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On Illustrating Carnot's General Proposition by Means of Reversible Stirling Engines

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Abstract
Carnot's general proposition, also referred to as one of Carnot's principles, states that the work producing potential of heat—harvested by reversible heat engines—is *independent* on the working fluid and on engine internal details, being only a function of the temperatures of the reservoirs with which the engine exchanges heat. This concept, usually presented to ME students in the context of the second law of thermodynamics, is usually proven by contradiction, using second law concepts and abstractions, without concrete examples, even though Carnot's proposition mentions concrete things such as working fluids and engine internal details. This work proposes to document the usage of reversible Stirling engine models that take the engine arrangement and fluid properties into account towards illustrating the validity of Carnot's general proposition.

Keywords
Carnot principles — Carnot general proposition — Second law of Thermodynamics — Reversible Stirling engine models — Thermal efficiency

Highlights
Revisits Carnot's 'fundamental' and 'general' propositions — Identifies a link between Carnot's general proposition and the later Kelvin-Planck statement of the second law of Thermodynamics — Draws insights and points out connections between published models and thermodynamics theory.

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Contents

1. Introduction

The subject of the second law of thermodynamics figures amidst the most philosophical teaching topics in mechanical engineering. Many studies have been produced around the theme of pedagogical improvements in presenting the second law of thermodynamics, under a variety of strategies [12, 8, 2, 5, 10, 1].

One of the key concepts covered in textbook presentations of the second law are the so-called Carnot principles [17] or corollaries [11], or claim [3, p. 88], or even deductions than make up one principle [7]—different sources attribute different names and list a varying number of what Carnot has mostly called *propositions* [15], without labeling them, except for the ones that he named "fundamental" and "general".

Carnot's fundamental proposition is a statement of a necessary condition for heat engines to attain "maximum" work production from heat [15, p. 56]:

> *"[...] that in the bodies employed to realize the motive power of heat there should not occur any change of temperature which may not be due to a change of volume."*

Recall his work appears many years before the second law of thermodynamics had been established, and the properties we now use, such as entropy, had been invented. In modern terms, Carnot's fundamental proposition can be paraphrased as "the temperature of the working fluid can only change during isentropic changes in volume", recalling that both friction and heat transfer irreversibilities would cause the fluid to experience temperature changes that could not be explained by adiabatic and reversible change in volume.

Carnot's general proposition, on the other hand, is the generalization of his fundamental proposition, and states that [15, p. 68]:

> *"The motive power of heat is independent of the agents employed to realize it; its quantity is fixed solely by the temperatures of the bodies between, which is effected, finally, the transfer of the caloric."*

Meaning, in current thermodynamic terms, that the maximum work producing potential of heat is independent of the working fluid substance employed in its production, being determined only by the temperatures[1] in which heat is being transferred. Modern texts make it explicit that such work producing potential is also independent on the engine's internal details [17], or, equivalently, sequence of logical [13] processes [11] of the reversible machines.

Carnot's general proposition is typically presented to mechanical engineering students in the context of the second law of thermodynamics in the form of a statement followed by a proof by contradiction [17, 11, 7], using second law concepts and abstractions, i.e., by proposing a violation of the general proposition and showing in the conceptual realm that it leads to a violation of a statement of the second law of thermodynamics—a method that is both sufficient and general from a mathematical viewpoint.

Despite the mathematical fitness of this approach, it lacks concrete examples. In one hand, one may argue that lack of concrete examples may pose unnecessary extra difficulties on learners; on the other hand, the proposition's phrasing seems to beg for concrete examples, as it mentions concrete things such as working fluids and engine internal details.

Keen and attentive learners may correctly wonder that both heat intake and net work output done by reversible models of different real engines may be written as complicated functions of the engines' internal details, working fluid properties, and reservoir temperatures—and wonder how all these things may fit together under the restrictions of Carnot's general proposition.

[1] Note the plural which embeds what would later on make up the Kelvin-Planck statement of the second law.

This work focuses in documenting one such analysis, and thus becoming a reference that can be presented opportunely—not in place of, but alongside the usual proof by contradiction—to inquiring learners, or whenever an instructor see to be fitting. Moreover, it is hoped that the concrete aspects of Carnot's general proposition are met, at least in the few instances provided herein.

Well-known engineering textbook reversible cycles include (i) the Carnot, (ii) the Stirling, and (iii) the Erickson ones [17]. The Stirling cycle is of peculiar interest for the task due to (a) the present time interest displayed towards it; (b) the fact that many reversible models are available in the literature; and (c) the existence of three different such engine configuration types: the α-, β-, and γ-ones—so that variations on engine internal details and operating parameters and on working fluid can be performed in the context of illustrating Carnot's general proposition.

Historically, reversible Stirling engine cycles have been proposed by Schmidt [14], Finkelstein [6], Walker [16] and Kirkley [9]. The focus of early works has been investigating optimal engine construction and operating parameters.

More recently, Cheng and Yang [4] developed a reversible, dimensionless, and parametric Stirling engine model that accounts not only for working fluid and for engine operating parameter variations, but also for engine configuration alteration, i.e., among the α-, the β-, and the γ-types. Moreover, they provide exact analytical expressions for net work output and for heat inlet that are written in terms of the various model parameters and engine internal configurations.

Following reasons (b) and (c) given above, the existing work of Cheng and Yang [4] is deemed ideally suited for the purpose of illustrating Carnot's general proposition "in action" based on a concrete example—of reversible Stirling engines of various mechanical arrangements. Therefore, this work expands knowledge by the *connections that are made* and the *insights that are registered* in association to the aimed illustration, also in its *production* and *documentation*, rather than developing another Stirling engine model or one that is beyond Cheng and Yang's work.

2. Reversible Stirling Engine Model

In this section, Cheng and Yang's reversible Stirling engine model [4] is presented. Only a concise subset of the model that is useful for the goal of the present work is being presented.

Moreover, the model nomenclature is kept the same as the one used in the original reference [4], so as to allow readers to easily lookup in the original work the parts herein abridged. There is only a minute change in the employed nomenclature for conciseness, which consists in annotating the Greek letter of the engine configuration above a symbol, so as to specify that symbol's expression for that particular engine configuration, starting on Eq. (1).

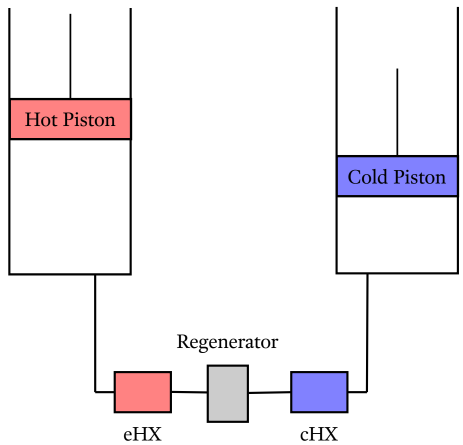

Figure 1. Schematic representation of an ideal Stirling engine of the α-type. The 'eHX' and the 'cHX' are the 'expansion' and 'compression' heat exchangers, respectively of high- and low- temperatures. The power pistons and the regenerator are indicated. Volumes are not to scale.

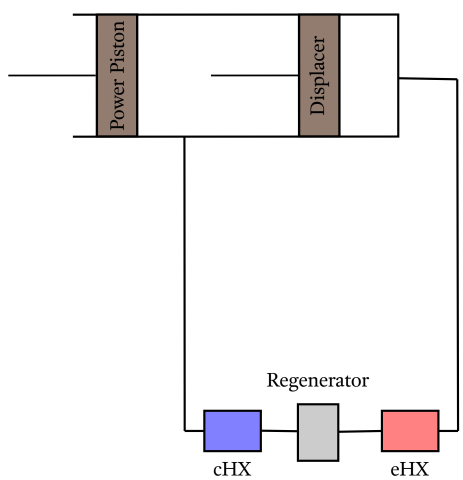

Figure 2. Schematic representation of an ideal Stirling engine of the β-type. The 'eHX' and the 'cHX' are the 'expansion' and 'compression' heat exchangers, respectively of high- and low- temperatures. The power and displacement pistons and the regenerator are indicated. Volumes are not to scale.

Let V_{cs} and V_{es} be piston and displacer sweep volumes in the operation of a reversible Stirling engine of either α-, β-, or γ-type—in Stirling engine nomenclature, it is common to name high- and low- temperature spaces and adjacent moving parts as 'expansion', and 'compression' ones, respectively, see Figures 1–3 that illustrate the 'expansion', and 'compression' heat exchangers; hence the 'c' and 'e' subscripts.

Let further V_d be the so-called engine "dead volume", which is composed by piston and displacer minimum clearances and regenerator volumes, i.e., $V_d = V_{emin} + V_{cmin} + V_r$, respectively.

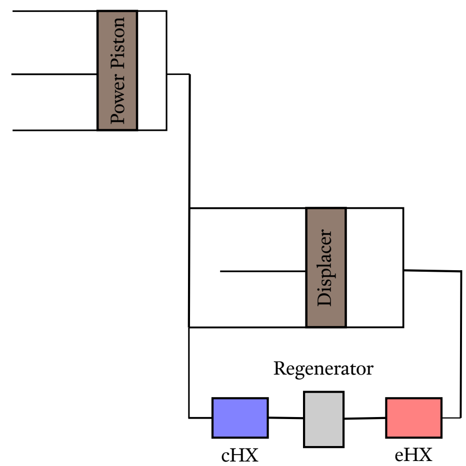

Figure 3. Schematic representation of an ideal Stirling engine of the γ-type. The 'eHX' and the 'cHX' are the 'expansion' and 'compression' heat exchangers, respectively of high- and low- temperatures. The power and displacement pistons and the regenerator are indicated. Volumes are not to scale.

Let ϕ be the engine's crankshaft angle, and α be the expansion-to-compression piston-piston (for the α-type) or piston-displacer (for the β- and γ-types) phase angle. For all engine configurations the expansion volume, $V_e(\phi)$, is:

$$\begin{aligned} V_e(\phi) &= V_e^{\alpha}(\phi) = V_e^{\beta}(\phi) = V_e^{\gamma}(\phi) \\ &= V_{emin} + \frac{1}{2} V_{es}(1 + \sin(\phi + \alpha)). \end{aligned} \tag{1}$$

The compression volume, $V_c(\phi)$, is configuration dependent:

$$V_c^\alpha(\phi) = V_{cmin} + \frac{1}{2}V_{es}\kappa(1+\sin(\phi)), \tag{2}$$

$$V_c^\beta(\phi) = V_{cmin} + \frac{1}{2}V_{es}(\sqrt{1-2\kappa\cos\alpha+\kappa^2} + \kappa\sin\phi - \sin(\phi+\alpha)), \tag{3}$$

$$V_c^\gamma(\phi) = V_{cmin} + \frac{1}{2}V_{es}(1+\kappa+\kappa\sin\phi - \sin(\phi+\alpha)), \tag{4}$$

where $\kappa \equiv V_{cs}/V_{es}$ is the engine sweep volume ratio.

It is worth noting that each engine has its very own distinct volume dynamics, making them suitable to verifying Carnot's general proposition from the point of view of "different internal details".

Let the total engine volume—the instantaneous volume occupied by the engine's working fluid,

$$V(\phi) = V_e(\phi) + V_c(\phi) + V_r, \tag{5}$$

be written in terms of a configuration dependent dimensionless function most generally written as $\Phi(\phi|\chi,\kappa,\alpha)$, i.e., in terms of parameters χ, κ and α, so that

$$V(\phi) = V_{es}\Phi(\phi|\chi,\kappa,\alpha), \tag{6}$$

with:

$$\Phi^\alpha(\phi|\chi,\kappa,\alpha) = \chi + \frac{1}{2}(1+\kappa) + \frac{1}{2}(\sin(\phi+\alpha)+\kappa\sin\phi), \tag{7}$$

$$\Phi^\beta(\phi|\chi,\kappa,\alpha) = \chi + \frac{1}{2}\kappa\sin\phi + \frac{1}{2}(1 + \sqrt{1-2\kappa\cos\alpha+\kappa^2}), \tag{8}$$

$$\Phi^\gamma(\phi|\chi,\kappa) = \chi + \frac{1}{2}(2+\kappa) + \frac{1}{2}\kappa\sin\phi. \tag{9}$$

where $\chi \equiv V_d/V_{es}$ is the engine dead volume ratio. It is worth noting that for the γ-type engine, parameter α is absent from the Φ function, thus showing that different engine configuration may also lead to different mathematical "signature" of intermediate functions.

The working fluid pressure, p, is assumed to be uniform throughout the engine cavity—as pressure-drops introduce irreversibilities that would prevent one from verifying Carnot's general proposition in such models. Assuming, for the sake of simplicity, while still allowing for different fluids to be used, ideal gas p-V-T behavior of the working fluid, one has:

$$p = \frac{mR}{\frac{V_e}{T_e}+\frac{V_c}{T_c}+\frac{V_d}{T_d}} = \frac{mRT_e}{V_{es}}\Psi(\phi|\alpha,\kappa,\tau,\chi), \tag{10}$$

where T_e, T_c, and $T_d \equiv (T_e+T_c)/2$ are the expansion, compression, and dead space temperatures, respectively, with $T_e > T_c$ for heat engine operation; and $\Psi(\phi|\alpha,\kappa,\tau,\chi)$ is the dimensionless engine pressure function of ϕ with parameters α,κ,τ,χ, in which $\tau \equiv T_c/T_e$ is the compression-to-expansion reservoir temperature ratio. It is worth noting that if Carnot's general proposition is to hold, the thermal efficiency of the exemplified Stirling engines must be a function of τ only.

The dimensionless pressure function is a multi-term, engine configuration dependent expression:

$$\frac{1}{\Psi^\alpha(\phi)} = \frac{2\chi}{1+\tau} + \frac{\kappa+\tau}{2\tau} + \frac{\kappa\sin\phi}{2\tau} + \frac{\sin(\phi+\alpha)}{2}, \tag{11}$$

$$\frac{1}{\Psi^\beta(\phi)} = \frac{2\chi}{1+\tau} + \frac{\tau+\sqrt{1-2\kappa\cos\alpha+\kappa^2}}{2\tau} + \frac{\kappa\sin\phi}{2\tau} + \frac{(\tau-1)\sin(\phi+\alpha)}{2\tau}, \tag{12}$$

$$\frac{1}{\Psi^\gamma(\phi)} = \frac{2\chi}{1+\tau} + \frac{1+\tau+\kappa}{2\tau} + \frac{\kappa\sin\phi}{2\tau} + \frac{(\tau-1)\sin(\phi+\alpha)}{2\tau}. \tag{13}$$

The RT_e-normalized (i) specific net cycle work, $\bar{W}$, and (ii) specific inlet cycle heat, $\bar{Q}_{in}$, are dimensionless quantities that can be expressed as:

$$\bar{W} = \frac{\oint p\,\mathrm{d}(V/m)}{RT_e} = \int_0^{2\pi}\Psi(\phi)\frac{\partial\Phi(\phi|\chi,\kappa,\alpha)}{\partial\phi}\,\mathrm{d}\phi, \tag{14}$$

$$\bar{Q}_{in} = \frac{\oint p\,\mathrm{d}(V_e/m)}{RT_e} = \frac{1}{2}\int_0^{2\pi}\Psi(\phi)\cos(\phi+\alpha)\,\mathrm{d}\phi, \tag{15}$$

recalling that Φ has a different mathematical signature for γ-type engines, with respect to the α- and β-types, Eq. (7)–(9).

Equations (14) and (15) embody the argument made on the Introduction section of this work, meaning that both cycle net work and cycle inlet heat expressions are *complicated* and *different* functions of the engine's internal details—whether in terms of α-, β-, or of γ-type configurations, but also of construction/tuning parameters such as α, κ, and χ—and of the thermal reservoir temperatures, τ.

3. Illustration of Carnot's General Proposition

Solutions to the integrals of Eqs. (14) and (15) are given by Cheng and Yang [4], which applied the residual theorem. The resulting expressions are:

$$\bar{W} = \frac{2\pi\tau\kappa(1-\tau)\sin\alpha}{a^2+b^2}\cdot\Lambda, \tag{16}$$

$$\bar{Q}_{in} = \frac{2\pi\tau\kappa\sin\alpha}{a^2+b^2}\cdot\Lambda, \tag{17}$$

with

$$\Lambda = \left(\frac{\beta-\sqrt{\beta^2-(a^2+b^2)}}{\sqrt{\beta^2-(a^2+b^2)}}\right), \tag{18}$$

where a, b, and β are configuration dependent expressions, given as:

$$a^\alpha = \tau\sin\alpha, \tag{19}$$

$$a^\beta = -(1-\tau)\sin\alpha, \tag{20}$$

$$a^\gamma = -(1-\tau)\sin\alpha; \tag{21}$$

$$b^\alpha = \kappa+\tau\cos\alpha, \tag{22}$$

$$b^\beta = \kappa-(1-\tau)\cos\alpha, \tag{23}$$

$$b^\gamma = \kappa-(1-\tau)\cos\alpha; \tag{24}$$

$$\beta^\alpha = 4\tau\chi/(1+\tau)+\tau+\kappa, \tag{25}$$

$$\beta^\beta = 4\tau\chi/(1+\tau)+\tau + \sqrt{1-2\kappa\cos\alpha+\kappa^2}, \tag{26}$$

$$\beta^\gamma = 4\tau\chi/(1+\tau)+1+\tau+\kappa. \tag{27}$$

Although the cycle net work, $\bar{W}$, and inlet heat, $\bar{Q}_{in}$, are indeed *complicated* functions of the engine's configuration, internal details, and of the reservoir temperatures, as shown by Eqs. (16) and (17), one can realize, by mere inspection on these same equations, that the thermal efficiencies, $\eta_t \equiv \bar{W}/\bar{Q}_{in}$, of all these reversible engines:

1. Are indeed *the same*: $\eta_t^\alpha = \eta_t^\beta = \eta_t^\gamma$;
2. Are *equal to any reversible engine's* thermal efficiency, $\eta_{t,rev} = 1-\tau$, operating between the same temperature reservoirs; and
3. Are indeed a function of *only* the reservoirs' temperatures: $\eta_t : \eta_t(\tau)$.

Therefore, the reversible α-, β-, and γ-types Stirling engine models of Cheng and Yang [4], are shown to follow Carnot's general proposition, according to the second law of thermodynamics.

The worked out illustration serves thus as an instance of defined reversible engines with different internal details and working fluids of possibly different properties, whose thermal efficiencies are restricted by Carnot's general proposition, despite the fact that their net work and inlet heat are different and complicated functions of the various working parameters and engine configuration.

4. Conclusions

This work has illustrated the validity of Carnot's general proposition [15] by using a parametric, reversible Stirling engine model from the literature [4] that takes into account engine geometry and working fluid properties. Moreover, the analytic nature of the model, with its exact solution, helps in keeping the illustration concise and exact, and generic up to the model's modeling space.

In one hand, the example illustrated herein is less generic and less conceptual (i.e., more instantiated) than the canonical proof by contradiction using only general concepts, usually presented in thermodynamics courses and textbooks [17, 11, 7], and therefore, evidently does not serve as a general replacement of the proof by contradiction.

On the other hand, the illustration based on a reversible model of a *real* engine is more defined and hopefully more palpable and interesting to inquiring students. One can argue that at least the illustration complements the proof by contradiction and may boost interest by the more concrete approach.

Conflict of Interest

The authors declare that there is no conflict of interest in this work.

CRediT Author Statement

CN: Conceptualization, Methodology, Writing - Original Draft, Writing - Review & Editing, Project administration. **KAEB:** Methodology, Investigation, Writing - Original Draft (figures), Writing - Review & Editing. **JMSL:** Methodology, Resources, Writing - Review & Editing, Supervision, Funding acquisition.

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To YHWH God be the glory!

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