ISyE 4232 Spring 2013

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Solutions to Homework 3

1. (a) Let λ_i be the total arrival rate to station i, $i = 1, 2, 3$. Then we have the following traffic equations:

$$\lambda_1 = 1 + (0.1)\lambda_2 + (0.15)\lambda_3,$$
$$\lambda_2 = \lambda_1 + (0.1)\lambda_3,$$
$$\lambda_3 = (0.9)\lambda_2.$$

Solving the equations, one has $\lambda_1 \approx 1.348$, $\lambda_2 \approx 1.481$ and $\lambda_3 \approx 1.333$. Thus,

$$\rho_1 = \lambda_1 m_1 = 1.078, \quad \rho_2 = 1.037, \quad \rho_3 = 1.0667.$$

Then the system is unstable, and the number of units in the system goes to ∞ as $t \to \infty$. So, let $X_i(t)$ be the number of jobs at station i at time t. Then the long-run fraction of time that there are 2 jobs at station 1, 1 job at station 2 and 4 jobs at station 3 is equal to

$$\mathbb{P}\{X_1(\infty) = 2, X_2(\infty) = 1, X_3(\infty) = 4\} = 0$$

Because the probability of having a finite amount of units in the system goes to 0 as $t \to \infty$.

(b) Here you may be tempted to say that $N_3 = \infty$ because the system is not stable, but this is not actually true. Just because the system as a whole is unstable it doesn't necessarily mean that the queue at each station grows to infinity. Now, let's suppose station 1 is unstable, so it is busy 100% of the time and so the rate at which customers leave station 1 is $1.25/hr$ (the service rate). In this case we can view stations 2 and 3 as a separate system which has external arrivals at rate $1.25/hr$. Then the traffic equations are

$$\lambda_2 = 1.25 + (0.1)\lambda_3,$$
$$\lambda_3 = (0.9)\lambda_2.$$

Hence $\lambda_2 \approx 1.37363 < \mu_2$ and $\lambda_3 \approx 1.23626 < \mu_3$. Therefore

$$N_3 = \frac{\lambda_3}{\mu_3 - \lambda_3} = 89.9755$$

which is a lot, but it's finite.

One final check that is necessary, we assumed in the long run station 1 is busy all the time, this means we are assuming the arrival rate is indeed higher than the service rate. Let's check

$$\lambda_1 = 1 + 0.1\lambda_2 + 0.15\lambda_3 = 1.3228 > 1.25/hr$$

So everything works out.

(c) The total long-run average time each customer spends in the system also clearly goes to infinity, as each arriving customer goes to station 1 and in station 1 she has to wait for infinitely many to go before her. So

$$T = \infty$$

2. (a) Let γ_i be the throughput rate going through station i. Then we have traffic equations:

$$\gamma_1 = (0.1)\gamma_2 + \gamma_3, \tag{1}$$
$$\gamma_2 = \gamma_1, \tag{2}$$
$$\gamma_3 = (0.9)\gamma_2. \tag{3}$$

Equations (2) and (3) imply equation (1). Thus, equation (1) is redundant. (In general, in a closed network, there is always a redundant traffic equation.) Thus, one cannot solve γ from (1)-(3). However, if we can find γ_1 by using some method, then γ_2 and γ_3 follow readily. For the moment, we set

$$\gamma_1 = 1. \tag{4}$$

(Setting γ_1 to any other positive number is OK.) Then, $\gamma_2 = 1$ and $\gamma_3 = 0.9$. (Remember these are not true throughput rates. With γ set, one can define

$$\rho_1 = \gamma_1 m_1 = 0.8, \quad \rho_2 = 0.7, \quad \rho_3 = 0.72.$$

(**Warning:** These ρ's are not true utilizations of machines because γ's are not true throughput. But at least we know machine 1 is the mostly heavily utilized machine.)

Let $X_i(t)$ be the number of jobs at station i at time t. Then

$$\begin{aligned}
\mathbb{P}\{X_1(\infty) = 2, X_2(\infty) = 0, X_3(\infty) = 0\} &= C\rho_1^2\rho_2^0\rho_3^0 \\
\mathbb{P}\{X_1(\infty) = 1, X_2(\infty) = 1, X_3(\infty) = 0\} &= C\rho_1^1\rho_2^1\rho_3^0 \\
\mathbb{P}\{X_1(\infty) = 1, X_2(\infty) = 0, X_3(\infty) = 1\} &= C\rho_1^1\rho_2^0\rho_3^1 \\
\mathbb{P}\{X_1(\infty) = 0, X_2(\infty) = 1, X_3(\infty) = 1\} &= C\rho_1^0\rho_2^1\rho_3^1 \\
\mathbb{P}\{X_1(\infty) = 0, X_2(\infty) = 2, X_3(\infty) = 0\} &= C\rho_1^0\rho_2^2\rho_3^0 \\
\mathbb{P}\{X_1(\infty) = 0, X_2(\infty) = 0, X_3(\infty) = 2\} &= C\rho_1^0\rho_2^0\rho_3^2
\end{aligned}$$

These probabilities have to add up to 1. Thus, we have

$$C = 0.305. \tag{5}$$

Note we omitted the expressions $(1 - \rho_i)$ in each case above. Since these show up in every term, it's easier to just absorb them into C, it does not affect the answer. Hence,

$$\mathbb{P}\{X_1(\infty) = 0, X_2(\infty) = 1, X_3(\infty) = 1\} = 0.1537, \tag{6}$$
$$\mathbb{P}\{X_1(\infty) = 0, X_2(\infty) = 2, X_3(\infty) = 0\} = 0.1495, \tag{7}$$
$$\mathbb{P}\{X_1(\infty) = 0, X_2(\infty) = 0, X_3(\infty) = 2\} = 0.1581. \tag{8}$$

Note that these probabilities do *not* depend on the choice to γ_1 in (4). If one changes the choice of γ_1, C in (5) will change accordingly. From (6)–(8), the long-run fraction of time that machine 1 is idle is given by $0.1494 + 0.1537 + 0.1581 = .461$. Thus, the utilization of machine 1 is $\rho_1^* = 1 - 0.461 = 0.539$. (This is the **true** utilization of machine 1.) Therefore, the true throughput at station 1 is $\gamma_1^* = 0.539/m_1 = 0.539/0.8 = .6737$, and hence $\gamma_2^* = 0.6737$ and $\gamma_3^* = (0.9)\gamma_2^* = .6064$. Thus, the production rate of the system is 0.6064 jobs per hour. That is also the rate at which new jobs are introduced into the system. Using Little's law, $L = \lambda W$, we have that the average time in system per job is $2/0.6064 = 3.2982$ hours.

By the way, the true utilizations of machines are

$$\rho_1^* = 0.539, \quad \rho_2^* = \gamma_2^* m_2 = 4716, \quad \rho_3^* = \gamma_3^* m_3 = 0.4851.$$

(b) No. To double γ_3^*, one needs to double γ_1^*, and hence double ρ_1^*. Currently, machine 1 is more than 50% utilized, and one cannot double it.