MATH 2401 Name: ______________________

Quiz 3 (Feb 5, 2015) GT Id: ______________________

1. Find **T**, **N**, **B** and κ for the curve $\mathbf{r}(t) = 2\cos 2t\mathbf{i} + 3t\mathbf{j} + 2\sin 2t\mathbf{k}$. [1.5+1.5+2+2]

$$\vec{v} = \vec{r}\,'(t) = \langle -4\sin 2t,\ 3,\ 4\cos 2t \rangle$$

$$|\vec{v}| = \sqrt{16\sin^2 2t + 9 + 16\cos^2 2t} = \sqrt{16(\sin^2 2t + \cos^2 2t) + 9}$$

$$= \sqrt{16+9} = 5$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \left\langle -\frac{4}{5}\sin 2t,\ \frac{3}{5},\ \frac{4}{5}\cos 2t \right\rangle$$

$$\vec{T}\,' = \left\langle \frac{-8}{5}\cos 2t,\ 0,\ -\frac{8}{5}\sin 2t \right\rangle$$

$$|\vec{T}\,'| = \sqrt{\frac{64}{25}\cos^2 2t + 0 + \frac{64}{25}\sin^2 2t} = \sqrt{\frac{64}{25}} = \frac{8}{5}$$

$$\vec{N} = \frac{\vec{T}\,'}{|\vec{T}\,'|} = \langle -\cos 2t,\ 0,\ -\sin 2t \rangle$$

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{4}{5}\sin 2t & 3/5 & 4/5\cos 2t \\ -\cos 2t & 0 & -\sin 2t \end{vmatrix}$$

$$= \left(-\frac{3}{5}\sin 2t\right)\vec{i} - \left(\frac{4}{5}\sin^2 t + \frac{4}{5}\cos^2 t\right)\vec{j} + \left(\frac{3}{5}\cos 2t\right)\vec{k}$$

$$= \left\langle -3/5\sin 2t,\ -\frac{4}{5},\ \frac{3}{5}\cos 2t \right\rangle$$

$$\kappa = \frac{|\vec{T}\,'|}{|\vec{v}|} = \frac{8/5}{5} = \frac{8}{25}.$$

2. Let $\mathbf{r}(t) = (t+5)\,\mathbf{i} - 2t\,\mathbf{j} + t^2\,\mathbf{k}$ be the position of a particle in space at time t. Find the tangential component of the acceleration at $t = 1$. [3]

$$\vec{v} = \vec{r}\,'(t) = \langle 1, -2, 2t \rangle$$

$$|\vec{v}| = \sqrt{1 + 4 + 4t^2} = \sqrt{4t^2+5}$$

$$a_T = \frac{d}{dt}|\vec{v}| = \frac{d}{dt}\sqrt{4t^2+5} = \frac{1}{2}(4t^2+5)^{-1/2} \cdot 8t$$

$$= \frac{4t}{\sqrt{4t^2+5}}$$

At $t=1$, $a_T = \dfrac{4(1)}{\sqrt{4(1)^2+5}} = \dfrac{4}{\sqrt{9}} = \dfrac{4}{3}$.

MATH 2401
Quiz 4 (Feb 12, 2015)

Name: ____________________
GT Id: ____________________

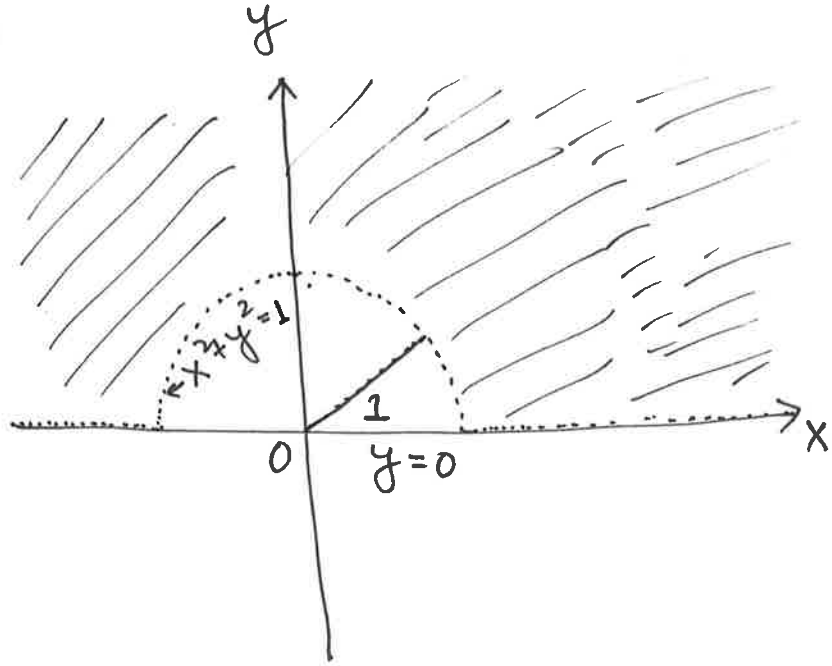
1. Let $f(x,y) = \dfrac{4+\ln y}{\sqrt{x^2+y^2-1}}$.

(a) Find and sketch the domain of f. [3]

Domain of f is determined by

$y>0, \quad x^2+y^2-1>0$

i.e. $y>0, \quad x^2+y^2>1$



(b) Is the domain of f open, closed, both or neither? [1]

Open

(c) Is the domain of f bounded or unbounded? [1]

Unbounded

2. Find the torsion τ of the space curve given by $\mathbf{r}(t) = 4\sin t\mathbf{i} + 4\cos t\mathbf{j} + 3t\mathbf{k}$. [5]

$$\vec{v}(t) = \vec{r}'(t) = \langle 4\cos t, -4\sin t, 3\rangle$$

$$|\vec{v}| = \sqrt{16\cos^2 t + 16\sin^2 t + 9} = \sqrt{16+9} = 5$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{4}{5}\cos t, -\frac{4}{5}\sin t, \frac{3}{5}\right\rangle$$

$$\vec{T}'(t) = \left\langle -\frac{4}{5}\sin t, -\frac{4}{5}\cos t, 0\right\rangle$$

$$|\vec{T}'| = \sqrt{\frac{16}{25}\sin^2 t + \frac{16}{25}\cos^2 t + 0} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|} = \langle -\sin t, -\cos t, 0\rangle$$

$$\vec{B} = \vec{T}\times\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{4}{5}\cos t & -\frac{4}{5}\sin t & \frac{3}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix}$$

$$= \left(\frac{3}{5}\cos t\right)\vec{i} - \left(\frac{3}{5}\sin t\right)\vec{j} + \left(-\frac{4}{5}\cos^2 t - \frac{4}{5}\sin^2 t\right)\vec{k}$$

$$= \left\langle \frac{3}{5}\cos t, -\frac{3}{5}\sin t, -\frac{4}{5}\right\rangle$$

$$\frac{d\vec{B}}{dt} = \left\langle -\frac{3}{5}\sin t, -\frac{3}{5}\cos t, 0\right\rangle$$

$$\tau = -\frac{d\vec{B}}{ds}\cdot\vec{N} = -\frac{d\vec{B}/dt}{ds/dt}\cdot\vec{N} = -\frac{1}{|\vec{v}|}\frac{d\vec{B}}{dt}\cdot\vec{N}$$

$$= -\frac{1}{5}\left\langle -\frac{3}{5}\sin t, -\frac{3}{5}\cos t, 0\right\rangle\cdot\langle -\sin t, -\cos t, 0\rangle$$

$$= -\frac{1}{5}\left(\frac{3}{5}\sin^2 t + \frac{3}{5}\cos^2 t + 0\right)$$

$$= -\frac{1}{5}\left(\frac{3}{5}\right) = -\frac{3}{25}$$

MATH 2401 **Name:** ______________________

Quiz 5 (March 5, 2015) **GT Id:** ______________________

Let $f(x,y) = 2 + y\cos x - xe^{xy}$.

1. Find the direction in which f increases most rapidly at $O(0,0)$. [2]

$$\nabla f = \langle f_x, f_y \rangle = \langle -y\sin x - e^{xy} - xye^{xy},\ \cos x - x^2 e^{xy} \rangle$$

$$\nabla f(0,0) = \langle 0 - e^0 - 0,\ \cos(0) - 0 \rangle = \langle -1, 1 \rangle$$

$$|\nabla f(0,0)| = \sqrt{1+1} = \sqrt{2}.$$

$\therefore$ The direction in which f increases most rapidly at $(0,0)$ is $\dfrac{\nabla f(0,0)}{|\nabla f(0,0)|} = \dfrac{\langle -1, 1 \rangle}{\sqrt{2}} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

2. Find the derivative of f at $O(0,0)$ in the direction of $3\mathbf{i} + 4\mathbf{j}$. [3]

The direction of $3\vec{i} + 4\vec{j}$ is $\vec{u} = \dfrac{\langle 3, 4 \rangle}{\sqrt{3^2+4^2}} = \langle 3/5, 4/5 \rangle$

$$\therefore D_{\vec{u}} f(0,0) = \nabla f(0,0) \cdot \vec{u}$$

$$= \langle -1, 1 \rangle \cdot \langle 3/5, 4/5 \rangle$$

$$= -3/5 + 4/5 = \frac{1}{5}.$$

3. Estimate how much the value of $f(x,y)$ will change as the point (x,y) moves 0.1 unit from $(0,0)$ straight toward the point $(3,4)$? [2]

Given direction is $\vec{u} = \dfrac{\langle 3-0,\ 4-0\rangle}{\sqrt{9+16}} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

$$df = D_{\vec{u}} f(0,0)\, ds$$

$$= \frac{1}{5}(0.1) \qquad \left(D_{\vec{u}} f(0,0) = \frac{1}{5} \text{ from } (2) \right)$$

$$= 0.02$$

4. Find the linearization of f at $O(0,0)$. [3]

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$

For $f(x,y) = 2 + y\cos x - x e^{xy}$,

$$f(0,0) = 2 + y - 0 = 2 + y$$

$\cancel{f_x = -y\sin x - e^{xy} - xye^{x}}$

From part (1) $f_x(0,0) = -1$, $f_y(0,0) = 1$.

$$\therefore L(x,y) = 2 + y + (-1)x + (1)y$$

$$= 2 + y - x + y$$

$$= 2 + 2y - x$$

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Quiz 6 (March 26, 2015) **GT Id:** ______________________

1. Evaluate the following integrals.

(a) $\displaystyle\iint_R \frac{2xy}{x^2+1}\,dA$, where $R = [0,1] \times [0,2]$. [2]

$$= \int_0^2 \int_0^1 \frac{2xy}{x^2+1}\,dx\,dy$$

$$= \int_0^2 y \ln(x^2+1)\Big]_{x=0}^{x=1} dy = \int_0^2 (y \ln 2 - y \ln 1)\,dy$$

$$= \int_0^2 (\ln 2)\, y\,dy = (\ln 2)\frac{y^2}{2}\Big]_0^2$$

$$= (\ln 2)(4/2 - 0)$$

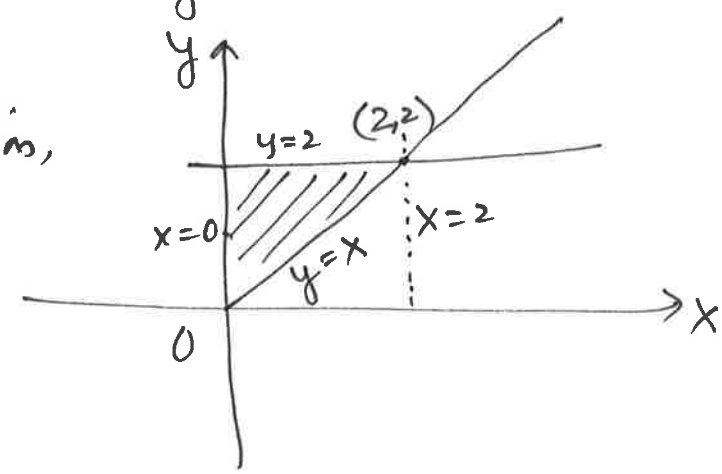
$$= 2 \ln 2$$

(b) $\displaystyle\int_0^2 \int_x^2 2y^2 e^{xy}\,dy\,dx$ [5]

The region of integration is bounded by

$y = x$, $y = 2$, $x = 0$ & $x = 2$.

Reversing the order of integration,



given integral $\displaystyle = \int_0^2 \int_0^y 2y^2 e^{xy}\,dx\,dy$

$$= \int_0^2 2y^{\cancel{2}} \frac{e^{xy}}{\cancel{y}}\Big]_{x=0}^{x=y} dy$$

$$= \int_0^2 (2y e^{y^2} - 2y)\,dy = (e^{y^2} - y^2)\Big|_{y=0}^{y=2}$$

$$= (e^4 - 4) - (1 - 0)$$

$$= e^4 - 5.$$

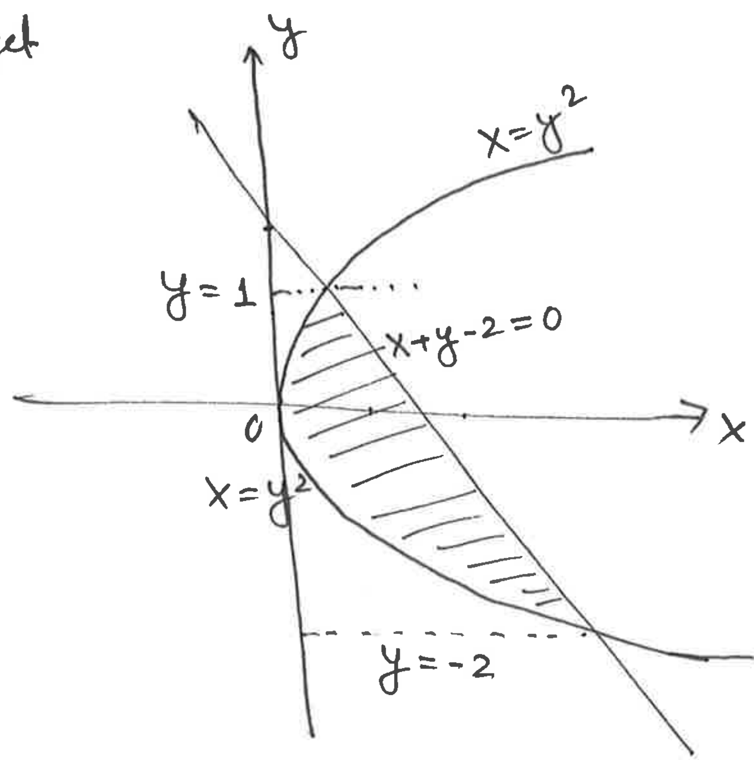
2. Set up an iterated double integral for the area of the region R in the xy-plane enclosed by the parabola $x = y^2$ and the line $x + y - 2 = 0$. (Do not evaluate.) [3]

Solving $x = y^2$ & $x + y - 2 = 0$, we get

$$y^2 + y - 2 = 0$$

or $(y+2)(y-1) = 0$

i.e. $y = -2, 1$



Area of R $= \iint_R dA$

$$= \int_{-2}^{1} \int_{y^2}^{2-y} dx\, dy$$

MATH 2401 Name: ______________
Quiz 7 (April 2, 2015) GT Id: ______________

This quiz contains 2 problems. Write neatly and show all your work.

1. Change the Cartesian integral into an equivalent polar integral. (Do not evaluate.) [5]

$$\int_0^1 \int_x^{\sqrt{2-x^2}} (1+x^2+y^2)\,dy\,dx$$

The region of integration is bounded by

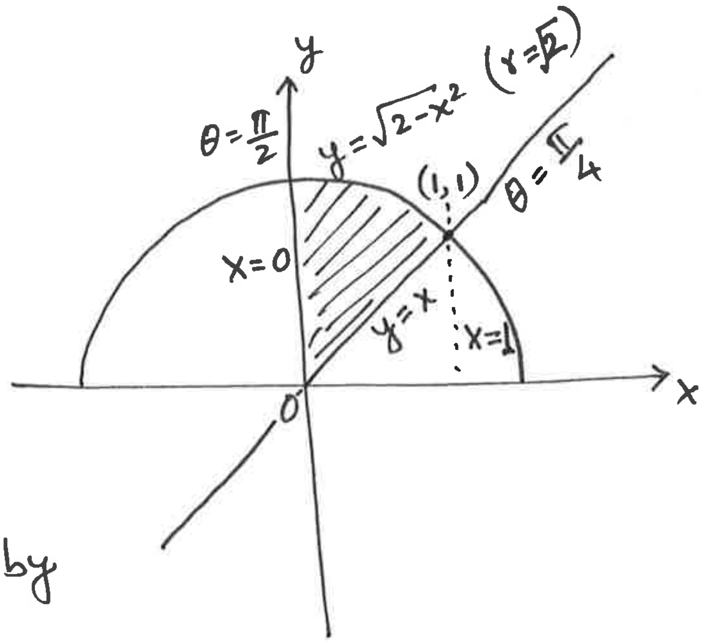
$y = x$,

$y = \sqrt{2-x^2}$ or $y^2 = 2-x^2$, $y \geq 0$

i.e. $x^2+y^2=2$, $y \geq 0$;

$x = 0$

and, $x = 1$.



In polar coordinates it is given by

$0 \leq r \leq \sqrt{2}$, $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$.

$\therefore$ Given integral can be written as

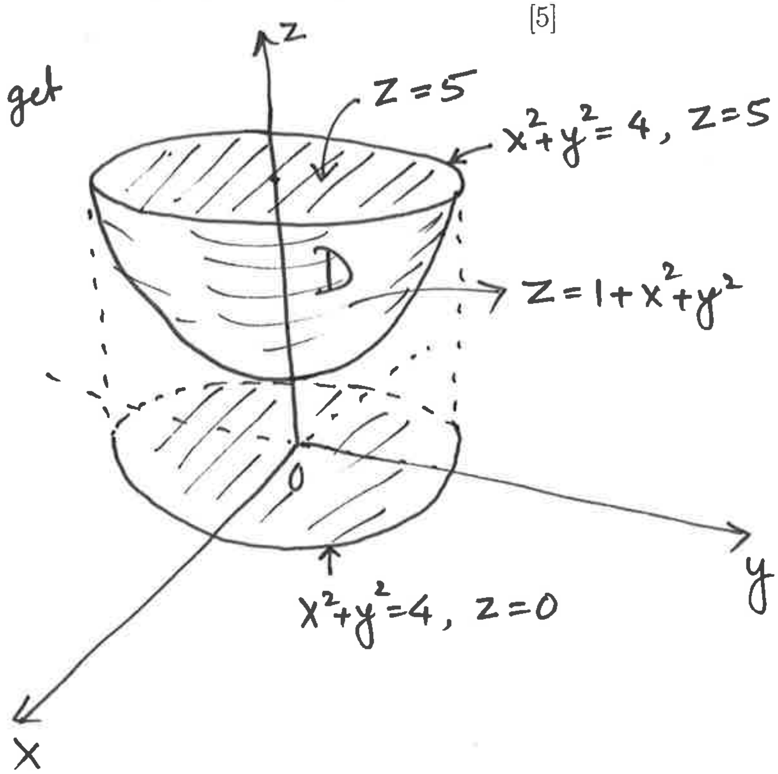
$$\int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} (1+r^2)\, r\, dr\, d\theta$$

2. Set up an iterated triple integral in **Cartesian coordinates** for the volume of the solid bounded below by the paraboloid $z = 1 + x^2 + y^2$ and above by the plane $z = 5$. (Do not evaluate.) [5]

Solving $z = 1 + x^2 + y^2$ & $z = 5$, we get

$$5 = 1 + x^2 + y^2$$

or, $x^2 + y^2 = 4$.



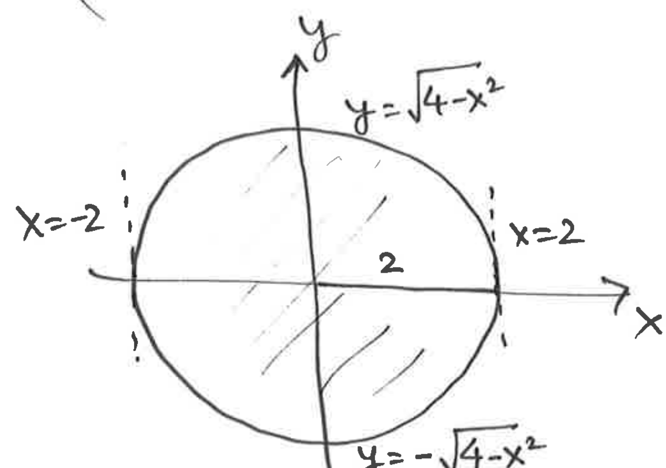
Volume of the solid D

$$= \iiint_D dV$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{1+x^2+y^2}^{5} dz\, dy\, dx$$

($x^2 + y^2 = 4 \Rightarrow y^2 = 4 - x^2$

$\Rightarrow y = \pm\sqrt{4-x^2}$



MATH 2401 Name: ______________________
Quiz 8 (April 16, 2015) GT Id: ______________________

This quiz contains 2 problems. Write neatly and show all your work.

1. Evaluate the following integrals along the curve C given by

$$\mathbf{r}(t) = 4t\mathbf{i} + 3\sin t\mathbf{j} + 3\cos t\mathbf{k},\ 0 \le t \le 2\pi.$$

(a) $\int_C y ds.$ [3]

Here $y = 3\sin t$, $ds = |\vec{r}'(t)|\, dt$

$= |4\vec{i} + 3\cos t\,\vec{j} - 3\sin t\,\vec{k}|\, dt$

$= \sqrt{16 + 9\cos^2 t + 9\sin^2 t}\; dt$

$= \sqrt{16+9}\; dt = 5\, dt$

$\therefore \int_C y\,ds = \int_0^{2\pi} 3\sin t\,(5)\,dt = -15\cos t\Big|_0^{2\pi} = -15(\cos 2\pi - \cos 0)$

$= -15(1-1)$

$= 0.$

(b) $\int_C \mathbf{F}.d\mathbf{r}$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. [3]

$= \int_0^{2\pi} F(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$

$= \int_0^{2\pi} (4t\,\vec{i} + 3\sin t\,\vec{j} + 3\cos t\,\vec{k}) \cdot (4\vec{i} + 3\cos t\,\vec{j} - 3\sin t\,\vec{k})\, dt$

$= \int_0^{2\pi} (16t + 9\sin t\cos t - 9\sin t\cos t)\, dt$

$= \int_0^{2\pi} 16t\, dt = 8t^2\Big]_0^{2\pi} = 8(4\pi^2) - 0 = 32\pi^2$

2. Find the flux of $\mathbf{F} = (x+y)\mathbf{i} + y\mathbf{j}$ across the circle $x^2+y^2=4$ in the xy-plane. [4]

The circle $C: x^2+y^2=4$ can be parameterized by

$$x = 2\cos t, \quad y = 2\sin t \quad (0 \le t \le 2\pi).$$

$$\therefore \text{Flux} = \oint_C M\,dy - N\,dx \quad \text{where} \quad \begin{cases} M = x+y = 2\cos t + 2\sin t \\ N = y = 2\sin t \\ dx = -2\sin t\,dt \\ dy = 2\cos t\,dt \end{cases}$$

$$= \int_0^{2\pi} (2\cos t + 2\sin t)\,2\cos t\,dt - 2\sin t\,(-2\sin t)\,dt$$

$$= \int_0^{2\pi} (4\cos^2 t + 4\sin t\cos t + 4\sin^2 t)\,dt$$

$$= \int_0^{2\pi} \left[4(\cos^2 t + \sin^2 t) + 4\sin t\cos t\right] dt$$

$$= \int_0^{2\pi} (4 + 4\cos t\sin t)\,dt$$

$$= \left(4t + \cancel{4}^{2} \frac{\sin^2 t}{\cancel{2}}\right)\Bigg|_0^{2\pi}$$

$$= (8\pi + 0) - (0 + 0)$$

$$= 8\pi$$