ISyE 2027C Probability with Applications
Homework "due" but NOT TO HAND IN and "NO QUIZ" 17 February 2015

Simple Calculation Problems

1. Roll a die repeatedly until you get a 3 or 6. What is the expected number of rolls?
2. Roll two dice. What is the probability distribution of the sum of the rolls given that the first roll is greater than 4? What is the expected value of the sum given that the first roll is greater than 4?
3. Roll two dice repeatedly until you get either a 7, an 11, or doubles (the two die have the same number), in which case you stop. Conditioned on your stopping on your 99th roll, what is the probability that your 9th roll was an 11? 1/7
4. Let X be uniformly distributed on the interval $[-10, 10]$. Find $E[X], E[333X - 17], E[X^2], E[\sqrt{X}]$. Use your intuition to find $E[X|X \geq 0]$ and $E[X|X \leq -8]$.
5. You order pizza for your family of 8. There are 16 possible toppings. You order a topping only if 0 or 1 family members don't like the topping. Each family member likes each topping with probability 0.9 independent of other toppings and people. What is the probability that you will order exactly 6 toppings? 13 toppings? 17 toppings? What is the expected number of toppings?
6. You are given k dice with probability $1/k$ for $k = 2, 3, 6$. You roll the dice and receive the sum of the values in dollars. What is the expected number of dollars you will get? Conditioned on the event that you got \$6, what is the probability that you were given 2 dice?

Word Problems

Example: You are diving for treasure in a lake. The probability that the lake has treasure is 0.4. Each time you dive, you have a .1 chance of finding treasure if treasure is there, independent of previous dives. You dive twice but find no treasure. What is the probability that the lake has treasure? Answer: let T be the event that there is treasure in the lake. Let F be the event that you find treasure in two dives. We seek $P(T|F^C)$. $P(F|T^C) = 0$ and $P(T) = 0.4$. $P(F^C|T) = 0.9^2 = 0.81$, hence $P(F|T) = 1 - .81 = .19$. Next we need $P(F)$. From the law of total probabilty,

$$P(F) = P(T)P(F|T) + P(T^C)P(F|T^C) = .4 \cdot .19 + 0 = .076 = 1 - P(F^C)$$

Then $P(T|F^C) = P(F^C|T)P(T)/P(F^C) = .81 \cdot .4/.924$. Another way to look at it: there are two ways you could fail to find treasure in two dives. The first way is because there is no treasure, with probabilty .6. The second is that there is treasure but both dives fail, with probability $.4 \cdot .9^2$. The conditional probablity that there is treasure is the probability of the 2nd way divided by the sum of the probabilities, $.4 \cdot .9^2/(.6 + .4 \cdot .9^2)$.

You plan to dive until you find treasure. What is the probability that you never stop diving? .6 You plan to dive until you find treasure. Given that you did find treasure, what is your expected number of dives? 10

You plan to dive until you find treasure or you have completed 4 dives. What is your expected number of dives? Answer: Let N be the number of dives. Then by the law of total probability for expected values, $E[N] = P(T)E[N|T] + P(T^C)E[N|T^C]$. Suppose there is treasure. Then $P(N = 1|T) = 0.1$, $P(N = 2|T) = 0.9 \cdot 0.1$; $P(N = 3|T) = 0.9^2 \cdot 0.1$, $P(N = 4|T) = .9^3$ since you stop after 4 dives whether or not you found treasure. Then $E[N|T] = .1 + 2 \cdot .09 + 3 \cdot .081 + 4 \cdot .729$. Obviously $P(N = 4|T^C) = 1$. Combine these numbers with $P(T) = 1 - P(T^C) = .4$.

You plan to dive until you find treasure or you have completed 4 dives. Given that you find treasure, is your expected number of dives less than 2.5, equal to 2.5, or more than 2.5? Explain how you know. Answer: less. Your best chance to find treasure is your first dive, namely .1 given that there is treasure. Your second dive has a .09 chance, which is less. The expected value is a weighted average of 1,2,3,4 with highest weight on 1 and least weight on 4. The expected number is

$$\frac{1 \cdot .1 + .09 \cdot 2 + .081 \cdot 3 + .0729 \cdot 4}{.1 + .9 + .081 + .0729}$$

GENERAL GUIDELINES
STEP 1: Define the pertinent events and/or random variables.
STEP 2: State the goal of the problem in terms of the events and/or random variables.
STEP 3: State the given information in terms of probabilities involving those events and/or the distribution (pmf or pdf or cdf) of the random variables.
STEP 4: Solve the problem.
STEP 5: Check your answer and your reasoning. Does it make sense if you consider an extreme case? Can you see the answer with hindsight in a glance?

1. An eccentric billionaire hires you to dive for a treasure off the Florida coast. The probability that treasure is there is 0.4. Each day that you dive, the probability is 0.05 that you will find the treasure if it is there, independent of previous dives. (This would not a realistic assumption if your diving were methodically planned, but the billionaire selects the diving location randomly each day.) You are paid \$500 per day for diving. If you find the treasure, the search terminates and you will get a \$50,000 bonus. There is one additional complication. On each day, if the billionaire does not hear the cry of a seagull nor see a dolphin, she takes it as a bad omen, flips a (fair) coin, and terminates the search if it comes up tails. . The probability of hearing a seagull on any day is 0.9. The probability of seeing a dolphin on any day is 0.25, independent of whether a seagull is heard.

   What is the expected number of days you will dive?
   What is the expected amount of money you will earn?
   What is the probability that you will find treasure?
   Given that the search terminated after exactly 3 days, what is the probability that you found treasure?
   Given that you have dived without finding treasure for 10 days, what is the expected amount of addtional money you will earn?
   Given that you have dived without finding treasure for 10 days, what is the expected number of additional days you will dive?
   Given that you have dived without finding treasure for 10 days, what is the expected amount of average daily earnings in the future? Why can or can't you answer this question by taking the ratio of the answers to the previous two questions?

2. A zombie is shuffling back and forth on a sidewalk. Each minute it moves one step east with probability .7 or one step west with complementary probability .3. Write an equation that, if solved, finds the probability that the zombie will ever reach the place one step west of its starting place.

3. A zombie is shuffling back and forth on a sidewalk. Each minute it moves **two** steps east with probability .4 or **one** step west with complementary probability .6. Write an equation that, if

solved, finds the probability that the zombie will ever reach the place one step west of its starting place.

4. You flip a fair coin repeatedly. Write an equation that, if solved, finds the probability that the number of tails will ever be more than three times the number of heads.

5. Repeat the previous problem if the coin comes up heads with probability p for the values $p = 1, p = .7, p = .3, p = 0$. For what value of p is it most difficult to figure out the answer?

6. There are 4 jittery assassins with loaded guns sitting around a table. An alarm goes off. Simultaneously, each assassin picks one of the other assassins at random and shoots him/her. What is the probability that all of them die? Suppose that a second alarm goes off 10 seconds later, and that if there are two or more survivors, they repeat the process. What is the probability that all of them die? Suppose that a third alarm goes off 20 seconds later, and if there are two or more survivors, they repeat the process. What is the probability that all of them die? What is the answer if there is a fourth alarm?

7. Suppose there are 55 students in the class. I give you a random amount of time to take the final exam, as follows. I number you 1 to 55. I give each of you a fair coin. 1 minute after the exam starts, student 1 flips their coin. If it is tails, the student has to give me the coin. If it is heads, the student keeps the coin. I repeat this procedure: every minute, the lowest-numbered student who still has a coin flips it and gives it to me if it comes up tails. When the 55th student gives me their coin, the exam is over. What is the expected number of minutes you will have for the exam? You should pretend that it takes 0 time to flip a coin.

8. Same as the previous problem, except that each minute I pick a student at random from among those who still have a coin.

9. Same as the previous previous problem, except that 30 of you secretly bring a coin of your own to the exam, and I naively trust you when you say that you still have a coin.

10. Same as the previous^3 question, except that 20 of you substitute an unfair coin which comes up heads with probability .8 for the coin that I give you.

11. There are 75 students in your class, even though fire regulations only permit 71 people (one teacher and 70 students). The fire marshall is suspicious of ISyE. He pays a surprise visit to the class and counts the number of people. If each student comes to class with probability .95, independent of what other students do, what is the probability that the fire marshall will find us in violation of the fire code?