ISyE 4803 D – Spring 2014 – Midterm – Solutions

Name (please print!)	**ID (please print!)**

Dear participants:

1. *I expect just answers, although you can attach your derivations as well.*
2. *To the best of my understanding, the test is easy.*

Good luck!

Problem 1 [20 points]. Mark by "P" in the below list the sets which are polyhedral.

1. [4 points] $X = \{[x_1; x_2] \in \mathbf{R}^2 : x_1 + \max[2x_1, x_1 - x_2] \leq 3\}$
2. [4 points] $X = \{[x_1; x_2] \in \mathbf{R}^2 : x_1 + \min[2x_1, x_1 - x_2] \leq 3\}$
3. [4 points] $X = \{[x_1; x_2] \in \mathbf{R}^2 : x_1^2 - 2x_1x_2 + x_2^2 \leq 1\}$
4. [4 points] $X = \{[x_1; x_2] \in \mathbf{R}^2 : x_1^2 - 2x_1x_2 + x_2^2 \geq 1\}$
5. [4 points] $X = \{[x_1; x_2] \in \mathbf{R}^2 : \max[x_1, x_2] + \min[x_1, x_2] \leq 1\}$.

Solution: Polyhedral sets are #1, #3 and #5. Indeed,

- in #1, $X = \{[x_1; x_2] : x_1 + [2x_1] \leq 3, x_1 + [x_2 - x_2] \leq 3\}$;
- in #3, $X = \{[x_1; x_2] : -1 \leq x_1 - x_2 \leq 1\}$ (since $x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2$)
- in #5, $X = \{[x_1; x_2] : x_1 + x_2 \leq 1\}$ (since $\min[x_1, x_2] + \max[x_1, x_2] = x_1 + x_2$).

Set #2 is $\{[x_1; x_2] : x_1 + [2x_1] \leq 3$ **or** $x_1 + [x_1 - x_2] \leq 3\}$, that is, X is the union of two half-planes $\{[x_1; x_2] : x_1 \leq 1\}$ and $\{[x_1; x_2] : 2x_1 - x_2 \leq 3\}$, and this union is non-convex (draw a picture!) and as such is not polyhedral.

Set #4 is $\{[x_1; x_2] : |x_1 - x_x| \geq 1\}$, which is the complement to the stripe $-1 < x_1 - x_2 < 1$; this complement is nonconvex and as such is not polyhedral.

Problem 2 [20 points]. A polyhedral set X in $\mathbf{R}^4$ is given by $n = 2014$ pairs of inequalities as follows:

$$X = \{[x_1; x_2; x_3; x_4] : -1 \leq a_i x_1 + b_i x_2 + c_i x_3 + x_4 \leq 1,\ i = 1, 2, ..., 2014\}$$

We apply the Fourier-Motzkin elimination to describe by linear inequalities in variables x_1, x_2, x_3 the projection

$$Y = \{[x_1; x_2; x_3] : \exists x_4 : [x_1; x_2; x_3; x_4] \in X\}$$

of X onto the space of x_1, x_2, x_3. How many inequalities we get?
Answer: The number of inequalities is

Solution: Every one of the 2014 pairs of original inequalities imposes one upper and one lower bound, expressed in terms of x_1, x_2, x_3, on x_4. According to the description of Fourier-Motzkin elimination, the description of the projection will be given by $n^2 = 4,056,196$ inequalities, specifically,

$$-1 - a_i x_1 - b_i x_2 - c_i x_3 \leq 1 - a_j x_1 - b_j x_2 - c_j x_3,\ 1 \leq i, j \leq n = 2014.$$

We could also observe that when $i = j$, the above inequalities are automatically valid and can be dropped (although this goes beyond the Fourier-Motzkin elimination *per se*); I would not mind to get the answer $n^2 - n = 4,054,182$, although treating the question literally (and this is how a mathematical question should be treated!), this answer is incorrect.

Problem 3 [20 points]. Write down the problems to follow as Linear Optimization programs (if you find the standard techniques, based on polyhedral representation, unapplicable, mark the problem by "N/A"):

1. [10 points]

$$\min_{x \in \mathbf{R}^3} \{x_1 + \max[x_2 - x_3, 0] : \min[x_1, x_2] + \min[x_2, x_3] + \min[x_3, x_1] \geq 5\}$$

   Solution: LO reformulation is, e.g.,

$$\min_{x,u} \left\{ x_1 + u_1 : \begin{array}{l} x_2 - x_3 \leq u_1, 0 \leq u_1 \\ u_{12} + u_{23} + u_{31} \geq 5, \\ u_{12} \leq x_1, u_{12} \leq x_2; u_{23} \leq x_2, u_{23} \leq x_3; u_{31} \leq x_3, u_{31} \leq x_1 \end{array} \right\}$$

2. [10 points]

$$\min_{x \in \mathbf{R}^3} \{x_1 + \max[x_2 - x_3, 0] : \min[x_1, x_2/*] + \min[x_2, x_3] + \min[x_3, x_1] \leq 5\}$$

   Solution: N/A.

Problem 4 [20 points]. $A_1, A_2, ..., A_{2014}$ are convex sets in $\mathbf{R}^{214}$. Mark by "T" the statements in the following list which definitely are true:

1. [4 points] If every 214 of the sets from the collection have a point in common, all the 2014 sets have a point in common.

2. [4 points] If every 215 of the sets from the collection have a point in common, then all the 2014 sets have a point in common.

3. [4 points] If A_1 is a line segment and every 214 sets from the collection have a point in common, then all the 2014 sets have a point in common.

4. [4 points] If A_1 is a line segment and every 3 sets from the collection have a point in common, then all the 2014 sets have a point in common.

5. [4 points] If A_1 is a line segment and every 2 sets from the collection have a point in common, then all the 2014 sets have a point in common.

Solution: The definitely true statements are ## 2,3,4. The validity of # 2 is the Helley Theorem. The validity of # 4 (and thus – of # 3) is given by the following reasoning. Let ℓ be the line to which A_1 belongs, and let $\bar{A}_i = A_i \cap A_1$, $i = 2, 3, ..., 2014$. Consider the collection of convex sets $A_1, \bar{A}_2, ..., \bar{A}_{2014}$ and observe that every two of them have a point in common. Indeed, the sets in the pair of the form $A_1, \bar{A}_j$ have a point in common since the sets A_1, A_i are so (note that a common point of A_1 and A_i is common for A_1 and $\bar{A}_i$ as well). The sets in the pair $\bar{A}_i, \bar{A}_j$ have a point in common since the sets A_1, A_i, A_j have a point in common (such a point clearly belongs to $\bar{A}_i$ and $\bar{A}_j$). Since the sets $A_1, \bar{A}_2, \bar{A}_3, ..., \bar{A}_{2014}$ belong to a one-dimensional affine plane (namely, the line ℓ) and every two of the sets have a point in common, all the sets have a point in common by Helley Theorem; this common point clearly is common for $A_1, A_2, ..., A_{2014}$.

To see that # 1 is false in general, take, as the first 214 of A_i's the sets

$$A_i = \{x \in \mathbf{R}^{214} : x_i \leq 0\}, i = 1, ..., 214,$$

and set $A_{215} = \{x \in \mathbf{R}^{214} : \sum_{i=1}^{214} x_i \geq 1\}$. Then take every one of the remaining $2014 - 215$ sets of A_i each equal to one of the already built sets (e.g., all equal to A_1). Clearly, every 214 sets in this collection have a point in common (since either every 2014 sets in question are among $A_1, ..., A_{214}$, and the common point is 0, or one of the sets is A_{215}, and the remaining 213 sets are among $A_1, ..., A_{214}$; in this case, a common point is the basic orth e_i, where i is *not* among the indexes of those A_i's, $1 \leq i \leq 214$, which are present in the collection under consideration). However, all 2014 sets in the collection have no point in common (since already the intersection of the first 215 sets clearly is empty).

To see that # 5 can be false, take a triangle in $\mathbf{R}^{214}$, take A_1, A_2, A_3 to be the sides of this triangle and then set A_4, A_5,...,A_{2014} to be equal to A_3. In this collection, A_1 is segment, every two of A_i's have a point in common, but all the sets (and already the first three of them) have no common point.

Problem 5 [20 points].

1. [10 points] Fill the table:

Set in $\mathbf{R}^{10}$	Linear subspace [Y/N]	Affine subspace [Y/N]	Convex [Y/N]	Dimension is equal to
$\emptyset$	N	N	Y	N/A
$\{x : \sum_i x_i = 0\}$				
$\{x : \sum_i x_i = 1\}$				
$\{x : \max[1 - \sum_i x_i, \sum_i x_i] \le 1\}$				
$\{x : \min[3 - \sum_i x_i, \sum_i x_i] \le 1\}$				

Solution:

Set in $\mathbf{R}^{10}$	Linear subspace [Y/N]	Affine subspace [Y/N]	Convex [Y/N]	Dimension is equal to
$\emptyset$	N	N	Y	N/A
$\{: \sum_i x_i = 0\}$	Y	Y	Y	9
$\{x : \sum_i x_i = 1\}$	N	Y	Y	9
$\{x : \max[1 - \sum_i x_i, \sum_i x_i] \le 1\}$	N	N	Y	10
$\{x : \min[3 - \sum_i x_i, \sum_i x_i] \le 1\}$	N	N	N	10

Comment: the last set is $\{x : \sum_i x_i \le 1$ **or** $\sum_i x_i \ge 2\}$. This set is non-convex and thus non-polyhedral.

2. [5 points] Mark by "T" the correct relations below:

   (a) $[1; 1] \in \mathrm{Aff}(\{[0; 0], [-1; -1]\})$

   (b) $[1; -1] \in \mathrm{Aff}(\{[0; 0], [-1; -1]\})$

   (c) $\mathrm{Aff}(\{[0; 0], [-1; -1]\}) = \{[u; v] \in \mathbf{R}^2 : u = v\}$

Solution:

   (a) $[1; 1] \in \mathrm{Aff}(\{[0; 0], [-1; -1]\})$ **T**

   (b) $[1; -1] \in \mathrm{Aff}(\{[0; 0], [-1; -1]\})$

   (c) $\mathrm{Aff}(\{[0; 0], [-1; -1]\}) = \{[u; v] \in \mathbf{R}^2 : u = v\}$ **T**

3. [5 points] What is the linear subspace L parallel to the affine plane

$$M = \{x \in \mathbf{R}^4 : x_1 + x_3 = 1, x_2 + x_4 = 1\}$$

Answer:

$L =$

Solution: $L = \{x \in \mathbf{R}^4 : x_1 + x_3 = 0,\ x_2 + x_4 = 0\}$.