Find an integrating factor which makes the equation exact.

   (c) Solve the equation.

2. Suppose that a given population can be divided into two parts: those who have a given disease and can infect others, and those who do not have it but are susceptible.
   Let x be the proportion of susceptible individuals and y the proportion of infectious individuals (then $x+y=1$). Assume that the disease spreads by contact between sick and well members of the population and the rate of spread is proportional to the number of such contacts. Further, assume that members of both groups move freely among each other, so the number of contacts is proportional to the product of x and y.

   (a) Calling α the positive proportionality factor and y_0 the initial proportional of infectious, write the autonomous equation associated to the problem.

   (b) Set $\alpha = 1$. Sketch the graph of $f(y)$ versus y.

   (c) Determine the critical points and classify them.

   (d) Draw the phase line.

   (e) Sketch several graphs of the solutions in the ty-plane.

3. Given the following system of differential equations

$$\begin{cases} x_1' = x_1 + x_2 + x_3 \\ x_2' = 2x_1 + x_2 - x_3 \\ x_3' = -8x_1 - 5x_2 - 3x_3 \end{cases} \qquad \begin{cases} x_1(0) = 0 \\ x_2(0) = 1 \\ x_3(0) = 0 \end{cases}$$

(a) Write the system in the matrix notation $x' = Ax$.

(b) Use the following factorization of the characteristic polynomial

$$det(A - \lambda I) = -(\lambda + 1)(\lambda^2 - 4)$$

in order to find the general solution of the problem.

(c) Write a fundamental matrix $X(t)$ for the problem.

(d) Find the solution of the initial value problem.

4. Given the following system of differential equations

$$x' = \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & -3 \end{bmatrix} x$$

(a) Find the general solution in terms of real-valued functions.

(b) List and classify any critical points.

(c) Draw the phase portrait.

(d) Describe how the solutions behave as $t \to \infty$.

5. Given the following homogeneous system

$$x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x$$

(a) Find the general solution in terms of real-valued functions.

(b) List and classify any critical points.

(c) Draw the phase portrait.

(d) Describe how the solutions behave as $t \to \infty$.

6. Given the following second order differential equation

$$y'' + 5y' - 6y = 22 + 18x - 18x^2 + 6e^{3x}$$

(a) Find two solutions of the complementary equation.

(b) Verify that the solutions you found form a fundamental set of solutions for the problem.

(c) Using the principle of superposition, find the general solution of the problem.

7. Given the following second order differential equation

$$x^2y'' + xy' - y = 2x^2 + 2$$

(a) Verify that $y_1 = x$ and $y_2 = \frac{1}{x}$ are two solutions of the complementary equation.

(b) Verify that $\{y_1, y_2\}$ is a fundamental set of solutions for the problem.

(c) Find a particular solution using the method of variation of parameters.

(d) Find the general solution of the problem.

8. Given the following initial value problem

$$\begin{cases} y'' - y = f(t) \\ y(0) = 3;\ y'(0) = -1 \end{cases} \quad \text{with } f(t) = \begin{cases} e^{2t} & 0 \le t < 2 \\ 1 & t \ge 2 \end{cases}$$

(a) Rewrite the problem using a unit step function.

(b) Find the solution of the initial value problem using the Laplace Transform.

9. Solve the following initial value problem

$$\begin{cases} y'' + y' - 2y = -10e^{-t} + 5\delta(t-1) \\ y(0) = 7;\ y'(0) = -9 \end{cases}$$

10. Given the initial value problem

$$\begin{cases} y'' + 6y' + 9y = f(t) \\ y(0) = 0; \ y'(0) = -2 \end{cases}$$

give a formula for the solution $y(t)$.

Laplace Transfom Table

$$1 \longleftrightarrow \frac{1}{s} \quad s > 0$$

$$e^{at} \longleftrightarrow \frac{1}{s-a} \quad s > a$$

$$t^n \longleftrightarrow \frac{n!}{s^{n+1}} \quad s > 0, n > 0 \text{ integer}$$

$$\sin(at) \longleftrightarrow \frac{a}{s^2+a^2} \quad s > 0$$

$$\cos(at) \longleftrightarrow \frac{s}{s^2+a^2} \quad s > 0$$

$$\sinh(at) \longleftrightarrow \frac{a}{s^2-a^2} \quad s > |a|$$

$$\cosh(at) \longleftrightarrow \frac{s}{s^2-a^2} \quad s > |a|$$

$$e^{at}\sin(bt) \longleftrightarrow \frac{b}{(s-a)^2+b^2} \quad s > a$$

$$e^{at}\cos(bt) \longleftrightarrow \frac{s-a}{(s-a)^2+b^2} \quad s > a$$

$$t^n e^{at} \longleftrightarrow \frac{n!}{(s-a)^{n+1}} \quad s > a, n > 0 \text{ integer}$$