ISyE 2027C Probability with Applications
Homework "due" but NOT TO HAND IN and "NO QUIZ" 17 February 2015

Simple Calculation Problems

1. Roll a die repeatedly until you get a 3 or 6. What is the expected number of rolls? $1/\frac{1}{3} = 3$.

2. Roll two dice. What is the probability distribution of the sum of the rolls given that the first roll is greater than 4? *Let X be the sum of the rolls.* $P(X = 6) = \frac{1}{12} = P(X = 12)$. The other probabilities for the values 7 through 11 are $1/6$. What is the expected value of the sum given that the first roll is greater than 4? $5.5 + 3.5 = 9$.

3. Roll two dice repeatedly until you get either a 7, an 11, or doubles (the two die have the same number), in which case you stop. Conditioned on your stopping on your 99th roll, what is the probability that your **99**th roll was an 11? $1/7$ *There was a typo. The* **99** *in boldface was a* 9. *Before I changed the* 9 *to* 99 *the correct answer was* 0.

4. Let X be uniformly distributed on the interval $[-10, 10]$. Find $E[X], E[333X - 17]$,$E[X^2], E[\sqrt{X}]$. Use your intuition to find $E[X|X \geq 0]$ and $E[X|X \leq -8]$. 0, -17 (use linearity of expectation to save time), $\frac{100}{3}$, either undefined if you don't like imaginary numbers or $\frac{1}{2}(1+i)\frac{2}{3}10\sqrt{10}$. Intuition: 5, -9.

5. You order pizza for your family of 8. There are 16 possible toppings. You order a topping only if 0 or 1 family members don't like the topping. Each family member likes each topping with probability 0.9 independent of other toppings and people. What is the probability that you will order exactly 6 toppings? 13 toppings? 17 toppings? What is the expected number of toppings? *Define* $p = .9^8 + 8 \cdot .9^7 \cdot .1$. *That is the probability that a particular topping gets ordered. The answers are then*
$$\binom{16}{6}p^6(1-p)^{10}; \binom{16}{13}p^{13}(1-p)^3; 0; 16p.$$

6. You are given k dice with probability $1/k$ for $k = 2, 3, 6$. You roll the dice and receive the sum of the values in dollars. What is the expected number of dollars you will get? Conditioned on the event that you got \$6, what is the probability that you were given 2 dice? 10.5. Let S be the event that you got \$6 and let $K_j : j = 2, 3, 6$ be the event that you got exactly j dice. Then
$$P(K_2|S) = P(S|K2)P(K_2)/P(S) = \frac{5/72}{5/(2\cdot 6^2) + 10/(3\cdot 6^3) + 1/(6\cdot 6^6)}.$$

Word Problems

Example: You are diving for treasure in a lake. The probability that the lake has treasure is 0.4. Each time you dive, you have a .1 chance of finding treasure if treasure is there, independent of previous dives. You dive twice but find no treasure. What is the probability that the lake has treasure? Answer: let T be the event that there is treasure in the lake. Let F be the event that you find treasure in two dives. We seek $P(T|F^C)$. $P(F|T^C) = 0$ and $P(T) = 0.4$. $P(F^C|T) = 0.9^2 = 0.81$, hence $P(F|T) = 1 - .81 = .19$. Next we need $P(F)$. From the law of total probabilty,
$$P(F) = P(T)P(F|T) + P(T^C)P(F|T^C) = .4 \cdot .19 + 0 = .076 = 1 - P(F^C)$$
Then $P(T|F^C) = P(F^C|T)P(T)/P(F^C) = .81 \cdot .4/.924$. Another way to look at it: there are two ways you could fail to find treasure in two dives. The first way is because there is no treasure, with

probabilty .6. The second is that there is treasure but both dives fail, with probability $.4 \cdot .9^2$. The conditional probablity that there is treasure is the probability of the 2nd way divided by the sum of the probabilities, $.4 \cdot .9^2/(.6 + .4 \cdot .9^2)$.

You plan to dive until you find treasure. What is the probability that you never stop diving? .6 You plan to dive until you find treasure. Given that you did find treasure, what is your expected number of dives? 10

You plan to dive until you find treasure or you have completed 4 dives. What is your expected number of dives? Answer: Let N be the number of dives. Then by the law of total probability for expected values, $E[N] = P(T)E[N|T] + P(T^C)E[N|T^C]$. Suppose there is treasure. Then $P(N = 1|T) = 0.1$, $P(N = 2|T) = 0.9 \cdot 0.1$; $P(N = 3|T) = 0.9^2 \cdot 0.1$, $P(N = 4|T) = .9^3$ since you stop after 4 dives whether or not you found treasure. Then $E[N|T] = .1 + 2 \cdot .09 + 3 \cdot .081 + 4 \cdot .729$. Obviously $P(N = 4|T^C) = 1$. Combine these numbers with $P(T) = 1 - P(T^C) = .4$.

You plan to dive until you find treasure or you have completed 4 dives. Given that you find treasure, is your expected number of dives less than 2.5, equal to 2.5, or more than 2.5? Explain how you know. Answer: less. Your best chance to find treasure is your first dive, namely .1 given that there is treasure. Your second dive has a .09 chance, which is less. The expected value is a weighted average of 1,2,3,4 with highest weight on 1 and least weight on 4. The expected number is

$$\frac{1 \cdot .1 + .09 \cdot 2 + .081 \cdot 3 + .0729 \cdot 4}{.1 + .9 + .081 + .0729}$$

GENERAL GUIDELINES

STEP 1: Define the pertinent events and/or random variables.
STEP 2: State the goal of the problem in terms of the events and/or random variables.
STEP 3: State the given information in terms of probabilities involving those events and/or the distribution (pmf or pdf or cdf) of the random variables.
STEP 4: Solve the problem.
STEP 5: Check your answer and your reasoning. Does it make sense if you consider an extreme case? Can you see the answer with hindsight in a glance?

1. An eccentric billionaire hires you to dive for a treasure off the Florida coast. The probability that treasure is there is 0.4. Each day that you dive, the probability is 0.05 that you will find the treasure if it is there, independent of previous dives. (This would not a realistic assumption if your diving were methodically planned, but the billionaire selects the diving location randomly each day.) You are paid \$500 per day for diving. If you find the treasure, the search terminates and you will get a \$50,000 bonus. There is one additional complication. On each day, if the billionaire does not hear the cry of a seagull and does not see a porpoise she takes it as a bad omen, flips a (fair) coin, and terminates the search if it comes up tails. . The probability of hearing a seagull on any day is 0.9. The probability of seeing a porpoise on any day is 0.25, independent of whether a seagull is heard. *Disambiguation: I interpret termination to mean that if there is diving on day i, there is a chance that the billionaire will terminate the search at the end of day i. In other words, the billionaire listens for seagulls and watches for porpoises throughout the day.*

   Let T be the event that there is treasure. $P(T) = .4$. Let D_i be the event that you dive on day $i : i = 1, 2, 3, \ldots$. Let F_i be the event that you find treasure on day i. $P(F_i|D_i \cap T) = .05$. Obviously $P(F_i|T^C) = 0$. Let X be the number of days you will dive. It probably won't be useful, but you should see that if you define X_i to equal 1 if D_i occurs and 0 if D_i^C occurs, then $X = \sum_{i=1}^{\infty} X_i$ and $E[X] = \sum_{i=1}^{\infty} P(D_i)$.

   Since we never are asked to distinguish among the coin coming up heads, the seagull being heard, and the dolphin being seen, we can do one calculation and use it as a black box afterwards. Let

$S1_i$ be the event of hearing a seagull cry on day i and $S2_i$ be the event of seeing a porpoise on day i. Let $S3_i$ be the event that the billionaire decides on day i to terminate the search. Then $P(S3_i|D_i \cap F_i^C) = P(S1_i^C|D_i \cap F_i^C)P(S2_i^C|D_i \cap F_i^C)\frac{1}{2} = .1 \cdot .75 \cdot .5 = .0375$. In words, if you dive on day i but don't find treasure, there is a 3.75% chance that the billionaire will call off the search at the end of the day.

What is the expected number of days you will dive?
$E[X] = .4E[X|T]+.6E[X|T^C]$. If there is no treasure, on each day of diving there is a 3.75% chance of termination. Remember the mean of a geometric distribution. Hence $E[X|T^C] = \frac{1}{.0375} = 80/3$. If there is treasure, the probability of termination is $P(F_i|D_i \cap T)+P(F_i^C|D_i \cap T).0375 = .05+.95 \cdot .0375 \equiv \beta$. Another way to compute β is with the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Apply this formula to $P(F_i \cup S3_i|D_i) = .05 + 0375 - (.05)(.0375) = \beta$. The answer is therefore $.4/\beta + 16$.

What is the expected amount of money you will earn?
Let $Y =$ the number of dollars you earn. Let $Z =$ the number of dollars you earn as a bonus. Then $Y = 500X+Z$. By linearity – I've told you that linearity is useful – $E[Y] = 500E[X]+E[Z]$. $E[X]$ is a value we've already found. $E[Z] = 50000P$(you find treasure). I think this equation obvious but if not, define F to be the event that you find treasure. Then $E[Z] = P(F)E[Z|F]+P(F^C)E[Z|F^C]$ $= 50000P(F) + 0P(F^C) = 50000P(F)$. On a test you can just write down the equation unless the test gives you the equation and asks you to explain it.

Back to the problem at hand, we have the answer except for $P(F)$. By the law of total probability, $P(F) = .4P(F|T) + .6P(F|T^C) = .4P(F|T)$. Of course $F = \cup_{i=1}^{\infty} F_i$ and since you can't find treasure twice the F_i are disjoint and $P(F) = \sum_{i=1}^{\infty} P(F_i)$ and $P(F|T) = \sum_{i=1}^{\infty} P(F_i|T)$.

So, to start off, let's figure out $P(F_1|T)$ and $P(F_2|T)$. Remember the chance of finding treasure, if treasure is there, is .05. So $P(F_1|T) = .05$. But $P(F_1|T)$ is smaller than .05 for two reasons. First, if F_1 happens, you don't dive on day 2. Second, even if F_1 happens, the bad luck seagull-porpoise-coinflip event could happen, preventing you from doing more diving. We already calculated that $P(D_2|F_1^C) = 1 - .0375 = .9625$. Calculate for F_2: $P(F_2) = P(F_1^C)P(F_2|F_1^C) = .95 \cdot .9625 \cdot .05 = .95 \cdot .048125$.

You can see this intuitively. The chance of treasure on day 2 is the chance of no treasure day 1, .95 times the chance the billionaire doesn't superstitiously terminate the dive, .9625, times .05 which is the chance of finding treasure on day 2 given that you dive then, all conditioned on the event that there is treasure. If at ths point you don't see the pattern you could calculate $P(F_3)$. But you ought to see that, if you dive on day i, day i behaves the same, probabilistically, as some other day j given that you dive on day j. You can think of this as a kind of symmetry. In probability we call this stationarity because the probabilities don't move around – they are stationary – through time.

Back to the problem, $P(F_3|T) = P(F_1^C \cap F_2^C|T)P(F_3|(F_1^C \cap F_2^C|T)$ because $F_3 = F_3 \cap F_1^C \cap F_2^C$. Therefore

$$P(F_3|T) = .95 \cdot .9625 \cdot .95 \cdot .9625 \cdot .05$$

In words, to find treasure on day 3, the following things have to happen, in order: unsuccessful dive day 1; seagull or porpoise or heads evening 1; unsuccessful dive day 2; seagull or porpoise or heads evening 2; successful dive day 3.

By now, you definitely should see the pattern. $P(F_i|T) = .95^{i-1}.9625^{i-1}.05$. Can you see this in a glance? The probability of finding treasure given T is therefore the sum of a geometric series with first term .05 and multiplication term $.95 \cdot .9625$, which equals $\frac{.05}{1-(.95 \cdot .9625)}$. Thus

$$P(F) = .4\frac{.05}{1-(.95 \cdot .9625)}.$$

Extra credit: why would it be an accidental event if the billionaire superstitiously terminates the diving? Answer: because she wouldn't do it on porpoise.

What is the probability that you will find treasure?
We had to figure this out already. One way to see the answer at a glance is this. $P(F|T) = .05 + .95 \cdot .9625 \cdot P(F|T)$. And of course $P(F) = .4P(F|T)$.

Given that the search terminated after exactly 3 days, what is the probability that you found treasure?
Let $M3$ denote the event $D_3 \cap D_4^C$, the event that diving terminated after exactly 3 days. As above F denotes the event of finding treasure. We seek $P(F|M3)$. This seems tricky because knowing that $M3$ occurs affects the chance that there is treasure.

There are three ways $M3$ could occur.

(a) Treasure is found on day 3. The probability is $.4 \cdot .95^2 \cdot .9625^2 \cdot .05 \equiv \alpha_1$.

(b) There is no treasure. The billionaire's superstition is triggered on day 3 (but not on days 1 or 2). The probability is $.6 \cdot .9625^2 \cdot .0375 \equiv \alpha_2$.

(c) There is treasure, it does not get found, and the billionaire's superstition is triggered on day 3 (but not on days 1 or 2). The probability is $.4 \cdot .95^3 \cdot .9625^2 \cdot .0375 \equiv \alpha_3$.

Use the formula $P(F|M3) = P(F \cap M3)¶(M3) = \frac{alpha_1}{\alpha_1+\alpha_2+\alpha_3}$.
Given that you have dived without finding treasure for 10 days, what is the expected amount of addtional money you will earn?
First, be sure you see why this is not simply $.9625E[Y]$ from above, even though there is stationarity. The reason is that not finding treasure for 10 days affects the probability that there is treasure when you start diving on day 11.

Let $N10$ be the event that you dive unsuccessfully for treasure for 10 days. Let Y' be the amount earned starting on day 11. We want to know $P(T|N10)$. That will give us $P(T^C|N10)$. Then we use those two numbers as weights for the values $E[Y'|T]$ and $E[Y'T^C]$, times .9625.

$P(T|N10) = P(N10|T)P(T)/P(N10)$.
$P(N10|T) = (.95)^{10}(.9625)^9 \equiv \gamma_1$.
$P(N10|T^C) = (.9625)^9 \equiv \gamma_2$.
$P(N10) = .4\gamma_1 + .6\gamma_2 \equiv \gamma_3$ by the law of total probability. $P(T|N10) = .4\gamma_1/\gamma_3$.
$P(T^C|N10) = 1 - P(T|N10)$.

If there is no treasure and you start diving your expected earnings are 500 times the expected number of days you dive. Given $N10 \cap T^C$, the chance of diving on day 11 is .9625. Each day you dive there is a .0375 chance that superstition will make that dive your last. The expected number of dives is therefore $.9625/.0375 \equiv \gamma_4$. This implies $E[Y'|T^C \cap N10] = 500\gamma_4$.

If there is treasure and you dive on day 11 your expected number of days diving is $1/(.05 + .95 \cdot .0375) \equiv \gamma_5$ because on any day you dive, if there is treasure, the chance of stopping is

$.05 + .95 \cdot .0375 \equiv \gamma_6$. The chance that you find treasure if you do continue diving on day 11 is $P(F_i|D_i \cap D_{i+1}^C \cap T) = \frac{.05}{\gamma_6}$.

Therefore your total expected earnings are

$$P(T|N10) \cdot .9675 \cdot (500\gamma_5 + 50000 \cdot .05/\gamma_6 + P(T^C|N10)\gamma_4.$$

Given that you have dived without finding treasure for 10 days, what is the expected number of additional days you will dive?

We had to figure this out to answer the previous qeustion. The answer is

$$P(T|N10) \cdot .9675 \cdot \gamma_5 + P(T^C|N10)\gamma_4.$$

Given that you have dived without finding treasure for 10 days, what is the expected amount of average daily earnings in the future? Why can or can't you answer this question by taking the ratio of the answers to the previous two questions?

You can't because the question asks you to weigh each future scenario by the probability it occurs, but the latter increases the weight on scenarios that last fewer days. If you don't see this, suppose the probability is .5 that you make \$10 in 1 day, and the probability is .5 that you make \$100 in 5 days. The expected daily earning is \$15 but total expected earnings divided by total expected days equals $55/3 > 15$.

2. A zombie is shuffling back and forth on a sidewalk. Each minute it moves one step east with probability .7 or one step west with complementary probability .3. Write an equation that, if solved, finds the probability that the zombie will ever reach the place one step west of its starting place. Let z be the value in question.

$$z = .3 + .7(\text{probability ever reaching 2 steps west of the place the zombie is now} = .3 + .7z^2$$

. Why? Let W be the event that the zombie's first move is to the west. Let A be the event that the zombie ever reaches the space one west of its starting space. So $P(W) = .3$ and $P(W^C)$ obviously is .7. Then $z = P(A) = P(A|W)P(W) + P(A|W^C)P(W^C)$. Obviously $P(A|W) = 1$, and by the fact that the zombie's future moves are not influenced by its previous moves (i.e. zombies have no memory), $P(A|W^C) = z^2$.

3. A zombie is shuffling back and forth on a sidewalk. Each minute it moves **two** steps east with probability .4 or **one** step west with complementary probability .6. Write an equation that, if solved, finds the probability that the zombie will ever reach the place one step west of its starting place. Let z be the probability in question.

$$z = .6 + .4z^3$$

4. You flip a fair coin repeatedly. Write an equation that, if solved, finds the probability that the number of tails will ever be more than three times the number of heads. *This is the same as a zombie moving 3 steps east w.p. .5 for heads and 1 step west w.p. .5 for tails in a problem like the previous one.* $z = .5 + .5z^4$.

5. Repeat the previous problem if the coin comes up heads with probability p for the values $p = 1, p = .7, p = .3, p = 0$. For what value of p is it most difficult to figure out the answer? $z = 0; z = .3 + .7z^4; z = .7 + .3z^4; z = 1$. At $p = .25$ it seems unclear what the answer is, but the equation $z = .75 + .25z^4$ is obviously satisfied at $z = 1$. So we should think that at $p = .25 + \epsilon$ the computation is difficult for small $\epsilon > 0$.

6. Four jittery assassins with loaded guns sit around a table. An alarm goes off. Simultaneously, each assassin picks one of the other assassins at random and shoots him/her. What is the probability that all of them die? $\frac{1}{9}$. *Hint: label them a,b,c,d. By symmetry you can assume that a shoots b. That reduces the number of possibilities from* 3^4 *to* 3^3.

   Suppose that a second alarm goes off 10 seconds later, and that if there are two or more survivors, they repeat the process. What is the probability that all of them die?

   The a priori *probability that 3 are left after the 1st alarm is 0. For 2 to be left, there are* $\binom{4}{2} = 6$ *possibly pairs of survivors. These 6 cases are symmetric, so analyze one case and multiply by 6. Suppose a and b survive. Then c must shoot d and d must shoot c. a and b each can shoot either c or d, giving 4 ways this can happen. So there are 24 ways out of 81 for 2 to be left, for a probability of 8/27. (We can't use the a shoots b assumption to take 81 possibilities down to 27 at the same time we use the a and b survive assumption, because those are not independent assumptions. In fact they can't both happen.) We already know that the probability is 3/27 that 0 are left. Hence the probability is* $16/27$ *that 1 is left. You can check this directly to be sure: 4 choices of survivor, 6 ways the other 3 kill each other and the survivor shoots any of those 3, 6 ways the survivor shoots someone no one else shoots, giving* $4(6 + 6)/81$.

   Therefore, after the 2nd alarm, the probability is $3/27 + 8/27 = 11/27$. *I am of course assuming that a jittery assassin would not shoot him/herself. The next two questions have the same answer,* $11/27$.

   Suppose that a third alarm goes off 20 seconds later, and if there are two or more survivors, they repeat the process. What is the probability that all of them die? What is the answer if there is a fourth alarm?

7. Suppose there are 55 students in the class. I give you a random amount of time to take the final exam, as follows. I number you 1 to 55. I give each of you a fair coin. 1 minute after the exam starts, student 1 flips their coin. If it is tails, the student has to give me the coin. If it is heads, the student keeps the coin. I repeat this procedure: every minute, the lowest-numbered student who still has a coin flips it and gives it to me if it comes up tails. When the 55th student gives me their coin, the exam is over. What is the expected number of minutes you will have for the exam? You should pretend that it takes 0 time to flip a coin. $55\frac{1}{1/2} = 110$ minutes.

8. Same as the previous problem, except that each minute I pick a student at random from among those who still have a coin. *same answer*

9. Same as the previous previous problem, except that 30 of you secretly bring a coin of your own to the exam, and I naively trust you when you say that you still have a coin.$85 \cdot 2 = 170$ minutes.

10. Same as the previous^3 question, except that 20 of you substitute an unfair coin which comes up heads with probability .8 for the coin that I give you.$35 \cdot 2 + 20 \cdot 5$.

11. There are 75 students in your class, even though fire regulations only permit 71 people (one teacher and 70 students). The fire marshall is suspicious of ISyE. He pays a surprise visit to the class and counts the number of people. If each student comes to class with probability .95, independent of

what other students do, what is the probability that the fire marshall will find us in violation of the fire code?

$$\sum_{j=0}^{4} \binom{75}{j} (.05)^j (.95)75 - j$$